INVESTIGATING SECONDARY SCHOOL STUDENTS' EPISTEMOLOGIES THROUGH A CLASS ACTIVITY CONCERNING INFINITY

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In this paper, we report findings from a pilot study investigating school students' epistemologies of mathematics by using novel mathematics definitions. Students aged 17 and 18-year-old in Italy and the UK were asked to complete a worksheet that used a numerical approach to determine the sizes of infinite sets and were, then, invited to attend focus group interviews about their experience with the material. Thematic analysis of the interviews reveals that this approach is useful to distinguish between naïve and advanced epistemologies and using unseen mathematical definitions can help enrich our understanding of epistemologies held by students of school age.

BACKGROUND TO THE STUDY

Students' beliefs about mathematics have often been connected to their engagement with the subject (Muis, 2004), their behaviour as problem solvers (Schoenfeld, 1989; Muis et al. 2015) and their self-regulation strategies (Muis, 2007). However, understanding what these beliefs are and how to best measure them has generated a lively methodological debate in the epistemological beliefs literature (see for example Limon, 2006). Many (e.g. Muis et al. 2014) find the most common questionnaires used so far, and in general quantitative methods alone, to be unsuitable for such investigations. Criticisms to the use of large scale surveys include the inability to ascertain that there is a shared meaning of key words between the researchers designing the surveys and the students filling them in (Muis et al. 2014), and doubts have been recently raised that large scale questionnaires cannot be used across diverse cultural contexts (Mogashana et al. 2012). In this pilot study, we tested a qualitative methodology for the investigation of school students' epistemological beliefs. We hypothesised that, by documenting the reactions of secondary school students when asked to work with a definition of infinity (a concept that they would have encountered at this point in their education) very different from the one they have been used to, we may gain insight into their epistemology of mathematics. We report preliminary findings from this pilot study and we suggest some directions for future research.

STUDENTS' EPISTEMOLOGIES OF MATHEMATICS

Francisco (2013) makes a strong argument for the need of more studies investigating secondary school students' epistemological beliefs about mathematics and observes that many findings regarding school students are assumed to be true only because

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they are found to be true for university students and not because they originate from empirical research involving school-age students. For example, Perry (1970) found that college students are likely to hold naïve epistemologies when they start their university studies and many researchers have therefore assumed this would be the case for school students too. Francisco (2013) also notices the disagreement on what are considered to be epistemological beliefs and how these can be studied. For the scope of our study, we adopt the definition of epistemological beliefs found in Hofer (2001): these are beliefs about knowledge and knowing, including:

> . . . the definition of knowledge, how knowledge is constructed, how knowledge is evaluated, where knowledge resides, and how knowing occurs. (Hofer 2001, p. 355)

This definition is only deceivingly simple, but it is one that has drawn widespread agreement amongst researchers in this field (Limon, 2006). A comprehensive review of the literature regarding epistemological beliefs about mathematics by Muis (2004) finds, among its main results, that epistemological beliefs about mathematics hinder rather than help students learning and that these beliefs have a clear impact on the students' academic progress. The author also reviews the evidence of the impact of such beliefs on problem solving activities and mathematics learning more in general and finds that, amongst the most non-availing beliefs school student hold, are that in mathematics there always exist one right answer and that every problem has one right answer only. A subsequent review of the literature by Depaepe et al. (2016) found similar results but noticed that in the years since Muis's (2004) review there has been much variety of methodologies employed to study students' epistemological beliefs well beyond the use of large scale quantitative surveys. This finding reflects the methodological issues raised at the start of this paper. Given that school students' epistemological beliefs about mathematics have been linked to many aspects of their engagement with the subject e.g. to problem solving habits (Schoenfeld, 1989), mathematical achievement and conceptual change (Mason, 2003), it seems important to have solid methodologies to investigate such beliefs. Hence, we ask the following research question:

RQ: What can the students' reactions to the introduction of an alternative approach to a familiar but difficult mathematical concept tell us about their epistemological beliefs about mathematics?

As familiar concept we selected infinity and we suggested an alternative definition of the measure of an infinite set as this definition is in stark contrast to what students would have encountered during their studies. Similar methods could however have been employed by choosing to use definitions from non-standard analysis, or by using the superreal number system proposed by Tall (1980). In the following paragraph, we summarise some research on students' understanding of infinity as some of these findings will also be reflected in our data.

STUDENTS AND INFINITY

Mathematics education has been preoccupied with the way in which students make sense of infinity because this concept is crucial both for the way in which it underpins several ideas from analysis and calculus; and for the understanding of set theory and the concept of cardinality. Many approaches have been used to make sense of students' understanding of infinity and, while it is beyond the scope of this paper to offer a comprehensive review of this literature, we will just mention a few ideas which will be useful for our analysis later on. Monaghan (2001) observes that students often perceive infinity as a process (the process of counting without ending, or a process that goes on and on – also defined as potential infinity, see also Kidron and Tall, 2015) while an object view of infinity would require students to regard infinite sets as completed totalities. Monaghan (2001) also points out that a process view of infinity is at odds with the classical concept of cardinality (actual infinity) and creates conflict when students encounter Cantorian set theory. In this setting students prove that a proper subset of a set and the set itself have the same cardinality if the two sets are countable and infinite. This creates conflict as it is obviously not the case for finite sets. Paradoxes are also used to elicit students' understanding of infinity. For example, Mamolo and Zazkis (2008) report that most difficulties with paradoxes concerning infinity are caused by the conflict of a potentialist (infinity perceived as a process that may go on forever such as counting) and an actualist (an object perceived in its entirety which has infinite size, such as the natural numbers) interpretation of infinity. They also notice that the experience that the students have of reality often gets in the way of the understanding of paradoxes.

MATERIALS

To construct materials for our investigation we introduced students to a numerical treatment of infinity due to Yaroslav Sergeyev (see Sergeyev (2003)). The basics of this treatment can be developed within a conservative extension of Peano Arithmetic, as shown in Lolli (2015). The intuitive idea behind Lolli's theory is that, within a model of arithmetic that contains infinitely large numbers, one may identify a cut-off point for N, the set of natural numbers. A new arithmetical term ① (read: gross-one) is used to denote this cut-off point. Suitable axioms then enable the construction of a theory of numerical measures of infinite parts of N. For instance, in view of these axioms, the initial segment of a model that is bounded by ① is such that any two subsets in bijective correspondence are assigned the same measure, which is smaller than ①. In particular, even and odd numbers are assigned the same measure, smaller than ① and denoted by ①/2. Thus, the whole part relation typical of finite collections is preserved for infinitely large ones.

METHODS

The study was carried out at two sites, in Italy and in the UK. At the first site participants were Year 11 to 13 students (aged between 16 and 18) in a private school in the

South of England. We first held a 90-minute session where they were asked to work in groups of 4 or 5 on the worksheet we designed. The worksheet guided them through five exercises involving grossone including: doing field arithmetic with ①; computing the sum of geometric progressions with an infinitely large number of terms as a strategy to study geometric series and investigating the Thomson lamp paradox (Berresford, 1981) without appealing to ① or by appealing to ①. After the session, we held 2 focus group interviews with nine participants. At the second site participants were students in 6 classes of fourth and fifth year of high school (aged between 17 and 19) attending 2 secondary schools in the south of Italy. There were 77 and 12 students who took part in sessions designed as the previous ones using the same worksheet, which had been translated by the second author of this paper. After the sessions, we held 6 focus group interviews (structured this time as class discussions and thus involving all students who had taken part in the activities). Altogether we collected 8 focus group interviews and observed 6 sessions. The focus group interviews were audio recorded. Thematic analysis (supported by analysis of the field notes taken during the observations) was carried out on the interviews transcripts with focus on the evidence of students' difficulties with the concept of infinity and hints of their epistemological beliefs concerning mathematics. The project was approved by the Research Ethics Committee of the institution where the second and third authors work.

THE DATA

The data were analysed both to investigate misconceptions that students hold about infinity (mainly through discussion of the Thompson Lamp task) and to look for indications of their epistemological beliefs about mathematics. During the analysis of the interview data we found agreement with many previous studies regarding students' understanding of infinity. For example, concerning the discussion on the Thompson lamp paradox, we observed how students' concrete intuitions interfered with the formulation and handling of the paradox, just as Mamolo and Zazkis (2008) found in their study. When a UK student was asked about her thoughts on the solution of the Thompson Lamp paradox she replied:

> Student (UK): The person would die before the end of the process!

We also observed evidence regarding students' tendency to reason in terms of potential (infinite counting) rather than actual infinity, in accordance to what Monaghan, (2001) found.

> Student (UK): Since infinity, there is no actual number for infinity, if you think there will always be 1 more…

Some of the students stated that using the new definition could remove some of what they perceived to be incongruences in the Cantorian approach, such as for example that in the case of infinite countable sets, a set and one of its proper subsets can have the same size.

Student (IT): It is a strange idea [*having various sizes of infinity*] but very intuitive. It allows us to understand a new concept of infinity. Before this we thought that infinity minus a quantity was infinity. Now we can see this better – that an infinity can be smaller than another infinity.

Therefore, students seem to engage in a meaningful way with this concept. Regarding students' epistemologies about mathematics we observed two distinct approaches amongst the students we interviewed: that of *rejection* of the new formulation of infinity or *acceptance* of this formulation. We argue here that these two stances are linked to students' views of knowledge and knowing in mathematics.

Rejection: I think this is a contradiction…

During the observations of the sessions with the students we noticed how all students engaged with the material and worked together through the exercises. However, the follow up interviews revealed that some of the students could not accept that there would be a different definition of a concept they had already encountered. The extract below is from one of the focus group interviews with the Italian students:

Student 1: I think this [*the definition of ①*] is a contradiction - it is a concept which I cannot make mine because it is in contradiction to what I know…

Interviewer: … contradictory because it has both characteristics of infinity and characteristics of finite numbers?

Student 1: Yes ...

Student 2: If you consider it as an infinite big number it is not contradictory because in the end this is not [*the*] infinity

Student 3: It is one of the characteristics of grossone… continuously increasing…

(IT focus group interview)

From this extract emerges a distinct sense of unease on the part of the students and especially of Student 1. They seem to be torn between being able to use formally a definition that they have been given (analysis of the written work produced during the group work sessions revealed that many students managed to find a solution for the Thompson Lamp using grossone) but being unable to accommodate this definition in their beliefs about mathematics. The quote below (collected in a separate focus group) can also be interpreted as manifestation of this unease.

Student (IT): I can't think of subtracting an infinitely large number from an infinitely large number - where do I get to? I don't get to zero for sure . . .

In this case we may argue that, for these students, mathematics is either right or wrong and that an alternative definition of a familiar concept cannot be accommodated because it appears to be in contradiction to what they have studied and taken to be right.

Acceptance: It does kind of work as *i*…

Unlike the previous group, other students not only appear able to accommodate this new concept in their knowledge about mathematics but could work with it without perceiving it as incompatible with what they already knew:

> Student (UK): It does kind of work as *i*, that you have your real part and your imaginary part and ...like i^2 would be minus one . . .

Or:

> Student (IT): It is like *i* - you don't know what is *i* but you know that i^2 is -1.

Indeed, the parallel that these students draw with the imaginary unit *i* is revealing. We know from the history of mathematics and Cauchy's famous remark that 'We completely repudiate the symbol $\sqrt{-1}$, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it' (Nahin, 2010) that the mathematics community took much time to accept this new mathematical object especially because it contradicted (or seemed to contradict) much of the mathematics known before. We interpret this ability to see the similarities between these objects, *i* and grossone, as evidence of an advanced view of what mathematics is. Moreover, another student remarked:

> Student (UK): Because […] I mean they say infinity isn't a number but then […] there is an argument for and against that.

In this extract, we can infer that this student is considering that perhaps there may be different ways of defining mathematical concepts and perhaps more than one interpretation is possible. This may be an indication of a more advanced mathematics epistemology, one where not every statement is true or false and that recognises mathematics as the product of a social construction.

DISCUSSION

The aim of this study was to test a novel qualitative methodology to investigate students' epistemological beliefs about mathematics. We tested whether asking secondary school students to work through a worksheet introducing a new conceptualisation of infinity, unseen and somewhat incompatible with some of their existing knowledge, could provide a strategy suitable to expose secondary school students' epistemological beliefs. We chose an alternative view of infinity and how to measure the size of infinite sets as this approach is in contrast to the way in which students have been exposed to the concept of infinity in their studies. How to understand and measure school students' epistemological beliefs about mathematics is an important topic as these beliefs impact on most aspects of their learning and engagement with the subject (Muis, 2004). Indeed, both Muis (2004) and Depaepe et al. (2016) found in their reviews that the beliefs held by students regarding mathematics were hindering rather than facilitating their learning, making the issue of measuring these beliefs (and eventually influencing them) all the more important. Through thematic analysis of the focus group interviews held after the class activities we found that we could distinguish at least two separate understandings of how mathematics is structured and operates, i.e. two different mathematics epistemologies held by the students participating in the study. Some students held a naïve view close to an absolutist position, according to which mathematics is perceived as a fixed body of knowledge that cannot change (Depaepe et a. 2016). This view manifested itself in the unease felt by the students who were able to work formally through the definitions and concepts given but could not accommodate those in their understanding of infinity because they perceived them to be in stark contrast with what they already knew. Other students held a more advanced view in line with a fallibilist view of mathematics, which perceives this subject as socially constructed hence open to revisions and changes. This view manifested itself in the parallel that some students drew between the introduction of grossone and the introduction of the imaginary unit *i*. These students were able to accommodate the idea that some mathematical definitions may change and that different (even contrasting) definitions of the same concept may exist in mathematics. Therefore the call for caution voiced by Francisco (2013) that not all school students hold naïve epistemologies of mathematics seems to be justified. This finding partially answers our research question by showing that such methods can potentially elicit students' epistemological beliefs and can help understanding their structure. Moreover, following the idea that epistemological beliefs impact on conceptual change and that more sophisticated epistemologies such as those related to fallibilist views of mathematics promote conceptual change (Pintrich, 1999), we would argue that our methodology can not only elicit such epistemologies but also stimulate re-thinking of previously held beliefs by kindling cognitive conflict in the students. More extensive data collection and testing the use of other concepts (such as the superreals, Tall, 1980) could refine this methodology and contribute to our understanding of students' epistemologies but also could potentially help students refine their own epistemologies of mathematics.

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