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Nonstandard utilities for lexicographically decomposable orderings

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June 29, 2015

Abstract

Using a basic theorem from mathematical logic, I show that there are field-extensions of $\mathbb{R}$ on which a class of orderings that do not admit any real-valued utility functions can be represented by uncountably large families of utility functions. These are the lexicographically decomposable orderings studied in [1]. A corollary to this result yields an uncountably large family of very simple utility functions for the lexicographic ordering

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of the real Cartesian plane. I generalise these results to the lexicographic ordering of $\mathbb{R}^n$, for every $n > 2$, and to lexicographic products of lexicographically decomposable chains. I conclude by showing how almost all of these results may be obtained without any appeal to the Axiom of Choice.

1 Non-Archimedean utility functions

Utility functions may be seen as strong homomorphisms from a complete preorder $\mathsf{Z} = \langle Z, \preccurlyeq \rangle$ into a numerical ordering. Although a customary choice for the codomain of a utility function is the set $\mathbb{R}$ of real numbers, alternatives that violate the Archimedean property have been studied since at least the 1950's (see e.g. [6]). Two common choices of a non-Archimedean codomain have established themselves: lexicographically ordered vector spaces, usually $\mathbb{R}^n$, for some natural number n (see for instance [6], [3], [9]), and suitable non-Archimedean extensions of the reals, obtained by an ultrapower construction (see for instance [5], [11], [14])[1]. Two main motivations have led to employing these alternatives to the reals: on the one hand, utility functions with values in some non-Archimedean structure may exist even when real-valued utility functions do not exist; on the other hand, certain qualitative setups are better modelled by means of non-Archimedean utilities, which, for instance, allow one to assign infinitely small numbers to negligible features of a given problem (for instance, [11] pursues this approach in order to discriminate between main issues and side issues in the context of expected utility theory). Both motivations come together when the reason why real-valued utility functions do not exist may be ascribed to the fact that the underlying preference exhibits features that cannot consistently be captured by the reals. It is then worth looking for codomains alternative to the reals in order to capture more faithfully the structure of the

[1] This last approach comes from Nonstandard Analysis, a field created by Abraham Robinson in the 1960's (see [13]).

preference to be represented. In this paper I adopt this perspective on a class of linear orders that lack real-valued utility functions and show that each of them has an uncountable family of utility functions on an arbitrary, elementary extension of the real field containing positive infinitesimals. As a consequence of this result, I also establish a connection between lexicographically ordered, real vector spaces and elementary extensions of the reals, the two more prominent choices of non-Archimedean codomain for utility functions. The best known linear order from the class I consider was introduced by Debreu in [4]: it is the lexicographic ordering of the real Cartesian plane, i.e., the chain $\mathsf{L}_2 = \langle \mathbb{R}^2, \preccurlyeq_2 \rangle$, where the binary relation $\preccurlyeq_2$ is defined by the condition:

$$\langle r, s \rangle \preccurlyeq_2 \langle r', s' \rangle \text{ iff } r < r' \text{ or } r = r' \text{ and } s \leq s'.$$

Seen as a vector space, L_2 is non-Archimedean, since there is no positive, integer multiple of $\langle 0, 1 \rangle$ that is greater than $\langle 1, 0 \rangle$. This suggests that a utility representation for L_2 should assign to every vector of the form $\langle 0, n \rangle$ a value that is infinitely close to that of $\langle 0, 0 \rangle$. The same argument naturally extends to encompass the class of lexicographically decomposable chains described in [2] and studied in [1]. Since these chains are isomorphic to certain lexicographic orders, and one may regard lexicographic orders as a linear arrangement of clusters of infinitely close points, it is natural to associate them with a utility function on a non-Archimedean structure. Theorem 3.2 shows that uncountable families of such functions always exist[2]. This result locally improves in several ways the general theorem, proved by Skala in [14] and more concisely presented by Narens in [12], to the effect that every transitive and complete relation has a utility function on a particular ultrapower extension of the reals. I show in section 3 that, with regard to lexicographically decomposable chains,

[2]To be precise, I restrict attention to the lexicographically decomposable chains of cardinality $2^{\aleph_0}$. This restriction is reasonable, in view of the fact that structures whose power is strictly greater than the power of the continuum are not of central significance to mathematical economics

it is possible to bypass the ultrapower construction altogether while obtaining utility functions on an arbitrary elementary extension of the real field containing a positive infinitesimal (i.e., no special ultrafilter has to be specified to select the relevant elementary extensions). Furthermore, the analytical form of these functions can be explicitly given in a remarkably simple way. This sheds further light on the significance of [1] and [2], since the lexicographically decomposable chains isolated and examined in these papers do not only constitute an important class of linear orders lacking real-valued utility functions, but also turn out to be a class of linear orders admitting uncountably large families of analytically specifiable utility functions on non-Archimedean extensions of the reals. In view of the last fact, it is also possible to introduce uncountable families of utility functions for certain lexicographic products of linear orders, in particular $\mathbb{R}^n$, for every $n > 2$. This establishes an explicit connection between the use of lexicographic, real vector spaces and elementary extensions of the reals within utility theory: any vector-valued utility function on $\mathbb{R}^n$ can be converted into transfinitely many utility function on an arbitrary extension of the real field containing positive infinitesimals. To the best of my knowledge, this connection has not been noted in the literature so far: its significance lies in the fact that it connects two typical choices of non-Archimedean codomain for a utility function, thus yielding some immediate extensions of existing theorems. For example, the result recently obtained by Herzberg in [9], which establishes the existence of certain utility functions with codomain the vector space $\mathbb{R}^{n+1}$ (without any appeal to Nonstandard Analysis) directly implies the existence of uncountably many utility functions on an arbitrary nonstandard extension of the reals. The structure of the paper is as follows: in section 2 I illustrate a basic, ultraproduct-free, construction of field-extensions of the reals; in section 3 I rely on it to prove the existence of utility functions on nonstandard extensions of the real field for lexicographically decomposable chains; in section 4 I extend the same approach to lexicographic products of linear orders; in section 5 I show how one could obtain most of the results in this paper without appealing to the Axiom of Choice.

2 Infinitesimals and L_2

The utility functions I appeal to in the remainder of the paper take values in certain field-extensions of the reals containing positive infinitesimals. These can be constructed from the expanded real field $\mathsf{R} = \langle \mathbb{R}, <, +, \cdot, \{r\}_{r\in\mathbb{R}} \rangle$, where 'expanded' refers to the fact that $\mathbb{R}$ is endowed with a distinct constant for each real number. A first-order language $\mathscr{L}$ for R contains in its alphabet the quantifier $\exists$, the connectives $\vee, \neg$ (the remaining quantifier and connectives can be defined in terms of those listed), open and closed brackets and the symbol for equality, together with a relation symbol denoting $<$, function symbols for the field operations and uncountably many constant symbols, each naming a distinct real number. Let $\mathcal{ED}(\mathsf{R})$ be the elementary diagram of R, i.e., the set of all $\mathscr{L}$-sentences true in R: in particular $\mathcal{ED}(\mathsf{R})$ contains the axioms for ordered fields. Now, consider the set $\mathcal{C}$ of uncountably many inequalities $\{r < c\}_{r\in\mathbb{R}}$, witch c a constant symbol not in $\mathscr{L}$. Since any finite subset of $\mathcal{ED}(\mathsf{R}) \cup \mathcal{C}$ is satisfied by R, the compactness theorem of first-order logic implies that this set of sentences has a model $^*\mathsf{R}$, which is an ordered field and whose domain will be denoted by $^*\mathbb{R}$. In fact, R is elementarily embeddable in $^*\mathsf{R}$ and the latter structure is, as a result, an extension of the real field, since it contains a copy of every real and an additional positive element named by c, which is greater than every real number. Call its multiplicative inverse ϵ: because $c > n$, for every n, it follows (since the ordinary laws of arithmetic are in $\mathcal{ED}(\mathsf{R})$) that $0 < \epsilon < \frac{1}{n}$, i.e., ϵ is a positive infinitesimal. More generally, for every positive real number

r, ϵr is a positive infinitesimal: there are therefore uncountably many of them and their additive inverses are the negative infinitesimals. Now, if r is any real and $\mathbb{I}$ the set of all infinitesimals, the set $r + \mathbb{I} = \mu(r)$ is called the monad of r. When $r = 0$, $\mu(r) = \mathbb{I}$ and the fact that $s < t$ implies $\epsilon s < \epsilon t$ shows how the ordering of $\mathbb{R}$ can be encoded into $\mu(0)$: by translation, this ordering can be encoded into any monad. Intuitively, it is as if we could attach to each real number an order-isomorphic copy of $\mathbb{R}$ collapsed into a monad. The immediate significance of this remark lies in the fact that L_2 may be regarded as a real line, to each point of which a copy of $\mathbb{R}$ has been attached. In particular, we may take the line in question to be $y = 0$: to each point on it, of coordinates $(r, 0)$, the vertical line $x = r$ is attached, which is order-isomorphic to $\mathbb{R}$ under the projection map $(r, s) \mapsto s$. By collapsing each vertical line into a monad, one obtains a ${}^*\mathbb{R}$-valued utility function for L_2. More precisely:

Lemma 2.1. *Let $\epsilon \in {}^*\mathbb{R}$ be a fixed, positive infinitesimal. Then L_2 has a continuous ${}^*\mathbb{R}$-valued utility function defined by the condition $u(\langle r, s\rangle) = r + \epsilon s$.*

Proof. That u is a utility function follows from Theorem 3.2 in section 3. Continuity holds with respect to the order topology on L_2 and the interval topology on ${}^*\mathsf{R}$. To see this, it suffices to note that, relative to u, the pre-image of any open ray in ${}^*\mathsf{R}$ is open in L_2. □

This result is based on the fact that $\mathbb{R}^2$ admits a partition into equivalence classes (vertical lines) that behave like monads in the lexicographic ordering. This feature of L_2 can be spelled out informally by observing that a lexicographic ordering models a preference based on several graded criteria: when two are at play, the first takes precedence over the second unless the alternatives being compared are identical relative to the first criterion. In other words, however significant the second criterion is, it can never modify a preference based on the first criterion. This behaviour is aptly captured by a utility function that scales the second criterion by an infinitesimal factor, making it negligible with respect to the first. Note that, since there are uncountably many positive infinitesimals, there are uncountably many choices of scaling factor and this gives rise to an uncountable family of continuous utility functions for L_2 of the form described in the lemma 1. The next section proves it as the corollary of a more general result, which can be directly obtained from the analysis of lexicographically decomposable chains offered in [1].

3 Lexicographic Decompositions

Suppose that $\mathsf{Z} = \langle Z, \prec \rangle$ is a countably bounded and connected chain. Countable boundedness amounts to the fact that there is a sequence $\{d_i\}_{i \in \mathbb{N}} \subseteq Z$ such that $Z = \bigcup_{n,m \in \mathbb{N}} (d_m, d_n)$ and connectedness is the familiar topological property, relative the order topology on Z. Theorem 2.3-(a) of [1] implies the existence of a sub-chain included in Z which is order-isomorphic to $\mathbb{R}$, the real line[3]. The set X plays the role of the x-axis in the discussion of L_2 from the previous section. Its importance lies in the fact that it has a $\mathbb{R}$-valued utility function and, furthermore, it determines a partition of Z, which can be defined as follows, if $[z]$ denotes the cell of the partition that contains $z \in Z$:

$$[z] = \bigcap_{x,y \in X \wedge x \prec z \prec y} (x, y).$$

Note that, in L_2, the cell $[\langle x, 0 \rangle]$ is the vertical line through $\langle x, 0 \rangle$. Thus, the partition of the lexicographic ordering of $\mathbb{R}^2$ is just a special case of a general type of partition. The reader familiar with Nonstandard Analysis will have noted that the definition of the cell $[z]$ resembles the definition of a monad in an internal topological space: there one considers the family of open neighborhoods of a standard point x and intersects their *-images to obtain $\mu(x)$. The partitions

[3] The original result is stated relative to $(0, 1)$, which is order-homeomorphic to $\mathbb{R}$.

of chains just described mirror this process and suggest that a natural utility function for an object like Z should take values on a numerical domain that contains infinitesimals. In order to show that this is indeed the case, it is possible to rely on Theorem 2.3-(b) of [1], to the effect that there is an order-isomorphism between Z and the lexicographic ordering on the set $\bigcup_{x \in X}(x \times [x])$. The last set may be called the lexicographic decomposition of Z. If each cell in the lexicographic decomposition had a $\mathbb{R}$-valued utility function, then it would be possible to represent Z on $^*\mathbb{R}$. As a result, one can convert Proposition 2.2 of [1], stated below, into an existence theorem for utility functions.

Lemma 3.1. *Let Z be a non-representable chain, and let X be a representable sub-chain $X \subset Z$ so that Z admits the decomposition $\bigcup_{x \in X}\{x\} \times [x]$. If [x] is representable for every $x \in X$, then Z is a planar chain.*

A planar chain is a linear ordering one of whose subsets is order-isomorphic to a subset of L_2 without a $\mathbb{R}$-valued utility function. Although the original motivation of the above proposition was simply to describe an ordering that has no $\mathbb{R}$-valued utility functions, the information it provides suffices to deduce an existence theorem. To state it, call a function $u : Z \longrightarrow {}^*\mathbb{R}$ a properly $^*\mathbb{R}$-valued utility function for Z iff the range of u is not a subset of $\mathbb{R}$. Then:

Theorem 3.2. *Let Z be a countably bounded chain with no largest and no smallest element. If every cell in its lexicographic decomposition has a $\mathbb{R}$-valued utility function, then Z has uncountably many $^*\mathbb{R}$-valued utility functions.Moreover, if Z is non-representable, then the utility functions are properly $^*\mathbb{R}$-valued.*

Proof. By earlier remarks in this section, Z is order-isomorphic to the lexicographical ordering $\mathsf{L}_X = \langle \bigcup_{x \in X}\{x\} \times [x], \preccurlyeq_X \rangle$. Here $x \in X$ and X is a subchain of Z that is order-isomorphic to $\mathbb{R}$. The ordering is defined by the following biconditional:

$$\langle x, y\rangle \preccurlyeq_X \langle x', y'\rangle \text{ iff either } x \prec x \text{ or } x = x' \text{ and } y \preccurlyeq y',$$

where $\prec, \preccurlyeq$ are the asymmetric and symmetric part of the linear order on Z respectively. First, fix an isomorphism $u : X \to \mathbb{R}$. Next, since each cell $[x]$ has a $\mathbb{R}$-valued utility function u_x, define a function u^* for L_X using the following condition:

$$u^*(\langle x, y\rangle) = u(x) + \epsilon u_x(y),$$

where $y \in [x]$ and ϵ is a fixed, positive infinitesimal. It remains to verify that u^* is indeed a utility function, i.e., that:

$$\langle x, y\rangle \preccurlyeq_X \langle x', y'\rangle \text{ iff } u^*(\langle x, y\rangle) \leq u^*(\langle x', y'\rangle).$$

Suppose first that $\langle x, y\rangle \preccurlyeq_X \langle x', y'\rangle$. By definition of lexicographic ordering this means that either (i) $x \prec x'$ or (ii) $x = x'$ and $y \preccurlyeq y'$. In case (i) $u(x) < u(x')$ iff $u(x') - u(x) > 0$, because u is an order-isomorphism. To verify $u(x) + \epsilon u_x(y) < u(x') + \epsilon u_{x'}(y')$ iff $u'(x) - u(x) > \epsilon(u_x(y) - u_{x'}(y'))$ it is enough to note that $u(x') - u(x)$ is a positive real, larger than any infinitesimal. Thus, certainly $u^*(\langle x, y\rangle) < u^*(\langle x', y'\rangle)$. In case (ii) $x = x'$ implies $y, y' \in [x] = [x']$. Since y, y' are in the same cell, $y \preccurlyeq y'$ iff $u_x(y) \leq u_x(y')$. Multiplication by a positive infinitesimal preserves inequalities, so $u(x) + \epsilon u_x(y) \leq u(x') + \epsilon u_x(y')$ and, again, $u^*(\langle x, y\rangle) < u^*(\langle x', y'\rangle)$. This shows that $\langle x, y\rangle \preccurlyeq_X \langle x', y'\rangle$ implies $u^*(\langle x, y\rangle) \leq u^*(\langle x', y'\rangle)$. In fact, the argument just given shows that strict inequalities are also preserved when $x \prec x$ (this fact will be relied upon later). The next step is to verify that $u^*(\langle x, y\rangle) \leq u^*(\langle x', y'\rangle)$ implies $\langle x, y\rangle \preccurlyeq_X \langle x', y'\rangle$. To this end, assume $u^*(\langle x, y\rangle) \leq u^*(\langle x', y'\rangle)$. By the law of excluded middle, either $x = x'$ or $x \neq x'$. In the former case $u(x) = u(x')$ and $y, y' \in [x]$ hold. Then $u(x)+\epsilon u_x(y) \leq u(x')+\epsilon u_x(y')$ implies $\epsilon u_x(y) \leq \epsilon u_x(y')$ iff $u_x(y) \leq u_x(y')$, since positive infinitesimals have multiplicative inverses. The fact that u_x is a strong homomorphism finally yields $y \preccurlyeq y'$ and $\langle x, y\rangle \preccurlyeq_x \langle x', y'\rangle$. If, on the other hand, $x' \neq x$, it suffices to rule out $x' \prec x$. But $x' \prec x$ implies $\langle x', y' \rangle \preccurlyeq_X \langle x, y \rangle$, which in turn implies $u^*(\langle x', y' \rangle) < u^*(\langle x, y \rangle)$, contradicting the hypothesis. It follows that $x \preccurlyeq x'$ and, since x, x' are distinct, $x \prec x'$ (X is a chain), which in turn leads to $\langle x, y \rangle \preccurlyeq_X \langle x', y' \rangle$. This concludes the verification that u^* is a utility function. Note that u^* depends on a choice of infinitesimal, and there are uncountably many possible choices. It follows that there are uncountably many ${}^*\mathbb{R}$-valued utility functions for the lexicographic ordering L_X. Since there is an order-isomorphism f from Z onto L_X, the composition $u^* \circ f$, for each of uncountably many possible u^*, is a ${}^*\mathbb{R}$-valued utility function for Z.

□

Lemma 2.1 from section 2 is a special case of Theorem 3.2 because L_2 is order-isomorphic (under the obvious function) to the lexicographic decomposition:

$$\bigcup_{x \in \mathbb{R}} \{x\} \times [\langle x, s \rangle]$$

which one can represent choosing u and u_x from Theorem 1 to be the identity and the projection function $\langle x, s \rangle \mapsto s$ respectively. In this special case it is clear that each cell in the partition has a $\mathbb{R}$-valued utility function, a fact that has been assumed in the statement of Theorem 3.2. This assumption can be replaced by sufficient conditions (given in Theorem 2.4 of [1]) under which a lexicographic decomposition of a chain into cells with $\mathbb{R}$-valued utility functions exists. In order to understand how these conditions work, it is important to bear in mind that, if Z is a countably bounded chain without extrema, then it admits a lexicographic decomposition based on a subchain X into cells that are intervals and, thus, connected in the interval topology. If these cells have no real-valued utility functions, one can find a larger subchain $X' \supset X$ that gives rise to a lexicographic decomposition whose cells split the cells of the initial decomposition. In this case X' is a refinement of X. The trick consists in finding a condition ensuring the existence of a maximal refinement whose cells, because they can't be further split, will have $\mathbb{R}$-valued utility functions. The relevant condition is stated in [1], p.101, where it is called σ-finiteness. It amounts to the fact that, for every X determining a lexicographic decomposition, every increasing $\subset$-chain of refinements is $\subset$-bounded by a countable subchain (i.e., there is a subchain $\mathcal{X} = \{X_i\}_{i\in\mathbb{N}}$ such that any X' in the chain is refined by an element of $\mathcal{X}$). With this condition in place, Theorem 3.2 immediately implies a counterpart of Theorem 2.4 from [1]:

Theorem 3.3. *Let* Z *be a countably bounded, connected and* σ*-finite chain that has no* $\mathbb{R}$*-real valued utility functions. Then* Z *has uncountably many* ${}^*\mathbb{R}$*-valued utility functions.*

4 Lexicographic Products

The conditions for the existence of a lexicographic decomposition described in the previous section make it possible to introduce ${}^*\mathbb{R}$-valued utility functions for chains that cannot be embedded into the ordered reals by a strong homomorphism. This may be seen as a deconstructive strategy: it begins with a particular type of chain Z and it produces an isomorphic copy that can be broken down into components with real-valued utility functions. In view of section 3, this strategy can be reversed: one can consider chains with $\mathbb{R}$-valued, or in fact ${}^*\mathbb{R}$-valued utility functions, and take their lexicographic product, which will turn out to have uncountably many ${}^*\mathbb{R}$-valued utility functions. If A, B are chains with domains $\mathbb{A}, \mathbb{B}$ respectively, their lexicographic product, denoted by $\mathsf{A} \circ_{\mathsf{L}} \mathsf{B}$, is the lexicographic ordering on the Cartesian product $\mathbb{A} \times \mathbb{B}$, determined by the corresponding orderings of the given chains. The results discussed in section 3 offer a sort of bootstrapping technique to obtain utility functions for lexicographic products, since now it is no longer necessary to describe decompositions into elements that have $\mathbb{R}$-valued utility functions but it is possible to construct lexicographic products of chains that have $^*\mathbb{R}$-valued utility functions. The relevant procedure requires specific reference to one subset of $^*\mathbb{R}$, namely the set $\mathbb{F}$ of finite numbers, defined as follows:

$$x \in \mathbb{F} \text{ iff there is } n \in \mathbb{N} \text{ such that } |x| < n.$$

In the present, special sense, finiteness is the property of not being infinitely large. The utility functions described in the previous sections are not only $^*\mathbb{R}$-valued but also $\mathbb{F}$-valued. This restriction is necessary to collapse the 'negligible' components of a lexicographic ordering into monads, since the product of an infinitesimal and an infinitely large number may be finite and not infinitesimal. With this notion of finiteness in mind, one can adapt the main argument of section 3 to obtain the following:

Lemma 4.1. *Let* A, B *be chains with domains* $\mathbb{A}, \mathbb{B}$ *respectively and suppose that* A *has a* $\mathbb{R}$*-valued utility function* f *and* B *has a* $\mathbb{F}$*-valued utility function* g*. Then there are uncountably many* $\mathbb{F}$*-valued utility functions for* $\mathsf{A} \circ_{\mathsf{L}} \mathsf{B}$ *of the form:* $f + \epsilon g$*, where* ϵ *is a positive infinitesimal.*

Proof. Let $\leq_{\mathsf{A}}, \leq_{\mathsf{B}}$ be the total orderings defined on $\mathbb{A}, \mathbb{B}$ respectively. Then the lexicographic product $\mathbb{A} \circ_{\mathsf{L}} \mathbb{B}$ is ordered by the relation $\leq_{\mathbb{AB}}$ defined as follows:

$$\langle a, b\rangle \leq_{\mathbb{AB}} \langle a', b'\rangle \text{ iff } a <_{\mathsf{A}} a' \text{ or } a = a' \text{ and } b \leq_{\mathsf{B}} b'.$$

It suffices to verify that:

$$\langle a, b\rangle \leq_{\mathbb{AB}} \langle a', b'\rangle \text{ iff } f(a) + \epsilon g(b) \leq f(a') + \epsilon g(b').$$

First, suppose that $\langle a, b\rangle \leq_{\mathbb{AB}} \langle a', b'\rangle$. If $a <_{\mathsf{A}} a'$, then $f(a) < f(a')$ and $f(a), f(a')$ differ by a real number. Since $g(b), g(b') \in \mathbb{F}$ by hypothesis, $\epsilon g(b), \epsilon g(b')$ are infinitesimals (because $\mathbb{I}$ is an ideal in the ring $\mathbb{F}$) and an argument already provided in Theorem 3.2 shows that $f(a)+\epsilon g(b) < f(a')+\epsilon g(b')$. If, on the other hand $a = a'$, then $f(a) = f(a')$ and the result follows from the fact that multiplication by a positive infinitesimal preserves inequalities. To prove the converse conditional, suppose that $f(a)+\epsilon g(b) \leq f(a')+\epsilon g(b')$ but $\langle a, b\rangle \leq_{\mathbb{AB}} \langle a', b'\rangle$ fails. In this case, either $a' <_{\mathsf{A}} a$ or $a' = a$ and $b' <_{\mathsf{B}} b$. By the previous part of the proof, the first possibility cannot arise, so $a = a'$. In this case $f(a) = f(a')$ and $\epsilon g(b) \leq \epsilon g(b')$: since ϵ has a positive, multiplicative inverse, $g(b) \leq g(b')$ iff $b \leq_{\mathsf{B}} b'$, a contradiction that concludes the proof. □

It follows from the last lemma that $\mathsf{L}_1 \circ_{\mathsf{L}} \mathsf{L}_2 = \mathsf{L}_3$, the lexicographic ordering of $\mathbb{R}^3$, is represented by uncountably many functions of the form $f + \epsilon g$, where one can take f to be the identity on $\mathbb{R}$ and choose g such that $g(\langle s, t\rangle) = s + \epsilon t$, in the light of Lemma 2.1. Varying the choice of ϵ in $f + \epsilon g$, one obtains uncountably many 'quadratic' utility functions for L_3. The choice of ϵ described above yields a utility function defined by the following condition: $u(\langle r, s, t\rangle) \mapsto r + \epsilon s + \epsilon^2 t$. The same strategy works on $\mathsf{L} \circ_{\mathsf{L}} \mathsf{L}_3 = \mathsf{L}_4$ and, after sufficiently many iterations, it applies to L_n, for every $n \in \mathbb{N}$. A connection thus arises between lexicographically ordered real vector spaces and nonstandard extension of the reals, which is spelled out in the following:

Corollary 4.2. L_n *has uncountably many, continuous utility functions on an arbitrary field-extension of the reals, each of which satisfies an equation of the form:*

$$u(\langle r_1, \ldots, r_n\rangle) = r_1 + \epsilon r_2 + \ldots + \epsilon^{n-1} r_n.$$

Here continuity can be established for the relevant interval topologies by noting that a strongly order-preserving function between two chains is continuous. The construction of utility functions for lexicographic orderings is not restricted to the real case, since one easily obtains a further corollary, which applies to the abstract setting described in section 3:

Corollary 4.3. *Let* Z *be as in theorem 3.3 and let* $X \subseteq Z$ *determine a maximal partition of* X *into representable cells. Then there are uncountably many* ${}^*\mathbb{R}$*-valued utility functions for* $X \circ_{\mathsf{L}} Z$*.*

A natural question is whether families of utility functions exist for lexicographic products of lexicographically decomposable chains like those satisfying the hypotheses of Theorem 3.3. The answer, in the affermative, is most clearly articulated by focusing at first on the lexicographic product $\mathsf{Z}_2 = \mathsf{Z} \circ_L \mathsf{Z} = \langle Z^2, \prec_{Z^2} \rangle$, where Z is a lexicographically decomposable chain. In view of Theorem 3.2, the following inequalities hold:

$$u^* \circ f(z) = u^*(\langle x_z, y_z \rangle) = g(x_z) + \epsilon h_{x_z}(y_z),$$

where $\langle x_z, y_z \rangle$ is the image of $z \in Z$ under some isomorphism f from Z into its lexicographic decomposition, $y_z \in [x_z]$ and g is real-valued. Keeping f fixed and letting $\langle z_1, z_2 \rangle$ be a generic element of Z^2, it is now possible to define a function u by the condition $u(\langle z_1, z_2 \rangle) \mapsto u^* f(z_1) + \epsilon^2 u^* f(z_2)$. It can be verified that u is a utility function for Z_2 by exploiting of the following equalities:

$$\begin{aligned} u(\langle z_1, z_2 \rangle) &= u^* f(z_1) + \epsilon^2 u^* f(z_2) \\ &= u^*(\langle x_{z_1}, y_{z_1} \rangle) + \epsilon^2 u^*(\langle x_{z_2}, y_{z_2} \rangle) \\ &= g(x_{z_1}) + \epsilon h_{x_{z_1}}(y_{z_1}) + \epsilon^2 (g(x_{z_2}) + \epsilon h_{x_{z_2}}(y_{z_2})), \end{aligned}$$

where $g, h_{x_{z_i}}$ $(i = 1, 2)$ are real-valued functions. The biconditional to be verified is then:

$$\langle z_1, z_2 \rangle \preccurlyeq_{Z^2} \langle w_1, w_2 \rangle \text{ iff } u(\langle z_1, z_2 \rangle) \leq u(\langle w_1, w_2 \rangle),$$

where the right-hand side in expanded form is:

$$\begin{gathered} g(x_{z_1}) + \epsilon h_{x_{z_1}}(y_{z_1}) + \epsilon^2 (g(x_{z_2}) + \epsilon h_{x_{z_2}}(y_{z_2})) \leq \\ g(x_{w_1}) + \epsilon h_{x_{w_1}}(y_{w_1}) + \epsilon^2 (g(x_{w_2}) + \epsilon h_{x_{w_2}}(y_{w_2})). \end{gathered}$$

The verification is tedious because it splits into several cases, but not substantially different from the proof of Theorem 3.2. For example, it may be assumed that $\langle z_1, z_2\rangle \prec_{Z^2} \langle w_1, w_2\rangle$ holds and that, in addition $z_1 \prec w_1$. In this case, since f is an isomorphism from Z onto its lexicographic decomposition, the inequality $f(z_1) \prec f(w_1)$ holds[4]. This can be rewritten as $\langle x_{z_1}, y_{z_1}\rangle \prec \langle x_{z_2}, y_{z_2}\rangle$. If $x_{z_1} \prec x_{z_2}$, then $g(x_{z_2}) - g(x_{z_1})$ is a positive real number and this suffices to conclude that, in this particular case, $u(\langle z_1, z_2\rangle) < u(\langle w_1, w_2\rangle)$. If $x_{z_1} = x_{z_2}$, then it follows that $y_{z_1} \prec y_{z_2}$ and that $y_{z_1}, y_{z_2} \in [x_{z_1}]$. Some field arithmetic eventually yields $u(\langle z_1, z_2\rangle) < u(\langle w_1, w_2\rangle)$. The other cases are dealt with similarly. In the light of the existence of utility functions for Z_2, one may adapt the argument just sketched to the lexicographic product $\mathsf{Z} \circ_{\mathsf{L}} \mathsf{Z}_2 = \mathsf{Z}_3$. The relevant family of utility functions is now described by the condition:

$$u(\langle z_1, z_2, z_3\rangle) = \\ g(x_{z_1}) + \epsilon h_{x_{z_1}}(y_{z_1}) + \epsilon^2(g(x_{z_2}) + \epsilon h_{x_{z_2}}(y_{z_2})) + \epsilon^4(g(x_{z_3}) + \epsilon h_{x_{z_3}}(y_{z_3})).$$

In general:

Lemma 4.4. *Let* Z *be as in Theorem 3.3 and* f *be an isomorphism from* Z *into its lexicographic decomposition. Then* Z_n *has uncountably many* ${}^*\mathbb{R}$*-valued utility functions satisfying the equality:*

$$u(\langle z_1, \dots, z_n\rangle) = g(x_{z_1}) + \epsilon h_{x_{z_1}}(y_{z_1}) + \epsilon^2 g(x_{z_2}) + \epsilon^3 h_{x_{z_2}}(y_{z_2}) + \dots + \\ \epsilon^{2(n-1)} g(x_{z_n}) + \epsilon^{2(n-1)+1} h_{x_{z_n}}(y_{z_n}).$$

5 Dispensing with Choice

The construction of a field-extenstion ${}^*\mathsf{R}$ relies on an application of the compactness theorem of first-order logic for uncountable languages. As such, it presupposes the Axiom of Choice (AC). The theorems from sections 3 and 4,

[4]Here $\prec$ is the order relation on the lexicographic decomposition of Z.

however, can still be proved without appealing to AC because they do not require field-extensions of the reals, strictly speaking, but only extensions with positive infinitesimals. In other words, one may renounce some field structure in order not to invoke AC, and rely on a numerical domain that, despite not being field, includes a copy of the real field as well as positive infinitesimals. This can be done in an easy and surprisingly fruitful way, pointed out and developed by James Henle in [7] and [8]. The framework described by Henle is known as Non-nonstandard analysis: only its basics are needed in the present context. Henle's leading idea amounts to breaking off the ultrapower construction of a nonstandard model of the real field just before introducing the ultrafilter, whose existence is guaranteed by AC. In other words, one may just work with the ring $\mathcal{R}$ of all real-valued sequences and determine an equivalence relation $\sim$ on them as follows:

$$a \sim b \text{ iff there is } n \text{ such that for every } m > n \; a_m = b_m$$

where a, b are two real-valued sequences. The $\sim$-equivalence classes determine the set of non-nonstandard reals ${}^*\mathcal{R}$. Although still a ring, ${}^*\mathcal{R}$ is not totally ordered when one defines order by the stipulation:

$$\mathbf{a} \leq \mathbf{b} \text{ iff there is } n \text{ such that for every } m > n \; a_m \leq b_m.$$

where $\mathbf{a}, \mathbf{b}$ are $\sim$-equivalence classes containing a, b respectively. Furthermore, ${}^*\mathcal{R}$ has zero divisors and thus is not a field. On the other hand, it contains positive infinitesimals: one of them is the equivalence class $\mathbf{e}$ containing the sequence defined by the condition $a_n = \frac{1}{n}$, which is positive according to the definition of ordering given above and smaller than any $\sim$-class containing a positive constant sequence, i.e., the counterpart of a standard real number in ${}^*\mathcal{R}$. The theorems from sections 3 and almost all theorems from section 4 continue to hold if one works with ${}^*\mathcal{R}$-valued functions. On the other hand, the proofs of continuity in lemma 2.1 and corollary 4.2 rely on the fact that the codomain of a utility function should be a linear order.

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