

CHARACTERISATION AND APPLICATIONS OF \mathbb{k} -SPLIT BIMODULES

VOLODYMYR MAZORCHUK, VANESSA MIEMIETZ AND XIAOTING ZHANG

ABSTRACT. We describe the structure of bimodules (over finite dimensional algebras) which have the property that the functor of tensoring with such a bimodule sends any module to a projective module. The main result is that all such bimodules are \mathbb{k} -split in the sense that they factor (inside the tensor category of bimodules) over \mathbb{k} -vector spaces. As one application, we show that any simple 2-category has a faithful 2-representation inside the 2-category of \mathbb{k} -split bimodules. As another application, we classify simple transitive 2-representations of the 2-category of projective bimodules over the algebra $\mathbb{k}[x, y]/(x^2, y^2, xy)$.

1. INTRODUCTION AND DESCRIPTION OF THE RESULTS

The structure and representation theory of 2-categories is a young and intensively studied area of modern mathematics which originated in [BFK, Kh, CR, Ro, KL]. The series of papers [MM1]–[MM6] initiated the study of the structure and representation theory of so-called *finitary 2-categories*, which are natural 2-analogues of finite dimensional associative algebras.

Classical representation theory of finite dimensional algebras is essentially based on the classification of simple algebras provided by the classical Artin-Wedderburn Theorem. For a special class of finitary 2-categories, called *strongly regular fiat 2-categories*, an analogue of the Artin-Wedderburn Theorem was obtained in [MM3]. Fiat 2-categories can be viewed as a vague 2-analogue of cellular algebras: they have a weak involution and adjunction morphisms which lead to the fact that, in each 2-representation, the involution gets interpreted as taking the adjoint functor. This involution plays a crucial role in all arguments of [MM3] and therefore it is unlikely that one could generalise these arguments to any wider class of 2-categories.

In the present paper we take a first step towards understanding the structure of arbitrary simple finitary 2-categories. The main idea motivating our study comes, fairly unexpectedly, from the results of [KiM2, KMMZ]. Following the ideas in the proof of [KMMZ, Theorem 2], which are based on the main result of [KiM2], we observe that every simple finitary 2-category can be faithfully represented using functorial actions in which all involved functors are right exact and have the property that they send any module to a projective module.

The main technical result of this article is a characterisation and description of such functors. In fact, in Theorem 1 we show that bimodules representing such functors belong to the additive closure of bimodules of the form $P \otimes_{\mathbb{k}} N$, where P is a projective left module and N is an arbitrary right module. Put differently, in the tensor category of bimodules, all such bimodules factor through \mathbb{k} -vector spaces. A precise formulation of Theorem 1 and its proof can be found in Section 2.

We also give two applications of Theorem 1. The first one, which can be found in Section 3, concerns the faithful representation of simple finitary 2-categories as explained

above. The second one, presented in Section 5, is of different nature. It concerns the problem of classifying simple transitive 2-representations for the 2-category of projective bimodules over the finite dimensional algebra $A := \mathbb{k}[x, y]/(x^2, y^2, xy)$. This kind of problem was studied, in different contexts, for many other 2-categories, see [MM5, MM6, Zi, MaMa, KMMZ, MT, MMT, MZ, Zh2]. The importance of this problem is supported by interesting applications, see e.g. [KiM1]. The case of the algebra A treated in this paper differs significantly from all previously studied cases. To start with, the 2-category of projective functors for A is not fiat, as A is not self-injective, cf. [MM1, Subsection 7.3]. However, simple transitive 2-representations for some non-fiat 2-categories have also been classified in [Zh2, MZ]. The main point of the algebra A is that this is the smallest algebra which does not have any non-zero projective-injective modules. Therefore the general approach outlined in [MZ], which is based on existence of a projective-injective module, is not applicable either. In Section 5 we propose a new approach to this problem which crucially uses our Theorem 1. In Section 4 we show that the decategorification of a fiat 2-category with strongly regular two-sided cells is a quasi-hereditary algebra with a simple preserving duality.

As the material discussed in Sections 2, 3 and 5 is of rather different nature, we do not provide a general list of notation and preliminaries but rather postpone all this to each individual section separately.

For some further examples and structural results on finitary 2-categories we refer the reader to [GM1, GM2, Xa, Zh1].

Acknowledgements: This research was partially supported by the Swedish Research Council, Knut and Alice Wallenberg Stiftelse and Göran Gustafsson Stiftelse. We thank Claus Michael Ringel for stimulating discussions.

2. \mathbb{k} -SPLIT BIMODULES FOR FINITE DIMENSIONAL ALGEBRAS

2.1. Main Theorem. Throughout the paper, we fix an algebraically closed field \mathbb{k} . For finite dimensional \mathbb{k} -algebras A and B , we denote by

- $A\text{-mod}$ the category of finite dimensional left A -modules;
- $\text{mod-}A$ the category of finite dimensional right A -modules;
- $A\text{-mod-}B$ the category of finite dimensional A - B -bimodules.

The main result of the paper is the following statement.

Theorem 1. *Let A and B be two basic finite dimensional \mathbb{k} -algebras and $Q \in A\text{-mod-}B$. Then the following conditions are equivalent.*

- (a) *The functor $Q \otimes_B - : B\text{-mod} \rightarrow A\text{-mod}$ maps any B -module to a projective A -module.*
- (b) *The functor $\text{Hom}_A(Q, -) : A\text{-mod} \rightarrow B\text{-mod}$ maps any short exact sequence in $A\text{-mod}$ to a split short exact sequence in $B\text{-mod}$.*
- (c) *The A - B -bimodule Q belongs to the additive closure, in $A\text{-mod-}B$, of all A - B -bimodules of the form $A \otimes_{\mathbb{k}} K$, where $K \in \text{mod-}B$.*

Bimodules satisfying the conditions in Theorem 1(c) will be called \mathbb{k} -split.

Note that, by considering the algebra $A \times B$, we can reduce Theorem 1 to the case $A = B$. So, in the proof which follows, we assume $A = B$.

2.2. Implication (a) \Rightarrow (b). Assume that condition (a) is satisfied. Then, in particular, $Q \otimes_A A \cong {}_A Q$ is projective and hence the functor $\text{Hom}_{A\cdot}(Q, -)$ is exact. Furthermore, for any $M \in A\text{-mod}$, the A -module $Q \otimes_A M$ is projective and therefore the functor

$$\text{Hom}_{A\cdot}(Q \otimes_A M, -) \cong \text{Hom}_{A\cdot}(M, \text{Hom}_{A\cdot}(Q, -)) : A\text{-mod} \rightarrow \mathbb{k}\text{-mod}$$

is exact.

For a short exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

in $A\text{-mod}$, application of the exact functor $\text{Hom}_{A\cdot}(Q, -)$ produces the short exact sequence

$$(1) \quad 0 \rightarrow \text{Hom}_{A\cdot}(Q, X) \rightarrow \text{Hom}_{A\cdot}(Q, Y) \xrightarrow{\alpha} \text{Hom}_{A\cdot}(Q, Z) \rightarrow 0.$$

If (1) splits, then its image under $\text{Hom}_{A\cdot}(M, -)$ is, clearly, split short exact, for any $M \in A\text{-mod}$. At the same time, if (1) does not split, then, for $M = \text{Hom}_{A\cdot}(Q, Z)$, the identity morphism on M is not in the image of the map

$$\text{Hom}_{A\cdot}(M, \text{Hom}_{A\cdot}(Q, Y)) \xrightarrow{\alpha \circ -} \text{Hom}_{A\cdot}(M, M).$$

Therefore the latter map is not surjective and, consequently, the sequence

$$0 \rightarrow \text{Hom}_{A\cdot}(M, \text{Hom}_{A\cdot}(Q, X)) \rightarrow \text{Hom}_{A\cdot}(M, \text{Hom}_{A\cdot}(Q, Y)) \rightarrow \text{Hom}_{A\cdot}(M, M) \rightarrow 0$$

is not exact. Thus the functor $\text{Hom}_{A\cdot}(Q \otimes M, -)$ is not exact either, a contradiction. Hence condition (b) is satisfied.

2.3. Implication (b) \Rightarrow (c). Assume that condition (b) is satisfied. In particular, the functor $\text{Hom}_{A\cdot}(Q, -)$ is exact and thus the left A -module ${}_A Q$ is projective. Denote by R the Jacobson radical $\text{Rad}(A)$ of A .

Applying the duality $* := \text{Hom}_{\mathbb{k}}(-, \mathbb{k})$ to the short exact sequence

$$0 \rightarrow R \rightarrow A \rightarrow \text{top}(A) \rightarrow 0$$

in $A\text{-mod-}A$, gives the short exact sequence

$$0 \rightarrow (\text{top}(A))^* \rightarrow A^* \rightarrow R^* \rightarrow 0$$

in $A\text{-mod-}A$. Applying $\text{Hom}_{A\cdot}(Q, -)$ to the latter short exact sequence results in the short exact sequence

$$(2) \quad 0 \rightarrow \text{Hom}_{A\cdot}(Q, (\text{top}(A))^*) \rightarrow \text{Hom}_{A\cdot}(Q, A^*) \rightarrow \text{Hom}_{A\cdot}(Q, R^*) \rightarrow 0.$$

By condition (b), this sequence is split in $A\text{-mod}$.

By adjunction, we have

$$(3) \quad \text{Hom}_{A\cdot}(Q, A^*) \cong \text{Hom}_{\mathbb{k}}(A \otimes_A Q, \mathbb{k}) \cong Q^*$$

and

$$(4) \quad \text{Hom}_{A\cdot}(Q, R^*) \cong \text{Hom}_{\mathbb{k}}(R \otimes_A Q, \mathbb{k}).$$

Moreover, as any A -homomorphism from Q to $(\text{top}(A))^*$ vanishes on RQ , we have

$$\text{Hom}_{A\cdot}(Q, (\text{top}(A))^*) \cong \text{Hom}_{A\cdot}(Q/RQ, (\text{top}(A))^*)$$

and then, by adjunction,

$$\text{Hom}_{A\cdot}(Q/RQ, (\text{top}(A))^*) \cong \text{Hom}_{\mathbb{k}}(\text{top}(A) \otimes_A (Q/RQ), \mathbb{k}).$$

Finally, we observe that Q/RQ is semi-simple as left A -module which yields an isomorphism $\text{top}(A) \otimes_A (Q/RQ) \cong Q/RQ$.

Plugging the latter into (2), using (3) and (4), and applying $*$, gives us the short exact sequence

$$0 \rightarrow R \otimes_A Q \rightarrow Q \rightarrow Q/RQ \rightarrow 0$$

which is split in $\text{mod-}A$. We denote by $\beta : Q/RQ \rightarrow Q$ the splitting morphism in $\text{mod-}A$.

Fix a decomposition of the A - A -bimodule Q/RQ into a direct sum $X_1 \oplus X_2 \oplus \cdots \oplus X_k$ of indecomposable A - A -bimodules. As Q/RQ is semi-simple, as a left A -module, we have that each X_i is isotypic as a left A -module and indecomposable as a right A -module. Let L_i denote the (unique) simple subquotient of the left A -module ${}_A X_i$ and P_i denote an indecomposable projective cover of L_i in $A\text{-mod}$. Let also e_i denote a primitive idempotent corresponding to P_i .

Consider the A - A -bimodule

$$\hat{Q} := \bigoplus_{i=1}^k P_i \otimes_{\mathbb{k}} X_i$$

and note that \hat{Q} belongs to the additive closure of bimodules described in condition (c). By adjunction, for any A - A -bimodule V , we have

$$(5) \quad \text{Hom}_{A-A}(\hat{Q}, V) \cong \bigoplus_{i=1}^k \text{Hom}_A(X_i, e_i V).$$

The homomorphism β induces, by equation (5), a homomorphism $\gamma : \hat{Q} \rightarrow Q$ of A - A -bimodules. By construction, the image of this homomorphism covers Q/RQ , that is the top of Q , considered as a left A -module. Therefore γ is surjective by the Nakayama Lemma. At the same time, we already know that Q is projective as a left A -module and that all X_i are isotypic as left A -modules. Compared to the construction of \hat{Q} , this implies the equality $\dim(\hat{Q}) = \dim(Q)$ and yields that γ is, in fact, an isomorphism. Condition (c) follows.

2.4. Implication (c) \Rightarrow (a). Assume that condition (c) is satisfied. Note that, for any $M \in A\text{-mod}$, the A -module $A \otimes_{\mathbb{k}} K \otimes_A M$ is just a direct sum of $\dim(K \otimes_A M)$ copies of A , in particular, it is projective. By additivity, $Q \otimes_A M$ is also projective, for any Q from the additive closure of the bimodules described in condition (c). Therefore condition (a) is satisfied.

3. APPLICATION: SIMPLE FINITARY 2-CATEGORIES

3.1. Finitary 2-categories. For generalities on 2-categories we refer to [McL, Le]. Following [MM1, Subsection 2.2], a *finitary 2-category over \mathbb{k}* is a 2-category \mathcal{C} such that

- \mathcal{C} has finitely many objects;
- each $\mathcal{C}(i, j)$ is equivalent to the category of projective modules over some finite dimensional associative \mathbb{k} -algebra (i.e. is a *finitary \mathbb{k} -linear category*);
- all compositions are biadditive and \mathbb{k} -bilinear, when applicable;
- identity 1-morphisms are indecomposable.

In particular, we have a finite set $\mathcal{S}[\mathcal{C}]$ of isomorphism classes of indecomposable 1-morphisms. Furthermore, by [MM2, Section 3], $\mathcal{S}[\mathcal{C}]$ has the natural structure of a multisemigroup (cf. [KuM]). We denote by \leq_L and \sim_L the corresponding *left order* and *left equivalence relation*, by \leq_R and \sim_R the corresponding *right order* and *right*

equivalence relation and by \leq_J and \sim_J the corresponding *two-sided order* and *two-sided equivalence relation*, see [MM2, Section 3] and [KuM, Subsection 4.1]. Equivalence classes for \sim_L , \sim_R and \sim_J are called *cells* (left, right or two-sided, respectively).

A 2-category which satisfies all the above conditions except for the last one will be called *weakly finitary*. Clearly, splitting idempotents in the endomorphism algebras of the identity 1-morphisms, every weakly finitary 2-category can be Morita reduced to a finitary 2-category (cf. [MM4] for general Morita theory of finitary 2-categories).

A finitary 2-category \mathcal{C} will be called *simple* provided that

- any non-zero 2-ideal of \mathcal{C} contains the identity 2-morphism for some non-zero 1-morphism;
- there is a unique two-sided cell, which we call \mathcal{J} , containing a 1-morphism that is not isomorphic to any of the identity 1-morphisms;
- the cell \mathcal{J} is the maximal two-sided cell with respect to \leq_J ;
- the cell \mathcal{J} is idempotent in the sense that $F \circ G \neq 0$ for some (possibly equal) 1-morphisms F and G in \mathcal{J} .

In particular, a simple 2-category is \mathcal{J} -simple in the sense of [MM2, Subsection 6.2].

We note that the above definition excludes the very easy situation when the only indecomposable 1-morphisms in \mathcal{C} are those isomorphic to the identity 1-morphisms. Such 2-categories are easy to construct and study, so we will ignore them.

Similarly to [MM2, Subsection 6.2], one shows that a finitary 2-category which satisfies the last three of the above conditions has a unique simple quotient.

3.2. Simple fiat strongly regular 2-categories. As in [MM1, Subsection 6.2], a finitary 2-category \mathcal{C} is called *fiat* provided that it has a weak anti-involution \star (reversing both 1- and 2-morphisms) and adjunction morphisms between any pair (F, F^\star) of 1-morphisms. A fiat 2-category \mathcal{C} is called *strongly regular* provided that no two left (right) cells inside the same two-sided cell are comparable, and that the intersection of each left and each right cell inside the same two-sided cell consists of exactly one element, see [MM1, Subsection 4.8] and [KiM2, Corollary 19].

2-categories which are, at the same time, simple, strongly regular and fiat, were classified in [MM3, Theorem 13]. Roughly speaking (up to the structure of the endomorphism algebra of identity 1-morphisms), they are biequivalent to the bicategory of projective bimodules for a finite dimensional, weakly symmetric \mathbb{k} -algebra A , or, equivalently to the 2-category \mathcal{C}_A defined as follows: Let $A = A_1 \times A_2 \times \cdots \times A_k$ be a decomposition of A into a direct sum of connected components. Then

- objects of \mathcal{C}_A are $1, 2, \dots, k$, where i should be thought of as a small category \mathcal{A}_i equivalent to A_i -mod;
- 1-morphisms in $\mathcal{C}_A(i, j)$ are functors from \mathcal{A}_i to \mathcal{A}_j corresponding to tensoring with bimodules from the additive closure of $A_j \otimes_{\mathbb{k}} A_i$, with the additional bimodule A_i , if $i = j$;
- 2-morphisms are all natural transformations of such functors.

It is natural to ask for a description of simple finitary 2-categories in general. Unfortunately, an easy generalisation of [MM3, Theorem 13] is too much to hope for.

3.3. Cell 2-representations. For any finitary 2-category \mathcal{C} , we can consider the 2-category $\mathcal{C}\text{-afmod}$ of *finitary 2-representations of \mathcal{C}* , where

- objects in $\mathcal{C}\text{-afmod}$ are 2-functors which represent every object in \mathcal{C} by a finitary \mathbb{k} -linear category, each 1-morphism by an additive functor and each 2-morphism by a natural transformation of functors;
- 1-morphisms in $\mathcal{C}\text{-afmod}$ are 2-natural transformations, see [MM3, Subsection 2.3];
- 2-morphisms in $\mathcal{C}\text{-afmod}$ are modifications.

An easy example of a 2-representation is the *principal* (Yoneda) 2-representation $\mathbf{P}_i := \mathcal{C}(i, -)$, defined for each $i \in \mathcal{C}$.

If \mathcal{L} is a left cell, then there is $i \in \mathcal{C}$ such that all 1-morphisms in \mathcal{L} start at i . In this case we can define the corresponding *cell 2-representation* $\mathbf{C}_{\mathcal{L}}$ of \mathcal{C} as a certain subquotient of \mathbf{P}_i , see [MM2, Subsection 6.5], and also [MM1, Section 4] for the original definition.

3.4. Main idea. Let now \mathcal{C} be a simple finitary 2-category. Fix a left cell \mathcal{L} inside the distinguished two-sided cell \mathcal{J} and consider the corresponding cell 2-representation $\mathbf{C}_{\mathcal{L}}$. By construction, the fact that \mathcal{J} is idempotent implies that $\mathbf{C}_{\mathcal{L}}$ has trivial annihilator. Therefore $\mathbf{C}_{\mathcal{L}}$ defines a faithful 2-functor from \mathcal{C} into some 2-category of right exact functors between modules categories of finite dimensional algebras. The main point of [MM3, Theorem 13] was to show that, on the level of 1-morphisms in \mathcal{J} , this embedding is, in fact, 2-full and each 1-morphism in \mathcal{J} is represented by a *projective functor*, that is by a functor isomorphic to tensoring with a projective bimodule.

If \mathcal{C} is fiat (but not necessarily strongly regular), then [KMMZ, Theorem 2] still asserts that, in the setup of the previous paragraph, each 1-morphism in \mathcal{J} is represented by a *projective functor*. However, outside strongly regular situation the 2-fullness statement is no longer true. An easy example is given by the small quotient of the 2-category of Soergel bimodules in Weyl type B_2 , see [KMMZ, Subsection 3.2]. This is a simple, fiat, but not strongly regular 2-category. Under $\mathbf{C}_{\mathcal{L}}$, the indecomposable 1-morphism corresponding to a simple reflection belonging to \mathcal{L} acts non-trivially on two simple modules of the abelianised cell 2-representation. Therefore this indecomposable 1-morphism is represented by a decomposable projective functor, which means that the representation 2-functor cannot be 2-full.

If \mathcal{C} is not fiat, then 1-morphisms in \mathcal{J} no longer act as projective functors in general. Later on we will use our Theorem 1 to describe what kind of functors appear in this case.

3.5. Simple transitive 2-representations. Cell 2-representations of finitary 2-categories have the following two properties:

- They are *transitive* in the sense that, for any pair X, Y of indecomposable objects in the underlying categories of the representation, there is a 1-morphism F such that X appears (up to isomorphism) as a direct summand in $F Y$.
- They are *simple* in the sense that the underlying categories do not have any proper ideals invariant under the action of \mathcal{C} .

In general, simple transitive 2-representations are natural 2-analogues of simple modules over \mathbb{k} -algebras, see [MM5, MM6].

If \mathbf{M} is a simple transitive 2-representation of \mathcal{C} , then, according to [CM, Subsection 3.2], there is a unique two-sided cell $\mathcal{J}_{\mathbf{M}}$ which is a maximal element (with respect to the two-sided order) in the set of all two-sided cells whose 1-morphisms are not annihilated by \mathbf{M} . The cell $\mathcal{J}_{\mathbf{M}}$ is called the *apex of \mathbf{M}* and is idempotent in the sense that it contains three (possibly equal) 1-morphisms F , G and H such that H appears, as a summand and up to isomorphism, in $F \circ G$.

3.6. Abelianisation. Instead of *additive* 2-representations of \mathcal{C} one can also study *abelian* 2-representations where

- each object in \mathcal{C} is represented by a category equivalent to the module category for some finite dimensional associative \mathbb{k} -algebra;
- each 1-morphism is represented by a right exact functor;
- each 2-morphism is represented by a natural transformation of functors.

The 2-category of abelian 2-representations is denoted $\mathcal{C}\text{-mod}$, see e.g. [MM2, Section 4]. There is a natural 2-functor $\overline{} : \mathcal{C}\text{-afmod} \rightarrow \mathcal{C}\text{-mod}, \mathbf{M} \mapsto \overline{\mathbf{M}}$, called *abelianisation 2-functor*, see [MM2, Subsection 4.2].

3.7. Main results of this section.

Proposition 2. *Let \mathcal{C} be a finitary 2-category and \mathbf{M} a simple transitive 2-representation of \mathcal{C} . Then, for any 1-morphism F in $\mathcal{J}_{\mathbf{M}}$, the functor $\overline{\mathbf{M}}(F)$ sends any object to a projective object.*

Proof. The claim follows directly from the proof of the first part of [KMMZ, Theorem 2] as this proof does not use fiatness of \mathcal{C} . \square

Let A be a finite dimensional associative \mathbb{k} -algebra with a fixed decomposition $A = A_1 \times A_2 \times \cdots \times A_k$ into a product of (not necessarily connected) components. For each $i = 1, 2, \dots, k$ fix a small category \mathcal{C}_i equivalent to $A_i\text{-mod}$ and a right A_i -module N_i . Let $\mathcal{C} = \{\mathcal{C}_i\}$ and $N = \{N_i\}$. Define the weakly finitary 2-category $\mathcal{C}_{A,\mathcal{C},N}$ as follows:

- The objects of $\mathcal{C}_{A,\mathcal{C},N}$ are $1, 2, \dots, k$, where \mathbf{i} should be thought of as \mathcal{C}_i ;
- 1-morphisms in $\mathcal{C}_{A,\mathcal{C},N}(\mathbf{i}, \mathbf{j})$ are all functors from \mathcal{C}_i to \mathcal{C}_j which are isomorphic to tensoring with $A_j\text{-}A_i$ -bimodules in the additive closure of $A_j \otimes_{\mathbb{k}} N_i$ and, additionally, the $A_i\text{-}A_i$ -bimodule A_i , if $i = j$;
- 2-morphisms are natural transformations of functors.

The main result of this section is the following statement which, roughly speaking, says that all simple finitary 2-categories are 2-subcategories of the categories of the form $\mathcal{C}_{A,\mathcal{C},N}$.

Theorem 3. *Let \mathcal{C} be a simple finitary 2-category. Then there are A , \mathcal{C} and N as above and a faithful 2-functor from \mathcal{C} to $\mathcal{C}_{A,\mathcal{C},N}$.*

Proof. Let \mathcal{C} be a simple finitary 2-category. For a left cell \mathcal{L} in \mathcal{J} , consider the left cell 2-representation $\overline{\mathbf{C}}_{\mathcal{L}}$ of \mathcal{C} . By Proposition 2, for any 1-morphism F in \mathcal{J} from \mathbf{i} to \mathbf{j} , the functor $\overline{\mathbf{C}}_{\mathcal{L}}(F)$ maps any object of $\overline{\mathbf{C}}_{\mathcal{L}}(\mathbf{i})$ to a projective object of $\overline{\mathbf{C}}_{\mathcal{L}}(\mathbf{j})$.

For each \mathbf{i} , let $A_{\mathbf{i}}$ denote the underlying algebra of $\overline{\mathbf{C}}_{\mathcal{L}}(\mathbf{i})$. We note that $A_{\mathbf{i}}$ does not have to be connected. By Theorem 1, there is a right $A_{\mathbf{i}}$ -module $N_{\mathbf{i}}$ such that any 1-morphism in \mathcal{J} from \mathbf{i} to any \mathbf{j} is represented, via $\overline{\mathbf{C}}_{\mathcal{L}}$, by a functor isomorphic to tensoring with a bimodule of the form $A_{\mathbf{j}} \otimes_{\mathbb{k}} N_{\mathbf{i}}$ (and, additionally, $A_{\mathbf{i}}$, if $i = j$).

Let $A := \prod_i A_i$. Since \mathcal{J} is idempotent, it coincides with the apex of $\mathbf{C}_{\mathcal{L}}$. Hence, thanks to simplicity of \mathcal{C} , the representation 2-functor $\mathbf{C}_{\mathcal{L}}$ is faithful on the level of 2-morphisms and thus provides an embedding of \mathcal{C} into the weakly finitary category $\mathcal{C}_{A, \{\overline{\mathbf{C}_{\mathcal{L}}(i)}\}, \{N_i\}}$. This completes the proof. \square

We note that, usually, the embedding of \mathcal{C} given by Theorem 3 will not be 2-full. Furthermore, A , \mathcal{C} and N in the formulation of Theorem 3 are not uniquely defined, even up to isomorphism/equivalence.

4. DECATEGORYIFICATION

Let $\mathcal{C} := \mathcal{C}_{A, \mathcal{C}, N}$ be as in Section 3.7. Let P_1, \dots, P_r be a complete list of projective indecomposable modules for A , and N_1, \dots, N_s a complete list of elements in N . Then a complete list of indecomposable 1-morphisms in $\mathcal{C}_{A, \mathcal{C}, N}$ is given by $\mathbb{1}_i$, $i \in \{1, \dots, \mathbf{k}\}$, and $\{F_{i,j}\}$ where $F_{i,j}$ is a functor isomorphic to tensoring with $P_i \otimes_{\mathbb{k}} N_j$. The structure of a multisemigroup with finite multiplicities (in the sense of [Fo]) on $\mathcal{S}[\mathcal{C}] \cap \mathcal{J}$ is given by $[F_{i,j}][F_{l,t}] = \dim(N_j e_l)[F_{i,t}]$.

Proposition 4. *Suppose $\text{add}(N_1 \oplus N_2 \oplus \dots \oplus N_s) \cong \text{add}(A)$.*

- (i) *The algebra $\mathbb{k} \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{S}[\mathcal{C}]$ has a filtration by two-sided ideals such that the lowest ideal is given by the span of $\{[F_{i,j}] | i, j = 1, \dots, r\}$, and the remaining subquotients are spanned by $[\mathbb{1}_i]$, taken in any order.*
- (ii) *The ideal J spanned by $\{[F_{i,j}] | i, j = 1, \dots, r\}$ in $\mathbb{k} \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{S}[\mathcal{C}]$ is, in the terminology of [KX2, Definition 3.3], isomorphic to the swich algebra of the algebra of $r \times r$ -matrices with respect to the matrix $(\dim(e_j A e_l))_{j,l=1}^r$.*
- (iii) *If A is weakly symmetric, the filtration in (i) is a cell filtration, where the involution is given by the action of $*$ on $\mathcal{S}[\mathcal{C}]$, which corresponds to interchanging the two subscripts on $F_{i,j}$.*

Proof. (i) and (ii) follow directly from the definitions.

To prove (iii), assume A is weakly symmetric, so \mathcal{C} is fiat. The involution $*$ induces an involution on $\mathbb{k} \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{S}[\mathcal{C}]$, fixing the $[\mathbb{1}_i]$, and sending $[F_{i,j}]$ to $[F_{j,i}]$ and hence, under the isomorphism to the swich algebra from (ii), acts as matrix transposition. Since $\dim e_i A e_j = \dim e_j A e_i$, it is easy to check that, for V the \mathbb{k} -span of $[F_{i,1}]$, where $i = 1, 2, \dots, r$, the morphism $\alpha : J \rightarrow V \otimes V$, $F_{j,i} \mapsto v_i \otimes v_j$, defines the structure of a cell ideal on J (cf. [KX1, Definition 3.2]).

Quotienting out by J , all remaining subquotients in the ideal filtration are one dimensional and idempotent, and hence cell ideals. \square

Corollary 5. *Let \mathcal{C} be a fiat 2-category such that all two-sided cells are strongly regular. Then $\mathbb{k} \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{S}[\mathcal{C}]$ is a quasi-hereditary algebra with simple-preserving duality.*

Proof. By induction with respect to the two-sided order, it follows immediately from Proposition 4(iii) that $\mathbb{k} \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{S}[\mathcal{C}]$ is a cellular algebra. Since each cell contains an idempotent, it is indeed quasi-hereditary. \square

5. APPLICATION: SIMPLE TRANSITIVE 2-REPRESENTATIONS OF PROJECTIVE BIMODULES FOR $\mathbb{k}[x, y]/(x^2, y^2, xy)$

5.1. Classification. In this section we consider the problem of classification of simple transitive 2-representations of the simple 2-category \mathcal{C}_A of projective bimodules for the 3-dimensional algebra $A = \mathbb{k}[x, y]/(x^2, y^2, xy)$. As A is connected, the 2-category \mathcal{C}_A has only one object. We call this object \mathfrak{i} . The main result of this section is the following.

Theorem 6. *Every simple transitive 2-representations of \mathcal{C}_A , for the algebra $A = \mathbb{k}[x, y]/(x^2, y^2, xy)$, is equivalent to a cell 2-representation.*

Under the additional assumption that all 1-morphisms in \mathcal{C}_A are represented by exact functors, the claim of Theorem 6 is proved in [MM5, Proposition 19]. Therefore, to prove Theorem 6, we just have to show that, in every simple transitive 2-representations of \mathcal{C}_A , all 1-morphisms in \mathcal{C}_A are indeed represented by exact functors. The rest of this section is devoted to the proof of Theorem 6.

5.2. Combinatorics of the action. We fix a simple transitive 2-representation \mathbf{M} of \mathcal{C}_A . Let X_1, X_2, \dots, X_k be a list of representatives of isomorphism classes of indecomposable objects in $\mathbf{M}(\mathfrak{i})$. Let F be an indecomposable 1-morphism in \mathcal{C}_A which is isomorphic to tensoring with $A \otimes_{\mathbb{k}} A$. Note that

$$(6) \quad F \circ F \cong F^{\oplus 3}.$$

For $i, j = 1, 2, \dots, k$, let $m_{i,j}$ denote the multiplicity of X_i in $F X_j$. Then $M = (m_{i,j})_{i,j=1}^k \in \text{Mat}_{k \times k}(\mathbb{Z}_{\geq 0})$, moreover, all $m_{i,j}$ are positive due to transitivity of \mathbf{M} , see [MM5, Subsection 7.1] for details.

From (6), it follows that, $M^2 = 3M$. Therefore, up to permutation of the X_i 's, M is one of the following matrices (again, see [MM5, Subsection 7.1] and [MZ, Proposition 10] for details):

$$M_1 = (3), \quad M_2 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

5.3. General part of the proof of Theorem 6. Let B be an associative algebra such that $\overline{\mathbf{M}}(\mathfrak{i}) \cong B\text{-mod}$, then we have $\mathbf{M}(\mathfrak{i}) \cong B\text{-proj}$. Let e_1, e_2, \dots, e_k be primitive idempotents of B corresponding to the objects X_1, X_2, \dots, X_k above. Note that $k \leq 3$ by Subsection 5.2. For $i = 1, 2, \dots, k$, we denote by L_i the simple top of X_i in $\overline{\mathbf{M}}(\mathfrak{i})$. We also denote by L'_i the corresponding *right* simple B -module, for $i = 1, 2, \dots, k$.

By Proposition 2, the functor $\overline{\mathbf{M}}(F)$ sends any object in $\overline{\mathbf{M}}(\mathfrak{i})$ to a projective object in $\overline{\mathbf{M}}(\mathfrak{i})$. Hence, by Theorem 1, there are right B -modules N_1, N_2, \dots, N_k such that $\overline{\mathbf{M}}(F)$ is isomorphic to tensoring with the B - B -bimodule

$$Be_1 \otimes_{\mathbb{k}} N_1 \oplus Be_2 \otimes_{\mathbb{k}} N_2 \oplus \dots \oplus Be_k \otimes_{\mathbb{k}} N_k.$$

Note that, for $i, j = 1, 2, \dots, k$, we have

$$Be_i \otimes_{\mathbb{k}} N_i \otimes_B Be_j \cong Be_i^{\oplus \dim_{\mathbb{k}}(N_i \otimes_B Be_j)}$$

and hence

$$m_{i,j} = \dim_{\mathbb{k}}(N_i \otimes_B Be_j) = \dim_{\mathbb{k}}(N_i e_j) = [N_i : L'_j].$$

Next we observe that the right B -module $N = N_1 \oplus N_2 \oplus \dots \oplus N_k$ is faithful. Indeed, if this were not the case, the annihilator of this module would be, due to the form of the functor $\overline{\mathbf{M}}(F)$, annihilated by $\overline{\mathbf{M}}(F)$ and hence would generate a non-zero proper

\mathcal{C}_A -stable ideal of \mathbf{M} . This is, however, not possible thanks to simplicity of \mathbf{M} . Consequently, as the sum of entries in each row of the above matrices is at most 4, the Loewy length of N is at most 4, and we have that $\text{Rad}(B)^4 = 0$.

As mentioned above, we just need to show that $\overline{\mathbf{M}}(\mathbb{F})$ is exact or, equivalently, that N is projective. If B is semi-simple, then N is automatically projective, so in what follows we may assume that $\text{Rad}(B) \neq 0$.

Finally, we will need the following.

Lemma 7. *The 2-functor $\overline{\mathbf{M}}$ induces an embedding of the algebra A into the center $Z(B)$ of the algebra B .*

Proof. The 2-functor $\overline{\mathbf{M}}$ gives a map from $A \cong \text{End}_{\mathcal{C}_A}(\mathbb{1})$ to the endomorphism algebra of the B - B -bimodule B . The latter is isomorphic to $Z(B)$ and injectivity of this map follows from simplicity of \mathcal{C}_A . \square

Now we have to go into a case-by-case analysis.

5.4. Case 1: $M = M_1$. In this case we have $k = 1$ and $N = N_1$ has dimension 3. If $\text{Rad}(B)^2 \neq 0$, then N must be uniserial and hence $B \cong \mathbb{k}[x]/(x^3)$. This means that N is projective and we are done.

If $\text{Rad}(B)^2 = 0$, then we have two possibilities. The first one is $B \cong \mathbb{k}[x]/(x^2)$ and $N = B \oplus \mathbb{k}$, which immediately contradicts Lemma 7. The second possibility which is left is $B \cong A$. In this case, as N is faithful, it must be either the projective indecomposable or the injective indecomposable B -module. In the projective case we are done, so let us assume that N is injective. To get a contradiction in this case we will need to do more subtle computations.

We have $A = B = \mathbb{k}[x, y]/(x^2, y^2, xy)$. Then the A - A -bimodule $A \otimes_{\mathbb{k}} A$ has the following basis:

$$1 \otimes 1, 1 \otimes x, 1 \otimes y, x \otimes 1, y \otimes 1, x \otimes x, x \otimes y, y \otimes x, y \otimes y.$$

The A - A -bimodule $A \otimes_{\mathbb{k}} N \cong A \otimes_{\mathbb{k}} A^*$ has the following basis:

$$1 \otimes x^*, 1 \otimes y^*, 1 \otimes 1^*, x \otimes x^*, x \otimes y^*, y \otimes x^*, y \otimes y^*, x \otimes 1^*, y \otimes 1^*.$$

Now it is easy to check that $\dim_{\mathbb{k}} \text{Hom}_{A-A}(A, A \otimes_{\mathbb{k}} A) = 4$, where the generator 1 of A can be mapped to any of the elements

$$x \otimes x, x \otimes y, y \otimes x, y \otimes y.$$

At the same time, $\dim_{\mathbb{k}} \text{Hom}_{A-A}(A, A \otimes_{\mathbb{k}} A^*) = 3$, where the generator 1 of A can be mapped to any of the elements

$$1 \otimes 1^* + x \otimes x^* + y \otimes y^*, x \otimes 1^*, y \otimes 1^*.$$

As \mathcal{C}_A is simple, $\text{Hom}_{A-A}(A, A \otimes_{\mathbb{k}} A)$ should be embeddable into the homomorphism spaces between the functors $\overline{\mathbf{M}}(\mathbb{1}_1)$ and $\overline{\mathbf{M}}(\mathbb{F})$, but the above calculation shows that this is not possible. This completes the proof in Case 1.

5.5. Case 2: $M = M_2$. In this case $k = 2$, $[N_1 : L'_i] = 2$ and $[N_2 : L'_i] = 1$, for $i = 1, 2$. The endomorphism algebra of the multiplicity-free module N_2 is a direct sum of copies of \mathbb{k} . As N is faithful, it follows that A embeds into $\text{End}_B(N_1)$. Both simples appear in N_1 with multiplicity 2. This implies that the image of $Z(B)$ in the endomorphism algebra of each of the two indecomposable projective B -modules has dimension at most 2. Therefore both these images must have dimension 2 and the corresponding kernels must be different.

As A embeds into the endomorphism algebra of N_1 , we have $\dim_{\mathbb{k}}(\text{End}_B(N_1)) \geq 3$. In particular, N_1 cannot have simple top or socle for in that case the dimension of its endomorphism algebra would be at most 2, that is the multiplicity of the top (or socle) simple module in N_1 . As N_1 cannot be semi-simple (since A cannot be embedded into a direct sum of copies of \mathbb{k}), it follows that N_1 has Loewy length two with both top and socle isomorphic to $L'_1 \oplus L'_2$. Consequently, $\text{Rad}(B)^2 = 0$.

It follows that the quiver of B has one loop at each vertex. Furthermore, we could also have at most two arrows in each direction between the two different vertices (and, in total, at most three such arrows). Thus, if B is not connected, then $B = Z(B) \cong \mathbb{k}[x]/(x^2) \oplus \mathbb{k}[y]/(y^2)$. If B is connected, then $Z(B) \cong A$.

Consider first the case when B is not connected. In this case the above discussion implies $N_1 \cong B$ while $N_2 \cong \mathbb{k}_x \oplus \mathbb{k}_y$, the direct sum of two non-isomorphic simple B -modules corresponding to the x - and y -components of B , respectively. Setting $D_x := \mathbb{k}[x]/(x^2)$ and $D_y := \mathbb{k}[y]/(y^2)$, we obtain that $\overline{\mathbf{M}}(\mathbb{F})$ is represented by the bimodule

$$(7) \quad (D_x \otimes_{\mathbb{k}} D_x) \oplus (D_x \otimes_{\mathbb{k}} D_y) \oplus (D_y \otimes_{\mathbb{k}} \mathbb{k}_x) \oplus (D_y \otimes_{\mathbb{k}} \mathbb{k}_y).$$

The dimension of the homomorphism space from the B - B -bimodule B to the bimodule in (7) is 3, where two linearly independent homomorphisms come from homomorphisms from D_x to $D_x \otimes_{\mathbb{k}} D_x$ and one more linearly independent homomorphism comes from the map from D_y to $D_y \otimes_{\mathbb{k}} \mathbb{k}_y$. As mentioned above, $\dim_{\mathbb{k}} \text{Hom}_{A-A}(A, A \otimes_{\mathbb{k}} A) = 4$ and we get a contradiction.

If B is connected, the analogue of the bimodule (7) will look more complicated. However, the new quiver will be obtained from the case of disconnected B by adding some new arrows between different vertices. The restriction of the center of this new B to each vertex still coincides with the case considered in the previous paragraph. Therefore any homomorphism from B to this new bimodule will restrict, by forgetting all new arrows, to a homomorphism from the previous paragraph. As generators for our bimodules remain the same, this restriction map is injective. Of course, new arrows might come with new conditions, so there is no obvious guarantee that the restriction map is surjective. In any case, the dimension of the homomorphism space from our new B to this new bimodule will be at most 3, which again leads to a contradiction. This completes the proof in Case 2.

5.6. Case 3: $M = M_3$. In this case $k = 2$, $[N_i : L'_1] = 2$ and $[N_i : L'_2] = 1$, for $i = 1, 2$. As N is faithful, we obviously have $\text{Rad}(B)^3 = 0$. Moreover, as N is faithful and L'_2 appears with multiplicity 1 in both N_1 and N_2 , it follows that $\text{End}(P_2) \cong \mathbb{k}$ and hence Lemma 7 gives an embedding of A into $\text{End}(P_1)$. As N has only 2 direct summands and L_1 appears with multiplicity 2 in both, it follows that $A \cong \text{End}(P_1)$ and that the generators x and y of A can be chosen such that $\text{End}(N_1 e_1) \cong A/(x)$ under the natural map from A to $\text{End}(N_1 e_1)$.

The 2-functor $\overline{\mathbf{M}}$ induces a map from the 4-dimensional space $\text{Hom}_{A-A}(A, A \otimes_{\mathbb{k}} A)$ to the space of B - B -bimodule homomorphisms from B to $P_1 \otimes_{\mathbb{k}} N_1 \oplus P_2 \otimes_{\mathbb{k}} N_2$. Due to

the previous paragraph, the image of the generator $1 \in B$ belongs to

$$e_1 P_1 \otimes_{\mathbb{k}} N_1 e_1 \oplus e_2 P_2 \otimes_{\mathbb{k}} N_2 e_2.$$

The space $e_2 P_2 \otimes_{\mathbb{k}} N_2 e_2$ has dimension 1 by the above. The space $e_1 P_1 \otimes_{\mathbb{k}} N_1 e_1$ can be identified with $A \otimes_{\mathbb{k}} (A/(x))$. Even on the level of A - A -bimodules, the image of 1 in $A \otimes_{\mathbb{k}} A/(x)$ has to belong to the 2-dimensional subspace generated by $x \otimes y$ and $y \otimes y$. Therefore we obtain an injection of a 4-dimensional space into a space of dimension at most 3, a contradiction. The proof of Case 3 is now complete.

5.7. Case 4: $M = M_4$. In this case $k = 3$ and $[N_i : L'_j] = 1$, for all $i, j = 1, 2, 3$. In particular, all N_i 's are multiplicity free and hence the endomorphism algebra of each N_i is a direct sum of copies of \mathbb{k} . Thus \overline{M} cannot embed A into $Z(B)$, contradicting Lemma 7. This completes the proof in Case 4 and thus of the whole Theorem 6.

REFERENCES

- [BFK] J. Bernstein, I. Frenkel, M. Khovanov. A categorification of the Temperley-Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors. *Selecta Math. (N.S.)* **5** (1999), no. 2, 199–241.
- [CM] A. Chan, V. Mazorchuk. Diagrams and discrete extensions for finitary 2-representations. Preprint arXiv:1601.00080.
- [CR] J. Chuang, R. Rouquier. Derived equivalences for symmetric groups and \mathfrak{sl}_2 -categorification. *Ann. of Math. (2)* **167** (2008), no. 1, 245–298.
- [Fo] L. Forsberg. Multisemigroups with multiplicities and complete ordered semi-rings. Preprint arXiv:1510.01478, to appear in *Beiträge zur Alg. Geom.*
- [GM1] A. L. Gensing, V. Mazorchuk. Categorification of the Catalan monoid. *Semigroup Forum* **89** (2014), no. 1, 155–168.
- [GM2] A. L. Gensing, V. Mazorchuk. Categorification using dual projection functors. Preprint arXiv:1501.00095. To appear in *Commun. Contemp. Math.*
- [Kh] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.* **101** (2000), no. 3, 359–426.
- [KL] M. Khovanov, A. Lauda. A categorification of a quantum \mathfrak{sl}_n . *Quantum Topol.* **1** (2010), 1–92.
- [KiM1] T. Kildetoft, V. Mazorchuk. Parabolic projective functors in type A . *Adv. Math.* **301** (2016), 785–803.
- [KiM2] T. Kildetoft, V. Mazorchuk. Special modules over positively based algebras. *Documenta Math.* **21** (2016) 1171–1192.
- [KMMZ] T. Kildetoft, M. Mackaay, V. Mazorchuk, J. Zimmermann. Simple transitive 2-representations of small quotients of Soergel bimodules. Preprint arXiv:1605.01373.
- [KX1] S. König, C. Xi. On the structure of cellular algebras. *Algebras and modules, II* (Geiranger, 1996), 365–386, CMS Conf. Proc., **24**, Amer. Math. Soc., Providence, RI, 1998.
- [KX2] S. Koenig, C. Xi. Affine cellular algebras. *Adv. Math.* **229** (2012), no. 1, 139–182.
- [KuM] G. Kudryavtseva, V. Mazorchuk. On multisemigroups. *Port. Math.* **72** (2015), no. 1, 47–80.
- [Le] T. Leinster. *Basic bicategories*, Preprint arXiv:math/9810017.
- [McL] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 1998.
- [MaMa] M. Mackaay, V. Mazorchuk. Simple transitive 2-representations for some 2-subcategories of Soergel bimodules. *J. Pure Appl. Algebra* **221** (2017), no. 3, 565–587.
- [MT] M. Mackaay, D. Tubbenhauer. Two-color Soergel calculus and simple transitive 2-representations. Preprint arXiv:1609.00962.
- [MMMT] M. Mackaay, V. Mazorchuk, V. Miemietz and D. Tubbenhauer. Simple transitive 2-representations via (co)algebra 1-morphism. Preprint arXiv:1612.06325.
- [MM1] V. Mazorchuk, V. Miemietz. Cell 2-representations of finitary 2-categories. *Compositio Math* **147** (2011), 1519–1545.
- [MM2] V. Mazorchuk, V. Miemietz. Additive versus abelian 2-representations of fiat 2-categories. *Moscow Math. J.* **14** (2014), No.3, 595–615.
- [MM3] V. Mazorchuk, V. Miemietz. Endmorphisms of cell 2-representations. *Int. Math. Res. Notes*, Vol. **2016**, No. 24, 7471–7498.
- [MM4] V. Mazorchuk, V. Miemietz. Morita theory for finitary 2-categories. *Quantum Topol.* **7** (2016), no. 1, 1–28.

- [MM5] V. Mazorchuk, V. Miemietz. Transitive 2-representations of finitary 2-categories. *Trans. Amer. Math. Soc.* **368** (2016), no. 11, 7623–7644.
- [MM6] V. Mazorchuk, V. Miemietz. Isotypic faithful 2-representations of \mathcal{J} -simple fiat 2-categories. *Math. Z.* **282** (2016), no. 1-2, 411–434.
- [MZ] V. Mazorchuk, X. Zhang. Simple transitive 2-representations for two non-fiat 2-categories of projective functors. Preprint arXiv:1601.00097.
- [Ro] R. Rouquier. 2-Kac-Moody algebras. Preprint arXiv:0812.5023.
- [Xa] Q. Xantcha. Gabriel 2-quivers for finitary 2-categories. *J. Lond. Math. Soc. (2)* **92** (2015), no. 3, 615–632.
- [Zh1] X. Zhang. Duflo involutions for 2-categories associated to tree quivers. *J. Algebra Appl.* **15** (2016), no. 3, 1650041, 25 pp.
- [Zh2] X. Zhang. Simple transitive 2-representations and Drinfeld center for some finitary 2-categories. Preprint arXiv:1506.02402.
- [Zi] J. Zimmermann. Simple transitive 2-representations of Soergel bimodules in type B_2 . *J. Pure Appl. Algebra* **221** (2017), no. 3, 666–690.

Vo. Ma.: Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, SWEDEN, email: mazor@math.uu.se

Va. Mi.: School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK, v.miemietz@uea.ac.uk

X. Z.: Department of Mathematics, Uppsala University, Box. 480, SE-75106, Uppsala, SWEDEN, email: xiaoting.zhang@math.uu.se