Doi-Peliti path integral methods for stochastic systems with partial exclusion

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***Highlights (for review)**

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- Doi-Peliti methods for stochastic models with partial exclusion
- Paragrassmannian path integral actions constructed with the aid of Magnus expansions
- Carring capacity birth-death processes have exact perturbative expansions

Doi-Peliti Path Integral Methods for Stochastic Systems with Partial Exclusion

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Abstract

Doi-Peliti methods are developed for stochastic models with finite maximum occupation numbers per site. We provide a generalized framework for the different Fock spaces reported in the literature. Paragrassmannian techniques are then utilized to construct path integral formulations of factorial moments. We show that for many models of interest, a Magnus expansion is required to construct a suitable action, meaning actions containing a finite number of terms are not always feasible. However, for such systems, perturbative techniques are still viable, and for some examples, including carrying capacity population dynamics, and diffusion with partial exclusion, the expansions are exactly summable.

Keywords: Doi-Peliti, Path Integral, Partial exclusion, Carrying Capacity, Population Dynamics

1. Introduction

This work is concerned with parallels between quantum field theory (QFT) and population dynamics. QFT was developed [1], [2] to model interactions of subatomic particles. These interactions result in particle populations that vary in size and position. Classical population dynamics also model populations that vary in size, via mechanisms such as birth-death processes, for example. These populations can also vary in 'position', where position can be interpreted as a continuous feature of interest, such as physical location of a molecule, the size of a cell, or the age of individuals, for example. Doi [3], [4] was the first to notice this parallel and used QFT machinery to model molecular reactions.

The path integral formulation of quantum mechanics was introduced by Dirac, further developed and popularized by Feynman [5]. Peliti [6] adapted these ideas, using functional integration techniques to construct path integral formulations of the Doi paradigm. These techniques have seen a range of applications including molecular reactions [3], [4], birth-death processes on lattices [6], [7], branching random walks [8], percolation [9], phylogenetics [10], algebraic probability [11], knot theory [12], and age dependent population dynamics [13], to name a few.

These works have all been concerned with bosonic forms of QFT, applied to systems with no restriction in occupation number. It is natural to consider analogous applications of fermionic QFT, used to describe quantum systems where there can be no more than one particle in a given state. These techniques can be adapted to population dynamics by modeling classical motions of particles on a lattice, where each site is exclusive, being restricted to single maximum occupancy. Such an approach has been successfully used to model a range of systems such as aggregation processes [14], Ising models [15], and lattice diffusion [16], for example. Exclusive dynamics have also been achieved within a bosonic framework [17]. Grassmannian path integral techniques can also be adapted to such systems [18], [19].

In addition to bosonic (unrestricted) and fermionic (single occupancy) statistics, QFT has been developed for states with limited occupation number. This was first developed by Green [20] and has since been well characterized with the aid of generalized paragrassmannian variables [21], although no fundamental particles of this nature have been observed to date, and path integral formulations for these methods are not widespread [22], [23]. The Doi framework using parafermi QFT techniques for stochastic systems with partial exclusion has developed for cyclic chemical reactions [24], and for diffusion [25], [26], although path integral techniques have not previously been considered. We turn to this problem and address this deficit with the work presented.

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We also mention that significant work in renormalization with Doi-Peliti techniques have also been developed [7], [27], [28], [8], [29], [30], although such methods are not explored in this work. A recent review of Doi-Peliti approaches can be found in [31].

The systems that we shall apply these methods to are partially excluded lattice diffusion [26], [25], [16], where maximum particle numbers are fixed over a lattice of sites, and birth-death processes with a carrying capacity, where population size is limited over a single site. The term carrying capacity usually refers to biological species, representing the maximum population size that can be supported given the available resources (e.g. food, space, competition etc). Such birth-death processes are also known as stochastic logistic growth or Verhulst models [32], [33], and are characterized by birth and death rates β_n and μ_n which depend upon population size n in some capacity limiting fashion. A linear birth rate $\beta_n = p - n$, for example, reduces as the population capacity p is approached. Such linear systems can be analyzed using classical techniques [34], [35]. However, the per individual rate $\frac{\beta_n}{n} = \frac{p}{n} - 1$ is not very natural. A birth rate $\beta_n = n(p - n)$ has a linear per individual birth rate, and approaches zero as full capacity is reached. Although more natural, the quadratic nature makes this difficult to analyze analytically [34], [36]. A death rate $\mu_n = \mu n$ has a constant death rate per individual, and approaches zero as the population empties, so is reasonably natural and the approach we take, although quadratic death rates could similarly be considered.

The work is organized as follows. Section 2 develops a generalized Fock system system suitable for stochastic systems with partial exclusion, explaining the different Fock spaces found in the literature [25], [24], [26]. Section 3 describes how generalized paragrassmannian algebras can be used to construct coherent states. Section 4 develops a coherent state path integral representation, demonstrating that the non-commutative nature of paragrassmannian variables means Magnus expansions [37], [38] are required to construct path integral actions. Section 5 considers applications to birth-death processes and diffusion. Conclusions in Section 6 complete the work.

2. Fock Spaces

2.1. General Structure

We assume in all that follows that the maximum occupancy of any site is p. We also assume, until otherwise stated, that we are dealing with a single site, with occupancy n. We let a and $a^\dagger$ represent annihilation and creation operators for a single site. The Green parafermi relations then take the form [20], [21]

$$[a, [a^\dagger, a]] = 2a. \tag{1}$$

When Green introduced parastatistics, he used what is now referred to as the Green representation. In this formulation we have p distinct occupational 'bins', the i^{th} associated with standard Pauli operators a_i and $a_i^\dagger$. These obey standard anti-commutation relations

$$\{a_i, a_i^\dagger\} = 1, \qquad \{a_i, a_i\} = \{a_i^\dagger, a_i^\dagger\} = 0. \tag{2}$$

These operators commute for distinct i, j, so $[a_i, a_j^\dagger] = 0$, for example. One can then show that operator $a = \sum_i a_i$ satisfies the Green relation of Eq. 1.

Next we introduce states $|n\rangle$ with $n \in \{0, 1, \ldots, p\}$ such that

$$a^\dagger |n\rangle = p_n |n+1\rangle, \qquad a |n\rangle = q_n |n-1\rangle, \tag{3}$$

where p_n, q_n are normalization factors that will later be specified. Repeated application of these recurrences results in

$$|n\rangle = \frac{(a^\dagger)^n}{\prod_{i=0}^{n-1} p_i} |0\rangle, \qquad a^n |n\rangle = \prod_{i=1}^{n} q_i |0\rangle. \tag{4}$$

Now, the commutation relations can be applied to show that $a^n (a^\dagger)^n |0\rangle = (n!)^2 \binom{p}{n} |0\rangle$. We thus find from Eq. 4 that $(n!)^2 \binom{p}{n} = \prod_{i=1}^{n} p_{i-1} q_i$, which results in the expression

$$p_{n-1} q_n = n(p - n + 1). \tag{5}$$

Fock Space No.	$a^\dagger \|n\rangle = p_n \|n+1\rangle$	$a\|n\rangle = q_n \|n-1\rangle$	$\langle s\|a^r\|\psi\rangle$	$\langle n\|n\rangle$
1	$(p-n)\|n+1\rangle$	$n\|n-1\rangle$	$\sum_n \psi_n (p-n+r)_r$	$\binom{p}{n}^{-1}$
2	$(n+1)\|n+1\rangle$	$(p-n+1)\|n-1\rangle$	$\sum_n \psi_n (n)_r$	$\binom{p}{n}$
3	$\sqrt{(n+1)(p-n)}\|n+1\rangle$	$\sqrt{n(p-n+1)}\|n-1\rangle$	$\sum_n \psi_n \sqrt{(n)_r (p-n+r)_r}$	1
4	$\|n+1\rangle$	$n(p-n+1)\|n-1\rangle$	$\sum_n \psi_n$	$(n!)^2\binom{p}{n}$
5	$(n+1)(p-n)\|n+1\rangle$	$\|n-1\rangle$	$\sum_n \psi_n (n)_r (p-n+r)_r$	$(n!)^{-2}\binom{p}{n}^{-1}$

Table 1: Fock space alternatives satisfying $p_n q_n = n(p-n+1)$. Terms $(n)_r = n(n-1)\ldots(n-r+1)$ are Pochhammer symbols.

This offers a range of possibilities for normalization factors p_n, q_n, five obvious choices of which are described in Table 1. For the fermionic case ($p = 1$) these choices are identical, whereas for fully parafermionic systems ($p > 1$) they differ. The third Fock space was used for diffusion and a three species chemical reaction model in [25], [24]. The second Fock space was used for diffusion in [26]. It can be seen from a comparison between [25] and [26] that the second Fock space is algebraically easier to deal with than the third.

Next we introduce the number operator

$$N = \frac{1}{2}(p + [a^\dagger, a]) = a^\dagger \cdot a = \sum_i a_i^\dagger a_i. \tag{6}$$

For all the Fock spaces in Table 1, the states only differ in magnitude and satisfy the same eigenstate equation $N|n\rangle = n|n\rangle$. The number operator is thus identical across all Fock spaces.

Using this formalism, we let $|s\rangle = \sum_{n=0}^{p} |n\rangle$ (for the second Fock space, this is equivalent to the standard expression $|s\rangle = e^{a^\dagger}|0\rangle$), and we let ψ_n represent the probability that the site is occupied by n individuals. Then, for all Fock spaces, we represent the state of the system as

$$|\psi\rangle = \sum_{n=0}^{p} \psi_n |n\rangle \langle n|n\rangle^{-1}. \tag{7}$$

With this formalism, we can recover statistical features of interest. For example, in all cases the probability of n-fold occupation is given by

$$\psi_n = \langle n|\psi\rangle. \tag{8}$$

For the second Fock space, we find that the r^{th} factorial moment is

$$\langle (n)_r \rangle_\psi (t) = \sum_n (n)_r \psi_n(t) = \langle s|a^r|\psi(t)\rangle, \tag{9}$$

where $(n)_r = n(n-1)\ldots(n-r+1)$ denotes the Pochhammer symbol. This is the standard form usually observed for moments using bosonic Doi-Peliti methods [3], [4], [6], and is the main advantage of defining $\langle s|$ as above. Note that for the remaining Fock spaces, the moment equations will differ (see Table 1). However, from now on, we shall just be using the algebraically more compact second Fock space.

2.2. Liouvillians

To model the stochastic dynamics of interest, we convert the corresponding master equation into the following form, where $\mathcal{L}$ denotes a suitable *Liouvillian* operator:

$$\frac{d|\psi\rangle}{dt} = \mathcal{L}|\psi\rangle. \tag{10}$$

From this formalism dynamic equations of interest can be readily obtained. For example, utilizing Eq. 8, the master equation is recovered via

$$\frac{\partial \psi_n}{\partial t} = \langle n|\mathcal{L}|\psi\rangle. \tag{11}$$

Similarly, from Eq. 9, we find factorial moment dynamic equation

$$\frac{\partial \langle (n)_r \rangle_\psi}{\partial t} = \langle s|a^r \mathcal{L}|\psi(t)\rangle = \langle s|[a^r, \mathcal{L}]|\psi(t)\rangle\,, \tag{12}$$

where probability conservation $\langle s|\,\mathcal{L} = 0$ has been used in the latter form.

We note finally that Eq. 10 has formal solution

$$|\psi(t)\rangle = e^{t\mathcal{L}}\,|\psi(0)\rangle\,. \tag{13}$$

This form will later be used to construct path integral representations of factorial moments of interest, offering an alternative approach to solving the dynamic equation of Eq. 12. Next, however, we consider the dynamic form for some applications of interest.

2.3. Applications

The three applications we consider are a birth-death processes with linear rates, one with quadratic rates, and a lattice diffusion process.

Firstly, then, consider a birth-death process where the population birth rate is $\beta_n = \beta(p-n)$ and death rate is $\mu_n = \mu n$. This is perhaps the simplest model of a birth-death system with carrying capacity, where the population is restricted in size between 0 and p. The corresponding birth-death master equation is

$$\frac{d\psi_n}{dt} = -\psi_n(\beta_n + \mu_n) + \psi_{n+1}\mu_{n+1} + \psi_{n-1}\beta_{n-1}. \tag{14}$$

We convert this into bra-ket formalism as follows. Firstly, note that we can use Eq. 7 to convert the left hand side of Eq. 14 via $\sum_n \frac{d\psi_n}{dt}\,|n\rangle\binom{p}{n}^{-1} = \frac{d|\psi\rangle}{dt}$. The birth and death terms $\psi_{n-1}\beta_{n-1}$ and $\psi_{n+1}\mu_{n+1}$ are similarly transformed as:

$$\sum_n \psi_{n-1}\beta(p-n+1)\,|n\rangle\binom{p}{n}^{-1} = \sum_n \psi_{n-1}\beta n\,|n\rangle\binom{p}{n-1}^{-1} = \sum_n \psi_{n-1}\beta a^\dagger\,|n-1\rangle\binom{p}{n-1}^{-1} = \beta a^\dagger\,|\psi\rangle\,, \tag{15}$$

$$\sum_n \psi_{n+1}\mu(n+1)\,|n\rangle\binom{p}{n}^{-1} = \sum_n \psi_{n+1}\mu(p-n)\,|n\rangle\binom{p}{n+1}^{-1} = \sum_n \psi_{n+1}\mu a\,|n+1\rangle\binom{p}{n+1}^{-1} = \mu a\,|\psi\rangle\,. \tag{16}$$

The remaining term converts as $\sum_n \psi_n(\beta_n+\mu_n)\,|n\rangle\binom{p}{n}^{-1} = (\beta\overline{N} + \mu N)\,|\psi\rangle$, where $\overline{N} = p - N$. Finally, from Eq. 10 we find the master equation has a corresponding Liouvillian operator

$$\mathcal{L} = \beta(a^\dagger - \overline{N}) + \mu(a - N). \tag{17}$$

From the commutation relations we find $[a, \mathcal{L}] = -2\beta N - \nu a + \beta p$, where $\nu = \mu - \beta$ and so, using Eq. 12 and $\gamma = \mu + \beta$, we find mean occupancy satisfies

$$\frac{\partial \langle n\rangle_\psi}{\partial t} = -\gamma\,\langle n\rangle_\psi + \beta p. \tag{18}$$

If we assume an initial population of size n this results in solution

$$\langle n\rangle_\psi = \frac{\beta p}{\gamma}(1 - e^{-\gamma t}) + n e^{-\gamma t}. \tag{19}$$

For a second application, we consider a birth-death model with quadratic birth rate $\beta_n = \beta n(p-n)$ and linear death rate $\mu_n = \mu n$. This results in the two following Liouvillians, both equally valid:

$$\mathcal{L} = -\beta N(p-N) - \mu N + \beta a^\dagger N + \mu a, \tag{20}$$
$$\mathcal{L} = -\beta a^\dagger a - \mu N + \beta a^\dagger N + \mu a. \tag{21}$$

Although distinct operators, the moment equation $\frac{\partial \langle n \rangle_\psi}{\partial t} = \langle s|[a,\mathcal{L}]|\psi\rangle$ in both cases gives:

$$\frac{\partial \langle n \rangle_\psi}{\partial t} = -\langle n \rangle_\psi (\mu - \beta(p-1)) - \beta \langle n^2 \rangle_\psi \, , \tag{22}$$

which implicates the second moment. We could similarly derive an equation for the second moment, which would implicate higher moments, and so on, making this system more awkward to solve than the linear process analyzed above. We later turn to path integrals to provide some insight.

Finally, we consider diffusion on a lattice, with particles transferring from site I to neighboring site J at rate $\nu n_I(p - n_J)$, where n_I, n_J are the associated occupation numbers. Then the Liouvillian takes the form [26]

$$\mathcal{L} = \nu \sum_{I,J} \left(N_I \overline{N}_J - a_I a_J^\dagger \right), \tag{23}$$

where ordered pairs (I, J) label neighboring lattice sites. Note the subscripts I, J are now referring to sites rather than bin indices i, j of Eq. 2. Using the third Fock space in Table 1 results in a somewhat more complicated Liouvillian [25]. One can use the commutation relations to derive dynamic equations for moments using equations such as Eq. 12. Equations for the first two moments are given below (the mean was observed in [25]), both taking the following discretized forms of diffusion, where $I(J)$ and $I(K)$ index neighbors of sites J and K, respectively:

$$\frac{\partial \langle n_K \rangle_\psi}{\partial t} = \nu p \sum_{I(K)} (\langle n_K \rangle_\psi - \langle n_I \rangle_\psi), \tag{24}$$

$$\frac{\partial \langle n_J n_K \rangle_\psi}{\partial t} = \nu p \sum_{I(J)} \langle (n_I - n_J) n_K \rangle_\psi + \nu p \sum_{I(K)} \langle n_J (n_I - n_K) \rangle_\psi + \begin{cases} -2\nu \sum_{I(K)} \langle n_I n_K \rangle_\psi \, , & J = K, \\ -2\nu \langle n_J n_K \rangle_\psi - \nu p \langle n_J + n_K \rangle_\psi \, , & |J-K| = 1, \\ 0, & |J-K| > 1. \end{cases}$$

We are thus able to obtain dynamic equations for our examples using the Fock formalism directly. Solutions from these equations can then be sought. We now turn to alternative approaches afforded by path integral construction. In order to do this we first need the machinery of paragrassmannian algebras.

3. Paragrassmannian Algebra

Next we introduce generalized paragrassmannian algebras that are required for coherent state path integral construction. Paragrassmannian vectors ξ with components ξ_i, $i \in \{1, 2, \ldots, p\}$ are defined by [21]

$$\xi_i \xi_j = \eta_{i,j} \xi_j \xi_i, \tag{25}$$

where the *signature* η satisfies $\eta_{i,i} = -1$ and $\eta_{i,j} = \eta_{j,i} \in \{-1, +1\}$. For our applications, we shall be interested in the parafermionic case, where $\eta_{i,j} = +1$ ($i \neq j$). The same commutation relations apply if one of the grassmannian variables, say ξ_i, is replaced with operator a_i or $a_i^\dagger$ (e.g. $\xi_i a_i^\dagger = -a_i^\dagger \xi_i$).

We introduce coherent states as:

$$|\xi\rangle = e^{\xi \cdot a^\dagger} |\phi\rangle \, , \qquad \langle \xi^* | = \langle \phi | \, e^{a \cdot \xi^*}, \tag{26}$$

where we utilize dot product representations such as $\xi.a^\dagger = \sum_{i=1}^{p} \xi_i a_i^\dagger$. One can then use the commutation relations to establish the following eigenfunction and normalization properties:

$$\langle \xi^* | \, a_i^\dagger = \langle \xi^* | \, \xi_i^*, \qquad a_i \, |\xi\rangle = \xi_i \, |\xi\rangle \, , \qquad \langle \xi^* | \xi \rangle = e^{\xi^* \cdot \xi}. \tag{27}$$

A coherent state path integral requires a resolution of the identity between time slices. Integration with respect to paragrassmannians is thus required, which is identical to grassmannian integration for each component:

$$\int \mathrm{d}\xi_i \, 1 = 0, \qquad \int \mathrm{d}\xi_i \, \xi_i = 1. \tag{28}$$

Note that integration acts from the left. For example, $\int \mathrm{d}\xi_i\, \zeta_i \xi_i = -\int \mathrm{d}\xi_i\, \xi_i \zeta_i = -\zeta_i$, where ζ_i is a distinct paragrassmannian. Now, from the definition of integration and coherent states, one can obtain the following resolution of identity (see appendices of [21] for a derivation):

$$I = \iint \mathrm{d}\xi^* \mathrm{d}\xi\, e^{-\xi^*\cdot\xi} \left|\xi\right\rangle \left\langle \xi^*\right|, \tag{29}$$

where $\mathrm{d}\xi = \prod_{i=1}^{p} \mathrm{d}\xi_i$ and $\mathrm{d}\xi^* = \prod_{i=1}^{p} \mathrm{d}\xi_i^*$. Note that the commutator relations $[\xi_i, \xi_j] = [\xi_i^*, \xi_j^*] = 0$ for $i \neq j$ means the order of integration within each product is not important. However, $\mathrm{d}\xi^*$ and $\mathrm{d}\xi$ do not commute and their order matters.

4. Path Integral Construction

We next use the paragrassmannians to construct a path integral, assuming the second Fock space of Table 1 in all that follows. The r^{th} factorial moment, with resolutions of the identity, can be written as the following, where z_0 and z_T are initial and final paragrassmannian vectors:

$$\langle (n)_r \rangle (T) = \sum_m (m)_r \psi_m(T) = \langle s | a^r e^{T\mathcal{L}} | \psi(0) \rangle = \iint \mathrm{d}z_T^* \mathrm{d}z_T \mathrm{d}z_0^* \mathrm{d}z_0 e^{-z_T^*\cdot z_T - z_0^*\cdot z_0} \left\langle s \right| a^r \left| z_T \right\rangle \left\langle z_T^* \right| e^{T\mathcal{L}} \left| z_0 \right\rangle \langle z_0^* | \psi(0) \rangle . \tag{30}$$

This gives us an initial term $\langle z_0^* | \psi(0) \rangle$, a final term $\langle s | a^r | z_T \rangle$, and a time evolution factor $\langle z_T^* | e^{T\mathcal{L}} | z_0 \rangle$ to calculate.

For the initial state we use $|\psi(0)\rangle = |n\rangle \binom{p}{n}^{-1}$ to represent an initial population size of n. Using the commutation relations, this gives us an initial term of the form $\langle z_0^* | \psi(0) \rangle = \binom{p}{n}^{-1} S_n(z_0^*)$, where $S_n(z_0^*)$ is composed of the sum of products of terms in all subsets of size n from the set $\{z_1^*, \ldots, z_p^*\}$ of components of paragrassmannian vector z_0^*. For example, with $p = 4$, $S_2(z^*) = z_1^* z_2^* + z_1^* z_3^* + z_1^* z_4^* + z_2^* z_3^* + z_2^* z_4^* + z_3^* z_4^*$. For the final term we find $\langle s | a^r | z_T \rangle = z_T^r e^{z_T}$, where z_T is shorthand for $\sum_i (z_T)_i$.

To evaluate the time evolution factor, we first assume the Liouvillian operator $\mathcal{L}(a^\dagger, a)$ has been put into normal form via the commutation relations, with creation operators left of annihilation operators. Complications arising from the general situation are described in [21], however, for the models we consider, this is straightforwardly done. From this we can attempt to construct a coherent state path integral in the usual fashion. There is, however, a significant adjustment that we need to be aware of that arises due to the non-commutative nature of paragrassmannian variables. Now, we write the time evolution factor as a product of time slices with the aid of Eq. 29 to give:

$$\langle z_T | e^{T\mathcal{L}(a^\dagger, a)} | z_0 \rangle = \lim_{\epsilon \to 0} \prod_{\alpha=1}^{N} \iint \mathrm{d}z_\alpha^* \mathrm{d}z_\alpha\, e^{-z_\alpha^* \cdot z_\alpha} \langle z_\alpha^* | e^{\epsilon \mathcal{L}(a^\dagger, a)} | z_{\alpha-1} \rangle = \iint \mathcal{D}z^* \mathcal{D}z\, e^{-\int_0^T \mathrm{d}t \left(z^*(t) \cdot \frac{\partial z(t)}{\partial t} \right) - z^*(0)\cdot z(0)} \Pi(T), \tag{31}$$

where $T = N\epsilon$, z_α represent paragrassmannian vectors for each α, $z(t)$ is a time dependent paragrassmannian vector, and

$$\Pi(T) = \lim_{\epsilon \to 0} \prod_{\alpha=1}^{N-1} e^{\epsilon \mathcal{L}(z_\alpha^*, z_{\alpha-1})}. \tag{32}$$

Note that terms such as $z_\alpha^* \cdot z_{\alpha-1}$ commute with all paragrassmannian variables and have been collected into a single exponential $\exp\left(\int_0^T \left(z^*(t) . \frac{\partial z(t)}{\partial t} \right) - z^*(0) \cdot z(0) \right)$. It is standard methodology to try the same with terms $e^{\epsilon \mathcal{L}(z_\alpha^*, z_{\alpha-1})}$. For some stochastic models of interest, however, these terms do not commute (see next section) and the continuous Baker-Campbell-Hausdorff theorem is required [38]. This results in the following time ordered Dyson series expansion, where $X(t) = \mathcal{L}(z^*(t), z(t))$, τ is the time ordering operator, and Δ_n is the triangular region $T > t_1 > \cdots > t_n > 0$:

$$\Pi(T) = \tau \left\{ \exp \int_0^T X(t)\, \mathrm{d}t \right\} = 1 + \int_0^T X(t_1)\, \mathrm{d}t_1 + \int_{\Delta_2} X(t_1) X(t_2)\, \mathrm{d}t_1 \mathrm{d}t_2 + \cdots + \int_{\Delta_n} X(t_1) \ldots X(t_n)\, \mathrm{d}\mathbf{t}_n + \cdots . \tag{33}$$

Now, we are using paragrassmannian variables, thus to write the series $\Pi(T) = e^{\Omega(T)}$ as an exponential and construct an action requires a Magnus series [37]. There are many different formulations for these series [38], the most explicit given by Saenza and Suarez [39]:

$$\Omega(T) = \int_0^T X(t_1)\,\mathrm{d}t_1 + \frac{1}{2}\int_{T>t_1>t_2>0} [X(t_1), X(t_2)]\,\mathrm{d}t_1\mathrm{d}t_2 + \sum_{n\geq 3}\frac{1}{n}\int_{\Delta_n} L_n[\dots[X(t_1), X(t_2)], \dots, X(t_n)]\,\mathrm{d}\mathbf{t}_n, \tag{34}$$

where, given standard step function $\theta(t)$,

$$L_n = \sum_{i=1}^{n}\frac{(-1)^{i+1}}{i}\sum_{0<j_1<j_2\cdots<j_{n-i}<n}\prod_{m=1}^{n-i}\theta(t_{j_m} - t_{j_m+1}). \tag{35}$$

In the case that $X(t)$ is commutative (e.g. for bosonic systems) $\Omega(T) = \exp\int_0^T X(t)\,\mathrm{d}t$, resulting in the form usually observed in path integral actions [6]. Although the case of commutative $[X(t), X(s)]$ is tractable (see below), the general case results in a complicated action involving an infinite number of terms.

We next consider these path integrals for specific examples.

5. Applications

5.1. Linear Birth and Death Process with Carrying Capacity

Consider next the linear birth-death model with the Liouvillian given in Eq. 17. We are interested in a path integral formulation for the r^{th} factorial moment, $\langle (n)_r\rangle$, which should be equivalent to the expression given in Eq. 19 for $r = 1$.

Now, for paragrassmannian variables x^*, x, y^* and y, the commutator

$$[\mathcal{L}(x^*, x), \mathcal{L}(y^*, y)] = 2(\mu x + \beta x^*)\cdot(\mu x + \beta x^*), \tag{36}$$

is commutative, meaning the Magnus expansion $\Omega(t)$ contains two terms and we find, using $\nu = \mu - \beta$, and shorthand notation $z_t = z(t)$,

$$\begin{aligned}\langle (n)_r\rangle(T) \;=\; & \binom{p}{n}^{-1} e^{-p\beta T}\iint \mathcal{D}z^*\mathcal{D}z\, z_T^r e^{z_T}\exp\left\{-\int_0^T \mathrm{d}t\,(z_t^*\cdot(\partial_t z_t + \nu z_t)) - z_0^*\cdot z_0\right\}\cdot \\ & \exp\left\{\int_0^T \mathrm{d}t\,(\beta z_t^* + \mu z_t) + \int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\,(\beta z_t^* + \mu z_t)\cdot(\beta z_s^* + \mu z_s)\right\} S_n(z_0^*).\end{aligned} \tag{37}$$

Note that terms such as z_t in dot products represent paragrassmannian vectors, otherwise it is shorthand for $\sum_i (z_t)_i$. Now, the first and third exponentials in the path integral contain non-commutative terms and the Baker-Cambell-Hausdorff theorem would be required to construct a single action for this path integral. However, we shall treat the first and third terms perturbatively. To do this we require a generating functional of the following form, rearranged using standard completing the square techniques [2]:

$$Z(\eta^*, \eta) = \iint \mathcal{D}z^*\mathcal{D}z e^{[z^*D^+z]+[z^*\eta]+[\eta^*z]} = e^{-[\eta^*D^-\eta]}\iint \mathcal{D}z^*\mathcal{D}z e^{[z^*D^+z]} = e^{-[\eta^*D^-\eta]}Z(0,0). \tag{38}$$

Here we have used the shorthand notation $[z^*D^+z] = \int_0^T\int_0^T \mathrm{d}t\mathrm{d}s\, z^*(t)\cdot D^+(t,s)z(s)$ and $[z^*\eta] = \int_0^T \mathrm{d}t\, z^*(t)\cdot\eta(t)$, where $D^+_{i,j}(s,t) = \delta_{i,j}\delta(t-s)(\partial_t + \nu + \delta(t))$ and the inverse $D^-_{i,j}(t,s) = \delta_{i,j}\theta(t-s)e^{-\nu(t-s)}$ satisfy equations $\sum_{j=1}^p\int_0^T \mathrm{d}u\, D^{\pm}_{i,j}(t,u)D^{\mp}_{j,k}(u,s) = \delta_{i,k}\delta(t-s)$, and η^*, η are indeterminant paragrassmannian vectors.

To simplify the expansion, we make the following change of variables:

$$\left.\begin{aligned} y &= \alpha^{-1}(\beta z^* + \mu z) \\ y^* &= \alpha^{-1}(\beta z^* - \mu z)\end{aligned}\right\} \iff \left\{\begin{aligned} z &= \tfrac{\alpha}{2\mu}(y - y^*), \\ z^* &= \tfrac{\alpha}{2\beta}(y + y^*).\end{aligned}\right. \tag{39}$$

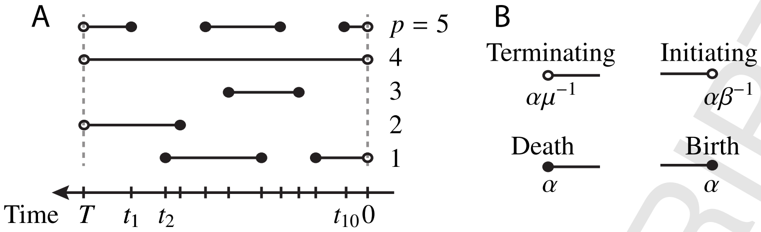

Figure 1: Feynman diagram for linear birth death model. A) Sample diagram. B) Diagram nodes.

The factor $\alpha = \sqrt{2\beta\mu}$ ensures the transformation has unit Jacobian, meaning that the path integral measure $\iint \mathcal{D}z^*\mathcal{D}z = \iint \mathcal{D}y^*\mathcal{D}y$ is preserved. Thus we get:

$$\langle (n)_r\rangle = \binom{p}{n}^{-1} e^{-p\beta T} \iint \mathcal{D}y^*\mathcal{D}y\, z_T^r e^{z_T} e^{[y^*Ey]} e^{\alpha\int_0^T \mathrm{d}t\, y_t + \alpha^2 \int_0^T\int_0^t \mathrm{d}t\mathrm{d}s\, y_t\cdot y_s} S_n(z_0^*), \tag{40}$$

where $[y^*Ey]$ represents $[z^*Dz]$ following substitution with Eq. 39. We also find that with the transformation

$$\left.\begin{array}{l} \zeta = \alpha^{-1}(\beta\eta^* + \mu\eta) \\ \zeta^* = \alpha^{-1}(\beta\eta^* - \mu\eta) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \eta = \frac{\alpha}{2\mu}(\zeta - \zeta^*), \\ \eta^* = \frac{\alpha}{2\beta}(\zeta + \zeta^*), \end{array}\right. \tag{41}$$

the generating functional can be written as

$$Z(\eta^*, \eta) = \hat{Z}(\zeta^*, \zeta) = Z(0,0)\exp\left\{\frac{1}{2}\int_0^T\int_0^T \mathrm{d}t\mathrm{d}s\,(\zeta(t) + \zeta^*(t))\cdot D^-(t,s)(\zeta(s) - \zeta^*(s))\right\}, \tag{42}$$

which gives rise to four classes of propagator ($\delta\zeta$ is shorthand for functional derivative $\frac{\partial}{\partial\zeta}$)

$$\begin{aligned}
G^{i,j}_{yy}(t,s) &= \iint \mathcal{D}y^*\mathcal{D}y\, y_i(t)y_j(s)e^{[y^*Ey]} = \delta\zeta_i^*(t)\delta\zeta_j^*(s)\hat{Z}(\zeta^*,\zeta)|_{\zeta\equiv\zeta^*\equiv 0} = \frac{1}{2}D^-_{i,j}(t,s), \\
G^{i,j}_{yy^*}(t,s) &= \iint \mathcal{D}y^*\mathcal{D}y\, y_i(t)y_j^*(s)e^{[y^*Ey]} = -\delta\zeta_i^*(t)\delta\zeta_j(s)\hat{Z}(\zeta^*,\zeta)|_{\zeta\equiv\zeta^*\equiv 0} = \frac{1}{2}D^-_{i,j}(t,s), \\
G^{i,j}_{y^*y}(t,s) &= \iint \mathcal{D}y^*\mathcal{D}y\, y_i^*(t)y_j(s)e^{[y^*Ey]} = -\delta\zeta_i(t)\delta\zeta_j^*(s)\hat{Z}(\zeta^*,\zeta)|_{\zeta\equiv\zeta^*\equiv 0} = -\frac{1}{2}D^-_{i,j}(t,s), \\
G^{i,j}_{y^*y^*}(t,s) &= \iint \mathcal{D}y^*\mathcal{D}y\, y_i^*(t)y_j^*(s)e^{[y^*Ey]} = \delta\zeta_i(t)\delta\zeta_j(s)\hat{Z}(\zeta^*,\zeta)|_{\zeta\equiv\zeta^*\equiv 0} = -\frac{1}{2}D^-_{i,j}(t,s).
\end{aligned} \tag{43}$$

Next consider the perturbative expansion. All the internal nodes of the associated Feynman diagram arise from expansion of the third exponential of Eq. 37 into a Dyson series of Eq. 33, which has a general term $\alpha^n \int_{\Delta_n} y(t_1)\ldots y(t_n)d\mathbf{t}_n$ to consider. Note, therefore, that the internal nodes are just associated with variable y (rather than y^*).

Now, the initiating terms from $S_n(z_0^*)$ arise from components of the paragrassmannian vector $z_0^* = \frac{\alpha}{2\beta}(y_0 + y_0^*)$. We see from above that propagators with initiating nodes y_0 or y_0^* will give the same value (e.g $G^{i,j}_{yy^*}(t,s) = G^{i,j}_{yy}(t,s)$). We can thus represent this as a single y node with factor $\frac{\alpha}{\beta}$. Similarly, we can replace any terminating nodes arising from $z_T e^{z_T}$ by y nodes with factors $\frac{\alpha}{\mu}$. All nodes are thus of type y and $G^{i,j}_{yy}(t,s)$ is the only propagator we need to consider.

From Eq. 43 we note that propagators between node pairs in bins i, j are only non-zero if $i = j$, and so are represented as horizontal intervals $([t_1, t_2], i)$ sitting within $[0, T] \times \{1, \ldots, p\}$. Furthermore, the anti-commutative nature of paragrassmanian components with the same bin number i means that time intervals of propagators do not overlap for each i.

The collections of intervals for each bin i group into four classes depending upon whether they contain initiating or terminating nodes. Consider then the Laplace transform $\hat{I}_{01}(s)$ of the sum of all propagator products, within a single bin, of the class with one terminating node and no initiating node (e.g. the propagators in Fig. 1A are the contribution from one diagram for $i = 2$). This is a sum of convolutions of propagators $e^{-\nu t}$ (segments) and the value 1 (gaps), giving a Laplace transform and inverse of the following form, where $\gamma = \beta + \mu$, and k counts segment/gap pairs:

$$\hat{I}_{01}(s) = \frac{1}{\mu}\sum_{k=1}^{\infty}\frac{(\beta\mu)^k}{s^k(s+\nu)^k} = \frac{\beta}{(s-\beta)(s+\mu)}, \qquad I_{01}(T) = \frac{\beta}{\gamma}(e^{\beta T} - e^{-\mu T}). \tag{44}$$

The other three cases are found similarly, giving $I_{10}(T) = \frac{\mu}{\beta}I_{01}(T)$, $I_{00}(T) = \frac{\mu}{\gamma}e^{\beta T} + \frac{\beta}{\gamma}e^{-\mu T}$ and $I_{11}(T) = \frac{\beta}{\gamma}e^{\beta T} + \frac{\mu}{\gamma}e^{-\mu T}$. Now the sum of terms across all Feynman diagrams for each bin $i \in \{1, 2, \ldots, p\}$ contains one of these four options as a factor in a product of p terms. Given n initial nodes, which are counted by copies of $I_{10}(T)$ and $I_{11}(T)$, we sum over the possibilities to give the r^{th} factorial moment as a hypergeometric sum

$$\langle (n)_r \rangle_\psi = Z(0,0)e^{-p\beta t}\sum_{a=0}^{p-n}\sum_{b=0}^{n}(a+b)^r\binom{p-n}{a}\binom{n}{b}I_{01}{}^{a}I_{00}{}^{p-n-a}I_{11}{}^{b}I_{10}{}^{n-b}. \tag{45}$$

If we calculate this for the 0^{th} mean ($\langle 1 \rangle_\psi = 1$) to find $Z(0,0)$, the mean ($r = 1$) can then be found, which reduces to the formulation given in Eq. 19 after some algebra, as one might expect. Higher moments can be found similarly.

5.2. *Quadratic Birth Death Process*

Next consider the quadratic birth-death process, where we found associated Liouvillians in Eq. 20, 21. To construct a path integral approach, we note that $[\mathcal{L}(x^*, x), \mathcal{L}(y^*, y)] = 2(\beta(x^* \cdot x)x^* + \mu x) \cdot (\beta(y^* \cdot y)y^* + \mu y)$, using the Liouvillian of Eq. 20. This is a commutative object meaning we have a finite Magnus expansion with two terms. Note that this is not the case with the Liouvillian of Eq. 21, where the innocuous looking term $a^\dagger a$ results in a complicated Magnus expansion. Using the former results in the following factorial moment path integral, where $\nu = \mu + \beta(p-1)$,

$$\begin{aligned}\langle (n)_r \rangle (T) \quad = \quad & \binom{p}{n}^{-1}\iint \mathcal{D}z^*\mathcal{D}z\, z_T^r e^{z_T}\exp\left\{-\int_0^T \mathrm{d}t\,(z_t^* \cdot (\partial_t z_t + \nu z_t)) - z_0^* \cdot z_0 + \beta\int_0^T \mathrm{d}t\,(z_t^* \cdot z_t)^2\right\} \cdot \\ & \exp\left\{\int_0^T \mathrm{d}t(\beta(z_t^* \cdot z_t)z_t^* + \mu z_t) + \int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\,(\beta(z_t^* \cdot z_t)z_t^* + \mu z_t) \cdot (\beta(z_s^* \cdot z_s)z_s^* + \mu z_s)\right\} S_n(z_0^*).\end{aligned} \tag{46}$$

We treat this with perturbative expansion, much the same as the previous example. The main difference is that instead of a birth term βz^*, we have a birth term $\beta(z^* \cdot z)z^*$, and we have a new term of the form $\beta(z^* \cdot z)^2$ to consider. We also have the same death term μz. This gives three classes of internal node in corresponding Feynman diagrams, as exemplified in Fig. 2C.

The death term $\mu z = \mu \sum_i z_i$ results in a simple terminating node, where a node in bin i corresponds to term μz_i. The term $\beta(z^* \cdot z)^2$ is composed of terms $2\beta z_i^* z_i z_j^* z_j$ with distinct i, j. This results in the vertical connections (labeled neutral in Fig. 2C) given in the sample diagram Fig. 2A. The birth term consists of terms of the form $\beta z_i^* z_i z_j^*$ with distinct i, j, represented as birth nodes with connections from an existing propagator in bin i to a new propagator in bin j (e.g time t_3 in Fig. 2A has a birth term for $i = 1$, $j = 2$). The propagator takes the same form $e^{-\nu t}$ as the previous section, albeit with a different ν. Now the technique utilized in the previous section summed the Laplace transform of propagators for each bin $i \in \{1, \ldots, p\}$ and then amalgamated the results. Now, however, the links between segments of differing bins arising from birth and neutral terms make this impossible and we need to change the approach, and instead will sum all diagrams collectively.

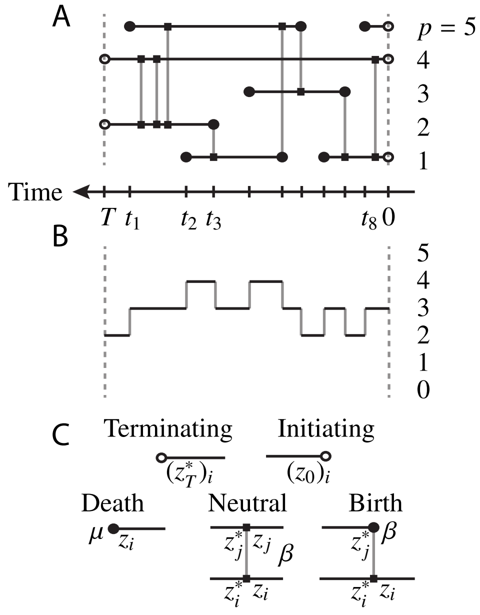

Figure 2: Feynman diagram for quadratic birth death model. A) Sample diagram. B) Total propagator profile corresponding to A). C) Diagram nodes.

In Fig. 2B we see a diagram counting the number of propagators bridging any point in time. Note that in any time interval, any number of neutral connectors can be placed between two propagators without affecting the propagator count in that interval. For a time interval of length t such that the propagator count m is constant for each point of time (a flat portion of the path in Fig. 2B), we sum the Laplace transform of the product of propagators $e^{-\nu mt}$ of all these possibilities to give the following, where ℓ indexes the number of neutral connections,

$$\sum_{\ell=0}^{\infty} \binom{m}{2}^{\ell} \frac{(2\beta)^{\ell}}{(s+m\nu)^{\ell+1}} = \frac{1}{s+m\nu-\beta m(m-1)} = \frac{1}{s+\mu_m+\beta_m}. \tag{47}$$

Note the following characteristics of Fig. 2B. The count profile can be represented as a path that can move one position up or down at a time corresponding to internal birth or death nodes, starting at the initial number of occupied bins ($n = 3$ in Fig. 2B). The final value indicates the number of terminating nodes in the diagram ($n = 2$ in our example). There are $\binom{p}{n}$ possible ways of selecting the n initial nodes, which cancels the factor $\binom{p}{n}^{-1}$ in Eq. 46. If the path is at height m prior to an upward step representing term $\beta z_i^* z_i z_j^*$, there are $p-m$ choices for the internal birth node (i.e. selecting an unoccupied bin j), and m choices for occupied bin i. With weight β these choices are encapsulated by factor $\beta_m = \beta m(p-m)$. If the path is at height m prior to a drop representing term μz_i, there are m choices (i.e. select an occupied bin i to terminate), each associated with death factor μ. These choices are encapsulated by factor $\mu_m = \mu m$. Now each time segment of height m has a Laplace transform $\frac{1}{s+\mu_m+\beta_m}$ corresponding to a time dependent term $e^{-(\mu_m+\beta_m)t}$. Then the Laplace transform for all diagrams corresponding to a particular path with k steps is given by:

$$\mathcal{L}\left\{\int_{0=t_0<t_1<\cdots<t_k=T} \mathrm{d}\mathbf{t}_k \prod_{i=1}^{k} e^{-(\mu_{m_i}+\beta_{m_i})(t_i-t_{i-1})}\right\} = \prod_{i=1}^{k} \frac{1}{s+\mu_{m_i}+\beta_{m_i}}. \tag{48}$$

Next we introduce $f_m^{(k)}$ as the sum of Laplace transforms for all paths ending at height m in k steps. Note that $k-1$ counts the number of internal birth-death (i.e. non-neutral) nodes at time points separating steps. The implicit initial height n is fixed throughout the discussion. Then we can construct the following recurrence:

$$f_m^{(k+1)} = \frac{1}{s+\mu_m+\beta_m}\left[\beta_{m-1} f_{m-1}^{(k)} + \mu_{m+1} f_{m+1}^{(k)}\right]. \tag{49}$$

Next introduce $f_m = \sum_{k\geq 1} f_m^{(k)}$. Then summing over the recurrence gives

$$f_m = f_m^{(1)} + \frac{1}{s+\mu_m+\beta_m}\left[\beta_{m-1} f_{m-1} + \mu_{m+1} f_{m+1}\right]. \tag{50}$$

Noting the initial condition $f_m^{(1)} = \delta_{mn}\frac{1}{s+\mu_m+\beta_m}$, we can write this as:

$$(s+\mu_m+\beta_m) f_m = \delta_{m,n} + \beta_{m-1} f_{m-1} + \mu_{m+1} f_{m+1}. \tag{51}$$

Now the r^{th} factorial moment $\langle (n)_r\rangle(T)$ of the population size can be written as a path integral with terminating nodes described by the factor $z_T^r e^{z_T}$. Then we can write this as an inverse Laplace transform:

$$\langle (n)_r\rangle(T) = Z(0,0)\mathcal{L}_T^{-1}\left\{\sum_m (m)_r f_m\right\}. \tag{52}$$

For the case $r = 0$ we sum Eq. 51 over m to find $\sum_m f_m = s^{-1}$ resulting in $Z(0,0) = 1$. Now if we write Eq. 51 in matrix form as $(sI - B)\mathbf{f} = \mathbf{e}_n$ with tridiagonal matrix B we can write the mean as follows, where λ_i represent eigenvalues of matrix B:

$$\langle n\rangle_{\psi}(T) = \mathcal{L}_T^{-1}\left\{\sum_m m f_m\right\} = \mathcal{L}_T^{-1}\left\{\sum_m m\,\{\mathrm{Adj}(sI-B)\}_{mn}\,|sI-B|^{-1}\right\} = \sum_i \frac{\sum_m m\,\{\mathrm{Adj}(\lambda_i I-B)\}_{mn}}{\prod_{j\neq i}(\lambda_i-\lambda_j)} e^{\lambda_i t}. \tag{53}$$

Now if vector ψ represents the population probability distribution, where component $\psi_m(T)$ is the probability of population size m, the master equation is $\frac{\partial\psi}{\partial t} = B\psi$, which has a solution of the form $\mathbf{p} = e^{TB}\mathbf{e}_n$. Diagonalizing B results in precisely the solution given in Eq. 53, and we find that the perturbative expansion is equivalent to diagonalization of the original system [40].

5.3. Diffusion on a Lattice

We lastly point out that these path integral techniques can be extended to lattice methods. Factorial moment path integral formulation is similar to the previous example, so we just highlight the salient points. Firstly, from Eq. 23 we separate the Liouvillian $\mathcal{L} = \mathcal{L}^0 + \mathcal{L}^1$ into a quadratic part $\mathcal{L}^0 = -p\nu \sum_{I,J} N_I$ which we shall deal with non-perturbatively, and $\mathcal{L}^1 = \nu \sum_{I,J}(N_I N_J + a_I a_J^\dagger)$, which we deal with perturbatively.

We rewrite Eq. 31 as follows, where $\mathcal{L}_\alpha^{0/1} = \mathcal{L}^{0/1}(z_\alpha^*, z_{\alpha-1})$, $X(t) = \mathcal{L}^1(z_t^*, z_t)$, z_α is a paragrassmannian vector over bins and sites (α indexes time slices), and z_t is shorthand for $z(t)$:

$$\begin{aligned}\langle z_T|e^{T\mathcal{L}(a^\dagger,a)}|z_0\rangle &= \lim_{\epsilon\to 0}\prod_{\alpha=1}^{N}\iint \mathrm{d}z_\alpha^*\mathrm{d}z_\alpha\, e^{-z_\alpha^*\cdot z_\alpha}\langle z_\alpha^*|e^{\epsilon\mathcal{L}(a^\dagger,a)}|z_{\alpha-1}\rangle = \iint \mathcal{D}z^*\mathcal{D}z \lim_{\epsilon\to 0}\prod_{\alpha=1}^{N}\langle z_\alpha^*|z_{\alpha-1}\rangle(1+\epsilon\mathcal{L}_\alpha^0+\epsilon\mathcal{L}_\alpha^1)\\ &= \iint \mathcal{D}z^*\mathcal{D}z \lim_{\epsilon\to 0}\prod_{\alpha=1}^{N}\langle z_\alpha^*|z_{\alpha-1}\rangle(1+\epsilon\mathcal{L}_\alpha^0)(1+\epsilon\mathcal{L}_\alpha^1) = \iint \mathcal{D}z^*\mathcal{D}z e^{-\int z_t^*\cdot\left[\frac{\partial z_t}{\partial t}+pd\nu z_t\right]-z_0^*\cdot z_0}\Pi(T), \quad (54)\end{aligned}$$

where d represents the number of neighbours of each site, and

$$\Pi(T) = \lim_{\epsilon\to 0}\prod_{\alpha=1}^{N-1} e^{\epsilon\mathcal{L}^1(z_\alpha^*,z_{\alpha-1})} = \tau\left\{\exp\int_0^T X(t)dt\right\} = \sum_{n=0}^{\infty}\int_{\Delta_n} X(t_1)\dots X(t_n)d\mathbf{t}_n. \quad (55)$$

This produces a path integral with form similar to the previous section, except propagators take the form $e^{\nu p d t}$, we have two types of Feynman diagram nodes corresponding to the two types of terms in $\mathcal{L}^1$, and now there are p bins for each site of the lattice.

Now, each term arising from $\nu\sum_{I,J} N_I N_J$ takes the form $\nu z_{I,\kappa}^* z_{I,\kappa} z_{J,\ell}^* z_{J,\ell}$ linking a propagator with label (I,κ) to (J,ℓ), where I and J and neighboring sites, and κ and ℓ index bins. These are analogous to the vertical segments representing neutral terms in the previous section. If $\mathbf{n}$ is a vector indexing the (assumed finite) number of occupied bins across sites, the number of possible links is $\pi_{\mathbf{n}} = \frac{1}{2}\sum_{I,J} n_I n_J$. Then, much like Eq. 47, the Laplace transform of the sum of such terms across any time segment with total occupation number $n = \sum_I n_I$ is

$$L(s) = \sum_{m=1}^{\infty}\frac{(\nu\pi_{\mathbf{n}})^{m-1}}{(s+n\nu pd)^m} = \frac{1}{s+n\nu pd-\nu\pi_{\mathbf{n}}}. \quad (56)$$

Now we assume that $\mathbf{n}^0$ is the initial distribution across the lattice. We let $f_{\mathbf{n}}^{(k)}$ denote the sum of Laplace transformed diagrams representing $k-1$ non-neutral nodes arising from $\nu\sum_{I,J} a_I a_J^\dagger$ that start with distribution $\mathbf{n}^0$ and end with distribution $\mathbf{n}$. Then if $\mathbf{n}^{+I,-J}$ is the vector $\mathbf{n}$ with the $I^{\text{th}}(J^{\text{th}})$ component increased(decreased) by one unit, we have recurrence:

$$f_{\mathbf{n}}^{(k+1)} = \frac{\nu}{s+n\nu pd-\nu\pi_{\mathbf{n}}}\sum_{I,J} f_{\mathbf{n}^{+I,-J}}^{(k+1)}(n_I+1)(p-n_J+1). \quad (57)$$

Then defining $f_{\mathbf{n}} = \sum_{k=1}^{\infty} f_{\mathbf{n}}^{(k)}$ and noting that $(s+n\nu pd-\nu\pi_{\mathbf{n}})f_{\mathbf{n}}^{(1)} = \delta_{\mathbf{n},\mathbf{n}^0}$ results in a recurrence, analogous to Eq. 51,

$$(s+n\mu+\pi_{\mathbf{n}})f_{\mathbf{n}} + \sum_{I,J}\nu f_{\mathbf{n}^{+I,-J}}(n_I+1)(p-n_J+1) = \delta_{\mathbf{n},\mathbf{n}^0}. \quad (58)$$

This is a finite (albeit high dimensional) system of equations for a finite population, which can be solved exactly, meaning moments can then be obtained in much the same way as the previous section.

6. Conclusions

In this paper a generalized Fock space has been developed, explaining the different Fock spaces seen in the literature for stochastic systems with partial exclusivity. The machinery of generalized paragrassmannian algebras has also been utilized to construct path integral formulations for features of interest, a methodology not explored elsewhere. The non-commutative nature of paragrassmannians means that a Magnus expansion is required to construct a single action, which can be complex in nature, resulting in actions with an infinite number of terms for certain systems. This adjustment also applies to fermionic path integrals, an issue not discussed in the literature [18], [19], [30]. This means exact techniques will in general be difficult to implement, although perturbative techniques have been shown to be feasible, albeit involving rather intricate calculations. An alternative approach to fermionic techniques for exclusive systems was developed in [17], using a bosonic framework. This yields path integral actions with a finite number of terms, although the terms involved are exponential (i.e. non-polynomial) in nature. This offers certain advantages in terms of exact transformations of the action (see details in [17]), although perturbative techniques are likely to be complicated, requiring expansion of both exponential terms in the action and the (exponentiated) action itself. The techniques in [17] were not applied to systems with partial exclusion, although such an approach seems feasible.

These parafermionic methods have been applied to birth-death processes with a carrying capacity, and diffusion on an occupation limited lattice, producing results consistent with those obtained via classical methods. Finding alternative perturbative and non-perturbative expansion schemes to offer alternative solution formulations, and make greater use of the Magnus expansion, remains a future direction of research. An obvious approach is to examine the effect of the Doi shift on expressions. Although the quantity $\hat{z} = z + 1$ is not a paragrassmannian number, $\int dz(a + bz)$ is invariant to such a substitution and a Doi shift is valid. However, the fact that the path integrals are not easily expressed as a single action seems to limit the usefulness of such an approach, and other ideas are needed.

References

[1] S. Weinberg, The quantum theory of fields, Vol. 1, Cambridge university press, 1995.

[2] F. Mandl, G. Shaw, Quantum field theory, John Wiley & Sons, 2010.

[3] M. Doi, Second quantization representation for classical many-particle system, Journal of Physics A: Mathematical and General 9 (9) (1976) 1465.

[4] M. Doi, Stochastic theory of diffusion-controlled reaction, Journal of Physics A: Mathematical and General 9 (9) (1976) 1479.

[5] R. P. Feynman, A. R. Hibbs, Quantum mechanics and path integrals, Vol. 2, McGraw-Hill New York, 1965.

[6] L. Peliti, Path integral approach to birth-death processes on a lattice, Journal de Physique 46 (9) (1985) 1469–1483.

[7] L. Peliti, Renormalisation of fluctuation effects in the a+ a to a reaction, Journal of Physics A: Mathematical and General 19 (6) (1986) L365.

[8] J. L. Cardy, U. C. Täuber, Field theory of branching and annihilating random walks, Journal of statistical physics 90 (1) (1998) 1–56.

[9] H.-K. Janssen, U. C. Täuber, The field theory approach to percolation processes, Annals of Physics 315 (1) (2005) 147–192.

[10] P. D. Jarvis, J. Bashford, J. Sumner, Path integral formulation and feynman rules for phylogenetic branching models, Journal of Physics A: Mathematical and General 38 (44) (2005) 9621.

[11] J. Ohkubo, Algebraic probability, classical stochastic processes, and counting statistics, Journal of the Physical Society of Japan 82 (8) (2013) 084001.

[12] C. M. Rohwer, K. K. Müller-Nedebock, Operator formalism for topology-conserving crossing dynamics in planar knot diagrams, Journal of Statistical Physics 159 (1) (2015) 120–157.

[13] C. D. Greenman, A path integral approach to age dependent branching processes, Journal of Statistical Mechanics: Theory and Experiment 2017 (3) (2017) 033101.

[14] S. Sandow, S. Trimper, Aggregation processes in a master-equation approach, EPL (Europhysics Letters) 21 (8) (1993) 799.

[15] M. Schulz, S. Trimper, Fock-space approach to the kinetic ising model, Physical Review B 53 (13) (1996) 8421.

[16] M. Schulz, S. Trimper, Exclusive dynamics and nonlinear diffusion equation, Physics Letters A 227 (3-4) (1997) 172–176.

[17] F. van Wijland, Field theory for reaction-diffusion processes with hard-core particles, Physical Review E 63 (2) (2001) 022101.

[18] M. Schulz, P. Reineker, Exact substitute processes for diffusion–reaction systems with local complete exclusion rules, New Journal of Physics 7 (1) (2005) 31.

[19] É. M. Silva, P. T. Muzy, A. E. Santana, Fock space for fermion-like lattices and the linear glauber model, Physica A: Statistical Mechanics and its Applications 387 (21) (2008) 5101–5109.

[20] H. S. Green, A generalized method of field quantization, Physical Review 90 (2) (1953) 270.

[21] Y. Ohnuki, S. Kamefuchi, Quantum field theory and parastatistics, Springer-Verlag, 1982.

[22] A. P. Polychronakos, Path integrals and parastatistics, Nuclear Physics B 474 (2) (1996) 529–539.

[23] O. Greenberg, A. Mishra, Path integrals for parastatistics, Physical Review D 70 (12) (2004) 125013.

[24] M. Schulz, S. Trimper, Three-state model and chemical reactions, Journal of Physics A: Mathematical and General 29 (20) (1996) 6543.

[25] M. Schulz, S. Trimper, Parafermi statistics and p-state models, Physics Letters A 216 (6) (1996) 235–239.

[26] G. Schütz, S. Sandow, Non-abelian symmetries of stochastic processes: Derivation of correlation functions for random-vertex models and disordered-interacting-particle systems, Physical Review E 49 (4) (1994) 2726.

[27] B. P. Lee, J. Cardy, Scaling of reaction zones in the a+b 0 diffusion-limited reaction, Physical Review E 50 (5) (1994) R3287.

[28] B. P. Lee, J. Cardy, Renormalization group study of thea+ bŁ diffusion-limited reaction, Journal of statistical physics 80 (5) (1995) 971–1007.

[29] U. C. Täuber, M. Howard, B. P. Vollmayr-Lee, Applications of field-theoretic renormalization group methods to reaction–diffusion problems, Journal of Physics A: Mathematical and General 38 (17) (2005) R79.

[30] U. C. Täuber, Critical dynamics: a field theory approach to equilibrium and non-equilibrium scaling behavior, Cambridge University Press, 2014.

[31] M. F. Weber, E. Frey, Master equations and the theory of stochastic path integrals, Reports on Progress in Physics 80 (4) (2017) 046601.
[32] A. T. A. T. Bharucha-Raid, et al., Elements of the theory of markov processes and their applications, Tech. rep. (1960).
[33] W. Feller, Die grundlagen der volterraschen theorie des kampfes ums dasein in wahrscheinlichkeitstheoretischer behandlung, Acta Biotheoretica 5 (1) (1939) 11–40.
[34] D. G. Kendall, On the generalized" birth-and-death" process, The annals of mathematical statistics (1948) 1–15.
[35] M. Takashima, Note on evolutionary processes, Bulletin of Mathematical Statistics 7 (1) (1956) 18–24.
[36] D. G. Kendall, Stochastic processes and population growth, Journal of the Royal Statistical Society. Series B (Methodological) 11 (2) (1949) 230–282.
[37] W. Magnus, On the exponential solution of differential equations for a linear operator, Communications on pure and applied mathematics 7 (4) (1954) 649–673.
[38] S. Blanes, F. Casas, J. Oteo, J. Ros, The magnus expansion and some of its applications, Physics Reports 470 (5) (2009) 151–238.
[39] L. Saenz, R. Suarez, A combinatorial approach to the generalized baker–campbell–hausdorff–dynkin formula, Systems & control letters 45 (5) (2002) 357–370.
[40] F. R. Gantmakher, The theory of matrices, Vol. 131, American Mathematical Soc., 1998.