Coalition bargaining in repeated games*

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Abstract

We consider an intertemporal framework where different coalitions interact repeatedly over time. *Both* the terms of trade and the endogenous cooperation structure are characterized, in a protocol-free manner, when:

(C1) A coalition is formed with positive probability if, and only if, the shares obtained by its members weakly exceed their respective share expectations.

(C2) Each matched coalition distributes the entire surplus among its members.

(C3) Members of any coalition are treated symmetrically with respect to their share expectations.

We show, in particular, that the cooperation structure and the shares are unique when the game ends each date with vanishing probability.

Keywords repeated multi-coalitional games; coalition formation; network games

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1 Introduction

The lion's share of economic activity takes place in groups and organizations that cooperate repeatedly. For example, in production economies employees meet regularly in their respective firms and each meeting results in the production of some output. Similarly, goods and services are repeatedly exchanged among a given set of traders. Our objective in this paper is to provide a general approach to these *intertemporal* situations that, abstracting from the varying features and contrasting implications of specific matching and bargaining protocols, provides a unified and robust understanding of the main issues involved.

Formally, our model considers an environment where multiple productive coalitions of agents can meet over time, each of them limited to participating in one coalition at a time.[1] Then, each agent demands a share of the created (and transferable) surplus taking into account her opportunities in other coalitions. Such demanded shares in turn determine the probability with which coalitions may form over time. As stressed, our approach to the problem abstracts from the particular mechanism that may be at work in the process of formation of the different coalitions and their internal bargaining. Instead, our aim is to characterize the outcome (i.e. payoff shares and coalitions formed) under the assumption that, independently of the aforementioned procedural details, the following conditions hold at any point in the process:

1. Coalitions formed with positive probability are those, and only those, in which each member obtains at least her expected share.

2. Each matched coalition distributes the entire surplus (as shares) among its members.

[1] Therefore, non-intersecting coalitions can meet at each date and produce a separate surplus.

3. Members of any coalition are treated symmetrically with respect to their expected shares when the surplus of this coalition is distributed.

The first condition determines the cooperation structure given some expected payoff profile over the whole game. It asserts that a coalition will meet whenever it generates enough surplus to satisfy the expectations of all its members. The second condition requires that all surplus is distributed among any set of agents who decide to form a coalition. Importantly, this condition rules out transfers to non-members. The third condition embodies fairness in the sense of egalitarian treatment of (rational) expectations

As implicitly suggested in our motivation of the model, we can think of the payoff shares that satisfy the former three conditions as the outcome of dynamic negotiations undertaken by several parties prior to the actual repeated cooperation. Relatedly, they can be thought of as embodying some appealing normative criteria that agents will abide by, and insist upon, when choosing the way in which to allocate their resources. Both views are largely reflected in the following quote from Shapley and Shubik (1972, p. 116):

"A prudent 'economic' man [...] would be loath to enter a partnership for a stated share of the proceeds until he had satisfied himself that more favorable terms could not be obtained elsewhere. We can imagine that each player would set a price on his participation, and that no contracts would be signed until the prices [...] are in harmony."

There is a vast literature on surplus sharing and coalition formation. One branch of this literature is based on non-cooperative models, as surveyed by Ray (2007). Our approach can be seen as complementary to that pursued by this branch as it abstracts from the details of the matching and bargaining procedures and turns instead its attention to the properties that any outcome should satisfy independently of the specific protocol. This is relevant because it is well-known that, in general, small changes to the postulated rules can have a major impact on equilibrium outcomes. There is, therefore, the concern that such a theoretical approach may not be robust to minor modelling details. By way of illustration, the models considered in Baron and Ferejohn (1989), Chatterjee et al. (1993), Hart and Mas-Colell (1996), Krishna and Serrano (1996), or Okada (2011) lead to quite different outcomes (e.g. the Shapley value, the nucleolus, or a point belonging to the core) depending on the specific features of the matching and bargaining environment contemplated in each case.

Protocol-dependence is not an issue for the strand of the literature that relies on cooperative game theory and, as in our present approach, is axiomatic and outcome-based. The stable set (von Neumann–Morgenstern, 1944), core (Gillies, 1953; Shapley, 1953), Shapley value (Shapley, 1953) and the bargaining set (Aumann and Maschler, 1964) are prominent examples in the long tradition concerned with such an axiomatic characterization of solutions to cooperative games.[2] These concepts, however, are not designed to predict coalition formation. They answer the fundamentally different question of how the coalitional gains should be distributed provided that the grand coalition (or some other given coalition structure) has formed. Moreover, these concepts offer solutions for one-shot games rather than games that are played repeatedly. There is, however, some literature that adds an intertemporal dimension to cooperative games (e.g., Lehrer, 2003; Predtetchinski et al., 2006; Lehrer and Scarsini, 2013). This literature is concerned with a context very different from ours in that the payoffs accrue to coalitions over time and any coalition that forms is required to be

[2] Some cooperative solutions allow players/coalitions to contemplate alternative paths of the game as in Chwe (1994) or in Ray and Vohra (2015), who modified the concept of the stable set to incorporate farsighted behavior. The sequentiality of the contemplated moves is, however, hypothetical and not factual as in our game.

robust to “internal" deviations at intermediate stages of the payoff-accumulation process. Its main focus is also on extensions of the static solution concepts of the classical cooperative literature to such a context.

Although our framework shares with the classical cooperative game theory its axiomatic approach, it differs from its standard solutions in two important respects. Firstly, it is explicitly designed to predict *both* coalition formation and surplus sharing. Secondly and more importantly, it is driven by the *dynamic* structure of our game and is not restricted by an (arbitrary) time limit. It is precisely this structure that allows for the joint computation of the matching probabilities and shares, relying alone on the Conditions 1-3 that embody their mutual consistency.

In particular, our Conditions 1 and 3 formalize the idea that expectations should play a prominent role in the computation of players' shares. “Rational" (in the sense of consistent) expectations have, of course, long been an integral part of equilibrium analysis in non-cooperative game theory. Recently, they have also found their way into cooperative game theory (see e.g., Jordan, 2006; Dutta and Vohra, 2017). There is also substantial empirical evidence that expectations, as much as fairness, are important drivers of human behavior (see, e.g., Sunder, 1995; Harrison and McCabe, 1996; Bohnet and Zeckhauser 2004). In particular, expectations can induce (or be the result of) prevailing social conventions – see Acemoglu and Jackson (2015) for a model of this reciprocal dependence. It is natural, therefore, to think of expectations as setting a benchmark. The idea that agents compare a bargaining outcome to their respective benchmarks relates our work to the solution concepts based on aspirations (e.g., Albers, 1979; Benett, 1983) and claims (e.g., Chun and Thomson, 1992; Marco, 1995).

In summary, this paper is a novel approach to repeated coalitional interactions. It proposes a solution to the ubiquitous problem of repeated matching and bargaining without imposing an arbitrary time limit or protocols on these processes. As a positive concept, this solution aims at providing predictions that are not sensitive to modelling details. Alternatively, it can be also interpreted as a normative concept used, e.g., by a mediator to induce outcomes with some desirable properties.

The rest of the paper is organized as follows. In Section 2, we introduce the repeated coalitional game with random termination times, show that there always exist shares and probabilities that satisfy Conditions 1-3 (Proposition 2) and characterize them in terms of the expected shares (Proposition 1). In the limit case of vanishing termination probability, we obtain the unique set of cooperating coalitions and unique shares that satisfy the aforementioned conditions and provide implicit formulae for their computation (Theorem 1). Proposition 3 (and Proposition 5 in the Appendix) provide then a number of interesting properties that our solution concept satisfies in the limit case. Next, in Section 3 we apply our abstract theoretical framework to a trading context where agents display heterogeneous valuations and can buy or sell an homogeneous good repeatedly as allowed by some underlying network. When trading possibilities are unrestricted (i.e. the underlying network is complete), we show that all transactions are carried out at a uniform price (Proposition 4). However, we also show that when the underlying network is incomplete, trading at different prices can arise with some agents playing the role of (endogenous) arbitrageurs who buy and sell the good. Section 4 concludes with a summary. All proofs of our results, and some complementary results, are found in the Appendix.

2 Consistent Probabilities and Shares

We denote by $N = \{1, ..., n\}$ a finite set of players, by 2^N the set of all subsets of N and we set $\Theta \equiv 2^N \backslash \emptyset$. The value function $v : 2^N \to \mathbb{R}$ is zero-normalized, $v(\emptyset) = 0$, and for each coalition $C \in \Theta$, it assigns the (possibly negative) total surplus $v(C)$ that C generates if actually formed. We consider an intertemporal game[3] $\Gamma(v, \delta)$ that unfolds over an infinite number of discrete dates. At the start of each date, the game either terminates with probability $1 - \delta$ or continues with the complementary probability $\delta \in [0, 1]$.[4] In the former case, all players leave the game with zero payoffs. Otherwise, at most[5] one (possibly empty) coalition $C \in 2^N$ may form with probability π^C and create the surplus $v(C)$ to be divided among the members $i \in C$ according to the shares $\{\varphi_i^C\}_{i \in C}$. Each share φ_i^C can be interpreted as the price that i receives for her input or cooperation in coalition C.

Probabilities $\{\pi^C\}_{C \in \Theta}$, where $\pi^\emptyset = 1 - \sum_{C \in \Theta} \pi^C \geq 0$, and shares $\{\varphi_i^C\}_{C \in \Theta, i \in C}$ are assumed publicly known and stationary, i.e. constant over time. Jointly, these shares and probabilities allow us to compute, for each agent $i \in N$, a reference payoff that will play a central role in our analysis. This is the share that i expects to receive in her *first* subsequent cooperation in some active coalition taking into account that the game may terminate before this happens. *Ex ante*, a first such active role may materialize at different points in the future, provided the process continues to operate. Or, it may not materialize if the game

[3] We refer to the described intertemporal interaction as a game although, strictly speaking, it does not fit the conventional definitions in non-cooperative game theory.

[4] Alternatively, δ can also be regarded as a time discount factor, applied to future utilities or, more generally, as the product of the continuation probability and the discount factor. Nothing substantial changes in our analysis by this reinterpretation.

[5] This assumption simplifies the notation. Our results hold when any number of non-intersecting coalitions can meet each period.

ends before i becomes active. In this latter case, agent i assumes a payoff equal to zero when computing the expectation. Accordingly, the *expected share*, which we simply denote by φ_i, is defined as follows:

$$\text{(1)} \qquad \begin{aligned} \varphi_i &\equiv \delta \left[\textstyle\sum_{C\in\Theta:i\in C} \pi^C \varphi_i^C + (1 - \sum_{C\in\Theta:i\in C} \pi^C)\varphi_i \right] \\ &= \sum_{C\in\Theta:i\in C} \frac{\delta\pi^C}{\delta \sum_{C\in\Theta:i\in C} \pi^C + 1 - \delta}\, \varphi_i^C . \end{aligned}$$

The first line in (1) defines the expected share φ_i per future cooperation for player i and contemplates two possibilities. One of them, occurring with probability $1 - \delta$, is that the game ends before the next period arrives. This would result in all players earning zero payoffs and is omitted in that expression for the sake of simplicity. Alternatively, with the complementary probability δ, the game continues. In this case, player i envisages one of two possibilities:

- she cooperates in one of the active coalitions C where she is a member and obtains the share φ_i^C with corresponding probability π^C;
- she remains inactive and anticipates holding an unchanged share expectation φ_i thereupon.

Note that the former derivations rely on our stationarity assumption on matching probabilities and shares. The second line in (1) follows by simple algebraic manipulation and expresses φ_i as a weighted average of shares that i receives in coalitions where she is a member. The corresponding weights are proportional to the matching probabilities and they sum up to less than (equal to) unity when $\delta < 1$ ($\delta = 1$).[6] One can think of φ_i as a benchmark by which i decides whether to participate or not in a coalition and the minimal

[6] If we have $\delta = 1$ and $\pi^C = 0$ for all $C \in \Theta$ such that $i \in C$, then we set $\varphi_i = 0$ by convention.

demand she makes in case of participation.

As anticipated in Section 1, three conditions alone characterize the probabilities $\{\pi^C\}_{C\in\Theta}$ and shares $\{\varphi_i^C\}_{C\in\Theta,i\in C}$ that define our proposed solution to the intertemporal game. These conditions appear below:

(C1) *Coalition formation compatible with payoff expectations*:

$$\forall C \in \Theta,\ \ \pi^C > 0 \Leftrightarrow [\forall i \in C,\, \varphi_i^C \geq \varphi_i]. \tag{2}$$

(C2) *Local efficiency*:

$$\forall C \in \Theta,\ \ v(C) = \textstyle\sum_{k\in C} \varphi_k^C. \tag{3}$$

(C3) *Egalitarian treatment relative to payoff expectations*:

$$\forall C \in \Theta,\, \forall i,k \in C,\ \ \varphi_i^C - \varphi_i = \varphi_k^C - \varphi_k. \tag{4}$$

Axiom (C1) guarantees that a non-empty coalition is formed with positive probability if, and only if, the shares φ_i^C proposed for each of its members $i \in C$ do not fall below their expectations φ_i. Heuristically, we may envisage this situation as follows: In the interim stage, when coalition C considers forming, if any of the players $i \in C$ finds her share φ_i^C unacceptable (i.e. below her expectations) she walks away and the coalition is, in fact, not formed. In this sense, we can say that (C1) guarantees individual rationality. Condition (C2) requires that, if a coalition is formed, the entire surplus (loss) is distributed among its members. This condition is violated, for example, when taxes are imposed on coalitional production. Axiom (C3) posits egalitarian treatment of players' expectations in the sense that after fulfilling their expectations, all agents in a coalition should receive an equal share of the excess surplus. This condition can be generalized by assigning player-specific weights α_i to each agent $i \in N$ and, then, requiring that

$$\alpha_i(\varphi_i^C - \varphi_i) = \alpha_k(\varphi_k^C - \varphi_k), \quad \forall i, k \in C, \forall C \in \Theta. \tag{5}$$

Most of our qualitative results would still hold in that case but then the shares would, of course, depend on players' weights.

As anticipated in the introduction, (C1)-(C3) allow for different interpretations. On the one hand, they can be regarded as a set of rules that a mediator would adopt if motivated by considerations of *intertemporal fairness* with respect to expectations. In this interpretation, the termination probability (or the discount factor) matters as it is instrumental in defining expectations (1). Although we will focus below on the limit scenario as δ approaches one, it is instructive to reflect upon the polar case of $\delta = 0$. Then, the expected shares are zero for every player and, therefore, our condition (C3) implies that the division of surplus for every coalition follows the classical Nash Bargaining Solution (NBS) under a uniform disagreement point of zero. The NBS is, therefore, embedded in our intertemporal model as a particular case when, in effect, time plays no relevant role.

Alternatively, conditions (C1)-(C3) provide a protocol-free prediction for a general process of coalition formation and bargaining. In this interpretation, axiom (C3) implies that symmetric (focal) outcomes will be selected. If there are asymmetries (e.g., with respect to different bargaining powers), we can replace (C3) by a generalized formulation (5) with asymmetric agents' weights without affecting our qualitative results.

Probabilities $\{\pi^C\}_{C\in\Theta}$, shares $\{\varphi_i^C\}_{C\in\Theta, i\in C}$ and the corresponding expected shares $\{\varphi_i\}_{i\in N}$, computed by (1), are said to be (mutually) *consistent* if they jointly satisfy (C1)-

(C3). Given some consistent shares and probabilities, we will refer to the set,

$$A^{\varphi} \equiv \{C \in \Theta : \pi^{C} > 0\}, \tag{6}$$

as *cooperation structure*. We shall also use the terms *active* and *inactive* to distinguish between the coalitions that belong to the set A^{φ} from those that do not. As the notation A^{φ} suggests, in order to determine whether a particular coalition C is active or not it is enough to know the expected consistent shares $\{\varphi_i\}_{i\in C}$. It turns out that these shares also suffice to compute the actual shares of $v(C)$ that the members of C obtain when this coalition is matched. Therefore, the expected consistent shares (ECS) contain the essential information on the equilibrium outcome as summarized in the following result. (Recall that all proofs are included in the Appendix.)

PROPOSITION 1 *The expected consistent shares $\varphi = \{\varphi_i\}_{i\in N}$ determine the cooperation structure and consistent shares in all coalitions:*

$$\forall C \in \Theta, \quad \pi^{C} > 0 \Leftrightarrow v(C) \geq \textstyle\sum_{k\in C} \varphi_k, \tag{7}$$

$$\forall C \in \Theta, \forall i \in C, \quad \varphi_i^{C} = \varphi_i + \frac{v(C) - \sum_{k\in C} \varphi_k}{\#C}, \tag{8}$$

where $\#C$ is the cardinality of the set C.

From (7) follows that φ specifies uniquely the cooperation structure A^{φ} but not the precise (positive) matching probabilities. On the other hand, the shares computed in (8) reflect fairness or equal bargaining power in the following sense: whenever the surplus of the coalition exceeds the sum of expected payoffs (i.e. what agents will minimally insist upon to form the coalition), the difference is split up evenly among its members. While members of an active coalition receive the shares (8) when this coalition is matched, for inactive coalitions these shares are purely hypothetical.

Proposition 1 suggests that consistent magnitudes can be conceived of as either "primitive" or "derived" in two somewhat reciprocal ways: On the one hand, given the coalition formation probabilities $\{\pi^C\}_{C\in\Theta}$ and the consistent shares $\{\varphi_i^C\}_{C\in\Theta,i\in C}$, unique cooperation structure A^φ and unique ECS $\{\varphi_i\}_{i\in N}$ follow directly. On the other, we can use Proposition 1 to obtain from the ECS $\{\varphi_i\}_{i\in N}$ (now conceived as primitives) the consistent shares $\{\varphi_i^C\}_{C\in\Theta,i\in C}$ and the cooperation structure A^φ. We cannot, however, obtain from the ECS the precise values of the matching probabilities $\{\pi^C\}_{C\in\Theta}$. For a stark illustration of this indeterminacy, consider the extreme case where $\delta = 0$. Then, the ECS $\varphi_i = 0$ for all $i \in N$. This implies that, for every coalition C and player $i \in C$, the consistent share $\varphi_i^C = v(C)/\#C$, i.e., φ_i^C coincides with the share induced by the classical NBS for a uniform disagreement point equal to zero. Furthermore, the unique coalition structure A^φ contains all non-empty coalitions C for which $v(C) \geq 0$. However, these ECS allow for an arbitrary distribution of consistent probabilities $\{\pi^C\}_{C\in\Theta}$ that are positive if and only if $C \in A^\varphi$.

In the remainder of this section, we examine the existence, uniqueness, and other properties of consistent probabilities and shares. First, we settle the existence issue.

Proposition 2 *Consistent probabilities and shares exist in any game* $\Gamma(v,\delta)$.

Unsurprisingly, under our mild conditions, there is generally a multiplicity of probabilities and shares that qualify as consistent. The next example illustrates this multiplicity.

Example 1 *Consider a single buyer (player 1) and two identical sellers (players 2 and 3), where each buyer-seller pair can generate one unit of surplus. Formally,* $N = \{1,2,3\}$, $v(\{1,2\}) = v(\{1,3\}) = 1$ *and* $v(C) = 0$ *for any other subset* $C \subseteq N$. *It can be easily verified that the following two different probabilities and shares are consistent if* $\delta = 1/2$:

- $\pi^{\{1,2\}} = \pi^{\{1,3\}} = \frac{1}{2}, \quad \varphi_1^{\{1,2\}} = \varphi_1^{\{1,3\}} = \frac{4}{7}, \quad \varphi_2^{\{1,2\}} = \varphi_3^{\{1,3\}} = \frac{3}{7},$

- $\pi^{\{1,2\}} = \frac{2}{3}, \quad \pi^{\{1,3\}} = \frac{1}{3}, \quad \varphi_1^{\{1,2\}} = \frac{43}{78}, \quad \varphi_1^{\{1,3\}} = \frac{46}{78}, \quad \varphi_2^{\{1,2\}} = \frac{35}{78}, \quad \varphi_3^{\{1,3\}} = \frac{32}{78},$

where $\pi^C = 0$ *for all other subsets* $C \subseteq N$*. Thus, while the first alternative is symmetric across the two sellers, the second one provides seller* 2 *with a higher (expected) consistent share than seller* 3*. The cooperation structure is the same in both cases,* $A^{\varphi} = \{\{1,2\}, \{1,3\}\}$*, with equal matching probabilities for* $\{1,2\}$ *and* $\{1,3\}$ *in the first case and asymmetric probabilities in the second.*

In view of the above, our main focus turns to situations where the termination probability is arbitrarily small. Formally, we consider a sequence of games $\{\Gamma(v, \delta_k)\}_{k \in \mathbb{N}}$, where the continuation probabilities $\{\delta_k\}_{k \in \mathbb{N}}$ converge to one, i.e.

$$\lim_{k \to \infty} \delta_k = 1, \quad \text{or simply} \quad \{\delta_k\}_{k \in \mathbb{N}} \to 1.$$

Then, for each game $\Gamma(v, \delta_k)$ in this sequence, we consider some consistent shares and probabilities, which we denote by $\{\varphi_i^{C,\delta_k}\}_{C \in \Theta, i \in C}$ and $\{\pi^{C,\delta_k}\}_{C \in \Theta}$, respectively. Such shares and probabilities always exist by Proposition 2. To avoid inessential technicalities, we shall require that, for every $C \in \Theta$, any strictly positive subsequence $\{\pi^{C,\delta_{k_\tau}}\}_{\tau \in \mathbb{N}}$ remains bounded away from zero. Formally,

$$\forall \{\delta_{k_\tau}\}_{\tau \in \mathbb{N}} \to 1, \quad \left[\pi^{C,\delta_{k_\tau}} > 0, \forall \tau \in \mathbb{N}\right] \Rightarrow \liminf_{\tau \to \infty} \pi^{C,\delta_{k_\tau}} > 0, \tag{9}$$

Essentially, this condition embodies the idea that every coalition that forms with positive probability when δ_k is arbitrarily close to one should also do so in the limit.

Given any sequences of coalitional shares and probabilities as specified above, we may rely on (1) to obtain the corresponding sequence of expected consistent shares $\{\{\varphi_i^{\delta_k}\}_{i \in N}\}_{k \in \mathbb{N}}$.

The limit values of these shares as $\delta_k \to 1$ will be called *Limit Expected Consistent Shares* (LECS). Our main result below establishes that the LECS exist, are unique, and admit a simple characterization.

THEOREM 1 *A sequence of expected consistent shares and corresponding coalition-formation probabilities* $\{\{\varphi_i^{\delta_k}\}_{i\in N}\, ,\, \{\pi^{C,\delta_k}\}_{C\in\Theta}\}_{k\in\mathbb{N}}$ *that satisfies (9) always exists. Under this condition, as* $\delta_k \to 1$ *the sequence of shares* $\{\{\varphi_i^{\delta_k}\}_{i\in N}\}_{k\in\mathbb{N}}$ *converges to the LECS* $\{\widetilde{\varphi}_i\}_{i\in N}$*, which are the unique solution to the following inequality-constrained quadratic program:*

$$\min_{\{x_k\}} \textstyle\sum_{k\in N} x_k^2 \quad s.t. \quad \sum_{k\in C} x_k \geq v(C), \quad \forall C \subseteq N. \tag{10}$$

The previous result establishes that, as the termination probability $(1-\delta)$ becomes arbitrarily small, the expected consistent shares approach some limit values that can be computed as the unique solution of a simple optimization problem.[7] In fact, as we shall see in Proposition 3 below, such limit values do not just capture agents' (accurate or rational) expectations of their respective shares in their next cooperation. They also approximate the shares the agents actually receive as part of any active coalition they belong to, when the termination probability is low.

Interestingly, the quadratic program (10) also determines unique limit payoffs (for symmetric players) in the bargaining model proposed by Nguyen (2015). However, his non-cooperative framework is quite different from ours as it follows a specific protocol with fixed matching probabilities. Moreover, players leave the game and are replaced by "clones" upon agreement in his game. The formal similarity between the two solutions stems from

[7] We note that the computation of the LECS from (10) is reminiscent of the coalitional Nash bargaining solution in Compte and Jehiel (2010). Their solution (defined for games with non-empty cores) is identified with the core allocation for which the product of players' payoffs is maximal.

the fact that the limit payoffs in Nguyen (2015) are computed from a system of equations that resembles conditions for the computation of the LECS in our model (see the proofs of Proposition 2 and Theorem 1 in the Appendix).

On a more technical vein, it is worth mentioning that the LECS obtained as the termination probability vanishes can also be seen as a robust selection (or refinement) among the multiplicity of possible consistent solutions that generally arise when $\delta = 1$. To illustrate starkly that, in general, such a selection procedure is not trivial (i.e. a wide multiplicity can indeed arise), consider the simple example of two agents ($N = \{1, 2\}$) and a value function with $v(N) = 1$ and $v(C) = 0$ for $C \neq \{1, 2\}$. Then, for the game $\Gamma(v, 1)$, any configuration with $\pi^N > 0$ and shares $\varphi_1^N = 1 - \varphi_2^N = \gamma$ is consistent for any $\gamma \in [0, 1]$. In contrast, the unique LECS are equal to $\widetilde{\varphi}_i = \widetilde{\varphi}_i^N = 1/2$ for $i = 1, 2$.

As we have repeatedly emphasized, the theoretical framework that underlies LECS is inherently dynamic and this sets it apart from the axiomatic approach typically pursued by cooperative game theory. One may wonder, however, whether there is any parallelism or similarity between LECS and some of the standard solution concepts for the corresponding (static) cooperative games. The following example illustrates that this is not the case, and indeed the reasons for this are grounded in the dynamic underpinning of our concept.

EXAMPLE 2 *(Example 1 cont.) In the game defined in example 1, let us replace* $v(N) = 0$ *by* $v(N) = 1$ *and keep all other values unchanged. Then, the sum of LECS*

$$\widetilde{\varphi}_1 = \frac{2}{3}, \quad \widetilde{\varphi}_2 = \widetilde{\varphi}_3 = \frac{1}{3},$$

exceeds the value of the grand coalition, which contrasts with most of the solutions proposed in the cooperative game literature. For example, the Shapley value of the corresponding static game yields the imputation vector $(4/6, 1/6, 1/6)$ *and the singleton core consists only of* $(1, 0, 0)$.

Moreover, as the game considered is an assignment game (see Shapley and Shubik, 1971), its kernel and nucleolus lie in the core (Schmeidler, 1969, and Driessen, 1998, respectively), while its bargaining set coincides with the core (Solymosi and Raghavan, 2001). Hence, LECS are different from all of these concepts. Also the d-core (Albers, 1979) or the aspiration core (Bennett, 1983) coincide with the core in this game. In Section 3, we generalize this example to a trading game with many players and heterogeneous valuations.

The next Proposition 3 gives some useful properties of the LECS $\widetilde{\varphi}$, the *limit cooperation structure* $A^{\widetilde{\varphi}}$ and the *limit shares* $\{\widetilde{\varphi}_i^C\}_{C\in\Theta,i\in C}$ obtained from $\widetilde{\varphi}$ by Proposition 1. (Additional interesting properties are included in Proposition 5 in the Appendix.)

Proposition 3 *LECS $\widetilde{\varphi}$ display the following features.*

(i) (Rational expectations) $\widetilde{\varphi}_i^C = \widetilde{\varphi}_i$ for all $C \in A^{\widetilde{\varphi}}$ and $i \in C$.

(ii) (Blocking-proof) $\sum_{k\in C} \widetilde{\varphi}_k \geq v(C)$, $\forall C \subseteq N$, and $\sum_{k\in C} \widetilde{\varphi}_k = v(C)$, $\forall C \subseteq A^{\widetilde{\varphi}}$.

(iii) (Pareto efficiency) There are no probabilities $\{\widehat{\pi}^C\}_{C\in\Theta}$ and shares $\{\widehat{\varphi}_i^C\}_{C\in\Theta,i\in C}$ such that,

$$\widehat{\varphi}_i \equiv \sum_{C\in\Theta:i\in C} \frac{\widehat{\pi}^C}{\sum_{C\in\Theta:i\in C}\widehat{\pi}^C}\widehat{\varphi}_i^C \geq \widetilde{\varphi}_i, \quad \forall i \in N, \quad \text{and,} \tag{11}$$

$$\textstyle\sum_{i\in N} \widehat{\varphi}_i > \sum_{i\in N} \widetilde{\varphi}_i.$$

The above proposition highlights three interesting properties of the LECS concept. Item (i) was already anticipated: every coalition in the limit cooperation structure allows each of its constituent players to fulfil exactly their expected payoff. Next, Item (ii) specifies that every coalition C has to contemplate (if C is not in the limit cooperation structure, only hypothetically) the implementation of shares that satisfy the payoff expectations of all its members. This, as suggested by our labelling, can be interpreted as assuming that any subset of agents in C whose expectations are not met can block the formation (and hence even the meaningful consideration) of that coalition. Note that the second part of this condition implies that all coalitions in the limit cooperation structure satisfy *exactly* the expectations of their members.

Finally, Item (iii) has the following interpretation. Suppose that, given LECS $\{\widetilde{\varphi}_i\}_{i\in N}$, a planner considers the possibility of implementing an alternative (limit) pattern of coalition-formation probabilities $\{\widehat{\pi}^C\}_{C\in\Theta}$ and coalition-based shares $\{\widehat{\varphi}_i^C\}_{C\in\Theta, i\in C}$, which are not necessarily derived from consistent sequences. This in turn yields some corresponding limit expected shares (again, not necessarily consistent), denoted by $\{\widehat{\varphi}_i\}_{i\in N}$. It is immediate to see that the induced limit expected shares $\{\widehat{\varphi}_i\}_{i\in C}$ in (11) obtain from (1) as $\delta \to 1$. Then, the property (iii) simply asserts that, even if the planner can implement a pattern of matching probabilities and shares free from the requirement of consistency, the expectation profile embodied by LECS cannot be dominated in the Pareto sense.

3 Repeated Trade (in Networks)

In this section, we apply our framework to a model of recurring trade in some homogeneous good, where agents can assume endogenous roles of buyers or sellers depending on their current partner. We consider, specifically, the following trading scenario. Each player $i \in N$ has the valuation $\omega_i \geq 0$ for one unit of the good. We can think of this good as a (financial) asset and of ω_i as i's price expectation for this asset over a certain period. Alternatively, i can be the representative merchant of a good in country i and ω_i the price of this good in country i. Whatever the interpretation, the key feature of the model is that, when matched with another agent, player i will be willing to buy (sell) the good only if she can do it at a price below (above) ω_i. For simplicity, we assume that at most one unit[8] of the good is traded in each transaction and neither short-selling nor liquidity constraints are binding i.e., a matched pair will always exchange the good if this is profitable for both parties. We call an agent buyer (seller) if she buys (sells) the good in all her transactions. If a player assumes different roles in her transactions, we will call her an arbitrageur. In order to avoid uninteresting technicalities, we assume different valuations that we sort (without the loss of generality) in an increasing order, $\omega_1 < \omega_2 < ... < \omega_n$.

We model the underlying trade connections through a network $G = \{N, L\}$, in which pairs of agents from the set N are connected by links in the set $L \subseteq \{\{i,k\} : i,k \in N,\ i \neq k\}$. For each connected pair, we define their trade surplus as the total gain available from transferring a unit of good from the low- to the high-valuation agent, i.e.

$$v(\{i,k\}) = |\omega_i - \omega_k|, \quad \forall \{i,k\} \in L, \tag{12}$$

while any other coalition of agents does not generate any surplus:

$$v(C) = 0, \quad \forall C \notin L. \tag{13}$$

First, we focus on the case of unrestricted bilateral trade, i.e., we assume that the network is *complete*. That is, $L = \{\{i,k\} : i,k \in N,\ i \neq k\}$, which implies that every pair of agents can trade. The next proposition shows that, under these circumstances, the LECS entail a unique (and thus uniform) price for the good in all transactions.

PROPOSITION 4 *Let $\omega = (\omega_1, ..., \omega_n)$ be the distinct valuations of players in the set N and let the function v be defined by (12)-(13) for the complete connection structure $L = \{\{i,k\} : i,k \in N,\ i \neq k\}$. Then, the LECS satisfy $\widetilde{\varphi}_i = |\omega_i - p|$ for all $i \in N$, where $p = \frac{1}{n}\sum_{k=1}^{n} \omega_k$.*

[8] The results go through when each pair can trade $k < \infty$ units.

This proposition establishes that each active pair trades the good at the common price p that is equal to the average valuation. This price partitions the set N into two sets, depending on whether their valuation lies above or below the average valuation. Each agent in the first set acts as a buyer and everyone in the second as a seller. Thus, in the complete network, there are no arbitrageurs. Typically, unless p happens to be a Walrasian price (and n is even), the number of buyers will be different from that of sellers. Hence, in general, the "market" will not clear, with demand not being equal to supply. Furthermore, only links connecting buyers to sellers will be active (and, therefore, matched with a positive probability). In the language of graph theory, the set of active links induces a complete *bipartite* subnetwork of G, the set of buyers and sellers defining the two parts.

Next, we relax the assumption that the network is complete and turn our attention to the impact of an *incomplete* connection structure L. Basically, we want to assess the ability of L to replicate the outcome obtained under the complete network, for which all players trade at the same price. We refer to this benchmark as the Uniform-Price (UP) outcome. The following example discusses several contrasting cases for a simple four-player context where the trading network is a line. Subsequently, we discuss the problem more widely through a numerical investigation for the six connected network architectures that may arise with four nodes.

EXAMPLE 3 *Consider the network* $\{N, L\}$*, where* $N = \{1, ..., 4\}$ *and* $L = \{\{1,2\}, \{2,3\}, \{3,4\}\}$*. The network is therefore a line with three links. We compute the LECS in this network for the value function (12)-(13) and for different valuation profiles. The results are collected in Table 1. For the profile* $\omega = (3, 1, \frac{1}{2}, 0)$*, the corresponding LECS* $(1, 1, \frac{1}{4}, \frac{1}{4})$ *imply that only two pairs,* $\{1, 2\}$ *and* $\{3, 4\}$*, trade at the respective prices* 2 *and* $\frac{1}{4}$*. There is no exchange in* *the link $\{2,3\}$ although $\omega_2 \neq \omega_3$. For $\omega = (\frac{1}{2}, 3, 0, \frac{5}{2})$, all players trade at the common price $\frac{3}{2}$ that is equal to the average valuation. Hence, the restrictions on the connection structure are irrelevant and the UP outcome is attained. A situation with arbitrageurs occurs for $\omega = (3,2,1,0)$. In this case, player 1 is a seller and 4 a buyer, while players 2 and 3 are arbitrageurs and both sell and buy the good. Finally, the profile $\omega = (1,3,\frac{1}{2},0)$ illustrates the possibility that a player is excluded from trade (player 4 in this case), even though all actual trade that does occur takes place at a uniform price.*

$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3, \boldsymbol{\omega}_4)$	$(\widetilde{\varphi}_1,\widetilde{\varphi}_2,\widetilde{\varphi}_3,\widetilde{\varphi}_4)$	**Roles**	**Prices**
$(3,1,\frac{1}{2},0)$	$(1,1,\frac{1}{4},\frac{1}{4})$	(B,S,B,S)	$(2,-,\frac{1}{4})$
$(\frac{1}{2},3,0,\frac{5}{2})$	$(1,\frac{3}{2},\frac{3}{2},1)$	(S,B,S,B)	$(\frac{3}{2},\frac{3}{2},\frac{3}{2})$
$(3,2,1,0)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	(B,A,A,S)	$(\frac{5}{2},\frac{3}{2},\frac{1}{2})$
$(1,3,\frac{1}{2},0)$	$(\frac{1}{2},\frac{3}{2},1,0)$	(S,B,S,I)	$(\frac{3}{2},\frac{3}{2},-)$

Table 1: LECS $(\widetilde{\varphi}_1,\widetilde{\varphi}_2,\widetilde{\varphi}_3,\widetilde{\varphi}_4)$ computed for different valuation profiles $(\omega_1,\omega_2,\omega_3,\omega_4)$ from (12)-(13) with $L = \{\{1,2\},\ \{2,3\},\ \{3,4\}\}$. The LECS imply different roles and prices for the four nodes-agents. The possible roles are: buyer (B), seller (S), arbitrageur (A), and inactive (I). The prices reflect the implicit terms of trade in each of the three links, where "$-$" means no trade.

As the last example illustrates, an incomplete network can be compatible with different outcomes, including the UP outcome, depending on valuations. Next, we conduct a short simulation exercise where we illustrate further, in a more extensive manner, the role that the network structure may have on the achievement or not of the UP outcome. We do so by considering the following six connected network architectures that are possible with four nodes,

$$L1 = \{\{1,2\},\{1,3\},\{1,4\}\}, \quad L2 = \{\{1,2\},\{2,3\},\{3,4\}\},$$

$$L3 = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\}\}, \quad L4 = \{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\},$$

$$L5 = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}, \quad L6 = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}.$$

The numerical exploration is carried out as follows. First, for each of the network architectures considered, Lx $(x = 1, ..., 6)$, we draw a random sample ω of four valuations independently from the uniform distribution on the interval $[0, 1]$. Then, we compute the LECS obtained from these valuations (given the value function (12)-(13)) and each architecture. Finally, we verify whether the LECS are consistent with the UP outcome, i.e., whether,

$$\widetilde{\varphi}_i = |\omega_i - \frac{1}{4}\sum_{k=1}^{4}\omega_k|, \quad i = 1, ..., 4. \tag{14}$$

After repeating this procedure 10,000 times, we compute for each architecture Lx the relative frequency $\mu(Lx)$ of LECS that satisfy the condition (14). This frequency can be interpreted as an index of market incompleteness of Lx with lower values indicating a more incomplete market. This index attains the minimum of zero for a disconnected network and the maximum value of one for the complete network. The index values for each of the architectures depicted in Figure 1 are as follows:

$$\mu(L1) = 0.11, \quad \mu(L2) = 0.16, \quad \mu(L3) = 0.35,$$

$$\mu(L4) = 0.30, \quad \mu(L5) = 0.62, \quad \mu(L6) = 1.$$

Note, for example, that networks with the same number of links can differ significantly in their indices of market incompleteness.

4 Conclusions

We propose an equilibrium notion for intertemporal coalitional bargaining that is protocol-free and is solely characterized by three natural conditions on the probabilities with which different coalitions form and on the shares that players earn in each of them. Probabilities and shares satisfying these conditions are called consistent and are shown to exist in any game. For games with vanishing termination probability, the expected consistent shares are computed as the unique solution to an inequality-constrained quadratic program. We derive some of their properties and apply the general framework to a trading game, where productive coalitions are restricted to be pairs connected by an underlying network. When this network is complete, trade takes place at a uniform price and agents can be categorized into buyers or sellers. In contrast, if the network is incomplete, trade may occur at different prices and there may be some agents (arbitrageurs) who buy and sell, depending on their matched partners, the homogeneous good.

5 Appendix

The function $v(\cdot)$ and the set Θ in the proofs below have been defined in Section 2.

LEMMA 1 *The following function $g^C : R^n \rightarrow R$ is convex.*

$$g^C(\mathbf{x}) \equiv (\max\{\Delta^C(\mathbf{x}), 0\})^2, \tag{15}$$

$$where, \quad \Delta^C(\mathbf{x}) \equiv v(C) - \textstyle\sum_{k \in C} x_k, \quad C \in \Theta. \tag{16}$$

Proof: Follows from *the fact that the maximum of linear functions is convex and that the composition of a convex function with a convex and non-decreasing function is convex.*

■

Lemma 2 *We consider the system,*

$$x_i = \sum_{C\in\Theta: i\in C} \frac{\delta\cdot\omega^{C,\delta}}{\#C(1-\delta)} \max\{\Delta^C(\mathbf{x}), 0\}, \quad \forall i \in N, \tag{17}$$

where $\#C$ *is the cardinality of the set* C*,* $\omega^{C,\delta} \in [0,1]$ *and* $\Delta^C(\mathbf{x})$ *is defined in (16).*

(1) The system (17) has a unique uniformly bounded solution $\mathbf{x}^\delta$ *for any* $\delta \in [0,1)$.

(2) If $\omega^{C,\delta} \in [\epsilon, 1]$*,* $0 < \epsilon < 1$ *for all* $C \in \Theta$ *and* $\delta \in [0,1)$*, then* $\widetilde{\mathbf{x}} = \lim_{\delta\to 1}\mathbf{x}^\delta$ *is the unique solution to the inequality-constrained quadratic program,*

$$\min_{\mathbf{x}} \textstyle\sum_{k\in N} x_k^2, \quad s.t. \quad \sum_{k\in C} x_k \geq v(C), \quad \forall C \in \Theta. \tag{18}$$

Proof: (1) In Lemma 1, we show that the function $g^C(.)$ defined in (15) is convex. Then, for any $\delta \in [0,1)$ the function $g^\delta : R^n \to R$,

$$g^\delta(\mathbf{x}) \equiv \frac{1}{2}\textstyle\sum_{k\in N} x_k^2 + \sum_{C\in\Theta} \frac{\delta\cdot\omega^{C,\delta}}{2\cdot\#C(1-\delta)} g^C(\mathbf{x}), \tag{19}$$

is a sum of a (strictly) convex functions and is strictly convex itself. This function has then a unique (strict) minimum at some point $\mathbf{x}^\delta \in R^n_+$ with $0 \leq x_i^\delta \leq \max_{C\subseteq N} v(C)$ for all $i \in N$. We note that the partial derivatives,

$$\frac{\partial g^C(\mathbf{x})}{\partial x_k} = -2\max\{\Delta^C(\mathbf{x}), 0\},$$

are well-defined for all $\mathbf{x} \in R^n$ (in particular, for any $\mathbf{x}$ such that $\Delta^C(\mathbf{x}) = 0$, the right and the left partial derivatives are both equal zero). Then, it can be readily verified that the first order conditions $\bigtriangledown g^\delta(\mathbf{x}) = \mathbf{0}$ that determine the unique minimizer $\mathbf{x}^\delta$ are identical with the system (17).

(2) As the solution $\mathbf{x}^\delta$ to (17) is unique, uniformly bounded and continuous in $\delta \in [0,1)$, the limit solution $\widetilde{\mathbf{x}} \equiv \lim_{\delta\to 1} \mathbf{x}^\delta$ is well-defined. After multiplying (17) by $1-\delta > 0$,

$$(1-\delta)x_i = \textstyle\sum_{C\in\Theta: i\in C} \frac{\delta\cdot\omega^{C,\delta}}{\#C} \max\{\Delta^C(\mathbf{x}), 0\}, \quad \forall i \in N,$$

we observe that the l.h.s. of the last system converges to zero as $\delta \to 1$ because each x_i^δ is bounded. The same must then hold for the r.h.s., which for $\omega^{C,\delta} \geq \epsilon > 0$ implies in the limit,

$$\max\{\Delta^C(\widetilde{\mathbf{x}}), 0\} = 0 \Rightarrow v(C) \leq \textstyle\sum_{k\in C} \widetilde{x}_k, \quad \forall C \in \Theta. \tag{20}$$

We conclude, therefore, that $\widetilde{\mathbf{x}} \in \Phi \equiv \{\mathbf{z} \in R^n : v(C) \leq \sum_{k\in C} z_k, \forall C \in \Theta\}$ and note that for any $\delta < 1$,

$$g^\delta(\widetilde{\mathbf{x}}) = \frac{1}{2}\sum_{k\in N} \widetilde{x}_k^2, \quad \text{because} \quad v(C) \leq \sum_{k\in C} \widetilde{x}_k \Rightarrow g^C(\widetilde{\mathbf{x}}) = 0.$$

Now, we show that $g^\delta(\mathbf{x}^\delta)$ converges to $\sum_{k\in N} \widetilde{x}_k^2/2$ as $\delta \to 1$. For the minimizer $\mathbf{x}^\delta$ of $g^\delta(.)$, we compute the following limit by the algebraic limit theorem,

$$\begin{aligned}\lim_{\delta\to 1}\sum_{C\in\Theta} \frac{\delta\cdot\omega^{C,\delta}}{2\cdot\#C(1-\delta)} g^C(\mathbf{x}^\delta) &\leq \sum_{C\in\Theta}\lim_{\delta\to 1}\left(\frac{\delta}{2\cdot\#C(1-\delta)}\max\{\Delta^C(\mathbf{x}^\delta),0\}^2\right) \\ &= \sum_{C\in\Theta}\frac{1}{2\cdot\#C}\left(\lim_{\delta\to 1}\frac{\sqrt{\delta(1-\delta)}\max\{\Delta^C(\mathbf{x}^\delta),0\}}{1-\delta}\right)^2.\end{aligned}$$

Furthermore,

$$\lim_{\delta\to 1}\frac{\sqrt{\delta(1-\delta)}\max\{\Delta^C(\mathbf{x}^\delta),0\}}{1-\delta} = \lim_{\delta\to 1}\sqrt{\delta(1-\delta)}\lim_{\delta\to 1}\frac{\max\{\Delta^C(\mathbf{x}^\delta),0\}}{1-\delta} = 0. \tag{21}$$

The last equality follows because the solution $\mathbf{x}^\delta$ to (17) and, hence, the r.h.s. of (17) are bounded. Then,

$$\lim_{\delta\to 1} g^\delta(\mathbf{x}^\delta) = \lim_{\delta\to 1}\sum_{k\in N}\frac{(x_k^\delta)^2}{2} + \lim_{\delta\to 1}\sum_{C\in\Theta}\frac{\delta\cdot\omega^{C,\delta}\cdot g^C(\mathbf{x})}{2\cdot\#C(1-\delta)} = \lim_{\delta\to 1}\sum_{k\in N}\frac{(x_k^\delta)^2}{2} = \sum_{k\in N}\frac{\widetilde{x}_k^2}{2}.$$

In order to prove that $\widetilde{\mathbf{x}}$ minimizes $\sum_{k\in N} z_k^2/2$ over Φ, we assume, for the sake of contradiction, that

$$\exists \widetilde{\mathbf{z}} \in \Phi : \sum_{k\in N} \widetilde{x}_k^2/2 > \sum_{k\in N} \widetilde{z}_k^2/2.$$

Then, for sufficiently high $\delta < 1$, $\mathbf{x}^\delta$ becomes arbitrarily close to $\widetilde{\mathbf{x}}$ and $g^\delta(\mathbf{x}^\delta)$ to $\sum_{k\in N} \widetilde{x}_k^2/2$,

$$g^\delta(\mathbf{x}^\delta) \approx \sum_{k\in N} \widetilde{x}_k^2/2 > \sum_{k\in N} \widetilde{z}_k^2/2 = g^\delta(\widetilde{\mathbf{z}}),$$

which contradicts that $\mathbf{x}^\delta$ is the global minimizer of $g^\delta(.)$. ■

Proof of Proposition 1: First, we prove each of the implications in

$$\pi^C > 0 \Leftrightarrow v(C) \geq \sum_{k\in N} \varphi_k :$$

$$\Rightarrow: \pi^C > 0 \Rightarrow \varphi_i^C \geq \varphi_i, \forall i \in C \Rightarrow \sum_{i\in C} \varphi_i^C = v(C) \geq \sum_{i\in C} \varphi_i,$$

where we used, consecutively, C1 and C2.

$$\Leftarrow: v(C) = \sum_{i\in C} \varphi_i^C \geq \sum_{i\in C} \varphi_i \Rightarrow \varphi_i^C \geq \varphi_i, \forall i \in C \Rightarrow \pi^C > 0,$$

where we used, consecutively, C2, C3 and C1.

Then, we show the second part of the proposition,

$$\varphi_i^C - \varphi_i = \frac{v(C) - \sum_{k\in C} \varphi_k}{\#C},$$

by combining C2 and C3:

$$\varphi_i^C - \varphi_i = \varphi_k^C - \varphi_k \equiv \frac{\Delta^C(\boldsymbol{\varphi})}{\#C} \Rightarrow \sum_{i\in C} \varphi_i^C - \sum_{i\in C} \varphi_i = v(C) - \sum_{i\in C} \varphi_i = \Delta^C(\boldsymbol{\varphi}). \quad ■$$

Proof of Proposition 2: First, we prove the claim for a fixed $\delta < 1$. Let $\hat{\pi}^{C,\delta} = 1/\#\Theta$ for all $C \in \Theta$. Then, there is a unique solution $\boldsymbol{\varphi}^\delta$ to the system,

$$\varphi_i = \sum_{C\in\Theta: i\in C} \frac{\delta \cdot \hat{\pi}^{C,\delta}}{\#C(1-\delta)} \max\{\Delta^C(\boldsymbol{\varphi}), 0\}, \quad \forall i \in N, \tag{22}$$

by Lemma 2(1). This solution does not change if in (22) we replace $\hat{\pi}^{C,\delta}$ by

$$\pi^{C,\delta} = \frac{1}{\#\Theta} \cdot I(\Delta^C(\boldsymbol{\varphi}^\delta)), \quad \forall C \in \Theta, \tag{23}$$

where $I(x) = 1$ if $x \geq 0$ and $I(x) = 0$ otherwise. For the solution $\boldsymbol{\varphi}^\delta$ and each $C \in \Theta$ and $i \in C$, we compute the corresponding shares by Proposition 1,

$$\varphi_i^{C,\delta} = \varphi_i^\delta + \frac{\Delta^C(\boldsymbol{\varphi}^\delta)}{\#C}. \tag{24}$$

It is readily verified that φ_i^δ, $\pi^{C,\delta}$ and $\varphi_i^{C,\delta}$ computed from (22), (23) and (24), respectively, jointly satisfy (1) and (C1)-(C3). In particular, the system (22) is equivalent to (1),

$$\varphi_i = \sum_{C\in\Theta: i\in C} \frac{\delta \cdot \pi^{C,\delta}}{\#C(1-\delta)} \Delta^C(\boldsymbol{\varphi}) = \delta \sum_{C\in\Theta: i\in C} \pi^{C,\delta}(\varphi_i + \frac{\Delta^C(\boldsymbol{\varphi})}{\#C}) + \delta(1 - \sum_{C\in\Theta: i\in C} \pi^{C,\delta})\varphi_i,$$

while (23) implements C1 by Proposition 1,

$$\pi^{C,\delta} > 0 \Leftrightarrow v(C) \geq \textstyle\sum_{i\in N} \varphi_i^\delta \Leftrightarrow \Delta^C(\boldsymbol{\varphi}^\delta) \geq 0,$$

and (24) obtains by combining C2 and C3.

For $\delta = 1$, let $\boldsymbol{\varphi}^1 = \{\varphi_i^1\}_{i\in N}$ be the unique solution to the optimization problem (18) established by Lemma 2. Then, it is straightforward to verify that the expected shares $\{\varphi_i^1\}_{i\in N}$, together with the coalitional shares $\varphi_i^{C,1} = \varphi_i^1$ for each $i \in C \in \Theta$ and the probabilities $\pi^{C,1} = \frac{1}{\#\Theta} \cdot I(\Delta^C(\boldsymbol{\varphi}^1))$ jointly satisfy (1) and (C1)-(C3). ■

Proof of Theorem 1: First, we show that sequences of coalitional shares and probabilities satisfying (9) always exist. Recall that, in the proof of Proposition 2, we rely on the unique solution $\boldsymbol{\varphi}^\delta$ to the system (22) to obtain consistent shares $\{\boldsymbol{\varphi}^{C,\delta}\}_{C\in\Theta}$ (from (24)) and consistent probabilities $\{\pi^{C,\delta}\}_{C\in\Theta}$ (from (23)) for any $\delta \in [0, 1)$. This construction implies that $\pi^{C,\delta} = 1/\#\Theta$ whenever $\pi^{C,\delta} > 0$. Hence, for any sequence $\{\delta_k\}_{k\in\mathbb{N}} \to 1$, the corresponding ECS sequence $\{\boldsymbol{\varphi}^{\delta_k}\}_{k\in\mathbb{N}}$ is computed by (1) from shares $\{\{\boldsymbol{\varphi}^{C,\delta_k}\}_{C\in\Theta}\}_{k\in\mathbb{N}}$ and probabilities $\{\{\pi^{C,\delta_k}\}_{C\in\Theta}\}_{k\in\mathbb{N}}$ that satisfy (9).

Next, we show that LECS exist. This requires proving that the ECS sequence $\{\boldsymbol{\varphi}^{\delta_k}\}_{k\in\mathbb{N}}$ converges to some well-defined limit $\widetilde{\boldsymbol{\varphi}}$. By (1) and (24), each $\boldsymbol{\varphi}^{\delta_k}$ must solve the system:

$$
\begin{aligned}
(25) \qquad \varphi_i^{\delta_k} &= \delta_k \sum_{C\in\Theta:i\in C} \pi^{C,\delta_k}(\varphi_i^{\delta_k} + \frac{\Delta^C(\boldsymbol{\varphi}^{\delta_k})}{\#C}) + \delta_k(1 - \sum_{C\in\Theta:i\in C} \pi^{C,\delta_k})\varphi_i^{\delta_k} \\
&= \sum_{C\in\Theta:i\in C} \frac{\delta_k \cdot \pi^{C,\delta_k}}{\#C(1-\delta_k)} \max\{\Delta^C(\boldsymbol{\varphi}^{\delta_k}), 0\}, \quad \forall i \in N.
\end{aligned}
$$

The solution to this system does not change if expression (25) is changed so that for all those $C \in \Theta$ for which $\pi^{C,\delta_k} = 0$, we have instead $\pi^{C,\delta_k} = 1/\#\Theta$. This is because, by Proposition 1,

$$\pi^{C,\delta_k} = 0 \Leftrightarrow v(C) < \textstyle\sum_{i\in N} \varphi_i^{\delta_k} \Leftrightarrow \max\{\Delta^C(\boldsymbol{\varphi}^{\delta_k}), 0\} = 0.$$

After this substitution, by (9), $\pi^{C,\delta_k} \geq \epsilon$ for some $\epsilon > 0$ and all $C \in \Theta$, $\delta_k < 1$. Then, by Lemma 2(2), the solution to (25) converges to some LECS $\widetilde{\boldsymbol{\varphi}}$ when $(\delta_k)_{k\in\mathbb{N}} \to 1$, as desired. Finally, to establish uniqueness of LECS simply note that the preceding argument does not depend on the particular sequences of coalitional shares and probabilities considered. Therefore, any consistent sequences that satisfy (9) must have their induced ECS converge to the same limit. ■

Proof of Proposition 3: Properties of the LECS $\widetilde{\varphi}$:

(i) (rational expectations) $\widetilde{\varphi}_i^C = \widetilde{\varphi}_i$ for all $C \in A^{\widetilde{\boldsymbol{\varphi}}}$ and $i \in C$: This claim follows from (8) in Proposition 1 and the property (ii) below.

(ii) (Blocking-proof) $\sum_{k\in C} \widetilde{\varphi}_k \geq v(C)$, $\forall C \subseteq N$, and $\sum_{k\in C} \widetilde{\varphi}_k = v(C)$, $\forall C \subseteq A^{\widetilde{\boldsymbol{\varphi}}}$: The first part follows from the restrictions in the quadratic program (10). This part, combined with (7) in Proposition 1 and the definition (6) of cooperation structure, yields the second claim for all $C \subseteq A^{\widetilde{\boldsymbol{\varphi}}}$.

(iii) (Pareto efficiency): For the sake of contradiction assume that there are probabilities $\{\widehat{\pi}^C\}_{C\in\Theta}$ and shares $\{\widehat{\varphi}_i^C\}_{C\in\Theta,i\in C}$ such that,

$$\widehat{\varphi}_i \equiv \sum_{C\in\Theta:i\in C} \frac{\widehat{\pi}^C}{\sum_{C\in\Theta:i\in C}\widehat{\pi}^C}\widehat{\varphi}_i^C \geq \widetilde{\varphi}_i, \quad \forall i \in N, \quad \textit{and}, \quad \sum_{i\in N}\widehat{\varphi}_i > \sum_{i\in N}\widetilde{\varphi}_i.$$

We define an $\#N \times \#\Theta$ matrix Ψ with the typical element $\psi_{i,C} \equiv \widehat{\pi}^C(\widehat{\varphi}_i^C - \widetilde{\varphi}_i)$ in row $i \in N$ and column $C \in \Theta$, where $\psi_{i,C} = 0$ if $i \notin C$. We note that the sum of each column C is non-positive,

$$\textstyle\sum_{i\in N}\psi_{i,C} = \sum_{i\in C}\psi_{i,C} = \widehat{\pi}^C(\sum_{i\in C}\widehat{\varphi}_i^C - \sum_{i\in C}\widetilde{\varphi}_i) \leq \widehat{\pi}^C(v(C) - \sum_{i\in C}\widetilde{\varphi}_i) \leq 0,$$

by the fact that $\sum_{i\in C}\widehat{\varphi}_i^C \leq v(C)$ and the property (ii) above. On the other hand, the sum of each row i is non-negative for all $i \in N$ and strictly positive for some $i \in N$,

$$\sum_{C\in\Theta}\psi_{i,C} = \sum_{C\in\Theta:i\in C}\psi_{i,C} = \sum_{C\in\Theta:i\in C}\widehat{\pi}^C(\widehat{\varphi}_i^C - \widetilde{\varphi}_i) = (\sum_{C\in\Theta:i\in C}\widehat{\pi}^C)(\widehat{\varphi}_i - \widetilde{\varphi}_i) \geq 0,$$

by the Pareto conditions $\widehat{\varphi}_i \geq \widetilde{\varphi}_i$ and $\sum_{i\in N}\widehat{\varphi}_i > \sum_{i\in N}\widetilde{\varphi}_i$. Hence, we reach a contradiction as the sum of elements of Ψ is non-positive when summing up the columns but it is strictly positive when summing up the rows. ■

Proof of Proposition 4: Surplus sharing by an active pair in our trading game can be interpreted as a transaction, in which one unit of the good is exchanged at a price. For the sake of contradiction, we assume that two pairs of players, $ab \in L$ and $cd \in L$, trade the good in the limit equilibrium at two different prices, p_{ab} and p_{cd}, respectively. Without the loss of generality, we assume,

$$\omega_a \geq p_{ab} \geq \omega_b, \quad \omega_c \geq p_{cd} \geq \omega_d, \quad \text{and } p_{ab} > p_{cd}.$$

Hence, a, c (b, d) buy (sell) the good at the respective prices p_{ab} and p_{cd} earning the shares,

$$\widetilde{\varphi}_a = \widetilde{\varphi}_a^{\{a,b\}} = \omega_a - p_{ab}, \quad \widetilde{\varphi}_c = \widetilde{\varphi}_c^{\{c,d\}} = \omega_c - p_{cd}, \tag{26}$$

$$\widetilde{\varphi}_b = \widetilde{\varphi}_b^{\{a,b\}} = p_{ab} - \omega_b, \quad \widetilde{\varphi}_d = \widetilde{\varphi}_d^{\{c,d\}} = p_{cd} - \omega_d,$$

where the expected and actual shares are equal by Proposition 3(i). We note that,

$$\widetilde{\varphi}_a + \widetilde{\varphi}_b = |\omega_a - \omega_b| = v(\{a,b\}),$$

$$\widetilde{\varphi}_c + \widetilde{\varphi}_d = |\omega_c - \omega_d| = v(\{c,d\}),$$

as stipulated by Proposition 3(ii) for each active coalition. Then, Proposition 3(ii) implies a contradiction for the link (coalition) $ad \in L$,

$$v(\{a,d\}) \leq \widetilde{\varphi}_a + \widetilde{\varphi}_d = \omega_a - p_{ab} + p_{cd} - \omega_d = |\omega_a - \omega_d| - (p_{ab} - p_{cd})$$
$$= v(\{a,d\}) - (p_{ab} - p_{cd}) < v(\{a,d\}),$$

where the second equality and the last inequality follow from our assumption $\omega_a \geq p_{ab} > p_{cd} \geq \omega_d$. We conclude, therefore, that all transactions in the complete network occur at the same price p.

Computation of p: The price p partitions the set of players into the set of sellers $S = \{i : \omega_i < p\}$ and the set of buyers $B = \{i : \omega_i \geq p\}$. We can compute the total payoff to each set by Proposition 5(f) below,

$$\forall s \in S, \quad \widetilde{\varphi}_s = \textstyle\sum_{k\in B} \dfrac{z^v(\{s,k\})}{2} \Rightarrow \textstyle\sum_{s\in S} \widetilde{\varphi}_s = \textstyle\sum_{s\in S}\sum_{k\in B} \dfrac{z^v(\{s,k\})}{2},$$
$$\forall b \in B, \quad \widetilde{\varphi}_b = \textstyle\sum_{k\in S} \dfrac{z^v(\{b,k\})}{2} \Rightarrow \textstyle\sum_{b\in B} \widetilde{\varphi}_b = \textstyle\sum_{b\in B}\sum_{k\in S} \dfrac{z^v(\{b,k\})}{2},$$

The r.h.s. in these two equations are equal and, hence,

$$\textstyle\sum_{s\in S} \widetilde{\varphi}_s = \sum_{b\in B} \widetilde{\varphi}_b \Rightarrow \sum_{s\in S}(p - \omega_s) = \sum_{b\in B}(\omega_b - p) \Rightarrow p = \dfrac{1}{n}\sum_{k=1}^{n} \omega_k. \quad \blacksquare$$

We end the Appendix with a result that complements Proposition 3 with some additional properties of LECS, followed by a discussion on the formal and conceptual relationship between our solution concept and the decomposition of Shapley values into the so-called Harsanyi dividends.

Proposition 5 *In addition to those listed in Proposition 3, LECS* $\widetilde{\varphi}$ *and the corresponding limit cooperation structure* $A^{\widetilde{\boldsymbol{\varphi}}}$ *display the following properties.*

(a) (degree-1 homogeneity) $\widetilde{\boldsymbol{\varphi}}(\alpha \cdot v) = \alpha \cdot \widetilde{\boldsymbol{\varphi}}$ *for all* $\alpha \in R_+$.

(b) (null player) $\widetilde{\varphi}_i = 0$ *if* $v(C \cup \{i\}) = v(C)$ *for all* $C \subseteq N \backslash \{i\}$.

(c) (symmetry) $\widetilde{\varphi}_i = \widetilde{\varphi}_k$ *if* $v(C \cup \{i\}) = v(C \cup \{k\})$ *for all* $C \subseteq N \backslash \{i, k\}$.

(d) (marginal contribution) $v(T \cup \{i\}) - v(T) \leq \widetilde{\varphi}_i \leq v(C) - v(C \backslash \{i\}), \forall T, C \in A^{\widetilde{\boldsymbol{\varphi}}}$, $i \notin T$, $i \in C$.

(e) (feasibility) $\widetilde{\varphi}_i > 0 \Rightarrow \exists C \in A^{\widetilde{\boldsymbol{\varphi}}} : i \in C$.

(f) (decomposition into unique coalitional net values)

$$\widetilde{\varphi}_i = \sum_{C \in \Theta : i \in C} \frac{z^v(C)}{\#C},$$

where $\#C$ *is the cardinality of the set* C *and the coalitional net values* $z^v(C) \in [0, v(C)]$ *solve the system:*

$$(27) \qquad z^v(C) = v(C) - \sum_{T \in \Theta : T \neq C} \#(C \cap T) \frac{z^v(T)}{\#T}, \quad \forall C \in A^{\widetilde{\boldsymbol{\varphi}}},$$

$$z^v(C) = 0, \quad \forall C \in \Theta \backslash A^{\widetilde{\boldsymbol{\varphi}}}.$$

Proof:

(a) (degree-1 homogeneity) $\widetilde{\varphi}(\alpha \cdot v) = \alpha \cdot \widetilde{\boldsymbol{\varphi}}$ for all $\alpha \in R_+$: The following representation of $\widetilde{\boldsymbol{\varphi}}$ as a sum of unique Lagrange multipliers λ_T,

$$(28) \qquad \widetilde{\varphi}_i = \sum_{T \in \Theta : i \in T} \lambda_T, \quad \lambda_T \geq 0,$$

$$\lambda_T (v(T) - \sum_{t \in T} \widetilde{\varphi}_t) = 0, \quad \forall T \in \Theta,$$

follows from the KKT conditions for the inequality-constrained quadratic program (18). Then, the $\alpha{\cdot}\widetilde{\boldsymbol{\varphi}}$ in the game $\alpha{\cdot}v$ are obtained from the unique Lagrange multipliers $\lambda_T(\alpha{\cdot}v) = \alpha \cdot \lambda_T$.

(b) (null player) $\widetilde{\varphi}_i = 0$ if $v(C \cup \{i\}) = v(C)$ for all $C \subseteq N\backslash\{i\}$: For the sake of contradiction assume $\widetilde{\varphi}_i > 0$ for a dummy player i. Hence, by Proposition 5(e), there is a coalition $T \subseteq N$ such that $\sum_{t\in T} \widetilde{\varphi}_t + \widetilde{\varphi}_i = v(T \cup \{i\}) > 0$ (note that $\widetilde{\varphi}_i \geq 0$ for all $i \in N$ by Proposition 5(f)). By Proposition 3(ii) and the definition of the dummy player,

$$\textstyle\sum_{t\in T} \widetilde{\varphi}_t \geq v(T) = v(T \cup \{i\}) = \sum_{t\in T} \widetilde{\varphi}_t + \widetilde{\varphi}_i,$$

which implies $\widetilde{\varphi}_i = 0$.

(c) (symmetry) $\widetilde{\varphi}_i = \widetilde{\varphi}_k$ if $v(C \cup \{i\}) = v(C \cup \{k\})$ for all $C \subseteq N\backslash\{i, k\}$: For the sake of contradiction and w.l.o.g. assume $\widetilde{\varphi}_i > \widetilde{\varphi}_k \geq 0$. Hence, by Proposition 5(e), there is a coalition $T \subseteq N$ such that $\sum_{t\in T} \widetilde{\varphi}_t + \widetilde{\varphi}_i = v(T \cup \{i\}) > 0$. Then, by our assumption $\widetilde{\varphi}_i > \widetilde{\varphi}_k$, Proposition 3(ii) and the definition of the symmetric player,

$$v(T \cup \{i\}) = \textstyle\sum_{t\in T} \widetilde{\varphi}_t + \widetilde{\varphi}_i > \sum_{t\in T} \widetilde{\varphi}_t + \widetilde{\varphi}_k \geq v(T \cup \{k\}) = v(T \cup \{i\}),$$

which yields a contradiction.

(d) (marginal contribution) $v(T \cup \{i\}) - v(T) \leq \widetilde{\varphi}_i \leq v(C) - v(C\backslash\{i\}), \;\; \forall T, C \in A^{\widetilde{\boldsymbol{\varphi}}}$, $i \notin T, i \in C$: This follows directly from Proposition 3(ii),

$$\textstyle\sum_{t\in T} \widetilde{\varphi}_t + \widetilde{\varphi}_i \geq v(T \cup \{i\}) \Rightarrow \widetilde{\varphi}_i \geq v(T \cup \{i\}) - \sum_{t\in T} \widetilde{\varphi}_t = v(T \cup \{i\}) - v(T),$$

$$\textstyle\sum_{t\in C\backslash\{i\}} \widetilde{\varphi}_t + \widetilde{\varphi}_i = v(C) \Rightarrow \widetilde{\varphi}_i = v(C) - \sum_{t\in C\backslash\{i\}} \widetilde{\varphi}_t \leq v(C) - v(C\backslash\{i\}).$$

(e) (feasibility) $\widetilde{\varphi}_i > 0 \Rightarrow \exists C \in A^{\widetilde{\boldsymbol{\varphi}}} : i \in C$ follows from Proposition 5(f) as $\lambda_T = 0$ for any $T \notin A^{\widetilde{\boldsymbol{\varphi}}}$.

(f) (decomposition into net values) The following representation of $\widetilde{\boldsymbol{\varphi}}$ as a sum of unique Lagrange multipliers λ_T,

$$\text{(29)} \qquad \widetilde{\varphi}_i = \sum_{T\in\Theta:i\in T} \lambda_T, \quad \lambda_T \geq 0,$$
$$\lambda_T(v(T) - \sum_{t\in T} \widetilde{\varphi}_t) = 0, \quad \forall T \in \Theta,$$

follows from the KKT conditions for the inequality-constrained quadratic program (18). On the other hand, by Proposition 3(ii),

$$\text{(30)} \qquad v(C) = \sum_{k\in C} \widetilde{\varphi}_k, \quad \forall C \in A^{\widetilde{\boldsymbol{\varphi}}}.$$

By combining (29) and (30), we obtain for each active coalition $C \in A^{\widetilde{\boldsymbol{\varphi}}}$,

$$\text{(31)} \qquad v(C) = \sum_{k\in C} \widetilde{\varphi}_k = \sum_{k\in C} \sum_{T\in\Theta:k\in T} \lambda_T = \sum_{T\in\Theta} \#(C\cap T)\lambda_T$$
$$= \#C\frac{z_C^v}{\#C} + \sum_{T\in\Theta:T\neq C} \#(C\cap T)\frac{z_T^v}{\#T}, \quad \textit{where}, \quad z_X^v = \#X \cdot \lambda_X. \quad \blacksquare$$

As advanced, there is a striking similarity of the property stated in Proposition 5(f) with the decomposition of Shapley values into Harsanyi dividends (Harsanyi, 1959). The latter can be defined recursively as follows:

$$d^v(C) = v(C), \quad \#C = 1,$$
$$d^v(C) = v(C) - \sum_{K\subset C} d^v(C), \quad \#C > 1.$$

The dividend $d^v(C)$ is the value that the coalition C creates in excess of the values generated in all strict subsets of it. Is is well known that Shapley values are linear combinations of Harsanyi dividends,

$$Sh_i = \sum_{C\subseteq N:i\in C} \frac{d^v(C)}{\#C}.$$

Hence, if each coalition C brings forth a dividend $d^v(C)$ and each member of C owns an equal share of it, then each player earns her Shapley value. In a similar vein, $\widetilde{\varphi}_i$ can be decomposed into equal shares of net values (27) of the coalitions to which i belongs. These values internalize the opportunity cost of inputs (which can be used in at most one coalition at each date). The value $z^v(C)$ captures the net surplus of the coalition C after taking into account that the input of each $i \in C$, used in the production of $v(C)$, could have been used to produce surplus in other coalitions. For instance, the net values in Example 2 are computed by (27) as,

$$z^v(\{1,2\}) = z^v(\{1,3\}) = 2/3 < v\{\{1,2\}\} = v\{\{1,3\}\} = 1.$$

In particular, $z^v(\{1,2\})$ obtains by subtracting from $v(\{1,2\})$ player 1's contribution to $z^v(\{1,3\})$, which does not materialize when player 1 cooperates in coalition $\{1,2\}$.

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