PSEUDO-EXPONENTIAL MAPS, VARIANTS, AND QUASIMINIMALITY

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ABSTRACT. We give a construction of quasiminimal fields equipped with pseudo-analytic maps, generalising Zilber's pseudo-exponential function. In particular we construct pseudo-exponential maps of simple abelian varieties, including pseudo-$\wp$-functions for elliptic curves. We show that the complex field with the corresponding analytic function is isomorphic to the pseudo-analytic version if and only the appropriate version of Schanuel's conjecture is true and the corresponding version of the strong exponential-algebraic closedness property holds. Moreover, we relativize the construction to build a model over a fairly arbitrary countable subfield and deduce that the complex exponential field is quasiminimal if it is exponentially-algebraically closed. This property asks only that the graph of exponentiation have non-trivial intersection with certain algebraic varieties but does not require genericity of these points. Furthermore Schanuel's conjecture is not required as a condition for quasiminimality.

CONTENTS

1. Introduction

1.1. **Exponential fields.** The field $\mathbb{C}$ of complex numbers is well-known to be *strongly minimal*, that is, any subset of $\mathbb{C}$ definable in the ring language is either finite or cofinite. Consequently the model theory of $\mathbb{C}$ is very tame: there is a very well-understood behaviour of the models (one model of each uncountable cardinality, known as uncountable categoricity) and of the definable sets (they have finite Morley rank and we can understand them geometrically in terms of algebraic varieties). The other most important mathematical field, the field $\mathbb{R}$ of real numbers, is *o-minimal* which means that although the class of models is not well-behaved (not classifiable) there is a very good geometric understanding of the definable sets (they are the semialgebraic sets). Remarkably, Wilkie showed that when the real exponential function e^x is adjoined, the structure $\mathbb{R}_{\exp}$ is still o-minimal [Wil96]. Adjoining the complex exponential function e^z to $\mathbb{C}$ gives the structure $\mathbb{C}_{\exp}$ which cannot be well-behaved in terms of the class of models or the definable sets because it interprets the ring $\mathbb{Z}$. However, Zilber suggested that in the model $\mathbb{C}_{\exp}$ itself, the influence of $\mathbb{Z}$ might only extend to the countable subsets of $\mathbb{C}$. He made the following conjecture.

Conjecture 1.1 (Zilber's weak quasiminimality conjecture)**.** *The complex exponential field* $\mathbb{C}_{\exp} = \langle \mathbb{C}; +, \cdot, \exp \rangle$ *is* quasiminimal, *that is, every subset of $\mathbb{C}$ definable in $\mathbb{C}_{\exp}$ is either countable or co-countable.*

A slightly stronger version of the conjecture which avoids reference to definable sets is that every automorphism-invariant subset is countable or co-countable. As far as we are aware, all known approaches to the conjecture would give this stronger result anyway. If the conjecture is true then the solution sets of exponential polynomial equations, which we can call complex exponential varieties, would be expected to have good geometric properties similar to those of algebraic varieties, provided

we avoid some exceptional cases like $\mathbb{Z}$. If the conjecture is false, another possibility is that $\mathbb{R}$ is definable as a subset of $\mathbb{C}$. The field $\mathbb{R}$ with $\mathbb{Z}$ as a definable subset is so-called *second-order arithmetic*, and the definable sets are extremely wild, with no geometric properties in general.

As one approach to his conjecture, Zilber [Zil00], [Zil05b] showed how to construct a quasiminimal exponential field we call $\mathbb{B}$ using a variant of Hrushovski's predimension method from [Hru93]. He called $\mathbb{B}$ a *pseudo-exponential field* with the idea that the exponential map is a *pseudo-analytic* function.

Zilber's approach was to prove that a certain list of axioms $\mathrm{ECF}_{\mathrm{SK,CCP}}$ in the infinitary logic $L_{\omega_1,\omega}(Q)$ behaves in an analogous way to a strongly minimal first-order theory. In particular, all its models are quasiminimal and it is uncountably categorical.

Theorem 1.2. *Up to isomorphism there is exactly one model of the axioms $\mathrm{ECF}_{\mathrm{SK,CCP}}$ of each uncountable cardinality, and it is quasiminimal.*

This theorem appears in [Zil05b]. Some gaps in the proof were filled in the unpublished note [BK13], which this paper supersedes. In this paper we give a new construction of $\mathbb{B}$ and hence a complete proof of Theorem 1.2. The theorem suggests a stronger form of the quasiminimality conjecture which evidently implies Conjecture 1.1.

Conjecture 1.3 (Strong quasiminimality conjecture). $\mathbb{C}_{\exp}$ *is isomorphic to the unique model $\mathbb{B}$ of $\mathrm{ECF}_{\mathrm{SK,CCP}}$ of cardinality continuum.*

The axioms in $\mathrm{ECF}_{\mathrm{SK,CCP}}$ will be explained in Section 8, but briefly there are two algebraic axioms which are obviously true in $\mathbb{C}_{\exp}$ and then three more axioms: Schanuel's conjecture, strong exponential-algebraic closedness, and the countable closure property. Schanuel's conjecture is a conjecture of transcendental number theory which can be seen as saying that certain systems of exponential polynomial equations do not have solutions. Strong exponential-algebraic closedness roughly says that a system of equation has solutions (even generic over any given finite set) unless that would contradict Schanuel's conjecture. The countable closure property says roughly that such systems of equations which are *balanced*, in the sense of having the same number of equations as variables, have only countably many solutions. Zilber proved the countable closure property for $\mathbb{C}_{\exp}$, so we have the following reformulation.

Theorem 1.4. *Conjecture 1.3 is true if and only if Schanuel's conjecture is true and $\mathbb{C}_{\exp}$ is strongly exponentially-algebraically closed.*

Theorems 1.2 and 1.4 together imply that if $\mathbb{C}_{\exp}$ satisfies Schanuel's conjecture and is strongly exponentially-algebraically closed then it is quasiminimal. Schanuel's conjecture is considered out of reach, since even the very simple consequence that the numbers e and π are algebraically independent is unknown. Proving strong exponential-algebraic closedness involves finding solutions of certain systems of equations and then showing they are generic, the latter step usually done using Schanuel's conjecture. A weaker condition is *exponential-algebraic closedness* which requires the same systems of equations to have solutions, but says nothing about their genericity. We are able to remove the dependence on Schanuel's conjecture completely from Conjecture 1.1.

Theorem 1.5. *If $\mathbb{C}_{\exp}$ is exponentially-algebraically closed then it is quasiminimal.*

1.2. **A more general construction: Γ-fields.** Our construction is more general and we can use it to construct also a pseudo-analytic version of the Weierstrass

$\wp$-functions, the exponential maps of simple abelian varieties, and more generally other pseudo-analytic subgroups of the product of two commutative algebraic groups. For example, we prove an analogous form of Theorems 1.2 and 1.4 for $\wp$-functions. The list of axioms $\wp\mathrm{CF}_{\mathrm{SK,CCP}}(E)$ and the other notions used in the statement of the theorem will be explained in section 9.3 of the paper.

Theorem 1.6. *Given an elliptic curve E over a number field $K_0 \subseteq \mathbb{C}$, the list $\wp\mathrm{CF}_{\mathrm{SK,CCP}}(E)$ of axioms is uncountably categorical and every model is quasiminimal. Furthermore, if $\wp$ is the Weierstrass function associated to $E(\mathbb{C})$, so $\exp_E = [\wp : \wp' : 1] : \mathbb{C} \to E(\mathbb{C})$ is the exponential map of $E(\mathbb{C})$, then $\mathbb{C}_\wp := \langle \mathbb{C}; +, \cdot, \exp_E \rangle \models \wp\mathrm{CF}_{\mathrm{SK,CCP}}(E)$ if and only if the analogue of Schanuel's conjecture for $\wp$ holds and $\mathbb{C}_\wp$ is strongly $\wp$-algebraically closed.*

In the most general form, we consider what we call Γ-fields, which are fields F of characteristic zero equipped with a subgroup $\Gamma(F)$ of a product $G_1(F) \times G_2(F)$ where G_1 and G_2 are commutative algebraic groups. The complete definition is given in section 3.1, where we also explain how the examples we consider fit into cases (EXP), generalizing the exponential and Weierstrass $\wp$-functions above, (COR), generalizing analytic correspondences between non-isogenous elliptic curves, and (DEQ), generalizing the solution sets of certain differential equations.

Hrushovski used Fraïssé's amalgamation method which produces countable structures. Zilber wanted uncountable structures so he instead framed his constructions in terms of existentially closed models within a certain category. He gave a framework of *quasiminimal excellent classes* [Zil05a], building on Shelah's notion of an excellent $L_{\omega_1,\omega}$-sentence [She83], to prove the uniqueness of the uncountable models. The second author showed [Kir10b] that the quasiminimal excellence conditions can be checked just on the countable models, and with Hart, Hyttinen and Kesälä we proved in [BH²K²14] that the most complicated of the conditions to check, excellence, follows from the other conditions. So in this paper we recast the construction in 4 stages.

1. We start with a suitable base Γ-field F_{base}, and describe a category $\mathcal{C}(F_{\mathrm{base}})$ of so-called *strong extensions* of F_{base}.
2. We apply a suitable version of *Fraïssé's amalgamation theorem* to the category to produce a countable model $M(F_{\mathrm{base}})$.
3. We check that $M(F_{\mathrm{base}})$ satisfies the conditions to be part of a *quasiminimal class*, and deduce there is a unique model of cardinality continuum we denote by $\mathbb{M}(F_{\mathrm{base}})$.
4. We give the *axioms* $\Gamma\mathrm{CF}_{\mathrm{CCP}}(F_{\mathrm{base}})$ describing the class.

As a more general form of Theorem 1.2 we prove:

Theorem 1.7. *Given an essentially finitary Γ-field F_{base} of type (EXP), (COR), or (DEQ), the list of axioms $\Gamma\mathrm{CF}_{\mathrm{CCP}}(F_{\mathrm{base}})$ is uncountably categorical and every model is quasiminimal.*

Our notion of Γ-fields is algebraic and not every example is related to an analytic prototype. However cases (EXP) and (COR) do have many analytic examples, given in Definitions 3.1 and 3.2. We call these *analytic Γ-fields*. For these we are able to prove the countable closure property, extending Zilber's result for $\mathbb{C}_{\exp}$ and the equivalent result in [JKS16] for $\wp$-functions.

Theorem 1.8. *Let $\mathbb{C}_\Gamma$ be an analytic Γ-field. Then $\mathbb{C}_\Gamma$ satisfies the countable closure property.*

One of the key ideas of this paper is that the amalgamation construction is done over a base Γ-field F_{base}, and that everything is done relative to that base. Pushing

this idea further, we can also work over a base which is closed with respect to the quasiminimal pregeometry on the model F_{base}. This involves modifying the amalgamation construction so we only consider extensions of F_{base} in which F_{base} remains closed with respect to the pregeometry. This idea comes from differential fields, where the base would be the field of constants and one often wants to consider differential field extensions with no new constants. One advantage of this approach for us is that Ax's versions of Schanuel's conjecture then apply to say that Schanuel's conjecture is true relative to the base in the analytic Γ-fields.

In the paper [KZ14] it was shown that, assuming the Conjecture on Intersections with Tori (CIT, also known as the multiplicative Zilber-Pink conjecture), any exponential field satisfying Schanuel's conjecture and exponential-algebraic closedness is actually strongly exponentially-algebraically closed. In this paper we are able to adapt that idea to show unconditionally that the difference between Γ-closedness and strong Γ-closedness (the analogues of exponential-algebraic closedness and strong exponential-algebraic closedness) disappears if we consider the generic version, meaning relative to a closed base. Instead of the CIT we use a theorem which we call the *horizontal semiabelian weak Zilber-Pink*, a theorem about intersections of families of algebraic varieties with cosets of algebraic subgroups of semiabelian varieties. This method allows us to prove Theorem 1.5 and a more general version:

Theorem 1.9. *Let $\mathbb{C}_\Gamma$ be an analytic Γ-field. If $\mathbb{C}_\Gamma$ is Γ-closed then it is quasiminimal.*

1.3. **An overview of the paper.** In section 2 we explain our conventions on viewing algebraic varieties and their profinite covers in a model-theoretic way. We also explain the relationship between subgroups and endomorphisms of the commutative algebraic groups we study.

In section 3 we define our Γ-fields and their finitely generated extensions. We prove that finitely generated extensions of suitable (so-called *essentially finitary*) Γ-fields are determined by good bases, and that these good bases exist and are determined by finite data from a countable range of possibilities. This is the key step in proving the form of $\aleph_0$-stability which is essential for the existence of quasiminimal models. The main tool here is Kummer theory over torsion for abelian varieties.

In section 4 we introduce the predimension notion and use it to define which extensions of Γ-fields are strong. We also use it to define a pregeometry on Γ-fields. Then we show that there is a unique *full-closure* of an essentially finitary Γ-field, and classify the strong finitely generated extensions of Γ-fields and of full Γ-fields. This completes stage 1 of the construction as described above.

Section 5 covers stage 2 of the construction. We recall a category-theoretic version of Fraïssé's amalgamation theorem which is suitably general for us. Then, starting with a suitable base Γ-field F_{base}, we consider the category $\mathcal{C}(F_{\text{base}})$ of strong extensions of F_{base} and apply the amalgamation theorem to get a countable Fraïssé limit $M(F_{\text{base}})$. We also consider a variant amalgamating only the Γ-algebraic extensions and another variant where we consider only extensions which are purely Γ-transcendental over F_{base}.

In section 6 we show that the Fraïssé limit models we have produced are quasiminimal pregeometry structures, and hence give rise to uncountably categorical classes. In this way we get the uncountable models, in particular the model $\mathbb{M}(F_{\text{base}})$ of cardinality continuum. This is stage 3.

In section 7 we give a classification of the finitely generated strong extensions of Γ-fields, and in section 8 we use that to give axiomatizations of our models and prove Theorem 1.7. This completes stage 4.

In section 9 we consider specific instances of our Γ-fields including pseudo-exponentiation, pseudo Weierstrass $\wp$-functions, and others, and prove Theorem 1.2 and half of Theorem 1.6.

In section 10 we compare our models to the complex analytic prototypes. For Weierstrass $\wp$-function we relate the Schanuel property to the André-Grothendieck conjecture on the periods of 1-motives, using work of Bertolin, finishing the proof of Theorem 1.6. We briefly discuss the literature on steps towards proving the strong Γ-closedness and Γ-closedness properties for analytic Γ-fields. Then we prove Theorem 1.8.

In section 11 we consider Γ-fields which may not be Γ-closed but are generically so. These are the Γ-fields produced by the variant construction in which the base F_{base} remains closed with respect to the pregeometry. We state and prove the horizontal semiabelian weak Zilber-Pink, and then prove Theorems 1.5 and 1.9.

2. Algebraic background

2.1. **Algebraic varieties and groups.** We will use the standard model-theoretic foundations for algebraic varieties and algebraic groups, as described by Pillay [Pil98], roughly following Weil. In particular, we will work in the theory ACF_0 with parameters for a field K_0. Any variety V is considered as a definable set, and using elimination of imaginaries it is in definable bijection with a constructible subset of affine space. We always assume we have chosen such a bijection, although we will not mention it explicitly. Given any field extension F of K_0, we write $V(F)$ for the points of V all of whose coordinates lie in F. In this way, V is a functor from the category of field extensions of its field of definition to the category of sets. Similarly, given any subset $A \subseteq V(F)$, we can form the subfield of F which is generated by (the co-ordinates of) the points in A.

In the same way, a commutative algebraic group G, defined over K_0, is considered as a functor from the category of field extensions of K_0 to the category of abelian groups. If G is an algebraic $\mathcal{O}$-module, that is, the ring $\mathcal{O}$ acts on G via regular endomorphisms, defined over K_0, we can also consider it as a functor to the category of $\mathcal{O}$-modules.

$\mathbb{G}_{\text{a}}$ denotes the additive group, $\mathbb{G}_{\text{a}}(F) = \langle F; + \rangle$, and $\mathbb{G}_{\text{m}}$ denotes the multiplicative group, $\mathbb{G}_{\text{m}}(F) = \langle F^\times; \cdot \rangle$.

An algebraic group is *connected* if it has no proper finite index algebraic subgroups. The *connected component* G^o of an algebraic group G is the largest connected algebraic subgroup.

We write $G[m]$ for the m-torsion subgroup of an algebraic group G.

If G is a commutative algebraic group over a field of characteristic 0, we write LG for the (commutative) Lie algebra of G, the tangent space at the identity considered as an algebraic group. So $LG \cong \mathbb{G}_{\text{a}}^{\dim(G)}$. If $\theta : G \to G'$ is an algebraic group homomorphism, then $L\theta : LG \to LG'$ is the derivative at the identity.

For algebraic groups over $\mathbb{C}$, these definitions agree with the usual definitions for complex Lie groups.

2.2. **Subgroups and endomorphisms.** Any connected algebraic subgroup H of a power $\mathbb{G}_{\mathrm{m}}^n$ of the multiplicative group can be defined by a system of monomial equations, $H = \ker(M)$ for some integer square matrix $M \in \mathrm{Mat}_n(\mathbb{Z})$ acting multiplicatively. Then $LH \leq L\mathbb{G}_{\mathrm{m}}^n = \mathbb{G}_{\mathrm{a}}^n$ is the kernel of the same matrix acting additively.

As we observe in the following lemma, the picture is almost the same when we replace $\mathbb{G}_{\mathrm{m}}$ with an abelian variety G and $\mathbb{Z}$ with its endomorphism ring $\mathrm{End}(G)$: up to finite index, subgroups are defined by $\mathcal{O}$-linear equations, namely those which define the corresponding Lie subalgebra. With a few self-contained exceptions, this lemma is essentially all we will use of the theory of abelian varieties.

Lemma 2.1. *Suppose G is $\mathbb{G}_{\mathrm{m}}$ or an abelian variety over a field of characteristic 0, and $\mathcal{O} = \mathrm{End}(G)$ is its endomorphism ring. Then*

  (i) *any connected algebraic subgroup $H \leq G^n$ is the connected component of the kernel of an endomorphism $\eta \in \mathrm{End}(G^n) \cong \mathrm{Mat}_n(\mathcal{O})$,*

$$H = \ker(\eta)^o;$$

  (ii) *$LH \leq LG^n$ is then the kernel of $L\eta \in \mathrm{End}(LG^n)$.*

Proof.   (i) By Poincaré's complete reducibility theorem [Mum70, p173], there exists an algebraic subgroup H' such that the summation map $\Sigma : H \times H' \to G^n$ is an isogeny, that is, a surjective homomorphism with finite kernel. Let m be the exponent of the kernel of Σ. Then $\theta(\Sigma(h, h')) := (mh, mh')$ defines an isogeny $\theta : G^n \to H \times H'$. Let $\pi_2 : H \times H' \to H'$ be the projection. Then $(\pi_2 \circ \theta \circ \Sigma)(h, h') = mh'$, so $\ker(\pi_2 \circ \theta)^o = (\Sigma(H \times H'[m]))^o = (H + H'[m])^o = H$.

  (ii) Since $H \subseteq \ker(\eta)$, the derivative $L\eta$ of η at 0 vanishes on LH, so $LH \leqslant \ker(L\eta)$. Also $\ker(L\eta) = L\ker(\eta)$, so we have

$$
\begin{aligned}
\dim(\ker(L\eta)) &= \dim(L\ker(\eta)) \\
&= \dim(\ker(\eta)) \quad \text{since 0 is a smooth point} \\
&= \dim H \quad \text{since H has finite index in $\ker(\eta)$} \\
&= \dim LH \quad \text{again since 0 is a smooth point.}
\end{aligned}
$$

So LH has finite index in $\ker(L\eta)$, but $\ker(L\eta) \leqslant LG^n$ which is torsion-free, so $\ker(L\eta)$ is connected, so $LH = \ker(L\eta)$. $\square$

2.3. **Division points and the profinite cover.**

Definition 2.2. Let G be a commutative group and let $a \in G$. A *division point* of a in G is any $b \in G$ such that, for some $m \in \mathbb{N}^+$, $mb = a$.

A *division sequence* for a in G is a sequence $(a_m)_{m \in \mathbb{N}^+}$ in G such that $a_1 = a$ and for all $m, n \in \mathbb{N}^+$ we have $na_{nm} = a_m$.

If $(a_m)_{m \in \mathbb{N}^+}$ is a division sequence for a in G we can define a group homomorphism $\theta : \mathbb{Q} \to G$ by $\theta(\frac{r}{m}) = ra_m$ for $r \in \mathbb{Z}$ and $m \in \mathbb{N}^+$. This gives a bijective correspondence between division sequences for a in G and group homomorphisms $\theta : \mathbb{Q} \to G$ such that $\theta(1) = a$.

Definition 2.3. The *profinite cover* $\widehat{G}$ of a commutative group G is the group of all homomorphisms $\mathbb{Q} \to G$, with the group structure defined pointwise in G. We write $\rho_G : \widehat{G} \to G$ for the evaluation homomorphism given by $\rho_G(\theta) = \theta(1)$.

Thus the set of division sequences for a in G is in bijective correspondence with $\rho_G^{-1}(a)$, and we think of elements of $\widehat{G}$ both as homomorphisms from $\mathbb{Q}$ and as division sequences.

The group $\widehat{G}$ itself is divisible and torsion-free. The image of ρ_G is the subgroup of divisible points of G, and ρ_G is injective if and only if G is torsion-free. In general, $\ker(\rho_G)$ is a profinite group built from the torsion of G (in fact it is, the product over primes l of the l-adic Tate modules of G).

For an element $a \in G$, we will often use the notation $\widehat{a}$ for a chosen element of $\widehat{G}$ such that $\rho_G(\widehat{a}) = a$. Of course $\widehat{a}$ is determined by a only when ρ_G is injective, that is, when G is torsion-free.

If $f : G \to H$ is a group homomorphism, we can lift it to a homomorphism $\widehat{f} : \widehat{G} \to \widehat{H}$ defined by $\theta \mapsto f \circ \theta$. In particular, if $G \subseteq H$ is a subgroup then $\widehat{G}$ is naturally a subgroup of $\widehat{H}$. (In category-theoretic language, $\widehat{}$ is a covariant representable functor and in fact $\rho_G : \widehat{G} \to G$ is the universal arrow from the category of divisible, torsion-free abelian groups into G.)

When G is a commutative algebraic group we think of $\widehat{G}$ also as a functor, so we write $\widehat{G}(F)$ rather than $\widehat{G(F)}$ for the group of division sequences of the group $G(F)$.

Model-theoretically we think of $\widehat{G}$ as the set of division sequences from G, which is a set of infinite tuples satisfying the divisibility conditions. It can be seen as an inverse limit of definable sets, sometimes called a pro-definable set [Kam07].

Remark 2.4. Suppose that G is a Lie group, for example the complex points of a complex algebraic group, and $\exp : LG \to G$ is the exponential map. For $a \in LG$, the sequence $(\exp(a/m))_{m \in \mathbb{N}^+}$ is a division sequence in G. In fact, the division sequences which arise this way are precisely those which converge topologically to the identity of G [BHP14, Remark 2.20].

3. Γ-FIELDS

3.1. **Γ-fields.** In this section we describe the analytic examples we are studying and give the definition of a Γ-field which is intended to capture and generalize the model-theoretic algebra of the examples.

Definition 3.1 (Analytic Γ-fields of type (EXP)). The graph of the usual complex exponential function is a subgroup of $\mathbb{G}_a(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C})$. Similarly, if $A(\mathbb{C})$ is a complex abelian variety (or more generally any commutative complex algebraic group) of dimension d, then the graph Γ of the exponential map of A is a subgroup of $LA(\mathbb{C}) \times A(\mathbb{C})$. Here $LA(\mathbb{C})$ is the Lie algebra of A and we can identify it with the group $\mathbb{G}_a^d(\mathbb{C})$. In this paper we will only consider the cases when A is $\mathbb{G}_m$ or A is a simple abelian variety of dimension d. We combine these by saying A is a *simple semiabelian variety*.

We write $\mathcal{O}$ for the ring $\mathrm{End}(A)$ of algebraic endomorphisms of A. In many cases $\mathcal{O} = \mathbb{Z}$, but sometimes, for example if A is an elliptic curve with *complex multiplication*, then $\mathcal{O}$ properly extends $\mathbb{Z}$. Any $\eta \in \mathcal{O}$ acts on LA as the derivative $d\eta$, a linear map. Thus $\mathcal{O}$ naturally acts on $LA(\mathbb{C})$ as a subring of $\mathrm{GL}_d(\mathbb{C})$, and Γ is an $\mathcal{O}$-submodule of $LA(\mathbb{C}) \times A(\mathbb{C})$.

In the case where A is an elliptic curve E, embedded in projective space $\mathbb{P}^2$ in the usual way via its Weierstrass equation, then the exponential map of $E(\mathbb{C})$ is written in homogeneous coordinates as $z \mapsto (\wp(z) : \wp'(z) : 1)$, where $\wp$ is the Weierstrass $\wp$-function associated with E.

We call all of these examples *analytic Γ-fields of type (EXP)*.

Definition 3.2 (Analytic Γ-fields of type (COR))**.** The exponential map of a complex elliptic curve factors through $\mathbb{G}_{\mathrm{m}}(\mathbb{C})$, giving an analytic map $\theta : \mathbb{G}_{\mathrm{m}}(\mathbb{C}) \to E(\mathbb{C})$. More generally there are analytic correspondences between semiabelian varieties. We take G_1 and G_2 both to be simple complex semiabelian varieties of the same dimension d, and assume G_1 and G_2 are not isogenous. Suppose $\mathrm{End}(G_1)$ and $\mathrm{End}(G_2)$ are both isomorphic to a ring $\mathcal{O}$, and furthermore there is a $\mathbb{C}$-vector space isomorphism $\psi : LG_1 \to LG_2$ which respects the actions of $\mathcal{O}$. We choose such a ψ and take Γ to be the image of the graph of ψ under $\exp_{G_1 \times G_2} : LG_1(\mathbb{C}) \times LG_2(\mathbb{C}) \to G_1(\mathbb{C}) \times G_2(\mathbb{C})$. Then Γ is an $\mathcal{O}$-submodule of $G_1(\mathbb{C}) \times G_2(\mathbb{C})$, and a complex Lie-subgroup. The graph of the map θ is an example of such a Γ, but more generally Γ will not be the graph of a function.

We call these examples *analytic Γ-fields of type (COR)*. By an *analytic Γ-field* we mean one of type (EXP) or type (COR).

Definition 3.3 (Differential equation examples)**.** If $f(t)$ is a holomorphic function in a neighbourhood of $0 \in \mathbb{C}$, then the pair $(x, y) = (f(t), \exp(f(t)))$ satisfies the differential equation $Dx = \frac{Dy}{y}$, where $D = \frac{\mathrm{d}}{\mathrm{d}t}$. We can consider the set Γ of solutions of the differential equation not just in a field of functions but in a differentially closed field F. Then Γ is a subgroup of $\mathbb{G}_{\mathrm{a}}(F) \times \mathbb{G}_{\mathrm{m}}(F)$. The paper [Kir09] studies this situation for the differential equations satisfied by the exponential maps of semiabelian varieties S. While these S do not have to be simple, they do have to be defined over the constant field C of F. In these cases the group Γ is quite closely related to the graph of the exponential map and can be analysed via a similar amalgamation construction.

We capture all of these three types of examples in the notion of a Γ-field. We next give the assumptions we will use on the algebraic groups, and then define Γ-fields. The assumptions we make are not the most general possible, but they are what we will use throughout this paper.

Definition 3.4 (Conventions for K_0, G_2, $\mathcal{O}$, and $k_{\mathcal{O}}$)**.** We take K_0 to be a countable field of characteristic 0, which must be a number field except in case (DEQ) below. Let G_2 be a simple semiabelian variety defined over K_0.

We write $\mathcal{O}$ for the ring $\mathrm{End}(G_2)$ of algebraic (that is, regular) group endomorphisms of G_2 and assume that they are also all defined over K_0. Let $k_{\mathcal{O}}$ denote the ring $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}$.

Remarks 3.5. The ring $\mathcal{O}$ has no zero divisors because G_2 is simple. So $\mathcal{O}$ embeds in $k_{\mathcal{O}}$. If $\mathcal{O} = \mathbb{Z}$ then $k_{\mathcal{O}}$ is just $\mathbb{Q}$. Every non-zero algebraic group endomorphism of a simple abelian variety is an isogeny, so becomes invertible in $k_{\mathcal{O}}$. Hence $k_{\mathcal{O}}$ is a division ring, and the $\mathcal{O}$-torsion of any $\mathcal{O}$-module is exactly the $\mathbb{Z}$-torsion.

Definition 3.6 (Conventions for G_1, G, and the torsion)**.** We consider two cases for the choice of G_1, corresponding to the above analytic examples.

>   **Case (EXP):** We take $G_1 = \mathbb{G}_{\mathrm{a}}^d$, where $d = \dim G_2$. We identify G_1 with the Lie algebra LG_2, that is, the tangent space at the identity of G_2. As in the analytic case, this identification makes G_1 into an algebraic $\mathcal{O}$-module, that is, an $\mathcal{O}$-module in which every element of $\mathcal{O}$ acts as a regular map.

>   **Case (COR):** G_1 is also a simple semiabelian variety defined over K_0, and with all its algebraic endomorphisms defined over K_0. We assume G_1 is not isogenous to G_2, but $\mathrm{End}(G_1) \cong \mathcal{O}$ and we choose an isomorphism, so $G_1 \times G_2$ becomes an algebraic $\mathcal{O}$-module over K_0.

Let $G = G_1 \times G_2$, and write $\pi_i : G \to G_i$ for the projection maps of the product, for $i = 1, 2$. We will write the groups G_1, G_2 and G additively.

For $i = 1, 2$ the torsion of G_i is contained in $G_i(K_0^{\mathrm{alg}})$, and hence is bounded. We write Tor_i for the torsion of $G_i(F)$ for any F such that $G(F)$ contains $\mathrm{Tor}(G(K_0^{\mathrm{alg}}))$. The torsion of $G(F)$ is written $\mathrm{Tor}(G)$. It is equal to $(\mathrm{Tor}_1 \times \mathrm{Tor}_2) \cap G(F)$.

Remarks 3.7. Note that for any algebraically closed field F extending K_0 the groups $G_i(F)$ and $G(F)$ are divisible $\mathcal{O}$-modules. Furthermore, $G(F)/\mathrm{Tor}(G)$ is divisible and torsion-free, and hence is a $k_{\mathcal{O}}$-vector space.

Definition 3.8 (Γ-fields). A Γ-*field* (with respect to the $\mathcal{O}$-module G) is a field extension A of K_0 equipped with a divisible $\mathcal{O}$-submodule $\Gamma(A)$ of $G(A)$ such that

    (1) A is generated as a field by $\Gamma(A)$.
    (2) The projection $\pi_i(\Gamma(A))$ in $G_i(A)$ contains Tor_i for $i = 1, 2$.

We write $\Gamma_i(A)$ for the projections $\pi_i(\Gamma(A))$. The Γ-field A is *full* if, in addition, A is algebraically closed and the projections $\Gamma_i(A)$ are equal to $G_i(A)$.

The *kernels* of a Γ-field A are defined to be

$$\ker_1(A) := \{x \in G_1(A) \mid (x, 0) \in \Gamma(A)\},$$

$$\ker_2(A) := \{y \in G_2(A) \mid (0, y) \in \Gamma(A)\}.$$

When A is full and $\ker_2(A)$ is trivial, $\Gamma(A)$ will be the graph of a surjective $\mathcal{O}$-module homomorphism from $G_1(F)$ to $G_2(F)$ with kernel $\ker_1(F)$ as in the analytic examples of type (EXP). However the case (EXP) for our Γ-fields is more general.

The most difficult part of this paper uses the Kummer theory of semiabelian varieties over number fields. This is not needed for the differential equations examples, or more generally in the following variant.

Definition 3.9 (Case (DEQ)). A Γ-field in case (DEQ) is the same as above except that we require the full torsion group $\mathrm{Tor}(G)$ to be contained in $\Gamma(A)$, and we relax the assumption that K_0 is a number field so it can be any countable field of characteristic 0.

Definition 3.10 (Extensions of Γ-fields). An *extension* of a Γ-field A is a Γ-field B together with an inclusion of fields $A \subseteq B$ over K_0 such that $\Gamma(A) \subseteq \Gamma(B)$. We also say that A is a Γ-*subfield* of B. We say an extension $A \subseteq B$ *preserves the kernels*, and that A and B *have the same kernels*, if $\ker_i(A) = \ker_i(B)$, for $i = 1, 2$.

In this paper, we will only consider extensions of Γ-fields which preserve the kernels.

Remarks 3.11 (Γ-fields as model-theoretic structures).
    (1) Model-theoretically, we consider a Γ-field as a structure in the 1-sorted first-order language $L_\Gamma = \langle +, \cdot, -, \Gamma, (c_a)_{a \in K_0} \rangle$, where Γ is a relation symbol of appropriate arity to denote a subset of the group G, and we have parameters for the field K_0. Later we will also be adding parameters for a base Γ-field F_{base}.
    (2) However, our notion of Γ-field extension corresponds to an injective L_Γ-homomorphism, not necessarily an L_Γ-embedding. Specifically, it is not necessary in an extension $A \hookrightarrow B$ of Γ-fields that $\Gamma(B) \cap G(A) = \Gamma(A)$, although in most cases we will consider later that will be true.
    (3) Although we use the 1-sorted language with the sort being that of the underlying field, we will also refer to elements of Γ as being from the sort Γ, rather than from the definable set Γ. Model theorists used to working with L^{eq} will see there is no important difference.

(4) By definition, the Γ-field A is determined by the submodule $\Gamma(A)$ of $G(F)$. Furthermore, an extension $A \hookrightarrow B$ is determined by the inclusion of submodules $\Gamma(A) \hookrightarrow \Gamma(B)$. Thus, if F is a monster model of ACF_0, the category of Γ-fields is equivalent to the category of divisible $\mathcal{O}$-submodules of $G(F)$ (whose projections contain Tor_1 and Tor_2), with embeddings, in a first-order language with relation symbols for all of the Zariski-closed subsets of $G(F)$ which are defined over K_0. This is more-or-less Zilber's setting in [Zil05b].

Remark 3.12. One might also consider the case that G_1 and G_2 are equal (or, which comes to essentially the same thing, isogenous). Γ can then be considered as the graph of a new (quasi)endomorphism of G_1. The situation is complicated by the need to consider the extension of the algebraic endomorphism ring generated by Γ. Analytic examples include raising to a complex power on $\mathbb{G}_{\mathrm{m}}$, which is analysed with a different setup in [Zil03] and [Zil11].

In an earlier draft of this paper we tried to incorporate this into our setup, and in fact produced an example where Γ was the graph of a multivalued endomorphism θ on $\mathbb{G}_{\mathrm{m}}$, lifting to a generic action of the ring $\mathbb{Q}[\theta, \theta^{-1}]$ on the profinite cover $\widehat{\mathbb{G}_{\mathrm{m}}}$. However this is subtly different from giving an action of the field $\mathbb{Q}(\theta)$ which is what occurs for complex powers.

While we expect that such Γ can be treated along the lines of this paper, much as we expect that the simplicity assumption on the semiabelian variety could be relaxed, these elaborations are left to future work.

3.2. Finitely generated extensions.

Definition 3.13. Let B be a Γ-field, and let $\{A_j \mid j \in J\}$ be a set of Γ-subfields of B, each with the same kernels as B. We define $\bigwedge_{j \in J} A_j$ to be the Γ-subfield A of B such that $\Gamma(A) = \bigcap_{j \in J} \Gamma(A_j)$.

Lemma 3.14. $\bigwedge_{j \in J} A_j$ *is a Γ-subfield of B.*

The proof is straightforward, but we give the details because they show exactly where all the hypotheses of the definitions are used.

Proof. Let $A = \bigwedge_{j \in J} A_j$. Since $\Gamma(A)$ is defined as the intersection of a set of $\mathcal{O}$-submodules of $\Gamma(B)$, it is also an $\mathcal{O}$-submodule of $\Gamma(B)$. A is defined as the subfield of B generated by the coordinates of the points in $\Gamma(A)$, so $\Gamma(A)$ is an $\mathcal{O}$-submodule of $G(A)$.

If $a \in \ker_1(B)$, then $(a, 0) \in \Gamma(A_j)$ for all $j \in J$ because $\ker_1(A_j) = \ker_1(B)$, so $(a, 0) \in \Gamma(A)$. So $\ker_1(A) = \ker_1(B)$ and similarly $\ker_2(A) = \ker_2(B)$.

If $a \in \mathrm{Tor}_1 = \mathrm{Tor}_1(B)$ then there is $b \in G_2(B)$ such that $(a, b) \in \Gamma(B)$. Furthermore for any $b' \in G_2(B)$ we have $(a, b') \in \Gamma(B)$ if and only if $b' - b \in \ker_2(B)$. For each $j \in J$, A_j is a Γ-subfield of B, so $\Gamma_1(A_j)$ contains $\mathrm{Tor}_1(B)$, so there is b' such that $(a, b') \in \Gamma(A_j)$. But $\ker_2(A_j) = \ker_2(B)$ by assumption, so we have $(a, b) \in \Gamma(A_j)$, and since this holds for all j we have $(a, b) \in \Gamma(A)$. Thus $\Gamma_1(A)$ contains $\mathrm{Tor}_1(B)$, and similarly $\Gamma_2(A)$ contains $\mathrm{Tor}_2(B)$.

In particular, $\Gamma(A)$ contains all the torsion from $\Gamma(B)$, so since it is the intersection of divisible $\mathcal{O}$-submodules, it is itself divisible as an $\mathcal{O}$-submodule of $G(A)$. Hence A is a Γ-subfield of B. $\qquad\square$

Definition 3.15. Let B be a Γ-field and $X \subseteq \Gamma(B)$ a subset. We say that

$$A = \bigwedge \{A', \text{ a } \Gamma\text{-subfield of } B \text{ with the same kernels as } B \mid X \subseteq \Gamma(A')\}$$

is the Γ-subfield *generated by* X, and write it as $\langle X \rangle_B$ or, more usually with the B suppressed, as $\langle X \rangle$.

We say A is a *finitely generated* Γ-field if $\Gamma(A)$ is of finite rank as an $\mathcal{O}$-module, or equivalently as a $\mathbb{Z}$-module. Equivalently, A is generated by a finite subset and $\ker_1(A)$ and $\ker_2(A)$ are of finite rank.

Note that a finitely generated Γ-field will not usually be finitely generated as a field, because we insist that $\Gamma(A)$ is a divisible $\mathcal{O}$-submodule.

If Y is a subset of $\Gamma(A)$, we say that A is *finitely generated over Y* if there is a finite subset X of $\Gamma(A)$ such that A is the Γ-subfield of itself generated by $X \cup Y$. In particular, for Y a Γ-subfield of A, we have the notion of a finitely-generated extension of Γ-fields. It is easy to see that an extension $A \hookrightarrow B$ of Γ-fields is finitely generated if and only if $\mathrm{ldim}_{k_{\mathcal{O}}}(\Gamma(B)/\Gamma(A))$ is finite.

Definition 3.16. The intersection of full Γ-subfields of B (with the same kernels as B) is again a full Γ-subfield. Thus we can define a full Γ-field A to be *finitely generated as a full Γ-field* if there is a finite subset X of A such that

$$A = \bigwedge \{A', \text{ a full } \Gamma\text{-subfield of } A \text{ with the same kernels as } A \mid X \subseteq \Gamma(A')\}$$

Likewise there is the notion of being *finitely generated as a full Γ-field extension*.

Except in trivial cases, a finitely generated full Γ-field will not be finitely generated as a Γ-field, and a finitely generated full Γ-field extension will not be finitely generated as a Γ-field extension.

Definition 3.17. Recall that an $\mathcal{O}$-submodule H of G is *pure* in G if whenever $x \in G$ and $nx \in H$ for some $n \in \mathbb{N}^+$, then $x \in H$.

Lemma 3.18. *If A is the Γ-subfield of B generated by X, then $\Gamma(A)$ is the pure $\mathcal{O}$-submodule of $\Gamma(B)$ generated by $X \cup \pi_1^{-1}(\mathrm{Tor}_1) \cup \pi_2^{-1}(\mathrm{Tor}_2)$.*

Proof. This pure $\mathcal{O}$-submodule together with the field it generates is a Γ-subfield of B with the same kernels as B, so it suffices to see that it is contained in $\Gamma(A_j)$ for any A_j in the definition.

$\Gamma(A_j)$ contains X by definition, and since $\pi_i(\Gamma(A_j)) = \mathrm{Tor}_i$ and A_j has the same kernels as B, it also contains $\pi_i^{-1}(\mathrm{Tor}_i)$. Hence it also contains $\mathrm{Tor}(G) \cap \Gamma(B)$. Since it is divisible, it follows that it is pure in $\Gamma(B)$. $\qquad\square$

3.3. **Good bases.** Let A be a Γ-field, and B a finitely generated Γ-field extension of A. So the linear dimension $\mathrm{ldim}_{k_{\mathcal{O}}}(\Gamma(B)/\Gamma(A))$ is finite. Thus we can find a *basis* for the extension, by which we mean a tuple $b = (b_1, \ldots, b_n) \in \Gamma(B)^n$ of minimal length n such that $b \cup \Gamma(A)$ generates $\Gamma(B)$, or equivalently such that $b_1 + \Gamma(A), \ldots, b_n + \Gamma(A)$ is a basis for the quotient $k_{\mathcal{O}}$-vector space $\Gamma(B)/\Gamma(A)$.

We consider the locus $\mathrm{Loc}(b/A)$ of b, that is, the smallest Zariski-closed subset of G, defined over A and containing b.

Definition 3.19. A basis $b \in \Gamma(B)^n$ for a finitely generated extension $A \hookrightarrow B$ of Γ-fields is *good* if the isomorphism type of the extension is determined up to isomorphism by the locus $\mathrm{Loc}(b/A)$. That is, whenever B' is another extension of A which is generated by a basis b' such that $\mathrm{Loc}(b'/A) = \mathrm{Loc}(b/A)$ then there is an isomorphism of Γ-fields $B \cong B'$ fixing A pointwise, which takes b to b'.

Proposition 3.20. *Suppose we are in case (DEQ), that is $\mathrm{Tor}(G) \subseteq \Gamma$. Let $A \hookrightarrow B$ be a finitely generated extension of Γ-fields. Then every basis of the extension is good.*

Proof. Suppose b is a basis of B over A, and we have another extension B' of A with basis b' such that $\mathrm{Loc}(b'/A) = \mathrm{Loc}(b/A)$. There is a (not necessarily unique) field isomorphism $\theta : B \cong B'$ over A which takes b to b'. Now for an element $c \in G(B)$ we have $c \in \Gamma(B)$ if and only if there is $m \in \mathbb{N}$ such that mc is in the

$\mathcal{O}$-linear span of $\Gamma(A)$ and b, because $\Gamma(B)$ is divisible and contains all the torsion of G. It follows that $c \in \Gamma(B)$ if and only if $\theta(c) \in \Gamma(B')$, so θ is an isomorphism of Γ-field extensions. So b is a good basis. $\qquad\square$

In the proof it is critical that $\text{Tor}(G) \subseteq \Gamma$ since otherwise some division points of the basis will be in Γ but others will not. In general we can specify an extension B of A by specifying a choice of division sequence $\widehat{b}$ for a basis b, such that $\widehat{b} \in \widehat{\Gamma}(B)$.

Definition 3.21. A Γ-field is *essentially finitary* if it is finitely generated or if it is a finitely generated extension of a countable full Γ-field.

Proposition 3.22 (Existence of good bases)**.**
Let A be an essentially finitary Γ-field, and let B be a finitely generated Γ-field extension of A (with the same kernels as A). Let b be a basis for the extension. Then there is $m \in \mathbb{N}^+$ such that any m^{th} division point of b in $\Gamma(B)$ is a good basis. Furthermore in case (DEQ) we may take $m = 1$, so every basis is good, and we may even remove the assumption that A is essentially finitary.

The bulk of the proof is contained in the following Kummer-theoretic results.

Definition 3.23. For a commutative algebraic group H we write $\widehat{T}(H)$ for the kernel of the map $\rho_H : \widehat{H} \to H$. So $\widehat{T}(H)$ is the group of division sequences of the identity of H (which is the product over primes l of the l-adic Tate modules $T_l(H)$ of H, hence the notation).

Proposition 3.24. *Let $H = A \times \mathbb{G}_m^r$ be the product of an abelian variety and an algebraic torus.*

Suppose that A is defined over a number field K_0, and moreover that every endomorphism of A is also defined over K_0.

Let D be either $\text{Tor}(H)$ or $H(L)$ for an algebraically closed field extension L of K_0 and let K be a finitely generated field extension of $K_0(D)$.

Let $a \in H(K)$ and suppose that a is free in H over D, that is, in no coset $H' + \gamma$ for a proper algebraic subgroup H' of H and $\gamma \in D$.

Let $\widehat{a} = (a_m)_{m \in \mathbb{N}^+}$ be a division sequence for a in $\widehat{H}(K^{\text{alg}})$ and consider the Kummer map $\xi_a : \text{Gal}(K^{\text{alg}}/K) \to \widehat{T}(H)$ given by

$$\xi_a(\sigma) = (\sigma(a_m) - a_m)_{m \in \mathbb{N}^+}.$$

Then ξ_a does not depend on the choice of division sequence $\widehat{a}$, so is well-defined, and the image of ξ_a is of finite index in $\widehat{T}(H)$.

Remark 3.25. For the groups $\widehat{T}(H)$ which occur in this theorem, the finite index subgroups are precisely those which are open in the profinite topology, so the conclusion of the proposition is that $\xi_a(\text{Gal}(K^{\text{alg}}/K))$ is open in $\widehat{T}(H)$.

Proof of Proposition 3.24. It is straightforward that ξ_a is well-defined.

First suppose $D = \text{Tor}(H)$ and K is a *finite* extension of $K_0(D)$ - so by increasing K_0, we may assume $K = K_0(D)$.

The result then follows from Kummer theory for abelian varieties. For the case $H = A$, we refer to [Ber11, Theorem 5.2], and for the generalization to $H = A \times \mathbb{G}_m^n$ we refer to [BHP14, Proposition A.9].

Suppose now that $D = H(L)$ where L is an algebraically closed field. In this case, the result has a Galois-theoretic proof given as [BGH14, Section 3, Claim 2]. In the case that $K = K_0(D, a)$, the result follows directly from that claim; in general, it follows on noting that

$$\xi_a(\text{Gal}(K^{\text{alg}}/K)) \cong \text{Gal}(K(\widehat{a})/K) \cong \text{Gal}(K_0(D, \widehat{a})/K \cap K_0(D, \widehat{a})),$$

and $K \cap K_0(D, \widehat{a})$ is a finite extension of $K_0(D, a)$.

See also references in the introduction of [BGH14] for alternative proofs, and [Ber11, Theorem 5.3] for an analytic proof.

Finally, suppose $D = \mathrm{Tor}(H)$ and K is a finitely generated extension of $K_0(D)$.

The result in this case follows from the first two cases. This can be seen model-theoretically in the context of [BHP14] as a matter of transitivity of atomicity, but we give here a direct argument.

Say B is the minimal algebraic subgroup of H such that, writing $\theta : H \to H/B$ for the quotient map, we have $\theta(a) \in (H/B)(\mathbb{Q}^{\mathrm{alg}})$. Let $K' = K \cap \mathbb{Q}^{\mathrm{alg}}$, so K is a regular extension of K' and K' is a finite extension of $K_0(D)$. Consider the following diagram

$$
\begin{array}{ccc}
1 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Gal}(K^{\mathrm{alg}}/\mathbb{Q}^{\mathrm{alg}}(K)) & \xrightarrow{\xi_a} & \widehat{T}(B) \\
\downarrow & & \downarrow \\
\mathrm{Gal}(K^{\mathrm{alg}}/K) & \xrightarrow{\xi_a} & \widehat{T}(H) \\
\downarrow & & \downarrow \\
\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/K') & \xrightarrow{\xi_{\theta(a)}} & \widehat{T}(H/B) \\
\downarrow & & \downarrow \\
1 & & 0
\end{array}
$$

where the middle horizontal map is the Kummer map for a, the top map is its restriction, and the bottom map is the Kummer map for $\theta(a)$. The vertical sequences are exact.

Say $a \in a_B + H(\mathbb{Q}^{\mathrm{alg}})$, where $a_B \in B$. By minimality of B, we have that a_B is free in B over $B(\mathbb{Q}^{\mathrm{alg}})$. Now the top map agrees with the Kummer map ξ_{a_B} in B, and so by the second case above, the map has finite index image.

Now since a is free in H over $\mathrm{Tor}(H)$, we have that $\theta(a)$ is free in H/B over $\mathrm{Tor}(H/B)$, so by the first case applied to H/B, the bottom map in the above diagram also has finite index image.

It follows that the central map has finite index image, as required. $\square$

Now we prove that good bases exist.

Proof of Proposition 3.22. Let $\widehat{b} \in \widehat{\Gamma}(B)^n$ be a division sequence of the basis b and write $\widehat{b} = (b_m)_{m \in \mathbb{N}^+}$. Then $\Gamma(B)$ is precisely the $\mathcal{O}$-linear span of $\Gamma(A)$ and the b_m, so to specify B up to isomorphism it is enough to specify the ACF-type of $\widehat{b}$ over A. A is an essentially finitary Γ-field, so it is either finitely generated or a finitely generated extension of a countable full Γ-field A_0. In the former case, let $D = \mathrm{Tor}(G)$ and and in the latter case let $D = G(A_0)$. For $i = 1, 2$, write $b_i = \pi_i(b)$ and $D_i = \pi_i(D)$, and let a_i be a $k_{\mathcal{O}}$-basis for $\pi_i(\Gamma(A))$ over D_i.

We consider the different cases in turn.

Case (EXP) Since the extensions are kernel-preserving, (a_2, b_2) is $k_{\mathcal{O}}$-linearly independent over D_2, and so is free in G_2^{n+k} over D_2.

So, by Proposition 3.24, $\xi_{a_2,b_2}(\mathrm{Gal}(K_0(D, a, b)^{\mathrm{alg}}/K_0(D, a, b)))$ has finite index in $\widehat{T}(G_2^{n+k})$. In particular, its intersection with $0 \times \widehat{T}(G_2^n)$ is of finite index.

Since A is generated as a field by $K_0(D, a)$ and the division points of a_2, it follows that $\Xi := \xi_{b_2}(\mathrm{Gal}(A(b)^{\mathrm{alg}}/A(b)))$ has finite index in $\widehat{T}(G_2^n)$. So if m is the exponent of the finite quotient $\widehat{T}(G_2^n)/\Xi$, then $m\widehat{T}(G_2^n)$ is a subgroup of Ξ.

Hence, if b' is an mth division point of b we have $\xi_{b_2'}(\mathrm{Gal}(A(b')^{\mathrm{alg}}/A(b'))) = \widehat{T}(G_2^n)$. So all division sequences of b' have the same ACF-type over $A(b')$, and hence b' is a good basis for B over A.

Case (COR)   Again, since the extensions are kernel-preserving, (a_i, b_i) is free over D_i for $i = 1, 2$.

Since G_1 and G_2 are simple and non-isogenous, every algebraic subgroup of G^{k+n} is of the form $H_1 \times H_2$ for H_i a subgroup of G_i^{k+n}, so it follows that (a, b) is free in G^{k+n} over D.

Since A is generated as a field by $K_0(D)$ and the division points of a_1 and of a_2, we conclude as in case (EXP).

Case (DEQ)   was covered in Proposition 3.20.

<div style="text-align: right">□</div>

Corollary 3.26. *If A is an essentially finitary Γ-field there are, up to isomorphism, only countably many finitely generated kernel-preserving extensions of A.*

Proof. Each extension B has a good basis b, and is determined by $\mathrm{Loc}(b/A)$. Since A is countable there are only countably many algebraic varieties defined over it.   □

4. Predimension and strong extensions

4.1. **Predimension.** We define a predimension function δ as follows.

Definition 4.1. Let $A \subseteq B$ be Γ-fields. For any Γ-subfield X of B that is finitely generated over A, let

$$\delta(X/A) := \mathrm{trd}(X/A) - d\,\mathrm{ldim}_{k_\mathcal{O}}(\Gamma(X)/\Gamma(A))$$

where recall $d = \dim G_1 = \dim G_2$.

Note that since X is assumed to be finitely generated over A, the linear dimension $\mathrm{ldim}_{k_\mathcal{O}}(\Gamma(X)/\Gamma(A))$ is finite, and, since $\mathcal{O}$ acts by K_0-definable functions and X is the field generated by $\Gamma(X)$, $\mathrm{trd}(X/A)$ is also finite. Hence the predimension is well-defined.

As a convention, for any finite $b \subset \Gamma(B)$, we set

$$\delta(b/A) := \delta(X/A),$$

where $X = \langle Ab \rangle$, the Γ-subfield of B generated by $b \cup A$.

Note that $\delta(b/A) = \mathrm{trd}(b/A) - d\,\mathrm{ldim}_{k_\mathcal{O}}(b/\Gamma(A))$.

Lemma 4.2. *Let $A \subseteq B$ be Γ-fields.*

(1) *(Finite character for δ)*
   If $b \subseteq \Gamma(B)$ is finite, there is a finitely generated Γ-subfield A_0 of A such that for any intermediate Γ-field $A_0 \subseteq A' \subseteq A$, we have $\delta(b/A) = \delta(b/A')$.

(2) *(Addition formula for δ)*
   Let X, Y be Γ-subfields of B finitely generated over A with $X \subseteq Y$. Then

$$\delta(Y/A) = \delta(Y/X) + \delta(X/A).$$

(3) *(Submodularity of δ)*
   Suppose X, Y are Γ-subfields of B with X finitely generated over $X \wedge Y$. Then abbreviating $\langle X \cup Y \rangle$ by XY we have

$$\delta(XY/Y) \leqslant \delta(X/X \wedge Y).$$

Proof. (1) Immediate since transcendence degree and $k_{\mathcal{O}}$-linear dimension have finite character.

(2) Note that the addition formula holds with transcendence degree or linear dimension in place of δ, so it also holds for δ by linearity.

(3) The submodularity condition is true when δ is replaced by transcendence degree. Linear dimension is modular, which means

$$\mathrm{ldim}_{k_{\mathcal{O}}}(\Gamma(XY)/\Gamma(Y)) = \mathrm{ldim}_{k_{\mathcal{O}}}(\Gamma(X)/\Gamma(X \wedge Y)),$$

so by subtracting we get the required submodularity of δ. $\qquad\square$

4.2. **Strong extensions.**

Definition 4.3. An extension $A \subseteq B$ of Γ-fields is said to be a *strong extension* if

(1) the extension preserves kernels; and
(2) for every Γ-subfield X of B that is finitely generated over A, $\delta(X/A) \geqslant 0$.

In this case, we also say that A is a *strong Γ-subfield* of B, and write $A \lhd B$.

For arbitrary Γ-fields A, B, an embedding $A \hookrightarrow B$ is said to be a *strong embedding* if the image of A is a strong Γ-subfield of B. To denote that an embedding is strong we use the notation $A \overset{\lhd}{\hookrightarrow} B$.

The method of predimensions and strong (also known as self-sufficient) extensions has been widely used since it was introduced by Hrushovski [Hru93]. We now give a few basic results which are well-known in general, but fundamental to the later development so it would be inappropriate to omit them. Some of the proofs are slightly more involved for this setting than the more well-known settings, especially those where no field is present.

Lemma 4.4. *The composition of strong embeddings is strong.*

Proof. Suppose $A \lhd B$ and $B \lhd C$. Clearly the kernels of C are the same as those of A, since both are the same as those of B. Let $X \subseteq C$ be finitely generated over A. Then $\delta(X/A) = \delta(X/X \wedge B) + \delta(X \wedge B/A)$ by the addition formula. We have $\delta(X/X \wedge B) \geqslant \delta(XB/B)$ by submodularity, and $\delta(XB/B) \geqslant 0$ because $B \lhd C$. Also $\delta(X \wedge B/A) \geqslant 0$ because $A \lhd B$. So $\delta(X/A) \geqslant 0$. $\qquad\square$

Given a strong extension $A \lhd B$ of Γ-fields, and an intermediate Γ-field X, finitely generated over A, it follows that $A \lhd X$ but it may not be the case that $X \lhd B$. However, as Y varies over finitely generated extensions of X inside B, the predimension $\delta(Y/A)$ takes integer values bounded below by 0 because $A \lhd B$. Thus we can replace X by a finitely generated extension X' of X, inside B, such that $\delta(X'/A)$ is minimal, and from the addition formula for δ it follows that $X' \lhd B$.

The next lemma shows that we can find this X' in a canonical way. It is crucial for understanding the finitely generated Γ-fields we will amalgamate, and it will allow us to understand the types in our models and prove there are only countably many of them.

Lemma 4.5. *Suppose B is a Γ-field and for each $j \in J$, A_j is a strong Γ-subfield of B. Then $\bigwedge_{j \in J} A_j$ is also strong in B.*

Proof. The kernels of $\bigwedge_{j \in J} A_j$ are the same as those of B since they are for each A_j, so it remains to consider the predimension condition.

First we prove that if $A_1, A_2 \lhd B$ then $A_1 \wedge A_2 \lhd A_1$. So suppose X is a finitely generated Γ-field extension of $A_1 \wedge A_2$, inside A_1. Then

$$\delta(X/A_1 \wedge A_2) = \delta(X/X \wedge A_2) \geqslant \delta(XA_2/A_2) \geqslant 0$$

using submodularity and the fact that $A_2 \lhd B$. So $A_1 \wedge A_2 \lhd A_1$, but $A_1 \lhd B$ so, by Lemma 4.4, $A_1 \wedge A_2 \lhd B$. It follows by induction that if J is finite, $\bigwedge_{j \in J} A_j \lhd B$.

Now suppose that J is infinite and that X is a Γ-subfield of B which is finitely generated as an extension of $A = \bigwedge_{j \in J} A_j$. Then we have

$$A = A \wedge X = \bigwedge_{j \in J} (A_j \wedge X).$$

Each Γ-field $A_j \wedge X$ is in the lattice of Γ-fields intermediate between A and X. This lattice is isomorphic to the lattice of vector subspaces of the finite-dimensional vector space $\Gamma(X)/\Gamma(A)$ and so has no infinite chains. Thus there is a finite subset J_0 of J such that writing $A_{J_0} = \bigwedge_{j \in J_0} A_j$ we have

$$A = \bigwedge_{j \in J} (A_j \wedge X) = \bigwedge_{j \in J_0} (A_j \wedge X) = A_{J_0} \wedge X.$$

Now using the result for finite intersections we have that $A_{J_0} \lhd B$, so using also submodularity we have

$$\delta(X/A) = \delta(X/A_{J_0} \wedge X) \geqslant \delta(A_{J_0}X/A_{J_0}) \geqslant 0$$

and hence $A \lhd B$ as required. $\qquad \square$

Consider again a strong extension $A \lhd B$ of Γ-fields, and an intermediate Γ-field X.

Definition 4.6. We define the *hull* of X in B, $\lceil X \rceil_B$ (also known as the strong closure of X or the self-sufficient closure of X) by

$$\lceil X \rceil_B = \bigwedge \{ Y \text{ a strong } \Gamma\text{-subfield of } B \mid X \subseteq Y \}.$$

The previous lemma shows that $\lceil X \rceil_B$ is indeed strong in Y, and we observe also that if X is finitely generated as an extension of A then so is $\lceil X \rceil_B$. Furthermore, if $B \lhd C$ then it is immediate that $\lceil X \rceil_C = \lceil X \rceil_B$, so often we will drop the subscript B.

Lemma 4.7. *The hull operator has finite character. That is, if $A \lhd B$ and X is an intermediate Γ-field,*

$$\lceil X \rceil_B = \bigcup \{ \lceil X_0 \rceil_B \mid X_0 \subseteq X \text{ and } X_0 \text{ is a finitely generated extension of } A \}.$$

Proof. Let U be the union in the statement of the lemma. It is immediate from the definition of the hull that if $X_0 \subseteq X$ then $\lceil X_0 \rceil_B \subseteq \lceil X \rceil_B$. It follows that $U \subseteq \lceil X \rceil_B$. Also $X \subseteq U$. Now U is a directed union of strong Γ-subfields of B, and since δ has finite character, it follows that $U \lhd B$. So $\lceil X \rceil_B \subseteq U$, as required. $\quad \square$

Finally in this section we give a useful lemma giving a simple sufficient condition for an extension of a strong Γ-subfield also to be strong.

Lemma 4.8. *If $A \lhd B$ and $A \subseteq A' \subseteq B$ with $\delta(A'/A) = 0$ then $A' \lhd B$.*

Proof. Let $X \subseteq B$ be a finitely generated extension of A'. Then

$$\delta(X/A') = \delta(X/A) - \delta(A'/A) = \delta(X/A) \geqslant 0.$$

$\square$

4.3. **Pregeometry.** In this section, F is any full Γ-field strongly extending a Γ-subfield F_{base}. We will use the predimension function δ to define a pregeometry on F. We could drop the assumptions that F is full and that F strongly extends some F_{base} and give a definition along the lines of that done for exponential fields in [Kir10a] and for Weierstrass $\wp$-functions in [JKS16]. However, it is sufficient for our purposes and much more straightforward to do it this way.

Definition 4.9. A Γ-subfield A of F, extending F_{base}, is Γ-*closed in* F, written $A \lhd_{\text{cl}} F$, if for any $A \subseteq B \subseteq F$ with B finitely generated over A and $\delta(B/A) \leqslant 0$ we have $B = A$.

Lemma 4.10.   *(1) If $A \lhd_{\text{cl}} F$ then $A \lhd F$.*
      (2) If $A \lhd_{\text{cl}} F$ then A is a full Γ-subfield of F.
      (3) If $A_j \lhd_{\text{cl}} F$ for $j \in J$ and $A = \bigwedge_{j \in J} A_j$ then $A \lhd_{\text{cl}} F$.

Proof.   (1) Immediate.
   (2) Suppose $a \in G_1(F)$ is algebraic over A. Since F is full, there is $b \in G_2(F)$ with $(a,b) \in \Gamma(F)$. We have $\text{trd}((a,b)/A) = \text{trd}(b/A) \leqslant \dim G_2 = d$, so $\delta((a,b)/A) \leqslant d - d\,\text{ldim}_{k_{\mathcal{O}}}((a,b)/\Gamma(A))$. If $\text{ldim}_{k_{\mathcal{O}}}((a,b)/\Gamma(A)) = 1$ then $\delta((a,b)/A) \leqslant 0$, so since A is closed in F we have $(a,b) \in \Gamma(A)$. Otherwise $\text{ldim}_{k_{\mathcal{O}}}((a,b)/\Gamma(A)) = 0$ so again $(a,b) \in \Gamma(A)$. Similarly if $b \in G_2(F)$ is algebraic over A. Since $G_1(A)$ contains all points of G_1 that are algebraic over A, A is an algebraically closed field. Thus A is a full Γ-field.
   (3) Suppose $\delta(B/A) \leqslant 0$. By submodularity and Lemma 4.5, for each j we have $\delta(BA_j/A_j) \leqslant \delta(B/A_j \wedge B) = \delta(B/A) - \delta(A_j \wedge B/A) \leqslant 0$, so $B \subseteq A_j$. Thus $B \subseteq A$.
$\square$

This notion of Γ-closedness induces a closure operator on the field F.

Definition 4.11. If $A \subseteq F$ is any subset the Γ-*closure of A in F* is defined to be the smallest Γ-closed Γ-subfield containing A,

$$\Gamma\text{cl}^F(A) = \bigwedge \{ B \lhd_{\text{cl}} F \mid A \subseteq B \} .$$

$\Gamma\text{cl}^F(A)$ is a Γ-subfield of F, and in particular a subset of F, so Γcl^F induces a map $\mathcal{P}F \to \mathcal{P}F$ which we also denote by Γcl^F.

Lemma 4.12. *For any Γ-subfield A of F, we have $\Gamma\text{cl}^F(A) = \bigcup \mathcal{B}$, where $\mathcal{B}$ is the set of all Γ-subfields $B \subseteq F$ such that B is a finitely generated Γ-field extension of $\lceil A \rceil_F$ and $\delta(B/\lceil A \rceil_F) = 0$.*

Proof. Since $\Gamma\text{cl}^F(A) \lhd F$ we have $\lceil A \rceil_F \subseteq \Gamma\text{cl}^F(A)$. So $\Gamma\text{cl}^F(A) = \Gamma\text{cl}^F(\lceil A \rceil_F)$, and thus we may assume $A \lhd F$. Let $C = \bigcup \mathcal{B}$. Using the submodularity of δ it is easy to see that the system $\mathcal{B}$ of Γ-subfields of F is directed, so its union C is a Γ-subfield of F.

Suppose that b is a finite tuple from $\Gamma(F)$ such that $\delta(b/C) \leqslant 0$. Then by the finite character of δ and directedness of the union defining C, there is a finitely generated extension B of A inside C such that $\delta(B/A) = 0$ and $\delta(b/B) = \delta(b/C)$. Using the addition formula,

$$0 \geqslant \delta(b/B) = \delta(b/A) - \delta(B/A) = \delta(b/A) \geqslant 0.$$

So $\delta(b/A) = 0$ and hence $b \in \Gamma(C)$. Thus C is Γ-closed, so $\Gamma\text{cl}^F(A) \subseteq C$.

Now suppose B is a finitely generated Γ-field extension of A with $\delta(B/A) = 0$ and $A \subseteq D \lhd_{\mathrm{cl}} F$. Then

$$
\begin{aligned}
\delta(BD/D) &\leqslant \delta(B/B \wedge D) \\
&= \delta(B/A) - \delta(B \wedge D/A) \\
&\leqslant 0
\end{aligned}
$$

because $A \lhd F$ so $\delta(B \wedge D/A) \geqslant 0$ and so $B \subseteq D$. Hence $B \subseteq \Gamma\mathrm{cl}^F(A)$, and so $C \subseteq \Gamma\mathrm{cl}^F(A)$. $\qquad\square$

The predimension function δ is a function depending on the sort Γ, but Γ-closure will be shown to be a pregeometry on the field sort. The next lemma allows us to move from one sort to the other.

Lemma 4.13. *If $A \lhd F$ and $a \in F \smallsetminus \Gamma\mathrm{cl}^F(A)$ there is $\alpha \in \Gamma(F)$ such that $\pi_1(\alpha) \in G_1(F)$ is interalgebraic with a over A and $\delta(\alpha/A) = 1$, and the Γ-subfield $\langle A\alpha \rangle$ of F generated by A and α satisfies $\langle A\alpha \rangle \lhd F$. We can choose α such that a is rational over $A(\pi_1(\alpha))$, and if A is essentially finitary also such that α is a good basis. Furthermore, the locus $\mathrm{Loc}(\alpha, a/A)$ can be taken to be a particular algebraic curve defined over K_0, not depending on A or a.*

Proof. We have $\mathrm{trd}(a/A) = 1$ (since otherwise $a \in \Gamma\mathrm{cl}^F(A)$). We have fixed an identification of G_1 with a constructible subset of affine space $\mathbb{A}^N$ for some N, defined over K_0. Let f be the projection map from G_1 to the first coordinate where the projection is dominant, and choose a constructible curve $X \subseteq G_1$ by fixing the values of all the other coordinates to be values in K_0. Then X and the map f are defined over K_0. Choose $\alpha_1 \in X(F)$ with $f(\alpha_1) = a$. Then α_1 is interalgebraic with a over A, with a rational over $A(\alpha_1)$, and the locus $\mathrm{Loc}(\alpha_1, a/A)$ is defined over K_0.

Since F is full, there is $\alpha \in \Gamma(F)$ with $\pi_1(\alpha) = \alpha_1$. Then $\alpha \in \Gamma\mathrm{cl}^F(Aa)$ and $a \in \Gamma\mathrm{cl}^F(A\alpha)$. Then $\mathrm{ldim}_{k_{\mathcal{O}}}(\alpha/\Gamma(A)) = 1$ and so since $\delta(\alpha/A) \neq 0$ by Lemma 4.12, $\mathrm{trd}(\alpha/A) = d + 1$ and $\delta(\alpha/A) = 1$. If there were $B \supseteq \langle A\alpha \rangle$ with $\delta(B/A\alpha) < 0$ then $\delta(B/A) \leqslant 0$ which contradicts $a \notin \Gamma\mathrm{cl}^F(A)$. So $\langle A\alpha \rangle \lhd F$. If A is essentially finitary then by Lemma 3.22 we can divide α by some $m \in \mathbb{N}^+$ to ensure it is a good basis. Since α_2 is generic in $G_2(A)$ over α_1, and G_2 is defined over K_0, we deduce that $\mathrm{Loc}(\alpha, a/A)$ is defined over K_0. $\qquad\square$

Proposition 4.14. *The Γ-closed subsets of F are the closed sets of a pregeometry on F.*

Proof. It is immediate that for any subsets $A \subseteq B$ of F we have $A \subseteq \Gamma\mathrm{cl}^F(A)$, $\Gamma\mathrm{cl}^F(\Gamma\mathrm{cl}^F(A)) = \Gamma\mathrm{cl}^F(A)$ and $\Gamma\mathrm{cl}^F(A) \subseteq \Gamma\mathrm{cl}^F(B)$.

For finite character, suppose $b \in \Gamma\mathrm{cl}^F(A)$. By Lemma 4.12 there is a finitely generated extension $\lceil A \rceil_F \subseteq B$ in F such that $\delta(B/\lceil A \rceil_F) = 0$ and $b \in B$. Then there is a finite tuple $\beta \in \Gamma(F)$ with b rational over β and $\delta(\beta/A) = 0$. By finite character of δ from Lemma 4.2, there is a finitely generated Γ-subfield A_0 of $\lceil A \rceil_F$ such that for any A' with $A_0 \subseteq A' \subseteq \lceil A \rceil_F$ we have $\delta(\beta/A') = 0$. So by Lemma 4.12 again, $b \in \Gamma\mathrm{cl}^F(A_0)$.

The hull operator is a closure operator which by Lemma 4.7 has finite character. We have $A_0 \subseteq \lceil A \rceil_F$, so there is a finite subset A_{00} of A such that $\lceil A_0 \rceil_F = \lceil A_{00} \rceil_F$. Hence $b \in \Gamma\mathrm{cl}^F(A_{00})$, and so $\Gamma\mathrm{cl}^F$ has finite character.

For exchange, suppose $A \lhd_{\mathrm{cl}} F$ and that $a, b \in F \smallsetminus A$ with $b \in \Gamma\mathrm{cl}^F(Aa)$. Using Lemma 4.13, we choose $\alpha, \beta \in \Gamma(F)$ corresponding to a and b respectively.

Now $\beta \in \Gamma\mathrm{cl}^F(A\alpha)$ so there is a finitely generated Γ-field extension $A \subseteq B$ inside F with $\beta, \alpha \in \Gamma(B)$ and $\delta(B/A\alpha) = 0$. Then we have

$$\begin{aligned}
\delta(B/A\beta) &= \delta(B/A) - \delta(\beta/A) \\
&= \delta(B/A) - 1 \\
&= \delta(B/A) - \delta(\alpha/A) \\
&= \delta(B/A\alpha) = 0
\end{aligned}$$

so $\alpha \in \Gamma\mathrm{cl}^F(A\beta)$, or equivalently $a \in \Gamma\mathrm{cl}^F(Ab)$. $\qquad\square$

We write $\Gamma\mathrm{dim}^F$ for the dimension with respect to the pregeometry $\Gamma\mathrm{cl}^F$. However, if F_1 and F_2 are both full Γ-fields with $F_1 \lhd_{\mathrm{cl}} F_2$ and $A \subseteq F_1$ then $\Gamma\mathrm{cl}^{F_1}(A) = \Gamma\mathrm{cl}^{F_2}(A)$. So from now on we will usually drop the superscript F and just write $\Gamma\mathrm{cl}$ and $\Gamma\mathrm{dim}$ except where it might cause confusion.

We have the usual connection between the dimension and the predimension function.

Lemma 4.15. *Suppose that $A \lhd F$ and B is a finitely generated Γ-field extension of A in F. Then:*

  (1) $\Gamma\mathrm{dim}(B/A) = \min\{\delta(C/A) \mid B \subseteq C \subseteq F\}$*, and*
  (2) $B \lhd F$ *if and only if* $\Gamma\mathrm{dim}(B/A) = \delta(B/A)$*.*

Proof. Since $\lceil B \rceil_F \subseteq \Gamma\mathrm{cl}^F(B)$, we have $\Gamma\mathrm{dim}(B/A) = \Gamma\mathrm{dim}(\lceil B \rceil_F/A)$. Now it follows from the addition formula that $\delta(\lceil B \rceil_F/A) = \min\{\delta(C/A) \mid B \subseteq C \subseteq F\}$, so statement 1 reduces to the left-to-right direction of statement 2. To prove that, first assume $B \lhd F$.

Let $n = \Gamma\mathrm{dim}(B/A)$ and let $b_1, \ldots, b_n$ be a $\Gamma\mathrm{cl}$-basis for B over A. Applying Lemma 4.13, we get $\beta_i \in \Gamma(F)$ in the closure of B with β_i corresponding to b_i. Let D be the Γ-subfield of F generated by A and $\beta_1, \ldots, \beta_n$. Then $\delta(D/A) = n$ and $D \lhd F$. Furthermore $\Gamma\mathrm{cl}^F(D) = \Gamma\mathrm{cl}^F(B)$.

Since $D \subseteq \Gamma\mathrm{cl}^F(B)$, there is $C \supseteq B \cup D$ such that $\delta(C/B) = 0$. Then

$$\delta(B/A) = \delta(C/A) - \delta(C/B) = \delta(C/A) \geqslant \delta(D/A) = n$$

using that $D \lhd F$. Reversing the roles of B and D, the same argument shows that $\delta(B/A) \leqslant \delta(D/A)$, and so $\delta(B/A) = n = \Gamma\mathrm{dim}(B/A)$ as required.

The right-to-left direction of statement 2 now follows from statement 1 and the addition property. $\qquad\square$

Remarks 4.16.   (1) In the sense of the pregeometry $\Gamma\mathrm{cl}$, the set $\Gamma(F)$ is d-dimensional. Thus when $d = 1$ such as in pseudo-exponentiation and pseudo-$\wp$ we actually get a pregeometry directly on $\Gamma(F)$.

  (2) In the case of pseudo-exponentiation or a pseudo-$\wp$-function, $G_1(F) = \mathbb{G}_{\mathrm{a}}(F) = F$, and Γ is the graph of a function $\exp$, so we have a bijection $\varphi : F \to \Gamma(F)$ given by $x \mapsto (x, \exp(x))$. The predimension usually considered for exponentiation, for example in [Zil05b], is a function on tuples from the field sort, and in fact is just the composite $\delta \circ \varphi$ of the predimension function described here with φ.

  (3) It is possible to define a predimension function directly on the field sort, even in our generality. Given any subfield A of F we write $\Gamma(A)$ for $\Gamma(F) \cap G(A)$. For any subset X of F (in the field sort) we write X^{alg} for the field-theoretic algebraic closure of $F_{\mathrm{base}}(X)$ in F.

  Given subsets X, Y of F, with $\mathrm{trd}\left((X \cup Y)^{\mathrm{alg}}/X^{\mathrm{alg}}\right) < \infty$, define

$$\eta(Y/X) = \mathrm{trd}\left((X \cup Y)^{\mathrm{alg}}/X^{\mathrm{alg}}\right) - \mathrm{ldim}_{k_{\mathcal{O}}}\left(\Gamma((X \cup Y)^{\mathrm{alg}})/\Gamma(X^{\mathrm{alg}})\right)$$

which takes values in $\mathbb{Z} \cup \{-\infty\}$.

The predimension functions η and δ are closely related, and we could write $\eta(Y/X) = \delta((X \cup Y)^{\mathrm{alg}}/X^{\mathrm{alg}})$ except that X^{alg} will usually fail to be a Γ-subfield of F by our definition, because as a field it will not usually be generated by the coordinates of the points in $\Gamma(X^{\mathrm{alg}})$. It may not even be algebraic over the field generated by those points.

It is possible to define the notion of strong embeddings of Γ-fields using this predimension function instead. Some things are easier with this approach, because the predimension is defined on a 1-dimensional sort. However we choose to work in the sort Γ because it is a vector space and hence has a modular geometry, which makes other things much easier.

4.4. **Full closures.** The following theorem and proof follow [Kir13, Theorem 2.18].

Theorem 4.17. *If A is a Γ-field then there is a full Γ-field extension A^{full} of A such that $A \lhd A^{\mathrm{full}}$ and A^{full} is generated as a full Γ-field by A. Furthermore if A is essentially finitary then A^{full} is unique up to isomorphism as an extension of A.*

Proof. First we prove existence. Embed A in a large algebraically closed field F. Choose a point $a \in G_1(F)$ which is algebraic over A and not in $\pi_1(\Gamma(A))$, if such exists. Choose a division sequence $\widehat{a} \in \widehat{G}_1(F)$ for a. Let $b \in G_2(F)$ be generic over A and choose a division system $\widehat{b}$ for it. (Up to field isomorphism over A, $\widehat{b}$ is unique.) Let A' be the field generated by A and the division sequences $\widehat{a} = (a_m)_{m \in \mathbb{N}^+}$ and $\widehat{b} = (b_m)_{m \in \mathbb{N}^+}$, and define $\Gamma(A')$ to be the $\mathcal{O}$-submodule of $G(A')$ generated by $\Gamma(A)$ and the points (a_m, b_m) for $m \in \mathbb{N}^+$. Then A' is a Γ-field extension of A. Since $\pi_1(\Gamma(A))$ already contains the torsion of G_1, the extension preserves the kernels. We have

$$\delta(A'/A) = \mathrm{trd}(b/A) - d\,\mathrm{ldim}_{k_\mathcal{O}}(b/A) = d - d = 0,$$

so it is a strong extension. Similarly if there is $b \in G_2(F)$ which is algebraic over A but not in $\pi_2(\Gamma(A))$ we can form a similar strong extension. Iterating these constructions, a strong full extension A^{full} of A is readily seen to exist.

Now we prove uniqueness under the additional hypothesis that A is essentially finitary. Suppose that B and B' both satisfy the conditions for A^{full}. Since A is essentially finitary it is countable, and then the construction above shows that we can take B to be countable as well. Enumerate $\Gamma(B)$ as $(s_n)_{n \in \mathbb{N}^+}$ such that for each n, either $\pi_1(s_n)$ or $\pi_2(s_n)$ is algebraic over $A \cup \{s_1, \ldots, s_{n-1}\}$. This is possible since B is generated as a full Γ-field by A.

We will inductively construct a chain of strong Γ-subfields $A_n \lhd B$, each a finitely generated Γ-field extension of A such that $A_0 = A$ and $s_n \in \Gamma(A_n)$. We will also construct a chain of strong embeddings $\theta_n : A_n \overset{\lhd}{\hookrightarrow} B'$. Assume we have A_n and θ_n. Let A_{n+1} be the Γ-subfield of B generated by A_n and s_{n+1}. As a field, A_{n+1} is generated by A_n and the division points of s_{n+1}. If $s_{n+1} \in \Gamma(A_n)$, then we have $A_{n+1} = A_n$ and can just take $\theta_{n+1} = \theta_n$. Otherwise, we have $\mathrm{ldim}_{k_\mathcal{O}}(\Gamma(A_{n+1})/\Gamma(A_n)) \geqslant 1$. By hypothesis, one of $\pi_1(s_{n+1})$ or $\pi_2(s_{n+1})$ is algebraic over A_n, say $\pi_1(s_{n+1})$. Thus $\mathrm{trd}(A_{n+1}/A_n) = \mathrm{trd}(s_{n+1}/A_n) \leqslant d$. By inductive hypothesis $A_n \lhd B$, so we have $\delta(A_{n+1}/A_n) \geqslant 0$. It follows that $\mathrm{ldim}_{k_\mathcal{O}}(\Gamma(A_{n+1})/\Gamma(A_n)) = 1$ and $\mathrm{trd}(A_{n+1}/A_n) = d$, so $\delta(A_{n+1}/A_n) = 0$. Thus by Lemma 4.8, $A_{n+1} \lhd B$. Also $\pi_2(s_{n+1})$ is generic in G_2 over A_n.

Since A_n is a finitely generated Γ-field extension of A, by Proposition 3.22 there is $m \in \mathbb{N}$ such that $\{s_{n+1}/m\}$ is a good basis for the extension $A_n \lhd A_{n+1}$. Replacing s_{n+1} by s_{n+1}/m, we may assume $m = 1$. Now let W be the locus of $\pi_1(s_{n+1})$ over A_n (a variety of dimension 0, irreducible over A_n, but not necessarily absolutely irreducible), and let w be any point in W^{θ_n}, the corresponding subvariety of $G_1(B')$.

Choose $v \in G_2(B')$ such that $(w, v) \in \Gamma(B')$. Since w is algebraic over $\theta_n(A_n)$ which is strong in B', the same predimension argument as above shows that v is generic in G_2 over $\theta_n(A_n)$, so $\mathrm{Loc}(w, v/\theta_n(A_n)) = W^{\theta_n} \times G_2 = (\mathrm{Loc}(s_{n+1}/A_n))^{\theta_n}$. Since s_{n+1} is a good basis over A_n, we can extend θ_n to a field embedding $\theta_{n+1} : A_{n+1} \to B'$ with $\theta(s_{n+1}) = (w, v)$, and using again that $\theta_n(A_n) \lhd B'$ we get that $\Gamma(\theta_{n+1}(A_{n+1}))$ is generated by (w, v) over $\Gamma(\theta_n(A_n))$ and hence θ_{n+1} is a Γ-field embedding.

Also $\delta(\theta_{n+1}(A_{n+1})/\theta_n(A_n)) = 0$ so θ_{n+1} is a strong embedding by Lemma 4.8.

Now $B = \bigcup\{A_n \mid n \in \mathbb{N}\}$ and $\bigcup\{\theta_n(A_n) \mid n \in \mathbb{N}\}$ is a full Γ-subfield of B' containing A, so it must be B'. Hence $\bigcup\{\theta_n \mid n \in \mathbb{N}\}$ is an isomorphism $B \cong B'$. So A^{full} is unique, up to isomorphism as an extension of A. $\square$

Proposition 4.18. *Let A be a countable full Γ-field. Then there are only countably many finitely generated strong full Γ-field extensions of A, up to isomorphism.*

Proof. Let $A \lhd B$ be such an extension and let b be a finite tuple generating B over A as a full Γ-field, such that $B_0 := \langle Ab \rangle \lhd B$, and of minimal length such. Then by Proposition 3.22 we may replace b by b/m for some $m \in \mathbb{N}^+$ to ensure that b is a good basis for the extension $A \lhd B_0$. Then $B = B_0^{\mathrm{full}}$ which by Proposition 4.17 is determined uniquely up to isomorphism by B_0, and by Corollary 3.26 there are only countably many choices for B_0. $\square$

5. The canonical countable model

5.1. **The amalgamation theorem.** We use the definition of *amalgamation category* from [Kir09], slightly extending Droste and Göbel [DG92] who were themselves abstracting from Fraïssé's amalgamation theorem. We restrict to the countable case. We will apply the general theory to various categories of Γ-fields with strong embeddings as morphisms. The notions of *finitely generated*, *universal* and *saturated* all have category-theoretic translations which we give first.

Definition 5.1. Given a category $\mathcal{K}$, an object A of $\mathcal{K}$ is said to be $\aleph_0$-*small* if and only if for every ω-chain (Z_i, γ_{ij}) in $\mathcal{K}$ with direct limit Z_ω, any arrow $A \xrightarrow{f} Z_\omega$ factors through the chain, that is, there is $i < \omega$ and $A \xrightarrow{f^*} Z_i$ such that $f = \gamma_{i\omega} \circ f^*$. We write $\mathcal{K}^{<\aleph_0}$ for the full subcategory of $\aleph_0$-small objects of $\mathcal{K}$ and $\mathcal{K}^{\leqslant \aleph_0}$ for the full subcategory of the limits of ω-chains of $\aleph_0$-small objects of $\mathcal{K}$.

Definition 5.2. Given a category $\mathcal{K}$ and a subcategory $\mathcal{K}'$, an object U of $\mathcal{K}$ is said to be $\mathcal{K}'$-*universal* if for every object A of $\mathcal{K}'$ there is an arrow $A \longrightarrow U$ in $\mathcal{K}$. U is $\mathcal{K}'$-*saturated* if for every arrow $A \xrightarrow{f} B$ in $\mathcal{K}'$ and every arrow $A \xrightarrow{g} U$ in $\mathcal{K}$, there is an arrow $B \xrightarrow{h} U$ in $\mathcal{K}$ such that $g = h \circ f$. U is $\mathcal{K}'$-*homogeneous* if for every object A of $\mathcal{K}'$ and every pair of arrows $A \xrightarrow{f,g} U$ in $\mathcal{K}$, there exists an isomorphism $U \xrightarrow{h} U$ in $\mathcal{K}$ such that $g = h \circ f$.

Some authors refer to $\mathcal{K}'$-saturation as *richness* with respect to the objects and arrows from $\mathcal{K}'$.

Definition 5.3. A category $\mathcal{K}$ is an *amalgamation category* if the following hold.

AC1. Every arrow in $\mathcal{K}$ is a monomorphism.
AC2. $\mathcal{K}$ has direct limits (unions) of ω-chains.
AC3. $\mathcal{K}^{<\aleph_0}$ has at most $\aleph_0$ objects up to isomorphism.
AC4. For each object $A \in \mathcal{K}^{<\aleph_0}$ there are at most $\aleph_0$ extensions of A in $\mathcal{K}^{<\aleph_0}$, up to isomorphism.

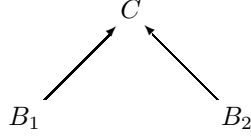AC5. $\mathcal{K}^{<\aleph_0}$ has the *amalgamation property* (AP), that is, any diagram of the form
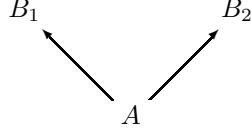
$$B_1 \qquad\qquad B_2$$
$$\nwarrow \qquad \nearrow$$
$$A$$

can be completed to a commuting square

$$C$$
$$\nearrow \qquad \nwarrow$$
$$B_1 \qquad\qquad B_2$$
$$\nwarrow \qquad \nearrow$$
$$A$$
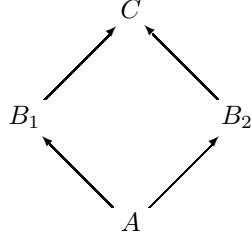
in $\mathcal{K}^{<\aleph_0}$.

AC6. $\mathcal{K}^{<\aleph_0}$ has the *joint embedding property* (JEP), that is, for every $B_1, B_2 \in \mathcal{K}^{<\aleph_0}$ there is $C \in \mathcal{K}^{<\aleph_0}$ and arrows

$$C$$
$$\nearrow \qquad \nwarrow$$
$$B_1 \qquad\qquad B_2$$

in $\mathcal{K}^{<\aleph_0}$.

The point of the definition is that the following form of Fraïssé's amalgamation theorem holds.

Theorem 5.4 ([Kir09, Theorem 2.18]). *If $\mathcal{K}$ is an amalgamation category then there is an object $U \in \mathcal{K}^{\leqslant\aleph_0}$, the "Fraïssé limit", which is $\mathcal{K}^{\leqslant\aleph_0}$-universal and $\mathcal{K}^{<\aleph_0}$-saturated.*

Furthermore, if $A \in \mathcal{K}^{\leqslant\aleph_0}$ is $\mathcal{K}^{<\aleph_0}$-saturated then $A \cong U$.

Remark 5.5. It follows from saturation and a back-and-forth argument that U is also $\mathcal{K}^{<\aleph_0}$-homogeneous.

5.2. **Amalgamation of Γ-fields.** We fix a Γ-field F_{base} which is either finitely generated as a Γ-field, or is a countable full Γ-field.

The identity map on a Γ-field is obviously a strong embedding, hence from Lemma 4.4 we have a category of strong Γ-field extensions of F_{base}, with strong embeddings as the arrows. We write $\mathcal{C}(F_{\text{base}})$ for this category, but will usually abbreviate it to $\mathcal{C}$. We also consider the following full subcategories of $\mathcal{C}$.

Notation 5.6.
- $\mathcal{C}^{\text{full}}$ (or $\mathcal{C}^{\text{full}}(F_{\text{base}})$) consists of the full strong Γ-field extensions of F_{base}.
- $\mathcal{C}^{\text{fg}}$ consists of the strong Γ-field extensions of F_{base} which are finitely generated.
- $\mathcal{C}^{\text{fg-full}}$ consists of the strong Γ-field extensions of F_{base} which are full and finitely generated as full extensions.
- $\mathcal{C}^{\leqslant\aleph_0}$ consists of the strong Γ-field extensions of F_{base} which are countable.
- $\mathcal{C}^{\text{full},\leqslant\aleph_0}$ consists of the strong Γ-field extensions of F_{base} which are full and countable.

For our categories $\mathcal{C}$ and $\mathcal{C}^{\text{full}}$, it is immediate that $\aleph_0$-small just means finitely generated in the appropriate sense, and a (full) Γ-field is the union of an ω-chain of finitely generated (full) Γ-fields if and only if it is countable.

We will construct our canonical model as the Fraïssé limit of $\mathcal{C}^{\text{fg}}$. In fact it is also the Fraïssé limit of $\mathcal{C}^{\text{fg-full}}$.

In proving the amalgamation property we actually prove a stronger result, asymmetric amalgamation, which will be necessary when we come to axiomatize our models. However, the asymmetric property holds only in the case of full Γ-fields, not for $\mathcal{C}^{\text{fg}}$. We also observe that our amalgams are disjoint.

Proposition 5.7. *The categories $\mathcal{C}^{\text{full}}$ and $\mathcal{C}^{\text{full},\leqslant\aleph_0}$ have the disjoint asymmetric amalgamation property. That is, given full Γ-fields $A_0, A_L, A_R \in \mathcal{C}^{\text{full}}$, an embedding $A_0 \hookrightarrow A_L$ and a strong embedding $A_0 \overset{\triangleleft}{\hookrightarrow} A_R$, there exist $A \in \mathcal{C}^{\text{full}}$ and dashed arrows making the following diagram commute;*

$$
\begin{array}{ccc}
 & A_L & \\
\nearrow & & \searrow \\
A_0 & & A \\
\searrow & & \nearrow \\
 & A_R &
\end{array}
$$
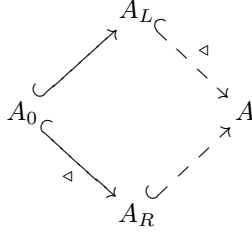
moreover, if the embedding $A_0 \hookrightarrow A_L$ is also strong, then so is the embedding $A_R \hookrightarrow A$; furthermore, identifying A_0, A_L and A_R with their images in A, we have that $A_L \cap A_R = A_0$.

Proof. Since A_0 is algebraically closed as a field, we may form the free amalgam A_1 of A_L and A_R over A_0 as fields, that is, the unique (up to isomorphism) field compositum of A_L and A_R in which they are algebraically independent over A_0. We identify A_L and A_R as subfields of A_1 so, in particular, $A_L \cap A_R = A_0$. We make A_1 into a Γ-field by defining $\Gamma(A_1)$ to be the $\mathcal{O}$-submodule $\Gamma(A_L) + \Gamma(A_R)$ of $G(A_1)$.

Then $\Gamma(A_L)$ and $\Gamma(A_R)$ are $\mathcal{O}$-submodules of $\Gamma(A_1)$.

Suppose that $a \in \ker_1(A_1)$, that is, $(a, 0) \in \Gamma(A_1)$. Then there are $(a_L, b_L) \in \Gamma(A_L)$ and $(a_R, b_R) \in \Gamma(A_R)$ such that $(a, 0) = (a_L, b_L) + (a_R, b_R)$. Then $b_L = -b_R$, so

$$b_L, b_R \in \Gamma_2(A_L) \cap \Gamma_2(A_R) \subseteq G_2(A_L) \cap G_2(A_R) = G_2(A_0).$$

Since A_0 is a full Γ-field there is $a_0 \in G_1(A_0)$ such that $(a_0, b_L) \in \Gamma(A_0)$. Then $a_L - a_0 \in \ker_1(A_L) = \ker_1(A_0)$, so $a_L \in \Gamma_1(A_0)$. Similarly, $a_R \in \Gamma_1(A_0)$, so $a \in \Gamma_1(A_0)$.

Thus $\ker_1(A_1) = \ker_1(A_0)$. The same argument shows that $\ker_2(A_1) = \ker_2(A_0)$, and hence the inclusions of A_L and A_R into A_1 preserve the kernels.

Let us check that the inclusion $A_L \hookrightarrow A_1$ is strong. Let X be a Γ-subfield of A_1 which is finitely generated over A_L. Choose a basis b for the extension, say of length n. Translating by points in $\Gamma(A_L)$, we may assume that $b \in \Gamma(A_R)^n$. Now $\delta(b/A_0) \geqslant 0$ since $A_0 \triangleleft A_R$, so $\text{trd}(b/A_0) \geqslant d \,\text{ldim}_{k_{\mathcal{O}}}(g/\Gamma(A_0) = dn$.

Since A_R is ACF-independent from A_L over A_0, we have $\text{trd}(b/A_L) = \text{trd}(b/A_0)$, and we also have $\text{ldim}_{k_{\mathcal{O}}}(b/\Gamma(A_L)) = n$ by assumption, so

$$\delta(X/A_L) = \text{trd}(b/A_L) - d\,\text{ldim}_{k_{\mathcal{O}}}(b/\Gamma(A_L)) = \delta(b/A_0) \geqslant 0$$

as required. Thus $A_L \triangleleft A_1$. The same argument shows that if the embedding $A_0 \hookrightarrow A_L$ is strong, then so is the embedding $A_R \hookrightarrow A_1$.

Now take $A = A_1^{\text{full}}$, which exists and is a strong extension of A_1, by the existence part of Theorem 4.17. Note that if A_L and A_R are countable then so is A. $\square$

Corollary 5.8. *The category $\mathcal{C}^{\text{fg}}$ has the amalgamation property. That is, given $A_0, A_L, A_R \in \mathcal{C}^{\text{fg}}$ and strong embeddings $A_0 \overset{\triangleleft}{\hookrightarrow} A_L$ and $A_0 \overset{\triangleleft}{\hookrightarrow} A_R$ as in the*

following diagram, there exist $A \in \mathcal{C}^{\mathrm{fg}}$ and dashed arrows making the diagram commute.

$$
\begin{array}{ccc}
 & A_L & \\
 \nearrow & & \searrow \\
A_0 & & A \\
 \searrow & & \nearrow \\
 & A_R &
\end{array}
$$
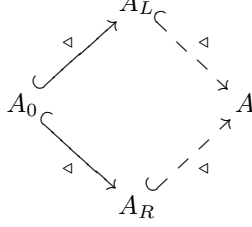
Proof. Let A_0, A_L, A_R be as in the statement. By the existence part of Theorem 4.17, we can extend each of the three Γ-fields to its full closure.

$$
\begin{array}{ccc}
 & A_L \hookrightarrow A_L^{\mathrm{full}} & \\
 \nearrow & & \\
A_0 \hookrightarrow A_0^{\mathrm{full}} & & \\
 \searrow & & \\
 & A_R \hookrightarrow A_R^{\mathrm{full}} &
\end{array}
$$

Then because we have $A_0 \lhd A_L^{\mathrm{full}}$ and $A_0 \lhd A_R^{\mathrm{full}}$, by the uniqueness part of the same theorem there are embeddings as in the following diagram, which are strong by Lemma 4.8 and finite character of δ.
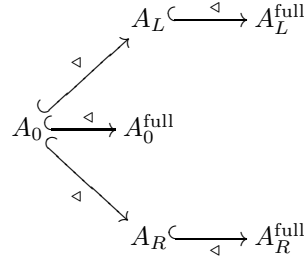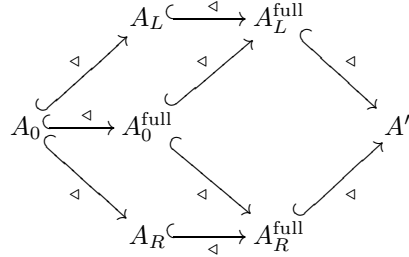
$$
\begin{array}{ccc}
 & A_L \hookrightarrow A_L^{\mathrm{full}} & \\
 \nearrow & \nearrow & \\
A_0 \hookrightarrow A_0^{\mathrm{full}} & & \\
 \searrow & \searrow & \\
 & A_R \hookrightarrow A_R^{\mathrm{full}} &
\end{array}
$$
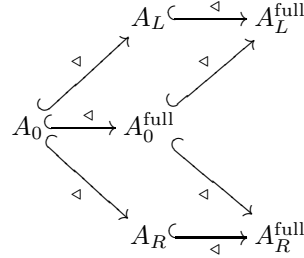
By Theorem 5.7, we can complete the diagram to

$$
\begin{array}{cccc}
 & A_L \hookrightarrow A_L^{\mathrm{full}} & & \\
 \nearrow & \nearrow & \searrow & \\
A_0 \hookrightarrow A_0^{\mathrm{full}} & & & A' \\
 \searrow & \searrow & \nearrow & \\
 & A_R \hookrightarrow A_R^{\mathrm{full}} & &
\end{array}
$$

and then we can take A to be the Γ-subfield of A' generated by $A_L \cup A_R$, which is in $\mathcal{C}^{\mathrm{fg}}$. $\qquad\qquad\square$

Theorem 5.9. *The categories $\mathcal{C}$ and $\mathcal{C}^{\mathrm{full}}$ are amalgamation categories, with the same Fraïssé limit.*

Proof. Strong embeddings are injective functions, so monomorphisms. Hence AC1 holds. It is clear that the union of a chain of (full) Γ-fields is a (full) Γ-field, so AC2 holds. AC4 is given by Corollary 3.26 for $\mathcal{C}$ and Proposition 4.18 for $\mathcal{C}^{\mathrm{fg\text{-}full}}$. The amalgamation property AC5 is proved in Proposition 5.7 and Corollary 5.8.

Since every Γ-field in $\mathcal{C}$ is an extension of F_{base}, and every full Γ-field in $\mathcal{C}^{\text{full}}$ is an extension of $(F_{\text{base}})^{\text{full}}$, properties AC3 and AC6 follow from AC4 and AC5 respectively.

Thus $\mathcal{C}$ and $\mathcal{C}^{\text{full}}$ are both amalgamation categories. Let M be the Fraïssé limit of $\mathcal{C}^{\text{full}}$. If $A \in \mathcal{C}^{\leqslant \aleph_0}$ then $A^{\text{full}} \in \mathcal{C}^{\text{full},\leqslant \aleph_0}$, so as M is $\mathcal{C}^{\text{full},\leqslant \aleph_0}$-universal there is a strong embedding $A^{\text{full}} \lhd M$, which restricts to a strong embedding $A \lhd M$. Hence M is $\mathcal{C}^{\leqslant \aleph_0}$-universal. Similarly, using Proposition 4.17 and the $\mathcal{C}^{\text{fg-full}}$-saturation of M we can see that M is also $\mathcal{C}^{\text{fg}}$-saturated. Hence M is also the Fraïssé limit of $\mathcal{C}^{\text{fg}}$. $\qquad\square$

Notation 5.10. We write $M(F_{\text{base}})$ for the Fraïssé limit in $\mathcal{C}$.

5.3. Γ-algebraic extensions.

Definition 5.11. Let $A \lhd B$ be a strong extension of Γ-fields. The extension is Γ-*algebraic* if for all finite tuples b from $\Gamma(B)$ there is a finite tuple $c \in \Gamma(B)$ containing b such that $\delta(c/A) = 0$.

Remark 5.12. From Lemma 4.15 we see that if F is a full Γ-field such that $B \lhd F$ then the extension $A \lhd B$ is Γ-algebraic if and only if $B \subseteq \Gamma\text{cl}^F(A)$.

Let $\mathcal{C}^{\text{alg}}$ be the subcategory of $\mathcal{C}$ consisting of the Γ-algebraic extensions of F_{base}.

Proposition 5.13. $\mathcal{C}^{\text{alg}}$ *is an amalgamation category.*

Proof. The proof of Theorem 5.9 goes through, except we also have to show that the amalgam of Γ-algebraic extensions is Γ-algebraic. So suppose we have the amalgamation square

$$
\begin{array}{ccc}
 & A_L & \\
\nearrow & & \searrow \\
A_0 & & A \\
\searrow & & \nearrow \\
 & A_R &
\end{array}
$$

as in Corollary 5.8 with A_L and A_R both Γ-algebraic over A_0, A' a full Γ-field and A the Γ-subfield of A' generated by $A_L \cup A_R$. Then by remark 5.12, we have $A_L \cup A_R \subseteq \Gamma\text{cl}^{A'}(A_0)$ and so $A \subseteq \Gamma\text{cl}^{A'}(A_0)$, so $A_0 \lhd A$ is Γ-algebraic. $\qquad\square$

Write M_0 (or $M_0(F_{\text{base}})$) for the Fraïssé limit of $\mathcal{C}^{\text{alg}}$.

Definition 5.14. A Γ-field F strongly extending F_{base} is $\aleph_0$-*saturated for Γ-algebraic extensions over F_{base}* if whenever $F_{\text{base}} \lhd A \lhd F$ with A finitely generated over F_{base} and $A \xrightarrow{\lhd} B$ is a finitely generated Γ-algebraic extension then B embeds (necessarily strongly) into F over A.

Proposition 5.15. $M_0(F_{\text{base}})$ *is the unique countable full Γ-field strongly extending F_{base} which is Γ-algebraic over F_{base} and $\aleph_0$-saturated for Γ-algebraic extensions.*

Proof. Immediate from the uniqueness part of the amalgamation theorem and Proposition 5.13. $\qquad\square$

5.4. Purely Γ-transcendental extensions.
In contrast with Γ-algebraic extensions are those we will call purely Γ-transcendental extensions. We discuss amalgamation of these which gives rise to some variant constructions.

Definition 5.16. Let $A \lhd B$ be a strong extension of Γ-fields. The extension is *purely Γ-transcendental* if for all tuples b from $\Gamma(B)$, either $\delta(b/A) > 0$ or $b \subseteq \Gamma(A)$.

Remark 5.17. If $A \lhd B$ is an extension of full Γ-fields then it is purely Γ-transcendental if and only if A is Γ-closed in B.

Definition 5.18. When F_{base} is a full countable Γ-field, we define $\mathcal{C}_{\Gamma\text{-tr}}(F_{\text{base}})$ (usually abbreviated to $\mathcal{C}_{\Gamma\text{-tr}}$) to be the full subcategory of $\mathcal{C}$ consisting of the strong purely Γ-transcendental extensions of F_{base}.

Lemma 5.19. *If $A \in \mathcal{C}_{\Gamma\text{-tr}}$ then $A^{\text{full}} \in \mathcal{C}_{\Gamma\text{-tr}}$.*

Proof. Consider the case when $(a_1, a_2) \in \Gamma(A^{\text{full}}) \smallsetminus \Gamma(A)$ with $a_1 \in G_1(A^{\text{full}})$ algebraic over A. Since $A \lhd A^{\text{full}}$ we have $\text{trd}(a_2/A) = d$. If $\delta((a_1, a_2)/F_{\text{base}}) \leqslant 0$ then $\text{trd}(a_1, a_2/F_{\text{base}}) \leqslant d$, which implies that $\text{trd}(a_1/F_{\text{base}}) = 0$. Since F_{base} is full, that implies $(a_1, a_2) \in \Gamma(F_{\text{base}})$, a contradiction.

Replacing A by the Γ-subfield of A^{full} generated by $A \cup \{(a_1, a_2)\}$ and iterating appropriately, we see that $A^{\text{full}} \in \mathcal{C}_{\Gamma\text{-tr}}$. $\qquad\square$

We will show that $\mathcal{C}_{\Gamma\text{-tr}}$ is an amalgamation category by showing that the free amalgam of purely Γ-transcendental extensions is purely Γ-transcendental, using a lemma on stable groups.

Lemma 5.20. *Let H be a commutative algebraic group defined over an algebraically closed field C. Suppose $a_1, a_2, a_3 \in H$ are pairwise algebraically independent over C and $a_1 + a_2 + a_3 = 0$. Then there is a connected algebraic subgroup U of H and cosets $c_i + U$ defined over C such that a_i is a generic point of $c_i + U$ over C, for each $i = 1, 2, 3$. In particular, $\text{trd}(a_i/C) = \dim U$ for each i.*

Proof. This is the special case for algebraic groups of a result about stable groups due to Ziegler [Zie06, Theorem 1]. $\qquad\square$

Theorem 5.21. *If F_{base} is a full countable Γ-field then $\mathcal{C}_{\Gamma\text{-tr}}$ and $\mathcal{C}_{\Gamma\text{-tr}}^{\text{full}}$ are amalgamation categories.*

Notation 5.22. We write $M_{\Gamma\text{-tr}}(F_{\text{base}})$ for the Fraïssé limit in $\mathcal{C}_{\Gamma\text{-tr}}(F_{\text{base}})$.

Proof of Theorem 5.21. Axioms AC1, AC3 and AC4 follow immediately from the fact that $\mathcal{C}_{\Gamma\text{-tr}}$ and $\mathcal{C}_{\Gamma\text{-tr}}^{\text{full}}$ are full subcategories of $\mathcal{C}$. AC2 and AC6 are also immediate. It remains to prove AC5, the amalgamation property.

Using Lemma 5.19, the same argument as for Corollary 5.8 allows us to reduce the amalgamation property for $\mathcal{C}_{\Gamma\text{-tr}}$ to the amalgamation property for $\mathcal{C}_{\Gamma\text{-tr}}^{\text{full}}$. So suppose we have full Γ-fields
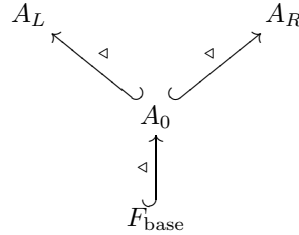
with A_0, A_L and A_R all purely Γ-transcendental extensions of F_{base}. Let A_1 be the free amalgam of A_L and A_R over A_0 as in the proof of Proposition 5.7. We must show that A_1 is a purely Γ-transcendental extension of F_{base}.

So let B be a Γ-subfield of A_1 properly containing F_{base} and finitely generated over it. It remains to show that $\delta(B/F_{\text{base}}) \geqslant 1$. If $B \wedge A_R \neq F_{\text{base}}$ then we have

$$
\begin{aligned}
\delta(B/F_{\text{base}}) &= \delta(B/B \wedge A_R) + \delta(B \wedge A_R/F_{\text{base}}) \\
&\geqslant \delta(BA_R/A_R) + \delta(B \wedge A_R/F_{\text{base}}) \qquad \text{by submodularity} \\
&\geqslant 0 + 1 = 1,
\end{aligned}
$$

the last line because $A_R \lhd A_1$ and A_R is purely Γ-transcendental over F_{base}. So in this case we are done, and similarly if $B \wedge A_L \neq F_{\text{base}}$.

So we may assume that $B \wedge A_R = B \wedge A_L = F_{\text{base}}$. Choose a basis $b = (b^1, \ldots, b^n)$ of B over F_{base}. For each i, choose $b^i_L \in \Gamma(A_L)$ and $b^i_R \in \Gamma(A_R)$ such that $b^i = b^i_L + b^i_R$. Let B_L and B_R be the Γ-field extensions of F_{base} generated by $b_L := (b^1_L, \ldots, b^n_L)$, and $b_R := (b^1_R, \ldots, b^n_R)$ respectively.

We claim that $B_L \wedge A_R = F_{\text{base}}$ and that $A_L \wedge B_R = F_{\text{base}}$. To see this, suppose that $v \in \Gamma(B_L \wedge A_R) = \Gamma(B_L) \cap \Gamma(A_R)$. Since $v \in \Gamma(B_L)$ there are $s_i \in k_{\mathcal{O}}$ and some $a \in \Gamma(F_{\text{base}})$ such that $v = \sum_{i=1}^n s_i b^i_L + a$. Let $u_L = v - a = \sum_{i=1}^n s_i b^i_L$, let $u_R = \sum_{i=1}^n s_i b^i_R$, and let $u = u_L + u_R = \sum_{i=1}^n s_i b^i$. Then $v, a \in \Gamma(A_R)$, so $u_L \in \Gamma(A_R)$, and also $u_R \in \Gamma(A_R)$, hence $u \in \Gamma(A_R)$. But $u \in \Gamma(B)$ and we have $B \wedge A_R = F_{\text{base}}$. So $u \in \Gamma(F_{\text{base}})$ and so each $s_i = 0$, and thus $v \in \Gamma(F_{\text{base}})$. So $B_L \wedge A_R = F_{\text{base}}$. The same argument shows that $A_L \wedge B_R = F_{\text{base}}$, and in particular $B_L \wedge B_R = F_{\text{base}}$.

Let C be the Γ-subfield of A_1 generated by $B \cup B_L$, and note that it is also generated by $B_L \cup B_R$. We have $B \wedge B_L = F_{\text{base}} = B_L \wedge B_R$, so applying modularity of linear dimension to the squares

we get

$$\text{ldim}_{k_{\mathcal{O}}}(\Gamma(B_R)/\Gamma(F_{\text{base}})) = \text{ldim}_{k_{\mathcal{O}}}(\Gamma(C)/\Gamma(B_L)) = \text{ldim}_{k_{\mathcal{O}}}(\Gamma(B)/\Gamma(F_{\text{base}})) = n,$$

and so b_R is $k_{\mathcal{O}}$-linearly independent over $\Gamma(F_{\text{base}})$, and hence, since $B_R \wedge A_0 = F_{\text{base}}$, over $\Gamma(A_0)$. We have

$$\text{trd}(b/F_{\text{base}}) \geqslant \text{trd}(b/A_0) \geqslant \text{trd}(b/A_0 b_L) = \text{trd}(b_R/A_0 b_L) = \text{trd}(b_R/A_0) \geqslant dn$$

with the last three (in)equalities holding because $b = b_L + b_R$, b_R is algebraically independent from b_L over A_0, and because $A_0 \lhd A_1$ and b_R is $k_{\mathcal{O}}$-linearly independent over $\Gamma(A_0)$. Similarly $\text{trd}(b/A_0) \geqslant \text{trd}(b_L/A_0) \geqslant dn$.

Suppose for a contradiction that $\delta(B/F_{\text{base}}) \leqslant 0$. Then we must have

$$\text{trd}(b/F_{\text{base}}) = \text{trd}(b/A_0) = \text{trd}(b_L/A_0) = \text{trd}(b_R/A_0) = dn,$$

and then we also have

$$\text{trd}(b, b_L/A_0) = \text{trd}(b, b_R/A_0) = \text{trd}(b_L, b_R/A_0) = \text{trd}(b, b_L, b_R/A_0) = 2dn$$

so b, b_L, b_R are pairwise algebraically independent over A_0.

We apply Lemma 5.20 with $H = G^n = G_1^n \times G_2^n$, $a_1 = -b$, $a_2 = b_L$ and $a_3 = b_R$ to get a connected algebraic subgroup U of G^n of dimension dn such that b is in an A_0-coset of U. Since $\text{trd}(b/F_{\text{base}}) = \text{trd}(b/A_0) = \dim U$, and F_{base} is an algebraically closed field, the coset is actually defined over F_{base}.

G_1 and G_2 are non-isogenous and so U is of the form $U_1 \times U_2$ where each U_i is a connected subgroup of G_i^n. Since $\dim U = dn$, if $U_2 = G_2^n$ then U_1 is the trivial subgroup of G_1^n, so $\pi_1(b) \in G_1^n(A_0)$. But A_0 is a full Γ-field and so $b \in \Gamma(A_0)^n$ which contradicts $\text{trd}(b/A_0) = dn$ (and $n > 0$). So U_2 must be a proper subgroup of G_2^n. Since G_2 is simple, it follows that $\pi_2(b)$ satisfies an $\mathcal{O}$-linear equation $\sum_{i=1}^n s_i \pi_2(b^i) = c$ with $c \in G_2(F_{\text{base}})$. Then, since $b \in \Gamma(B)^n$ and F_{base} is a full

Γ-field we have $\sum_{i=1}^{n} s_i b^i \in \Gamma(F_{\text{base}})$, which contradicts b being a basis for B over F_{base}.

So we have $\delta(B/F_{\text{base}}) \geqslant 1$, and thus A_1 is a purely Γ-transcendental extension of F_{base}, as required. $\qquad\square$

6. Categoricity

6.1. **Quasiminimal pregeometry structures.** This definition of quasiminimal pregeometry structures comes from [BH²K²14].

Definition 6.1. Let M be an L-structure for a countable language L, equipped with a pregeometry cl (or cl_M if it is necessary to specify M). Write qftp for the quantifier-free L-type. We say that M is a *quasiminimal pregeometry structure* if the following hold:

QM1. The pregeometry is determined by the language. That is, if a, a' are singletons, b, b' are tuples, $\text{qftp}(a, b) = \text{qftp}(a', b')$, and $a \in \text{cl}(b)$ then $a' \in \text{cl}(b')$.

QM2. M is infinite-dimensional with respect to cl.

QM3. (Countable closure property) If $A \subseteq M$ is finite then $\text{cl}(A)$ is countable.

QM4. (Uniqueness of the generic type) Suppose that $C, C' \subseteq M$ are countable closed subsets, enumerated such that $\text{qftp}(C) = \text{qftp}(C')$. If $a \in M \smallsetminus C$ and $a' \in M \smallsetminus C'$ then $\text{qftp}(C, a) = \text{qftp}(C', a')$ (with respect to the same enumerations for C and C').

QM5. ($\aleph_0$-homogeneity over closed sets and the empty set)
Let $C, C' \subseteq M$ be countable closed subsets or empty, enumerated such that $\text{qftp}(C) = \text{qftp}(C')$, and let b, b' be finite tuples from M such that $\text{qftp}(C, b) = \text{qftp}(C', b')$, and let $a \in \text{cl}(C, b)$. Then there is $a' \in M$ such that $\text{qftp}(C, b, a) = \text{qftp}(C', b', a')$.

We say M is a *weakly quasiminimal pregeometry structure* if it satisfies all the axioms except possibly QM2.

Definition 6.2. Given M_1 and M_2 both weakly quasiminimal pregeometry L-structures, we say that an L-embedding $\theta : M_1 \hookrightarrow M_2$ is a *closed embedding* if for each $A \subseteq M_1$ we have $\theta(\text{cl}_{M_1}(A)) = \text{cl}_{M_2}(\theta(A))$. In particular, $\theta(M_1)$ is closed in M_2 with respect to cl_{M_2}. We write $M_1 \lhd_{\text{cl}} M_2$ for a closed embedding.

Definition 6.3. Given a quasiminimal pregeometry structure M, let $\mathcal{K}(M)$ be the smallest class of L-structures which contains M and all its closed substructures and is closed under isomorphism and under taking unions of directed systems of closed embeddings. We call any class of the form $\mathcal{K}(M)$ a *quasiminimal class*.

The purpose of these definitions is the categoricity theorem, which is Theorem 2.3 in [BH²K²14].

Fact 6.4. *If $\mathcal{K}$ is a quasiminimal class then every structure $A \in \mathcal{K}$ is a weakly quasiminimal pregeometry structure, and up to isomorphism there is exactly one structure in $\mathcal{K}$ of each cardinal dimension. In particular, $\mathcal{K}$ is uncountably categorical. Furthermore, $\mathcal{K}$ is the class of models of an $L_{\omega_1, \omega}(Q)$ sentence.*

We will verify axioms QM1–QM5 for the Fraïssé limits we constructed. We first make some general observations which simplify what we have to verify.

Proposition 6.5. *Suppose that M is a countable L-structure. Then it satisfies QM1–QM5 if and only if it satisfies the following axioms.*

QM1'. *If a and b are finite tuples and $\text{qftp}(a) = \text{qftp}(b)$ then $\dim(a) = \dim(b)$.*

QM2. *M is infinite-dimensional with respect to cl.*

QM4. *(Uniqueness of the generic type)*

QM5a. ($\aleph_0$-*homogeneity over the empty set*)
  If a and b are finite tuples from M and qftp$(a) =$ qftp(b) *then there is $\theta \in \mathrm{Aut}(M)$ such that $\theta(a) = b$.*
QM5b. (*Non-splitting over a finite set*)
  If $C \lhd_{\mathrm{cl}} M$ and $b \in M$ is a finite tuple then there is a finite tuple $c \in C$ such that qftp(b/C) *does not split over c. That is, for all finite tuples $a, a' \in C$, if* qftp$(a/c) =$ qftp(a'/c) *then* qftp$(a/cb) =$ qftp(a'/cb).

Proof. QM1$'$ is equivalent to QM1, because $a \in \mathrm{cl}(b)$ if and only if $\dim(a, b) = \dim(b)$, so if quantifier-free types characterize the dimension they also characterize the closure operation, and vice versa.

QM3, the countable closure property, is immediate for a countable M.

Axiom QM5 with $C = \emptyset$ gives a back-and-forth condition which is equivalent to $\aleph_0$-homogeneity using the standard back-and-forth argument together with QM1 and QM4. Since M is countable, the back-and-forth construction gives QM5a. The converse is immediate.

Then [BH^{2}K^{2}14, Corollary 5.3] shows the case of QM5 with C closed is equivalent to QM5b. $\square$

Remark 6.6. All the axioms refer to quantifier-free types with respect to a particular language, and from QM5a we get the conclusion that if two finite tuples from M have the same quantifier-free type then they actually have the same complete type (even the same $L_{\infty,\omega}$-type, and furthermore they lie in the same automorphism orbit, that is, they have the same Galois-type). Since M is not necessarily a saturated model of its first-order theory, it does not follow that every definable set is quantifier-free definable. Nonetheless, identifying the language which works allows us to understand the types which are realised in M.

6.2. **Verification of the quasiminimal pregeometry axioms.** Recall that our language of Γ-fields is $L_\Gamma = \langle +, \cdot, -, \Gamma, (c_a)_{a \in K_0} \rangle$, where Γ is a relation symbol of suitable arity to denote a subset of G. We start by defining the expansion L^{QE} of L_Γ in which we will have the form of quantifier-elimination described in the previous remark.

Let W be any subvariety of $G^n \times \mathbb{A}^r$ defined over F_{base}, for some $n, r \in \mathbb{N}$. (It suffices to consider those W which are the graphs of rational maps $f : W' \to \mathbb{A}^r$, with $W' \subseteq G^n$.) Let $\varphi_W(x, y)$ name the subset of $G(M)^n \times M^r$ given by

$$(x, y) \in W \quad \& \quad x \in \Gamma^n \quad \& \quad x \text{ is } \mathcal{O}\text{-linearly independent over } \Gamma(F_{\mathrm{base}})$$

and let $\psi_W(y)$ be the formula $\exists x \varphi_W(x, y)$.

Definition 6.7. We define L^{QE} to be the expansion of L_Γ by parameters for F_{base} and relation symbols for all the formulas $\varphi_W(x, y)$ and $\psi_W(y)$.

Remark 6.8. Note that the formulas $\varphi_W(x, y)$ are are always expressible in $L_{\omega_1,\omega}(L_\Gamma)$ (with parameters in F_{base}), so a priori this is an expansion of $L_\Gamma(F_{\mathrm{base}})$ by $L_{\omega_1,\omega}$-definitions. However, if the ring $\mathcal{O}$ and its action on G are definable and $\Gamma(F_{\mathrm{base}})$ is either of finite rank (which is true for example in pseudo-exponentiation) or is otherwise an L_Γ-definable set, then L^{QE} is just an expansion of $L_\Gamma(F_{\mathrm{base}})$ by first-order definitions.

For the rest of this section we use tuples both from the field sort of a model M and from $\Gamma(M)$, so to distinguish them we will use Latin letters for tuples from M and Greek letters for tuples from $\Gamma(M)$.

Theorem 6.9. *Take M to be either $M(F_{\mathrm{base}})$ or $M_{\Gamma\text{-tr}}(F_{\mathrm{base}})$, the latter only if F_{base} is a full Γ-field. Then, considered in the language L^{QE} and equipped with Γcl, M is a quasiminimal pregeometry structure.*

Proof. We verify the axioms from Proposition 6.5. The main difficulty is that the axioms refer to the field sort whereas the construction of M was done in the sort Γ, and there is no canonical way to go from one sort to the other in either direction. However the sort Γ has rank d with respect to the pregeometry Γcl, so as we want to include the case $d > 1$ we have to verify the axioms with respect to the field sort.

First we prove QM2. For any $n \in \mathbb{N}$, there is a strong Γ-field extension A_n of F_{base} generated by a tuple $\alpha \in \Gamma(A_n)^n$ such that α is generic in G^n over F_{base}. Then $\delta(\alpha/F_{\text{base}}) = dn$. This A_n embeds strongly in M by the universality property of the Fraïssé limit, so $\Gamma\dim^M(\alpha) = dn$ by Lemma 4.15. Hence M is infinite-dimensional.

Now we prove QM1' and QM5a together.

Suppose $a, b \in M^r$ with $\text{qftp}_{L^{\text{QE}}}(a) = \text{qftp}_{L^{\text{QE}}}(b)$. Choose a strong Γ-subfield $A \lhd M$ which is a finitely-generated extension of F_{base} such that $a \in A^r$ and $\delta(A/F_{\text{base}})$ is minimal such. Let $\alpha \in \Gamma(A)^n$ be a good basis for A over F_{base}, such that a is in the field $F_{\text{base}}(\alpha)$, let $W = \text{Loc}(\alpha, a/F_{\text{base}})$, and let $V = \text{Loc}(\alpha/F_{\text{base}})$. Then $M \models \psi_W(a)$, so also $M \models \psi_W(b)$. So there is $\beta \in \Gamma(M)^n$ such that $M \models \varphi_W(\beta, b)$. In particular, $\beta \in V \cap \Gamma(M)^n$, $k_{\mathcal{O}}$-linearly independent over $\Gamma(F_{\text{base}})$. We claim that $W = \text{Loc}(\beta, b/F_{\text{base}})$. Suppose not, so $W' := \text{Loc}(\beta, b/F_{\text{base}})$ is a proper subvariety of W. We have $M \models \psi_{W'}(b)$, so since a and b have the same quantifier-free L^{QE}-type, $M \models \psi_{W'}(a)$. So there is some $\alpha' \in \Gamma(M)^n$ such that $M \models \varphi_{W'}(\alpha', a)$. W' is irreducible over F_{base}, so $\dim W' < \dim W$. Since a is rational over $F_{\text{base}}(\alpha)$, $\dim W = \dim V$, and so α' lies in a subvariety V' of V with $\dim V' < \dim V$. But then setting $A' = \langle F_{\text{base}}, \alpha' \rangle$ we have $a \in A'^r$ and $\delta(A'/F_{\text{base}}) = \delta(\alpha'/F_{\text{base}}) = \dim V' - \text{ldim}_{k_{\mathcal{O}}}(\alpha'/\Gamma(F_{\text{base}})) < \delta(A/F_{\text{base}})$, which contradicts the choice of A. Thus $\text{Loc}(\beta, b/F_{\text{base}}) = W$, and in particular $\text{Loc}(\beta/F_{\text{base}}) = V$. Let $B = \langle F_{\text{base}}, \beta \rangle$. We further deduce that $B \lhd M$, since if not the same proof would show that $\delta(A/F_{\text{base}})$ were not minimal.

By Lemma 4.15, we have $\Gamma\dim(A) = \delta(A/F_{\text{base}}) = \delta(B/F_{\text{base}}) = \Gamma\dim(B)$. Now $\Gamma\dim(a) = \Gamma\dim(A)$ by the minimality of $\delta(A/F_{\text{base}})$, since A could be taken within $\Gamma\text{cl}(a)$, and $\Gamma\dim(b) \leq \Gamma\dim(B)$, so $\Gamma\dim(b) \leq \Gamma\dim(a)$. By symmetry, $\Gamma\dim(a) = \Gamma\dim(b)$, so QM1' is proved.

Since α is a good basis, there is an isomorphism of Γ-fields $\theta_0 : A \to B$ over F_{base}, with $\theta_0(\alpha) = \beta$. Then also $\theta_0(a) = b$. Since M is $\mathcal{C}^{\text{fg}}$-homogeneous (or $\mathcal{C}^{\text{fg}}_{\Gamma\text{-tr}}$-homogeneous), θ_0 extends to an automorphism θ of M. That proves QM5a.

QM4: Suppose that $C_1, C_2 \lhd_{\text{cl}} M$ with the same quantifier-free L^{QE}-type according to some enumeration, and let $\theta : C_1 \cong C_2$ be the isomorphism given by the enumeration. Suppose also that $b_1 \in M \smallsetminus C_1$ and $b_2 \in M \smallsetminus C_2$.

Using Lemma 4.13 we get $\beta_1, \beta_2 \in \Gamma(M)$ such that $b_i \in C_i(\beta_i)$ and $\text{Loc}(\beta_1, b_1/C_1)$ is defined over K_0 and is equal to $\text{Loc}(\beta_2, b_2/C_2)$. Also, setting $B_i := \langle C_i, \beta_i \rangle$ we have $B_i \lhd M$ and β_i is a good basis for B_i over C_i. By the definition of a good basis, the isomorphism θ extends to $\theta_1 : B_1 \cong B_2$ with $\theta_1(\beta_1) = \beta_2$ and hence $\theta_1(b_1) = b_2$.

Let $F_{\text{base}} \lhd A_1 \lhd C_1$ with A_1 finitely generated over F_{base}, and let $A_2 = \theta(A_1)$. Then θ_1 restricts to an isomorphism $\theta_0 : \langle A_1 \beta_1 \rangle \cong \langle A_2 \beta_2 \rangle$. Also $\langle A_i \beta_i \rangle \lhd M$ since $\delta(\langle A_i \beta_i \rangle / A_i) = 1$ and $\beta_i \notin \Gamma\text{cl}(A_i)$. Since M is $\mathcal{C}^{\text{fg}}$-homogeneous (or $\mathcal{C}^{\text{fg}}_{\Gamma\text{-tr}}$-homogeneous), θ_0 extends to an automorphism of M. So $\text{qftp}_{L^{\text{QE}}}(A_1 b_1) = \text{qftp}_{L^{\text{QE}}}(A_2 b_2)$ and thus, as A_1 ranges over strong Γ-subfields of C_1 finitely generated over F_{base}, we deduce that $\text{qftp}_{L^{\text{QE}}}(C_1 b_1) = \text{qftp}_{L^{\text{QE}}}(C_2 b_2)$ as required.

QM5b: Let $C \lhd_{\text{cl}} M$ and let $b \in M$ be a finite tuple. Let B be a finitely generated Γ-field extension of C such that $B \lhd M$ and $b \in B$, and let $\beta \in \Gamma(B)^n$ be a good basis for B over C with $b \in C(\beta)$.

Now choose a finitely generated Γ-field extension C_0 of F_{base} in C with $C_0 \lhd C$, and a good basis γ for C_0, such that $\text{Loc}(\beta, b/C)$ is defined over $F_{\text{base}}(\gamma)$.

Suppose that finite tuples $a, a' \in C$ have $\text{qftp}_{L^{\text{QE}}}(a/\gamma) = \text{qftp}_{L^{\text{QE}}}(a'/\gamma)$. By QM5a, there is a Γ-field automorphism $\theta \in \text{Aut}(M/F_{\text{base}}(\gamma))$ such that $\theta(a) = a'$. Let A be a strong Γ-subfield of $\Gamma\text{cl}(C_0, a)$ which is finitely generated over C_0 and contains a, and let $A' = \theta(A)$. Then $A' \lhd C$.

Let $V = \text{Loc}(\beta/C)$. Then $\text{Loc}(\beta/A) = \text{Loc}(\beta/A') = V$ because V is defined over $F_{\text{base}}(\gamma)$. So, since β is a good basis, the isomorphism $\theta_0 : A \cong A'$ extends to $\theta_1 : \langle A\beta \rangle \cong \langle A'\beta \rangle$.

We claim that $\langle A\beta \rangle \lhd B$. To see this, suppose that $X \subseteq B$ is a finitely generated extension of $\langle A\beta \rangle$, and let $X_0 = X \wedge C$. Let ξ be a basis of X_0 over A. Then $\xi \cup \beta$ is a basis for X over A, since β is a basis for B over C and hence for X over $X \wedge C = X_0$. So

$$
\begin{aligned}
\delta(X/A\beta) &= \text{trd}(\xi/A\beta) - d\,\text{ldim}_{k_{\mathcal{O}}}(\xi/\Gamma(A), \beta) \\
&= \text{trd}(\xi/A) - d\,\text{ldim}_{k_{\mathcal{O}}}(\xi/\Gamma(A)) \\
&= \delta(\xi/A) \geqslant 0
\end{aligned}
$$

because β is algebraically and linearly independent from C over A and $\xi \in C$. Since $B \lhd M$ we have $\langle A\beta \rangle \lhd M$. The same argument shows that $\langle A'\beta \rangle \lhd M$.

Thus, since M is $\mathcal{C}^{\text{fg}}$-homogeneous (or $\mathcal{C}^{\text{fg}}_{\Gamma\text{-tr}}$-homogeneous), θ_1 extends to an automorphism θ_2 of M. Now θ_2 fixes b and γ and $\theta_2(a) = a'$, so $\text{qftp}_{L^{\text{QE}}}(a/b\gamma) = \text{qftp}_{L^{\text{QE}}}(a'/b\gamma)$. Taking $c = \gamma$, considered as a tuple from the field sort of C, we see that $\text{tp}(b/C)$ does not split over c, as required. $\qquad\square$

Remark 6.10. A more complete analysis of splitting for pseudo-exponentiation was carried out in the PhD thesis of Robert Henderson [Hen14].

We conclude this section by showing that the Γ-algebraic types over finite tuples are isolated.

Proposition 6.11. *Suppose that a, b are finite tuples in M and that $b \in \Gamma\text{cl}^M(a)$. Then $\text{tp}(b/a)$ is isolated by an L^{QE}-formula.*

Proof. Choose a finitely generated Γ-field $B \lhd M$ with $B \subseteq \Gamma\text{cl}(a)$, and a good basis β for B such that $a, b \in F_{\text{base}}(\beta)$. Let $W = \text{Loc}(\beta, a^\frown b/F_{\text{base}})$.

Then $M \models \varphi_W(\beta, a^\frown b)$ and $M \models \psi_W(a^\frown b)$. Suppose $M \models \psi_W(a^\frown c)$. Then there is a tuple γ from $\Gamma(M)$ such that $M \models \varphi_W(\gamma, a^\frown c)$. So $\text{Loc}(\gamma/F_{\text{base}}) \subseteq V$ but $F_{\text{base}} \lhd M$ and $\text{ldim}_{k_{\mathcal{O}}}(\gamma/\Gamma(F_{\text{base}})) = \dim V$ by the definition of φ_W, so γ is generic in V over F_{base}. Thus $\text{Loc}(\gamma/F_{\text{base}}) = \text{Loc}(\beta/F_{\text{base}})$ so, since β is a good basis, the Γ-field C generated by γ is isomorphic to B via an isomorphism $\theta : B \to C$ such that $\theta(\beta) = \gamma$, and then necessarily $\theta(a) = a$ and $\theta(b) = c$.

Using Lemma 4.15 repeatedly,

$$\delta(C/F_{\text{base}}) = \delta(B/F_{\text{base}}) = \Gamma\dim(B) = \Gamma\dim(a) \leqslant \Gamma\dim(C)$$

and so $\delta(C/F_{\text{base}}) = \Gamma\dim(C)$, and $C \lhd M$. Thus θ extends to an automorphism of M, so $\text{tp}(c/a) = \text{tp}(b/a)$, so the formula $\psi_W(a^\frown x)$ isolates $\text{tp}(b/a)$. $\qquad\square$

7. CLASSIFICATION OF STRONG EXTENSIONS

We next give a classification of the finitely generated strong extensions. In the next section we will use it to give axiomatizations of the classes of Γ-closed fields we have constructed, generalizing Zilber's axioms for pseudo-exponentiation.

Since G is an $\mathcal{O}$-module, each matrix $M \in \text{Mat}_n(\mathcal{O})$ defines an $\mathcal{O}$-module homomorphism $G^n \xrightarrow{M} G^n$ in the usual way. If $V \subseteq G^n$, we write $M \cdot V$ for its image. Note that if V is a subvariety of G^n then $M \cdot V$ is a constructible set, and since the

$\mathcal{O}$-module structure is defined over K_0, if V is defined over A then $M \cdot V$ is defined over $K_0 \cup A$. If V is irreducible then $M \cdot V$ is also irreducible.

We have $G^n = (G_1 \times G_2)^n$ and we write $x_1, \ldots, x_n$ for the coordinates in G_1 and $y_1, \ldots, y_n$ for the coordinates in G_2.

Definition 7.1. Let V be an irreducible subvariety of G^n. Then V is G_1-*free* if V does not lie inside any subvariety defined by an equation $\sum_{j=1}^n r_j x_j = c$ for any $r_j \in \mathcal{O}$, not all zero, and any $c \in G_1$. We define G_2-*free* the same way. We say V is *free* if it is both G_1-free and G_2-free.

V is *rotund* (for G as an $\mathcal{O}$-module) if for every matrix $M \in \mathrm{Mat}_n(\mathcal{O})$ we have

$$\dim(M \cdot V) \geqslant d \, \mathrm{rk}\, M$$

where dim means dimension as an algebraic variety or constructible set, $\mathrm{rk}\, M$ is the rank of the matrix M, and recall that $d = \dim G_1$.

V is *strongly rotund* if for every *non-zero* matrix $M \in \mathrm{Mat}_n(\mathcal{O})$ we have

$$\dim(M \cdot V) > d \, \mathrm{rk}\, M.$$

A reducible subvariety V of G^n is defined to be free / rotund / strongly rotund if at least one of its (absolutely) irreducible components is free / rotund / strongly rotund respectively. If we say that such a V is *free and (strongly) rotund* then we mean that the same irreducible component is free and (strongly) rotund.

So V is free if it is "free from $\mathcal{O}$-linear dependencies", and it is rotund if all its images under suitable homomorphisms are of large dimension.

Lemma 7.2. *An irreducible subvariety $V \subseteq G^n$ is G_2-free if and only if $\pi_2(V)$ does not lie in a coset of a proper algebraic subgroup of G_2^n. If $\mathcal{O} = \mathrm{End}(G_1)$ then V is G_1-free if and only if $\pi_1(V)$ does not lie in a coset of a proper algebraic subgroup of G_1^n.*

Proof. This is an immediate consequence of Lemma 2.1(i). $\square$

Proposition 7.3. *Suppose that A is a full Γ-field, $A \subseteq B$ is a finitely generated extension of Γ-fields, preserving the kernels, and that $b \in \Gamma(B)^n$ is a basis for the extension. Let $V = \mathrm{Loc}(b/A)$.*

Then V is free. Furthermore the extension is strong if and only if V is rotund, and it is purely Γ-transcendental if and only if V is strongly rotund.

Proof. If V is not G_1-free then, writing $b = (b_1^1, \ldots, b_1^n, b_2^1, \ldots, b_2^n) \in G_1^n \times G_2^n$ we have $\sum_{j=1}^n r_j b_1^j = c_1 \in G_1(A)$. Let $c_2 = \sum_{j=1}^n r_j b_2^j$. Then $(c_1, c_2) \in \Gamma(B)$ and since A is full and the extension preserves the kernels we have $(c_1, c_2) \in \Gamma(A)$. That contradicts b being a basis for the extension. So V is G_1-free and, symmetrically, G_2-free.

For $M \in \mathrm{Mat}_n(\mathcal{O})$ we have $M \cdot b \in \Gamma(B)^n$ with $\mathrm{ldim}_{k_\mathcal{O}}(M \cdot b / \Gamma(A)) = \mathrm{rk}\, M$. Furthermore, every finite tuple from $\Gamma(B)$ generates the same Γ-field extension of A as some tuple $M \cdot b$, because b is a basis. Thus the extension is strong if and only if for all M we have $\mathrm{trd}(M \cdot b / A) \geqslant d \, \mathrm{rk}\, M$, if and only if for all M we have $\dim(M \cdot V)^{\mathrm{Zar}} \geqslant d \, \mathrm{rk}\, M$, if and only if V is rotund.

Similarly any finite tuple from $\Gamma(B)$ which is not in $\Gamma(A)$ generates the same extension of A as a tuple $M \cdot b$ for some non-zero matrix M, and V is strongly rotund if and only if all such tuples have $\delta(M \cdot b / A) > 0$. $\square$

Corollary 7.4. *Suppose that $A \lhd B$ is a finitely generated strong extension of essentially finitary Γ-fields, that $A^{\mathrm{full}} \wedge B = A$, that $b \in \Gamma(B)^n$ is a basis for the extension, and that $V = \mathrm{Loc}(b/A)$. Then V is free and rotund, and it is strongly rotund if and only if B is a purely Γ-transcendental extension of A.*

Proof. First note that, since A and B are essentially finitary, by Theorem 4.17, B^{full} is uniquely determined up to isomorphism and A^{full} is uniquely determined as a Γ-subfield of B^{full}, so the condition that $A^{\text{full}} \wedge B = A$ makes unambiguous sense. Now the proof of Proposition 7.3 goes through with this weaker condition in place of $A = A^{\text{full}}$. □

8. Axiomatization of Γ-closed fields

Recall we have fixed a ring $\mathcal{O}$, algebraic $\mathcal{O}$-modules G_1 and G_2, both of dimension d, defined over a countable field K_0, we have $G = G_1 \times G_2$, and we consider structures in the language $L_\Gamma = \langle +, \cdot, -, \Gamma, (c_a)_{a \in K_0} \rangle$, where Γ is a relation symbol of appropriate arity to denote a subset of G. We are given an essentially finitary Γ-field F_{base} containing K_0, of type (EXP), (COR), or (DEQ). We add parameters for F_{base} to the language to get a language $L_{F_{\text{base}}}$. We also have an expanded language L^{QE}.

Definition 8.1. A model in the quasiminimal class $\mathcal{K}(M(F_{\text{base}}))$ will be called a Γ-*closed field* (with the countable closure property, on the base F_{base}).

Theorem 8.2. *An $L_{F_{\text{base}}}$ structure F is a Γ-closed field if and only if it satisfies the following list of axioms which we denote by $\Gamma\text{CF}_{\text{CCP}}(F_{\text{base}})$.*

1. **Full Γ-field:** *F is an algebraically closed field containing K_0 and $\Gamma(F)$ is an $\mathcal{O}$-submodule of $G(F)$ such that the projections of $\Gamma(F)$ to $G_1(F)$ and $G_2(F)$ are surjective.*
2. **Base and kernels:** *F satisfies the full atomic diagram of F_{base}. (In some examples we will discuss how this can be weakened.) Also $\ker_i(F) = \ker_i(F_{\text{base}})$ for $i = 1, 2$.*
3. **Predimension inequality (generalised Schanuel property):** *The predimension function*

   $$\delta(x/F_{\text{base}}) := \text{trd}(x/F_{\text{base}}) - d\,\text{ldim}_{k_\mathcal{O}}(x/\Gamma(F_{\text{base}}))$$

   satisfies $\delta(x/F_{\text{base}}) \geqslant 0$ for all tuples x from $\Gamma(F)$.
4. **Strong Γ-closedness:** *For every irreducible subvariety V of G^n defined over F and of dimension dn, which is free and rotund for the $\mathcal{O}$-module structure on G, and every finite tuple a from $\Gamma(F)$, there is $b \in V(F) \cap \Gamma(F)^n$ such that b is $k_\mathcal{O}$-linearly independent over $\Gamma(F_{\text{base}}) \cup a$ (that is, no non-zero $k_\mathcal{O}$-linear combination of the b_i lies in the $k_\mathcal{O}$-linear span of $\Gamma(F_{\text{base}}) \cup a$).*
5. **Countable Closure Property:** *For each finite subset X of F, the Γ-closure $\Gamma\text{cl}^F(X)$ of X in F is countable.*

Observe that if F is a Γ-field for the $\mathcal{O}$-module G, and in particular if it is full so satisfies axiom 1, then it satisfies axioms 2 and 3 if and only if $F_{\text{base}} \lhd F$.

Lemma 8.3. *If F satisfies axioms 1–3 then it also satisfies axiom 4 if and only if it is $\aleph_0$-saturated for Γ-algebraic extensions (in the sense of Definition 5.14).*

Proof. First assume that F satisfies axioms 1–4. Suppose $A \lhd F$ is finitely generated over F_{base} and $A \overset{\lhd}{\hookrightarrow} B$ is a finitely generated Γ-algebraic extension. We have assumed that F_{base} is essentially finitary, so A^{full} and B^{full} are unique up to isomorphism as extensions of A and B respectively, so A^{full} embeds (strongly) into B^{full}. Choose an embedding. Since A^{full} embeds in F we have $A^{\text{full}} \wedge B$ embedding

(strongly) into F, as summarised in the diagram below.
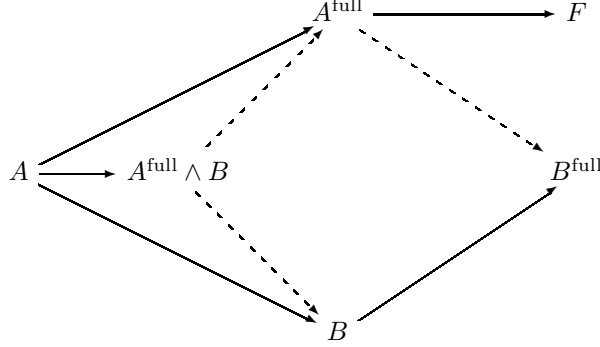
So it remains to embed B in F over $A^{\mathrm{full}} \wedge B$, so we may assume $A = A^{\mathrm{full}} \wedge B$. Let a be a basis for A over F_{base} and let $b \in \Gamma(B)^n$ be a good basis for B over A, which exists by Proposition 3.22. Let $V = \mathrm{Loc}(b/A)$, a subvariety of G^n. Then V is free and rotund by Corollary 7.4. Since B is Γ-algebraic over A we have $\delta(b/A) = 0$, so $\dim V = dn$.

Then, by axiom 4 applied to an absolutely irreducible component of V, there is $c \in \Gamma(F)^n \cap V(F)$, $k_{\mathcal{O}}$-linearly independent over $\Gamma(F_{\mathrm{base}}) \cup \{a\}$. Since $A \lhd F$ we have $\delta(c/A) \geqslant 0$, so $\mathrm{trd}(c/A) = dn = \dim V$. Thus c is generic in V over A. Let C be the Γ-subfield of F generated by A and c. Then c is a good basis of C over A because this is a property of $\mathrm{Loc}(c/A)$, that is, of V. So, by the definition of a good basis, C is isomorphic to B over A. So F is $\aleph_0$-saturated for Γ-algebraic extensions over F_{base}.

For the converse, suppose that F is $\aleph_0$-saturated for Γ-algebraic extensions over F_{base}. Let V be a free and rotund absolutely irreducible subvariety of G^n which is defined over F and of dimension dn, and let a be a finite tuple from $\Gamma(F)$. Extending a if necessary, we may assume that $A = \langle F_{\mathrm{base}}, a \rangle \lhd F$ and that V is defined over a.

Consider a Γ-field extension B of A, generated by a tuple $b \in \Gamma(B)$ such that $\mathrm{Loc}(b/A) = V$. By Proposition 7.3 the extension is strong. Since V is free, $\mathrm{ldim}_{k_{\mathcal{O}}}(b/\Gamma(A)) = n$ and so $\delta(b/A) = \dim V - dn = 0$. So B is a Γ-algebraic extension. Thus B embeds into F over A and so we have $b \in V(F) \cap \Gamma(F)^n$ which is $k_{\mathcal{O}}$-linearly independent over $\Gamma(F_{\mathrm{base}}) \cup \{a\}$ as required. $\qquad\square$

Proof of Theorem 8.2. Suppose F is a Γ-closed field. Then, by definition, F is (isomorphic to) a closed substructure of the canonical model M or is obtained from M (and its closed substructures) as the union of a directed system of closed embeddings. If F is a closed substructure of M then certainly it is a full Γ-field strongly extending F_{base}, so it satisfies axioms 1–3, and it is countable so satisfies axiom 5.

For axiom 4, suppose $A \lhd F$ is finitely generated and $A \xrightarrow{\lhd} B$ is a finitely generated and Γ-algebraic extension. Since M is $\mathcal{C}^{\mathrm{fg}}$-saturated, B embeds strongly into M over A and since F is closed in M, $B \subseteq F$. So by Lemma 8.3, F satisfies axiom 4.

So closed substructures of M satisfy axioms 1–5. Axioms 1–4 are preserved under unions of directed systems of strong embeddings, and all the axioms are preserved under unions of directed systems of closed embeddings, hence all Γ-closed fields satisfy all 5 axioms of $\Gamma\mathrm{CF}_{\mathrm{CCP}}(F_{\mathrm{base}})$.

Suppose now that F satisfies axioms 1–5. Since it satisfies axioms 1–3, we have the pregeometry $\Gamma\mathrm{cl}^F$ on F. If F_0 is a finite-dimensional substructure of F then F_0 satisfies axioms 1–3 and 5 immediately and, using Lemma 8.3, also axiom 4. Let

$\bar{a}$ be a $\Gamma\mathrm{cl}^F$-basis for F_0. Using Lemma 4.13, for each $a_i \in \bar{a}$, choose $\alpha_i \in \Gamma(F_0)$, interalgebraic with a_i over F_{base}. Let $C = \langle F_{\mathrm{base}}, \alpha_1, \ldots, \alpha_n \rangle$. Then F_0 is Γ-algebraic over C and is saturated for Γ-algebraic extensions so, by Proposition 5.15, $F_0 \cong M_0(C)$. Now choose an embedding of C into M and note that $\Gamma\mathrm{cl}^M(C)$ is also Γ-algebraic over C and is saturated for Γ-algebraic extensions so is also isomorphic to $M_0(C)$. Hence F_0 is a Γ-closed field.

Now F is the union of the directed system of all its finite-dimensional closed substructures, which by CCP are countable, and the class of Γ-closed fields is closed under such unions by definition, hence F is a Γ-closed field. $\square$

We can now prove Theorem 1.7.

Proof of Theorem 1.7. By Theorem 6.9, $M(F_{\mathrm{base}})$ is a quasiminimal pregeometry structure, so by Fact 6.4 the class $\mathcal{K}(M(F_{\mathrm{base}}))$ is uncountably categorical and every model is quasiminimal. By Theorem 8.2, the list of axioms $\Gamma\mathrm{CF}_{\mathrm{CCP}}(F_{\mathrm{base}})$ axiomatizes the class $\mathcal{K}(M(F_{\mathrm{base}}))$. $\square$

Remarks 8.4.   (1) It is easy to show that axioms 1–4 are $L_{\omega_1,\omega}$-expressible, and axiom 5 is expressible as an $L_{\omega_1,\omega}(Q)$-sentence.
   (2) If we add an ($L_{\omega_1,\omega}$-expressible) axiom stating that F is infinite dimensional to axioms 1–4, the only countable model is M and so we get an $\aleph_0$-categorical, and hence complete, $L_{\omega_1,\omega}$-sentence.

9. SPECIFIC APPLICATIONS OF THE GENERAL CONSTRUCTION

We list several instances of Γ-fields that are of interest, starting with the original example.

9.1. **Pseudo-exponentiation.** We take $K_0 = \mathbb{Q}$, $G_1 = \mathbb{G}_{\mathrm{a}}$ and $G_2 = \mathbb{G}_{\mathrm{m}}$. Take $\mathcal{O} = \mathbb{Z}$. Let τ be transcendental, and take F_{base} to be the field $\mathbb{Q}^{\mathrm{ab}}(\tau)$, where $\mathbb{Q}^{\mathrm{ab}}$ is the extension of $\mathbb{Q}$ by all roots of unity. For each $m \in \mathbb{N}^+$, choose a primitive m^{th} root of unity ω_m, such that for all $m, n \in \mathbb{N}^+$ we have $(\omega_{mn})^n = \omega_m$. We take $\Gamma(F_{\mathrm{base}})$ to be the graph of a homomorphism from the $\mathbb{Q}$-linear span of τ to the roots of unity such that $\tau/m \mapsto \omega_m$ for each $m \in \mathbb{N}^+$. (This F_{base} is called SK in the paper [Kir13].)

Then the construction gives a class of fields F with a predicate $\Gamma(F)$ defining the graph of a surjective homomorphism from $\mathbb{G}_{\mathrm{a}}(F)$ to $\mathbb{G}_{\mathrm{m}}(F)$, with kernel $\tau\mathbb{Z}$, which we denote by exp. The predimension inequality is precisely Schanuel's conjecture, and the strong existential closedness axiom is known as strong exponential-algebraic closedness. Thus we obtain a proof of Theorem 1.2, which we restate in explicit form.

Theorem 9.1. *Up to isomorphism, there is exactly one model $\langle F; +, \cdot, \exp \rangle$ of each uncountable cardinality of the following list $\mathrm{ECF}_{\mathrm{SK,CCP}}$ of axioms.*
   1. **ELA-field:** *F is an algebraically closed field of characteristic zero, and exp is a surjective homomorphism from $\mathbb{G}_{\mathrm{a}}(F)$ to $\mathbb{G}_{\mathrm{m}}(F)$.*
   2. **Standard kernel:** *the kernel of exp is an infinite cyclic group generated by a transcendental element τ.*
   3. **Schanuel Property:** *The predimension function*
$$\delta(\bar{x}) := \mathrm{trd}(\bar{x}, \exp(\bar{x})) - \mathrm{ldim}_{\mathbb{Q}}(\bar{x})$$
   satisfies $\delta(\bar{x}) \geqslant 0$ for all tuples $\bar{x}$ from F.
   4. **Strong exponential-algebraic closedness:** *If V is a rotund, additively and multiplicatively free subvariety of $\mathbb{G}_{\mathrm{a}}^n \times \mathbb{G}_{\mathrm{m}}^n$ defined over F and of dimension n, and $\bar{a}$ is a finite tuple from F, then there is $\bar{x}$ in F such that*

$(\bar{x}, e^{\bar{x}}) \in V$ and $\bar{x}$ is $\mathbb{Q}$-linearly independent over $\bar{a}$ (that is, no non-zero $\mathbb{Q}$-linear combination of the x_i lies in the $\mathbb{Q}$-linear span of the a_i).

5. **Countable Closure Property:** *For each finite subset X of F, the exponential algebraic closure $\mathrm{ecl}^F(X)$ of X in F is countable.*

Proof. We apply Theorem 1.7, but note that axioms 2 and 3 are slightly different from the axioms given in the statement of Theorem 8.2. The Schanuel property holds on our choice of F_{base} because τ is transcendental, and it follows from the addition property for δ that the two versions of axiom 3 are equivalent in this case. Since τ is transcendental and the kernel is standard, it follows that F_{base} embeds strongly in F, so the two versions of axiom 2 are also equivalent. $\square$

We denote the canonical model of cardinality continuum by $\mathbb{B}$.

9.2. **Incorporating a counterexample to Schanuel's conjecture.** We proceed as in the previous example, except now we choose an irreducible polynomial $P(x,y) \in \mathbb{Z}[x,y]$ and take (ϵ, τ) to be a generic zero of the polynomial $P(x,y)$. (We assume that P is such that neither ϵ nor τ is zero.) Choose a division sequence (ϵ_m) for ϵ, that is, numbers such that $\epsilon_1 = \epsilon$ and $(\epsilon_{mn})^n = \epsilon_m$ for all $m, n \in \mathbb{N}^+$. Now take K to be the field $\mathbb{Q}^{\mathrm{ab}}(\tau, (\epsilon_m)_{m \in \mathbb{N}^+})$, and define $\Gamma(K)$ to be the graph of a homomorphism from the $\mathbb{Q}$-linear span of τ and 1, with $\tau/m \mapsto \omega_m$ as above and $1/m \mapsto \epsilon_m$.

Now the construction gives us a canonical model $\mathbb{B}_P$, the unique model of cardinality continuum of almost the same list of axioms as those for $\mathbb{B}$, except that Schanuel's conjecture has this exception with the formal analogues ϵ and τ of e and $2\pi i$ being algebraically dependent via the polynomial P. More precisely, the predimension axiom is replaced by an axiom scheme stating that $\exp(1)$ and τ are transcendental, that $P(\exp(1), \tau) = 0$, and the condition that for all tuples $\bar{a}$, $\mathrm{trd}(\bar{a}, \exp(\bar{a})/\tau, \exp(1)) - \mathrm{ldim}_{\mathbb{Q}}(\bar{a}/\tau, 1) \geqslant 0$.

More generally, we can take any finitely generated partial exponential field with standard kernel (that is, a finitely generated Γ-field for the appropriate groups and kernels) as F_{base} and do the same construction to build a quasiminimal exponential field $\mathbb{M}(F_{\mathrm{base}})$ of size continuum with counterexamples to the Schanuel property within a finite-dimensional $\mathbb{Q}$-vector space, but with the Schanuel property holding over that vector space. Each $\mathbb{M}(F_{\mathrm{base}})$ is unique up to isomorphism as a model of appropriate axioms, just as $\mathbb{B}$ is. One could conjecture that $\mathbb{C}_{\exp}$ is isomorphic to one of these. Several people have asked us if it might be possible to prove Schanuel's conjecture easily by some method showing that $\mathbb{C}_{\exp}$ must be isomorphic to $\mathbb{B}$, just because $\mathbb{B}$ is categorical. Examples such as these show that soft methods which ignore transcendental number theory and analytic considerations cannot hope to work.

9.3. **Pseudo-Weierstrass $\wp$-functions.** Let E be an elliptic curve over a number field K_0. Choose a Weierstrass equation for E

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3$$

with $g_2, g_3 \in K_0$, which fixes an embedding of E into projective space $\mathbb{P}^2$, with homogeneous coordinates $[X : Y : Z]$. Apart from the point $O = [0 : 1 : 0]$ at infinity, we can identify E with its affine part, given by solutions of the equation

$$y^2 - 4x^3 - g_2 x - g_3 = 0$$

in $\mathbb{A}^2$.

For our construction we take $G_1 = \mathbb{G}_a$ and $G_2 = E$, and we take $\mathcal{O} = \mathrm{End}(E)$, so $\mathcal{O} = \mathbb{Z}$ if E does not have complex multiplication (CM) and $\mathcal{O} = \mathbb{Z}[\tau]$ if E has CM by the imaginary quadratic τ. In the CM case, we assume that $\tau \in K_0$ (and adjoin

it if not). Take ω_1 transcendental over K_0 and ω_2 transcendental over $K_0(\omega_1)$ if E does not have CM, or $\omega_2 = \tau\omega_1$ if E has CM by τ.

As a field, we define $F_{\text{base}} = K_0(\text{Tor}(E), \omega_1, \omega_2)$, where $\text{Tor}(E)$ means the full torsion group of E, which is contained in $E(\mathbb{Q}^{\text{alg}})$. We define $\Gamma(F_{\text{base}})$ to be the graph of a surjective $\mathcal{O}$-module homomorphism from $\mathbb{Q}\omega_1 + \mathbb{Q}\omega_2$ to $\text{Tor}(E)$, with kernel $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. While this may not specify F_{base} up to isomorphism, we will see that Serre's open image theorem allows us to specify F_{base} with only a finite amount of extra information.

In a model M, $\Gamma(M)$ will be the graph of a surjective homomorphism $\exp_{E,M} : \mathbb{G}_{\text{a}}(M) \to E(M)$ with kernel Λ. Using our chosen embedding of E into $\mathbb{P}^2$, we can identify the components of the function $\exp_{E,M}$ with functions $\wp, \wp' : M \to M \cup \{\infty\}$, where $\exp_{E,M}(a) = [\wp(a) : \wp'(a) : 1]$. We call the function $\wp$ a "pseudo-Weierstrass $\wp$-function".

Note that in our model M, Λ is definable by the formula $\exp_{E,M}(x) = O$. In the non-CM case, $\mathbb{Z}$ is definable by the formula $\forall y[y \in \Lambda \to xy \in \Lambda]$, so $\mathbb{Q}$ is also definable as the field of fractions. In the CM case, these formulas define the rings $\mathbb{Z}[\tau]$ and $\mathbb{Q}[\tau]$.

Following [Gav08], we will apply the following version of Serre's open image theorem to show that only a finite amount of extra information is required to specify F_{base} as a Γ-field.

Fact 9.2. *Let E be an elliptic curve defined over a number field K_0. Then there exists an $m \in \mathbb{N}$ such every $\text{End}(E)$-module automorphism of the torsion $\text{Tor}(E)$ which fixes the m-torsion $E[m]$ pointwise is induced by a field automorphism over K_0, that is, the natural homomorphism*

$$\text{Gal}(K_0^{\text{alg}}/K_0(E[m])) \to \text{Aut}_{\text{End}(E)}(\text{Tor}(E)/E[m])$$

is surjective.

Proof. When $\text{End}(E) \cong \mathbb{Z}$, this is Serre's open image theorem [Ser72, Introduction (3)]. When E has complex multiplication, it is the analogous classical open image theorem [Ser72, Section 4.5, Corollaire]. $\qquad\square$

Unfortunately the proof does not give an effective bound for m, so given an explicit K_0 and E we do not know how to compute it.

We can now prove the first half of Theorem 1.6, which we restate precisely.

Theorem 9.3. *Let E be an elliptic curve over a number field $K_0 \subseteq \mathbb{C}$. Up to isomorphism, there is exactly one model of each uncountable cardinality of the following list $\wp\text{CF}_{\text{SK,CCP}}(E)$ of axioms, and these models are all quasiminimal.*

1. **Full $\wp$-field:** *M is an algebraically closed field of characteristic zero, and Γ is the graph of a surjective homomorphism from $\mathbb{G}_{\text{a}}(M)$ to $E(M)$, which we denote by $\exp_{E,M}$. We add parameters for the number field K_0.*

2. **Kernel and base (non-CM case):** *There exist $\omega_1, \omega_2 \in \mathbb{G}_{\text{a}}(M)$, $\mathbb{Q}$-linearly independent, such that the kernel of $\exp_{E,M}$ is of the form $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and, for the number m specified by Fact 9.2, the algebraic type of the pair $(\exp_{E,M}(\omega_1/m), \exp_{E,M}(\omega_2/m)) \in E[m]^2$ over the parameters K_0 is specified.*

2. **Kernel and base (CM case):** *There exists a non-zero $\omega_1 \in \mathbb{G}_{\text{a}}(M)$ such that the kernel of $\exp_{E,M}$ is of the form $\mathbb{Z}[\tau]\omega_1$, and for the number m specified by Fact 9.2, the algebraic type of $\exp_{E,M}(\omega_1/m) \in E[m]$ over the parameters K_0 is specified.*

3. **Predimension inequality:** *The predimension function*

$$\delta(\bar{x}) := \text{trd}(\bar{x}, \exp_{E,M}(\bar{x})) - \text{ldim}_{k_\mathcal{O}}(\bar{x})$$

satisfies $\delta(\bar{x}) \geqslant 0$ *for all tuples* $\bar{x}$ *from* M *where* $k_{\mathcal{O}} = \mathbb{Q}$ *or* $\mathbb{Q}(\tau)$ *as appropriate.*

4. **Strong $\wp$-algebraic closedness:** *The specific case of strong Γ-closedness for this choice of G, $\mathcal{O}$ and F_{base}, from Theorem 8.2.*

5. **Countable Closure Property:** *as in Theorem 8.2.*

Proof. Again we must show that these axioms are equivalent to those given in Theorem 8.2. As in the exponential case, we have the absolute form of the predimension inequality here, which is equivalent to the relative statement over the base together with the assertion that ω_1 is transcendental and, in the non-CM case, that ω_2 is transcendental over $K_0(\omega_1)$. It remains to show that the axioms here specify the atomic diagram of F_{base}.

So suppose that M and M' are both models of the axioms, and their bases, that is the Γ-subfields generated by the kernels, are F_{base} and F_{base}'. We have K_0 as a common subfield, and the axioms give us kernel generators $(\omega_1, \omega_2) \in M^2$ and $(\omega_1', \omega_2') \in M'^2$ such that $(\alpha_1, \alpha_2) := (\exp_{E,M}(\omega_1/m), \exp_{E,M}(\omega_2/m))$ and $(\alpha_1', \alpha_2') := (\exp_{E,M'}(\omega_1'/m), \exp_{E,M'}(\omega_2'/m))$ have the same algebraic type over K_0. (In the CM case we define $\omega_2 = \tau\omega_1$ and $\omega_2' = \tau\omega_1'$ to treat the two cases at the same time.) The points $\alpha_i \in E$ generate the m-torsion subgroup $E[m]$ of E. So we can define a field isomorphism $\sigma_1 : K_0(E[m](M)) \to K_0(E[m](M'))$ by $\alpha_i \mapsto \alpha_i'$ for $i = 1, 2$. Then we extend σ_1 arbitrarily to a field isomorphism $\sigma_2 : K_0(\text{Tor}(E)(M)) \to K_0(\text{Tor}(E)(M'))$.

Now define an $\text{End}(E)$-module automorphism of $\text{Tor}(E)(M)$ by

$$\exp_{E,M}\left(\frac{\omega_1}{n_1} + \frac{\omega_2}{n_2}\right) \mapsto \sigma_2^{-1}\left(\exp_{E,M'}\left(\frac{\omega_1'}{n_1} + \frac{\omega_2'}{n_2}\right)\right)$$

for all $n_1, n_2 \in \mathbb{Z}$. By construction of σ_2 this automorphism fixes $E[m]$ pointwise, so by Fact 9.2 it extends to a field automorphism σ_3 of $K_0(\text{Tor}(E)(M))$. So defining $\sigma_4 = \sigma_2 \circ \sigma_3$ we get a field isomorphism $\sigma_4 : K_0(\text{Tor}(E)(M)) \to K_0(\text{Tor}(E)(M'))$ such that $\sigma_4\left(\exp_{E,M}\left(\frac{\omega_1}{n_1} + \frac{\omega_2}{n_2}\right)\right) = \exp_{E,M'}\left(\frac{\omega_1'}{n_1} + \frac{\omega_2'}{n_2}\right)$ for all $n_1, n_2 \in \mathbb{Z}$.

The predimension inequality implies that (ω_1, ω_2) and (ω_1', ω_2') have the same field-theoretic type over $\mathbb{Q}^{\text{alg}}$, so we can extend σ_4 to a field isomorphism σ_5 by defining $\sigma_5(\omega_i) = \omega_i'$ for $i = 1, 2$, and this σ_5 is a Γ-field isomorphism $F_{\text{base}} \to F_{\text{base}}'$ as required. $\square$

Later in Proposition 10.1 we will show that the predimension inequality above is the appropriate form of Schanuel's conjecture for the $\wp$-functions, and we will thereby complete the proof of Theorem 1.6.

9.4. Variants on $\wp$-functions.
As in the exponential case, we can do variant constructions by changing the base field F_{base} to a different finitely generated Γ-field, to incorporate some counterexamples to the predimension inequality. We can also do constructions of "pseudo-analytic" homomorphisms $\mathbb{G}_{\text{a}}(M) \to E(M)$ which have no complex-analytic analogue. For example, choose an elliptic curve E without complex multiplication and take the kernel lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for $\omega_1/\omega_2 \in \mathbb{R}$, totally real (that is algebraic and such that it and all its conjugates are real), for example real quadratic. The construction still works perfectly well to produce a unique quasiminimal model, but no embedding of Λ into $\mathbb{C}$ can be the kernel of a meromorphic homomorphism because it is dense on the line $\mathbb{R}\omega_1$.

9.5. Exponential maps of simple abelian varieties.
This is the algebraic setup corresponding to the complex example described in Definition 3.1. Take $G_1 = \mathbb{G}_{\text{a}}^d$ and G_2 a simple abelian variety of dimension d, defined over a number field K_0.

Take $\mathcal{O} = \text{End}(G_2)$, and suppose these endomorphisms are also defined over K_0. Fix an embedding of K_0 into $\mathbb{C}$.

Let $\omega_1, \ldots, \omega_{2d} \in \mathbb{C}^d$ be generators of a lattice Λ such that $\mathbb{C}^d/\Lambda$ is isomorphic to $G_2(\mathbb{C})$ as a complex $\mathcal{O}$-module manifold.

We take F_{base} to be the field generated by the ω_i together with $\text{Tor}(G_2)$, and $\Gamma(F_{\text{base}})$ to be the graph of an $\mathcal{O}$-module homomorphism from $\mathbb{Q}\Lambda$ onto $\text{Tor}(G_2)$.

For abelian varieties of dimension greater than 1 there is no non-conjectural analogue of Serre's open image theorem so we cannot be more specific about an axiomatization of the atomic diagram of the base. So we have no improvement on the statement of Theorem 8.2 in this case.

9.6. Factorisations via $\mathbb{G_m}$ of elliptic exponential maps.
The examples so far have all been of the exponential type, case (EXP). Here, we give an example in case (COR). Let $G_1 = \mathbb{G}_m$ and $G_2 = E$, an elliptic curve without complex multiplication, defined over a number field K_0. Let $\mathcal{O} = \mathbb{Z}$.

Let ω be transcendental. As a field $F_{\text{base}} = K_0(\omega, \text{Tor}(\mathbb{G}_m), \text{Tor}(E))$ and we define $\Gamma(F_{\text{base}})$ to be the graph of a surjective homomorphism from $\mathbb{Q}\omega + \text{Tor}(\mathbb{G}_m)$ onto $\text{Tor}(E)$ with kernel $\mathbb{Z}\omega$. Then for $\mathbb{M} = \mathbb{M}(F_{\text{base}})$, $\Gamma(\mathbb{M})$ is the graph of a surjective homomorphism $\theta_{\mathbb{M}} : \mathbb{G}_m(\mathbb{M}) \to E(\mathbb{M})$.

In the complex case, the exponential map of E factors through the exponential map of $\mathbb{G}_m$ as

$$\mathbb{G_a}(\mathbb{C}) \xrightarrow{[\wp : \wp' : 1]} E(\mathbb{C})$$

with $\exp$ going down to $\mathbb{G_m}(\mathbb{C})$ and θ the diagonal map.

and this pseudo-analytic map $\theta_{\mathbb{M}}$ is an analogue of the complex map θ. Since $E \times \mathbb{G}_m$ is not simple, the methods of this paper do not suffice to build a field F equipped with a map θ and pseudo-exponential maps of $\mathbb{G}_m$ and E together, in which the analogue of the above commutative diagram would hold together with a suitable predimension inequality and a categoricity theorem for a reasonable axiomatization. However, it seems likely that this is achievable by combining the methods of this paper with those of [Kir09].

Question 9.4. The main obstacle to stability for the first-order theory of $\mathbb{B}$ is the kernel. In this case the kernel is just a cyclic subgroup of $\mathbb{G}_m$, and it is known that $\mathbb{G}_m$ equipped with such a group is superstable. So it is natural to ask whether the first-order theory of $\mathbb{M}$ in this case is actually superstable. One could even ask if any construction of type (B), say with finite rank kernels, produces a structure with a superstable first-order theory.

9.7. Differential equations.
We now give an example of type (DEQ). Let K_0 be a countable field of characteristic 0, let G_2 be any simple semiabelian variety of dimension d defined over K_0 and let $G_1 = \mathbb{G}_a^d$. Let $\mathcal{O} = \text{End}(G_2)$. Let C be a countable algebraically closed field extending K_0, and define $\Gamma(C) = G(C)$.

Now consider the amalgamation construction using C as the base but considering only purely Γ-transcendental extensions of C, that is using the category $\mathcal{C}_{\Gamma\text{-tr}}(C)$ in place of $\mathcal{C}(C)$. Theorem 6.9 shows we have a quasiminimal pregeometry structure, and hence a canonical model in each uncountable cardinality. The models we obtain are quasiminimal and $\Gamma\text{cl}(\emptyset) = C$.

This construction is also considered in [Kir09] where it is shown that the first-order theory of these models is $\aleph_0$-stable. In that paper it is also shown that if $\langle F; +, \cdot, D \rangle$ is a differentially closed field and we define $\Gamma(F)$ to be the solution set

to the exponential differential equation for G_2 then the reduct $\langle F; +, \cdot, \Gamma \rangle$ is a model of the same first-order theory, and C is the field of constants for the differential field. The paper [Kir09] also considers the situation with several different groups Γ_S relating to the exponential differential equations of different semiabelian varieties S, which do not have to be simple, but they do have to be defined over the constant field C.

10. Comparison with the analytic models

Zilber conjectured that $\mathbb{C}_{\exp} \cong \mathbb{B}$ and it seems reasonable to extend the conjecture to the exponential map of any simple complex abelian variety, and indeed to other analytic functions such as the function θ from section 9.6. In each example, axioms 1 and 2 are set up to describe properties we know about these analytic functions, so verifying the conjecture amounts to verifying the other three axioms. We consider the progress towards each of the axioms in turn.

10.1. **The predimension inequalities.** For the usual exponential function, the predimension inequality states that for all tuples a from $\mathbb{C}$, $\operatorname{trd}(a, e^a) \geqslant \operatorname{ldim}_{\mathbb{Q}}(a)$. This is precisely Schanuel's conjecture.

In the case of an elliptic curve E defined over a number field, the predimension inequality states for all tuples a from $\mathbb{C}$, $\delta_E(a) = \operatorname{trd}(a, \exp_{E,\mathbb{C}}(a)) - \operatorname{ldim}_{k_\mathcal{O}}(a) \geqslant 0$.

Proposition 10.1. *The predimension inequality above for the exponential map of an elliptic curve follows from the André-Grothendieck conjecture on the periods of 1-motives.*

The following proof was explained to us by Juan Diego Caycedo, and follows the proof of a related statement in section 3 of [CZ14].

Proof. By Théorème 1.2 of [Ber02], with $s = 0$ and $n = 1$, a special case of André's conjecture (building on Grothendieck's earlier conjecture) states that if $j(E)$ is the j-invariant of E, ω_1 and ω_2 are the periods of E, η_1 and η_2 are the quasiperiods of E, $P_1, \ldots, P_n \in E(\mathbb{C})$, a_i is the integral of the first kind associated with P_i, and d_i is the integral of the second kind associated with P_i then:

$$(1) \qquad \operatorname{trd}(2\pi i, j(E), \omega_1, \omega_2, \eta_1, \eta_2, \bar{P}, a, d) \geqslant 2 \operatorname{ldim}_{k_\mathcal{O}}(a/\omega_1, \omega_2) + 4[k_\mathcal{O} : \mathbb{Q}]^{-1}.$$

In this case, we have that $P_i = [\wp(a_i) : \wp'(a_i) : 1] = \exp_{E,\mathbb{C}}(a_i)$. Since our E is defined over a number field, $j(E)$ is algebraic. The Legendre relation states $\omega_1 \eta_2 - \omega_2 \eta_1 = 2\pi i$, so $j(E)$ and $2\pi i$ do not contribute to the above inequality.

If we assume that $a_1, \ldots, a_n \in \mathbb{C}$ are $k_\mathcal{O}$-linearly independent over ω_1, ω_2 we can discard the integrals of the second kind to get the bound

$$(2) \qquad \operatorname{trd}(\omega_1, \omega_2, \eta_1, \eta_2, a, \exp_{E,\mathbb{C}}(a)) \geqslant \operatorname{ldim}_{k_\mathcal{O}}(a/\omega_1, \omega_2) + 4[k_\mathcal{O} : \mathbb{Q}]^{-1}.$$

Consider the case where there is no CM, so $k_\mathcal{O} = \mathbb{Q}$. Throwing away η_1 and η_2 we get

$$(3) \qquad \operatorname{trd}(\omega_1, \omega_2, a, \exp_{E,\mathbb{C}}(a)) \geqslant \operatorname{ldim}_{k_\mathcal{O}}(a/\omega_1, \omega_2) + 2.$$

From the case $n = 0$ we see that $\operatorname{trd}(\omega_1, \omega_2) = 2$ and since ω_1 and ω_2 are $\mathbb{Q}$-linearly independent we have $\delta_E(\omega_1, \omega_2) = 0$. Then (3) implies for any a we have $\delta_E(a/\omega_1, \omega_2) \geqslant 0$, and putting these two statements together we deduce that $\delta_E(a) \geqslant 0$.

Where E does have CM (and is defined over a number field) Chudnovsky's theorem [Chu80, Theorem 1 and Corollary 2] gives us $\operatorname{trd}(\omega_1, \omega_2, \eta_1, \eta_2) = \operatorname{trd}(\omega_1, \pi) = 2$, so in particular $\operatorname{trd}(\omega_1) = 1$. We also have $k_\mathcal{O} = \mathbb{Q}(\tau)$ with $[k_\mathcal{O} : \mathbb{Q}] = 2$ and $\omega_2 = \omega_1 \tau$, so we can discard ω_2, η_1 and η_2 from (2) to obtain

$$(4) \qquad \operatorname{trd}(\omega_1, a, \exp_{E,\mathbb{C}}(a)) \geqslant \operatorname{ldim}_{k_\mathcal{O}}(a/\omega_1) + 1.$$

The same argument now shows that $\delta_E(a) \geqslant 0$ for any tuple a.   □

Proof of Theorem 1.6. Theorem 9.3 shows that the axioms $\wp\mathrm{CF}_{\mathrm{SK},\mathrm{CCP}}(E)$ are uncountably categorical and that every model is quasiminimal. The analytic structure $\mathbb{C}_\wp$ is a model of the first two axioms by construction. Proposition 10.1 shows that the predimension inequality given in axiom 3 is the appropriate analogue of Schanuel's conjecture for $\wp$-functions. Axiom 5, the countable closure property was proved in this case in [JKS16].   □

We will give another proof of the countable closure property in Theorem 1.8 in this paper.

Our understanding of the periods conjecture uses Bertolin's translation to remove the motives, which she did only in the cases of elliptic curves and $\mathbb{G}_\mathrm{m}$. For abelian varieties of dimension greater than 1 we suspect that the predimension inequality axiom again follows from the André-Grothendieck periods conjecture, but there are more complications because the Mumford-Tate group plays a role and so we have not been able to verify it.

10.2. **Strong Γ-closedness.** In the case of the usual exponentiation for $\mathbb{G}_\mathrm{m}$, Mantova [Man14] currently has the best result towards proving the Strong Γ-closedness in the complex case. He only considers the case of a variety $V \subseteq G^n$ where $n = 1$. A free and rotund $V \subseteq G^1$ is just the solution set of an irreducible polynomial $p(x, y) \in \mathbb{C}[x, y]$ which depends on both x and y, that is, the partial derivatives $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ are both non-zero.

Fact 10.2. *Suppose $p(x, y) \in \mathbb{C}[x, y]$ depends on both x and y. Then there are infinitely many points $x \in \mathbb{C}$ such that $p(x, e^x) = 0$. Furthermore suppose Schanuel's conjecture is true and let a be a finite tuple from $\mathbb{C}$. Then there is $x \in \mathbb{C}$ such that (x, e^x) is a generic zero of p over a.*

The observation that there are infinitely many solutions and the whole statement in the case that p is defined over $\mathbb{Q}^{\mathrm{alg}}$ is due to Marker [Mar06]. The general case stated above is due to Mantova [Man14, Theorem 1.2].

10.3. **Γ-closedness.** In section 11 we will see that for some purposes strong Γ-closedness can be weakened to Γ-closedness.

Definition 10.3. A full Γ-field F is Γ-*closed* if for every irreducible subvariety V of G^n defined over F and of dimension dn, which is free and rotund, $V(F) \cap \Gamma(F)^n$ is Zariski-dense in V.

Using the classical Rabinovich trick, one can easily show this axiom scheme is equivalent to the existence of a single point $\beta \in V(F) \cap \Gamma(F)^n$, for every such V. For the usual exponentiation, Γ-closedness is known as exponential-algebraic closedness. In this direction, Brownawell and Masser [BM16, Proposition 2] have the following.

Fact 10.4. *If $V \subseteq (\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m})^n(\mathbb{C})$ is an algebraic subvariety of dimension n which projects dominantly to $\mathbb{G}_\mathrm{a}^n$ then there is $a \in \mathbb{C}^n$ such that $(a, e^a) \in V$.*

In this case V can be taken free without loss of generality, and the condition of projecting dominantly to $\mathbb{G}_\mathrm{a}^n$ implies rotundity. However it is much stronger than rotundity. Another exposition of this theorem is given in [DFT16].

10.4. **The pregeometry and the countable closure property.** To compare the pregeometry of our constructions such as $\mathbb{B}$ with the complex analytic models such as $\mathbb{C}_{\exp}$ we have to define the appropriate pregeometry on the complex field. Given a Γ-field F, we defined a Γ-subfield A of F to be Γ-closed in F if whenever $A \subseteq B \subseteq F$ with $\delta(B/A) \leqslant 0$ then $B \subseteq A$. One can construct Γ-fields with no proper Γ-closed subfields. Fortunately we are able to show unconditionally that there is a countable Γ-subfield of $\mathbb{C}$ which is Γ-closed in $\mathbb{C}$.

For $\mathbb{C}_{\exp}$, this was done in [Kir10a] by adapting the proof of Ax's differential forms version of Schanuel's conjecture. A similar proof was given in [JKS16] for elliptic curves. The same method ought to work for any semiabelian varieties, but here we give a different approach, applying the main result of [Ax72] directly to generalise a theorem of Zilber in the exponential case [Zil05b, Theorem 5.12].

Let $\mathbb{C}_{\Gamma}$ be an analytic Γ-field, which recall means a Γ-field from Definition 3.1 or 3.2. Then Γ is a complex Lie subgroup of $G(\mathbb{C})$, and $L\Gamma \leq LG(\mathbb{C})$ is the graph of a $\mathbb{C}$-linear isomorphism between $LG_1(\mathbb{C})$ and $LG_2(\mathbb{C})$. Thus Γ^n is a complex Lie subgroup of $G^n(\mathbb{C})$, so has a complex topology. It might not be a closed subgroup, so the topology might not be the subspace topology.

Definition 10.5. Given an algebraic subvariety $V \subseteq G^n$, write V^{isol} for the set of all isolated points of $V(\mathbb{C}) \cap \Gamma^n$ with respect to the complex topology on Γ^n. For any subset $A \subseteq \mathbb{C}$ we define $\Gamma\mathrm{cl}'(A)$ to be the subfield of $\mathbb{C}$ generated by the union of all V^{isol} where V ranges over the algebraic subvarieties $V \subseteq G^n$ which are defined over $K_0(A)$. We consider $\Gamma\mathrm{cl}'(A)$ as a Γ-subfield of $\mathbb{C}$ by defining $\Gamma(\Gamma\mathrm{cl}'(A)) := \Gamma \cap G(\Gamma\mathrm{cl}'(A))$.

Lemma 10.6. $\Gamma\mathrm{cl}'$ *is a closure operator on* $\mathbb{C}$ *and for any* $A \subseteq \mathbb{C}$, $\Gamma\mathrm{cl}'(A)$ *is a full* Γ-*subfield of* $\mathbb{C}$ *of cardinality* $|A| + \aleph_0$.

Proof. For transitivity, suppose x is a finite tuple from $\Gamma\mathrm{cl}'(A)$ and $y \in \Gamma\mathrm{cl}'(Ax)$. We may reduce to the case that $\alpha \in W(\mathbb{C}) \cap \Gamma^n$ is isolated and the tuple x lists the co-ordinates of α, and $\beta \in V(\mathbb{C}) \cap \Gamma^m$ is isolated and y is a co-ordinate of β, with W defined over $K_0(A)$ and V over $K_0(Ax)$. Then V can be written as a fibre $V'(\alpha)$ of a subvariety $V' \subseteq G^{m+n}$ over $K_0(A)$ projecting to $W \subseteq G^n$. Then $\beta\alpha \in V' \cap \Gamma^{m+n}$ is an isolated point, so $\beta \in \Gamma\mathrm{cl}'(A)$.

Now let $A \subseteq \mathbb{C}$, and set $A' := \mathrm{acl}^{\mathbb{C}}(K_0(A))$. Let $\alpha_0 \in G_1(A')$, and let $V_0 \subseteq G_1$ be the set of its conjugates, i.e. the 0-dimensional locus of α_0 over $K_0(A)$, and let $V := V_0 \times G_2 \subseteq G$. Then $V \cap \Gamma$ is non-empty, since $\pi_1 : \Gamma \twoheadrightarrow G_1(\mathbb{C})$ is surjective, and it consists of isolated points since $\ker(\pi_1)$ does. This shows that $A' \subseteq \Gamma\mathrm{cl}'(A)$, and hence in particular that $\Gamma\mathrm{cl}'$ is a closure operator, and furthermore this argument shows that $\Gamma\mathrm{cl}'(A)$ is full.

Finally, for the cardinality calculation, note there are only $(|A| + \aleph_0)$-many algebraic varieties V defined over A and for each there can be only countably many isolated points in $V(\mathbb{C}) \cap \Gamma^n$. $\square$

Proposition 10.7. *For an analytic* Γ-*field* $\mathbb{C}_{\Gamma}$, *the closure operators* $\Gamma\mathrm{cl}$ *and* $\Gamma\mathrm{cl}'$ *are the same. In particular,* $\Gamma\mathrm{cl}'$ *is a pregeometry on* $\mathbb{C}$.

To prove this we will use a lemma and Ax's theorem on the transversality of intersections between analytic subgroups and algebraic varieties.

Lemma 10.8. *If* $H \leq G^n$ *is a connected algebraic subgroup which is free then the analytic subgroup* $H(\mathbb{C}) + \Gamma^n$ *is equal to* $G^n(\mathbb{C})$.

Proof. Since G_2 is simple and not isogenous to G_1, every algebraic subgroup of G^n is of the form $H_1 \times H_2$ with H_i a subgroup of G_i^n, and since H is free it is G_2-free. Since $\mathcal{O} = \mathrm{End}(G_2)$ it follows that H is of the form $H_1 \times G_2^n$. Now since $\pi_1(\Gamma^n) = G_1^n(\mathbb{C})$ we see $H(\mathbb{C}) + \Gamma^n = G^n(\mathbb{C})$. $\square$

Fact 10.9 ([Ax72, Corollary 1]). *Suppose that $\mathcal{G}$ is a complex algebraic group, A is a connected analytic subgroup of $\mathcal{G}$, U is open in $\mathcal{G}$ and X is an irreducible analytic subvariety of U such that $X \subseteq A$, X^{Zar} is the Zariski closure of X and H is the smallest algebraic subgroup of $\mathcal{G}$ containing X. Then*

$$\dim(H + A) \leqslant \dim X^{\mathrm{Zar}} + \dim A - \dim X.$$

Proof of Proposition 10.7. Suppose that $A \subseteq \mathbb{C}$ is $\Gamma\mathrm{cl}$-closed. Let $\alpha \in V^{\mathrm{isol}}$ for some V defined over $K_0(A)$. Replacing α with a subtuple if necessary, we may assume α is $k_{\mathcal{O}}$-linearly independent over $\Gamma(A)$. Then α is a smooth point of V and of Γ^n, so by considering tangent spaces we see that $\mathrm{trd}(\alpha/A) \leqslant \dim V \leqslant nd$, and hence $\delta(\alpha/A) \leqslant 0$. So $\alpha \in \Gamma^n(A)$ and hence A is $\Gamma\mathrm{cl}'$-closed.

Now suppose A is $\Gamma\mathrm{cl}'$-closed, and that $A \subseteq B \subseteq \mathbb{C}$ is a proper finitely generated Γ-field extension in $\mathbb{C}_\Gamma$. Let $b \in \Gamma^n(B)$ be a basis for the extension and let $V = \mathrm{Loc}(b/A)$. We will show that $\delta(b/A) > 0$.

Let X be an irreducible analytic component containing b of the analytic subset $V(\mathbb{C}) \cap \Gamma^n$ of the complex Lie group Γ^n. Since A is $\Gamma\mathrm{cl}'$-closed and $b \notin A$, $\dim X \geqslant 1$.

We claim that X has some point in A. To see this, let e be a smooth point of X, and take regular local co-ordinates η_i at e in G^n such that X is locally the graph of a function from the first $\dim X$ co-ordinates to the rest. A is algebraically closed as a field, so is topologically dense in $\mathbb{C}$. So there is a point $a \in X$ close to e such that the first $\dim X$ co-ordinates are in A. Let W be the intersection of V with $\eta_i = a_i$ for $i = 1, ..., \dim X$. Then W is defined over A and a is an isolated point of $W(\mathbb{C}) \cap \Gamma^n$, hence a is in $G^n(A)$ as required.

Suppose X^{Zar} is not G_1-free, so say $(x, y) \in X^{\mathrm{Zar}}$ implies a non-trivial $\mathcal{O}$-linear equation $\sum_{j=1}^n r_j x_j = c$. Then this equation holds for $\pi_1(a)$, so $c \in G_1(A)$. But then since $b \in X$, already $(x, y) \in V$ implies this equation, so V is not G_1-free, a contradiction since V is the locus of a basis over A. The same proof shows that X^{Zar} is G_2-free, so it is free.

Let H be the algebraic subgroup of G^n generated by $X^{\mathrm{Zar}}(\mathbb{C}) - b$. Then $X^{\mathrm{Zar}}(\mathbb{C}) - b$ is free, so H is free. So by Lemma 10.8, the subgroup $H + \Gamma^n$ is equal to G^n.

Applying Fact 10.9 we get

$$\dim(H + \Gamma^n) \leqslant \dim(X^{\mathrm{Zar}} - b) + \dim \Gamma^n - \dim(X - b)$$

which gives

$$2dn \leqslant \dim X^{\mathrm{Zar}} + dn - \dim X$$

but $\dim X > 0$ and $X^{\mathrm{Zar}} \subseteq V$ so we deduce that $\dim V > nd$. Thus $\delta(b/A) > 0$. So A is $\Gamma\mathrm{cl}$-closed, as required. $\square$

Proof of Theorem 1.8. Proposition 10.7 shows that $\Gamma\mathrm{cl} = \Gamma\mathrm{cl}'$, and so Lemma 10.6 shows that $\Gamma\mathrm{cl}$ has the countable closure property. $\square$

Remark 10.10. In [Kir10a], an algebraic version of the isolated points closure $\Gamma\mathrm{cl}'$ was given, using the fact that a solution to a system of $2n$ equations of analytic functions in $2n$ variables is isolated if and only if a certain Jacobian matrix does not vanish at the point. So this gives a definition of a closure operator ecl which makes sense on any exponential field, and it was shown in [Kir10a] that ecl-closed sets are strong and agree with the $\Gamma\mathrm{cl}$-closed sets as we have defined them here, and in particular that ecl is always a pregeometry. This algebraic definition of the closure operator was suggested by Macintyre [Mac96] although it had previously been used in the real and complex cases by Khovanskii and by Wilkie.

11. Generically Γ-closed fields

In this section we consider Γ-fields which may not be strongly Γ-closed but are generically strongly Γ-closed. Using the variant of the amalgamation construction from Section 5.4, we show that such Γ-fields are also quasiminimal and that the *strong* part of strong Γ-closedness becomes redundant in this generic case.

Let K be a full Γ-field. Recall from section 5.4 that an extension $K \subseteq A$ of K is *purely Γ-transcendental* if and only if $K \lhd_{\mathrm{cl}} A$, if and only if for all tuples b from $\Gamma(A)$, either $\delta(b/K) > 0$ or $b \subseteq \Gamma(K)$. Clearly an extension A of K is purely Γ-transcendental if and only if all of its finitely generated sub-extensions are.

11.1. **Generic Γ-closedness.** Recall from Theorem 8.2 that a full Γ-field F is *strongly Γ-closed* if for every irreducible subvariety V of G^n defined over F and of dimension dn, which is free and rotund for the $\mathcal{O}$-module structure on G, and every finite tuple a from $\Gamma(F)$, there is $b \in V(F) \cap \Gamma(F)^n$ such that b is $k_{\mathcal{O}}$-linearly independent over $\Gamma(F_{\mathrm{base}}) \cup a$.

Assuming the Schanuel property, this is equivalent to requiring that b is generic in V over $F_{\mathrm{base}}(a)$. Recall also the weaker form of Γ-closedness from Definition 10.3:

F is *Γ-closed* if for every irreducible subvariety V of G^n defined over F and of dimension dn, which is free and rotund, $V(F) \cap \Gamma(F)^n$ is Zariski-dense in V.

For the concept of generic Γ-closedness with respect to a subfield K, we need to consider extensions of the form $K \lhd_{\mathrm{cl}} A \lhd B$ where A and B are finitely generated as extensions of the full Γ-field K. Say α is a basis of A over K and β is a basis of B over A, and that $V := \mathrm{Loc}(\beta/A)$ and $W := \mathrm{Loc}(\alpha, \beta/K)$. We also assume that $A^{\mathrm{full}} \wedge B = A$. Then by Corollary 7.4 both V and W are free, V is rotund, and W is strongly rotund.

Definition 11.1 (Generic Γ-closedness). Let F be a full Γ-field and $K \lhd_{\mathrm{cl}} F$, $K \neq F$. Suppose $V \subseteq G^n$ is free and rotund, irreducible and of dimension dn. Suppose also that there is $\alpha \in \Gamma(F)^r$, $k_{\mathcal{O}}$-linearly independent over $\Gamma(K)$ such that V is defined over $K(\alpha)$ and for $\beta \in V$, generic over $K(\alpha)$, the variety $W := \mathrm{Loc}(\alpha, \beta/K)$ is strongly rotund.

We say that F is *generically Γ-closed over K* (GΓC over K) if, for all such V, we have that $V(F) \cap \Gamma^n(F)$ is Zariski-dense in V.

F is *generically strongly Γ-closed over K* (GSΓC over K) if, whenever V and α are as above, there is $\gamma \in V(F) \cap \Gamma^n(F)$, $k_{\mathcal{O}}$-linearly independent over $\Gamma(K) \cup \alpha$.

We say F is GΓC or GSΓC without reference to K to mean G(S)ΓC over $\Gamma\mathrm{cl}^F(\emptyset)$.

Proposition 11.2. *Suppose F is a full Γ-field and $K \lhd_{\mathrm{cl}} F$, $K \neq F$. Then F is GSΓC over K if and only if F is $\aleph_0$-saturated for Γ-algebraic extensions which are purely Γ-transcendental over K.*

Proof. This is essentially the same as the proof of Lemma 8.3. $\square$

It is immediate from the definitions that GSΓC over K implies GΓC over K. In [KZ14] it was shown that if the Conjecture on Intersections with Tori (CIT, now also known as the multiplicative case of the Zilber-Pink conjecture) is true, then any exponential field satisfying the Schanuel property which is exponentially-algebraically closed is also strongly exponentially algebraically closed. We use similar ideas here to prove that GΓC over K implies GSΓC over K. The Schanuel property is replaced by the assumption that K is Γ-closed in F, and instead of the Zilber-Pink conjecture it is enough to use the weak version which is a theorem.

Given any variety S and subvarieties W, V of S, the typical dimension of $W \cap V$ is $\dim W + \dim V - \dim S$. If X is an irreducible component of $W \cap V$ it is said to

have *atypical dimension* (for the intersection) if

$$\dim X > \dim W + \dim V - \dim S.$$

We also say that X is an *atypical component* of the intersection. For the multiplicative group, the *weak Zilber-Pink* theorem is known as *weak CIT* and is due to Zilber [Zil02, Corollary 3]. The semiabelian form of the statement is the following theorem [Kir09, Theorem 4.6].

Fact 11.3 ("Semiabelian weak Zilber-Pink", basic version)**.** *Let S be a semiabelian variety, defined over an algebraically closed field of characteristic 0. Let $(W_b)_{b \in B}$ be a constructible family of constructible subsets of S. That is, B is a constructible set and W is a constructible subset of $B \times S$, with W_b the obvious projection of a fibre. Then there is a finite set $\mathcal{H}_W$ of connected proper algebraic subgroups of S such that for any $b \in B$ and any coset $c+J$ of any connected algebraic subgroup J of S, if X is an irreducible component of $W_b \cap c+J$ of atypical dimension with $c \in X$ then there is $H \in \mathcal{H}_W$ such that $X \subseteq c+H$.*

We also need a version for subvarieties not of S but of varieties of the form $U \times S$, which is sometimes called a "horizontal" family of semiabelian varieties.

Theorem 11.4 ("Horizontal semiabelian weak Zilber-Pink")**.** *Let S be a semiabelian variety and let U be any variety. Let $(W_b)_{b \in B}$ be a constructible family of constructible subsets of $U \times S$. Then there is a finite set $\mathcal{H}_W$ of connected proper algebraic subgroups of S such that for any $b \in B$ and any coset $c+J$ of any connected algebraic subgroup J of S, if X is an irreducible component of $W_b \cap (U \times c+J)$ of atypical dimension with $c \in X$ then there is $H \in \mathcal{H}_W$ such that $X \subseteq U \times c+H$. Furthermore H can be chosen such that we have*

$$(*) \qquad \dim X \leqslant \dim\big(W_b \cap (U \times c+H)\big) + \dim(H \cap J) - \dim H.$$

Proof. First suppose U is a point, so $U \times S \cong S$. The main part of the statement is then Fact 11.3. For the "furthermore" part, suppose $(*)$ does not hold for the H we chose from $\mathcal{H}_W$. Then rename $\mathcal{H}_W$ as $\mathcal{H}_W^1$. We give an inductive argument to find a new $\mathcal{H}_W$ which suffices. We have the irreducible X as a component of the intersection $(W_b \cap c+H) \cap c+(H \cap J)$, and the failure of $(*)$ says that X is atypical as a component of this intersection considered as an intersection of subvarieties of $c+H$. Translating everything by c, we get that $X-c$ is an atypical component of the intersection $(W_b-c \cap H) \cap (H \cap J)$ inside H. Now apply Fact 11.3 again with H in place of S to get a proper connected algebraic subgroup H' of H from the finite set $\mathcal{H}_W^2 := \mathcal{H}_W^1 \cup \bigcup_{H \in \mathcal{H}_W^1} \mathcal{H}_{(W_b-c \cap H)_{b,c}}$ such that $X \subseteq c+H'$. If necessary we can iterate this construction and since $\dim H' < \dim H$ it stops after finitely many steps, and the $\mathcal{H}_W$ we eventually find is still finite.

Now consider arbitrary U. Suppose first that all fibres of the co-ordinate projection $\pi : W_b \to S$ have the same dimension k, constant with respect to b. Take $\mathcal{H}_W$ to be the finite set $\mathcal{H}_{(\pi(W_b))_{b \in B}}$ given by this theorem with U a point. We will see that this works as required.

Indeed, let $c+J$ be a coset in S, let X be an atypical component of $W_b \cap (U \times c+J)$, let Y be any irreducible component of $\pi(W_b) \cap c+J$ containing $\pi(X)$, and let $H \in \mathcal{H}_{(\pi(W_b))_{b \in B}}$ be as given by the theorem.

Then by considering dimensions of fibres, we have

$$
\begin{aligned}
\dim X \;&=\; \dim(\pi(X)) + k \\
&\leqslant\; \dim Y + k \\
&\leqslant\; \dim(\pi(W_b) \cap c+H) + \dim(H \cap J) - \dim H + k \\
&=\; \dim\big(W_b \cap (U \times c+H)\big) + \dim(H \cap J) - \dim H
\end{aligned}
$$

Now for a general family $W \subseteq B \times U \times S$, write $\pi : U \times S \to S$ for the projection and define

$$W^k = \left\{ (b, u, s) \in W \mid \dim(W_b \cap \pi^{-1}(s)) = k \right\}$$

for each $k = 0, \ldots, \dim W$. By the definability of dimension, these W^k are all constructible subsets of W, partitioning it, and each W^k satisfies the above constancy condition on fibres. For any $c+J$, any component X of $W_b \cap (U \times c+J)$ contains a dense constructible subset which lies in some piece W_b^k. So we can take $\mathcal{H}_W$ to be $\bigcup_k \mathcal{H}_{W^k}$. $\qquad\square$

Now we can prove that GSΓC and GΓC are equivalent.

Proposition 11.5. *Suppose F is a full Γ-field and $K \lhd_{\mathrm{cl}} F$, $K \neq F$. Then F is GSΓC over K if and only if it is GΓC over K.*

Proof. As remarked earlier, it is immediate that GSΓC over K implies GΓC over K. So assume F is GΓC over K. Let V, α, β, and W be as given in Definition 11.1. Let $V_{\alpha,\mathrm{dep}}$ be the set of points of $V(F)$ which are $k_{\mathcal{O}}$-linearly dependent over $\Gamma(K) \cup \alpha$. We shall find a proper Zariski-closed subset of V containing $V_{\alpha,\mathrm{dep}}$.

We first work in case (EXP), so G_2 is a simple semiabelian variety of dimension d and $G_1 = \mathbb{G}_{\mathrm{a}}^d$, which we identify with the Lie algebra LG_2 of G_2.

For a $d(r+n)$-square matrix M and an $d(r+n)$-column vector c, let $\Lambda_{M,c} \subseteq \mathbb{G}_{\mathrm{a}}^{d(r+n)}$ be given by $x \in \Lambda_{M,c}$ if and only if $Mx = c$. So as M and c vary, we get the family of all possible affine linear subspaces. Let $U_{M,c} = W \cap (\Lambda_{M,c} \times G_2^{r+n})$.

Now suppose $\xi \in V_{\alpha,\mathrm{dep}} \cap \Gamma(F)^n$. Let $\zeta = (\alpha, \xi) \in \Gamma(F)^{r+n}$. We write ζ also as $\zeta = (\zeta_1, \zeta_2) \in G_1^{r+n} \times G_2^{r+n}$. Let J be the smallest algebraic subgroup of G_2^{r+n} such that ζ_2 lies in a K-coset of J, say $\zeta_2 \in c_2' + J$ with $c_2' \in G_2(K)^{r+n}$. Since $\xi \in V_{\alpha,\mathrm{dep}}$, we see that J is a proper algebraic subgroup of G_2^{r+n}.

By Lemma 2.1, there is $M \in \mathrm{Mat}_{r+n}(\mathcal{O})$ such that $J = (\ker(M))^o$ and $LJ = \ker(M)$. Since K is a full Γ-field, there is $c_1' \in G_1^{r+n}(K)$ such that $c' := (c_1', c_2') \in \Gamma(K)^{r+n}$. Then $M(\zeta - c') \in \Gamma(K)^{r+n}$ since $K \leq F$ preserves the kernels. Now $\Gamma(K)$ is a $k_{\mathcal{O}}$-subspace of $\Gamma(F)$, so in particular is existentially closed as an $\mathcal{O}$-submodule, so there exists $c'' \in \Gamma(K)^{r+n}$ such that $Mc'' = M(\zeta - c')$. Set $c = (c_1, c_2) := c' + c'' \in \Gamma(K)^{r+n}$, so $M(\zeta - c) = 0$. Then $\zeta - c$ is divisible in $\ker(M)$, since $\Gamma(F)$ is divisible and torsion-free, so $\zeta - c \in LJ \times J$.

Now we have $\zeta \in U_{M,c_1} \cap (G_1^{r+n} \times c_2+J)$. Let X be the irreducible component of this intersection containing ζ.

We next show that X has atypical dimension for the intersection. From the definition of the predimension δ and its relationship with Γdim we have

$$
\begin{aligned}
\dim X \geqslant \mathrm{trd}(\zeta/K) &= \delta(\zeta/K) + d\,\mathrm{ldim}_{\mathcal{O}}(\zeta/\Gamma(K)) \\
&= \delta(\zeta/K) + \dim J \\
&\geqslant \Gamma\mathrm{dim}^F(\zeta/K) + \dim J
\end{aligned}
$$

$$(5) \qquad\qquad \dim X \;\geqslant\; \Gamma\mathrm{dim}^F(\alpha/K) + \dim J.$$

Since $\alpha \in \Gamma(F)^r$ was chosen $k_{\mathcal{O}}$-linearly independent over $\Gamma(K)$ and such that $\langle K, \alpha \rangle \lhd F$, we have

$$(6) \quad \Gamma\mathrm{dim}^F(\alpha/K) = \delta(\alpha/K) = \mathrm{trd}(\alpha/K) - d\,\mathrm{ldim}_{\mathcal{O}}(\alpha/\Gamma(K)) = \mathrm{trd}(\alpha/K) - dr.$$

Since $W = \mathrm{Loc}(\alpha, \beta/K)$, and $V = \mathrm{Loc}(\beta/K(\alpha))$ has dimension dn, using (6) we have

$$
\begin{aligned}
(7) \qquad \dim W &= \dim V + \mathrm{trd}(\alpha/K) \\
&= d(r+n) + \Gamma\mathrm{dim}^F(\alpha/K)
\end{aligned}
$$

From (5) and (7),

$$
\begin{aligned}
\dim X \;\;\geqslant\;\; & \dim W + \dim J - d(r+n) \\
=\;\; & \dim W + (\dim J + d(r+n)) - 2d(r+n) \\
=\;\; & \dim W + \dim(G_1^{r+n} \times c_2 + J) - \dim(G^{r+n})
\end{aligned}
$$

but W is free, so $\dim U_{M,c_1} < \dim W$ and so

$$
(8) \qquad \dim X > \dim U_{M,c_1} + \dim(G_1^{r+n} \times c_2 + J) - \dim(G^{r+n}).
$$

So X has atypical dimension.

Applying Theorem 11.4 there is a proper algebraic subgroup H of G_2^{r+n} from the finite set $\mathcal{H}_U$ such that $X \subseteq G_1^{r+n} \times c + H$. We have $\zeta \in X$, so $\zeta_2 \in c + H$. J was chosen as the smallest algebraic subgroup of S such that ζ_2 lies in a K-coset of J, so $J \subseteq H$ and hence $H \cap J = J$. So, from the "furthermore" clause of Theorem 11.4 we have

$$
(9) \qquad \dim X \leqslant \dim\left(U_{M,c_1} \cap (G_1^{r+n} \times c_2 + J)\right) + \dim J - \dim H.
$$

We write $TJ = LJ \times J$ and $TH = LH \times H$, thinking of them as the tangent bundles. Then we have

$$
\begin{aligned}
U_{M,c_1} \cap (G_1^{r+n} \times c_2 + J) \;\;=\;\; & W \cap (c_1 + LJ \times G_2^{r+n}) \cap (G_1^{r+n} \times c_2 + J) \\
=\;\; & W \cap c + TJ \\
=\;\; & W \cap \zeta + TJ \\
\subseteq\;\; & W \cap \zeta + TH
\end{aligned}
$$

so

$$
(10) \qquad \dim\left(U_{M,c_1} \cap (G_1^{r+n} \times c_2 + J)\right) \leqslant \dim\left(W \cap \zeta + TH\right).
$$

Combining (9), (10), and (5) we get

$$
\begin{aligned}
\Gamma\dim^F(\alpha/K) + \dim J \;\;\leqslant\;\; & \dim\left(W \cap \zeta + TH\right) + \dim J - \dim H \\
(11) \qquad \Gamma\dim^F(\alpha/K) + \dim H \;\;\leqslant\;\; & \dim\left(W \cap \zeta + TH\right).
\end{aligned}
$$

By Lemma 2.1, there is $M \in \mathrm{Mat}_{r+n}(\mathcal{O})$ such that $H = \ker(M)^o$ and $LH = \ker(M)$, so $TH = \ker(M)^o$ for the action of $\mathcal{O}$ on G. So $M : G^{r+n} \to G^{r+n}$ factors as $G^{r+n} \twoheadrightarrow G^{r+n}/TH \to G^{r+n}$, where the first homomorphism is the quotient map, and the kernel of the second homomorphism is the finite group $\ker(M)/\ker(M)^o$. Now let $\theta_H : W \twoheadrightarrow W/TH$ be the restriction of the quotient map $G^{r+n} \twoheadrightarrow G^{r+n}/TH$. Then $\dim(W/TH) = \dim(M \cdot W)$. Now H is a proper subgroup of G_2^{r+n}, and $\dim(H) = \dim(\ker(M)) = d(r+n-\mathrm{rk}\,M)$, and so M is non-zero. Then since W is strongly rotund, $\dim(W/TH) = \dim(M \cdot W) > d\,\mathrm{rk}\,M = d(r+n) - \dim H$.

So using the fibre dimension theorem, the dimension of a typical fibre of θ_H is

$$
\begin{aligned}
\dim(\text{typical fibre}) \;\;=\;\; & \dim W - \dim(W/TH) \\
<\;\; & \left(d(r+n) + \Gamma\dim^F(\alpha/K)\right) - \left(d(r+n) - \dim H\right) \\
(12) \qquad =\;\; & \Gamma\dim^F(\alpha/K) + \dim H
\end{aligned}
$$

The fibre of θ_H in which ζ lies is $W \cap \zeta + TH$, so (11) says exactly that ζ lies in a fibre of θ_H of atypical dimension. By the fibre dimension theorem, there is a proper Zariski-closed subset W_H of W, defined over K, containing all the fibres of θ_H of atypical dimension.

Since α is generic in the projection of W, and hence of W_H, the subset $V_{H,\alpha} := \{y \in V \mid (\alpha, y) \in W_H\}$ is proper Zariski-closed in V. Let $V_\alpha := \bigcup_{H \in \mathcal{H}_W} H_{H,\alpha}$.

Then V_α is also a proper Zariski-closed subset of V, and we have shown that $V_{\alpha,\text{dep}} \subseteq V_\alpha$.

So since F is GΓC over K, there is a point $\beta \in \Gamma(F)^n \cap V(F) \smallsetminus V_\alpha(F)$. Since $\beta \notin V_{\alpha,\text{dep}}$, β is $k_{\mathcal{O}}$-linearly independent over $\Gamma(K) \cup \alpha$. Hence F is GSΓC over K as required.

The proof for case (COR) is very similar, but instead of $\zeta \in (c_1 + LJ) \times (c_2 + J)$ we have subgroups $J_1 \subseteq G_1^{r+n}$ and $J_2 \subseteq G_2^{r+n}$ which correspond to each other in the sense that they are connected components of solutions to the same system of $\mathcal{O}$-linear equations. So we get $\zeta \in (c_1 + J_1) \times (c_2 + J_2)$ with $\dim J_1 = \dim J_2 = \text{ldim}_{k_{\mathcal{O}}}(\zeta/\Gamma(K)) < r + n$. Then a similar calculation shows that ζ lies in a component of the intersection $W \cap c + (J_1 \times J_2)$ of atypical dimension, and we apply the weak Zilber-Pink for the semiabelian variety G^{r+n} and proceed as in case (EXP). $\qquad\square$

11.2. Sufficient conditions for quasiminimality.

Theorem 11.6. *Suppose F is a full Γ-field with the countable closure property which is generically Γ-closed over some countable $K \lhd_{\text{cl}} F$. Then F is quasiminimal.*

Proof. We take $F_{\text{base}} = K$ and consider the category $\mathcal{C}_{\Gamma\text{-tr}}(K)$. By Theorem 5.21 it is an amalgamation category so we have a Fraïssé limit $M_{\Gamma\text{-tr}}(K)$. By Theorem 6.9, $M_{\Gamma\text{-tr}}(K)$ is a quasiminimal pregeometry structure, so defines a quasiminimal class $\mathcal{K}(M_{\Gamma\text{-tr}}(K))$. Substituting Proposition 11.2 for Lemma 8.3, the proof of Theorem 8.2 shows that the models in this class are precisely the full Γ-fields which are purely Γ-transcendental extensions of K, are $\aleph_0$-saturated with respect to the Γ-algebraic extensions which are purely Γ-transcendental over K, and satisfy the the countable closure property. Hence by Propositions 11.2 and 11.5, F is in $\mathcal{K}(M_{\Gamma\text{-tr}}(K))$ and hence is quasiminimal. $\qquad\square$

If F is the complex field, in practice it might be difficult or impossible to identify a countable Γ-closed K and prove directly that F is generically Γ-closed over K. Thus the following corollaries may be more useful.

Corollary 11.7. *Suppose F is a full Γ-field with the countable closure property which is Γ-closed. Then F is quasiminimal.*

Proof. Clearly Γ-closedness implies generic Γ-closedness. $\qquad\square$

Since $\mathbb{C}_{\exp}$ has the countable closure property by Proposition 10.7, this completes the proof of Theorem 1.5. We can do slightly better.

Corollary 11.8. *Suppose F is a full Γ-field with the countable closure property which is almost Γ-closed. That is, for all but countably many free and rotund, irreducible subvarieties $V \subseteq G^n$ of dimension dn, $\Gamma^n(F) \cap V(F)$ is Zariski-dense in V. Then F is quasiminimal.*

Proof. Suppose F is almost Γ-closed, and take K_1 to be a countable subfield of F over which all the countably many exceptional varieties V are defined. Take $K = \Gamma\text{cl}^F(K_1)$. Then F is generically Γ-closed over K. $\qquad\square$

Overall we have proved the following generalization of Theorem 1.5, which applies to the exponential function, the Weierstrass $\wp$-functions, the exponential maps of simple abelian varieties, and more.

Theorem 11.9. *Let $\mathbb{C}_\Gamma$ be an analytic Γ-field. If $\mathbb{C}_\Gamma$ is almost Γ-closed then it is quasiminimal.*

Proof. Combine 11.8 with Proposition 10.7 which gives CCP. $\qquad\square$

Since being Γ-closed implies being almost Γ-closed, Theorem 1.9 follows. Theorem 1.5 is a special case.

Remark 11.10. We do not know if almost Γ-closedness is a necessary condition for quasiminimality. For example in the exponential case, is it possible to build an uncountable quasiminimal exponential field F with a definable family $(V_p)_{p \in P}$ of rotund and free varieties such that for only countably many p (perhaps none) there is $(\bar{x}, e^{\bar{x}}) \in V_p(F)$?

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