# Efficient hybrid algorithms to solve mixed discrete-continuous optimization problems: A comparative study

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**Abstract:**

*Purpose*
– In real world cases, it is common to encounter mixed discrete-continuous problems where some or all of the variables may take only discrete values. To solve these non-linear optimization problems, it is very time-consuming in use of finite element methods. The purpose of this paper is to study the efficiency of the proposed hybrid algorithms for the mixed discrete-continuous optimization, and compares it with the performance of Genetic Algorithms (GA).

*Design/methodology/approach*
– In this paper, the enhanced multipoint approximation method (MAM) is utilized to reduce the original nonlinear optimization problem to a sequence of approximations. Then, the Sequential Quadratic Programming (SQP) technique is applied to find the continuous solution. Following that, the implementation of discrete capability into the MAM is developed to solve the mixed discrete-continuous optimization problems.

*Findings*
– The efficiency and rate of convergence of the developed hybrid algorithms outperforming GA are examined by six detailed case studies in the ten-bar planar truss problem and the superiority of the Hooke-Jeeves assisted MAM algorithm over the other two hybrid algorithms and GAs is concluded.

*Originality/value*
– The authors propose three efficient hybrid algorithms: the rounding-off, the coordinate search, and the Hooke-Jeeves search assisted MAMs, to solve nonlinear mixed discrete-continuous optimization problems. Implementations include the development of new procedures for sampling discrete points, the modification of the trust region adaptation strategy, and strategies for solving mix optimization problems. To improve the efficiency and effectiveness of metamodel construction, regressors φ defined in this paper can have the form in common with the empirical formulation of the problems in many engineering subjects.

**Keywords:** Discrete-continuous design optimization; Hybrid algorithms; Multipoint approximation method; Direct search

## 1. Introduction

Problems with mixed discrete-continuous design variables are a class of complicated optimization problems that commonly exist in practical engineering design work. These non-linear optimization problems usually share the common features: the objective and constraint functions are prohibitively expensive to be examined in use of finite element methods or could be impossible to be evaluated at some combinations of design variables (e.g., nodal displacements, stresses, strains). In order to improve the computational efficiency and accuracy of the solvers dedicated for such structural optimization problems, the Multipoint Approximation Method (MAM) was developed (Haftka *et al*. 1987, Toropov 1989). The advantage of the MAM technique is presented to replace the original optimization problem with a succession of simpler mathematical programming in a scalable trust-region of design space and utilize high quality explicit approximations to reduce the total number of calls for analysis needed for the complicated optimization problems. Details on the development of the MAM technique for various applications are discussed in (Fadel *et al*. 1990, Wang and Grandhi 1995, Polynkin *et al* 1996, Keulen and Toropov 1997, Polynkin and Toropov 2012). To enhance the accuracy of metamodels used to approximate the real responses, metamodel assembly approach is implemented in MAM to construct an assembly of multiple surrogates into a single surrogate using linear regression (Viana and Haftka 2008, Acar and Rais-Rohani 2009, Liu and Toropov 2012).

In practical engineering designs, some or all of the design variables have discrete or integer values because the available values for those design variables are limited to a set of standard sizes, for example, the thickness of the laminated composite structure, the diameter of the cross-section in a truss structure, the wind farm layout design, etc. In general, it is only allowed to perform response function evaluations for points that have discrete values of the design variables (Balabanov and Venter 2004). To address such mixed discrete-continuous optimization problems, Genetic Algorithm (Goldberg 1989, Holland 1992, Michalewicz 1992, Shrestha and Ghaboussi 1998, Ghasemi *et al* 1999, Mendez *et al* 2006, Abdoun and Abouchabaka 2011, Carr 2014, Lemonge *et al* 2015) is considered a popular tool for locating the global optimum solution. GA is a population-based evolutionary algorithm and inspired by genetic evolutions. It employs the genetic operators of reproduction, mutation, crossover, and fitness selection as part of its evolutionary strategy. The mechanism of GA can be described as a combination of an artificial survival of the fittest and genetic operators from nature to iteratively improve the fitness of the population. However, function evaluations in GAs or other evolutionary algorithms (Dorigo *et al* 1996, Schutte and Groenwold 2003, Sadollah *et al* 2012, Yang and Gandomi 2012, Lakshmi and Rao 2013, Miguel and Lopez 2013, Baghlani *et al* 2014, Kaveh and Mahdavi 2014, Yilmaz *et al* 2015, Yu *et al* 2015, Chakri *et al* 2017, Duarte *et al* 2017) are the most time-consuming part as they are usually executed by finite element analysis or computational fluid dynamics with high computational demands, for example, a single function evaluation could be of the order of hours or days. Taking into account this limitation, it is necessary to develop an efficient algorithm for solving mixed discrete-continuous design problems with less computational effort. The enhanced MAM with the implementation of the discrete capability such as sampling, discrete properties within a trust region (Cheng *et al* 2015, Yuan 2015, Larson and Billups 2016), and the trust region adaptation strategy, has been proved as a better approach to tackle these problems (Liu and Toropov 2016). The obtained optimal results have demonstrated MAM with the implementation of the coordinate search technique can solve the complicated optimization problems with high efficiency and effectiveness.

In current research, a comparative study of three hybrid algorithms, which combine the metamodel-based MAM and direct search techniques (Kolda *et al* 2003), is presented. The developed hybrid algorithms are the rounding-off technique assisted MAM, the coordinate search technique assisted MAM, and the Hooke-Jeeves technique assisted MAM. These three discrete forms of the direct search methods are implemented within the MAM to search for the optimal discrete values in the sub-space of the discrete variables only starting from the optimal continuous values obtained by the Sequential Quadratic Programming method (SQP) on the approximated functions in a current trust region. Then, the values for continuous design variables are updated accordingly by the optimization based on the reduced design space for continuous variables only. Full details are given in the section *Direct Search Techniques for Discrete Optimization* for the description of the developed hybrid algorithms. Based on this study, the advantages of hybrid algorithms outperforming GA are concluded by the well-established benchmark examples in terms of the computational cost, rate of convergence, and the quality of solutions. The capability of these proposed optimization methods has also been assessed by the case studies with different discreteness intervals (0.2 and 1.0) for discrete design variables. Finally, the conclusion on the superiority of the Hooke-Jeeves assisted MAM algorithm over the other two hybrid algorithms and GAs is drawn.

## 2. Brief Review of Multipoint Approximation Method (MAM)

Many engineering applications involve numerical simulations of response functions. These evaluations might either suffer from numerical noise or the intensive computational time. The MAM, based on response surface methodology, aims at constructing mid-range approximations (Wang and Grandhi 1995, Keulen and Toropov 1997, Sun *et al* 2014, Yan *et al*, 2014, An *et al* 2016) and is suitable to solve large-scale optimization problems by producing better quality approximations that are sufficiently accurate in a current trust region and inexpensive in term of computational costs required for their building. These approximation functions have a relatively small number ($N$+1 where $N$ is number of design variables) of regression coefficients to be determined and the corresponding least squares problem can be solved easily. The feature of such approximations allows applying them to large scale optimization problems with the number of design variables in the order of hundreds.

In general, an optimization problem can be formulated as

$$\min F_0(\boldsymbol{x}), \quad F_j(\boldsymbol{x}) \le 1 \ \ (j = 1, \dots, M), \quad A_i \le x_i \le B_i \ \ (i = 1, \dots, N) \qquad (1)$$

where $\boldsymbol{x}$ refers to the vector of design variables; $A_i$ and $B_i$ are the given lower and upper bounds on the design variable $\boldsymbol{x}_i$; $N$ is the total number of the design variables; $F_0(\boldsymbol{x})$ is an objective function; $F_j(\boldsymbol{x})$ is the constraint function and $M$ is the total number of the constraint functions.

In order to present the detailed physical model using the response functions and reduce the number of calls for the response function evaluations, the MAM replaces the optimization problem by a sequence of approximate optimization problems:

$$\min \tilde{F}_0^k(\boldsymbol{x}), \ \tilde{F}_j^k(\boldsymbol{x}) \le 1 \ (j = 1, \dots, M), \ A_i^k \le x_i \le B_i^k, A_i^k \ge A_i, \ B_i^k \le B_i \ (i = 1, \dots, N) \qquad (2)$$

where $\tilde{F}_0^k(\boldsymbol{x})$ and $\tilde{F}_j^k(\boldsymbol{x})$ are the functions which approximate the functions $F_0(\boldsymbol{x})$ and $F_j(\boldsymbol{x})$ defined in Eq.(1), $A_i^k$ and $B_i^k$ are the side constraints of a trust sub-region, $k$ is the iteration number.

The selection strategy of the approximate response functions $\tilde{F}_j^k(\boldsymbol{x})$ $(j = 0,\ldots,M)$ outlines that their evaluations are inexpensive as compared to the evaluation of the actual response functions $F_j(\boldsymbol{x})$ and are intended to be adequate in a current trust region. This is achieved by appropriate planning of numerical experiments and use of the trust region defined by the side constraints $A_i^k$ and $B_i^k$.

## 3. Design of Experiments, Trust Region Strategy, and Weight Coefficients

Design of Experiments (DOE) is a powerful statistical technique to study the effect of multiple variables simultaneously. By applying this technique, the time required for experimental investigations or computer experiments can be significantly reduced. To solve the mixed discrete-continuous design problems proposed in this paper, DOE should generate sampling points, which represent the designs with mixed design variables having both continuous and discrete values. Also, all sampling points are checked for calculability of the response function. A new set of points is generated until a required number of sampling points (all passing the check) are obtained, if the checks for some points fail. A strategy to improve the quality of sampling points - constraint enhanced DOE rather than random DOE - is applied in this paper. It generates the sample points with a constraint on the minimal distance between the points using Eq. 3:

$$\frac{dist^{p}}{Diag} \geq r \tag{3}$$

where $$Diag = \sqrt{\sum_{i=1}^{N}\left(B_i^k - A_i^k\right)^2},\ dist^e = \sqrt{\sum_{i=1}^{N}\left(x_i^e - x_i^p\right)^2} \qquad (e, p = 1,\ldots,P; e \neq p),$$

*Diag* is a characteristic size of the trust region, $e$ is a number of a new sampling point, $p$ is a number of a previously generated point, $P$ is the total number of sampling points in the trust region, $N$ is the total number of the design variables, $r$ is the parameter and initially set to 0.9. Imposing such a constraint in Eq. 3 on the selection of points, the uniform distribution of sampling points across the design space is guaranteed. Based on DOE design points, the approximation models are built to efficiently evaluate the responses of the interests.

Trust region method (Ong *et al* 2003, Cheng *et al* 2015, Leifssona *et al* 2015, Yuan 2015, Kamandi *et al* 2017) is a class of numerical algorithms for solving nonlinear optimization problems. Inspired by this method, the aim of the trust region strategy in the MAM approach is to control the metamodel quality (Keulen and Toropov 1997). The trust region strategy in this paper includes: Once the optimization problem defined in Eq. 2 has been solved in each MAM iteration, an updated search region with new dimensions and location must be redefined according to the current sub-optimal value; The size of the trust region should be further reduced, if the approximation gets better; and if the sub-optimum point does not pass the check for calculability of the response functions, the trust region is reduced and the approximated problem is solved again. Hence, the track of the trust region indicates a search path from the initial starting point to the optimum over the entire searching domain. The only essential assumption made in the trust region strategy is that all functions of the optimization problem exist at the starting point. It should be noted that for the discrete variables there is a restriction on the trust region size reduction, i.e. the corresponding size of the trust region has to contain at least three levels of a discrete variable. A too small size for the trust region may result in the ill-conditioning of the matrix constructed by DOE points.

In most cases, the optimum point usually locates on the boundary of the feasible region. This requires that the approximation functions at the points located in the region which is quite close to the boundary of the feasible domain should be more accurate. Taking into account this situation, greater weights are assigned to the plan points close to the boundary of the feasible design domain. In this paper, the weight coefficients influenced by the value of a constraint function at a point are taken as $w_{pj} = e^{-4\left|F_j(\mathbf{x}_p)-1\right|}$ (see Eq. 1).

## 4. Build Approximations by a Two-Step Regression Procedure

Based on the research on multiple metamodels (Viana and Haftka 2008, Acar and Rais-Rohani 2009, Liu and Toropov 2012), a two-step regression procedure is used to build the metamodel, which is represented by the assembly of different approximations below

$$\tilde{F}(\boldsymbol{x}) = \sum_{l=1}^{NF} b_l \varphi_l(\boldsymbol{x}), \tag{4}$$

where $\varphi_l(x)$ means different approximate models,

$NF$ is the number of regressors in the approximate models { $\varphi_l(x)$ },

$b_l$ is the corresponding regression coefficient and determined by

$$\min \quad \sum_{p=1}^{P} w_{pj}\left[F_j(\boldsymbol{x}_p) - \tilde{F}_j\left(\boldsymbol{x}_p, \boldsymbol{b}_l\right)\right]^2 . \tag{5}$$

The above weighted least squares problem (Eq. 5) leads to solving a linear system of $NF$ equations with unknowns $\boldsymbol{b}_l$ . $P$ means the number of points used in a specified DOE (see Section *Design of Experiments*). The parameters $w_{pj}$ refer to the weights that reflect the inequality of data obtained at different sampling points. Before the Eq. 5 has been solved, the minimization defined in Eq. 6 below should be performed to determine $N$+1 tuning parameters $\boldsymbol{a}_j$ by matching the approximate response function $\varphi_j$ with the actual function $F_j$ .

$$\min \quad \sum_{p=1}^{P} w_{pj}\left[F_j(\boldsymbol{x}_p) - \varphi_j\left(\boldsymbol{x}_p, \boldsymbol{a}_j\right)\right]^2 \tag{6}$$

By applying the above two-step regression strategy, the approximation model with very high accuracy is produced. The evaluation of the regressors $\varphi_l$ is based on the data from the sampling points currently located in the trust region. To reduce the computational cost, inexpensive approximate models for objective and constraint functions are built using minimum required number of sampling points. The simplest case is a linear function and more complex ones are intrinsically linear functions (Box and Draper 1987) that have been successfully used for a variety of design optimization problems. Intrinsically linear functions are nonlinear but they can be led to linear ones by simple transformations. In some engineering subjects, for example, civil engineering, empirical or semiempirical formulae (Rahman *et al* 2010) are widely used to efficiently evaluate the solutions for many complex engineering problems. These equations are generated by curve fitting experimental data and are especially important in the fields (e.g., prediction of critical impact energy on the structures, fundamental vibration period determination of buildings subject to seismic excitation, etc.) due to the complexity of the phenomena reflecting the physical processes. It is often desirable to use closed-form expressions for the calculation of the response of the interest. Based on these facts, a user-defined regressor $\varphi_l$ is implemented in the model bank { $\varphi_l(x)$ } of the MAM framework and its form is defined as same as the empirical formulation, but the coefficients are determined by the proposed two-step regression analysis. Consequently, implementation of the user-defined regressor $\varphi_l$ into the model bank { $\varphi_l(x)$ } will significantly improve the efficiency and accuracy of the enhanced MAM for solving complex optimization problems. In this paper, the intrinsically linear functions considered in the model bank { $\varphi_l(x)$ } are:

$$\begin{aligned}
&\varphi(\boldsymbol{x}) = a_0 + \sum_{i=1}^{N} a_i x_i \\
&\varphi(\boldsymbol{x}) = a_0 \prod_{i=1}^{N} x_i^{\,a_i} \\
&\varphi(\boldsymbol{x}) = a_0 + \sum_{i=1}^{N} a_i / x_i \\
&\varphi(\boldsymbol{x}) = a_0 + \sum_{i=1}^{N} a_i x_i^2 \\
&\varphi(\boldsymbol{x}) = a_0 + \sum_{i=1}^{N} a_i / x_i^2
\end{aligned} \tag{7}$$

## 5. Direct Search Techniques for Discrete Optimization

Three hybrid algorithms, combining the metamodel-based Multipoint Approximation Method (MAM) and different direct search techniques, are presented in this paper to find the optimum discrete representation of the discrete variables. These three developed algorithms are a rounding-off technique assisted MAM, a coordinate search technique assisted MAM, and the Hooke-Jeeves technique assisted MAM (HJ-MAM). The flowchart for the proposed hybrid algorithms is shown in Figure 1 and the detailed methodology consists three steps below:

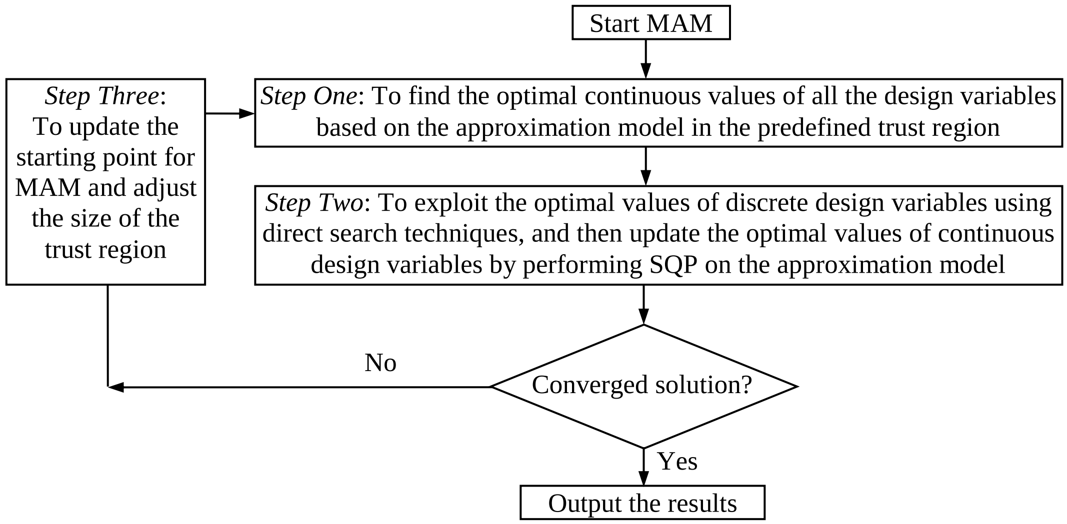


Figure 1 Flowchart of the hybrid algorithms

*Step One*: Before exploiting the sub-optimal solution for the discrete optimization in use of direct search techniques, a continuous optimization solution in each iteration of MAM should be initially sought in a current trust region using the SQP method applied to the approximate optimization formulation defined in Eq. 2.

*Step Two*: Based on *Step One*, detailed procedures for seeking the sub-optimal solution of the discrete optimization problems by local exploitations have been given with regards to each of the three direct search techniques:

(1) Rounding-off technique assisted MAM is applied to obtain the sub-optimal solution for the discrete form of the optimization problem (Eq. 2). The flowchart for this technique is shown in Figure 2 and its strategy consists of: rounding the continuous variables to the nearest discrete ones for the discrete variables, fixing these values, and solving a continuous optimization problem again (with the reduced number of design variables to include only continuous variables). This is the simplest strategy to solve an optimization problem with the discrete variables. Since the discrete variables have been assigned the rounded values from the continuous solution, the optimal values for the remaining continuous variables must be updated accordingly by performing a dimensionality-reduction continuous optimization again. It should be noted that the efficiency of this traditional search algorithm is quite low due to its simple mechanism of seeking the optimal solutions and the total computational time required for the converged solution in the entire optimization process is expected to be very long.

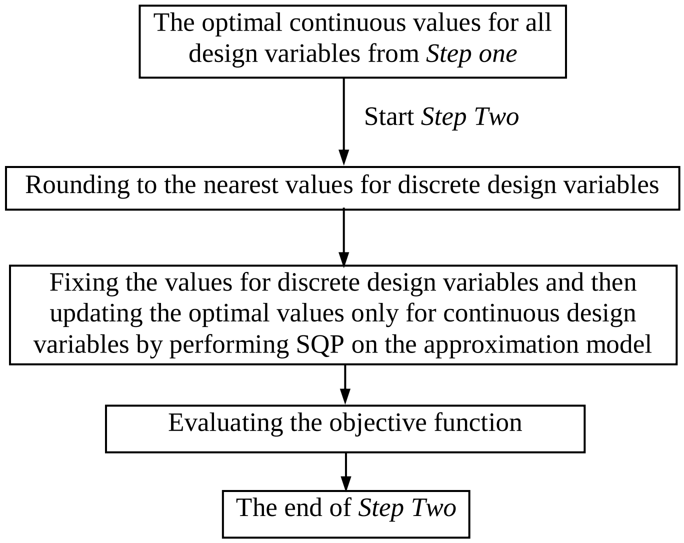


Figure 2 Flowchart for the rounding-off technique in Step Two

(2) Coordinate search technique assisted MAM is a more complicated algorithm and its flowchart is given in Figure 3. Based on the optimal values of continuous optimization in *Step One*, the rounding is performed to obtain the nearest discrete values for discrete design variables. Then, points near the rounding-off solution are examines by perturbing discrete design variables by a perturbation $\Delta_i$ (positive or negative) - one discrete variable at a time - until the point with the lowest objective function value (that will be penalized in case of violated constraints) is identified among all the tested points. There is a similarity between this technique and one previously suggested in (Balabanov and Venter 2004), however, the type of approximations applied to solve the optimization problems is completely different. The coordinate search technique assisted MAM begins with the starting point (the rounding-off solution), as well as $2N_d$ coordinate points for function evaluations using the approximation model, where $N_d$ is the number of discrete design variables or coordinates and the coefficient '2' means a pair of positive and negative perturbations $\Delta_i$ for each coordinate. As an example, the $i^{th}$ pair of coordinate points differs from the starting point only in the $i^{th}$ coordinate. The size of $\Delta_i$ is determined by the spacing of the $i^{th}$ discrete design variable. Again, it is noted that optimal values for the continuous variables must be updated accordingly at each coordinate point and the point with the lowest objective function value along $i^{th}$ coordinate will be selected as the new starting point for the next $(i+1)^{th}$ coordinate search until all the coordinate points are evaluated to determine the sub-optimal solution in the current iteration. The efficiency of coordinate search technique assisted MAM is much improved as compared to rounding-off technique assisted MAM and as expected, the number of iterations for the optimization process are reduced.

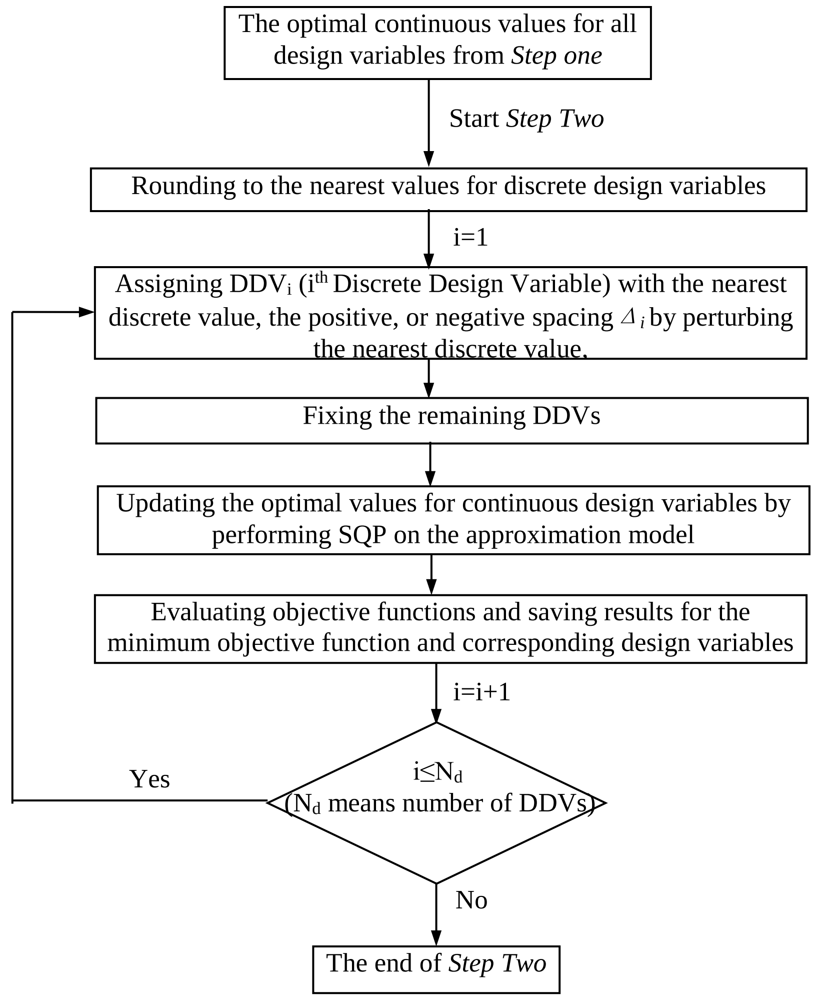


Figure 3 Flowchart for the coordinate search technique in Step Two

(3) Hooke-Jeeves direct search technique (Kolda *et al* 2003, Garcia *et al* 2006, Brauna *et al* 2015, Liu 2016), a more robust approach than the first two algorithms, can more efficiently and effectively exploit the design space of discrete variables for the optimal solutions. The flowchart for the Hooke-Jeeves direct search technique is depicted in Figure 4. This technique examines points near the current point (representing the rounded values for discrete design variables) by perturbing design variables along one coordinate at a time until an improved point is found in that coordinate direction. The Hooke-Jeeves technique and the coordinate search algorithm have the common feature seen to determine the search path for the optimal solution, however the Hooke-Jeeves technique has more efficient search strategy. Based on the search strategy of the coordinate search technique assisted MAM, the point with the lowest objective function value along $i^{th}$ coordinate will be saved and then, a further perturbation $\Delta_i$ along the preceding search direction is taken into account as well as optimal values for the continuous variables are updated accordingly. This search process will be terminated until a worse objective function value is identified in this pre-defined search direction. Then, the saved sub-optimal values are assigned to the $i^{th}$ discrete design variables as the optimum. The search strategy is applied again for the next $(i+1)^{th}$ coordinate search until all the coordinates representing discrete design variables are considered in the current iteration of MAM. Compared with the other two hybrid algorithms, HJ-

MAM has the maximum efficiency to seek the optimum and its rate of convergence is fastest according to the effective search mechanism.

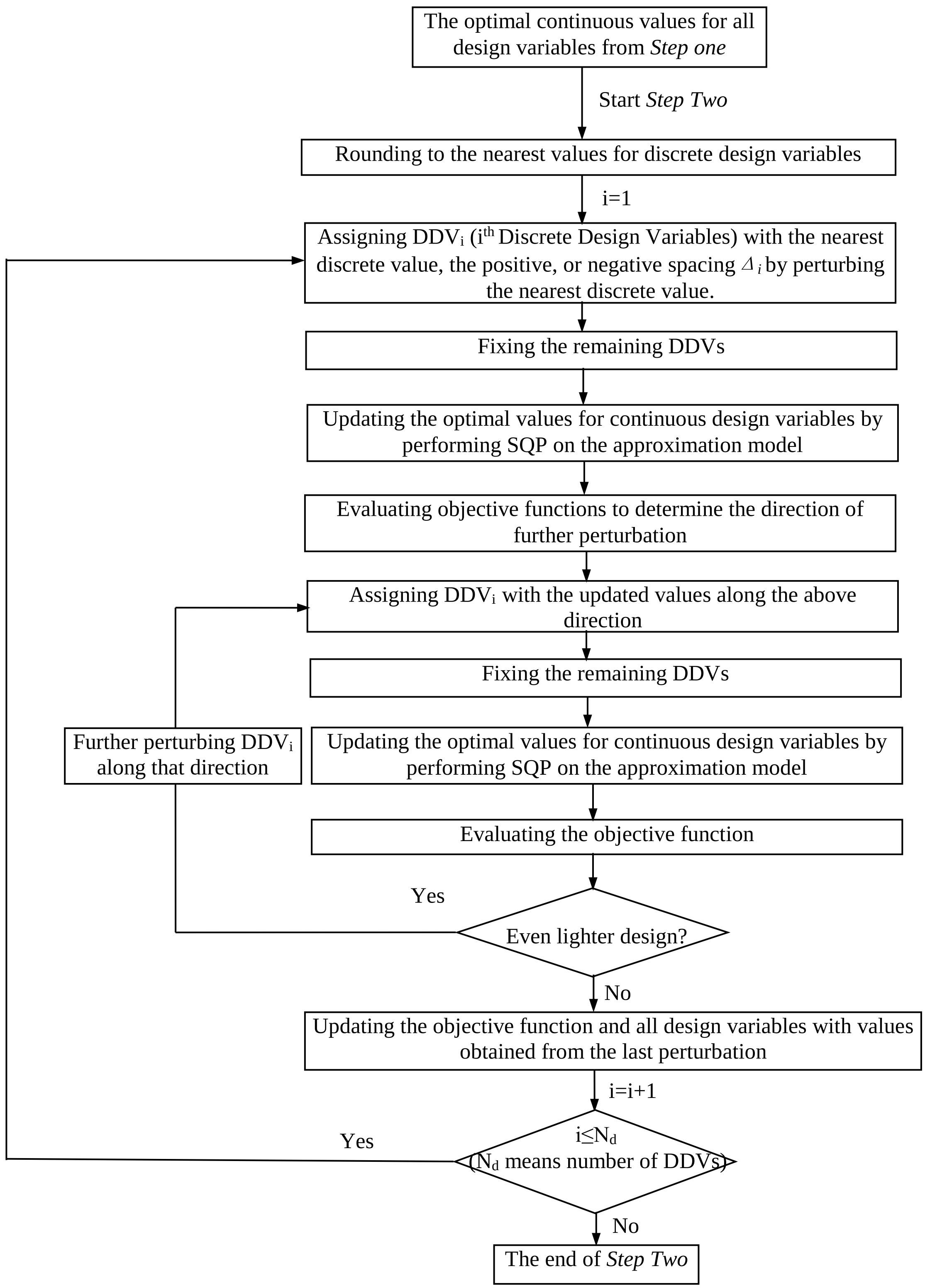


Figure 4 Flowchart for the Hooke-Jeeves direct search technique in Step Two

*Step Three: The* sub-optimal solution from local exploitations in *Step Two* becomes a starting guess for the next iteration of MAM and then, the *Step one* is repeated in an updated trust region to seek the sub-optimum until the optimal solution converges in the whole optimization process.

As direct search algorithms (Kolda *et al* 2003) are known as unconstrained optimization techniques, an exterior penalty function is used to accommodate the constraints by penalizing unfeasible solutions as follows:

$$f(\boldsymbol{x}) = \frac{F_0(x)}{F_0^*} + \alpha \sum_{i=1}^{m} \max[1, F_i(x)]^{\beta} \tag{8}$$

where $f(\boldsymbol{x})$ is the objective function penalized is case any of the constraints is violated,

$F_0(\boldsymbol{x})$ is the objective function,

$F_0^*$ is the initial value of objective function at the starting point,

$\alpha, \beta$ are the penalty parameters, here $\alpha = 0.5$ and $\beta = 1$ are suggested

$F_i(x)$ is the *i*-th constraint function, $i = 1, \cdots m$,

$m$ is the total number of constraints.

In summary, when the discrete variables are modified by the discrete optimizer, it is necessary to make an adjustment of the remaining, continuous design variables so that the optimal values for both discrete and continuous design variables are obtained at the current iteration. This is achieved by the use of SQP for the optimal values for continuous variables only while the discrete variables are assigned by the current discrete optimum. Therefore, the overall local optimization process is performed in two levels: The outer optimization loop deals with the discrete variables only in use of direct search techniques, and the inner optimization loop adjusts the continuous design variables using SQP.

## 6. Case Studies

### 6.1 Ten-bar Planar Truss Structure

The developed hybrid algorithms are tested on the ten-bar planar truss benchmark problem (Haftka and Gürdal 1992) shown in Figure 5 that had been used and further investigated by many researchers (Mahfouz 1999, Kripakaran *et al* 2007, Eskandar *et al*, 2013). For a more complex truss bridge problem, Hasançebi (2007) obtained the optimal design by an evolution strategy-based algorithm.

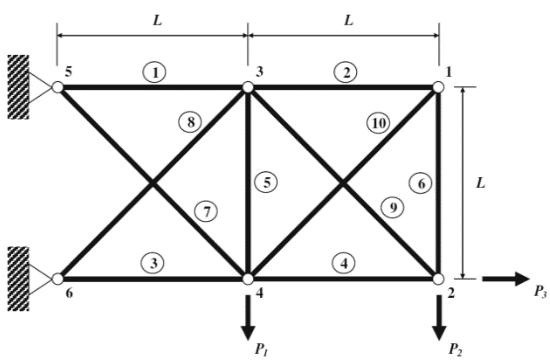


Fig. 5. Ten-bar planar truss structure

**Example 1**

The optimization formulation of this problem is defined to minimize the weight of the structure by varying the cross-sectional areas (from $0.1\,in^2$ to $12.7\,in^2$) of the ten-bar truss members subject to stress constraints. The allowable stress in each truss member is the same in tension and compression and is set to $25\,ksi$ for all members except member 9 for which it is $75\,ksi$. The density of the truss material is $0.1\,lb/in^3$, the member size L = $360\,in$, the loads P1 = P2 = 100 $Kips$ and P3 = 0. For the discrete optimization, the discreteness interval of 0.2 is used.

Optimal weight designs (continuous and discrete results) of the ten-bar planar truss structure are presented in Table 1 and information on the constraint values is listed in Table 2. The number of sampling points used to build metamodels in each iteration of MAM is assigned the value 15. The lightest weight (1497) of the truss structure can be obtained by continuous optimization of the benchmark example due to a largest space domain of design variables. This optimal solution can be deemed as a feasible design because only a slight violation of constraint (0.4%) is identified in Table 2. In the discrete optimization, the results obtained by three proposed hybrid algorithms and a binary GA are compared to demonstrate the potentials of these algorithms. In this research, the following settings of the binary GA were used in the examples in this Section: a population size of 200, 40 generations, elite as 10% of the population, tournament selection, single point crossover, and 1% mutation rate. This was performed on a personal computer with 8 cores (E5-2670, 2.6GHz) and 24G RAM. Since a GA is naturally suitable to the discrete optimization and is also a global search algorithm, a better result from GA could be expected. On the contrary, optimization by HJ-MAM has achieved the lighter weight (1546.0) with no violation of constraints and the other two hybrid algorithms, rounding-off and coordinate search assisted MAMs, have achieved the same and lightest weight (1525.6), but the violation of constraints by 0.8% has been observed (see Table 2). This can explain why the optimal design by HJ-MAM has a slightly heavier weight than the result by the other two hybrid algorithms. However, all of these optimal designs are lighter than the weight obtained by GA, which is the best result out of five separate GA runs.

In this paper, the number of iterations means the number of the repetitive processes to find a converged solution described in Figure 1, while the number of response analyses depicts the total number of times that the analytical model is called for the purpose of function evaluations by FEM or CFD simulations throughout the entire optimization process, see Table 1. In each iteration of the optimization process, there will be only few of numbers of response analyses involved to build the metamodels for seeking the sub-optimal solution. For a given physical problem, the time for each function evaluation of the analytical model is almost identical and this takes up most of the actual execution time. It can be obviously concluded that the less number of response analyses or iterations, the less the total execution time required to obtain the final solution in the whole optimization process. Therefore, the numbers of iterations and response analyses by the optimizers with different techniques, which are the three proposed hybrid algorithms and GA, are used as quality indicators to evaluate the effectiveness and efficiency of the algorithms in the following comparative study.

As expected, the numbers of iterations and response analyses in the optimization by a binary GA are significantly increased. The numbers of iterations and response analyses (values in the round bracket) in use of hybrid algorithms (rounding-off, coordinate search, and Hooke-Jeeves technique assisted MAMs) are 48 (721), 24 (361), and 11 (166), respectively. With the rounding-off assisted MAM algorithm, the number of iteration, 48, is reduced by 20% from the total number of 57 required by GA. It can be seen that the developed hybrid algorithms outperform GA in terms of the computational efficiency, the rate of convergence, and the quality of the results.

**Example 2**

The only difference between this example and *Example 1* is the allowable stress for the member 9 in the ten-bar truss problem is limited to 25 κsi. Optimal weight designs using GAs and three hybrid algorithms are presented in Table 3. It is expected that the lightest weight of the truss structure should be obtained by the continuous optimization. Results from the discrete optimizations using the hybrid algorithms have been compared with the ones by GAs (Mahfouz 1999, HyperStudy 2012) shown in Table 3. Optimizations by the coordinate search and HJ-MAMs have achieved the same result, which reflects the lightest weight design of 1610.1 as compared with a slightly heavier weight (1617.3) by the rounding-off assisted MAM algorithm and the result (1627.5) by GAs. It is also noted that the numbers of response analyses used in the hybrid algorithms have been significantly reduced by an order magnitude in comparison with the results from other published papers (Mahfouz 1999, HyperStudy 2012) and the numbers of iterations required are also less than the ones by GAs. This reflects the fact that the developed hybrid algorithms are very effective approaches with the fast rate of convergence and suitable to solve discrete optimization problems. In Table 4, the constraint values for the optimal designs by discrete optimizations are provided and feasible solutions with no violation of constraints are observed. It is concluded that the computational efficiency of the three developed hybrid algorithms for the discrete optimization is quite higher than GAs in terms of the rate of convergence, the number of response analyses, and the quality of the solutions.

**Example 3**

In this example, a mixed discrete-continuous design problem is studied and the stress constraints are the same as those formulated in *Example 1*. However, two types of design variables bounded from 0.1 $in^2$ to 12.7 $in^2$ are defined: Continuous cross-sectional areas for truss members 1 to 6 (horizontal and vertical members) and discrete cross-sectional areas for truss member 7 to 10 (diagonal members) with the increment of 0.2.

In Table 5, the weights of optimal designs by the coordinate search and HJ-MAMs are same (1506.7), but the optimal values for continuous variables are different because this is an optimization problem with multiple optimal solutions. As compared with the result obtained by GA (1583.3), the optimal weight is reduced by 5%. This should result in more constraints being activated for the lighter weight design shown in Table 6. The number of response analyses for GA (14901) has been reduced by 96.8% to 481, which is the result by HJ-MAM. The similar conclusion on the number of iterations can also be drawn. In comparison of the results obtained by these three hybrid algorithms, the fast rate of convergence (11) is identified using the round-off assisted MAM algorithm for the heaviest weight (1534.3), while the lightest weight can be obtained by the coordinate search and HJ-MAMs with acceptable more numbers of iterations (35 and 32, respectively). In Table 6, all the optimal solutions are feasible designs and no constraint violation happens in these discrete optimizations. Considering the computational efficiency, the hybrid algorithms for the mixed discrete-continuous optimization outperform GA in terms of the quality of the obtained solutions and numbers of iterations and response analyses. It can be concluded that taking into account the computational efficiency and quality of the solution, the HJ-MAM is the best approach among these three hybrid algorithms for solving the optimization problem with mixed design variables.

**Example 4**

In this challenging study, the discreteness interval of 1.0 for discrete design variables are applied to the above three examples in order to examine the effects of the discreteness interval on the performance (the quality of the solution and computational efficiency) of three developed hybrid algorithms. Therefore, the discrete design variables vary over the range of 0.1 $in^2$, 1.1 $in^2$, … , 12.1 $in^2$, 12.7 $in^2$.

In the foregoing Example 1, the optimal designs by GA and three hybrid algorithms are presented in Table 7. Optimizations using the coordinate search and HJ-MAMs have achieved the same objectives (the lightest weight, 1612.6), as compared with the result (1684.6) by the simple rounding-off assisted MAM and the solution (1663.5) by GA. Regarding the numbers of iterations required, only 21 iterations are needed for the lightest weight design by HJ-MAM, while four times more iterations (86) are observed to achieve the same weight design using the coordinate search assisted MAM. Although the numbers of iterations by the coordinate search (86) and the rounding-off assisted MAMs (120) are more than the result by GA (57), the total numbers of response analyses (1291 and 1801, respectively) are far less than the number required by GA (6900). In comparison with the weights achieved by the hybrid algorithms and GA, the overall quality of the optimal designs is well kept, whereas the maximum constraint violation shown in Table 8 for the lightest weight design in use of the hybrid algorithms is 0.8%, which is deemed to be within an acceptable tolerance. The reason for GA achieving a slightly heavier weight is explained as no constraints are violated in the optimal design.

Based on the preceding Example 2, both the effects of the discreteness interval of 1 for discrete variables on the optimal design and the performance of the hybrid algorithms and GA are investigated in Table 9. The coordinate search assisted MAM and simple rounding-off assisted MAM have the potential to identify the lighter designs (1786.4 and 1801.3, respectively), as compared to the result by GA (1858.4). The numbers of response analyses are reduced from 6764 (by GA) to 1711 (by the simple rounding-off assisted MAM), further to 376 (by the coordinate search assisted MAM). As expected, the minimum number of response analyses (181) for the lightest design can be found by HJ-MAM. In terms of the number of iterations, a twofold decrease can be observed when comparing the results of these three hybrid algorithms themselves. In Table 10, all four optimal designs are feasible solutions and no constraint violation has been observed. Conclusions can be drawn that compared with GA, the developed hybrid algorithms can solve the optimization problem more efficiently as well as produce a lighter design. They outperform GA for a better feasible design in terms of the structural weight, the rate of convergence, and the number of analyses. Among these three hybrid algorithms, HJ-MAM is the most efficient approach to solve the optimization problem.

The challenging, mixed discrete-continuous optimization problem defined in Example 3 is reformulated in light of the discreteness interval of 1.0 for discrete design variables. In Table 11, the number of response analyses by GA (12375) has been drastically reduced to 526 by the coordinate search assisted MAM and a further reduction to 436 by HJ-MAM. A similar conclusion can also be drawn on the objective function (the weight). Using the rounding-off assisted MAM, a slightly heavier design (1677.7) is achieved as compared to the result (1666.6) by GA, but the number of analyses used to seek the optimal solution is significantly reduced from 12375 to 1396. The weights of the optimal designs are further reduced by 4.6% and 5.2% using the coordinate search assisted MAM and HJ-MAM, respectively. The lightest weight (1591.1) is achieved by HJ-MAM and the minimum number of iterations (29) required for this optimal design can be observed as well in Table 11. In terms of constraint information listed in Table 12, it is noted that all four optimal solutions are feasible designs with no constraint violation.

**Discussions**

Based on the preceding fours examples, the performance of three developed hybrid algorithms in terms of computational efficiency, the rate of convergence, and the quality of solutions has been investigated by comparing the results with GAs and between themselves. Also, the effect of the different discreteness interval on the optimal designs has been evaluated.

For the mild cases considering the size (0.2) of the discreteness for discrete design variables shown in Tables 5, 7, and 9, the hybrid algorithms can generate, on average, better designs than GAs and the computational efficiency is also higher than the latter. The rounding-off assisted MAM has capability to solve the optimization problems with the acceptable accuracy of solutions, but with the comparison of the coordinate search assisted MAM and HJ-MAM, its optimal design is heavier while less iterations and response analyses are normally needed during the optimization process. The conclusions can be drawn that in the mild cases, the rounding-off assisted MAM is fast, but its solutions are less accurate. To seek the near optimal solution with less computational cost, the rounding-off assisted MAM is recommended for solving the optimization problem. No distinct difference in the optimal designs (the same level of quality of weights) can be observed by both the coordinate search assisted MAM and HJ-MAM, but the latter can solve the optimization problem with slightly higher computational efficiency.

As the discreteness interval is increased to 1.0, the heavier weights of the optimal designs can be observed because the smaller design space in the optimization process can be explored than the one in the foregoing mild cases. The hybrid algorithms have the ability to efficiently solve these optimization problems with sufficient accuracy. The optimal designs are lighter than the ones by GAs and the computational efficiency is also higher than the latter. As expected, the rounding-off assisted MAM has less capability to obtain the solutions with the sufficient accuracy in such challenging optimization problems. Both the better designs and the lower computational cost for solving the optimization problems are achieved by the coordinate search assisted MAM and HJ-MAM (see Table 7, 9 and 11). It turns out that using HJ-MAM, the lightest design are obtained with the minimum number of iterations required during the optimization process. In the most complicated case study (the mixed discrete-continuous optimization problem in Example 4), HJ-MAM can demonstrate its capacity superior to the other two hybrid algorithms and GAs with regard to computational efficiency and the quality of solutions.

In summary, comparing with the results by GAs, the optimal designs by the hybrid algorithms are, on average, lighter. The rounding-off assisted MAM can not solve the optimization problems as efficiently and precisely as the coordinate search and HJ-MAMs due to its simple search mechanism. HJ-MAM is the most robust method and its superiority over the other two hybrid algorithms and GA in solving complex optimization problems is demonstrated on the four examples, especially when the discreteness interval for discrete design variables is relatively large, for example the value of 1.0.

## 6.2 Design of a Pressure Vessel

Design optimization of the cylindrical pressure vessel capped at both ends by hemispherical heads shown in Fig. 6 is considered as the fifth example to demonstrate the efficiency and accuracy of three developed hybrid algorithms. This vessel design has been discussed by many researchers (Sandgren 1990, Kannan and Kramer 1994, Deb 1997, Coello 2000, Coello and Montes 2003, He and Wang 2007, Montes and Coello 2008, Kaveh and Talatahari 2010). The objective of the optimization problem is to minimize the total manufacturing cost of the vessel based on the combination of welding, material and forming costs. The vessel is designed for a working pressure of 3000 *psi* and a minimum volume of 750 $ft^3$ regarding the provisions of ASME boiler and pressure vessel code. The shell and head thicknesses should be multiples of 0.0625 *in*. The thickness of the shell and head is restricted to 2 *in*. The shell and head thicknesses must not be less than 0.625 *in* and 0.25 *in*, respectively. The design variables of the problem are $x_1$ as the shell thickness ($T_s$), $x_2$ as the spherical head thickness ($T_h$), $x_3$ as the radius of cylindrical shell ($R$), and $x_4$ as the shell length ($L$). The problem formulation is as follows:

$Minimize\ \ cost(x) = 0.6224x_3x_1x_4 + 1.7781x_3{}^2x_2 + 3.1611x_1{}^2x_4 + 19.8621x_3x_1{}^2$

Design variables: $\{x_1, x_2, x_3, x_4\}$, $x_1$ and $x_2$ are integer multiples of 0.0625

Subject to:

$$g_1(x) = 0.0193x_3 - x_1 \leq 0$$
$$g_2(x) = 0.00954x_3 - x_2 \leq 0$$
$$g_3(x) = 1296000 - (\pi x_3{}^2x_4 + \frac{4}{3}\pi x_3{}^3) \leq 0$$
$$g_4(x) = x_4 - 240 \leq 0$$

Bounds on the design variables: $0.625 \leq x_1 \leq 2,\ 0.25 \leq x_2 \leq 2,\ 10 \leq x_3 \leq 240,\ 10 \leq x_4 \leq 240$

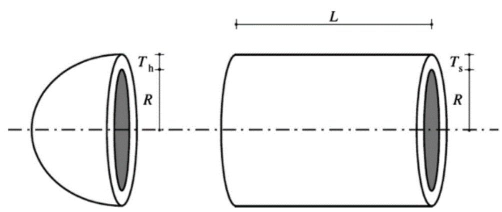


Fig. 6 The pressure vessel with indication of design variables

The comparison of results obtained by three developed hybrid algorithms and other methods has been shown in Table 13. The cost computed using HJ-MAM has been reduced to 5991.517 by 1.12% from 6059.0888, which was the best design referred by Kaveh and Talatahari (2010). In the optimization process, all these three hybrid MAM algorithms outperform GA not only in terms of the cost of the optimal design (the accuracy), but also the number of response analyses involved (the efficiency). It is noted that the optimized solution by HJ-MAM is the best feasible design (5991.517) since no violated constraints are observed in Table 14. The coordinate search assisted MAM has ability to seek the better design (6058.5456) than the solution (6092.8310) by rounding-off assisted MAM. However, the constraint $g_1$ is violated by 0.55% in the optimal design by the former shown in Table 14 and the numbers of iterations and response analyses have been also significantly increased, which are indicated in Table 13. Summarizing, all the three hybrid MAM algorithms have the superiority over GA in terms of the accuracy and efficiency. Rounding-off assisted MAM is the fastest algorithm with acceptable accuracy and HJ-MAM can find the best solution with a faster rate of convergence than coordinate search assisted MAM. It is advocated that the HJ-MAM with the overall best capability can be applied to effectively solve the complex mixed-variable optimization problems.

Table 13. Comparison of optimal designs of the pressure vessel

| Methods | $x_1(T_s)$ | $x_2(T_h)$ | $x_3(R)$ | $x_4(L)$ | cost | No. of iterations | No. of response analyses |
|---|---|---|---|---|---|---|---|
| Sandgren (1990) | 1.1250 | 0.625 | 47.7000 | 117.701 | 8129.8000 | - | - |
| Kannan and Kramer(1994) | 1.1250 | 0.625 | 58.2910 | 43.690 | 7198.2000 | - | - |
| Deb (1997) | 0.9375 | 0.500 | 48.3290 | 112.679 | 6410.3811 | - | - |
| Coello (2000) | 0.8125 | 0.4375 | 40.3239 | 200.000 | 6288.7445 | - | - |
| Coello and Montes (2003) | 0.8125 | 0.4375 | 42.0974 | 176.654 | 6059.9463 | - | - |
| He and Wang (2007) | 0.8125 | 0.4375 | 42.0913 | 176.747 | 6061.0777 | - | - |
| Montes and Coello (2008) | 0.8125 | 0.4375 | 42.0981 | 176.641 | 6059.7456 | - | - |
| Kaveh and Talatahari (2010) | 0.8125 | 0.4375 | 42.1036 | 176.573 | 6059.0888 | - | - |
| GA (HyperStudy 2012) | 0.8125 | 0.375 | 39.2436 | 215.720 | 6272.7006 | 40 | 2238 |
| Rounding-off assisted MAM | 0.8750 | 0.4375 | 45.3310 | 140.365 | 6092.8310 | 16 | 145 |
| Coordinate search assisted MAM | 0.8750 | 0.4375 | 45.6233 | 137.367 | 6058.5456 | 51 | 460 |
| Hooke-Jeeves technique assisted MAM | 0.750 | 0.375 | 38.1954 | 234.194 | 5991.517 | 28 | 253 |

Table 14. Constraint values of the mixed continuous-discrete optimization

| Methods | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| Sandgren (1990) | -0.204 | -0.170 | 0.000 | -122.299 |
| Kannan and Kramer(1994) | -2.9e-4 | -0.069 | -0.138 | -196.225 |
| Deb (1997) | -4.7e-3 | -0.0390 | -3652.877 | -127.321 |
| Coello (2000) | -3.4e-2 | -0.0528 | -27.106 | -40.000 |
| Coello and Montes (2003) | -2.0e-5 | -0.0359 | -27.886 | -63.346 |
| He and Wang (2007) | -1.4e-4 | -0.0359 | -116.383 | -63.254 |
| Montes and Coello (2008) | -7.0e-6 | -0.0371 | 2.94791 | -63.360 |
| Kaveh and Talatahari (2010) | 1.0e-5 | -0.0370 | -1.1823 | -63.427 |
| GA (HyperStudy 2012) | -5.5e-2 | -6.16e-4 | -865.498 | -24.280 |
| Rounding-off assisted MAM | -1.1e-4 | -0.0050 | -336.885 | -99.635 |
| Coordinate search assisted MAM | 5.5e-3 | -0.0023 | -50.591 | -102.633 |
| Hooke-Jeeves technique assisted MAM | -1.3e-2 | -0.0106 | -10777.078 | -5.806 |

## 7. Conclusions

Three proposed hybrid algorithms in this paper, combining the metamodel-based MAM and the different search techniques, have successfully solved the ten-bar planar truss structure with mixed discrete-continuous design variables. By comparing the optimal results with GAs and between themselves, the conclusions have been drawn that the developed hybrid algorithms have the ability to solve the complicated optimization problems with much higher computational efficiency than GAs. The influence of sizes of the discreteness interval (0.2 and 1.0) on the computational efficiency and preciseness of the solutions is analyzed in the ten-bar truss design example and the superiority of HJ-MAM algorithm over the other two hybrid algorithms and GA is also demonstrated by two case studies. HJ-MAM algorithm is the most robust method of three developed hybrid algorithms and as an ideal and efficient tool, it can be used to solve the complicated real-world problems in which it is essential to perform the mixed discrete-continuous optimization during the design process.

**Acknowledgement**

The author is thankful to Prof. Vassili Toropov from Queen Mary University of London for providing the necessary discussion for the preparation of the paper and the support from Natural Science Project of Education Office of Shaanxi Province, China (No.15JK1484).

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