Automorphisms and endomorphisms of first-order structures

A thesis submitted to the School of Mathematics at the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy

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Abstract

In this thesis, we consider questions relating to automorphisms and endomorphisms of countable, relational first-order structures $\mathcal{M}$, with a particular emphasis on bimorphism monoids.

We determine semigroup-theoretic results for three types of endomorphism monoid on $\mathcal{M}$, along with generation results when $\mathcal{M}$ is the random graph R or the discrete linear order $(\mathbb{N}, \leq)$. In addition, we introduce three types of partial map monoid of $\mathcal{M}$, and prove some semigroup-theoretic and generation results in these cases.

We introduce the idea of a permutation monoid, and characterise the closed submonoids of the infinite symmetric group $\mathrm{Sym}(\mathbb{N})$. Following this, we turn our attention the idea of oligomorphic transformation monoids, and expand on the existing results by considering a range of notions of homomorphism-homogeneity as introduced by Lockett and Truss in 2012. Furthermore, we show that for any finite group G, there exists an oligomorphic permutation monoid with group of units isomorphic to G.

The main result of the thesis is an analogue of Fraïssé's theorem covering twelve of the eighteen notions of homomorphism-homogeneity; this contains both Fraïssé's theorem, and a version of this for MM-homogeneous structures by Cameron and Nešetřil in 2006, as corollaries. This is then used to determine the extent to which some well-known countable homogeneous structures are also homomorphism-homogeneous.

Finally, we turn our attention to MB-homogeneous graphs and digraphs. We begin by classifying those homogeneous graphs that are also MB-homogeneous. We then determine an example of an MB-homogeneous graph not in this classification, and use the idea behind this construction to demonstrate $2^{\aleph_0}$ many non-isomorphic examples of MB-homogeneous graphs. We also give $2^{\aleph_0}$ many non-isomorphic examples of MB-homogeneous digraphs.

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Dedications

First, to my father, whose outstanding support in every aspect of my life made this thesis possible. Thanks Dad, I'm not done yet.

Second, to the memory of my friend Matthew Pyke, whose considerable influence on my life continues to drive my pursuit of knowledge.

1

Introduction

The study of automorphism groups of first-order structures has long been of interest to mathematicians due to connections with model theory and infinite permutation group theory [3, 37, 9, 43]. It follows from this connection that interesting examples of infinite permutation groups arise as automorphism groups of first-order structures with certain properties. For instance, if a countably infinite first-order structure $\mathcal{M}$ is $\aleph_0$-*categorical*, then $\mathrm{Aut}(\mathcal{M})$ has finitely many orbits on M^n for each $n \in \mathbb{N}$; in this instance, $\mathrm{Aut}(\mathcal{M})$ is called an *oligomorphic permutation group* [9]. Examples of $\aleph_0$-categorical structures often arise when considering the model-theoretic notion of *homogeneity* (or *ultrahomogeneity* in some sources). The celebrated theorem of Fraïssé [32] completely determines when a structure $\mathcal{M}$ is homogeneous, based on conditions on the class of finite substructures of $\mathcal{M}$. A large body of literature, in a range of subjects across mathematics, is devoted to the study of homogeneous structures [76, 49, 15, 54], and of properties of automorphism groups of homogeneous structures [79, 46, 5, 27, 45, 56].

We need not restrict ourselves to just automorphisms of a first-order structure. A semigroup-theoretic analogue of the automorphism group of a first-order structure $\mathcal{M}$ is the *endomorphism monoid* of $\mathcal{M}$. In the same fashion as above, the endomorphism monoid of a countably infinite structure provides interesting examples of infinite transformation monoids; this is another widely studied theme [7, 14, 59, 52, 23]. Furthermore, the concepts of oligomorphic permutation group and homogeneity have been generalised to semigroup cases; these are the concepts of an *oligomorphic transformation monoid* and *homomorphism-homogeneity* developed in [61] and [14, 53] respectively. Finding analogues of results about automorphism groups for endomorphism monoids is a motivating factor in this subject; examples of these include the idea of *generic endomorphisms* [52], and a version of Fraïssé's theorem for MM-homogeneous structures [14].

The automorphism group and the endomorphism monoid are not the only monoids associated to a countably infinite relational structure $\mathcal{M}$. In fact, by placing restrictions on the type of underlying partial function from the domain of $\mathcal{M}$ to itself, seven more monoids of self-maps preserving structure in $\mathcal{M}$ can be described. These are:

- Four endomorphism monoids contained between $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{End}(\mathcal{M})$, here called *intermediate monoids*:
  - $\mathrm{Bi}(\mathcal{M})$, the monoid of all bijective endomorphisms of $\mathcal{M}$ (the *bimorphism monoid* of $\mathcal{M}$);
  - $\mathrm{Emb}(\mathcal{M})$, the monoid of all embeddings of $\mathcal{M}$ (the *embedding monoid* of $\mathcal{M}$);
  - $\mathrm{Mon}(\mathcal{M})$, the monoid of all injective endomorphisms of $\mathcal{M}$ (the *monomorphism monoid* of $\mathcal{M}$), and
  - $\mathrm{Epi}(\mathcal{M})$, the monoid of all surjective endomorphisms of $\mathcal{M}$ (the *epimorphism monoid* of $\mathcal{M}$).
- Three *partial map monoids* of $\mathcal{M}$:
  - $\mathrm{Part}(\mathcal{M})$, the partial map monoid of all homomorphisms between substructures of $\mathcal{M}$ (the *partial endomorphism monoid* of $\mathcal{M}$);
  - $\mathrm{Inj}(\mathcal{M})$, the partial map monoid of all monomorphisms between substructures of $\mathcal{M}$ (the *partial monomorphism monoid* of $\mathcal{M}$), and
  - $\mathrm{Inv}(\mathcal{M})$, the partial map monoid of all isomorphisms between substructures of $\mathcal{M}$ (the *symmetric inverse monoid* of $\mathcal{M}$).

Together with the automorphism group $\mathrm{Aut}(\mathcal{M})$ and the endomorphism monoid $\mathrm{End}(\mathcal{M})$, they form a collection of nine monoids associated with a countably infinite relational structure $\mathcal{M}$; these are diagrammatically represented in Figure 1.1.

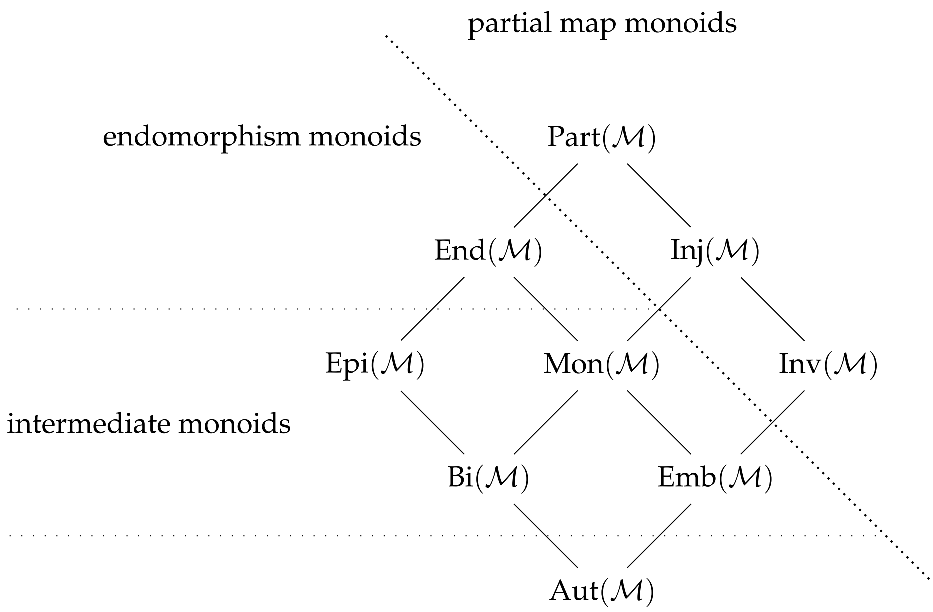


Figure 1.1: A diagram indicating transformation monoids of a countable, relational first-order structure $\mathcal{M}$. Containment is given by block lines. The thickly dotted line separates partial map and endomorphism monoids. Intermediate monoids lie between the three dotted lines.

Of these, the four intermediate monoids have been partially studied in diverse sources. For instance, epimorphisms of first-order structures preserve positive formulas [5], there exist generic endomorphisms of a countable set [52] and interesting monomorphism monoids arise from MM-homogeneous structures [14]. However, the general theory of these monoids for first-order structures is still a subject in its relative infancy.

Partial maps of a first-order structure $\mathcal{M}$ are a common subject in model theory as they are representative of local symmetry in $\mathcal{M}$. However, the idea of collecting these functions as partial map monoids has not been considered previously; certainly not from a semigroup-theoretic point of view. Much as the endomorphism monoid of a structure $\mathcal{M}$ is a structural analogue of the full transformation monoid on a set, it follows that Inv($\mathcal{M}$) and Part($\mathcal{M}$) are the structural analogues of the classical symmetric inverse monoid and the partial transformation monoid on a set respectively. This forms the basis for our interest.

Summary of results

The aim of this thesis is to develop the theory of selected intermediate and partial map monoids associated to countable, relational structures; with a particular emphasis on the bimorphism monoid Bi($\mathcal{M}$). We explore results relating to these monoids in the settings of semigroup theory, model theory and graph theory; preliminaries for each of these subjects can be found in Chapter 2. Because of the comparatively wide scope of this thesis, individual literature reviews about the different areas of study appear at the beginning of the relevant chapter. Similarly, any open questions that arise as a result of the work in a particular setting appear in that chapter, rather than at the end of the thesis.

Chapter 3 provides some background knowledge of cofinality, strong cofinality and the Bergman property (see [59]), as well as relative and Sierpiński ranks of semigroups (see [35, 64]). We then detail some results related to these concepts that will be useful in Chapters 4 and 5; including a novel result showing that if a semigroup S has a certain ideal structure, then S has countable strong cofinality (Proposition 3.1.11). The chapter concludes with an overview of the monomorphism monoid of a countable set; this summary informs work in Chapter 4.

In Chapter 4, we investigate those intermediate monoids of a countably infinite relational structure $\mathcal{M}$ that are made up of injective endomorphisms; these are Bi($\mathcal{M}$), Emb($\mathcal{M}$) and Mon($\mathcal{M}$). Because of the injectivity of maps in these monoids, they are right-cancellative semigroups; furthermore, as Bi($\mathcal{M}$) is a group-embeddable monoid, it is also left-cancellative. This in turn restricts the behaviour of the underlying maps; so we can consider relational structures in full generality, as opposed to specific examples. We determine some semigroup-theoretic results of these three intermediate monoids, including idempotents, ideal structure, and partial characterisations of Green's relations. These results are then used together with some of the work in Chapter 3 in order to show:

Theorem (Theorems 4.1.21, 4.1.25, 4.2.17, 4.2.21, 4.3.9, 4.3.12)**.** *Let R be the random graph and $T \in \{Bi(R), Emb(R), Mon(R)\}$. Then T has countable strong cofinality and* *does not have the Bergman property.*

Continuing the theme of semigroup-theoretic and generation results, Chapter 5 introduces the concept of a partial map monoid of a first-order structure. As with the intermediate monoids in Chapter 4, we consider the semigroup theory relating to the partial map monoids consisting of injective maps; that is, $\mathrm{Inv}(\mathcal{M})$ and $\mathrm{Inj}(\mathcal{M})$. The main result of this section (Theorem 5.2.1) presents a sufficient structural condition on $\mathcal{M}$ for the three partial map monoids to have uncountable strong cofinality (and hence the Bergman property), as well as demonstrating finite Sierpiński rank in these cases. We also examine cofinality results for the semilattice of idempotents $E(\mathrm{Inv}(\mathcal{M}))$ in Section 5.3.

We change tack slightly from the semigroup-theoretic first half of the thesis to consider model-theoretic results about intermediate monoids, with particular reference to homomorphism-homogeneity. In Chapter 6, we continue our work on bimorphisms of first-order structures by considering the notion of an *infinite permutation monoid* (Section 6.1); we prove some results about these in context of the topology on $\mathrm{Sym}(\mathbb{N})$ (Propositions 6.1.1 and 6.1.3). Following this, we build on work of Mašulovic and Pech [61] about oligomorphic transformation monoids by using the eighteen types of homomorphism-homogeneity introduced by Lockett and Truss [53]; these are summed up in an umbrella definition of XY-homogeneity (Definition 6.2.4). The main result of this section is:

Theorem (Theorem 6.2.8)**.** *If $\mathcal{M}$ is an XY-homogeneous countable first-order structure over a finite relational language, then $Y(\mathcal{M})$, the monoid of all maps of type Y, is an oligomorphic transformation monoid.*

This result motivates the construction of XY-homogeneous structures in order to find examples of oligomorphic transformation monoids. To this end, following the generalisation of Fraïssé's theorem by Cameron and Nešetřil for MM-homogeneous structures [14], Chapter 7 is devoted to demonstrating two Fraïssé-like theorems covering twelve of the eighteen possible instances of XY-homogeneity. In the first case, where the extended map need not be surjective, we require a forth-only construction similar to that of [14] and a single amalgamation property.

Theorem (Theorem 7.0.1, see Subsection 7.2.1)**.** *Let XY* $\in$ *{II, MI, MM, HI, HM, HH}.*

(1) If $\mathcal{M}$ *is an XY-homogeneous structure, then Age(*$\mathcal{M}$*) has the relevant amalgamation property.*

(2) If $\mathscr{C}$ *is a class of finite structures with countably many isomorphism types, is closed under isomorphisms and substructures, has the JEP and the relevant amalgamation property, then there exists a XY-homogeneous structure* $\mathcal{M}$ *with age* $\mathscr{C}$*.*

(3) Any two XY-homogeneous structures with the same age are equivalent up to a relevant notion of equivalence.

Definition 6.2.4 provides an explanation for II, MI, etc. in this theorem. The second case, where the extended map is surjective, requires a back-and-forth construction like Fraïssé's theorem itself. Here, the back portion of the back-and-forth is difficult to do because homomorphisms are not invertible in general; in fact, the converse of a homomorphism may not even be a function. Our method utilises a new notion of an *antihomomorphism* (Definition 7.1.1) to handle this case, where two distinct amalgamation properties are needed.

Theorem (Theorem 7.0.2, see Subsection 7.2.2)**.** *Let XZ* $\in$ *{IA, MA, MB, HA, HB, HE}.*

(1) If $\mathcal{M}$ *is an XZ-homogeneous structure, then Age(*$\mathcal{M}$*) has the two relevant amalgamation properties.*

(2) If $\mathscr{C}$ *is a class of finite structures with countably many isomorphism types, is closed under isomorphisms and substructures, has the JEP and the two relevant amalgamation properties, then there exists a XZ-homogeneous structure* $\mathcal{M}$ *with age* $\mathscr{C}$*.*

(3) Any two XZ-homogeneous structures with the same age are equivalent up to a relevant notion of equivalence.

As above, Definition 6.2.4 provides an explanation for IA, MA, etc. in this result. Following the work of Chapter 6, any structure on a finite relational language constructed in either of these results gives an example of an oligomorphic transformation monoid. We conclude the chapter by defining the notion of a *maximal homomorphism-homogeneity class* (Definition 7.3.3), and determine these for a selection of known homogeneous structures.

We narrow our focus to a particular instance of XY-homogeneity in Chapter 8; a structure $\mathcal{M}$ is *MB-homogeneous* if every finite partial monomorphism of $\mathcal{M}$ extends to a bimorphism of $\mathcal{M}$. In this chapter, we conduct an in-depth investigation into MB-homogeneous graphs and digraphs. This includes a classification of those MB-homogeneous graphs that are also homogeneous in the usual sense (Theorem 8.1.4), and a construction of an example of an MB-homogeneous graph (and digraph) that are *not* homogeneous in the usual sense (Examples 8.2.1, 8.3.6). We use strictly increasing sequences of natural numbers, along with families of pairwise non-embeddable finite graphs (and oriented graphs), to prove the main results of the section:

Theorem (Theorems 8.2.10, 8.3.11, 8.4.3, 8.4.6)**.** *There exist $2^{\aleph_0}$ many non-isomorphic countable MB-homogeneous graphs (oriented graphs, digraphs). Furthermore, there is a bijective homomorphism from each of these examples to the random graph R (generic oriented graph D, generic digraph D^*) and vice versa.*

For the graph case, this is a direct contrast to the countably many countable homogeneous graphs outlined in Lachlan and Woodrow's classification [49]. In this chapter, we also demonstrate the following theorem concerning oligomorphic permutation monoids:

Theorem (Theorem 8.2.11)**.** *For any finite group G, there exists an oligomorphic permutation monoid T such that the group of units of T is isomorphic to G.*

2

Preliminaries

Throughout the thesis, functions act on the right of their subjects, and composition of functions should be read from left to right.

2.1 Relations and functions

Let X be a set. An *n-tuple* $\bar{x}$ of X is some element of the set

$$X^n = \{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in X\}.$$

A *n-ary relation* R on X is some subset of X^n. Throughout, we consider *binary* relations on X; a subset of ordered pairs on X. We say that a binary relation R is

- *reflexive* if for all $x \in X$ it follows that $(x, x) \in R$;
- *irreflexive* if for all $x \in X$ we have that $(x, x) \notin R$;
- *symmetric* if for all $x, y \in X$, $(x, y) \in R$ implies that $(y, x) \in R$;
- *antisymmetric* if for all $x, y \in X$, $(x, y) \in R$ and $(y, x) \in R$ implies that $x = y$;
- *transitive* if for all $x, y, z \in X$, $(x, y), (y, z) \in R$ implies that $(x, z) \in R$;
- *total* if for all $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$.

A binary relation $\sim \subseteq X \times X$ is an *equivalence relation* of X if it is reflexive, symmetric, and transitive. A binary relation $\leq$ of X is a *partial order* on X (or that $\leq$ *partially orders* X) if $\leq$ is reflexive, antisymmetric and transitive; if this happens, write $(X, \leq)$ to mean the set X together with the partial order $\leq$. A partially ordered set $(X, \leq)$ is called a *linear order* if $\leq$ is also a total relation on X. Usually, we will write *poset* to mean partially ordered set.

We can also have binary relations between two sets X and Y; similar to above, this is some subset of $X \times Y$. For two relations $\phi \subseteq X \times Y$ and $\rho \subseteq Y \times Z$, the composition $\phi \circ \rho \subseteq X \times Z$ is defined to be the set $\{(x, z) : (\exists y \in Y)((x, y) \in \phi, (y, z) \in \rho)\}$

For two sets X and Y, we say that a relation $f \subseteq X \times Y$ is a *partial function* from X to Y if for all $x \in X$ and $y, z \in Y$, if $(x, y) \in f$ and $(x, z) \in f$ then $y = z$. If f is a partial function such that for all $x \in X$ there exists a $y \in Y$ such that $(x, y) \in f$, then we call f a *function*. Following standard convention, write $xf = y$ if $(x, y) \in f$ and $f : X \to Y$ if f is a function from X to Y. Denote the *domain* and *image* of f by dom f and im f respectively; in this case, dom $f = X$ and im $f = \{xf : x \in X\} \subseteq Y$. The *preimage* $W \subset Y$ under f^{-1} is defined to be the set $Wf^{-1} = \{x \in X : xf \in W\}$. Define the *kernel* of f to be the set ker $f = \{(x, y) \in X \times X : xf = yf\}$; this is an equivalence relation. The *kernel class* of an element x of X is the set $\{y \in X : (x, y) \in \text{ker } f\}$. If $Z \subseteq X$, we say that $Zf = \{xf : x \in Z\} \subseteq Y$ is the image of Z under f; it follows that $Xf = \text{im } f$. Define the *restriction of f to Z* to be the function $f|_Z : Z \to Y$. As functions are a special case of relations, we can compose functions using the relation composition above. The relation composition of two functions $f : X \to Y$ and $g : Y \to Z$ is a function $f \circ g : X \to Z$; this is well known as *function composition*. Often, we will write fg to be the composition of two such functions.

A function $f : X \to Y$ is *surjective* if and only if for all $y \in Y$ there exists $x \in X$ such that $xf = y$; equivalently, f is surjective if and only if im $f = Y$. Say that f is *injective* if for all $x, y \in X$, then $xf = yf$ in Y implies that $x = y$; equivalently, f is injective if its kernel classes are singletons. A function f is *bijective* if and only if f is both injective and surjective. We note that if $f : X \to Y$ and $g : Y \to Z$ are two surjective (injective, bijective) functions then their composition $h = fg : X \to Z$ is a surjective (injective, bijective) function. It is well known that a function $f : X \to Y$ is bijective if and only if there exists an inverse function $f^{-1} : Y \to X$ such that $ff^{-1} = 1_X$ and $f^{-1}f = 1_Y$, where $1_X, 1_Y$ are the identity functions on X and Y respectively.

2.1.1 Multifunctions

For a relation $\phi \subseteq X \times Y$, define the *converse* of ϕ to be the set $\phi^* = \{(y, x) : (x, y) \in \phi\} \subseteq Y \times X$. We say that a relation $f^* \subseteq Y \times X$ is a *partial multifunction* if $(y, x), (z, x) \in f^*$ implies that $y = z$; and that f^* is a *multifunction* if, in addition, for all $y \in Y$ there exists $x \in X$ such that $(y, x) \in f^*$. It is easy to see that f^* is a partial multifunction if and only if it is the converse of a partial function f, and that f^* is a multifunction if and only if the partial function f is surjective. A multifunction $f^* \subseteq Y \times X$ is *surjective* if for all $x \in X$ there exists $y \in Y$ such that $(y, x) \in f^*$. Consequently, f^* is a surjective multifunction if and only if it is the converse of a surjective function f. It is clear that a (partial) multifunction f^* is a (partial) function if and only if it is the converse of a (partial) injective function f. We adopt this asterisk notation throughout this chapter; if $f \subseteq X \times Y$ is a function, denote the multifunction given by the converse of f by $f^* \subseteq Y \times X$, and vice versa. Note that $(f^*)^* = f$.

Example 2.1.1. Let $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d, e\}$, and suppose that $f = \{(1, b), (2, b), (3, a), (4, c)\}$ is a function. Then the converse f^* of f is a partial multifunction given by $f^* = \{(b, 1), (b, 2), (a, 3), (c, 4)\}$ (see Figure 2.1).

By restricting the codomain Y of f to its image im f, the resulting function $g : X \to \text{im } f$ that behaves like f is a surjective function. In this case, the converse $g^* : \text{im } f \to X$ of g is a surjective and totally defined multifunction (see Figure 2.2). This technique will be used frequently in Chapter 7.

If $f^* \subseteq Y \times X$ is a multifunction, we will abuse notation and write $f^* : Y \to X$ where the context is clear. If $y \in Y$, define the set $yf^* = \{x \in X : (y, x) \in f^*\}$.

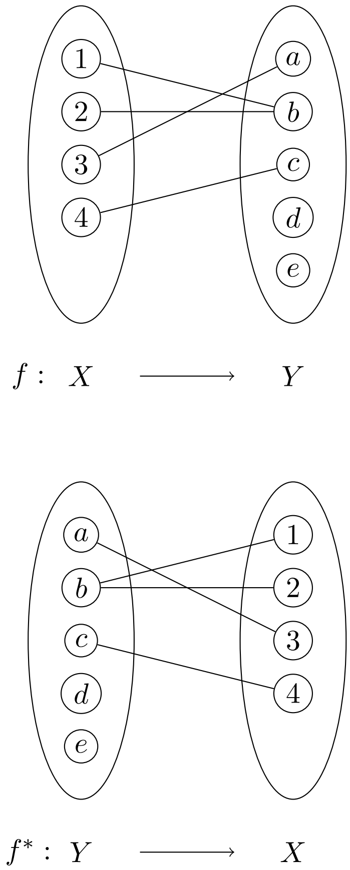


Figure 2.1: Example of a function f and its converse, the partial multifunction f^*

For a tuple $\bar{y} = (y_1, ..., y_n) \in Y^n$, write $\bar{y}f^*$ to be the following set of tuples

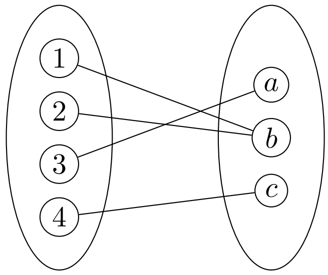
$$\bar{y}f^* = \{(x_1, ..., x_n) : x_i \in y_i f^* \text{ for all } 1 \leq i \leq n\}.$$

For a subset W of Y, we write

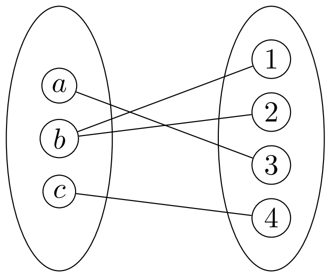
$$Wf^* = \{x \in X : (w, x) \in f^* \text{ for some } w \in W\} = \bigcup_{w \in W} wf^*.$$

For a multifunction $f^* : Y \to X$ and a subset $W \subseteq Y$, we say that the multifunction $f^*|_W : W \to X$ is the *restriction of* f^* *to* W. If $Y \subset B$ and $X \subset A$ are sets, and $f^* : Y \to X$ and $g^* : B \to A$ are two multifunctions, then we say that g^* *extends* f^* if $yf^* = yg^*$ for all $y \in Y$.

Throughout, we would like to be able to compose functions with multifunctions and vice versa; we achieve this by composing them as relations.



$$g: \quad X \quad \longrightarrow \quad \mathrm{im}\, f$$



$$g^* : \mathrm{im}\, f \quad \longrightarrow \quad X$$

Figure 2.2: Example of a surjective function g and its converse, the surjective multifunction g^*

Lemma 2.1.2. *Suppose that $f^* : X \to Y$ and $g^* : Y \to Z$ are multifunctions. Then $f^* \circ g^* \subseteq X \times Z$ is a multifunction.*

Proof. Suppose that $(x_1, z), (x_2, z) \in f^* \circ g^*$; so there exists $y_1, y_2 \in Y$ such that $(x_1, y_1), (x_2, y_2) \in f^*$ and $(y_1, z), (y_2, z) \in g^*$. As g^* is a multifunction, we have that $y_1 = y_2 = y$. As f^* is a multifunction, we have that $(x_1, y), (x_2, y) \in f^*$ implies that $x_1 = x_2$. $\square$

Remark. We previously noted that a function g is also a multifunction if and only if it is injective; so by this lemma, the composition of a multifunction f^* with an injective function g (or vice versa) is again a multifunction. Furthermore, the assumption of injectivity of the function g in this case is necessary for the composition $f^* \circ g$ to be a multifunction.

Lemma 2.1.3. *Let $f : A \to B$ and $g : B \to C$ be two functions, and suppose that $fg : A \to C$ is their composition. Then the converse map $(fg)^* : C \to A$ is equal to $g^* f^* : C \to A$, where g^* and f^* are composed as multifunctions.*

Proof. The proof is by containment both ways. Suppose that $(c, a) \in (fg)^*$; therefore, $(a, c) \in fg$. By definition of function composition, there exists $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$. As this happens, $(c, b) \in g^*$ and $(b, a) \in f^*$; so $(c, a) \in g^*f^*$. Now suppose that $(c, a) \in g^*f^*$; by relation composition, there exists $b \in B$ such that $(c, b) \in g^*$ and $(b, a) \in f^*$. Therefore, $(a, b) \in f$ and $(b, c) \in g$; implying that $(a, c) \in fg$. This means that $(c, a) \in (fg)^*$ and the proof is complete. □

2.2 Group and semigroup theory

The following definitions are standard, and can be found in [41] and [16].

2.2.1 Initial definitions

Let S be a set. A *binary operation* on S is any function $\star : S \times S \to S$; we write $(S, \star)$ to mean S together with the binary operation $\star$. We call $(S, \star)$ a *semigroup* if for all $a, b, c \in S$, then $(a \star b) \star c = a \star (b \star c)$. We omit the binary operation symbol where the meaning is clear, writing ab for $a \star b$, and writing a^n for the product of a with itself n times. A semigroup S is a *monoid* if there exists $1 \in S$ such that $1a = a = a1$ for all $a \in S$. Sometimes, it is necessary to turn a semigroup into a monoid; say that S^1 is the semigroup obtained by adjoining an identity element to S if necessary. A monoid G is a *group* if for all $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = 1 = g^{-1}g$. We say that $T \subseteq S$ is a *subsemigroup* if T is closed under the binary operation inherited from S, and a *subgroup* if T is a group with respect to this operation. For a semigroup S and a subset $U \subseteq S$, we define the *subsemigroup of S generated by U* to be the set $\langle U \rangle$ of all products of elements of U. If $\langle U \rangle = S$, then U is called a *generating set* for S and we say that U *generates* S.

An element e of a semigroup S is an *idempotent* if $e^2 = e$. We say that $a \in S$ is *regular* if there exists an $x \in S$ such that $axa = a$. Two elements $a, b \in S$ are *inverses* of each other if $aba = a$ and $bab = b$. An element s of a monoid S is a *unit* if there exists $t \in S$ such that $st = 1 = ts$. Say that U is the *group of units* of a monoid S if it is the largest subgroup of S containing the identity element 1; necessarily, U is the set of all units in S.

A semigroup S is a *regular semigroup* if all of its elements are regular; similarly, S is an *inverse semigroup* if there exists a unique inverse for every element of S. It follows that every inverse semigroup is regular. We say that a semigroup S is *left cancellative* if for all $a, b, x \in S$ we have that $xa = xb$ implies $a = b$. *Right cancellative* semigroups are defined in an analogous and dual manner, and a semigroup S is *cancellative* if it is both left and right cancellative. It is straightforward to see that any subsemigroup T of a (left, right) cancellative semigroup S is also (left, right) cancellative. In particular, as groups are cancellative, any subsemigroup of a group is cancellative; however, not every cancellative semigroup arises in this fashion [16]. Furthermore, it is easy to show that any finite cancellative semigroup is a group.

Say that a non-empty subset I of a semigroup S is a *left ideal* if for all $a \in S$ and $b \in I$, then $ab \in I$. We define *right* and *two sided* ideals in an analogous fashion. For shorthand, say that a subset I is an *ideal* if it is a two-sided ideal. We prove a basic lemma about ideals that will be important in the thesis.

Lemma 2.2.1. *Let T be a subsemigroup of a semigroup S, and suppose that V is an ideal of S. If $T \cap V$ is non-empty, then it is an ideal of T.*

Proof. Suppose that $t \in T$ and $v \in T \cap V$. As V is an ideal in S and $t \in S$ by definition, then $tv \in V$. But as $v \in T$ and $t \in T$, then $tv \in T$ as T is closed under multiplication. So $tv \in T \cap V$; as V is two sided, $vt \in T \cap V$ as well. $\square$

If a is an element of a semigroup S, define the set $Sa = \{sa : s \in S\}$; we can also define the sets aS and SaS in an analogous manner. The smallest left ideal containing a is the subset $Sa \cup \{a\} \subseteq S$; we denote this ideal by S^1a and say it is *the principal left ideal generated by* a. Similarly, $aS \cup \{a\} = aS^1$ is defined to be the *principal right ideal generated by* a, and $SaS \cup Sa \cup aS \cup \{a\} = S^1aS^1$ is the *principal two-sided ideal* (or just *principal ideal*) *generated by* a. We now define five equivalence relations central to the study of semigroup theory.

Definition 2.2.2. Let a and b be elements of a semigroup S. Then:

- a is *$\mathscr{L}$-related* to b (read $a\mathscr{L}b$) if and only if $S^1a = S^1b$;
- a is *$\mathscr{R}$-related* to b (read $a\mathscr{R}b$) if and only if $aS^1 = bS^1$;
- a is *$\mathscr{J}$-related* to b (read $a\mathscr{J}b$) if and only if $S^1aS^1 = S^1bS^1$;
- a is *$\mathscr{H}$-related* to b (read $a\mathscr{H}b$) if and only if $a\mathscr{L}b$ and $a\mathscr{R}b$.
- a is *$\mathscr{D}$-related* to b (read $a\mathscr{D}b$) if and only if there exists a c in S such that $a\mathscr{L}c$ and $c\mathscr{R}b$.

Together, these are known as *Green's relations*.

Remark. It can be shown that $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ (see [41]), and this composition of relations precisely describes $\mathscr{D}$. This result is crucial in determining that $\mathscr{D}$ is an equivalence relation; in fact, every type of Green's relation above is an equivalence relation. To this end, denote the $\mathscr{L}$-class ($\mathscr{R}$-class, $\mathscr{H}$-class, $\mathscr{D}$-class, $\mathscr{J}$-class) of an element $a \in S$ by L_a (R_a, H_a, D_a, J_a).

These definitions can be written in a form that emphasises their relation to divisibility of elements. For example, $a\mathscr{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b$ and $yb = a$. A similar definition can be written for $\mathscr{R}$-relations, in that $a\mathscr{R}b$ if and only if there exist $w, z \in S^1$ such that $aw = b$ and $bz = a$. Finally, we can write that $a\mathscr{J}b$ if and only if there exist $w, x, y, z \in S^1$ such that $wax = b$ and $ybz = a$. It is not hard to show that the two definitions given for these relations here are equivalent. The general containment of these relations is outlined in Figure 2.3.

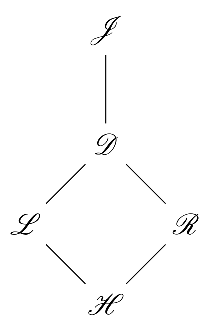


Figure 2.3: Containment of Green's relations

There are certain classes of semigroups where these relations coincide; for instance, if S is a group, then $\mathscr{H} = \mathscr{L} = \mathscr{R} = \mathscr{D} = \mathscr{J} = S \times S$. We investigate a generalisation of this case in our next result.

Lemma 2.2.3. *Let S be a monoid with group of units U, and suppose that $\alpha, \beta \in S$. If there exists $\gamma, \delta \in U$ such that $\gamma\alpha\delta = \beta$, then $\alpha\mathscr{D}\beta$.*

Proof. As δ is a unit, there exists $\delta^{-1} \in S$ such that $\delta\delta^{-1} = 1$. Therefore $(\alpha\delta)\delta^{-1} = \alpha$, and so $\alpha\mathscr{R}\alpha\delta$. Similarly, since γ is a unit, $\alpha\delta\mathscr{L}\gamma\alpha\delta = \beta$. So $\alpha\mathscr{D}\beta$ by definition. □

Remark. This lemma implies that $J_\alpha = D_\alpha$ in this case.

Finally, for semigroups S, T, we say that a function $\rho : S \to T$ is a *semigroup homomorphism* if for all $s, t \in S$ it follows that $(st)\rho = (s\rho)(t\rho)$. If S, T are monoids, we define a *monoid homomorphism* in the same way, with the additional caveat that $1_S\rho = 1_T$. A *group homomorphism* is a monoid homomorphism between two groups. A homomorphism $\rho : S \to T$ is an *embedding* if it is injective and an *isomorphism* if it is bijective.

2.2.2 Monoids of transformations

One of the main themes of semigroup theory is representing semigroups as collections of transformations, permutations or partial maps from a set to itself. For a set X, define the *symmetric group* $\mathrm{Sym}(X)$ to be the set of all bijective functions from X to itself, with the binary operation given by composition of functions. It is not hard to check that this is a group. Similarly, define the *full transformation monoid* (or endomorphism monoid) $\mathrm{End}(X)$ to be the set of all functions from X to itself, with the same binary operation as the symmetric group; it is easy to show this is a monoid. It is worth noting that if X is any set, $\mathrm{Sym}(X)$ is the group of units of $\mathrm{End}(X)$. If H is a subgroup of $\mathrm{Sym}(X)$ for some X, then we say that H is a *permutation group*. Similarly, if T is a subsemigroup (submonoid) of $\mathrm{End}(X)$ for some X, then we call T a *transformation semigroup (monoid)*. The following theorems from classical group and semigroup theory underline the importance of these constructs to their respective subjects.

Theorem 2.2.4 (Cayley, see [12])**.** *Let G be a group. Then there exists an embedding $\sigma : G \to Sym(G)$.* □

Theorem 2.2.5 (classical, see [41])**.** *Let S be a semigroup. Then there exists an embedding $\tau : S \to End(S^1)$.* □

Equivalently, these theorems mean that any group (semigroup) S can be represented as a permutation group (transformation semigroup) on the set S^1.

There are other self-map monoids of sets worth considering. For instance, the *monomorphism monoid* $\mathrm{Mon}(X)$ of X is the set of all injective functions (or *monomorphisms*) from X to itself with function composition. Similarly, define the *epimorphism monoid* of X to be the monoid $\mathrm{Epi}(X)$ of all surjective self-maps (or *epimorphisms*) of X. We quickly prove an easy lemma about cancellative properties of these monoids.

Lemma 2.2.6. *$Mon(X)$ is right cancellative, and $Epi(X)$ is left cancellative.*

Proof. Let $x \in X$. Suppose $\alpha, \beta, \gamma \in \mathrm{Mon}(X)$ and that $\alpha\gamma = \beta\gamma$. Then $x\alpha\gamma = x\beta\gamma$; since γ is injective, this implies that $x\alpha = x\beta$. Therefore, $\alpha = \beta$. Now suppose that $\delta, \epsilon, \zeta \in \mathrm{Epi}(X)$ and that $\zeta\delta = \zeta\epsilon$; then $x\zeta\delta = x\zeta\epsilon$. Take $y \in X$ to be such that $x\zeta = y$; then $y\delta = y\epsilon$. As ζ is surjective, this is true for all $y \in X$ and so $\delta = \epsilon$. □

We need not restrict ourselves to total functions of X. Consider the set $P(X)$ of all functions $p : \mathrm{dom}\ p \to \mathrm{im}\ p$, where $\mathrm{dom}\ p$, $\mathrm{im}\ p \subseteq X$. Note here that we include the unique function $\varepsilon : \varnothing \to \varnothing$; this is known as the *empty transformation*. Recall that the *converse* of p is the set $p^* = \{(z, y) : (y, z) \in p\}$. As the converse of a function, p^* is a multifunction and hence p^* is a function if and only if p is injective. Now, for functions p, q in $P(X)$, the domain and image of the function composition $p \circ q$ is:

$$\mathrm{dom}\ p \circ q = [\mathrm{im}\ p \cap \mathrm{dom}\ q]p^* \tag{2.1}$$

$$\mathrm{im}\ p \circ q = [\mathrm{im}\ p \cap \mathrm{dom}\ q]q. \tag{2.2}$$

This is proved in Proposition 1.4.3 of [41]. If im $p \cap$ dom $q = \varnothing$, then we say that $p \circ q = \varepsilon$. The set $P(X)$ together with this function composition forms a monoid Part(X); this is known as the *partial map monoid* of X. We can similarly take the collection of all partial bijections of X (with the empty transformation) together with this composition to define Inv(X), the *symmetric inverse monoid* of X. It can be shown that Part(X) is a regular semigroup, and that Inv(X) is an inverse subsemigroup of Part(X). Following the definition of inverse semigroup, this means that for every element $a \in$ Inv(X) there exists a unique $b \in$ Inv(X) such that ab is the identity map on dom a and ba is the identity map on dom b. The importance of the symmetric inverse semigroup as a generalisation of the symmetric group is underlined in another Cayley-esque theorem.

Theorem 2.2.7 (Vagner-Preston [80, 70], see Chapter 5 of [41])**.** *Let S be an inverse semigroup. Then there exists an embedding $\phi : S \to$ Inv(S).* □

For the final results of this section, we detail some semigroup-theoretic properties of Inv(X) that will be of use in the thesis. All three are basic results, and proofs for Lemma 2.2.8 and Lemma 2.2.10 can be found in Chapter 5 of [41]. Throughout, we write id_A for the identity map on some subset A of X.

Lemma 2.2.8. *Let $p \in$ Inv(X). Then p is an idempotent if and only if $p = id_A$ for some subset A of X.* □

Lemma 2.2.9. *Suppose that $\alpha, \beta \in$ Inv(X) for some set X.*

(1) If $\alpha\beta = \beta$, then $\alpha|_{dom\ \beta} = id_{dom\ \beta}$.

(2) If $\beta\alpha = \beta$, then $\alpha|_{im\ \beta} = id_{im\ \beta}$.

Proof. (1) Suppose that $\alpha\beta = \beta$. Here, dom $\beta =$ dom $\alpha\beta \subseteq$ dom α by Equation 2.1. Suppose $a \in$ dom β. Then $a\beta = a\alpha\beta$; as β is injective, it follows that $a = a\alpha$ for all $a \in$ dom β. Hence $\alpha|_{\text{dom}\ \beta} = id_{\text{dom}\ \beta}$.

(2) Suppose that $\beta\alpha = \beta$. It follows that im $\beta\alpha =$ im $\beta \subseteq$ dom α. Take $a \in$ dom β. By Equation 2.1, $a\beta \in$ im $\beta \cap$ dom α. In this case, $(a\beta)\alpha = a\beta\alpha = a\beta$, and so $\alpha|_{\text{im}\ \beta} = id_{\text{im}\ \beta}$. □

Lemma 2.2.10. *Let X be a set, and suppose that $f, g \in Inv(X)$. Then:*

(1) $f \mathscr{L} g$ if and only if im f = im g;

(2) $f \mathscr{R} g$ if and only if dom f = dom g;

(3) $f \mathscr{D} g$ if and only if $|$im $f| = |$ im $g|$, and;

(4) $\mathscr{D} = \mathscr{J}$. □

2.2.3 Monoid actions

A central topic in group theory is the idea of a group *acting* on some structure. As we will mainly be concerned with *monoids* acting on a structure, our definition must consider the idea of a monoid action. The following definitions on monoid actions can be found in [78]; references to partial monoid actions are given in the discussion.

Let X be a set and suppose that S is a monoid with identity element 1_S. We define a *right monoid action* of S on X to be a function $\alpha : X \times S \to X$, written $\alpha(x, s) = xs$, with the following properties:

- for all $x \in X$, $x1_S = x$, and;
- for all $x \in X$ and $s, t \in S$, $(xs)t = x(st)$.

Note that if S is a group, this is the definition of a *right group action* [12]. There is a one-to-one correspondence between monoid actions of S on the set X and monoid homomorphisms $\phi : S \to \text{End}(X)$ (see [78]); so these definitions can be used interchangeably. A right monoid action is *faithful* if $xs = ys$ implies that $x = y$ for all $x, y \in X$ and $s \in S$; equivalently, the action is faithful if the corresponding monoid homomorphism $\phi : S \to \text{End}(X)$ is injective.

Whenever we have an action of a monoid S on a set X, define the *forward orbit* of an element $x \in X$ to be the set

$$F(x) = \{y \in X \ : \ \exists s \in S, \ xs = y\}.$$

Define the *strong orbit* of an element $x \in X$ to be the set

$$S(x) = \{y \in X \ : \ \exists s, t \in S, \ xs = y \text{ and } yt = s\}.$$

If U is the group of units of S, define the *group orbit* of an element $x \in X$ to be the set

$$U(x) = \{y \in X \ : \ \exists u \in U, \ xu = y\}.$$

If S is a group acting on X, then we call $U(x)$ the *orbit* of x. Finally, we define the *pointwise stabilizer* of $x \in X$ to be the set

$$\text{Stab}(x) = \{s \in S \ : \ xs = x\}.$$

Say that a monoid action of S on X is *weakly transitive* if $F(x) = X$ for some $x \in X$, and *transitive* if $S(x) = X$ for some (and hence any) $x \in S$. Note that if S is a group acting transitively on X, then this coincides with the usual notion of transitivity for groups. The notion of a monoid action can be generalised to tuples of a set X.

Definition 2.2.11. Let X be a set, S be a monoid and $\alpha : X \times S \to X$ be a monoid action of S on X. Suppose that $\bar{x} = (x_1, ..., x_n) \in X^n$ is an n-tuple of X. Then the map $\bar{\alpha} : X^n \times S \to X^n$ given by

$$\bar{\alpha}((\bar{x}, s)) = (\alpha((x_1, s)), \alpha((x_2, s)), ..., \alpha((x_n, s))$$

is a monoid action on the set X^n. Say that $\bar{\alpha}$ is the *componentwise action of S with respect to α.*

When the context of the monoid action is clear (or not required), we say that this is the *componentwise action*. As the componentwise action of S on X^n is a monoid action, the definition of forward orbit, strong orbit, group orbit and pointwise stabilizer for an element $\bar{x} \in X^n$ all follow immediately. We now prove an easy lemma about containments of these different definitions of orbits.

Lemma 2.2.12. *Let S be a monoid acting on a set X, and consider the componentwise action of S on X^n. For any tuple $\bar{x} \in X^n$, we have the following:*

(1) $F(\bar{x}) = \bigcup_{\bar{y} \in F(\bar{x})} S(\bar{y})$.

(2) $S(\bar{x}) = \bigcup_{\bar{y} \in S(\bar{x})} U(\bar{y})$.

Proof. (1) As S is a monoid, we have that $F(\bar{x}) \subseteq \bigcup_{\bar{y} \in F(\bar{x})} S(\bar{y})$. To show the reverse containment, we need only prove that $S(\bar{y}) \subseteq F(\bar{x})$ for some tuple $\bar{y} \in F(\bar{x})$. Indeed, as $F(\bar{y}) \subseteq F(\bar{x})$ we have that $S(\bar{y}) \subseteq F(\bar{y}) \subseteq F(\bar{x})$.

(2) As the identity element of S is in U, it follows that $S(\bar{x}) \subseteq \bigcup_{\bar{y} \in S(\bar{x})} U(\bar{y})$. Suppose that $\bar{z} \in U(\bar{y})$, so there exists a $u \in U$ with $\bar{y}u = \bar{z}$. As u is a unit, we have $\bar{z}u^{-1} = \bar{y}$. Since $\bar{y} \in S(\bar{x})$, there exists $s, t \in S$ such that $\bar{x}s = \bar{y}$ and $\bar{y}t = \bar{s}$. Now, $\bar{x}su = \bar{z}$ and $\bar{z}u^{-1}t = \bar{x}$; therefore $\bar{z} \in S(\bar{x})$ by definition. □

We conclude this section with a discussion on *partial monoid actions*. There are many differing notions of a partial monoid action in the literature [31, 71, 39]; for the purposes of this thesis, it suffices to define the strongest of these. Let X be a set, and suppose that S is a monoid with identity element 1_S. We define a *right partial monoid action* to be a partial function $\pi : X \times S \to X$ given by $\pi((x, s)) = xs$ such that:

- for all $x \in X$, $(x, 1_S) \in \mathrm{dom}\,\pi$ and $x1_S = x$, and;
- if $x \in X$ and $s, t \in S$ and $(x, s) \in \mathrm{dom}\,\pi$, then $(xs, t) \in \mathrm{dom}\,\pi$ if and only if $(x, st) \in \mathrm{dom}\,\pi$, in which case $(xs)t = x(st)$.

If this happens, we say that S *partially acts* on X. This definition of a right partial monoid action is the one found in [71] as opposed to [31]. We choose this definition because it is equivalent to the existence of a monoid homomorphism $\psi : S \to \mathrm{Part}(X)$ [39]; this fact will be useful in Chapter 5.

We can extend this definition to that of a *right inverse monoid action*. Here, a right partial monoid action is a right inverse monoid action if $xs = ys$ implies that $x = y$ for all $x, y \in X$ and $s \in S$. This definition is equivalent to the existence of a monoid homomorphism $\chi : S \to \mathrm{Inv}(X)$.

Finally, we extend the notion of a componentwise action to partial monoid actions. Let X be a set and suppose that S is a monoid acting via a partial map π on the set X. Suppose that $\bar{x} = (x_1, ..., x_n) \in X^n$ is an n-tuple of X. Then the partial map $\bar{\pi} : X^n \times S \to X^n$, where $(\bar{x}, s) \in \text{dom } \bar{\pi}$ if and only if $(x_i, s) \in \text{dom } \pi$ for all $1 \leq i \leq n$, given by

$$\bar{\pi}((\bar{x}, s) = \bar{x}s = (\pi((x_1, s)), \pi((x_2, s)), ..., \pi((x_n, s))),$$

is a partial monoid action of S on X^n. Say that $\bar{\pi}$ is the *componentwise partial action with respect to* π. When the context of the partial monoid action is clear, we say that this is the *componentwise partial action*.

2.2.4 Topology on Sym($\mathbb{N}$), End($\mathbb{N}$)

We can also view symmetric groups and endomorphism monoids on a countably infinite set as topological spaces. Sources for these standard definitions are [67, 9, 43, 44].

Let X be a set. A *topology* $\mathcal{T}$ on X is a collection of subsets of X, containing $\varnothing$ and X, that is closed under arbitrary unions and finite intersections of elements of $\mathcal{T}$; we say that $(X, \mathcal{T})$ is a *topological space*. The elements of $\mathcal{T}$ are called *open sets*; the complement $X \smallsetminus Y$ of any open set Y is called a *closed set*. Note that subsets of X can both be open, closed, both, or neither. If Y is a subset of a topological space $(X, \mathcal{T})$, then the *subspace topology* on Y is given by the collection $\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$; this is a topology and we say that $(Y, \mathcal{T}_Y)$ is a *subspace* of $(X, \mathcal{T})$. If B is a collection of open sets from a topology $\mathcal{T}$ such that every element of $\mathcal{T}$ can be written as a union of elements in B, then we say that B is a *basis of open sets* for $\mathcal{T}$. A function f between two topological spaces X and Y is called *continuous* if the preimage of an open set of Y is an open set of X. If $x \in X$, then a *neighbourhood* of x is some open set U containing x. For a subset Z of a topological space X, say that $x \in X$ is a *limit point of* Z if $(U \smallsetminus \{x\}) \cap Z \neq \varnothing$ for any neighbourhood U of x. A subset Z of a topological space X is closed in X if and only if Z contains all its limit points. A topological space X is *perfect* if every point in X is a limit point.

There is a natural topology on the symmetric group on a countably infinite set Sym($\mathbb{N}$); known as the *pointwise convergence topology*. This is defined by saying that a sequence of permutations (g_n) tends to the limit g if and only if $kg_n = kg$ for any $k \in \mathbb{N}$ and sufficiently large n. A basis for open sets in this topology consists of the cosets of stabilizers of tuples; these are given by the sets $\{g \in \text{Sym}(\mathbb{N}) : \bar{a}g = \bar{b}\}$.

The pointwise convergence topology turns Sym($\mathbb{N}$) into a *topological group*; a group in which both multiplication and inversion are continuous functions. Furthermore, Sym($\mathbb{N}$) together with this topology is *completely metrizable*; that is, there exists a metric on Sym($\mathbb{N}$) that makes it a complete metric space. This, together with *separability* of Sym($\mathbb{N}$), makes it into a *Polish space*; a separable, completely metrizable space. Any non-empty perfect Polish space has cardinality $2^{\aleph_0}$ [44]; as an example, every point of Sym($\mathbb{N}$) is a limit point and so $|\text{Sym}(\mathbb{N})| = 2^{\aleph_0}$.

In addition, there is a natural topology on the endomorphism monoid End($\mathbb{N}$). This is the topology given by the basic open sets $\{f \in \text{End}(\mathbb{N}) : \bar{a}f = \bar{b}\}$. This means that the pointwise convergence topology on Sym($\mathbb{N}$) is the subspace topology of the topology of End($\mathbb{N}$).

We finish this section by stating a result equivalent to the definition of the subspace topology that will be useful in Chapter 6. If Y is a subspace of X, then A is *closed in* Y if both $A \subset Y$ and A is closed in the subspace topology of Y [67].

Theorem 2.2.13 (Theorem 17.2, [67])**.** *Let Y be a subspace of a topological space X. Then a set A is closed in Y if and only if $A = B \cap Y$, where B is some closed set of X.* □

2.3 Model theory

We now define a framework to investigate the different mathematical objects present in the thesis. For more background on this material, see [37] and [43].

2.3.1 Initial definitions

A *first-order signature* σ consists of a collection of *relations* $\{\bar{R}_i : i \in I\}$ where $\bar{R}_i$ is an n_i-ary relation for all $i \in I$, a collection of *functions* $\{\bar{f}_j : j \in J\}$ where $\bar{f}_j$ is an m_j-ary function for all $j \in J$, and a collection of elements $\{\bar{c}_k : k \in K\}$ called *constants*. We also stipulate that σ contains a list of variables $v_1, ..., v_n$ (ranging over some base set A) and logical operations, quantifiers and punctuation (=, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$, (,)) with the usual meanings. A *σ-structure* $\mathcal{A}$ consists of a set A (often called the *domain*), a set $R_i^{\mathcal{A}} \subseteq A^{n_i}$ interpreting each $\bar{R}_i$ of σ, a function $f_j^{\mathcal{A}} : A^{m_j} \rightarrow A$ interpreting $\bar{f}^j$ of σ and elements $c_k^{\mathcal{A}}$ of A interpreting each constant symbol $\bar{c}_k$ of σ. As in Section 2.1, the *n-tuple* of a σ-structure $\mathcal{A}$ is an element $\bar{a} = (a_1, ..., a_n)$ of A^n. For some relation $R_i \in \sigma$ with arity n_i and some n_i-tuple $\bar{a}$ of $\mathcal{A}$, say that $R_i(\bar{a})$ *holds in* $\mathcal{A}$ if and only if $\bar{a} \in R_i^{\mathcal{A}}$.

Let σ be some first-order signature, and let $\mathcal{A}$ be a σ-structure. A *σ-formula* ϕ is a finite string of symbols from σ. We say that $\mathcal{A}$ *satisfies* a formula ϕ if for each unquantified variable $v_1, ..., v_n$ in ϕ there exists an assignment of values $a_1, ..., a_n$ from A to these variables such that ϕ is true. If this happens, we write $\mathcal{A} \models \phi$. A *σ-sentence* is a σ-formula in which there are no unquantified variables. We say that a σ-structure $\mathcal{A}$ *models* a σ-sentence ϕ (or a set of σ-sentences Σ) if $\mathcal{A} \models \phi$. A set Σ of σ-sentences is called a *theory*; a theory Σ is *consistent* if there exists a model for Σ. For a σ-structure $\mathcal{A}$, define the *theory of* $\mathcal{A}$ to be the collection $\mathrm{Th}(\mathcal{A})$ of σ-sentences modelled by $\mathcal{A}$.

If a first-order signature σ has no other relations other than equality, we say that σ is a *functional signature* and the corresponding σ-structure is an *algebra*. If a signature σ has no function or constant symbols, we say that it is a *relational signature* and a corresponding σ-structure is a *relational structure*. We will mainly be concerned with relational structures throughout the thesis, so from now we proceed with definitions involving relational structures only.

Example 2.3.1. Let σ consist of a single binary relation symbol E. Let Σ be a set consisting of the single σ-sentence

$$(\forall x)(\forall y)(\neg E(x,x) \wedge (E(x,y) \rightarrow E(y,x)).$$

Any σ-structure Γ that models Σ is a *simple, undirected graph* (see Section 2.4).

Let $\mathcal{A}, \mathcal{B}$ be two relational σ-structures. Say that $\mathcal{A}$ is a *substructure* of $\mathcal{B}$ if and only if $A \subseteq B$ and for all $R_i \in \sigma$ and for all n_i-tuples $\bar{x}$ of $\mathcal{A}$ (where n_i is the arity of R_i), then $\bar{x} \in R_i^{\mathcal{A}}$ if and only if $\bar{x} \in R_i^{\mathcal{B}}$. If this happens, we can also say that $\mathcal{B}$ is an *extension* of $\mathcal{A}$. The *age* of a structure $\mathcal{M}$, denoted Age($\mathcal{M}$), is the class of all finite substructures of $\mathcal{M}$.

2.3.2 Maps between structures

If $f : A \rightarrow B$ is a function and $\bar{x}$ is an n-tuple of A, we say that $\bar{x}f = (x_1 f, ..., x_n f)$; this is the componentwise action outlined in Subsection 2.2.3. If $\mathcal{A}, \mathcal{B}$ are two relational σ-structures, say that a function $f : \mathcal{A} \rightarrow \mathcal{B}$ is a *homomorphism* if whenever $\bar{x} \in R_i^{\mathcal{A}}$, then $\bar{x}f \in R_i^{\mathcal{B}}$ for all $R_i \in \sigma$. It is straightforward to show that if we have two homomorphisms $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$, then their function composition $fg : \mathcal{A} \rightarrow \mathcal{C}$ is also a homomorphism.

Say that $f : \mathcal{A} \rightarrow \mathcal{B}$ is a *monomorphism* if f is an injective homomorphism. We say that f is an *embedding* if and only if f is a monomorphism and if $\bar{x} \notin R^{\mathcal{A}}$ then $\bar{x}f \notin R^{\mathcal{B}}$. If $f : \mathcal{A} \rightarrow \mathcal{B}$ is a bijective embedding, then we say that it is an *isomorphism*. It follows from these definitions that $f : \mathcal{A} \rightarrow \mathcal{B}$ is an isomorphism if and only if the inverse function f^{-1} of f is a homomorphism from $\mathcal{B}$ to $\mathcal{A}$.

The main focus of the thesis is collections of maps from some σ-structure $\mathcal{M}$ to itself. A homomorphism $\alpha : \mathcal{M} \rightarrow \mathcal{M}$ is called an *endomorphism of* $\mathcal{M}$. If α is surjective, call it an *epimorphism of* $\mathcal{M}$; if it is injective, say α is a *monomorphism of* $\mathcal{M}$. An *embedding of* $\mathcal{M}$ is a monomorphism of $\mathcal{M}$ that preserves non-relations. Say that α is a *bimorphism of* $\mathcal{M}$ if it is a bijective endomorphism. If α is an endomorphism that is also an isomorphism, we say that α is an *automorphism* of $\mathcal{M}$. Finally, we say that a structure $\mathcal{M}$ is *rigid* if Aut($\mathcal{M}$) $= \{e\}$, the identity map.

It follows from facts about function and homomorphism compositions stated above, for any type of endomorphism of $\mathcal{M}$ outlined above, the set of all such endomorphisms forms a monoid. Furthermore, the collections of all automorphisms of a structure $\mathcal{M}$ form a group. These are σ-structure analogues of various transformation monoids on a set as defined in Subsection 2.2.2. We detail the six endomorphism monoids on a σ-structure below:

- End($\mathcal{M}$), the endomorphism monoid of $\mathcal{M}$;
- Epi($\mathcal{M}$), the monoid of all surjective endomorphisms of $\mathcal{M}$ (the *epimorphism monoid* of $\mathcal{M}$);
- Mon($\mathcal{M}$), the monoid of all injective endomorphisms of $\mathcal{M}$ (the *monomorphism monoid* of $\mathcal{M}$);
- Bi($\mathcal{M}$), the monoid of all bijective endomorphisms of $\mathcal{M}$ (the *bimorphism monoid* of $\mathcal{M}$);
- Emb($\mathcal{M}$), the monoid of all embeddings of $\mathcal{M}$ (the *embedding monoid* of $\mathcal{M}$), and
- Aut($\mathcal{M}$), the automorphism group of $\mathcal{M}$.

It is well known (see [52]) that for any first-order structure $\mathcal{M}$, Aut($\mathcal{M}$) is the group of units of End($\mathcal{M}$). If Y($\mathcal{M}$) is any one of these monoids above, then there is a natural, faithful action $Y(\mathcal{M})$ on the domain M of $\mathcal{M}$ via the inclusion map $\iota : Y(\mathcal{M}) \to \mathrm{End}(M)$. We can define the componentwise action (see Definition 2.2.11) of $Y(\mathcal{M})$ on M^n with respect to this natural action. Unless stated otherwise, this is the action used throughout the thesis.

We now state two lemmas detailing scenarios where some of these monoids coincide.

Lemma 2.3.2 ([53])**.** *If $\mathcal{M}$ is a finite first-order structure, then Epi($\mathcal{M}$) = Mon($\mathcal{M}$) = Bi($\mathcal{M}$) = Emb($\mathcal{M}$) = Aut($\mathcal{M}$).* □

Lemma 2.3.3. *If $\mathcal{M}$ is a countably infinite set M, then Mon(M) = Emb(M) and Bi(M) = Aut(M).*

Proof. As there are no relations in M, there are no non-relations either; so any injective or bijective endomorphism must preserve non-relations of M. $\square$

We can also consider homomorphisms, monomorphisms and isomorphisms between substructures of a σ-structure $\mathcal{M}$; these sets form partial map monoids on a σ-structure $\mathcal{M}$ with composition rules similar to the partial map monoids in Subsection 2.2.2. We detail the three partial map monoids on a σ-structure $\mathcal{M}$ below:

- Part($\mathcal{M}$), the partial map monoid of all homomorphisms between substructures of $\mathcal{M}$ (the *partial endomorphism monoid* of $\mathcal{M}$);
- Inj($\mathcal{M}$), the partial map monoid of all monomorphisms between substructures of $\mathcal{M}$ (the *partial monomorphism monoid* of $\mathcal{M}$), and
- Inv($\mathcal{M}$), the partial map monoid of all isomorphisms between substructures of $\mathcal{M}$ (the *symmetric inverse monoid* of $\mathcal{M}$).

As is the case for the six endomorphism monoids of $\mathcal{M}$, each of the three partial map monoids above has a natural partial monoid action on the structure $\mathcal{M}$. In the case of Part($\mathcal{M}$), this partial action is given by the inclusion map ι : Part($\mathcal{M}$) $\to$ Part(M). In the case where $I(M) \in \{\text{Inv}(\mathcal{M}), \text{Inj}(\mathcal{M})\}$, this partial inverse action is given by the inclusion map $\epsilon : I(\mathcal{M}) \to \text{Inv}(M)$. As before, we can extend these to the componentwise partial action of Part($\mathcal{M}$) on M^n with respect to these natural actions. We use this action of partial map monoids of a first-order structure $\mathcal{M}$ on $\mathcal{M}$ throughout the thesis, unless otherwise stated.

We can present an analogue of Lemma 2.3.3 for partial map monoids:

Lemma 2.3.4. *If $\mathcal{M}$ is a set M, then Inj(M) = Inv(M).*

Proof. As there are no relations in M, there are no non-relations either; so any injective partial map must preserve non-relations of M. $\square$

Remark. Note that there is no analogue of Lemma 2.3.2 for partial map monoids; Inj($\mathcal{M}$) may not be the same as Inv($\mathcal{M}$) for some finite structure $\mathcal{M}$.

There is a natural containment order on the nine transformation monoids listed above; for instance, every epimorphism of $\mathcal{M}$ is also a partial endomorphism of $\mathcal{M}$, so $\mathrm{Epi}(\mathcal{M}) \subseteq \mathrm{Part}(\mathcal{M})$. A diagram illustrating the containment of all nine monoids listed in this section (for a σ-structure $\mathcal{M}$) is given in Figure 2.4.

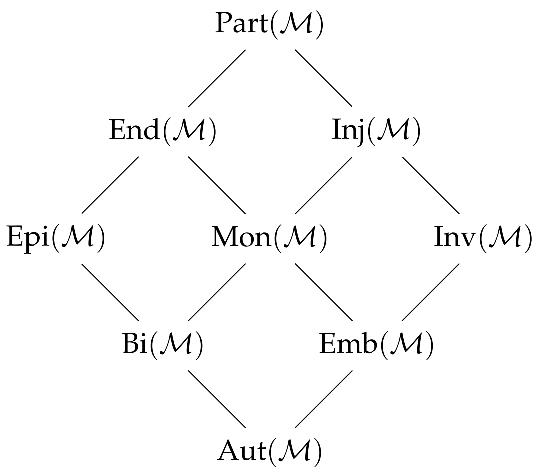


Figure 2.4: Some self-map monoids of a first-order structure $\mathcal{M}$

2.3.3 Automorphism groups, $\aleph_0$-categoricity and homogeneity

As 'structure is exactly what is preserved by automorphisms' [37], studying the automorphism group of a first-order structure $\mathcal{M}$ can tell us a great deal about the structure $\mathcal{M}$ itself. Furthermore, automorphism groups of first-order structures are interesting examples of (infinite) permutation groups; the first result given here characterises the closed subgroups of $\mathrm{Sym}(\mathbb{N})$ under the pointwise convergence topology outlined in Subsection 2.2.4.

Theorem 2.3.5 ([72], see [9]). *Let H be a subgroup of the infinite symmetric group $\mathrm{Sym}(\mathbb{N})$. Then H is closed under the pointwise convergence topology if and only if H is the automorphism group of some countably infinite first-order structure $\mathcal{M}$.*

We can also say something about the cardinalities of automorphism groups of first-order structures.

Theorem 2.3.6 (folklore, [9])**.** *Let $\mathcal{M}$ be a first-order structure. Then either $|Aut(\mathcal{M})| \leq \aleph_0$ or $|Aut(\mathcal{M})| = 2^{\aleph_0}$, the first alternative holding if and only if the stabilizer of some n-tuple of $\mathcal{M}$ is the identity element.*

Let Σ be a theory of σ-sentences. Say that Σ is $\aleph_0$-*categorical* if there exists a unique countable structure $\mathcal{M}$ that models Σ up to isomorphism. From this point, we say that a structure $\mathcal{M}$ is $\aleph_0$-categorical if $\mathrm{Th}(\mathcal{M})$ is $\aleph_0$-categorical. This strong condition on the theory of a first-order structure $\mathcal{M}$ is equivalent to a strong property of its automorphism group $\mathrm{Aut}(\mathcal{M})$; this is the Ryll-Nardzewski theorem [37].

Theorem 2.3.7 (Engeler, Ryll-Nardzewski, Svenonius, see [37])**.** *Let $\mathcal{M}$ be a countably infinite first-order structure. Then $\mathcal{M}$ is $\aleph_0$-categorical if and only if $Aut(\mathcal{M})$ has finitely many orbits on M^n for all $n \in \mathbb{N}$.* □

In this case, we say that $\mathrm{Aut}(\mathcal{M})$ is an *oligomorphic permutation group*. This equivalence indicates that we can study $\aleph_0$-categorical structures by studying oligomorphic permutation groups. Finding $\aleph_0$-categorical structures and hence oligomorphic permutation groups (or vice versa) is a central topic in model theory [30]. This task is made considerably easier by the connection between $\aleph_0$-categoricity and the model-theoretic notion of *homogeneity*, which we describe here.

Definition 2.3.8. Let $\mathcal{M}$ be a relational structure. Say that $\mathcal{M}$ is *homogeneous* if every isomorphism between finite substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

Remark. This definition is referred to as *ultrahomogeneity* in some sources.

Say that a theory Σ has *quantifier elimination* if every first-order sentence in Σ is logically equivalent to a first-order sentence without quantifiers.

Proposition 2.3.9 (see [37])**.** *(1) An $\aleph_0$-categorical structure $\mathcal{M}$ is homogeneous if and only if $Th(\mathcal{M})$ has quantifier elimination.*

(2) If σ is a finite signature and $\mathcal{M}$ is a homogeneous σ-structure, then $\mathcal{M}$ is $\aleph_0$-categorical.

□

So we can find oligomorphic permutation groups by finding homogeneous structures. This is made easier by the following famous results of Fraïssé ([32], see [3]). Recall that the *age* of a first-order structure $\mathcal{M}$ is the class of all finite substructures of $\mathcal{M}$. The following is known as the *extension property* (EP):

> (EP) For all $A, B \in \text{Age}(\mathcal{M})$ with $A \subseteq B$ and isomorphism $f : A \to \mathcal{M}$, there exists a isomorphism $g : B \to \mathcal{M}$ extending f.

The next theorem demonstrates that the (EP) is a necessary and sufficient condition for homogeneity.

Proposition 2.3.10 (Fraïssé [32], see [3])**.** *A countable structure $\mathcal{M}$ is homogeneous if and only if $\mathcal{M}$ has the EP.*

Sketch of proof. The forward direction is shown by extending a partial isomorphism of $\mathcal{M}$ and restricting. Using countability of $\mathcal{M}$, the converse direction follows after a standard *back-and-forth* argument. □

We now describe Fraïssé's main result. For a class $\mathscr{C}$ of finite first-order structures, Fraïssé described four properties that $\mathscr{C}$ can have to enable the construction of a countably infinite homogeneous structure $\mathcal{M}$ with age $\mathscr{C}$. Moreover, any two constructions made in this fashion are isomorphic. The four conditions are:

(1) $\mathscr{C}$ is closed under isomorphism.

(2) $\mathscr{C}$ is closed under substructures (the *hereditary property*).

(3) $\mathscr{C}$ has the *joint embedding property* (JEP):

> (JEP) For all $A, B \in \mathscr{C}$, there exists a structure $D \in \mathscr{C}$ such that A, B jointly embed in D.

(4) $\mathscr{C}$ has the *amalgamation property* (AP):

> (AP) For all $A, B_1, B_2 \in \mathscr{C}$ and embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$, there exists $D \in \mathscr{C}$ and embeddings $g_1 : B_1 \to D$ and $g_2 : B_2 \to D$ such that $f_1 g_1 = f_2 g_2$ (see Figure 2.5).

$$\begin{tikzcd}
 & D & \\
B_1 \arrow[ur, hook, dotted, "g_1"] & & B_2 \arrow[ul, hook', dotted, "g_2"'] \\
 & A \arrow[ul, hook, "f_1"] \arrow[ur, hook', "f_2"'] &
\end{tikzcd}$$

Figure 2.5: The amalgamation property (AP)

If $\mathscr{C}$ satisfies properties (1)-(4) above, we say that $\mathscr{C}$ is an *amalgamation class*.

Theorem 2.3.11 (Fraïssé's Theorem; [32], see [3])**.** *(1) If $\mathcal{M}$ is a countable homogeneous structure, then Age($\mathcal{M}$) is an amalgamation class.*

(2) (i) Let $\mathscr{C}$ be an amalgamation class. Then there exists a countable structure $\mathcal{M}$ with age $\mathscr{C}$ such that $\mathcal{M}$ is homogeneous.

(ii) Any two homogeneous structures $\mathcal{M}$ and $\mathcal{N}$ with the same age are isomorphic.

□

Sketch of proof. (1) It suffices to show the JEP and AP for Age($\mathcal{M}$). The union of two finite substructures of $\mathcal{M}$ is again a substructure of $\mathcal{M}$, showing the JEP. To show that Age($\mathcal{M}$) has the AP, extend a finite partial isomorphism of $\mathcal{M}$ and restrict accordingly.

(2) (i) The proof is an inductive construction using the JEP to guarantee that $\mathcal{M}$ exists and the AP to guarantee homogeneity. After the construction is complete, show that $\mathcal{M}$ has the EP; then $\mathcal{M}$ is homogeneous by Proposition 2.3.10.

(ii) This is via a back-and-forth argument, constructing the isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

□

Remark. Generalising these results on the model-theoretic connection between automorphism groups of first-order structures and infinite permutation groups to the case of endomorphism monoids and infinite transformation monoids is the focus of Chapters 6 and 7. A more rigorous proof of Fraïssé's theorem can be found in Subsection 7.2.2.

If $\mathcal{M}$ is a homogeneous structure with age $\mathscr{C}$, we say that $\mathcal{M}$ is the *Fraïssé limit* of the class of finite structures $\mathscr{C}$. To demonstrate the power of Fraïssé's theorem, we give two examples of homogeneous partial orders.

Example 2.3.12. Let $\mathscr{C}$ be the class of all finite linear orders. Then $\mathscr{C}$ is an amalgamation class and so there exists a countable homogeneous structure $\mathcal{M}$ with age $\mathscr{C}$. The Fraïssé limit of $\mathscr{C}$ is $\mathcal{M} = (\mathbb{Q}, <)$, the countable dense linear order without endpoints. Fraïssé's theorem (2) (ii) asserts that $(\mathbb{Q}, <)$ is unique up to isomorphism; re-proving a famous theorem of Cantor (see [3]).

Example 2.3.13. Let $\mathscr{C}$ be the class of all finite *partial* orders. Then $\mathscr{C}$ is an amalgamation class and so there exists a countable homogeneous structure $\mathcal{M}$ with age $\mathscr{C}$. The Fraïssé limit $\mathcal{M} = P$ of $\mathscr{C}$ is known as the *generic partial order*. [54].

In fact, these examples provide two of the only five cases in which a partially ordered set is homogeneous.

Theorem 2.3.14 (Schmerl [76], see [54]). *Let $\mathcal{P}$ be a homogeneous, countably infinite partially ordered set. Then $\mathcal{P}$ is isomorphic to one of the following:*

- $(\mathbb{Q}, <)$*, the countable dense linear order without endpoints;*
- *The infinite antichain A_ω;*
- *The disjoint union B_n of n many copies of $(\mathbb{Q}, <)$, where $n \geq 2$, with $(a, p) \leq (b, q)$ if and only if $a = b$ and $p < q$;*
- *The disjoint union C_n of n many copies of $(\mathbb{Q}, <)$, where $n \geq 2$, with $(a, p) \leq (b, q)$ if and only if $p < q$, or;*
- *The generic poset P.* □

Classification results are held in high esteem by model theorists; this is because of their value in immediate identification of properties of a given structure. For instance, if $\mathcal{Q}$ is a countable partial order not isomorphic to some $\mathcal{P}$ mentioned in Theorem 2.3.14, it is not homogeneous. We will mention two more celebrated classification results for graphs ([49], see Theorem 2.4.10) and digraphs ([15], see Theorem 2.4.11) in the next section.

2.4 Graph and digraph theory

The following are standard definitions from graph theory; a good source for these is [21].

If X is a set, define the set $[X]^2 = \{\{x, y\} \ : \ x \neq y \in X\}$. A *graph* $\Gamma = (V\Gamma, E\Gamma)$ is a set of *vertices* $V\Gamma$ together with a set of *edges* $E\Gamma \subseteq [V\Gamma]^2$. If $v, w \in V\Gamma$, we say that v and w are *adjacent* if $\{v, w\} \in E\Gamma$. Sometimes, we write $v \sim w$ to indicate when two vertices v, w are adjacent, and write $v \nsim w$ if they are not. Define the *neighbourhood* of v in Γ to be the set $N_\Gamma(v) = \{w \in V\Gamma : \{v, w\} \in E\Gamma\}$; the *extended neighbourhood* of $v \in \Gamma$ is the set $N_\Gamma(v) \cup \{v\}$. Say that the *degree* of a vertex v is $d_\Gamma(v) = |N_\Gamma(v)|$; say that a graph is *regular* (*n-regular*) if every vertex has the same degree (for some $n \in \mathbb{N}$). Often, when we are working in a single graph Γ, we write $N(v)$ and $d(v)$ for the neighbourhood and degree of a vertex $v \in V\Gamma$ respectively. A graph Γ is *finite* if $V\Gamma$ is a finite set, and *infinite* if $V\Gamma$ is infinite. A graph Γ is *locally finite* if for all $v \in V\Gamma$ then $d_\Gamma(v) = n$ for some $n \in \mathbb{N}$; that is, there are no vertices of infinite degree. The *handshake lemma* states that the sum of all degrees of vertices in a finite graph Γ is twice the number of total edges of Γ; a proof of this can be found in [21].

Let $\{v_0, v_1, ..., v_k\} = V \subseteq V\Gamma$, and suppose that $E = \{\{v_i, v_{i+1}\} : 0 \leq i \leq k - 1\} \subseteq E\Gamma$; then the graph $P = (V, E)$ is a *path* from v_0 to v_k. The *length* of the path is defined to be $|E|$. A graph Γ is *connected* if for any two vertices $v, w \in V\Gamma$ there exists some path P from v to w. If a graph Γ is not connected, we say it is *disconnected*. For a subset U of $V\Gamma$ and a vertex $v \in V\Gamma$, say that v is *connected to U* if for all $u \in U$, then $\{v, u\}$ in $E\Gamma$. Conversely, v is *independent of U* if the opposite occurs; that is, for all $u \in U$ then $\{v, u\} \notin E\Gamma$.

Let Γ, Δ be two graphs. If $V\Gamma \cap V\Delta = \varnothing$, then the *disjoint union* of Γ and Δ is the graph on the vertex set $\Gamma \cup \Delta$ together with edge set $E\Gamma \cup E\Delta$. We say that

- Δ is a *subgraph* of Γ if $V\Delta \subseteq V\Gamma$ and $E\Delta \subseteq E\Gamma$;
- Δ is a *spanning subgraph* of G if $V\Delta = V\Gamma$ and $E\Delta \subseteq E\Gamma$, and
- Δ is an *induced subgraph* of Γ if $V\Delta \subseteq V\Gamma$ and $E\Delta = [V\Delta]^2 \cap E\Gamma$.

In practice, the idea of an induced subgraph is more useful than a subgraph; this is because an induced subgraph is the correct notion of 'substructure' for graphs (see Section 2.3). From now on, any reference to 'subgraph' should be taken to mean 'induced subgraph'; except in the case of the phrase 'spanning subgraph'. For a graph Γ and a subset U of $V\Gamma$, we write $\Gamma(U)$ to be the induced subgraph on the vertex set U. As graphs are relational structures, every definition in Subsection 2.3.2 applies to graphs; for instance, a function $\phi : \Gamma \to \Delta$ is a *homomorphism* between two graphs if $\phi : V\Gamma \to V\Delta$ is a function and for all $\{v, w\} \in E\Gamma$, then $\{v\phi, w\phi\} \in E\Delta$.

For a graph Γ, define the *complement of* Γ to be the graph $\bar{\Gamma}$ with vertex set $V\Gamma$ and edge set $E\bar{\Gamma} = [V\Gamma]^2 \setminus E\Gamma$. If $n \in \mathbb{N} \cup \{\aleph_0\}$, define the *complete graph on n vertices* to the the graph K^n on $n = |VK^n|$ vertices with edge set $[VK^n]^2$; that is, every pair of vertices $\{v, w\}$ where $v \neq w \in VK^n$ is an edge. The complement $\bar{K}^n$ of K^n is known as the *null* or *empty graph on n vertices*; it is a graph on n vertices but with no edges. We say that a subset U of a graph Γ is a *clique* if $\Gamma(U) \cong K^{|U|}$; or that it is a *independent set* if $\Gamma(U) \cong \bar{K}^{|U|}$. Say that a clique (or independent set) U of Γ is a *maximum clique (independent set)* if there does not exist a subset W of Γ such that W is a clique or independent set with $|W| > |U|$.

As with any other first-order structure, graphs have automorphism groups and these automorphism groups are useful in determining the properties of a graph. For instance, an automorphism group acts transitively on the set of vertices only if the graph is regular [50]. We state a useful result about automorphism groups of graphs.

Theorem 2.4.1 (Frucht [33]). *Let G be a finite group. Then there exist countably many 3-regular graphs Γ such that $Aut(\Gamma) \cong G$.* $\square$

2.4.1 The random graph

We now define an object of central importance to the thesis.

Example 2.4.2. Let $\mathscr{C}$ be the class of all finite undirected graphs. Then $\mathscr{C}$ is closed under isomorphisms and substructures, and satisfies the JEP and AP, and as there are countably many finite graphs, it has countably many isomorphism types. Suppose that $A, B \in \mathscr{C}$. It is easy to see that $\mathscr{C}$ has the JEP; the disjoint union $A \cup B$ jointly embeds A and B. Now suppose $A, B_1, B_2 \in \mathscr{C}$, and that both B_1 and B_2 contain A as an induced subgraph. Define the graph $C = (B_1 \setminus A) \cup B_2$, with edge set given by $u \sim v$ if and only if $u \sim v$ in B_1 or $u \sim v$ in B_2. Note that there are no edges between $B_1 \setminus A$ and $B_2 \setminus A$. Then C satisfies all the requirements of the amalgamation property and so $\mathscr{C}$ is an amalgamation class. Define R, the Fraïssé limit of $\mathscr{C}$, to be the countable universal homogeneous graph; otherwise known as the *random graph*.

Remark. The unique countable homogeneous graph R is called the random graph due to a famous theorem of Erdős and Renyí [29]; this states that a random graph on a countably infinite set, formed by drawing in edges on pairs of vertices with some probability $0 < p < 1$, is almost surely isomorphic to R.

The random graph, due to its universality and the high amount of symmetry it possesses, has a range of interesting properties. We detail a selection of these here that will be useful through the thesis. The source for these, and for a lot more on the random graph, is [11]. First, we define *Alice's restaurant property* for a graph Γ.

> (ARP) For any finite, disjoint subsets $U, V \subseteq V\Gamma$, there exists $x \in V\Gamma$ such that $x \sim u$ for all $u \in U$ and $x \nsim v$ for all $v \in V$. (See Figure 2.6 for a pictorial representation.)

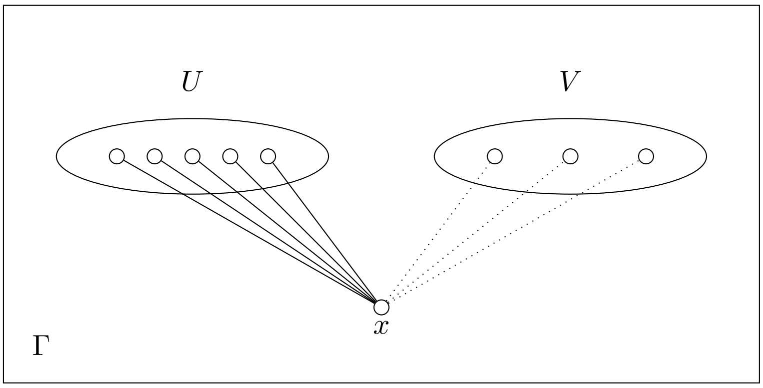


Figure 2.6: Alice's restaurant property in a countable graph Γ

This property turns out to be characteristic to the random graph.

Proposition 2.4.3 (Fact 2 [11]). *Let Γ be a countable graph with ARP. Then $G \cong R$.* □

The fact that R has the ARP can be used to prove the following useful properties of the random graph.

Theorem 2.4.4 (Proposition 2, [11]). *R has the property that you can add in any finite set of edges, or remove any finite set of vertices, and the resulting graph is isomorphic to R.*

Theorem 2.4.5 (Proposition 3, [11]). *Let $X_1 \cup ... \cup X_n$ be a partition of VR. Then the induced subgraph on at least one of these X_i is isomorphic to R.* □

Theorem 2.4.6 (Proposition 6, [11]). *R contains every countable graph as an induced subgraph.* □

Finally, the following is a consequence of Theorem 2.3.6.

Theorem 2.4.7 (Proposition 13, [11]). *$|Aut(R)| = 2^{\aleph_0}$.* □

2.4.2 Digraphs and oriented graphs

A *digraph* $D = (VD, AD)$ is a set of *vertices* VD together with a set $AD \subseteq VD^2$ of ordered pairs, called *arcs* of the digraph. For two vertices x, y of D, we write

$x \to y$ if $(x, y) \in AD$, and $x \| y$ if neither (x, y) nor (y, x) are in AD. Say that there is a *2-cycle* between x and y if and only if $x \to y$ and $y \to x$.

Most of the time, the digraphs in the thesis will be *loopless*; a digraph D is loopless if for all $x \in VD$ then $(x, x) \notin AD$. We say that a loopless digraph D is an *oriented graph* if for every pair of vertices $x, y \in VD$ then at most one of (x, y) and (y, x) are in AD; equivalently, a digraph D is an oriented graph if and only if it does not contain any 2-cycles. This class of digraphs is so named as you can 'orient' a graph Γ by adding directions to each edge of Γ; see Figure 2.7 below.

Figure 2.7: A graph Γ together with an orientation $\mathcal{G}$ of Γ

Say that a digraph D is a *tournament* if for every pair of vertices $x, y \in VD$ then exactly one of (x, y) or (y, x) is in AD; equivalently, this is an orientation of a complete graph K^n for some $n \in \mathbb{N} \cup \{\aleph_0\}$ (see Figure 2.8).

Figure 2.8: The complete graph K^4, together with an orientation $\mathcal{T}$, a tournament on 4 vertices

Let D be a digraph, let $X \subseteq VD$ and suppose that $y \in VD$. We define the following sets:

- $X^{\to}(y) = \{x \in X : x \to y\}$, the *in-neighbourhood* of y;
- $X^{\leftarrow}(y) = \{x \in X : y \to x\}$, the *out-neighbourhood* of y;
- $X^{\leftrightarrows}(y) = \{x \in X : y \to x \text{ and } x \to y\}$;
- $X^{\|}(y) = \{x \in X : x \| y\}$.

If $X = VD$, then these sets provide the digraph analogue of the *neighbourhood* of a vertex. If $X \neq VD$ and $y \in VD \setminus X$, then the union of all these sets is equal to X. If D is an digraph, then the *indegree* of a vertex $y \in VD$ is given by $|X^{\rightarrow}(y) \cup X^{\leftrightarrows}(y)|$. Analogously, the *outdegree* of $y \in VD$ is given by $|X^{\leftarrow}(y) \cup X^{\leftrightarrows}(y)|$.

We conclude this section with two examples of homogeneous digraphs. Note that these are both natural analogues of the random graph for digraphs (see Example 2.4.2) but they differ from each other slightly depending on whether digraphs are considered to have 2-cycles or not.

Example 2.4.8. Let $\mathscr{C}$ be the class containing all finite oriented graphs. Then $\mathscr{C}$ is closed under isomorphisms and substructures; as there are countably many finite oriented graphs, it has countably many isomorphism types. Using a similar argument to Example 2.4.2, we can show that $\mathscr{C}$ has the JEP and the AP; therefore, $\mathscr{C}$ is an amalgamation class. Define D, the Fraïssé limit of $\mathscr{C}$, to be the countable, universal homogeneous oriented graph, which we call the *generic oriented graph*. Note that in most sources [15, 1] this structure D is known as the *generic digraph*.

Using a similar argument to Proposition 2.4.3, it can be shown (see [1]) that D has a characteristic extension property which we call the *oriented Alice's restaurant property* (OARP). This is defined by:

> (OARP) For any finite and pairwise disjoint sets of vertices U, V, W of D, there exists a vertex x of D such that there is an arc from x to every element of U, an arc to x from every element of V, and x is independent of every vertex in W. (See Figure 2.9 for a diagram of an example.)

Example 2.4.9. Let $\mathscr{C}$ be the class containing all finite digraphs (which here, are permitted to have 2-cycles). Then $\mathscr{C}$ is an amalgamation class, for similar reasons to Example 2.4.8. Define D^*, the Fraïssé limit of $\mathscr{C}$, to be the countable, universal homogeneous digraph; which we call the *generic digraph*.

Again, using a similar argument to Proposition 2.4.3, it can be shown (see [62]) that D has a characteristic extension property which we call the *directed Alice's restaurant property* (DARP). This is defined by:

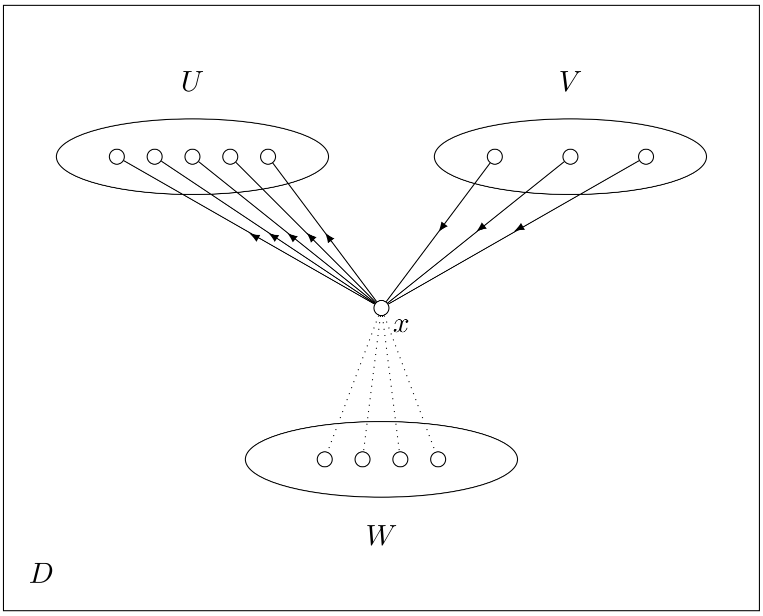


Figure 2.9: Oriented Alice's restaurant property in D

(DARP) For any finite and pairwise disjoint sets of vertices U, V, W, X of D^*, there exists a vertex z of D^* such that: there is an arc from z to every element of U, an arc to z from every element of V, a 2-cycle between z and every element of W, and z is independent of every vertex in X. (See Figure 2.10 for a diagram of an example.)

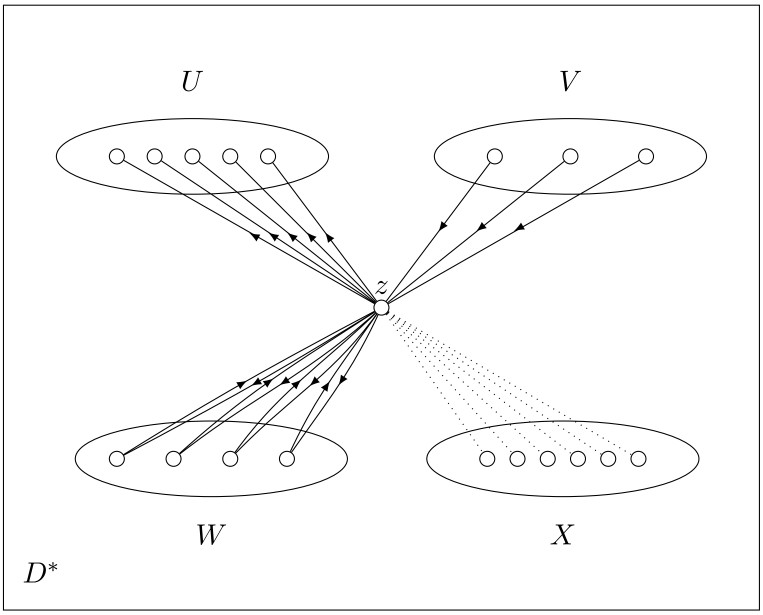


Figure 2.10: Directed Alice's restaurant property in D^*

2.4.3 Homogeneous graphs and digraphs

To conclude this section, we reproduce the seminal classifications of countably infinite homogeneous graphs (Lachlan and Woodrow [49]) and homogeneous digraphs (Cherlin, [15]). Examples of structures appearing in these catalogues that are used in the thesis have been defined previously (such as the random graph R, Example 2.4.2), or will be defined in the appropriate result.

Theorem 2.4.10 (Lachlan, Woodrow [49]). *Let Γ be a countably infinite graph. Then Γ is a homogeneous graph if and only if it is isomorphic to one of the following:*

- *The countably infinite disjoint union of complete graphs K^n where n is finite, or its complement;*
- *Any countable disjoint union of infinite complete graphs $K^{\aleph_0}$, or its complement;*
- *The generic graph omitting K^n where $n \geq 3$, or its complement; or,*
- *The random graph R.* □

We present Cherlin's classification of homogeneous digraphs in the style of Macpherson [54]. Note here that in this classification, a digraph is *not* considered to have 2-cycles.

Theorem 2.4.11 (Cherlin, [15]). *Let D be a countably infinite digraph. Then D is a homogeneous digraph if and only if it is isomorphic to one of the following:*

- *a homogeneous poset (see Theorem 2.3.14) when viewed as a digraph.*
- *a homogeneous tournament: the countable dense linear order without endpoints $(\mathbb{Q}, <)$, the local order $S(2)$, the generic tournament T.*
- *a Henson digraph: a digraph M_T as the Fraïssé limit of the class $\mathscr{C}_T$ of digraphs that do not embed any member of a set of pairwise non-embeddable finite tournaments T (where $|T| \geq 3$). Note that this definition includes omitting an empty set of tournaments; this is the generic digraph D.*
- *a I_n-free digraph ($n \geq 3$): the Fraïssé limit of the class $\mathscr{C}_n$ of digraphs that do not embed the independent set I_n.*

- *a member of a countable collection of homogeneous digraphs with imprimitive automorphism groups: including disjoint unions of homogeneous tournaments, or random orientations of homogeneous* n*-partite graphs.*
- *an exceptional case: either the myopic local order* $S(3)$*, or the homogeneous digraph* $\mathcal{P}(3)$*.* $\square$

3

Cofinality, strong cofinality and the Bergman property

As mentioned in the introduction, the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a first-order structure $\mathcal{M}$ provides many interesting examples of infinite permutation groups. More recent developments in infinite permutation group theory ask questions about the efficiency of generating these groups. This research stems from an influential paper of Bergman [2] in which it was proved that for any generating set U of $\mathrm{Sym}(\mathbb{N})$, there is some $n \in \mathbb{N}$ such that $\mathrm{Sym}(\mathbb{N}) = U \cup U^2 \cup \ldots \cup U^n$. This strong condition was subsequently generalised; an infinitely generated group G has the *Bergman property* if for any generating set U of G, there exists $n \in \mathbb{N}$ such that $G = U \cup U^2 \cup \ldots \cup U^n$. Droste and Holland [26] later described a connection between the Bergman property and the *cofinality* of a group, a notion previously studied by Sabbagh [74] and Macpherson and Neumann [55]. This result utilised a new notion of *strong cofinality*. As some automorphism groups of structures are examples of infinitely generated groups, cofinality and strong cofinality results for some $\mathrm{Aut}(\mathcal{M})$ have been considered by many authors, often from different perspectives [25, 18, 46].

There have also been extensive studies in the theory of generating infinite semigroups. The *rank* of a semigroup S is defined to be the size of the smallest generating set for S. Higgins, Howie and Ruškuc [36] defined the notion of a *relative rank* of a semigroup S modulo some subset X of S; further results on relative ranks by the same authors (together with Mitchell) were presented in [35]. Soon after cofinality, strong cofinality and the Bergman property was defined for the case of groups, the analogous notions for semigroups were developed by Maltcev, Mitchell and Ruškuc [59]. The connection made between these concepts in [27] was also generalised; the statement of this theorem is reproduced in Proposition 3.1.2. From here, and following work of Mesyan [63], they considered cofinality and generation results for examples of classical semigroups; an example of such a result is:

Theorem 3.0.1 (Theorem 4.1, [59]). *If X is an infinite set, then End(X), Inv(X), Part(X) all have uncountable strong cofinality and hence the Bergman property.* □

This was proved by showing that the semigroups in question are *strongly distorted*; a property which (with some extra work) implies uncountable strong cofinality [59, Lemma 2.4].

Furthermore, related to these properties is the notion of a *Sierpiński rank* of a semigroup S; this is the least natural number n (if it exists) in which every countable sequence of elements of S is contained in an n-generated semigroup [64]. It was in the process of investigating this property that Mitchell and Péresse [64] determined that both the monomorphism and epimorphism monoids on a countable set did not have the Bergman property. In the same paper, Mitchell and Péresse [64] noted that the Sierpiński rank of a strongly distorted semigroup is finite.

As with groups, examples of infinitely generated semigroups arise from first-order structures $\mathcal{M}$. Work of Dolinka [23, Theorem 2.2] extended the End(X) part of Theorem 3.0.1 to first-order structures, and determined those monoids that satisfied the conditions of Theorem 2.2 [23] had a Sierpiński rank of at most 3. By utilising techniques from category theory and the established literature on homogeneous structures, as well as using homomorphism-homogeneity (a notion of [14] and [60], see Chapter 6 and Chapter 7), he was able to determine a selection of Fraïssé limits whose endomorphism monoids had uncountable strong cofinality.

This brief chapter contains a summary of useful results concerning cofinality, strong cofinality and the Bergman property in Section 3.1, with an initial case study of the monomorphisms of a countable set in Section 3.2. Whilst these sections contain basic lemmas and restatements of previously known results from the literature (in particular [59] and [64]), these are included to aid the understanding of the work on intermediate monoids of a σ-structure $\mathcal{M}$ in Chapter 4. This is because Emb($\mathcal{M}$) and Mon($\mathcal{M}$) can be viewed as subsemigroups of the monomorphism monoid Mon(M) on the domain M of $\mathcal{M}$; properties described in Section 3.2 will guide the investigations into Emb($\mathcal{M}$) and Mon($\mathcal{M}$) in Section 4.2 and Section 4.3 respectively.

3.1 Definitions and general results

The following definitions can be found in [59]. We say that the *cofinality* of an infinitely generated semigroup S is the least cardinal λ such that there exists a chain of proper subsemigroups $(U_i)_{i<\lambda}$ where $\bigcup_{i<\lambda} U_i = S$. We denote the cofinality of S by cf(S), and we call the chain $(U_i)_{i<\lambda}$ a *cofinal chain* for S. The *strong cofinality* of an infinitely generated semigroup S is the least cardinal κ such that there exists a chain of proper subsets $(V_i)_{i<\kappa}$ such that for all $i < \kappa$ there exists a $j < \kappa$ such that $V_iV_i \subseteq V_j$ and $S = \bigcup_{i<\kappa} V_i$. We denote the strong cofinality of S by scf(S) and we call $(V_i)_{i<\kappa}$ a *strong cofinal chain*. It is clear that cf(S) $\geq$ scf(S).

Definition 3.1.1. Let S be a non-finitely generated semigroup. Say that S is *semigroup Cayley bounded* with respect to a set U that generates S as a semigroup if $S = U \cup U^2 \cup ... \cup U^n$ for some $n \in \mathbb{N}$. We say that S has the *semigroup Bergman property* if it is Cayley bounded for every generating set U of S.

Remarks. (i) As we do not consider the Bergman property for groups in this thesis, we refer to the semigroup Bergman property as just the Bergman property.

(ii) To show that S does not have the Bergman property, it is enough to find a 'bad' generating set U of S; that is, showing that there exists a generating set U of S that is not Cayley bounded.

As mentioned in the introduction, there is a connection between cofinality, strong cofinality and the Bergman property for a semigroup. This was first proved for the group case in [26]; this version is a result of [59].

Proposition 3.1.2 (Proposition 2.2, [59]). *Let S be a non-finitely generated semigroup. Then:*

(i) scf$(S) > \aleph_0$ if and only if S has the Bergman property and cf$(S) > \aleph_0$;

(ii) If scf$(S) > \aleph_0$, then scf$(S) =$ cf(S). □

For an non-finitely generated semigroup S, this theorem provides four possible cases regarding the cofinality of S and whether or not S has the Bergman property:

(a) $\mathrm{cf}(S) \geq \mathrm{scf}(S) > \aleph_0$ and S has the Bergman property;

(b) $\mathrm{cf}(S) > \mathrm{scf}(S) = \aleph_0$ and S does not have the Bergman property;

(c) $\mathrm{cf}(S) = \mathrm{scf}(S) = \aleph_0$ and S has the Bergman property;

(d) $\mathrm{cf}(S) = \mathrm{scf}(S) = \aleph_0$ and S does not have the Bergman property.

Examples of groups and semigroups that satisfy each of the above four cases can be found in the literature; we summarise some below. For example, any group that has uncountable strong cofinality also has the semigroup Bergman property by [59, Corollary 2.5], and so satisfies (a); groups like $\mathrm{Sym}(\mathbb{N})$ [2], the automorphism group of the countable dense linear order $\mathrm{Aut}(\mathbb{Q}, <)$, and the automorphism group of the random graph $\mathrm{Aut}(R)$ (both [26]). Semigroups that satisfy (a) include the full transformation monoid $\mathrm{End}(\mathbb{N})$, the symmetric inverse monoid on an infinite set $\mathrm{Inv}(\mathbb{N})$ (both [59]) and the endomorphism monoid of the random graph $\mathrm{End}(R)$ [69]. A semigroup that satisfies (b) is the *bounded symmetric group on* $\mathbb{Q}$ (see [25]); cofinality results are proved in [59], using results from [25]. Examples of semigroups satisfying (c) include the infinitely generated left/right zero semigroups and the infinitely generated rectangular band [59]. Additionally, there is also a group with countable cofinality and the Bergman property; this is a construction of Khelif given in [47]. Finally, groups and semigroups that satisfy (d) include free groups and semigroups of infinite rank, and the *Baer-Levi semigroup* on the natural numbers (given by all injective maps that leave out infinitely many elements from the image) [59].

Our first two cofinality results are basic and extend some ideas of [59].

Lemma 3.1.3. *Let S be an infinitely generated countable semigroup. Then cf$(S) = \aleph_0$.*

Proof. Suppose that $U = \{u_1, u_2, ...\}$ is a generating set for S; as S is countable, so is U. Now consider the chain of subsemigroups $V_1 \subseteq V_2 \subseteq V_3 \subseteq ...$ where V_i is the subsemigroup generated by $\langle u_1, u_2, ..., u_i\rangle$. As S is not finitely generated then V_i is a proper subsemigroup of S for all $i \in \mathbb{N}$. It follows that $\bigcup_{i\in\mathbb{N}} V_i$ is a cofinal chain for S. □

Lemma 3.1.4. *Let S be an infinitely generated semigroup. Suppose that T is an infinitely generated subsemigroup of S and I is an ideal of S such that $S = T \sqcup I$. Then cf$(S) \leq$ cf(T).*

Proof. Let $\bigcup_{i<\kappa} U_i = T$ be a cofinal chain for T. Then $\bigcup_{i<\kappa}(U_i \sqcup I) = S$ and so κ is an upper bound for the cofinality of S. □

Remark. With $T \leq S$ as in Lemma 3.1.4, we can note from this that if T has countable cofinality, then so does S.

We can link the idea of cofinality of an infinite semigroup to the relative rank of a semigroup; this is a definition of Higgins, Howie and Ruškuc [36].

Definition 3.1.5 ([36])**.** Suppose that S is a semigroup and A is a subset of S. The *relative rank* rank$(S : A)$ *of S modulo A* is the minimum cardinality of a set B such that $\langle A \cup B\rangle = S$.

For example, rank(End($\mathbb{N}$) : Sym($\mathbb{N}$)) $= 2$ [36], rank(Bin($\mathbb{N}$) : Mon($\mathbb{N}$)) $= 1$ and rank(Bin($\mathbb{N}$) : Epi($\mathbb{N}$)) $= 2$, where Bin($\mathbb{N}$) is the *binary relation monoid* on $\mathbb{N}$ (both [35]). We can use the concept of relative rank of a semigroup S modulo a subsemigroup T to connect the cofinality of T and S. The following is a special case of [68, Proposition 6.1].

Proposition 3.1.6 ([68]). *Let T be a subsemigroup of an infinitely generated semigroup S. If cf$(T) > \aleph_0$ and rank$(T : S)$ is finite then cf$(S) > \aleph_0$.* □

Example 3.1.7. As rank(End(ℕ) : Sym(ℕ)) $= 2$ [36], this result can be used to show that End(ℕ) has uncountable cofinality; re-proving a result of [63].

We now consider a different notion of rank on an infinitely generated semigroup.

Definition 3.1.8 ([64]). Let S be an infinitely generated semigroup. The *Sierpiński rank* of S is defined to be the smallest natural number n (if it exists) such that any countable sequence $(s_n)_{n\in\mathbb{N}}$ of elements in S is contained in an n-generated subsemigroup of S.

If there exists such an n, then say that S has finite Sierpiński rank; if it does not exist, then S has infinite Sierpiński rank. This property is so named due to a result of Sierpiński [77] in which he showed, for a countable set X, that any countable sequence of elements of End(X) is contained in a 4-generated subsemigroup of End(X); he later reduced this to a 2-generated subsemigroup (for more on the history on this, see [64]).

For each integer $m \geq 1$, there exists a semigroup with Sierpiński rank m: the infinite monogenic semigroup has rank 1 [64], the symmetric inverse semigroup on a countable set Inv(X) has rank 2 [42], the semigroup of increasing functions $f : [0, 1] \to [0, 1]$ has rank 3 [65], and Mon$(\aleph_n)$ has rank $n + 4$ for all $n \in \mathbb{N}_0$ [64]. Examples of semigroups with infinite Sierpiński rank include any infinitely generated countable semigroup, the Baer-Levi semigroup on ℕ and Mon$(\aleph_\omega)$ and Epi$(\aleph_\omega)$ (all [64]).

Definition 3.1.9 ([59]). A semigroup S is *strongly distorted* if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of natural numbers and $N_S \in \mathbb{N}$ such that for all sequences $(s_n)_{n\in\mathbb{N}}$ of elements from S there exist $t_1, t_2, ..., t_{N_S} \in S$ such that each s_n can be written as a product of length at most a_n in the elements $t_1, t_2, ..., t_{N_S}$.

Figure 3.1 helps to illustrate this definition. We say that

Element of $(s_n)_{n\in\mathbb{N}}$	s_1	s_2	s_3	...	s_n	...
Length of product of t_i's equal to s_n	a_1	a_2	a_3	...	a_n	...

Figure 3.1: Strong distortion

If S is strongly distorted then S has finite Sierpiński rank [64]. However, the converse is not true; it may take arbitrarily long products of elements from $t_1, t_2, ..., t_{N_S}$ to generate every element in the sequence $(s_n)_{n\in\mathbb{N}}$. The next lemma of [59] details the relationship between strong distortion and uncountable strong cofinality.

Lemma 3.1.10 (Lemma 2.4 [59]). *If S is non-finitely generated and strongly distorted, then scf$(S) > \aleph_0$.* □

Showing that a semigroup S is strongly distorted is a common way to show that S has uncountable strong cofinality (and hence the Bergman property by Proposition 3.1.2). Examples of strongly distorted semigroups include all monoids mentioned in Theorem 3.0.1 [59] and the endomorphism monoid of the random graph End(R) [69]. By generalising the example of End(R), Dolinka [23] determined sufficient conditions for endomorphism monoids of first-order structures to be strongly distorted. These conditions applied to several Fraïssé limits, including (amongst others) the generic poset P and any infinite-dimensional vector space over a finite field.

Finally in this section, we turn our attention to a result that implies *countable* strong cofinality of a semigroup S. This important proposition shows that if S contains a certain ideal structure, then we can form a countable strong cofinal chain for S.

Proposition 3.1.11. *Let S be an infinitely generated semigroup. Suppose that S has an infinite descending chain of ideals $S = I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$ and assume that $J = \bigcap_{i\in\mathbb{N}} I_i$ is non-empty. Let $L_i = I_i \setminus I_{i+1}$ and suppose also that $L_iL_j \subseteq (\bigcup_{n=0}^{h} L_n) \cup J$ for some $h \in \mathbb{N}$. Then scf$(S) = \aleph_0$.*

Proof. For any $a \in \mathbb{N}$, define $W_a = (\bigcup_{n=0}^{a} L_n) \cup J$. Note that W_a is a chain of proper subsets of S. Suppose that $x, y \in W_a$. Our aim is to show that there exists some $b \in \mathbb{N}$ where $xy \in W_b$. As J is an ideal, if x or y is in J then $xy \in J$; as each W_a contains J, it follows that $xy \in W_a$. Now, suppose that x and y are in $\bigcup_{n=0}^{a} L_n$. So $x \in L_i$ and $y \in L_j$ for some $i, j < k$; by our assumption, there exists a m such that $xy \in (\bigcup_{n=0}^{m} L_n) \cup J = W_m$. As a is finite we can choose $b \in \mathbb{N}$ such that $xy \in (\bigcup_{n=0}^{b}) L_n \cup J = W_b$ for all $x \in L_i$, $y \in L_j$ and $0 \leq i, j \leq k$. Therefore, for each $a \in \mathbb{N}$ there exists $b \in \mathbb{N}$ such that $W_a W_a \subseteq W_b$ and so scf$(S) = \aleph_0$. $\square$

3.2 Initial example: Mon$(\mathbb{N})$

This section provides a brief semigroup-theoretic overview of Mon$(\mathbb{N})$, the monomorphism monoid of a countable set. This includes basic facts such as regularity and Green's relations, and also considers cofinality and strong cofinality results. Whilst some of the results in this section are known, we include them to add insight to the later work on intermediate monoids of σ-structures in Sections 4.1, 4.2 and 4.3.

3.2.1 Semigroup-theoretic properties

We begin by stating a simple property of a monomorphism of $\mathbb{N}$.

Definition 3.2.1 ([40]). Let α be an element of Mon$(\mathbb{N})$. We define the *defect* of α to be the set $D(\alpha) = \mathbb{N} \smallsetminus \text{im } \alpha$.

Our first, fundamental lemma explains the defect of the composition of two monomorphisms. This result is folklore, but is mentioned in [16, Vol 2. Lemma 8.1]. The proof is included for completeness, and to illustrate similar techniques in later sections.

Lemma 3.2.2. *Let α and β be elements of Mon$(\mathbb{N})$. Then $D(\alpha\beta) = D(\beta) \cup D(\alpha)\beta$, and this is a disjoint union.*

Proof. We show containment both ways. Suppose $n \in D(\alpha\beta)$; so by definition n cannot be in im $\alpha\beta$. On one hand, assume that there exists an m such that

$m\beta = n$. If this occurs, then m cannot be in im α as then $m\beta = n \in$ im $\alpha\beta$, which is a contradiction. So m must be in $D(\alpha)$ and therefore $n \in D(\alpha)\beta$. On the other hand, if there is no such m such that $m\beta = n$, then n is not in im β by definition; hence $n \in D(\beta)$.

Conversely, assume that $n \in D(\beta)$; so there is no $m \in \mathbb{N}$ such that $m\beta = n$. As im $\alpha \subseteq \mathbb{N}$ there must be no $m \in$ im α such that this occurs; hence, $n \notin$ im $\alpha\beta$ and therefore $n \in D(\alpha\beta)$. Now suppose that $n \in D(\alpha)\beta$. Then there exists an $m \in D(\alpha)$ such that $m\beta = n$. As $m \notin$ im α and β is injective, it follows that $m\beta \notin$ im $\alpha\beta$ and so $n \in D(\alpha\beta)$.

Finally, as $D(\beta)$ consists of elements not in im β by definition and $D(\alpha)\beta$ consists of elements in the image of β, the union $D(\beta) \cup D(\alpha)\beta$ is disjoint. $\square$

Remark. Note that $\alpha \in \text{Mon}(\mathbb{N})$ is a bijection (and so $\alpha \in \text{Sym}(\mathbb{N})$) if and only if $D(\alpha) = \varnothing$.

The fact that Lemma 3.2.2 is a disjoint union leads us to two easy corollaries of the result.

Corollary 3.2.3. *(1) Mon*$(\mathbb{N})$ *is not regular.*

(2) The only idempotent element in Mon$(\mathbb{N})$ *is the identity element.*

Proof. (1) It is enough to show that any regular element is an element of Sym$(\mathbb{N})$. Suppose $\alpha \in \text{Mon}(\mathbb{N})$ is a regular element; so there exists a $\beta \in \text{Mon}(\mathbb{N})$ such that $\alpha\beta\alpha = \alpha$. By Lemma 3.2.2, it follows that

$$D(\alpha) = D(\alpha\beta\alpha) = D(\alpha) \cup D(\beta\alpha)\alpha.$$

As this is a disjoint union of sets, it follows that $D(\beta\alpha)\alpha = \varnothing$ and so $D(\beta\alpha) = \varnothing$. Using Lemma 3.2.2 again gives $D(\alpha) \cup D(\beta)\alpha = \varnothing$, and so $D(\alpha) = \varnothing$. This implies that $\alpha \in \text{Sym}(\mathbb{N})$.

(2) As every idempotent e of a semigroup is regular, it follows from (1) that $e \in \text{Sym}(\mathbb{N})$. But the only idempotent element of a group is the identity element, proving the claim. $\square$

Remark. In Lemma 2.2.6, we showed that Mon($\mathbb{N}$) is a right cancellative monoid; in fact, the only idempotent of a right cancellative monoid is the identity. Furthermore, a regular semigroup with a single idempotent is a group [16]; so a right cancellative monoid is regular if and only if it is a group.

If $D(\alpha)$ is finite, we denote the cardinality of $D(\alpha)$ by $d(\alpha) \in \mathbb{N}$. If $D(\alpha)$ is infinite, then we write $d(\alpha) = \infty$. Our next result is a special case of [35, Lemma 4.4].

Corollary 3.2.4. *Let α and β be elements of Mon$(\mathbb{N})$.*

(1) If $D(\alpha)$ and $D(\beta)$ are finite, then $d(\alpha\beta) = d(\alpha) + d(\beta)$.

(2) $d(\alpha\beta) = \infty$ if and only if $d(\alpha)$ or $d(\beta)$ is infinite.

Proof. (1) Lemma 3.2.2 asserts that $D(\beta) \cup D(\alpha)\beta$ is a disjoint union. As β is an injection we see that $|D(\alpha)\beta| = |D(\alpha)| = d(\alpha)$ and therefore $d(\alpha\beta) = d(\alpha) + d(\beta)$.

(2) The forward direction follows from part (1); the converse from Lemma 3.2.2. □

The previous result proves our next lemma concerning ideals of Mon($\mathbb{N}$).

Lemma 3.2.5. *For $k \in \mathbb{N}$, define the set $I_k = \{\beta \in Mon(\mathbb{N}) \mid d(\beta) \geq k\}$. Then I_k is an ideal of Mon$(\mathbb{N})$. Furthermore, $I_\infty = \{\gamma \in Mon(\mathbb{N}) \mid d(\alpha) = \infty\}$ is also an ideal of Mon$(\mathbb{N})$.*

Proof. Follows from Lemma 3.2.2 and Corollary 3.2.4. □

Remark. It is important to see that if $n \leq m$, then $I_m \subseteq I_n$; furthermore, $I_\infty \subseteq I_k$ for all $k \in \mathbb{N}$. So this lemma provides an infinite descending chain of ideals of Mon($\mathbb{N}$), where $\bigcap_{k\in\mathbb{N}} I_k = I_\infty$ is non-empty. We will say more on this in the subsection on generation results.

We now move on to characterising Green's relations in Mon($\mathbb{N}$). The next results provide a blueprint of the approach we will take in determining Green's relations for injective endomorphisms of σ-structures. Recall from Section 2.2 that the group of units U of Mon($\mathbb{N}$) is the symmetric group Sym($\mathbb{N}$).

Lemma 3.2.6. *Let $\alpha, \beta \in Mon(\mathbb{N})$.*

(1) Suppose that $\alpha \mathscr{L} \beta$. Then for all $\gamma, \delta \in Mon(\mathbb{N})$ such that $\gamma\alpha = \beta$ and $\delta\beta = \alpha$, the maps γ and δ are bijections.

(2) Suppose that $D(\alpha)$ and $D(\beta)$ are finite, and that $\alpha \mathscr{J} \beta$. For all $\gamma, \delta, \epsilon, \zeta \in Mon(\mathbb{N})$ such that $\gamma\alpha\delta = \beta$ and $\epsilon\beta\zeta = \alpha$, the maps $\gamma, \delta, \epsilon, \zeta$ are bijections.

Proof. (1) Suppose that $\gamma, \delta \in \mathrm{Mon}(\mathbb{N})$ are as in the statement. Therefore, $\gamma\delta\beta = \beta$ and $\delta\gamma\alpha = \alpha$. As $\mathrm{Mon}(\mathbb{N})$ is right-cancellative, it follows that $\gamma\delta = 1 = \delta\gamma$; by definition, $\gamma, \delta \in \mathrm{Sym}(\mathbb{N})$.

(2) Assume that $\gamma, \delta, \epsilon, \zeta \in \mathrm{Mon}(\mathbb{N})$ are such that $\gamma\alpha\delta = \beta$ and $\epsilon\beta\zeta = \alpha$. As the defects of α and β are finite, we can apply Corollary 3.2.4 twice to see that $d(\gamma\alpha\delta) = d(\gamma) + d(\alpha) + d(\delta)$ and $d(\epsilon\beta\zeta) = d(\epsilon) + d(\beta) + d(\zeta)$. Putting these equations together gives:

$$\begin{aligned} d(\beta) &= d(\gamma\alpha\delta) \\ &= d(\gamma) + d(\alpha) + d(\delta) \\ &= d(\gamma) + d(\epsilon) + d(\beta) + d(\zeta) + d(\delta). \end{aligned}$$

Here, $d(\gamma) + d(\epsilon) + d(\zeta) + d(\delta) = 0$ and so $d(\gamma) = d(\delta) = d(\epsilon) = d(\zeta) = 0$; therefore they are all bijections.

□

Remark. As part (2) of this lemma covers the $\mathscr{J}$ case, and $\mathscr{J} \supseteq \mathscr{D} \supseteq \mathscr{R}$ in general, a similar result holds for the $\mathscr{D}$ and $\mathscr{R}$ relations.

Example 3.2.7. Lemma 3.2.6 (1) does not hold for the $\mathscr{R}$-relation in general. For example, define $\alpha, \beta \in \mathrm{Mon}(\mathbb{N})$ by $n\alpha = 2n$ and $n\beta = 4n$ for all $n \in \mathbb{N}$. Furthermore, set

$$n\gamma = \begin{cases} 2n & \text{if } n \text{ even} \\ n & \text{if } n \text{ odd} \end{cases} \quad \text{and } n\delta = \begin{cases} n/2 & \text{if } n \equiv 0 \mod 4 \\ 2n - (2k+1) & \text{if } n = 4k + r,\ r = 1, 2, 3 \end{cases}$$

Here, $\gamma, \delta \in \mathrm{Mon}(\mathbb{N})$ are such that $\alpha\gamma = \beta$ and $\beta\delta = \alpha$; so $\alpha\mathscr{R}\beta$. But $d(\alpha) = d(\beta) = d(\gamma) = \infty$, and $d(\delta) = 0$.

Our next lemma is straightforward yet important; placing restrictions on the number of cases needed to characterise Green's relations in Mon($\mathbb{N}$).

Lemma 3.2.8. *Suppose that $\alpha, \beta \in Mon(\mathbb{N}) = S$ such that $D(\alpha)$ is finite and $D(\beta)$ is infinite. Then α and β are not $\mathscr{J}$-related.*

Proof. By assumption, β is in the ideal I_∞ (see Lemma 3.2.5) and so $S^1\beta S^1 \subseteq I_\infty$. But $\alpha \notin I_\infty$, so $\alpha \notin S^1\beta S^1$ and therefore α and β are not $\mathscr{J}$-related. □

We now have enough to describe the Green's relations in Mon($\mathbb{N}$).

Proposition 3.2.9. *Let α and β be monomorphisms of the natural numbers. Then:*

(1) $\alpha\mathscr{L}\beta$ if and only if $D(\alpha) = D(\beta)$;

(2) $\alpha\mathscr{R}\beta$ if and only if $d(\alpha) = d(\beta)$, and hence $\mathscr{L} = \mathscr{H}$;

(3) $\alpha\mathscr{J}\beta$ if and only if $d(\alpha) = d(\beta)$, and hence $\mathscr{R} = \mathscr{D} = \mathscr{J}$.

Proof. (1) Suppose that $\alpha\mathscr{L}\beta$. By Lemma 3.2.6 (1) there exist bijections γ and δ such that $\gamma\alpha = \beta$ and $\delta\beta = \alpha$. So then $D(\beta) = D(\gamma\alpha) = D(\alpha) \cup [D(\gamma)]\alpha$. But as $D(\gamma)$ is empty, it follows that $D(\beta) = D(\alpha)$.

Conversely, assume that $D(\alpha) = D(\beta)$. As this occurs, im α and im β are the same sets. So for each $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that $m\alpha = n\beta$ in im α = im β. As both α and β are injective, such an n is unique. Now, define a map $\gamma : \mathbb{N} \to \mathbb{N}$ that sends n to m whenever $n\beta = m\alpha$. This is a bijection with $\gamma\alpha = \beta$. We can use a similar argument to find a $\delta \in \mathrm{Sym}(\mathbb{N})$ such that $\delta\beta = \alpha$ and so $\alpha\mathscr{L}\beta$.

(2) Suppose that $\alpha\mathscr{R}\beta$; so there exist monomorphisms γ and δ such that $\alpha\gamma = \beta$ and $\beta\delta = \alpha$. If $D(\alpha)$ is finite, then $D(\beta)$ must also be finite by Lemma 3.2.8. By Lemma 3.2.6 (2), γ and δ must both be bijections. Using Lemma 3.2.2 gives $D(\beta) = D(\alpha\gamma) = D(\gamma) \cup [D(\alpha)]\gamma$. Since γ is a bijection, $D(\beta) = D(\alpha)\gamma$

and therefore $d(\alpha) = d(\beta)$. If $d(\alpha)$ is infinite, then by Lemma 3.2.8 $d(\beta)$ must also be infinite; and so $d(\alpha) = d(\beta) = \infty$.

On the other hand, assume that $d(\alpha) = d(\beta)$. Then as the defects are the same size, we can define a bijection f that takes $D(\alpha)$ to $D(\beta)$. We can then extend f to a map γ that sends $m\alpha$ to $m\beta$ for all $m \in \mathbb{N}$. As the defect and image are disjoint sets, this forms a bijection $\gamma : \mathbb{N} \to \mathbb{N}$ such that $\alpha\gamma = \beta$. As γ is a bijection, there exists a δ such that $\delta\gamma = 1 = \gamma\delta$. Using this gives $\beta\delta = \alpha\gamma\delta = \alpha$ and so $\alpha \mathscr{R} \beta$.

Furthermore, if $D(\alpha) = D(\beta)$ then $d(\alpha) = d(\beta)$, but the converse is not necessarily true. Therefore, $\mathscr{L} \subseteq \mathscr{R}$ and so $\mathscr{H} = \mathscr{L} \cap \mathscr{R} = \mathscr{L}$.

(3) Suppose that $\alpha \mathscr{J} \beta$. By definition, there exist monomorphisms γ, δ, ϵ and ζ such that $\gamma\alpha\delta = \beta$ and $\epsilon\beta\zeta = \alpha$. If $D(\alpha)$ is finite, then $D(\beta)$ is also finite by Lemma 3.2.8. In this case, γ, δ, ϵ and ζ are bijections by Lemma 3.2.6 (2). Using Lemma 3.2.2 twice and the fact that $D(\gamma) = D(\delta) = \varnothing$ gives:

$$\begin{aligned} D(\beta) = D(\gamma\alpha\delta) &= D(\delta) \cup [D(\gamma\alpha)]\delta \\ &= [D(\alpha) \cup [D(\gamma)]\alpha]\delta \\ &= [D(\alpha)]\delta. \end{aligned}$$

Hence $d(\alpha) = d(\beta)$ as in the $\mathscr{R}$-related case. Finally, if $d(\alpha) = \infty$, then $d(\beta) = \infty$ by Lemma 3.2.8 and we are done. Conversely, assume that $d(\alpha) = d(\beta)$. Then $\alpha \mathscr{R} \beta$ by part (2) of this result; and so $\alpha \mathscr{J} \beta$ by definition.

Finally, as $\mathscr{R} = \mathscr{J}$ in Mon($\mathbb{N}$) and $\mathscr{R} \subseteq \mathscr{D} \subseteq \mathscr{J}$ in general (see Figure 2.3), it follows that $\mathscr{R} = \mathscr{D}$ in Mon($\mathbb{N}$).

□

Remark. Note that by the construction in the proof of Proposition 3.2.9 (3), if two elements α, β of Mon($\mathbb{N}$) are $\mathscr{J}$-related then we can find bijections $\gamma, \delta \in$ Sym($\mathbb{N}$) such that $\alpha\gamma = \beta$ and $\beta\delta = \alpha$.

3.2.2 Generation results for Mon($\mathbb{N}$)

Most of this subsection comprises a restatement of previously known results from [64]; as with the rest of this section, we include them here as guidance for our approach to similar problems involving intermediate monoids of infinite σ-structures. Our first result states the relative rank of Mon($\mathbb{N}$) modulo the symmetric group Sym($\mathbb{N}$); this is due to Mitchell and Péresse [64].

Proposition 3.2.10 (Proposition 4.2 [64]). *rank*$(Mon(\mathbb{N}) : Sym(\mathbb{N})) = 2$. □

Following this, and the fact that cf(Sym($\mathbb{N}$)) $> \aleph_0$, we have enough information to determine the cofinality and strong cofinality of Mon($\mathbb{N}$). This corollary also re-proves an observation concerning the Bergman property of Mon($\mathbb{N}$) [64, Proposition 4.2].

Corollary 3.2.11. *cf*$(Mon(\mathbb{N})) > \aleph_0$ *and scf*$(Mon(\mathbb{N})) = \aleph_0$. *Furthermore, Mon*($\mathbb{N}$) *does not have the Bergman property.*

Proof. By Proposition 3.2.10, rank(Mon($\mathbb{N}$) : Sym($\mathbb{N}$)) $= 2$; therefore cf(Mon($\mathbb{N}$)) $=$ cf(Sym($\mathbb{N}$)) $> \aleph_0$ by Proposition 3.1.6. For the strong cofinality, the ideal structure outlined in Lemma 3.2.5 satisfies the conditions of Proposition 3.1.11 and so scf(Mon($\mathbb{N}$)) $= \aleph_0$. So Mon($\mathbb{N}$) does not have the Bergman property by Proposition 3.1.2. □

Finally, we conclude this section with a result concerning subsemigroups of Mon($\mathbb{N}$) intersecting with the ideal structure of Mon($\mathbb{N}$) given in Lemma 3.2.5, simplifying the conditions of Proposition 3.1.11 in these cases.

Proposition 3.2.12. *Let T be an infinitely generated subsemigroup of Mon*($\mathbb{N}$) *such that for all $i \in \mathbb{N}$ there exists an $\alpha \in T$ such that $d(\alpha) = i$, and there exists $\beta \in T$ such that $d(\beta) = \infty$. Then scf$(T) = \aleph_0$.*

Proof. Taking I_k as written in Lemma 3.2.5, define $J_k = T \cap I_k$ for all $k \in \mathbb{N}$; by Lemma 2.2.1, every such J_k is an ideal. Since $I_1 \subseteq I_2 \subseteq ...$ is a chain of ideals, and there exists an α in T such that $d(\alpha) = i$ for all $i \in \mathbb{N}$, none of the J_k are empty and hence $J_1 \subseteq J_2 \subseteq ...$ forms a chain of ideals. Furthermore,

$M_k = J_k \setminus J_{k+1}$ is nonempty; by Corollary 3.2.4, $M_k M_l \subseteq \bigcup_{i=0}^{k+l} M_i$. Finally, we notice that $I_\infty \cap T$ is non-empty as $d(\beta) = \infty$. So T has the ideal structure outlined in Proposition 3.1.11 and therefore $\mathrm{scf}(T) = \aleph_0$. $\square$

4

Intermediate monoids of first-order structures

Let $\mathcal{M}$ be a first-order structure. As we have seen in Subsection 2.2.2, there are four other monoids contained between $\text{Aut}(\mathcal{M})$ and $\text{End}(\mathcal{M})$. These are:

- $\text{Bi}(\mathcal{M})$, the monoid of all bijective endomorphisms of $\mathcal{M}$ (the *bimorphism monoid* of $\mathcal{M}$);
- $\text{Emb}(\mathcal{M})$, the monoid of all embeddings of $\mathcal{M}$ (the *embedding monoid* of $\mathcal{M}$);
- $\text{Mon}(\mathcal{M})$, the monoid of all injective endomorphisms of $\mathcal{M}$ (the *monomorphism monoid* of $\mathcal{M}$), and
- $\text{Epi}(\mathcal{M})$, the monoid of all surjective endomorphisms of $\mathcal{M}$ (the *epimorphism monoid* of $\mathcal{M}$).

We call these *intermediate monoids of* $\mathcal{M}$. Our aim for this chapter is to study the semigroup theory of intermediate monoids of a relational first-order structure $\mathcal{M}$ that are made up of injective endomorphisms of $\mathcal{M}$; that is, $\text{Bi}(\mathcal{M})$, $\text{Emb}(\mathcal{M})$ and $\text{Mon}(\mathcal{M})$. As bimorphisms and embeddings are special cases of monomorphisms, it makes sense to study $\text{Bi}(\mathcal{M})$ and $\text{Emb}(\mathcal{M})$ prior to the more general $\text{Mon}(\mathcal{M})$. Following Lemma 2.3.2, it makes no sense to talk about intermediate monoids of finite structures. To that end, we take $\mathcal{M}$ to be a first-order structure over a relational language $\sigma = \{R_i : i \in I\}$ on a countably infinite domain M throughout the chapter. Furthermore, if $\gamma \in \text{Mon}(\mathcal{M})$, we write $M\gamma$

for the image *set* of the function γ, and $\mathcal{M}\gamma$ for the *structure* induced by $\mathcal{M}$ on $M\gamma$.

The bimorphism monoid of a σ-structure $\mathcal{M}$ is of particular interest here. As a collection of bijective endomorphisms of $\mathcal{M}$, it is embeddable in the symmetric group on the domain M of $\mathcal{M}$ and so $\text{Bi}(\mathcal{M})$ is a *group-embeddable monoid*. The study of group-embeddable monoids was a principal interest of early semigroup theorists; a typical result of this study is Ore's Theorem [16]. A main focus of the thesis continues the study of group-embeddable monoids from the point of view of bimorphism monoids of relational first-order structures. Section 4.1 begins this study by considering the general semigroup theory of bimorphism monoids, as well as representing group-embeddable monoids as bimorphism monoids of σ-structures.

We will also look at results concerning cofinality, strong cofinality and the Bergman property for intermediate monoids of some σ-structure $\mathcal{M}$. Here, there is no guarantee that the intermediate monoids considered are distinct from each other; for instance, in the pure set case, $\text{Mon}(\mathbb{N}) = \text{Emb}(\mathbb{N})$ and $\text{Bi}(\mathbb{N}) = \text{Sym}(\mathbb{N})$. Following this observation, we consider the random graph R (see Example 2.4.2); here, $\text{Mon}(R) \neq \text{Emb}(R)$ and $\text{Bi}(R) \neq \text{Aut}(R)$ as a monomorphism of R need not be an embedding. Furthermore, there is a body of literature on R outlining useful properties that we can use; Subsection 2.4.1 contains some examples. To that end, we investigate generation results for intermediate monoids of R in Subsections 4.1.4, 4.2.3 and 4.3.2. To avoid only looking at graphs, we also consider cofinality and generation results for intermediate monoids of the discrete linear order $(\mathbb{N}, \leq)$; but as $\text{Epi}(\mathbb{N}, \leq) = \text{Bi}(\mathbb{N}, \leq) = \text{Aut}(\mathbb{N}, \leq) = \{e\}$, we only consider $\text{Mon}(\mathbb{N}, \leq) = \text{Emb}(\mathbb{N}, \leq)$ in Subsection 4.2.2.

The structure of this chapter is as follows. Section 4.1 presents an introduction to the semigroup theory of bimorphism monoids of σ-structures, including idempotents, ideals, and Green's relations. We then use the results established in Subsection 4.1.1 and Subsection 4.1.2 to investigate bimorphisms of graphs in Subsection 4.1.4, including cofinality results for the bimorphism monoid of the random graph R. Similarly, Section 4.2 gives an introduction to the semigroup theory of monoids of embeddings of σ-structures, including cofinality results for embeddings of the discrete linear order $(\mathbb{N}, \leq)$ (Subsection 4.2.2) and the random graph R (Subsection 4.2.3). Finally, we give a brief overview of the semigroup theory of monomorphism monoids of σ-structures in Section 4.3, concluding with cofinality results for Mon(R) in Subsection 4.3.2.

4.1 Bimorphisms of σ-structures

The first intermediate monoid of a σ-structure $\mathcal{M}$ we study in this chapter is the bimorphism monoid Bi$(\mathcal{M})$; this is the collection of all bijective endomorphisms of $\mathcal{M}$. Whereas automorphisms are bijective maps that preserve relations and non-relations, bimorphisms only preserve relations; they may change a number of non-relations to relations. Aside from presaging a future discussion on monomorphisms of $\mathcal{M}$, bimorphism monoids are an interesting topic to study in their own right for their similarities to groups; this is further explored in later chapters. This section provides an introduction to the topic from a semigroup-theoretic perspective.

4.1.1 Initial semigroup theory of Bi$(\mathcal{M})$

As mentioned in the introduction, each element of the bimorphism monoid is a bijection from the domain M of $\mathcal{M}$ to itself; therefore, Bi$(\mathcal{M})$ is a submonoid of Sym(M) and thus is a *group-embeddable monoid*. This fact leads on to our first result on bimorphism monoids of first-order structures.

Lemma 4.1.1. *Let $\mathcal{M}$ be a σ-structure. Then Bi$(\mathcal{M})$ is a cancellative monoid.*

Proof. As a collection of bijections, Bi$(\mathcal{M})$ is a group-embeddable monoid via some monoid embedding. As every subsemigroup of a group is cancellative (see Section 2.2), Bi$(\mathcal{M})$ is cancellative. □

The facts that any cancellative monoid has only one idempotent element (the identity) [16, Exercises §1.1], and any regular cancellative monoid is a group (Thierrin, see [16, Exercises §1.9]) yields the following easy corollary.

Corollary 4.1.2. *Let $\mathcal{M}$ be a σ-structure.*

(1) The only idempotent element in Bi($\mathcal{M}$) is the identity.

(2) Bi($\mathcal{M}$) is a regular monoid if and only if Bi($\mathcal{M}$) = Aut($\mathcal{M}$). □

As Aut($\mathcal{M}$) is the group of units of End($\mathcal{M}$), and Bi($\mathcal{M}$) $\subseteq$ End($\mathcal{M}$), we can deduce that:

Corollary 4.1.3. *For any σ-structure $\mathcal{M}$, the group of units of Bi($\mathcal{M}$) is the automorphism group Aut($\mathcal{M}$).* □

Remark. This statement holds for any intermediate monoid T such that Aut($\mathcal{M}$) $\leq$ $T \leq$ End($\mathcal{M}$) by a similar argument.

In our initial example on the intermediate monoid Mon($\mathbb{N}$), we used defects of monomorphisms to determine semigroup-theoretic properties of this monoid; such as the characterisation of Green's relations for Mon($\mathbb{N}$) in Proposition 3.2.9. Such an approach is not useful when working with bimorphisms; as bijections, every $\alpha \in$ Bi($\mathcal{M}$) has an empty defect. In order to study similar results, it is therefore necessary to introduce an analogous notion of defect on the level of relations rather than vertices. Our next definition codifies this, formalising the notion of a bimorphism "changing a non-relation to a relation".

Definition 4.1.4. For a bimorphism α of $\mathcal{M}$, define a σ-structure $\mathcal{A}(\alpha)$ with domain M and relations

$$\bar{a} \in R_i^{\mathcal{A}(\alpha)} \text{ if and only if } \bar{a} \notin R_i^{\mathcal{M}} \text{ and } \bar{a}\alpha \in R_i^{\mathcal{M}}$$

for all $i \in I$. We say that $\mathcal{A}(\alpha)$ is the *additional structure* of α. Define the *support* of α to be the set

$$S(\alpha) = \{x \in M : x \in \bar{a} \text{ and } \bar{a} \in R_i^{\mathcal{A}(\alpha)} \text{ for some } i \in I\}.$$

As $S(\alpha)$ is a subset of M, we can induce a structure $\mathcal{M}[S(\alpha)]$ on $S(\alpha)$ with relations from $\mathcal{M}$; call this the *support structure* of α.

Example 4.1.5. Let $\mathcal{M}$ be a graph with vertex set $\mathbb{Z}$ and adjacencies $i \sim j$ if and only if $i \leq 0$ and $j = i - 1$ (see Figure 4.1).

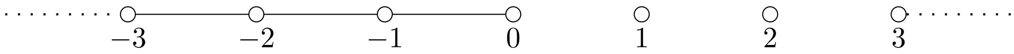


Figure 4.1: $\mathcal{M}$ as in Example 4.1.5

Consider the bimorphism $\alpha \in \mathrm{Bi}(\mathcal{M})$ defined by $i\alpha = i - 2$ for all $i \in \mathbb{Z}$. Then $\mathcal{A}(\alpha)$ is the graph on $\mathbb{Z}$ with the only two adjacencies given by $0 \sim 1$ and $1 \sim 2$, $S(\alpha)$ is the set $\{0, 1, 2\}$, and $\mathcal{M}[S(\alpha)]$ is the null graph induced by $\mathcal{M}$ on the vertex set $S(\alpha)$ (see Figure 4.2).

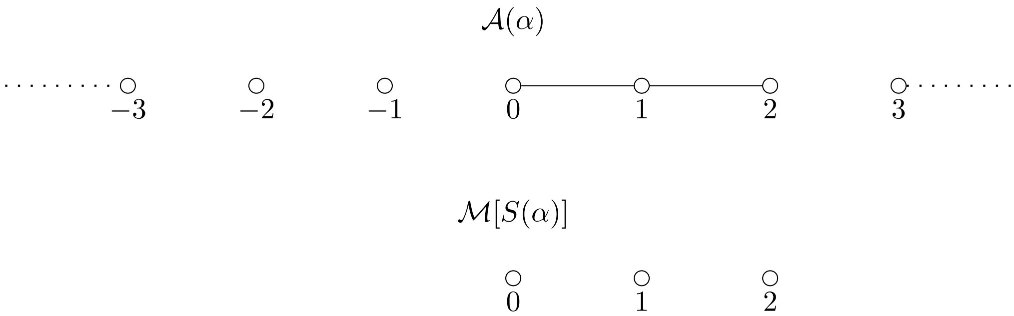


Figure 4.2: $\mathcal{A}(\alpha)$ and $\mathcal{M}(S(\alpha))$ for $\alpha \in \mathrm{Bi}(\mathcal{M})$ in Example 4.1.5

For two elements α, β of $\mathrm{Bi}(\mathcal{M})$ and some $R_i \in \sigma$ with arity n, define the set

$$R_i^{\mathcal{A}(\beta)}\alpha^{-1} = \{\bar{x} \in M^n : \bar{x}\alpha \in R_i^{\mathcal{A}(\beta)}\}.$$

We now consider a fundamental lemma, analogous to Lemma 3.2.2, that underpins many of the results in this section.

Lemma 4.1.6. *Suppose that $\alpha, \beta \in Bi(\mathcal{M})$ and $R_i \in \sigma$. Then $R_i^{\mathcal{A}(\alpha\beta)} = R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ and this is a disjoint union.*

Remark. The idea here is that the set of relations added in by the product $\alpha\beta$ is the same set of relations given by first applying α and then β. This is reflected in the two terms of the union; $R_i^{\mathcal{A}(\alpha)}$ is the set of relations added in by α, and $R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ is the set of relations added in by β after α has been applied.

Proof. The proof is by containment both ways. Suppose first that $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$; so $\bar{a} \notin R_i^{\mathcal{M}}$ but $\bar{a}\alpha \in R_i^{\mathcal{M}}$. As β preserves relations, $\bar{a}\alpha\beta \in R_i^{\mathcal{M}}$ and so $\bar{a} \in R_i^{\mathcal{A}(\alpha\beta)}$. Now suppose that $\bar{a} \in R_i^{\mathcal{A}(\beta)}\alpha^{-1}$; so $\bar{a}\alpha \notin R_i^{\mathcal{M}}$ and $\bar{a}\alpha\beta \in R_i^{\mathcal{M}}$. As α must preserve relations, it follows that $\bar{a} \notin R_i^{\mathcal{M}}$ and so $\bar{a} \in R_i^{\mathcal{A}(\alpha\beta)}$.

Conversely, assume that $\bar{a} \in R_i^{\mathcal{A}(\alpha\beta)}$. There are two cases to consider; either $\bar{a}\alpha \in R_i^{\mathcal{M}}$, or it isn't. If $\bar{a}\alpha$ is in $R_i^{\mathcal{M}}$, then $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$ and we are done. If it is not, then $a\alpha \notin R_i^{\mathcal{M}}$; but as $\bar{a} \in R_i^{\mathcal{A}(\alpha\beta)}$, we have that $\bar{a}\alpha\beta \in R_i^{\mathcal{M}}$. So $\bar{a}\alpha \in R_i^{\mathcal{A}(\beta)}$ and hence $\bar{a} \in R_i^{\mathcal{A}(\beta)}\alpha^{-1}$; therefore the sets $R_i^{\mathcal{A}(\alpha\beta)}$ and $R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ are equal.

It remains to show that this is a disjoint union. Assume for a contradiction that $\bar{a} \in R_i^{\mathcal{A}(\alpha)} \cap R_i^{\mathcal{A}(\beta)}\alpha^{-1}$. So $\bar{a}\alpha \in R_i^{\mathcal{M}}$ as $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$; but as $\bar{a} \in R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ we have that $\bar{a}\alpha \in R_i^{\mathcal{A}(\beta)}$ and so not in $R_i^{\mathcal{M}}$. This is a contradiction and so the sets are disjoint. □

Remark. Here, the two terms of the disjoint union $R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ describe the relations added in by α and β respectively.

The fact that this is a disjoint union provides an immediate corollary. For some bimorphism α of $\mathcal{M}$ and relation $R_i \in \sigma$, define $e_i(\alpha) = |R_i^{\mathcal{A}(\alpha)}|$, writing $e_i(\alpha) = \infty$ if $R_i^{\mathcal{A}(\alpha)}$ is infinite.

Corollary 4.1.7. *Let α, β be bimorphisms of a σ-structure $\mathcal{M}$, and suppose that $R_i \in \sigma$.*

(1) If both $e_i(\alpha)$ and $e_i(\beta)$ are finite then $e_i(\alpha\beta) = e_i(\alpha) + e_i(\beta)$.

(2) $e_i(\alpha\beta) = \infty$ if and only if at least one of $R_i^{\mathcal{A}(\alpha)}$ or $R_i^{\mathcal{A}(\beta)}$ is infinite.

Proof. (1) As α is a bijection, $|R_i^{\mathcal{A}(\beta)}\alpha^{-1}| = |R_i^{\mathcal{A}(\beta)}| = e_i(\beta)$. Since the union of $R_i^{\mathcal{A}(\alpha)}$ and $R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ is disjoint by Lemma 4.1.6, it follows that

$$|R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}| = |R_i^{\mathcal{A}(\alpha)}| + |R_i^{\mathcal{A}(\beta)}\alpha^{-1}| = e_i(\alpha) + e_i(\beta).$$

(2) As with Corollary 3.2.4 (2), the forward direction is by part (1); the converse follows from Lemma 4.1.6.

□

Following this, we can write $e_i(\alpha) + e_i(\beta) < \infty$ to signify that both $e_i(\alpha)$ and $e_i(\beta)$ are finite. On the other hand, a bimorphism that changes no non-relations to relations is an automorphism; hence we can say that $\alpha \in \text{Aut}(\mathcal{M})$ if and only if $R_i^{\mathcal{A}(\alpha)} = \varnothing$ (or that $e_i(\alpha) = 0$) for all $R_i \in \sigma$. We outline a simple application of Corollary 4.1.7.

Corollary 4.1.8. *Let $\mathcal{M}$ be a σ-structure, and let $R_i \in \sigma$ and $k \in \mathbb{N}$. and define $I(i,k) := \{\alpha \in Bi(\mathcal{M}) \mid e_i(\alpha) \geq k\}$. Then, if non-empty, $I(i,k)$ is an ideal of $Bi(\mathcal{M})$. Furthermore, if $I(i,\infty) := \{\alpha \in Bi(\mathcal{M}) \mid e_i(\alpha) = \infty\}$ is non-empty, then it also an ideal.*

Proof. Follows immediately from Corollary 4.1.7. □

4.1.2 Green's relations of Bi($\mathcal{M}$)

We now focus our attention on determining the Green's relations of Bi($\mathcal{M}$). As Bi($\mathcal{M}$) is a group-embeddable monoid, and the Green's relations for a group are trivial, a description of Green's relations for Bi($\mathcal{M}$) will depend on relation-preserving properties of the maps rather than the underlying maps themselves. To see this, note that if $a, b, c \in \text{Sym}(M)$ are such that $ac = b$, then $c = a^{-1}b$ is uniquely determined as Sym(M) is a group. As Bi($\mathcal{M}$) is embeddable in this group, if $\alpha, \beta \in \text{Bi}(\mathcal{M})$ and $\gamma \in \text{Sym}(M)$ such that $\alpha\gamma = \beta$, then γ is uniquely determined by the bijection $\alpha^{-1}\beta$; it is a bimorphism if and only if $\alpha^{-1}\beta$ is an endomorphism of $\mathcal{M}$. A similar result holds if $\delta\alpha = \beta$; here, δ is uniquely determined by $\beta\alpha^{-1}$. This rigidity in choice of map therefore places strict conditions on when two bimorphisms are $\mathscr{L}$ or $\mathscr{R}$-related in Bi($\mathcal{M}$). This allows us to obtain results for σ-structures in full generality. Our first result emphasises this point.

Lemma 4.1.9. *Let $\mathcal{M}$ be σ-structure and suppose that $\alpha, \beta \in Bi(\mathcal{M})$.*

(1) Suppose that $\alpha \mathscr{L} \beta$. Then for all $\gamma, \delta \in Bi(\mathcal{M})$ such that $\gamma\alpha = \beta$ and $\delta\beta = \alpha$, the maps γ and δ are automorphisms.

(2) Suppose that $\alpha \mathscr{R} \beta$. Then for all $\gamma, \delta \in Bi(\mathcal{M})$ such that $\alpha\gamma = \beta$ and $\beta\delta = \alpha$, the maps γ and δ are automorphisms.

(3) Suppose that $e_i(\alpha) + e_i(\beta) < \infty$ for all $R_i \in \sigma$, and $\alpha \mathscr{J} \beta$. For all $\gamma, \delta, \epsilon, \zeta \in$ Bi$(\mathcal{M})$ such that $\gamma\alpha\delta = \beta$ and $\epsilon\beta\zeta = \alpha$, the maps $\gamma, \delta, \epsilon, \zeta$ are automorphisms.

Proof. For (1), it follows from the assumptions that $\gamma\delta\beta = \beta$; as Bi$(\mathcal{M})$ is cancellative by Lemma 4.1.1, this implies that $\gamma\delta$ is the identity map. So by Corollary 4.1.3, γ and δ are both automorphisms. The proof of (2) is similar.

To prove (3), assume that $\gamma, \delta, \epsilon, \zeta \in$ Bi$(\mathcal{M})$ are as in the statement. Therefore $\gamma\epsilon\beta\zeta\delta = \beta$, and so $e_i(\gamma\epsilon\beta\zeta\delta) = e_i(\beta)$ for all $R_i \in \sigma$. As $e_i(\beta)$ is finite, so is $e_i(\gamma\epsilon\beta\zeta\delta)$; in particular, $\gamma, \delta, \epsilon, \zeta \in$ Bi$(\mathcal{M})$ do not add infinitely many relations for any $R_i \in \sigma$ by Corollary 4.1.7 (2). As this happens, we can use Corollary 4.1.7 (1) four times to get:

$$e_i(\gamma) + e_i(\epsilon) + e_i(\beta) + e_i(\zeta) + e_i(\delta) = e_i(\beta).$$

Since $e_i(\beta)$ is a natural number, this implies that $e_i(\gamma) = e_i(\epsilon) = e_i(\zeta) = e_i(\delta) = 0$ for all $R_i \in \sigma$, and so $\gamma, \delta, \epsilon, \zeta$ are automorphisms. □

An immediate consequence of this lemma is as follows.

Corollary 4.1.10. *Let $\mathcal{M}$ be a σ-structure and $\alpha \in$ Bi$(\mathcal{M})$. Suppose that $e_i(\alpha) < \infty$ for all $R_i \in \sigma$. Then $J_\alpha = D_\alpha$.*

Proof. The proof follows from Lemma 4.1.9 and Lemma 2.2.3. □

We now characterise $\mathscr{L}$ and $\mathscr{R}$ relations in the bimorphism monoid of a σ-structure $\mathcal{M}$.

Proposition 4.1.11. *Let $\alpha, \beta \in$ Bi$(\mathcal{M})$. Then $\alpha \mathscr{L} \beta$ if and only if $S(\alpha)\alpha = S(\beta)\beta$ and the bimorphism $\alpha\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$.*

Proof. First, suppose that $\alpha \mathscr{L} \beta$; so there exists $\gamma = \alpha\beta^{-1}$ such that $\gamma\beta = \alpha$. By Lemma 4.1.9, γ is an automorphism. Due to this and Lemma 4.1.6 (1):

$$R_i^{\mathcal{A}(\alpha)} = R_i^{\mathcal{A}(\gamma\beta)} = R_i^{\mathcal{A}(\gamma)} \cup R_i^{\mathcal{A}(\beta)}\gamma^{-1} = R_i^{\mathcal{A}(\beta)}\gamma^{-1}$$

for all $i \in \mathbb{N}$. By this, it follows that $S(\alpha) = S(\gamma\beta) = S(\beta)\gamma^{-1}$; as γ is an automorphism, $\mathcal{M}[S(\alpha)] \cong \mathcal{M}[S(\beta)]$ via this automorphism. It remains to show that

$S(\alpha)\alpha = S(\beta)\beta$; here, as $S(\alpha)\alpha\beta^{-1} = S(\alpha)\gamma = S(\beta)$, we have that $S(\alpha)\alpha\beta^{-1}\beta = S(\alpha)\alpha = S(\beta)\beta$.

For the converse direction, let $\alpha, \beta \in \text{Bi}(\mathcal{M})$. We need to find bimorphisms γ, δ such that $\gamma\beta = \alpha$ and $\delta\alpha = \beta$. To do this, we show that the uniquely determined bijection $\gamma = \alpha\beta^{-1}$ is a bimorphism of $\mathcal{M}$. Suppose that $\bar{a} \in R_i^{\mathcal{M}}$. There are two cases to consider; either $\bar{a} \in R_i^{\mathcal{M}[S(\alpha)]}$, or it is not.

In the first case, suppose that $\bar{a} \in R_i^{\mathcal{M}[S(\alpha)]}$. Then $\bar{a}\alpha \in R_i^{\mathcal{M}[S(\alpha)\alpha]}$ which, as $S(\alpha)\alpha = S(\beta)\beta$, means that $\bar{a}\alpha \in R_i^{\mathcal{M}[S(\beta)\beta]}$. Hence, $\bar{a}\alpha\beta^{-1} \in S(\beta)^n$; as $\alpha\beta^{-1}$ induces an isomorphism between $\mathcal{M}[S(\alpha)]$ and $\mathcal{M}[S(\beta)]$, we deduce that $\bar{a}\alpha\beta^{-1} \in R_i^{\mathcal{M}[S(\beta)]}$.

For the second case, note that if $\bar{a} \in R_i^{\mathcal{M}}$ and $\bar{a} \notin R_i^{\mathcal{M}[S(\alpha)]}$ then $\bar{a} \notin S(\alpha)^n$. Since this happens, $\bar{a}\alpha \notin (S(\alpha)\alpha)^n$ and hence $\bar{a}\alpha \notin (S(\beta)\beta)^n$. Therefore $\bar{a}\alpha\beta^{-1} \notin S(\beta)^n$; because of this $\bar{a}\alpha\beta^{-1} \notin R_i^{\mathcal{A}(\beta)}$. As $\bar{a}\alpha \in R_i^{\mathcal{M}}$ we have that $\bar{a}\alpha\beta^{-1} \in R_i^{\mathcal{M}}$ by definition of the support structure.

In both of these cases, $\alpha\beta^{-1}$ preserves relations and hence it is a bimorphism of $\mathcal{M}$. The proof that the map $\beta\alpha^{-1} = \delta$ is a bimorphism is similar and so $\alpha\mathscr{L}\beta$. $\square$

Remark. Note that these conditions imply that if two elements α and β are $\mathscr{L}$-related, then $\alpha\beta^{-1} : \mathcal{A}(\alpha) \to \mathcal{A}(\beta)$ is an isomorphism.

Proposition 4.1.12. *Suppose that $\alpha, \beta \in Bi(\mathcal{M})$. Then $\alpha\mathscr{R}\beta$ if and only if $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$.*

Proof. First, suppose that $\alpha\mathscr{R}\beta$; by definition and Lemma 4.1.9 there are automorphisms γ, δ such that $\alpha\gamma = \beta$ and $\beta\delta = \alpha$. By Lemma 4.1.6, we have that $R_i^{\mathcal{A}(\beta)} = R_i^{\mathcal{A}(\alpha\gamma)} = R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\gamma)}\alpha^{-1}$; as γ is an automorphism, $R_i^{\mathcal{A}(\gamma)} = \varnothing$ for all $i \in I$ and so $R_i^{\mathcal{A}(\beta)} = R_i^{\mathcal{A}(\alpha)}$ for all $i \in I$. This implies that $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$.

For the converse direction, let $\alpha, \beta \in \text{Bi}(\mathcal{M})$ with $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$. We need to find bimorphisms γ, δ such that $\alpha\gamma = \beta$ and $\beta\delta = \alpha$. To do this, we show that the uniquely determined bijection $\gamma = \alpha^{-1}\beta$ is an endomorphism of $\mathcal{M}$. Suppose that $\bar{a} \in R_i^{\mathcal{M}}$. There are two cases to consider; either $\bar{a} \in R_i^{\mathcal{M}[S(\alpha)\alpha]}$, or it is not.

For the first case, we have that $\bar{a} \in R_i^{\mathcal{M}[S(\alpha)\alpha]}$ and so $\bar{a} \in (S(\alpha)\alpha)^n$. As α is a bijection, $\bar{a}\alpha^{-1} \in S(\alpha)^n$ and therefore there are two further choices; either $\bar{a}\alpha^{-1} \in R_i^{\mathcal{M}}$ or it isn't. If it is the former, then $\bar{a}\alpha^{-1}\beta \in R_i^{\mathcal{M}}$ and so the relation is preserved. If it is the latter, then $\bar{a}\alpha^{-1} \in R_i^{\mathcal{A}(\alpha)}$ by definition. As $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$, we have that $\bar{a}\alpha^{-1} \in R_i^{\mathcal{A}(\beta)}$ and so $\bar{a}\alpha^{-1}\beta \in R_i^{\mathcal{M}}$.

In the second case, $\bar{a} \notin R_i^{\mathcal{M}[S(\alpha)\alpha)]}$ and so $\bar{a} \notin (S(\alpha)\alpha)^n$; implying that $\bar{a}\alpha^{-1} \notin S(\alpha)^n$. Now, if $\bar{a}\alpha^{-1} \notin R_i^{\mathcal{M}}$, then $\bar{a}\alpha^{-1} \in R_i^{\mathcal{A}(\alpha)}$ and so $\bar{a}\alpha^{-1} \in S(\alpha)^n$, a contradiction. So $\bar{a}\alpha^{-1} \in R_i^{\mathcal{M}}$ and therefore so is $\bar{a}\alpha^{-1}\beta$.

So γ preserves relations in both cases, and therefore $\alpha^{-1}\beta = \gamma$ is a bimorphism of $\mathcal{M}$. The proof that $\beta^{-1}\alpha = \delta$ is a bimorphism is similar and so $\alpha\mathscr{R}\beta$.

□

Remark. By definition of the $\mathscr{H}$-relation and the above two propositions, two bimorphisms α and β are $\mathscr{H}$ related if and only if $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$, $S(\alpha)\alpha = S(\beta)\beta$ and the bimorphism $\alpha\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$.

We can use the previous two results to characterise the $\mathscr{D}$ relation for Bi($\mathcal{M}$).

Theorem 4.1.13. *Let $\alpha, \beta \in Bi(\mathcal{M})$. Then $\alpha\mathscr{D}\beta$ if and only if there exists a bimorphism η such that: $\eta\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$, $\alpha^{-1}\eta$ induces an isomorphism from $\mathcal{M}[S(\alpha)\alpha]$ to $\mathcal{M}[S(\beta)\beta]$, and $S(\beta)\beta = S(\eta)\eta$.*

Proof. First, suppose that $\alpha\mathscr{D}\beta$; so there exists a bimorphism η such that $\alpha\mathscr{R}\eta$ and $\eta\mathscr{L}\beta$. By Lemmas 4.1.11 and 4.1.12 $S(\eta)\eta = S(\beta)\beta$, the bimorphism $\eta\beta^{-1}$ induces an isomorphism from that $\mathcal{M}[S(\eta)]$ to $\mathcal{M}[S(\beta)]$, and $\mathcal{A}(\alpha) = \mathcal{A}(\eta)$. As $\mathcal{A}(\alpha) = \mathcal{A}(\eta)$, it follows that $S(\alpha) = S(\eta)$ and so $\mathcal{M}[S(\eta)] = \mathcal{M}[S(\alpha)]$. Therefore $\eta\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$. Now as $\alpha\mathscr{R}\eta$, the uniquely defined bijection $\alpha^{-1}\eta$ is an automorphism by Lemma 4.1.9 and Proposition 4.1.12. Hence

$$\mathcal{M}[S(\alpha)\alpha]\alpha^{-1}\eta = \mathcal{M}[S(\alpha)\alpha\alpha^{-1}\eta] = \mathcal{M}[S(\alpha)\eta].$$

Now, as $S(\alpha) = S(\eta)$, and $S(\eta)\eta = S(\beta)\beta$:

$$\mathcal{M}[S(\alpha)\alpha]\alpha^{-1}\eta = \mathcal{M}[S(\alpha)\eta] = \mathcal{M}[S(\eta)\eta] = \mathcal{M}[S(\beta)\beta]$$

and so $\alpha^{-1}\eta$ induces an isomorphism from $\mathcal{M}[S(\alpha)\alpha]$ to $\mathcal{M}[S(\beta)\beta]$.

For the converse direction, we show that $\eta\mathscr{L}\beta$ and $\alpha\mathscr{R}\eta$ in that order. Note first that $\eta^{-1}\alpha$ induces an isomorphism from $\mathcal{M}[S(\beta)\beta]$ to $\mathcal{M}[S(\alpha)\alpha]$. Therefore, $\mathcal{M}[S(\alpha)\alpha] = \mathcal{M}[S(\beta)\beta]\eta^{-1}\alpha$; since $S(\eta)\eta = S(\beta)\beta$ by assumption, $\mathcal{M}[S(\alpha)\alpha] = \mathcal{M}[S(\eta)\eta]\eta^{-1}\alpha$. As $\eta^{-1}\alpha$ is an isomorphism,

$$\mathcal{M}[S(\eta)\eta]\eta^{-1}\alpha = \mathcal{M}[S(\eta)\eta\eta^{-1}\alpha] = \mathcal{M}[S(\eta)\alpha] = \mathcal{M}[S(\alpha)\alpha].$$

Since this occurs and α is bijective, $S(\eta) = S(\alpha)$. Now, as $S(\eta) = S(\alpha)$ and $\eta\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$, it follows that $\mathcal{M}[S(\eta)] \cong \mathcal{M}[S(\beta)]$ by the same isomorphism. From this, and our assumption that $S(\eta)\eta = S(\beta)\beta$, it follows that $\beta\mathscr{L}\eta$ by Proposition 4.1.11.

Now suppose that $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$; so $\bar{a} \in S(\alpha)$ by definition. Then $\bar{a}\alpha \in R_i^{\mathcal{M}}$ and $\bar{a}\alpha \in S(\alpha)\alpha$. As $\alpha^{-1}\eta$ describes an isomorphism from $\mathcal{M}[S(\alpha)\alpha)]$ to $\mathcal{M}[S(\beta)\beta]$, it follows that $\bar{a}\alpha\alpha^{-1}\eta = \bar{a}\eta \in R_i^{\mathcal{M}}$ and so $\bar{a} \in R_i^{\mathcal{A}(\eta)}$. We can use a similar argument to show that if $\bar{a} \in R_i^{\mathcal{A}(\eta)}$ then $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$; hence $\mathcal{A}(\alpha) = \mathcal{A}(\eta)$ and so $\alpha\mathscr{R}\eta$ by Proposition 4.1.12. □

Remark. If $e_i(\alpha) < \infty$ for all $R_i \in \sigma$, and $\beta \in \mathrm{Bi}(\mathcal{M})$ is such that there exists $R_i \in \sigma$ such that $e_i(\beta) = \infty$, then it follows from this result that α is not $\mathscr{D}$-related to β.

We can use Theorem 4.1.13 and Corollary 4.1.10 in order to give a partial classification for $\mathscr{J}$-relations in $\mathrm{Bi}(\mathcal{M})$; here, $D_\alpha = J_\alpha$ if α adds in finitely many relations for each $R_i \in \sigma$. On the other hand, if α adds in *infinitely* many relations, then there is no guarantee that $\beta \in J_\alpha$ is $\mathscr{J}$-related to α by automorphisms. Subsequently, this is a far more difficult question and one we leave open.

Question 4.1.14. *Classify Green's $\mathscr{J}$-relation in Bi($\mathcal{M}$).*

4.1.3 Representing group-embeddable monoids

It is a celebrated theorem of Frucht [33] that any finite group arises as the automorphism group of a finite undirected graph. Frucht's theorem was later extended to any group arising as the automorphism group of some infinite graph; this was independently proved by de Groot [20] and Sabidussi [75]. As bimorphism monoids provide natural examples of group-embeddable monoids, it is natural to ask: does every group-embeddable monoid arise as the bimorphism monoid of a structure?

This question is a natural generalisation of Frucht's theorem; this can be seen in considering the case where the group-embeddable monoid M is finite. If this happens, then M is a group as every finite cancellative semigroup is a group (see Section 2.2). As the bimorphism monoid $\mathrm{Bi}(\Gamma)$ of a finite graph Γ is the automorphism group by Lemma 2.3.2, then the question reduces to Frucht's theorem.

The infinite case is less straightforward. There are many examples of group-embeddable monoids in the literature; a widely studied example is the *free monoid* A^* on some set A. These are monoids with elements given by strings of elements of A (called *words*), with the identity element given by the *empty word* ε. The composition of two words is concatenation. Here, A^* naturally embeds in the *free group* on A [58]; the elements of which are reduced words over $A \cup A^{-1}$, where a word w is reduced if it does not contain $a^{-1}a$ or aa^{-1} for any $a \in A$.

It is well known that the free monoid on a singleton set is isomorphic to the infinite monogenic semigroup with identity $(\mathbb{N}_0, +)$. We present an example of a graph Γ such that $\mathrm{Bi}(\Gamma) \cong (\mathbb{N}_0, +)$.

Example 4.1.15. Consider a graph Γ with vertex set $\mathbb{Z} \times \{0, 1\}$, with adjacencies given by

- $(a, 0) \sim (b, 1)$ if and only if $a = b$;
- $(a, 0) \sim (b, 0)$ if and only if $|a - b| = 1$, and;
- $(a, 1) \sim (b, 1)$ if and only if $a \leq 0$ and $|a - b| = 1$.

This forms a graph given below in Figure 4.3.

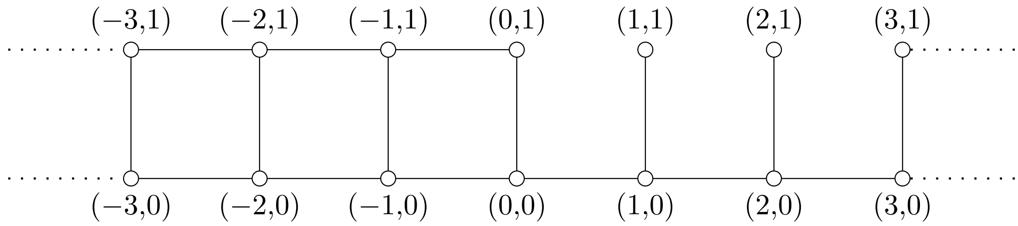


Figure 4.3: Construction of Γ

As $(0,1)$ is the only vertex of degree 2, any automorphism of Γ must fix $(0,1)$. In addition, an automorphism of Γ cannot swap $(0,0)$ and $(-1,1)$, as $(0,0)$ has a degree 1 vertex at distance 2 while $(-1,1)$ does not. So $(0,1)$ and $(0,1)$ are fixed, and it follows from this that every vertex is fixed and so Γ is rigid. However, there do exist bijections on $V\Gamma$ such that only edges are preserved. Consider the map $\alpha : \Gamma \to \Gamma$ given by $(a,x)\alpha = (a-1,x)$. This preserves all edges and sends the non-edge between $(0,1)$ and $(1,1)$ to the edge between $(-1,1)$ and $(0,1)$. We claim that the only bimorphisms of Γ are of the form α^n.

Claim. $\mathrm{Bi}(\Gamma) \cong (\mathbb{N}_0,+)$, the infinite monogenic semigroup with identity.

Proof of Claim. We show that the only bijective maps on vertices that preserve edges are of the form $\beta : \Gamma \to \Gamma$ such that $(b,x)\beta = (b-n,x)$; this is proved using a case analysis. Consider the vertex $(0,1)$. There is no bimorphism α sending some vertex $(a,0)$ to $(0,1)$; this is because $\deg((a,0)) > \deg((0,1))$ and every bimorphism preserves edges. Similarly, no bimorphism sends $(b,1)$ to $(0,1)$ for $b < 0$. So suppose that $\beta \in \mathrm{Bi}(\Gamma)$ maps some $(b,1)$ to $(0,1)$ for $b \geq 0$.

It follows that β must send $(b,0)$ to either $(1,-1)$ or $(0,0)$, in order to preserve the adjacency $(b,0) \sim (b,1)$. Suppose that $(b,0)\beta = (-1,1)$. This gives rise to two cases:

Case 1: This is where $(b+1,0)\beta = (-2,1)$ and $(b-1,0)\beta = (-1,0)$. We consider the image point of $(b-2,0)$ under β. There are two choices; either $(b-2,0)\beta$ is $(-2,0)$ or $(0,0)$. Suppose initially that $(b-2,0)\beta$ is $(-2,0)$. Since β preserves edges, the only potential image point for the two vertices $(b-3,0)$ and $(b-2,1)$ (both adjacent to $(b-2,0)$) is $(-3,0)$; as β is a bijection, this cannot happen. So now suppose that $(b-2,0)\beta = (0,0)$, the other potential choice. This means that $(1, 0)$ is the only potential image point for the vertices $(b - 3, 0)$ and $(b - 2, 1)$; another contradiction. Therefore, this case cannot occur.

Case 2: On the other hand, this is where $(b-1, 0)\beta = (-2, 1)$ and $(b+1, 0)\beta = (-1, 0)$. In this case, we consider the image point of $(b + 2, 0)$ under β; as above, this is either $(-2, 0)$ or $(0, 0)$. Assume that $(b+2, 0)\beta = (-2, 0)$. Here, this would leave $(-3, 0)$ as the only potential image point for both vertices $(b + 3, 0)$ and $(b + 2, 1)$; this is a contradiction as β is a bijection. So $(b + 2, 0)\beta = (0, 0)$, which means that $(1, 0)$ is the only potential image point for both $(b+3, 0)$ and $(b+2, 1)$. Therefore, this case also cannot occur; so it follows that $(b, 0)\beta$ cannot be $(-1, 1)$.

So $(b, 0)\beta = (0, 0)$ and hence β maps the edge $(b, 0) \sim (b, 1)$ to the edge $(0, 0) \sim (0, 1)$. We can use another, similar case analysis to show that $(b+1, 0)\beta = (1, 0)$. Finally, we can show that β cannot send $(a, 0)$ to $(b, 1)$ for $b < 0$ by another similar argument; this implies that β must preserve the infinite two way line $\{(a, 0) \, : \, a \in \mathbb{Z}\}$. All of these results together imply that β must be a shift map and so the only bijective homomorphisms are of the form $(b, x)\beta = (b - n, x)$, for $n \geq 0$. It is not hard to see that $\alpha^n = \beta$ and thus Bi(Γ) has a single generator. As mentioned above, any infinite semigroup that is generated by one element is isomorphic to $(\mathbb{N}, +)$; so Bi$(\Gamma) \cong (\mathbb{N}, +)$. $\square$

This automatically proves Green's relations of Bi(Γ); here, $\mathscr{H} = \mathscr{L} = \mathscr{R} = \mathscr{D} = \mathscr{J} = \{(a, a) \, : \, a \in \text{Bi}(\Gamma)\}$. It follows that Proposition 4.1.11, Proposition 4.1.12 and Theorem 4.1.13 all hold in this case; albeit in a rather trivial fashion.

It seems to be somewhat more difficult to represent A^* on larger sets as a bimorphism monoid on a graph; for instance, this graph must be rigid as the group of units of A^* consists solely of the empty word ε. However, by relaxing the restriction on the type of relational structure we consider, this problem can be made easier. For instance, defining a structure based on the Cayley graph of the free group on A (in this case, a labelled, directed graph), and adding extra relations on the part of the graph corresponding to the positive free monoid in that group, gives a rigid structure where each bimorphism may be representable as words in the free monoid. However, this remains to be proved, and would represent only a partial solution to our overreaching problem.

Question 4.1.16. *Does every countable group-embeddable monoid arise as the bimorphism monoid of some σ-structure $\mathcal{M}$? In particular, is this question true when $\mathcal{M}$ is restricted to the class of graphs?*

4.1.4 Bimorphisms of graphs

Recall from Example 2.3.1 that a simple, undirected graph can be expressed as a σ-structure Γ, where σ consists of a single binary relation E, and Γ models sentences expressing irreflexivity and symmetry for E. Because of this single relation E, most of the previously considered definitions and results about bimorphisms of structures are simplified in the case of graphs. To begin with, we rephrase Definition 4.1.4 in the language of graphs (see Section 2.4) for more details).

Definition 4.1.17. Let α be a bimorphism of a graph Γ. The *additional graph* of α, denoted $\mathcal{A}(\alpha)$, is the graph on $V\Gamma$ with adjacencies $\{a, b\} \in E\mathcal{A}(\alpha)$ if and only if $\{a\alpha, b\alpha\} \in E\Gamma$ and $\{a, b\} \notin E\Gamma$. The *support* of a bimorphism α of Γ is the vertex set defined by

$$S(\alpha) := \{x \in V\mathcal{A}(\alpha) : \deg_{\mathcal{A}(\alpha)}(x) \neq 0\}.$$

Finally, we denote the number of edges in $E\mathcal{A}(\alpha)$ by $e(\alpha)$; writing $e(\alpha) = \infty$ if α adds in infinitely many edges.

The following result is a rephrasing of Lemma 4.1.6 and Corollary 4.1.7 in the setting of graphs.

Corollary 4.1.18. *Let α and β be bimorphisms of a graph Γ.*

(1) $E\mathcal{A}(\alpha\beta) = E\mathcal{A}(\alpha) \cup E\mathcal{A}(\beta)\alpha^{-1}$, and this is a disjoint union.

(2) If $e(\alpha)$ and $e(\beta)$ are both finite, then $e(\alpha\beta) = e(\alpha) + e(\beta)$.

(3) $e(\alpha\beta) = \infty$ if and only if one of $e(\alpha)$ or $e(\beta)$ is infinite. □

Corollary 4.1.19. *Let k be a natural number and define $I_k := \{\alpha \in Bi(\Gamma) \mid e(\alpha) \geq k\}$. Then I_k is an ideal of $Bi(\Gamma)$. Furthermore, $I_\infty := \{\alpha \in Bi(\Gamma) \mid e(\alpha) = \infty\}$ is also an ideal.*

Proof. Follows immediately from Corollary 4.1.8. □

Generation properties of Bi(R)

Recall from Example 2.4.2 that the *random graph* R is the unique countable, universal, homogeneous graph. As Bi(R) contains the automorphism group of R, which has cardinality of the continuum by Theorem 2.4.7, it is certainly not finitely generated and therefore we can consider looking at cofinality results.

We begin by determining the strong cofinality of Bi(R). First, we use the ARP characteristic of R (see Proposition 2.4.3) to prove a strong statement about the random graph. The proof of this result is slightly more intricate than it needs to be, but it demonstrates an important property that we will need in determining the strong cofinality of Bi(R). Recall from the introduction that a first-order structure $\mathcal{M}$ is *MB-homogeneous* if every monomorphism between finite substructures of $\mathcal{M}$ extends to a bimorphism of $\mathcal{M}$.

Proposition 4.1.20. *The random graph R is MB-homogeneous.*

Proof. The idea for the proof is to extend a monomorphism $f : A \to B$ between two finite graphs to a bimorphism $\alpha \in$ Bi(R) using a back-and-forth argument. Set $f = f_0, A = A_0$ and $B = B_0$, and suppose that we have extended f to a monomorphism $f_k : A_k \to B_k$, where $A_i \subseteq A_{i+1}$ and $B_i \subseteq B_{i+1}$ for all $0 \leq i \leq k-1$. As R is countable, we can write $VR = \{v_0, v_1, ...\}$.

If k is even, select a vertex $v_i \in VR$ where i is the smallest number such that $v_i \notin$ dom f_k. Let V_i be the finite set $V_i = \{a \in A \ : \ a \sim v_i\}$, and note that $V_i f_k \subseteq B_k$. Using the ARP for R, there exists a vertex $w \in VR$ such that $w \sim af_k$ for all $af_k \in V_i f_k$ and $w \nsim x$ for all $x \in B_k \setminus V_i f_k$. Now, define $f_{k+1} : A \cup \{v_i\} \to B \cup \{w\}$ to be the map acting like f_k on A and sending v_i to w. This map extends f_k and is a monomorphism as every edge from v_i to A is preserved by f_{k+1}.

If k is odd, select a vertex $v_j \in VR$ where j is the smallest number such that $v_j \notin \text{im } f_k$. Let V_j be the finite set $V_j = \{b \in B \ : \ b \sim v_j\}$; it follows that $V_j f_k^{-1} \subseteq A_k$. Using the ARP for R, there exists a vertex $w \in VR$ such that $w \sim b f_k^{-1}$ for all $b f_k^{-1} \in V_j f_k^{-1}$ and $w \nsim y \in A_k \smallsetminus V_j f_k^{-1}$. Define $f_{k+1} : A \cup \{w\} \to B \cup \{v_j\}$ to be the map acting like f_k on A and sending w to v_j. This map extends f_k and is a monomorphism since every edge is preserved.

Repeating this process infinitely many times, ensuring that each $v_i \in VR$ appears at both an odd and even step, extends f to a bijective homomorphism $\alpha : R \to R$. □

Remarks. (i) As mentioned above, there is a reason for this very particular construction. Here, this method ensures that any monomorphism f between two finite graphs of R can be extended to a bimorphism α where the edges added by α are precisely those added by f; that is, α acts like an automorphism outside of f. This means that for all $n \in \mathbb{N}$, there exists a bimorphism β of R such that $e(\beta) = n$. Note that we cannot rely on Theorem 2.4.4 to demonstrate this directly. Changing a finite number of non-edges of R to edges produces a graph Γ on VR where $\Gamma \cong R$, but the identity map on VR between R and Γ is not an isomorphism.

(ii) The concept of MB-homogeneity is one that is discussed at length in Chapters 6 and 7. More examples of MB-homogeneous graphs are given in Chapter 8.

From the first remark, this means that the ideal $I_k \subseteq \text{Bi}(R)$ is non-empty for all $k \in \mathbb{N}$; furthermore, $L_k = I_k \smallsetminus I_{k+1}$ is non-empty for all $k \in \mathbb{N}$. It can also be shown that there exist bimorphisms of R that add in infinitely many edges. Following this, we can prove that:

Theorem 4.1.21. *scf(Bi(R))* $= \aleph_0$.

Proof. The ideal structure of $\text{Bi}(R)$ given in Corollary 4.1.19 is an infinite descending chain of ideals by Proposition 4.1.20. For some $k \in \mathbb{N}$, set $L_k = I_k \smallsetminus I_{k+1}$; this is the set of all bimorphisms β of R such that $e(\beta) = k$. It follows from Corollary 4.1.18 that $L_i L_j \subseteq L_{i+j}$. So Bi$(R)$ satisfies the conditions outlined in Proposition 3.1.11 and therefore scf(Bi$(R)) = \aleph_0$. $\square$

We now show that Bi(R) does not have the Bergman property by demonstrating the existence of a 'bad' generating set (see remarks following Definition 3.1.1). It is a consequence of Lemma 4.1.6 that we can generate a bimorphism with an arbitrary finite additional graph given the automorphisms and all the bimorphisms that add in one edge. The next lemma shows that we need only include one such bimorphism to generate them all. Recall from Example 2.4.2 that R is *homogeneous*; that is, any isomorphism between finite substructures of R extends to an automorphism of R.

Lemma 4.1.22. *The set $X = \{\alpha \in Bi(R) \mid e(\alpha) = 1\}$ forms a $\mathscr{J}$-class of Bi(R).*

Proof. Suppose that $\alpha \in$ Bi(R) is a bimorphism such that $e(\alpha) = 1$; it follows that $R[S(\alpha)] \cong \bar{K}_2$, the empty graph on two vertices and $R[S(\alpha)\alpha] \cong K_2$, the complete graph on two vertices. As $e(\alpha)$ is finite, it is enough to show that $X = D_\alpha$ by Corollary 4.1.10.

So suppose that β is any bimorphism of R where $e(\beta) = 1$; so $R[S(\alpha)] \cong R[S(\beta)]$ and $R[S(\alpha)\alpha] \cong R[S(\beta)\beta]$. Using homogeneity of R, extend the isomorphism $f : R[S(\alpha)] \to R[S(\beta)]$ to a automorphism γ of R, and define a bimorphism $\eta = \gamma\beta$. As γ is an automorphism, it has an inverse and so $\gamma^{-1}\eta = \beta$; therefore $\eta \mathscr{L} \beta$. Note also that the bimorphism η sends the non-edge $R[S(\alpha)]$ to the edge $R[S(\beta)\beta]$, and acts like an automorphism everywhere else.

We now show that $\alpha \mathscr{R} \eta$. Consider the uniquely defined bijective map $\eta^{-1}\alpha : R \to R$, and suppose that $\{x, y\}$ is an edge of R. There are two cases; either $\{x, y\}$ is the edge added by η or it isn't. If $\{x, y\} \notin R[S(\beta)\beta]$, then $\{x\eta^{-1}, y\eta^{-1}\}$ is an edge of R because η^{-1} acts like an automorphism outside of $R[S(\beta)\beta]$. As α is a bimorphism, $\{x\eta^{-1}\alpha, y\eta^{-1}\alpha\}$ is an edge of R. If $\{x, y\} = R[S(\beta)\beta]$, then $\{x\eta^{-1}, y\eta^{-1}\} = R[S(\alpha)]$ is an non-edge of R; but then $\{x\eta^{-1}\alpha, y\eta^{-1}\alpha\}$ is an edge. Therefore, $\eta^{-1}\alpha$ preserves edges and so $\eta^{-1}\alpha$ is a bimorphism such that $\eta(\eta^{-1}\alpha) = \alpha$. We can use a similar argument to show that the uniquely defined map $\alpha^{-1}\eta$ is a bimorphism such that $\alpha(\alpha^{-1}\eta) = \eta$ and so $\alpha\mathscr{R}\eta$. Therefore, $\alpha\mathscr{D}\beta$ and so $X = D_\alpha$. $\square$

This serves as a base case for our next result. In the style of Corollary 4.1.19, we denote the ideal of bimorphisms of R that add in infinitely many edges by I_∞. Note that as a consequence of Lemma 4.1.6 and Corollary 4.1.18, the set $\text{Bi}(R) \setminus I_\infty$ is a submonoid of $\text{Bi}(R)$.

Proposition 4.1.23. *Let $U = Aut(R) \cup \{\beta\}$, where $\beta \in Bi(R)$ such that $e(\beta) = 1$. Then U generates the monoid $Bi(R) \setminus I_\infty$.*

Proof. By Corollary 4.1.10 and the fact that $\text{Aut}(R) \subseteq U$, it is enough to show that we can generate at least one element α in each $\mathscr{J}$-class not contained in I_∞; this means generating a bimorphism with any finite additional graph on any support graph. To do this, we use proof by induction on $e(\alpha) = k$. The base case where $e(\alpha) = 0$ is easy as $\text{Aut}(R)$ is contained in U. The case where $e(\alpha) = 1$ is covered by Lemma 4.1.22.

For the inductive step, assume that we can generate any $\gamma \in \text{Bi}(R)$ such that $e(\gamma) = k$. Now suppose that $\alpha \in \text{Bi}(R) \setminus I_\infty$ such that $e(\alpha) = k + 1$, with additional graph $\mathcal{A}(\alpha)$ and support graph $R[S(\alpha)]$; our aim is to generate α. Let Γ be the graph created by removing an edge $\{a, b\}$ from $\mathcal{A}(\alpha)$. By the inductive hypothesis, we can generate a bimorphism γ such that $E\mathcal{A}(\gamma) = E\Gamma$. Note here that $S(\gamma) \cup \{a, b\} = S(\alpha)$.

As $\{a, b\}$ is not an edge in $\mathcal{A}(\gamma)$, it follows that $\{a\gamma, b\gamma\}$ is a non-edge of R. Using homogeneity of R and Lemma 4.1.22, we can find a bimorphism η of R with $e(\eta) = 1$ such that $\{a\gamma, b\gamma\} \in E\mathcal{A}(\eta)$. By Lemma 4.1.6, this means that $\{a, b\} \in \mathcal{A}(\gamma\eta)$. As $\{a, b\}$ is the only element of $E\mathcal{A}(\eta)\gamma^{-1}$, it follows that $\mathcal{A}(\gamma\eta)$ is the graph on VR with edges $E\mathcal{A}(\gamma) \cup \{a, b\}$. The support graph of $\gamma\eta$ is the graph $R[S(\gamma) \cup \{a, b\}]$. So we have generated a bimorphism $\gamma\eta$ with additional graph $\mathcal{A}(\alpha)$ and support graph $R[S(\alpha)]$, completing the proof of the inductive step. $\square$

This result leads to an immediate corollary.

Corollary 4.1.24. *cf*$(Bi(R) \setminus I_\infty) > \aleph_0$.

Proof. Proposition 4.1.23 shows that rank(Bi$(R) \setminus I_\infty$: Aut$(R)) = 1$. As Aut(R) has uncountable cofinality [26], it follows that cf(Bi$(R) \setminus I_\infty) > \aleph_0$ by Proposition 3.1.6. □

Finally in this section, we investigate. With U = Aut$(R) \cup \{\beta\}$ as in Proposition 4.1.23, define V to be the set $U \cup I_\infty$. Since U generates Bi$(R) \setminus I_\infty$ by Proposition 4.1.23, then it follows that V generates Bi(R). We show that Bi(R) is not Cayley bounded with respect to this generating set V.

Theorem 4.1.25. *Let $\tau = t_1t_2t_3...t_k$ be a product of bimorphisms from the set V = Aut$(R) \cup \{\beta\} \cup I_\infty$, where $\beta \in$ Bi(R) is such that $e(\beta) = 1$. Then either $e(\tau) \leq k$ or it is infinite.*

Proof. We note that if any of the t_i's (where $1 \leq i \leq k$) are in I_∞, then the product is in I_∞. It remains to show that if $\tau = t_1t_2t_3...t_k$ (where each $t_i \notin I_\infty$) then $e(\tau) \leq k$; the proof is by induction on length of product. In the base case where $k = 1$, then $\tau = t_1$; as a result τ is either an automorphism (in which case $e(\tau) = 0$) or $\tau = \beta$ and so $e(\tau) = 1$.

For the inductive step, suppose that the above statement is true. Multiply $\tau = t_1t_2t_3...t_k$ on the right by t_{k+1} to get $\tau t_{k+1} = t_1t_2...t_kt_{k+1}$. It follows from Corollary 4.1.18 that

$$e(t_1t_2...t_kt_{k+1}) = e(\tau t_{k+1}) = e(\tau) + e(t_{k+1}).$$

Now, either t_{k+1} is an automorphism of R or it is β. If $t_{k+1} \in$ Aut(R), then $e(t_{k+1}) = 0$ and so $e(\tau) + 0 \leq k$ by the inductive hypothesis. If $t_{k+1} = \beta$, then $e(t_{k+1}) = 1$ and so $e(\tau) + 1 \leq k + 1$ by the inductive hypothesis. □

So Bi(R) is not Cayley bounded with respect to the generating set V; proving that Bi(R) does not have the Bergman property. There is one question left over; we have not been able to determine the exact cofinality of Bi(R). Referring back to the list on page 54, as scf(Bi$(R)) = \aleph_0$ it follows that Bi(R) is in either case (b) or (d). We conjecture that cf(Bi(R)) is uncountable (i.e. case (b)) due to Corollary 4.1.24, but we leave this as an open problem.

Question 4.1.26. *Is cf(Bi(R)) $> \aleph_0$?*

4.2 Embeddings of σ-structures

As an injective endomorphism of a σ-structure $\mathcal{M}$, a general element $\alpha \in \text{Mon}(\mathcal{M})$ may change non-relations to relations and the image set $M\alpha$ of α may not be equal to M. We have studied maps $\beta \in \text{Mon}(\mathcal{M})$ such that $M\beta = M$ and where β may change non-relations to relations; these are bimorphisms of $\mathcal{M}$. In order to understand more about Mon($\mathcal{M}$), we now study those maps $\gamma \in \text{Mon}(\mathcal{M})$ where γ preserves non-relations, but $M\gamma$ is some infinite subset of M. These are the *embeddings* of $\mathcal{M}$; as in Subsection 2.3.2, we denote the monoid of embeddings of $\mathcal{M}$ by Emb($\mathcal{M}$). Recall from the introduction to this chapter that for some $\gamma \in \text{Mon}(\mathcal{M})$, we write $M\gamma$ for the image *set* of the function γ, and $\mathcal{M}\gamma$ for the *structure* induced by $\mathcal{M}$ on $M\gamma$. As γ in this case is an embedding, it follows that $\mathcal{M}\gamma \cong \mathcal{M}$.

4.2.1 Semigroup theory of Emb($\mathcal{M}$)

As each embedding of $\mathcal{M}$ is an injective map from the domain M of $\mathcal{M}$ to itself, it follows that Emb($\mathcal{M}$) embeds in Mon(M) via an inclusion mapping. As this happens, Emb($\mathcal{M}$) is a right-cancellative monoid for every σ-structure $\mathcal{M}$. As $\mathcal{M}$ in this chapter is countably infinite, we can use definitions and results from Section 3.2 to help describe the behaviour of embeddings of $\mathcal{M}$. The following is the σ-structure analogue of Definition 3.2.1.

Definition 4.2.1. Let $\mathcal{M}$ be a countably infinite σ-structure, and take $\alpha \in \text{Emb}(\mathcal{M})$. Define the *defect* of α to be the set $O(\alpha) = M \smallsetminus M\alpha$. We define the *omitted structure of* α to be $\mathcal{O}(\alpha) = \mathcal{M}[O(\alpha)]$. Furthermore, define $o(\gamma) = |O(\gamma)|$, writing $o(\gamma) = \infty$ when $O(\gamma)$ is infinite.

We immediately note that $O(\alpha) = \varnothing$ if and only if $o(\alpha) = 0$ if and only if α

is an automorphism. Definition 4.2.1 leads into a straightforward restatement of both Lemma 3.2.2 and Corollary 3.2.4.

Lemma 4.2.2. *Let $\mathcal{M}$ be a countably infinite σ-structure, and suppose that $\alpha, \beta \in$ Emb$(\mathcal{M})$.*

(1) $O(\alpha\beta) = O(\beta) \cup O(\alpha)\beta$ and this is a disjoint union.

(2) If both $o(\alpha)$ and $o(\beta)$ are finite, then $o(\alpha\beta) = o(\alpha) + o(\beta)$.

(3) $o(\alpha\beta) = \infty$ if and only if $o(\alpha)$ or $o(\beta)$ is infinite. □

Remark. We can use Lemma 4.2.2 to define the omitted structure of $\alpha\beta$; here, $\mathcal{O}(\alpha\beta) = \mathcal{M}[O(\beta) \cup O(\alpha)\beta]$. Note that in general,

$$\mathcal{M}[O(\beta) \cup O(\alpha)\beta] \neq \mathcal{M}[O(\beta)] \cup \mathcal{M}[O(\alpha)\beta].$$

In this case, equality only occurs if $\mathcal{O}(\beta)$ and $\mathcal{O}(\alpha)\beta$ are *independent* of each other in $\mathcal{M}$; that is, for all $R_i \in \sigma$, no tuple $\bar{x} \in M$ with $R_i^{\mathcal{M}}(\bar{x})$ meets both $\mathcal{O}(\beta)$ and $\mathcal{O}(\alpha)\beta$; see [56].

Following Lemma 4.2.2 and in a similar fashion to bimorphisms, we write $o(\alpha) + o(\beta) < \infty$ to mean that both $o(\alpha)$ and $o(\beta)$ are finite. We now summarise two results that immediately follow from the fact that Emb$(\mathcal{M})$ is a subsemigroup of Mon(M).

Corollary 4.2.3. *Let $\mathcal{M}$ be a countably infinite σ-structure.*

(1) The only idempotent element in Emb$(\mathcal{M})$ is the identity.

(2) If Emb$(\mathcal{M}) \neq$ Aut$(\mathcal{M})$, then Emb$(\mathcal{M})$ is not regular.

Proof. (1) Follows immediately from Corollary 3.2.3 (2).

(2) As every regular element of Mon(M) is a bijection by Corollary 3.2.3 (1), it follows that any regular element of Emb$(\mathcal{M})$ must be an automorphism.

□

Corollary 4.2.4. *Let $\mathcal{M}$ be a countably infinite σ-structure. For a natural number k, define $J_k := \{\epsilon \in Emb(\mathcal{M}) : o(\epsilon) \geq k\}$. Then, if non-empty, J_k is an ideal of $Emb(\mathcal{M})$. Furthermore, if non-empty, $J_\infty = \{\epsilon \in Emb(\mathcal{M}) : o(\epsilon) = \infty\}$ is an ideal of $Emb(\mathcal{M})$.*

Proof. Let $k \in \mathbb{N}$. As M is infinite, $\mathrm{Mon}(M)$ has ideals I_k identical to those in Lemma 3.2.5. In this case, $J_k = I_k \cap \mathrm{Emb}(\mathcal{M})$ and so J_k is an ideal of $\mathrm{Emb}(\mathcal{M})$ by Lemma 2.2.1. The same argument applies for J_∞. □

We can now move towards characterising Green's relations of $\mathrm{Emb}(\mathcal{M})$. Here, note that two elements α, β are $\mathscr{J}$-related in $\mathrm{Emb}(\mathcal{M})$ only if $\alpha \mathscr{J} \beta$ in $\mathrm{Mon}(M)$; however, we should not expect the converse to be true in general. This fact helps to prove our next result.

Lemma 4.2.5. *Let $\mathcal{M}$ be a σ-structure, and suppose that α, β in $Emb(\mathcal{M})$.*

(1) Suppose that $\alpha \mathscr{L} \beta$. Then for all $\gamma, \delta \in Emb(\mathcal{M})$ such that $\gamma\alpha = \beta$ and $\delta\beta = \alpha$, the maps γ and δ are automorphisms.

(2) Suppose that $o(\alpha) + o(\beta) < \infty$ and $\alpha \mathscr{J} \beta$. For all $\gamma, \delta, \epsilon, \zeta \in Emb(\mathcal{M})$ such that $\gamma\alpha\delta = \beta$ and $\epsilon\beta\zeta = \alpha$, the maps $\gamma, \delta, \epsilon, \zeta$ are automorphisms.

Proof. For both parts (1) and (2), if two embeddings of $\mathcal{M}$ are Green's related then they must also be Green's related as monomorphisms of the domain M. By Lemma 3.2.6, this only happens (in both cases) if the monomorphisms relating α and β are bijections. As a bijective embedding is an automorphism, we are done. □

As a consequence of this, we can immediately characterise the $\mathscr{L}$-relation in $\mathrm{Emb}(\mathcal{M})$.

Lemma 4.2.6. *Let $\alpha, \beta \in Emb(\mathcal{M})$. Then $\alpha \mathscr{L} \beta$ if and only if $O(\alpha) = O(\beta)$.*

Proof. Suppose that $\alpha \mathscr{L} \beta$. By Lemma 4.2.5, there exist automorphisms γ and δ such that $\gamma\alpha = \beta$ and $\delta\beta = \alpha$. By Lemma 4.2.2, it follows that $O(\gamma\alpha) = O(\alpha) \cup O(\gamma)\alpha$. As $O(\gamma) = \varnothing$, it follows that $O(\alpha) = O(\gamma\alpha) = O(\beta)$.

Conversely, suppose that $O(\alpha) = O(\beta)$; because of this, $M\alpha = M\beta$. As this happens, for every $m \in M$ there exists a unique $n \in M$ such that $m\alpha =$

$n\beta$. Define $\gamma = \alpha\beta^{-1}$ to be the unique bijection from M to itself sending m to n. As $\alpha : \mathcal{M} \to \mathcal{M}\alpha$ is an isomorphism, and $\mathcal{M}\alpha = \mathcal{M}\beta$, it follows that $\alpha : \mathcal{M} \to \mathcal{M}\beta$ is an isomorphism. Since $\beta : \mathcal{M} \to \mathcal{M}\beta$ is an isomorphism, there exists an inverse isomorphism $\beta^{-1} : \mathcal{M}\beta \to \mathcal{M}$. So $\gamma = \alpha\beta^{-1} : \mathcal{M} \to \mathcal{M}$ is an isomorphism, and hence an automorphism of $\mathcal{M}$. We can use a similar argument to show that there exists an automorphism δ of $\mathcal{M}$ such that $\delta\beta = \alpha$. $\square$

For the $\mathscr{R}$-relations, we take a slightly different approach to that of Proposition 3.2.9. Recall from Lemma 4.3.5 that $\mathscr{R} = \mathscr{D}$ in Mon(M); this fact transfers to Emb($\mathcal{M}$).

Proposition 4.2.7. *Let $\alpha, \beta \in$ Emb($\mathcal{M}$). Then $\alpha\mathscr{R}\beta$ if and only if $\alpha\mathscr{D}\beta$.*

Proof. Since $\mathscr{D}$ is the smallest equivalence relation containing $\mathscr{L}$ and $\mathscr{R}$, then $\alpha\mathscr{R}\beta$ implies that $\alpha\mathscr{D}\beta$.

Conversely, assume that $\alpha\mathscr{D}\beta$; so there exists $\gamma \in$ Emb($\mathcal{M}$) such that $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$. As $\mathscr{R}$ is a transitive relation, it will suffice to show that $\alpha\mathscr{R}\gamma$. To that end, write Emb($\mathcal{M}$) $= S$ and consider the right ideal αS^1. If ϵ is any embedding, then

$$\mathcal{O}(\alpha\epsilon) = \mathcal{M}[O(\epsilon) \cup O(\alpha)\epsilon]$$

and so $\mathcal{O}(\alpha\epsilon)$ embeds $\mathcal{O}(\alpha)$ as an induced substructure via ϵ. Therefore, every element of αS embeds $\mathcal{O}(\alpha)$. Now consider $\gamma\eta \in \gamma S^1$, where η is any embedding. Since $\alpha\mathscr{L}\gamma$ by assumption, it follows that $O(\gamma) = O(\alpha)$ by Lemma 4.2.6. So by Lemma 4.2.2,

$$O(\gamma\eta) = O(\eta) \cup O(\gamma)\eta = O(\eta) \cup O(\alpha)\eta.$$

Therefore, $\mathcal{O}(\gamma\eta) = \mathcal{M}[O(\eta) \cup O(\alpha)\eta]$ embeds $\mathcal{O}(\alpha)$ via the embedding η and so $\gamma\eta \in \alpha S^1$; proving that $\gamma S^1 \subseteq \alpha S^1$. The proof that $\alpha S^1 \subseteq \gamma S^1$ is symmetric and so $\alpha\mathscr{R}\gamma$. $\square$

Proposition 4.2.8. *Suppose that $\alpha, \beta \in$ Emb($\mathcal{M}$).*

(1) If there exists an isomorphism between $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$ that extends to an automorphism of $\mathcal{M}$, then $\alpha\mathscr{R}\beta$.

(2) Assume further that $o(\alpha) + o(\beta) < \infty$. If $\alpha\mathscr{R}\beta$, then there exists an isomorphism between $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$ that extends to an automorphism of $\mathcal{M}$.

(3) If $o(\alpha) < \infty$, then $L_\alpha = H_\alpha$ and $R_\alpha = D_\alpha = J_\alpha$.

Proof. (1) By assumption, extend the isomorphism between $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$ to an automorphism ν of $\mathcal{M}$. Define an embedding $\gamma = \alpha\nu$. Since ν is an automorphism we can write $\alpha = \gamma\nu^{-1}$ and thus $\gamma\mathscr{R}\alpha$. By Lemma 4.2.2, and the fact that ν is an automorphism sending $O(\alpha)$ to $O(\beta)$ gives

$$O(\gamma) = O(\alpha\nu) = O(\nu) \cup O(\alpha)\nu = O(\alpha\nu) = O(\beta).$$

We see that $O(\gamma) = O(\beta)$ and hence $\beta\mathscr{L}\gamma$ by Lemma 4.2.6. So $\beta\mathscr{D}\alpha$ and therefore $\beta\mathscr{R}\alpha$ by Proposition 4.2.7.

(2) Assume that $\alpha\mathscr{R}\beta$. As $o(\alpha) + o(\beta) < \infty$, Lemma 4.2.5 (2) implies that they must be $\mathscr{R}$-related by automorphisms. In particular, there exists an automorphism γ of $\mathcal{M}$ such that $\alpha\gamma = \beta$. It follows from Lemma 4.2.2 that $\mathcal{O}(\beta) = \mathcal{O}(\alpha\gamma) = \mathcal{M}[O(\gamma) \cup O(\alpha)\gamma]$. As $\gamma \in \mathrm{Aut}(\mathcal{M})$, $O(\gamma) = \varnothing$ and so $\mathcal{M}[O(\alpha)\gamma] = \mathcal{O}(\beta)$. Furthermore, we have that $\mathcal{M}[O(\alpha)]\gamma = \mathcal{O}(\beta)$. Therefore $\mathcal{O}(\alpha)\gamma = \mathcal{O}(\beta)$ and so there exists an isomorphism between $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$ that extends to an automorphism.

(3) Since $\mathscr{R} = \mathscr{D}$ in general in $\mathrm{Emb}(\mathcal{M})$ by Proposition 4.2.7, and since any element in J_α must be $\mathscr{J}$-related to α by automorphisms, we can apply Lemma 2.2.3 to see that $R_\alpha = D_\alpha = J_\alpha$ in this case. As $L_\alpha \subseteq D_\alpha = R_\alpha$, the fact that $H_\alpha = L_\alpha$ follows by definition.

□

Example 4.2.9. Note that the condition of Proposition 4.2.8 (1) that the two omitted structures must be isomorphic is necessary; providing a distinction between $\mathrm{Emb}(\mathcal{M})$ and $\mathrm{Mon}(M)$. For example, let $\mathcal{M}$ be the random graph R and let

$\alpha : R \to R\alpha$ be an embedding that omits an edge $\{a, b\}$, and $\beta : R \to R\beta$ that omits a non-edge $\{c, d\}$. By Theorem 2.4.4, both images are isomorphic to R, but there is no automorphism of R sending $R\alpha$ to $R\beta$ and $\{a, b\}$ to $\{c, d\}$.

As with the bimorphism case in Lemma 4.1.9, we are only able to give a partial characterisation of $\mathscr{J}$-relations in Emb($\mathcal{M}$); this is illustrated in Proposition 4.2.8 (2) and (3). It is a possibility, in some structure $\mathcal{M}$, that two elements $\alpha, \beta \in$ Emb($\mathcal{M}$) where $o(\alpha) = o(\beta) = \infty$ are $\mathscr{J}$-related by embeddings that add in infinitely many edges. As with the case of the bimorphisms, we leave this question open.

Question 4.2.10. *Classify Green's $\mathscr{R} = \mathscr{D}$ and $\mathscr{J}$-relations in Emb($\mathcal{M}$). In particular, does $\mathscr{R} = \mathscr{D} = \mathscr{J}$, as in Mon($\mathbb{N}$)?*

4.2.2 Embeddings of $(\mathbb{N}, \leq)$

The first of our two examples concerning embeddings of σ-structures is the *discrete linear order* $(\mathbb{N}, \leq)$, the natural numbers together with the natural ordering. This structure is rigid; the only order-preserving bijection of the natural numbers is the identity map. However, there do exist non-identity order-preserving monomorphisms of $\mathbb{N}$. As this structure is a linear order, there are no non-relations to change and so every order-preserving monomorphism of $\mathbb{N}$ is an embedding of $(\mathbb{N}, \leq)$. From this, Emb$(\mathbb{N}, \leq)$ is contained in the monomorphism monoid of the domain $\mathbb{N}$; this means we can use theory developed in both Subsection 4.2.1 and Section 3.2 to determine cofinality results for Emb$(\mathbb{N}, \leq)$. First, we begin with a fundamental lemma.

Lemma 4.2.11. *Let A be any set of natural numbers such that $|\mathbb{N} \smallsetminus A| = |\mathbb{N}|$. Then there exists a unique $\alpha \in$ Emb$(\mathbb{N}, \leq)$ such that $O(\alpha) = A$.*

Proof. Suppose that $\beta, \gamma \in$ Emb$(\mathbb{N}, \leq)$ are two different embeddings such that $O(\beta) = O(\gamma)$, and let k be the least natural number such that $k\beta \neq k\gamma$. Assume without loss of generality that $k\beta < k\gamma$. As $n\beta = n\gamma$ for all $n < k$, and β and γ preserve order, there must be no $a \geq k$ such that $a\gamma = k\beta$. This means that $k\beta \in O(\gamma)$. But $k\beta$ is in the image of β; therefore, $k\beta \notin O(\beta)$. This is a contradiction as we assumed that $O(\beta) = O(\gamma)$. $\square$

Let $J_k, J_\infty \subseteq \mathrm{Emb}(\mathbb{N}, \leq)$ be as in Corollary 4.2.4; these are both ideals of $\mathrm{Emb}(\mathbb{N}, \leq)$ by the same result.

Corollary 4.2.12. *(1)* $F = \mathit{Emb}(\mathbb{N}, \leq) \setminus J_\infty$ *is a countable submonoid of* $\mathit{Emb}(\mathbb{N}, \leq)$.

(2) $|J_\infty| = 2^{\aleph_0}$.

Proof. (1) Since F contains a unique function for each finite subset of the naturals by Lemma 4.2.11, it follows that there is a bijective correspondence between F and the set Y of all finite subsets of $\mathbb{N}$. As Y is countable, F must also be countable.

(2) Let $Z = \{A \subseteq \mathbb{N} : |A| = |\mathbb{N} \setminus A| = |\mathbb{N}|\}$. By Lemma 4.2.11, there exists a unique function in J_∞ for all A in Z; so $|J_\infty| = |Z| = 2^{\aleph_0}$.

$\square$

We now investigate generation results for $\mathrm{Emb}(\mathbb{N}, \leq)$. A generating set for this monoid must be uncountable as $\mathrm{Emb}(\mathbb{N}, \leq)$ is uncountable. The idea, as in the case of bimorphisms of the random graph R, is to show that $\mathrm{Emb}(\mathbb{N}, \leq)$ is not Cayley bounded with respect to some 'bad' generating set U; proving that $\mathrm{Emb}(\mathbb{N}, \leq)$ does not have the Bergman property. Before we begin, denote the unique map that has singleton defect $\{k\}$ by α_k.

Proposition 4.2.13. *Let* $F \subseteq \mathit{Emb}(\mathbb{N}, \leq)$ *be as in Corollary 4.2.12. Then any generating set for* F *contains* α_k *for all* $k \in \mathbb{N}$.

Proof. Suppose that B is any generating set for F; so B must generate α_k for every $k \in \mathbb{N}$. By Corollary 4.2.4, B must generate each α_k via elements β with $o(\beta) = 1$. By Lemma 4.2.2 (2), and the fact that the only bijection in $\mathrm{Emb}(\mathbb{N}, \leq)$ is the identity map, B must contain every α_k. $\square$

Proposition 4.2.13 forms a base case for our next result concerning generation of $\mathrm{Emb}(\mathbb{N}, \leq)$.

Proposition 4.2.14. *Let $K = \{\alpha_k : k \in \mathbb{N}\}$, and let e be the identity element of Emb$(\mathbb{N}, \leq)$. Then $X = K \cup J_\infty \cup \{e\}$ generates Emb$(\mathbb{N}, \leq)$.*

Proof. It is enough to show that X generates F; the proof, as in Proposition 4.1.23, is by induction. The case where $k = 0$ or $k = 1$ is easy as $X \supseteq K \cup \{e\}$. So assume that all functions with defect of size k have been generated; we want to show that we can generate every function with defect of size $k + 1$. Suppose that β is such a function with $O(\beta) = \{b_1, ..., b_k, b_{k+1}\}$. By the inductive hypothesis, we can generate γ, the unique map with defect $O(\gamma) = \{b_1, ..., b_k\}$. Since $b_{k+1} \notin O(\gamma)$, there exists some $c \in \mathbb{N}$ such that $c\gamma = b_{k+1}$. Now, take the product $\alpha_c\gamma$. Lemma 4.2.2 (1) implies that

$$\begin{aligned} O(\alpha_c\gamma) &= O(\alpha_c)\gamma \cup O(\gamma) \\ &= \{c\}\gamma \cup \{b_1, ..., b_k\} \\ &= \{b_1, ..., b_k, b_{k+1}\} = O(\beta). \end{aligned}$$

As β is the unique function with this defect, it follows that $\alpha_c\gamma = \beta$ and we are done. □

Our next result shows that Emb$(\mathbb{N}, \leq)$ is not Cayley bounded with respect to X.

Proposition 4.2.15. *Let X be as in Proposition 4.2.14, and let $t = t_1t_2...t_n$ be a product of elements of X. Then the size of the defect of t is either infinite or at most n.*

Proof. The proof is by induction on length of product. For the base case $t = t_1$, either $t_1 = e$, or $t_1 = \alpha_k$ for some $k \in \mathbb{N}$, or $t_1 \in J_\infty$; any choice here adheres to the conclusion of the statement. For the inductive step, assume that the statement holds for a product $t = t_1t_2...t_n$ and multiply on the right by an extra element $t_{n+1} \in X$. If $t_{n+1} = e$ then we are done by the inductive hypothesis. If $t_{n+1} \in J_\infty$, then as J_∞ is an ideal, $tt_{n+1} \in J_\infty$. So it remains to examine the case when $t_{n+1} = \alpha_k$ for some $k \in \mathbb{N}$. At most, α_k only adds a single element to the defect and therefore $o(tt_{n+1}) \leq n + 1$. □

This shows that $\mathrm{Emb}(\mathbb{N}, \leq)$ is not Cayley bounded with respect to X; as a consequence, $\mathrm{Emb}(\mathbb{N}, \leq)$ does not have the Bergman property. We can also determine the cofinality and strong cofinality of $\mathrm{Emb}(\mathbb{N}, \leq)$.

Theorem 4.2.16. *cf(Emb*$(\mathbb{N}, \leq)$*)* $=$ *scf(Emb*$(\mathbb{N}, \leq)$*)* $= \aleph_0$.

Proof. From Corollary 4.2.12, $\mathrm{Emb}(\mathbb{N}, \leq)$ is the disjoint union of a countable submonoid F and an uncountable ideal J_∞. Since F is not finitely generated by Proposition 4.2.13, Lemma 3.1.3 implies that $\mathrm{cf}(F) = \aleph_0$. We can use Lemma 3.1.4 to see that $\mathrm{cf}(\mathrm{Emb}(\mathbb{N}, \leq))$ is at most $\aleph_0$; as it cannot be less, we are done. By Proposition 3.1.2, $\mathrm{scf}(\mathrm{Emb}(\mathbb{N}, \leq)) = \aleph_0$. ☐

Remark. We could have used Proposition 3.2.12 to show that $\mathrm{scf}(\mathrm{Emb}(\mathbb{N}, \leq)) = \aleph_0$.

4.2.3 Generation results for Emb(R)

As $\mathrm{Emb}(R)$ contains the automorphism group of R (which is uncountable by Theorem 2.4.7) it must itself be uncountable; in particular it is not finitely generated. Following this, we can investigate cofinality and generation results for $\mathrm{Emb}(R)$.

Theorem 4.2.17. *scf(Emb*(R)*)* $= \aleph_0$.

Proof. We show that $\mathrm{Emb}(R)$ meets the conditions of Proposition 3.2.12. First, $\mathrm{Emb}(R)$ is a subsemigroup of $\mathrm{Mon}(VR)$. As outlined in Theorem 2.4.4, removing any finite number of vertices of R leaves a graph R' such that $R \cong R'$. So there exists an $\alpha \in \mathrm{Emb}(R)$ such that $M\alpha = VR'$. It follows that for each $n \in \mathbb{N}$, there exists $\beta \in \mathrm{Emb}(R)$ such that $o(\beta) = n$. Now, partition VR into two infinite pieces X_1 and X_2. By Theorem 2.4.5, the induced subgraph on one of these parts is isomorphic to R; without loss of generality, assume this is $R[X_1]$. Therefore, there exists an $\gamma \in \mathrm{Emb}(R)$ such that $\mathcal{M}\gamma = R[X_1]$; and so there is a $\gamma \in \mathrm{Emb}(R)$ such that $o(\gamma) = \infty$. Therefore, $\mathrm{Emb}(R)$ meets the conditions of Proposition 3.2.12 and so has countable strong cofinality. ☐

As with the bimorphism monoid of R, we show that $\mathrm{Emb}(R)$ does not have the Bergman property by fabricating a generating set for $\mathrm{Emb}(R)$ that is not Cayley bounded. Our first result uses Lemma 4.2.5 and Proposition 4.2.8 to detail an important fact.

Lemma 4.2.18. *The set $B = \{\alpha \in Emb(R) : o(\alpha) = 1\}$ forms a $\mathscr{J}$-class of $Emb(R)$.*

Proof. Let $\alpha, \beta \in B$. Since $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$ are just single vertices in R, they are isomorphic. By homogeneity of R, this partial isomorphism extends to an automorphism of R; so by Proposition 4.2.8 (2) and (3), α and β are $\mathscr{J}$-related. □

This lemma shows that with the automorphisms of R and just one embedding with singleton defect, we can generate all embeddings with a singleton defect. Similarly to Proposition 4.1.23, we use this as a base case in order to show that we can generate all embeddings with a finite defect. Recall that $J_\infty \subseteq \mathrm{Emb}(\mathcal{M})$ (as in Corollary 4.2.4) is the ideal of all embeddings of a σ-structure $\mathcal{M}$ with infinite defect. By Lemma 4.2.2 (2) the set $\mathrm{Emb}(R) \setminus J_\infty$ of all embeddings of R that omit finitely many vertices is a submonoid of $\mathrm{Emb}(R)$.

Proposition 4.2.19. *Let $C = Emb(R) \setminus I_\infty$, and let $\beta \in Emb(R)$ be an embedding such that $o(\beta) = 1$. Then $X = Aut(R) \cup \{\beta\}$ generates C.*

Proof. Let $\alpha \in C$; the proof is by induction on the size of defect $o(\alpha) = k$. The case when $k = 0$ is trivial as $\mathrm{Aut}(R) \subseteq X$. Here, Lemma 4.2.18 covers the case where $k = 1$; so we can generate every element of B using X.

For the inductive step, assume that we have generated all embeddings of R with defect of size k. Assume that $\beta \in C$ with $O(\beta) = \{b_1, b_2, ..., b_k, b_{k+1}\}$; our aim is to generate β. By Lemma 4.2.6, and the fact that $\mathrm{Aut}(R) \subseteq X$, it suffices to show that we can generate some $\delta \in \mathrm{Emb}(R)$ such that $O(\delta) = O(\beta)$. Using the inductive hypothesis, we can generate $\alpha \in \mathrm{Emb}(R)$ such that $O(\alpha) = \{b_1, ..., b_k\}$. As $b_{k+1} \notin O(\alpha)$, there exists $v \in VR$ such that $v\alpha = b_{k+1}$. Now, select some $\gamma \in$

$\mathrm{Emb}(R)$ with $O(\gamma) = \{v\}$, and set $\delta = \gamma\alpha$. Then, using Lemma 4.2.2 (1):

$$\begin{aligned} O(\delta) = O(\gamma\alpha) &= O(\gamma)\alpha \cup O(\alpha) \\ &= \{v\}\alpha \cup \{b_1, ..., b_k\} \\ &= \{b_1, ..., b_k, b_{k+1}\} = O(\beta) \end{aligned}$$

and we are done. □

As an immediate corollary of this result, we get:

Corollary 4.2.20. *cf*$(Emb(R) \setminus I_\infty) > \aleph_0$.

Proof. Similar to Corollary 4.1.24. □

Finally, suppose that X is as in Proposition 4.2.19. It follows that the set $Y = X \cup I_\infty$ generates $\mathrm{Emb}(R)$.

Theorem 4.2.21. *Let* $\tau = t_1t_2t_3...t_k$ *be a product of embeddings from* Y*. Then* $o(\tau)$ *is either less than* k *or it is infinite.*

Proof. The proof follows exactly the same steps as Theorem 4.1.25 with bimorphisms replaced by embeddings. □

By this result, $\mathrm{Emb}(R)$ is not Cayley bounded with respect to the generating set Y; hence $\mathrm{Emb}(R)$ does not have the Bergman property. As with the bimorphism case, we have not been able to determine the cofinality of $\mathrm{Emb}(R)$; following the observation of Corollary 4.2.20, we conjecture that $\mathrm{cf}(\mathrm{Emb}(R)) > \aleph_0$ and leave it as an open question.

Question 4.2.22. *Is cf*$(Emb(R)) > \aleph_0$*?*

4.3 Monomorphisms of σ-structures

Following the investigations into $\mathrm{Bi}(\mathcal{M})$ and $\mathrm{Emb}(\mathcal{M})$, our attention turns to the more general case of $\mathrm{Mon}(\mathcal{M})$. As monomorphisms are much more general than bimorphisms and embeddings, it follows that we will not be able to prove as many statements in full generality; this is reflected in the statements on Green's relations in Mon($\mathcal{M}$). Nevertheless, we can use the machinery of additional and omitted structures developed in these studies to help describe the basic semigroup theory of Mon($\mathcal{M}$).

4.3.1 Semigroup theory of Mon($\mathcal{M}$)

We begin with a straightforward generalisation of Definitions 4.1.4 and 4.2.1.

Definition 4.3.1. Let α be a monomorphism of a countably infinite σ-structure $\mathcal{M}$. Define the *defect* of α to be the set $O(\alpha) = M \smallsetminus M\alpha$ and the *omitted structure* to be $\mathcal{O}(\alpha) = \mathcal{M}[O(\alpha)]$. For a monomorphism α of $\mathcal{M}$, define a σ-structure $\mathcal{A}(\alpha)$ on M with $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$ if and only if $\bar{a} \notin R_i^{\mathcal{M}}$ and $\bar{a}\alpha \in R_i^{\mathcal{M}\alpha}$ for all $R_i \in \sigma$; call this the *additional structure* of α.

Furthermore, for $\alpha \in$ Mon($\mathcal{M}$), define the *support* of α to be the set:

$$S(\alpha) = \{x \in M : x \in \bar{a}, \bar{a} \in R_i^{\mathcal{A}(\alpha)} \text{ for some } R_i \in \sigma\}.$$

As $S(\alpha)$ is a subset of M, we can induce a structure $\mathcal{M}[S(\alpha)]$ on $S(\alpha)$ with relations from $\mathcal{M}$; call this the *support structure* of α.

In a similar fashion to Section 4.1 and Section 4.2, for some monomorphism α of $\mathcal{M}$, define $e_i(\alpha) = |R_i^{\mathcal{A}(\alpha)}|$ for all $R_i \in \sigma$ and $o(\alpha) = |O(\alpha)|$, writing $e_i(\alpha) = \infty$ if $R_i^{\mathcal{A}(\alpha)}$ is infinite and $o(\alpha) = \infty$ if $O(\alpha)$ is infinite.

Lemma 4.3.2. *Let $\mathcal{M}$ be a σ-structure, and suppose that $\alpha, \beta \in$ Mon($\mathcal{M}$). Then:*

(1)
- $O(\alpha\beta) = O(\beta) \cup O(\alpha)\beta$;
- $R_i^{\mathcal{A}(\alpha\beta)} = R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ *for all* $R_i \in \sigma$;
- *both of these are disjoint unions.*

(2)
- *If $o(\alpha)$ and $o(\beta)$ are both finite, then $o(\alpha\beta) = o(\alpha) + o(\beta)$.*
- *If $e_i(\alpha)$ and $e_i(\beta)$ are both finite for some $R_i \in \sigma$, then $e_i(\alpha\beta) = e_i(\alpha) + e_i(\beta)$.*
- *$o(\alpha\beta) = \infty$ if and only if $o(\alpha)$ or $o(\beta)$ is infinite; similarly, for $R_i \in \sigma$, $e_i(\alpha\beta)$ is infinite if and only if $e_i(\alpha)$ or $e_i(\beta)$ is infinite.*

Proof. For (1), Mon($\mathcal{M}$) is contained in Mon(M); so the conclusion that $O(\alpha\beta) = O(\beta) \cup O(\alpha)\beta$ (and this is a disjoint union) follow from Lemma 3.2.2. The proof that $R_i^{\mathcal{A}(\alpha\beta)} = R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ for all $R_i \in \sigma$ (and this is a disjoint union) is alike to that of Lemma 4.1.6. As both the unions in (1) are disjoint, the first two items follow. The proof of the third item is analogous to similar results in Corollary 4.1.7 and Lemma 4.2.2. □

Remarks. (i) Let $\alpha \in$ Mon($\mathcal{M}$). If $e_i(\alpha) = 0$ for all $R_i \in \sigma$, then α is an embedding. If $o(\alpha) = 0$, then α is a bimorphism of $\mathcal{M}$. If both happen, then α is an automorphism of $\mathcal{M}$.

(ii) As in Lemma 4.2.2, $\mathcal{O}(\alpha\beta) = \mathcal{M}[O(\beta) \cup O(\alpha)\beta]$ and in general,

$$\mathcal{M}[O(\beta) \cup O(\alpha)\beta] \neq \mathcal{M}[O(\beta)] \cup \mathcal{M}[O(\alpha)\beta].$$

As Mon($\mathcal{M}$) is a subsemigroup of Mon(M), we can write the same results for monomorphisms as we did for embeddings in Corollary 4.2.3 and Corollary 4.2.4.

Corollary 4.3.3. *Let $\mathcal{M}$ be a countably infinite σ-structure.*

(1) The only idempotent element in Mon($\mathcal{M}$) is the identity.

(2) If Mon($\mathcal{M}$) $\neq$ Aut($\mathcal{M}$), then Mon($\mathcal{M}$) is not regular.

Proof. (1) Follows immediately from Corollary 3.2.3.

(2) As every regular element of Mon(M) is a bijection by Corollary 3.2.3 (1), it follows that any regular element of Mon($\mathcal{M}$) must be an bimorphism. Corollary 4.1.2 (2) asserts that every regular element of Bi($\mathcal{M}$) is an automorphism.

□

This next corollary is important in showing strong cofinality results later in the section.

Corollary 4.3.4. *Let $\mathcal{M}$ be a countably infinite σ-structure. For $k \in \mathbb{N}$, define $\mathcal{I}_k = \{\epsilon \in Mon(\mathcal{M}) : o(\epsilon) \geq k\}$. Then, if non-empty, $\mathcal{I}_k$ is an ideal of $Mon(\mathcal{M})$. Furthermore, if non-empty, $\mathcal{I}_\infty = \{\epsilon \in Mon(\mathcal{M}) : o(\epsilon) = \infty\}$ is an ideal of $Mon(\mathcal{M})$.*

Proof. Similar to Corollary 4.2.4. □

Remark. This corollary does not describe all the ideals of Mon($\mathcal{M}$). For instance, the set $I(i,k) = \{\epsilon \in \text{Mon}(\mathcal{M}) : e_i(\epsilon) \geq k\}$ is also an ideal of Mon($\mathcal{M}$) by Lemma 4.3.2 (2).

We now look at Green's relations in Mon($\mathcal{M}$) in general. Writing down all-encompassing characterisations for Green's relations is somewhat difficult as we need to consider both ways in which a general monomorphism of $\mathcal{M}$ affect a structure. However, we can use the fact that Mon($\mathcal{M}$) is right-cancellative, as well as some previous results for bimorphisms and embeddings, to narrow our consideration slightly.

Lemma 4.3.5. *Let $\mathcal{M}$ be a σ-structure, with $\alpha, \beta \in Mon(\mathcal{M})$.*

(1) Suppose that $\alpha \mathscr{L} \beta$. Then for all $\gamma, \delta \in Mon(\mathcal{M})$ such that $\gamma\alpha = \beta$ and $\delta\beta = \alpha$, the maps γ and δ are automorphisms.

(2) Suppose that $o(\alpha) + o(\beta) < \infty$ and $\alpha \mathscr{J} \beta$. For all $\gamma, \delta, \epsilon, \zeta \in Mon(\mathcal{M})$ such that $\gamma\alpha\delta = \beta$ and $\epsilon\beta\zeta = \alpha$, the maps $\gamma, \delta, \epsilon, \zeta$ are automorphisms.

Proof. (1) Suppose that $\gamma, \delta \in \text{Mon}(\mathcal{M})$ are as in the statement. It follows $\gamma\delta\alpha = \alpha$; as Mon($\mathcal{M}$) is right-cancellative, we have that $\gamma\delta = 1$. Furthermore, the fact that $\delta\gamma\beta = \beta$ implies that $\delta\gamma = 1$. So both γ and δ are units; in other words, they are automorphisms.

(2) The proof of this is similar to both Lemma 4.1.9 (2) and Lemma 4.2.5 (2).

□

Following Lemma 4.3.5 (1), we can at least offer a characterisation of $\mathscr{L}$-relations in Mon($\mathcal{M}$).

Proposition 4.3.6. *Let $\alpha, \beta \in Mon(\mathcal{M})$. Then $\alpha \mathscr{L} \beta$ if and only if $O(\alpha) = O(\beta)$, $S(\alpha)\alpha = S(\beta)\beta$ and $\mathcal{M}[S(\alpha)] \cong \mathcal{M}[S(\beta)]$ via the isomorphism induced by $\alpha\beta^{-1}$.*

Proof. Suppose that $\alpha\mathscr{L}\beta$; so by Lemma 4.3.5 (1), there exist automorphisms γ, δ of $\mathcal{M}$ such that $\gamma\beta = \alpha$ and $\delta\alpha = \beta$. The fact that γ is an automorphism implies that $O(\gamma) = \varnothing$; using Lemma 4.3.2 (1), we get

$$O(\beta) = O(\gamma\beta) = O(\alpha) \cup O(\gamma)\beta = O(\beta).$$

As this happens, $M\alpha = M\beta$; so in a similar fashion to Lemma 4.2.6, define $\nu = \alpha\beta^{-1}$ as a bijection from M to itself. Note that ν is therefore an element of Sym(M). Since $\gamma, \alpha, \beta \in$ Mon(M), we have $\alpha = \gamma\beta = \nu\beta$; by right-cancellativity of Mon(M), it follows that that $\gamma = \nu$ and so ν is an automorphism. Now, by Lemma 4.3.2 (1), we have

$$R_i^{\mathcal{A}(\alpha)} = R_i^{\mathcal{A}(\gamma\beta)} = R_i^{\mathcal{A}(\gamma)} \cup R_i^{\mathcal{A}(\beta)}\gamma^{-1} = R_i^{\mathcal{A}(\beta)}\gamma^{-1}$$

for all $R_i \in \sigma$. From this, $S(\alpha) = S(\gamma\beta) = S(\beta)\gamma^{-1}$. Since $\gamma = \alpha\beta^{-1}$ is an automorphism, $\mathcal{M}[S(\alpha)] \cong \mathcal{M}[S(\beta)]$ via this automorphism. The proof that $S(\alpha)\alpha = S(\beta)\beta$ in this case is the same as the one given in Proposition 4.1.11.

Now assume the converse; that $O(\alpha) = O(\beta)$, $S(\alpha)\alpha = S(\beta)\beta$ and $\mathcal{M}[S(\alpha)] \cong \mathcal{M}[S(\beta)]$ via the isomorphism induced by $\alpha\beta^{-1}$. As $O(\alpha) = O(\beta)$ we can define the bijection $\alpha\beta^{-1} : M \to M$ as illustrated in the previous part of the proof. All that remains to show is that $\alpha\beta^{-1}$ is a monomorphism; the proof of this is almost exactly that of the similar direction of Proposition 4.1.11. Hence $(\alpha\beta^{-1})\beta = \alpha$. We can define $\beta\alpha^{-1}$ and show it is a monomorphism in a similar fashion to find that $(\beta\alpha^{-1})\alpha = \beta$; so $\alpha\mathscr{L}\beta$. $\square$

We now turn our attention to determining $\mathscr{R}$-relations in Mon$(\mathcal{M})$. The following is a generalised version of the $\mathscr{R}$-relations case of Proposition 3.2.9.

Proposition 4.3.7. *Let $\alpha, \beta \in$ Mon$(\mathcal{M})$. Then $\alpha\mathscr{R}\beta$ if and only if there exists a monomorphism $f : \mathcal{O}(\alpha) \to \mathcal{O}(\beta)$ that extends to a monomorphism η of $\mathcal{M}$ such that $\eta|_{\mathcal{M}\alpha} = \alpha^{-1}\beta : \mathcal{M}\alpha \to \mathcal{M}\beta$; and there exists a monomorphism $g : \mathcal{O}(\beta) \to \mathcal{O}(\alpha)$ that extends to a monomorphism θ of $\mathcal{M}$ such that $\theta|_{\mathcal{M}\beta} = \beta^{-1}\alpha : \mathcal{M}\beta \to \mathcal{M}\alpha$.*

Proof. Suppose that $\alpha\mathscr{R}\beta$ in Mon$(\mathcal{M})$. So there exist monomorphisms γ, δ of

$\mathcal{M}$ such that $\alpha\gamma = \beta$ and $\beta\delta = \alpha$. Then, $\gamma|_{\mathcal{M}\alpha} : \mathcal{M}\alpha \to \mathcal{M}\beta$ is a bijective monomorphism from $\mathcal{M}\alpha$ to $\mathcal{M}\beta$. Now, as $\alpha : M \to M\alpha$ and $\beta : M \to M\beta$ are bijections, there exists a unique bijection $\alpha^{-1}\beta : M\alpha \to M\beta$. So if $a\alpha \in M\alpha$ then $(a\alpha)\gamma = a\alpha\gamma = a\beta$ and therefore $\gamma|_{\mathcal{M}\alpha} = \alpha^{-1}\beta : \mathcal{M}\alpha \to \mathcal{M}\beta$. Also, γ sends $\mathcal{M} \setminus \mathcal{M}\alpha$ to $\mathcal{M} \setminus \mathcal{M}\beta$; so by Definition 4.3.1 γ sends $O(\alpha)$ to $O(\beta)$. The map $\gamma|_{\mathcal{O}(\alpha)} : \mathcal{O}(\alpha) \to \mathcal{O}(\beta)$ is a monomorphism that clearly extends to γ.

Now suppose the converse conditions, and let $\eta \in \text{Mon}(\mathcal{M})$ be as described above. Then $\alpha\eta : \mathcal{M} \to \mathcal{M}\beta$ first sends $a \in \mathcal{M}$ to $a\alpha$ in $\mathcal{M}\alpha$; then sends $a\alpha$ to $a\alpha\eta = a\alpha\alpha^{-1}\beta = a\beta$. So η is a monomorphism such that $\alpha\eta = \beta$. We can do the same in the other direction to see that θ is a monomorphism such that $\beta\theta = \alpha$; and so $\alpha\mathscr{R}\beta$. □

Remarks. (i) As mentioned above, if $\mathcal{M}$ is a countable set, then both of these propositions reduce to Proposition 3.2.9.

(ii) Note that if α, β are bijections (and hence bimorphisms), then Propositions 4.3.6 and 4.3.7 reduce to Propositions 4.1.11 and Proposition 4.1.12 respectively. Similarly, if α, β are embeddings then these two results reduce to Lemma 4.2.6 and Proposition 4.2.8 respectively.

(iii) If both $O(\alpha)$ and $O(\beta)$ are finite (and non-empty), then $\alpha\mathscr{R}\beta$ if and only if $\mathcal{O}(\alpha) \cong \mathcal{O}(\beta)$ and this isomorphism and its inverse both extend to bimorphisms of $\mathcal{M}$; this is a consequence of Lemma 3.2.6. If, in addition, $e_i(\alpha) + e_i(\beta) < \infty$ for all $R_i \in \sigma$, then the isomorphism between $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$ must extend to an automorphism of $\mathcal{M}$; this is by Lemma 4.3.2 (2).

We finish on the following questions, which we have not been able to determine in general due to time constraints. Note that we showed that $\mathscr{R} = \mathscr{D}$ in $\text{Mon}(\mathbb{N})$, the monomorphism monoid on a countable set, motivating the second question.

Question 4.3.8. *Characterise Green's $\mathscr{D}$ and $\mathscr{J}$-relations in Mon$(\mathcal{M})$. In particular, does $\mathscr{R} = \mathscr{D}$ in Mon$(\mathcal{M})$?*

4.3.2 Generation properties of Mon(R)

We can use similar techniques to those in Subsection 4.2.3 to determine generation and cofinality results for Mon(R). Let $\alpha \in$ Mon(R). As R is a graph, we write $e(\alpha)$ to mean the number of edges added in by α as in Subsection 4.1.4. Our first result is a straightforward application of Proposition 3.2.12.

Theorem 4.3.9. *scf(Mon(R))* $= \aleph_0$.

Proof. As Mon(R) contains Emb(R), it follows from Theorem 4.2.17 that Mon(R) contains maps γ_k such that $o(\gamma_k) = k$ for all $k \in \mathbb{N}$. Furthermore, there exists a $\delta \in$ Mon(R) such that $o(\delta) = \infty$. By Corollary 4.3.4, Mon(R) satisfies the conditions for Proposition 3.2.12 and so scf(Mon(R)) $= \aleph_0$. □

Following proofs that Bi(R) and Emb(R) do not have the Bergman property, we show that Mon(R) does not have the Bergman property. In a similar fashion to these cases, our aim is to demonstrate an inefficient generating set for Mon(R). To do this, we define a subset FMon(R) of Mon(R) that contains all monomorphisms of R with a finite additional and a finite omitted graph. By Lemma 4.3.2 (1), FMon(R) is a submonoid of Mon(R).

Theorem 4.3.10. *Let $Y =$ Aut$(R) \cup \{\beta\} \cup \{\epsilon\}$, where $\beta \in$ Bi(R) such that $e(\beta) = 1$ and $\epsilon \in$ Emb(R) such that $o(\epsilon) = 1$. Then Y generates FMon(R).*

Proof. It suffices to generate an element $\alpha \in$ FMon(R) with any additional graph (on any support graph) and any omitted graph. Suppose $\theta \in$ Bi(R) adds in finitely many edges and $\phi \in$ Emb(R) omits finitely many vertices. As Y contains all the automorphisms and a bimorphism that adds in a single edge, we can generate θ by Proposition 4.1.23. Similarly, since Y contains an embedding that omits a single vertex, we can generate ϕ by Proposition 4.2.19. Suppose that γ is an automorphism, and define a monomorphism $\alpha = \theta\gamma\phi \in$ FMon(R). Then by Lemma 4.3.2 (1):

$$\mathcal{A}(\alpha) = \mathcal{A}(\theta\gamma\phi) = \mathcal{A}(\theta) \cup \mathcal{A}(\gamma\phi)\theta^{-1} = \mathcal{A}(\theta)$$

as $\gamma\phi$ is an isomorphism. This means that $S(\alpha) = S(\theta)$. Similarly by Lemma 4.3.2 (1):

$$O(\alpha) = O(\theta\gamma\phi) = O(\phi) \cup O(\theta\gamma)\phi = O(\phi)$$

as $\theta\gamma$ is a bijection. So we have generated a monomorphism α with additional graph $\mathcal{A}(\theta)$, support $S(\theta)$, and omitted graph $\mathcal{O}(\phi)$. Now, for some automorphisms δ, ζ of R, the monomorphism $\delta\alpha\zeta$ has isomorphic additional, support and omitted graphs to α by Lemma 4.3.2 (1). As $\mathrm{Aut}(R) \subseteq Y$, we can generate any such monomorphism and we are done. □

Similar to Corollary 4.1.24 and Corollary 4.2.20, we have a straightforward corollary of Theorem 4.3.10.

Corollary 4.3.11. *cf(FMon(R))* $> \aleph_0$.

Proof. Theorem 4.3.10 shows that $\mathrm{rank}(\mathrm{FMon}(R) : \mathrm{Aut}(R)) = 2$. As $\mathrm{cf}(\mathrm{Aut}(R)) > \aleph_0$, the result follows from Proposition 3.1.6. □

Now, define the set $Z = \{\alpha \in \mathrm{Mon}(R) : e(\alpha) = \infty \text{ or } o(\alpha) = \infty\}$. By remarks following Lemma 4.3.2, this is an ideal of $\mathrm{Mon}(R)$. It follows that $\mathrm{Mon}(R) \setminus \mathrm{FMon}(R) = Z$, and so $Y \cup Z$ generates $\mathrm{Mon}(R)$ by Theorem 4.3.10. In the next result, we determine that $Y \cup Z$ is not a Cayley bounded generating set for $\mathrm{Mon}(R)$.

Theorem 4.3.12. *Let $\rho = \rho_1\rho_2...\rho_k$ be a product of elements from $Y \cup Z$. Then $e(\rho) + o(\rho) \leq k$ or it is infinite.*

Proof. We can see that if any of the ρ_i's are in Z, then $e(\rho)$ or $o(\rho)$ is infinite and we are done. So assume that each of the ρ_i's are in Y. We can now perform an induction on the length of product; the proof follows in the same fashion as Theorem 4.1.25 and Theorem 4.2.21. □

Similarly to $\mathrm{Bi}(R)$ and $\mathrm{Emb}(R)$, this final proposition shows that the generating set Z of $\mathrm{Mon}(R)$ is not Cayley bounded and therefore $\mathrm{Mon}(R)$ does not have the Bergman property. We have not been able to determine the cofinality of $\mathrm{Mon}(R)$; however, we conjecture that this is uncountable due to the uncountable cofinality of $\mathrm{Aut}(R)$ and Corollary 4.3.11. This needs to be verified and so we leave it as an open question.

Question 4.3.13. *Is cf$(Mon(R)) > \aleph_0$?*

Furthermore, whilst we have investigated semigroup-theoretic properties of injective endomorphisms in general, we have not considered the case of a general *surjective* endomorphism. A study along the lines of those conducted in this chapter would provide an interesting future direction for research.

Question 4.3.14. *Develop the theory of epimorphism monoids of first-order structures.*

5

Partial map monoids of first-order structures

The focus of Chapter 4 was the study of several types of endomorphism monoid on a σ-structure $\mathcal{M}$; determining semigroup-theoretic properties in general and generation properties of these monoids in some special cases. As mentioned in the introduction, there is a body of literature on semigroup-theoretic properties of endomorphism monoids of first-order structures [7, 8, 66, 24, 23]; studies have also been made with connections to constraint satisfaction problems [4, 5] and topological applications [52, 6].

Studies have been conducted into the partial map monoid and symmetric inverse monoid on a set X; ranging from semigroup-theoretic properties [41] to representation theory [34] and geometry [51]. Furthermore, there is literature on studying partial homomorphisms and isomorphisms of first-order structures, both in classical model theory [28, 37] and inverse semigroups acting on first-order structures [71]. To our knowledge however, no explicit study has been made on the semigroup theory of *partial map monoids of first-order structures*; which is the subject covered in this chapter.

The endomorphism monoids on $\mathcal{M}$ considered in Chapter 4 only represent a section of all self-map monoids on $\mathcal{M}$. For instance, as the composition of two homomorphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ is a homomorphism $fg : \mathcal{A} \to \mathcal{C}$ and this composition is associative, we can form a semigroup (with zero) of all homomorphisms between *substructures* of a first-order structure $\mathcal{M}$. This semigroup represents a natural analogue of the partial map monoid $\mathrm{Part}(X)$ on a set X; we say that this is the *partial homomorphism monoid* Part($\mathcal{M}$) of a first-order structure $\mathcal{M}$. In a similar fashion, we can define the *symmetric inverse monoid* $\text{Inv}(\mathcal{M})$ to be the partial map monoid consisting of all isomorphisms between substructures of $\mathcal{M}$. This is the structural analogue of the classical symmetric inverse monoid $\text{Inv}(X)$ on a set X. Moreover, $\text{Inv}(\mathcal{M})$ is also an inverse semigroup; as isomorphisms are invertible, it follows that for every element g in $\text{Inv}(\mathcal{M})$ there exists a unique h in $\text{Inv}(\mathcal{M})$ such that $ghg = g$ and $hgh = h$.

If $\mathcal{M}$ is a relational first-order structure, then a bijective endomorphism from $\mathcal{M}$ to itself may not be an automorphism; this formed the basis for the investigation of bimorphism monoids in Chapter 4. This can be generalised to all bijective homomorphisms between substructures of $\mathcal{M}$; these are not necessarily isomorphisms. As the composition of two injective maps is injective, it follows that the collection of partial monomorphisms of $\mathcal{M}$ forms a monoid Inj($\mathcal{M}$), which we shall call the *partial monomorphism monoid* of $\mathcal{M}$. Note that if $\mathcal{M}$ is a set, then Inj($\mathcal{M}$) $=$ $\text{Inv}(\mathcal{M})$. As every isomorphism is a monomorphism and every monomorphism is a homomorphism, it follows that $\text{Inv}(\mathcal{M}) \subseteq$ Inj($\mathcal{M}$) $\subseteq$ Part($\mathcal{M}$). Section 5.1 details some basic facts about these three partial map monoids including idempotents, cardinality, and Green's relations for $\text{Inv}(\mathcal{M})$ and Inj($\mathcal{M}$).

One of these results (Corollary 5.1.3) says that if $\mathcal{M}$ is a countably infinite structure, then each of the monoids Part($\mathcal{M}$), Inj($\mathcal{M}$) and $\text{Inv}(\mathcal{M})$ is uncountable. Because of this, they are infinitely generated and so there is a chance that these monoids have uncountable cofinality and strong cofinality and/or the Bergman property, depending on the structure $\mathcal{M}$. The objective of Section 5.2 is to modify Dolinka's approach in order to find a similar result to [23, Theorem 2.2] for partial map monoids of first-order structures.

However, Part($\mathcal{M}$), Inj($\mathcal{M}$) and $\text{Inv}(\mathcal{M})$ are not the only partial map monoids on a first-order structure. For any inverse semigroup S, there is a commutative subsemigroup $E(S)$ of S consisting of all idempotents of S; this is often referred to as the *semilattice of idempotents*. For the endomorphism monoids considered in Chapter 4, $E(S)$ consists of the identity endomorphism of $\mathcal{M}$ by Corollary 3.2.3

and Corollary 4.1.2. This is a different story for the symmetric inverse monoid on $\mathcal{M}$; the identity mapping on any *substructure* of $\mathcal{M}$ is an idempotent in Inv($\mathcal{M}$). By Lemma 5.1.2, these are the only idempotents of Inv($\mathcal{M}$). We perform a study on this monoid similar to any other self-map monoid considered in this thesis; this is the subject of Section 5.3.

As in Chapter 4, $\sigma = \{R_i : i \in I\}$ is a relational signature and $\mathcal{M}$ will be a σ-structure on a countable domain M throughout the chapter unless otherwise stated. We write the domain and image of a partial homomorphism α of $\mathcal{M}$ as dom α and im α respectively; we denote the substructures induced on dom α and im α by $\mathcal{M}$ as $\langle \text{dom } \alpha \rangle$ and $\langle \text{im } \alpha \rangle$ respectively. The identity map on some subset A of M is written as id_A. Any partial map monoid of a first-order structure $\mathcal{M}$ acts on n-tuples of $\mathcal{M}$ via the componentwise partial monoid action given in Subsection 2.2.3. For a bijective homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$, the inverse *function* will be written $\alpha^{-1} : B \to A$, regardless of whether or not α^{-1} is a homomorphism. If α is an isomorphism, then α^{-1} is the unique semigroup theoretic inverse for α.

5.1 Semigroup-theoretic properties

As Part($\mathcal{M}$) embeds in the partial map monoid Part(M) of the domain of $\mathcal{M}$, it follows that Equation 2.1 and Equation 2.2 (on page 26) hold for any $\alpha, \beta \in$ Part($\mathcal{M}$); the same is true for Inv($\mathcal{M}$) embedding in the symmetric inverse monoid Inv(M). As a collection of partial bijections, Inj($\mathcal{M}$) embeds in the symmetric inverse monoid Inv(M); so Equation 2.1 and Equation 2.2 hold in Inj($\mathcal{M}$) too. However, Inj($\mathcal{M}$) is *not* an inverse semigroup; for instance, a bimorphism of $\mathcal{M}$ (which is a partial monomorphism) sending a non-relation to a relation does not have an inverse by Corollary 4.1.7. This means that Inj($\mathcal{M}$) is an *inverse semigroup-embeddable monoid*; a partial map analogue of the bimorphism monoid of a first order structure $\mathcal{M}$.

Recall (from Section 2.1) that for a map $\alpha : \text{dom } \alpha \to \text{im } \alpha$, the *converse* $\alpha^* : \text{im } \alpha \to \text{dom } \alpha$ is a multifunction; furthermore, if α^* is a multifunction then

$\bar{b}\alpha^* = \{\bar{a} : \bar{a}\alpha = \bar{b}\}$. The next result generalises Proposition 1.4.5 of [41] to the case of partial map monoids of σ-structures.

Lemma 5.1.1. *Let $\mathcal{M}$ be a σ-structure, and suppose that $\alpha \in$ Part$(\mathcal{M})$. Then α^* is in Part$(\mathcal{M})$ if and only if α is a partial isomorphism.*

Proof. As α^* is in Part$(\mathcal{M})$, it must be a surjective map in its own right; this only happens if α is both injective and surjective (see Section 2.1). Now, let R_i be an n-ary relation of σ, and suppose $\bar{a} \in (\text{dom }\alpha)^n$. Since α is a partial homomorphism, if $\bar{a} \in R_i^{\langle \text{dom }\alpha\rangle}$ then $\bar{a}\alpha = \bar{b} \in R_i^{\langle \text{im }\alpha\rangle}$. Note that as α is injective, $\bar{a} = \bar{b}\alpha^*$ by definition. As $\alpha^* \in \text{Part}(\mathcal{M})$ by assumption, we have that $\bar{b}\alpha^*$ is in $R_i^{\langle \text{im }\alpha\rangle}$; and so $\bar{a} \in R_i^{\langle \text{dom }\alpha\rangle}$. Hence $\bar{a} \in R_i^{\langle \text{dom }\alpha\rangle}$ if and only if $\bar{a}\alpha \in R_i^{\langle \text{im }\alpha\rangle}$; therefore, α is an isomorphism.

Now suppose that α is a partial isomorphism. This means that α is bijective and so α^* is a bijective function by remarks in Section 2.1. We now check that α^* is a homomorphism. Suppose that $\bar{a}\alpha \in R_i^{\langle \text{dom }\alpha^*\rangle} = R_i^{\langle \text{im }\alpha\rangle}$; as α is a partial isomorphism, this implies that $\bar{a} \in R_i^{\langle \text{dom }\alpha\rangle} = R_i^{\langle \text{im }\alpha^*\rangle}$. Since $\bar{a}\alpha\alpha^* = \bar{a}$, it follows that α^* is a partial homomorphism and therefore $\alpha^* \in \text{Part}(\mathcal{M})$. □

Remark. Note here that if α is a partial bijection, then $\alpha^* = \alpha^{-1}$, the unique semigroup-theoretic inverse for α in Inv(M).

Recall that idempotents of the symmetric inverse monoid Inv(X) on a set X were characterised in Lemma 2.2.8; we can use this result to characterise idempotents in two of our three considered partial map monoids.

Lemma 5.1.2. *Let $\epsilon \in$ Inj$(\mathcal{M})$. Then ϵ is an idempotent if and only if ϵ is the identity map on some substructure of $\mathcal{M}$.*

Proof. As Inj$(\mathcal{M})$ embeds in the symmetric inverse monoid Inv(M), we can show that any idempotent in Inj$(\mathcal{M})$ is an idempotent of Inv(M) via a similar argument to that of Corollary 4.1.2. By Lemma 2.2.8, the idempotents of Inv(M) are precisely the identity maps on subsets. □

Remark. This result also holds in the case where $\epsilon \in$ Inv$(\mathcal{M})$. However, Lemma 5.1.2 may not hold for the case where $\epsilon \in$ Part$(\mathcal{M})$. For instance, the partial map monoid of the random graph $\mathrm{Part}(R)$ contains the endomorphism monoid $\mathrm{End}(R)$; this was shown to have $2^{\aleph_0}$ many primitive idempotents (notably, including examples that do not arise as identity maps on substructures) by Bonato and Delić [7].

Corollary 5.1.3. *Suppose that $\mathcal{M}$ is a countably infinite first-order structure with domain M. Then $|Inv(\mathcal{M})| = |Inj(\mathcal{M})| = |Part(\mathcal{M})| = 2^{\aleph_0}$.*

Proof. As the power set $\mathcal{P}(M)$ of a countable set is uncountable, the fact that $\mathrm{Inv}(\mathcal{M})$ contains an identity map for each subset of M implies that $|\mathrm{Inv}(\mathcal{M})| = 2^{\aleph_0}$. As $\mathrm{Inv}(\mathcal{M}) \subseteq \mathrm{Inj}(\mathcal{M}) \subseteq \mathrm{Part}(\mathcal{M})$, the result follows. □

Similarly to the bimorphism, embedding and monomorphism monoids of a first order structure $\mathcal{M}$, we can look at Green's relations for $\mathrm{Inv}(\mathcal{M})$ and $\mathrm{Inj}(\mathcal{M})$. Using classical semigroup-theoretic results together with Lemma 2.2.10, we can easily characterise Green's relations for the symmetric inverse monoid.

Lemma 5.1.4. *Let $\mathcal{M}$ be a σ-structure, and suppose $\alpha, \beta \in Inv(\mathcal{M})$. Then:*

(1) $\alpha \mathscr{L} \beta$ if and only if im α = im β;

(2) $\alpha \mathscr{R} \beta$ if and only if dom α = dom β;

(3) $\alpha \mathscr{D} \beta$ if and only if $\langle$im $\alpha\rangle \cong \langle$im $\beta\rangle$, and;

(4) $\mathscr{D} = \mathscr{J}$.

Proof. As $\mathrm{Inv}(\mathcal{M})$ is an inverse subsemigroup of $\mathrm{Inv}(M)$, it is also a regular subsemigroup of $\mathrm{Inv}(M)$. As this happens, the Green's relations of $\mathrm{Inv}(\mathcal{M})$ are the restrictions of the Green's relations of $\mathrm{Inv}(M)$. The results for $\mathscr{L}$ and $\mathscr{R}$ for $f, g \in \mathrm{Inv}(\mathcal{M})$, and the fact that $\mathscr{D} = \mathscr{J}$, follow directly from Lemma 2.2.10. From that same result two functions α, β in $\mathrm{Inv}(M)$ are $\mathscr{D}$-related if and only if $|\mathrm{im}\ \alpha| = |\mathrm{im}\ \beta|$; this means there is a bijection γ taking im α to im β. This only happens in $\mathrm{Inv}(\mathcal{M})$ if γ is also an isomorphism; so $\langle \mathrm{im}\ \alpha\rangle \cong \langle \mathrm{im}\ \beta\rangle$. □

Remark. We notice the similarities between this result and the characterisation of Green's relations for regular elements of endomorphism monoids of Δ-structures (not necessarily first-order) in Magill and Subbiah [57], and the corresponding special case for σ-structures $\mathcal{M}$ in [24, Lemma 2.5].

For Inj($\mathcal{M}$), the monoid of all partial monomorphisms of a σ-structure $\mathcal{M}$, slightly more care must be taken in determining Green's relations. As partial monomorphisms may change relations to non-relations, it follows that Inj($\mathcal{M}$) may not be a regular monoid (as it contains Bi($\mathcal{M}$); see Corollary 4.1.2 (2)) and hence is not a regular submonoid of Inv(M). However, as a partial map analogue of the bimorphism monoid, we can adapt some of our existing knowledge of bimorphisms from Section 4.1 to the setting of partial monomorphisms. Our first definition generalises Definition 4.1.4 to elements of Inj($\mathcal{M}$).

Definition 5.1.5. For a partial monomorphism α of $\mathcal{M}$, define a σ-structure $\mathcal{A}(\alpha)$ with domain dom α and relations given by $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$ if and only if $\bar{a} \notin R_i^{\langle \text{dom } \alpha \rangle}$ and $\bar{a}\alpha \in R_i^{\langle \text{im } \alpha \rangle}$ for all $i \in I$. We call this the *additional structure* of α.

For $\alpha \in$ Inj($\mathcal{M}$), define the *support* of α to be the set

$$S(\alpha) = \{x \in \text{dom } \alpha : x \in \bar{a} \text{ and } \bar{a} \in R_i^{\mathcal{A}(\alpha)} \text{ for some } i \in I\}.$$

As $S(\alpha)$ is a subset of dom α, we can induce a structure $\mathcal{M}[S(\alpha)]$ on $S(\alpha)$ with relations from $\mathcal{M}$; call this the *support structure* of α. For two elements α, β of Inj($\mathcal{M}$) and some $R_i \in \sigma$ with arity n, define the set

$$R_i^{\mathcal{A}(\beta)}\alpha^{-1} = \{\bar{x} \in (\text{dom } \alpha)^n : \bar{x}\alpha \in R_i^{\mathcal{A}(\beta)}\}.$$

The next result is a partial map analogue of Lemma 4.1.6.

Lemma 5.1.6. *Suppose that $\alpha, \beta \in$ Inj($\mathcal{M}$). Then*

$$R_i^{\mathcal{A}(\alpha\beta)} = (R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}) \cap (\text{dom } \alpha\beta)^n$$

and the first term of the intersection is a disjoint union.

Remark. As in Lemma 4.1.6, the idea here is that the set of relations added in by the product $\alpha\beta$ is the same set of relations given by first applying α and then

β. Here, $R_i^{\mathcal{A}(\alpha)}$ is the set of relations added in by α, and $R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ is the set of relations added in by β after α has been applied.

Proof. The proof is by containment both ways. We show that $(R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1}) \cap (\text{dom } \alpha\beta)^n \subseteq R_i^{\mathcal{A}(\alpha\beta)}$ by proving that both $R_i^{\mathcal{A}(\alpha)} \cap (\text{dom } \alpha\beta)^n$ and $R_i^{\mathcal{A}(\beta)}\alpha^{-1} \cap (\text{dom } \alpha\beta)^n$ are contained in $R_i^{\mathcal{A}(\alpha\beta)}$.

Suppose that $\bar{a} \in R_i^{\mathcal{A}(\alpha)} \cap (\text{dom } \alpha\beta)^n$. Since $\bar{a} \in (\text{dom } \alpha\beta)^n$, it follows that $\bar{a}\alpha\beta \in (\text{im } \alpha\beta)^n$. As $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$, we have that $\bar{a} \notin R_i^{\langle \text{dom } \alpha \rangle}$ but $\bar{a}\alpha \in R_i^{\langle \text{im } \alpha \rangle}$. Therefore, $\bar{a}\alpha\beta \in R_i^{\langle \text{im } \alpha\beta \rangle}$, since $\bar{a}\alpha\beta \in (\text{im } \alpha\beta)^n$. So $\bar{a} \in R_i^{\mathcal{A}(\alpha\beta)}$.

Now, assume that $\bar{a} \in R_i^{\mathcal{A}(\beta)}\alpha^{-1} \cap (\text{dom } \alpha\beta)^n$. As this happens, $\bar{a}\alpha \in R_i^{\mathcal{A}(\beta)}$; by definition, $\bar{a}\alpha \notin R_i^{\langle \text{dom } \alpha \rangle}$ but $\bar{a}\alpha\beta \in R_i^{\langle \text{im } \beta \rangle}$. Since $\bar{a} \in (\text{dom } \alpha\beta)^n = ((\text{im } \alpha \cap \text{dom } \beta)\alpha^*)^n$, we have that $\bar{a}\alpha \in (\text{im } \alpha \cap \text{dom } \beta)^n$, and so $\bar{a}\alpha \notin R_i^{\langle \text{im } \alpha \cap \text{dom } \beta \rangle}$. This means that $\bar{a}\alpha\beta \in R_i^{\langle \text{im } \alpha \cap \text{dom } \beta \rangle \beta} = R_i^{\langle \text{im } \alpha\beta \rangle}$, and so $\bar{a} \in R_i^{\mathcal{A}(\alpha\beta)}$.

For the reverse containment, suppose that $\bar{a} \in R_i^{\mathcal{A}(\alpha\beta)}$; so, $\bar{a} \in (\text{dom } \alpha\beta)^n = ((\text{im } \alpha \cap \text{dom } \beta)\alpha^*)^n$ by Equation 2.1. Similar to Lemma 4.1.6 there are two cases; either $\bar{a}\alpha \in R_i^{\langle \text{im } \alpha \cap \text{dom } \beta \rangle}$ or it is not. If $\bar{a} \in R_i^{\langle \text{im } \alpha \cap \text{dom } \beta \rangle}$, then $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$ and so this case is true. If $\bar{a}\alpha \notin R_i^{\langle \text{im } \alpha \cap \text{dom } \beta \rangle}$, then as $\bar{a}\alpha\beta \in R_i^{\langle \text{im } \alpha\beta \rangle}$ it follows that $\bar{a}\alpha \in R_i^{\mathcal{A}(\beta)}$. By definition, $\bar{a} \in R_i^{\mathcal{A}(\beta)}\alpha^{-1}$ and we are done.

The proof that $(R_i^{\mathcal{A}(\alpha)} \cup R_i^{\mathcal{A}(\beta)}\alpha^{-1})$ is a disjoint union is similar to the disjoint union portion of Lemma 4.1.6. $\square$

Remark. It is easy to see that Lemma 4.1.6 follows as a direct consequence of this result.

It is clear that $\alpha \in \text{Inj}(\mathcal{M})$ is a partial isomorphism (and hence in $\text{Inv}(\mathcal{M})$) if and only if $R_i^{\mathcal{A}(\alpha)} = \varnothing$ for all $R_i \in \sigma$. The next three results determine Green's $\mathscr{L}, \mathscr{R}$ and $\mathscr{D}$-relations for $\text{Inj}(\mathcal{M})$.

Proposition 5.1.7. *Let $\mathcal{M}$ be a σ-structure, and suppose $\alpha, \beta \in Inj(\mathcal{M})$. Then $\alpha \mathscr{L} \beta$ if and only if im α = im β, and the resulting map $\alpha\beta^{-1}$ is an isomorphism sending $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$, and $S(\alpha)\alpha = S(\beta)\beta$.*

Proof. For the converse direction, we need to find partial monomorphisms $\gamma, \delta \in \text{Inj}(\mathcal{M})$ such that $\gamma\beta = \alpha$ and $\delta\alpha = \beta$. As α and β are in $\text{Inj}(\mathcal{M})$, they are in $\text{Inv}(M)$ and so are partial bijections. As this occurs, and $\text{im}\,\alpha = \text{im}\,\beta$, we can uniquely define a bijection $\gamma = \alpha\beta^{-1} : \text{dom}\,\alpha \to \text{dom}\,\beta$. Using the assumptions that $\alpha\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$ and $S(\alpha)\alpha = S(\beta)\beta$, we can show that γ is also a monomorphism using a similar argument to Proposition 4.1.11. Similarly, the bijection $\delta = \beta\alpha^{-1} : \text{dom}\,\beta \to \text{dom}\,\alpha$ is uniquely defined and can be shown to be a monomorphism in the same fashion.

Now, suppose that $\alpha\mathscr{L}\beta$; so there exists $\gamma, \delta \in \text{Inj}(\mathcal{M})$ such that $\gamma\beta = \alpha$ and $\delta\alpha = \beta$. As $\alpha, \beta \in \text{Inv}(M)$, they have to be $\mathscr{L}$-related in $\text{Inv}(M)$; so Lemma 2.2.10 implies that $\text{im}\,\alpha = \text{im}\,\beta$. Now, as $\gamma\beta = \alpha$ and $\delta\alpha = \beta$, it follows that $\gamma\delta\alpha = \alpha$. By Lemma 2.2.9, this means that $\gamma\delta|_{\text{dom}\,\alpha} = id_{\text{dom}\,\alpha}$. So there exists a subset X of $\text{im}\,\gamma$ such that $\gamma|_{\text{dom}\,\alpha} : \text{dom}\,\alpha \to X$ composed with $\delta|_X : X \to \text{dom}\,\alpha$ is the identity element on $\text{dom}\,\alpha$. This implies that $\delta|_X$ is an inverse for $\gamma|_{\text{dom}\,\alpha}$; by the remark following Lemma 5.1.1, they are both isomorphisms. Similarly, we can show that as $\delta\gamma\beta = \beta$, then $\delta|_{\text{dom}\,\beta}$ is an isomorphism. Using the fact that $\text{im}\,\alpha = \text{im}\,\beta$, there exists a unique bijection $\alpha\beta^{-1} : \text{dom}\,\alpha \to \text{dom}\,\beta$ such that $(\alpha\beta^{-1})\beta = \alpha$. Since $\gamma\beta = \alpha$, we can conclude that $\gamma|_{\text{dom}}\,\alpha = \alpha\beta^{-1}$ and so $\alpha\beta^{-1}$ is an isomorphism. We can now show that $\alpha\beta^{-1}$ sends $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\alpha)]$ and that $S(\alpha)\alpha = S(\beta)\beta$ via a similar argument to Proposition 4.1.11, where we use Lemma 5.1.6 in place of Lemma 4.1.6. $\square$

Proposition 5.1.8. *Let $\mathcal{M}$ be a σ-structure, and suppose α and β are in Inj$(\mathcal{M})$. Then $\alpha\mathscr{R}\beta$ if and only if dom α = dom β and $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$.*

Proof. Suppose that $\text{dom}\,\alpha = \text{dom}\,\beta$ and $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$. As α, β are partial bijections and $\text{dom}\,\alpha = \text{dom}\,\beta$, we can find a uniquely defined bijection $\gamma = \alpha^{-1}\beta : \text{im}\,\alpha \to \text{im}\,\beta$ such that $\gamma\alpha = \beta$. We can use a similar argument to Proposition 4.1.12 in order to show that γ is a monomorphism from $\langle \text{im}\,\alpha \rangle$ to $\langle \text{im}\,\beta \rangle$. Similarly, the bijection $\delta = \beta^{-1}\alpha : \text{im}\,\beta \to \text{im}\,\alpha$ can be shown to be a monomorphism in the same fashion.

Conversely, suppose that $\alpha\mathscr{R}\beta$. Therefore, there exists $\gamma, \delta \in \text{Inj}(\mathcal{M})$ such that $\alpha\gamma = \beta$ and $\beta\delta = \alpha$. As α and β must be $\mathscr{R}$-related in $\text{Inv}(M)$, it follows that $\text{dom}\,\alpha = \text{dom}\,\beta$. Now, let $R_i \in \sigma$ and suppose that $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$; so in particular,

$\bar{a} \notin R_i^{\langle \mathrm{dom}\,\alpha \rangle} = R_i^{\langle \mathrm{dom}\,\beta \rangle}$ by assumption. Here, $\bar{a}\beta = \bar{a}\alpha\gamma$ and as $\bar{a} \in R_i^{\mathcal{A}(\alpha)}$, it follows that $\bar{a}\alpha \in R_i^{\langle \mathrm{im}\,\alpha \rangle}$. Since γ is a monomorphism, $\bar{a}\alpha\gamma = \bar{a}\beta \in R_i^{\langle \mathrm{im}\,\beta \rangle}$ and so $\bar{a} \in R_i^{\mathcal{A}(\beta)}$ by definition. Using a similar argument and the fact that δ is a monomorphism, we can show that any $\bar{b} \in R_i^{\mathcal{A}(\beta)}$ is in $R_i^{\mathcal{A}(\alpha)}$; and so $R_i^{\mathcal{A}(\alpha)} = R_i^{\mathcal{A}(\beta)}$ by containment both ways. This is true for all $R_i \in \sigma$ and so $\mathcal{A}(\alpha) = \mathcal{A}(\beta)$. □

Proposition 5.1.9. *Let $\mathcal{M}$ be a σ-structure, and suppose α and β are in Inj($\mathcal{M}$). Then $\alpha\mathscr{D}\beta$ if and only if there exists a partial monomorphism η such that*

- *dom α = dom η and im η = im β;*
- *$\eta\beta^{-1}$ induces an isomorphism from $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$, and $\alpha^{-1}\eta$ induces an isomorphism from $\mathcal{M}[S(\alpha)\alpha]$ to $\mathcal{M}[S(\beta)\beta]$, and;*
- *$S(\beta)\beta = S(\eta)\eta$.*

Proof. Suppose that $\alpha\mathscr{D}\beta$ in Inj($\mathcal{M}$); so there exists a partial monomorphism η such that $\alpha\mathscr{R}\eta$ and $\eta\mathscr{L}\beta$. By Proposition 5.1.7, it follows that $\mathrm{im}\,\eta = \mathrm{im}\,\beta$, and $\eta\beta^{-1}$ induces an isomorphism sending $\mathcal{M}[S(\eta)]$ to $\mathcal{M}[S(\beta)]$. By Proposition 5.1.8, $\mathrm{dom}\,\alpha = \mathrm{dom}\,\eta$ and $\mathcal{A}(\alpha) = \mathcal{A}(\eta)$. As this happens, it follows that $S(\eta) = S(\alpha)$, and so $\eta\beta^{-1}$ induces an isomorphism sending $\mathcal{M}[S(\alpha)]$ to $\mathcal{M}[S(\beta)]$. From this

$$S(\eta)\eta\beta^{-1} = S(\beta)$$

and so, as β is injective, $S(\eta)\eta = S(\beta)\beta$. It remains to show that $\alpha^{-1}\eta$ induces an isomorphism sending $\mathcal{M}[S(\alpha)\alpha]$ to $\mathcal{M}[S(\beta)\beta]$. We can show that $\alpha^{-1}\eta$ induces an isomorphism from $\mathcal{M}[S(\alpha)\alpha]$ to $\mathcal{M}[S(\beta)\beta]$ using a similar argument to Proposition 5.1.7. Finally, as $S(\alpha) = S(\eta)$, and $S(\eta)\eta = S(\beta)\beta$:

$$\mathcal{M}[S(\alpha)\alpha]\alpha^{-1}\eta = \mathcal{M}[S(\alpha)\eta] = \mathcal{M}[S(\eta)\eta] = \mathcal{M}[S(\beta)\beta]$$

and so $\alpha^{-1}\eta$ induces an isomorphism sending $\mathcal{M}[S(\alpha)\alpha]$ to $\mathcal{M}[S(\beta)\beta]$. This covers all conditions stated and so this direction is proved.

Now suppose that $\alpha, \beta \in \text{Inj}(\mathcal{M})$ and that there exists a $\eta \in \text{Inj}(\mathcal{M})$ such that the above conditions hold. The proof that $\alpha \mathscr{R} \eta$ and $\eta \mathscr{L} \beta$ is similar to that of Theorem 4.1.13. □

Remark. As a set of partial monomorphisms from $\mathcal{M}$ to itself, Bi($\mathcal{M}$) is a submonoid of Inj($\mathcal{M}$). Note that there are similarities between the characterisation of Green's relations in Inj($\mathcal{M}$) and the characterisation of Green's relations of the bimorphism monoid Bi($\mathcal{M}$) in Section 4.1. In fact, all of the extra conditions required for the partial monomorphism case were inherited from being related as partial maps in the symmetric inverse monoid Inv(M). This further underlines the viewpoint of Inj($\mathcal{M}$) as the partial map analogue of Bi($\mathcal{M}$).

Notice here that we have not examined the partial homomorphism monoid Part($\mathcal{M}$) of a first-order structure in much detail here; we leave some work on this as an open question.

Question 5.1.10. *Characterise Green's relations of Part($\mathcal{M}$).*

We note that the theory of partial map monoids of first-order structures is a nascent subject of semigroup theory; we feel that many more results on endomorphism monoids as exhibited in the introduction to this chapter may have some interesting analogues in the setting of partial map monoids.

Question 5.1.11. *Further develop the semigroup theory of partial map monoids of first-order structures.*

5.2 Generation results

We now aim to generalise the rest of Theorem 3.0.1 to the case of first-order structures. Our next result extends the ideas of [23, Theorem 2.2] to the case of partial map monoids of first-order structures. This is achieved by using similar structural conditions to [23, Theorem 2.2] to show that the semigroup in question is *strongly distorted* (see Definition 3.1.9). Throughout this section, we write ω to mean the set of natural numbers together with 0.

Theorem 5.2.1. *Let $\mathcal{M}$ be a countable first-order structure. Suppose that $\mathcal{M}$ has the following properties:*

(a) $\mathcal{M}$ contains substructures $\mathcal{M}_i$ (where $i \in \omega$) with $\mathcal{M}_i \cong \mathcal{M}$, and it also contains a substructure $\mathcal{N}_k = \bigcup_{i \geq k} \mathcal{M}_i$ such that for all $i \neq j$, we have that $M_i \cap M_j = \varnothing$;

(b) there exists an isomorphism between $\mathcal{N}_0$ and $\mathcal{N}_1$ mapping each $\mathcal{M}_i$ to $\mathcal{M}_{i+1}$, and

(c) for any countable sequence $(\hat{f}_i)_{i \in \omega}$ where each $\hat{f}_i$ is a partial isomorphism of $\mathcal{M}_i$, the union $\bigcup_{i \in \omega} \hat{f}_i : \bigcup_{i \in \omega}$ dom $\hat{f}_i \to \bigcup_{i \in \omega}$ im $\hat{f}_i$ is a partial isomorphism of $\mathcal{M}$.

Then Inv$(\mathcal{M})$ has uncountable strong cofinality. Furthermore, we have that when replacing partial isomorphism with partial homomorphism or partial monomorphism respectively in condition (c) above, Part$(\mathcal{M})$ and Inj$(\mathcal{M})$ respectively have uncountable strong cofinality.

Proof. Our aim is to prove that Inv$(\mathcal{M})$ is a strongly distorted semigroup. To do this, we need to show that there is $N \in \mathbb{N}$ and a sequence of natural numbers $(a_i)_{i \in \mathbb{N}}$ such that for every countable sequence of elements $f_0, f_1, \ldots \in$ Inv$(\mathcal{M})$, there exists $g_1, \ldots, g_N \in$ Inv$(\mathcal{M})$ such that each f_n can be written as a product of length at most a_n in the elements $g_1, \ldots, g_N$. Let $N = 5$ and $a_k = 2k + 3$ for all $k \in \mathbb{N}$. Here, it is important to note that N and the sequence $(a_i)_{i \in \mathbb{N}}$ depend on the structure $\mathcal{M}$ and not on the countable sequence of elements $f_0, f_1, \ldots \in$ Inv$(\mathcal{M})$. We present the argument for Inv$(\mathcal{M})$ and note that we can interchange Inv$(\mathcal{M})$ for Part$(\mathcal{M})$ (or Inj$(\mathcal{M})$) and partial isomorphism for partial homomorphism (or partial monomorphism) throughout to achieve the other results.

Let $g_1 : \mathcal{M} \to \mathcal{M}_0$ be any isomorphism; so g_1 is contained in Inv$(\mathcal{M})$. The existence of the isomorphism from $\mathcal{N}_0$ to $\mathcal{N}_1$ in condition (b) ensures that the relations between $\mathcal{M}_i$ and $\mathcal{M}_j$ (for $i, j \in \omega$) contained in $\mathcal{N}_0$ is preserved; therefore $\mathcal{N}_p \cong \mathcal{N}_q$ for any natural numbers p and q. Denote the isomorphism between $\mathcal{N}_0$ and $\mathcal{N}_1$ by g_2; an inductive argument shows that g_2^n is an isomorphism with domain $\mathcal{N}_0$ and image $\mathcal{N}_n$. It follows that their composition $g_1 g_2^n : \mathcal{M} \to \mathcal{M}_n$ is an element of Inv$(\mathcal{M})$. As both g_1 and g_2 are elements of Inv$(\mathcal{M})$, there exist unique semigroup-theoretic inverses to g_1 and g_2; call these g_4 and g_5 respectively. We can use an inductive argument to show that $g_5^n g_4 : \mathcal{M}_n \to \mathcal{M}$ defines the semigroup-theoretic inverse for $g_1 g_2^n$ in $\mathrm{Inv}(\mathcal{M})$.

It remains to define the partial isomorphism that contains enough information to recover any element of the countable sequence $(f_i)_{i\in\omega}$. By using the two isomorphisms $g_1 g_2^n : \mathcal{M} \to \mathcal{M}_n$ and $g_5^n g_4 : \mathcal{M}_n \to \mathcal{M}$, we define a partial isomorphism $\hat{f}_n = (g_5^n g_4) f_n (g_1 g_2^n)$ of $\mathcal{M}_n$. Here, the partial isomorphism $\hat{f}_n$ acts like f_n but has its domain and image in $\mathcal{M}_n$. Then as $M_i \cap M_j = \varnothing$ for all $i \neq j$ in ω, we have that dom $\hat{f}_i\cap$ dom $\hat{f}_j = \varnothing$ and im $\hat{f}_i\cap$ im $\hat{f}_j = \varnothing$. From this, define a function $g_3 = \bigcup_{i\in\omega} \hat{f}_i$ with domain $\bigcup_{i\in\omega}$ dom $\hat{f}_i$ and image $\bigcup_{i\in\omega}$ im $\hat{f}_i$, acting as $\hat{f}_i$ on each $\mathcal{M}_i$. As each $\hat{f}_i$ is a partial isomorphism of $\mathcal{M}_i$ it follows that g_3 is a partial isomorphism by condition (c) and hence $g_3 \in \mathrm{Inv}(\mathcal{M})$.

We now show that $f_k \in \langle g_1, g_2, g_3, g_4, g_5\rangle$ for any $k \in \omega$, and f_k can be written as a product of length $a_k = 2k + 3$. To do this, consider the product $g_1 g_2^k g_3 g_5^k g_4$; we claim that the domain and image of this product are identical to those of f_k and that it behaves like the partial isomorphism f_k. Using Equation 2.1, and that the converse $(g_1 g_2^k)^*$ of $g_1 g_2^k$ is its semigroup theoretic inverse $g_5^k g_4$, gives:

$$\text{dom } g_1 g_2^k g_3 g_5^k g_4 = [\text{im } g_1 g_2^k \cap \text{dom } g_3(g_5^k g_4)](g_5^k g_4).$$

Here, dom $g_3 g_5^k g_4 = [\text{im } g_3 \cap \text{ dom } g_5^k g_4] g_3^* = [\text{im } g_3 \cap M_k] g_3^* = [\text{im } \hat{f}_k] g_3^* =$ dom $\hat{f}_k$. Therefore, the equation becomes:

$$\begin{aligned}
\text{dom } g_1 g_2^k g_3 g_5^k g_4 &= [\text{im } g_1 g_2^k \cap \text{dom } \hat{f}_k] g_5^k g_4 \\
&= [M_k \cap \text{dom } \hat{f}_k] g_5^k g_4 \\
&= [\text{dom } \hat{f}_k] g_5^k g_4 \\
&= \text{dom } f_k.
\end{aligned}$$

Using Equation 2.2 we can prove that im $g_1 g_2^k g_3 g_5^k g_4$ is equal to im f_k in a similar fashion. All that remains to show is that the product $g_1 g_2^k g_3 g_5^k g_4$ reduces to f_k.

As im $g_1 g_k^2 = M_k$, it follows that g_3 acts like $\hat{f}_k$ in this product. Therefore:

$$g_1 g_2^k g_3 g_5^k g_4 = (g_1 g_2^k) \hat{f}_k (g_5^k g_4)$$

However $\hat{f}_k = (g_5^k g_4) f_k (g_1 g_2^k)$ as defined earlier. Using this,

$$g_1 g_2^k g_3 g_5^k g_4 = (g_1 g_2^k)(g_5^k g_4) f_k (g_1 g_2^k)(g_5^k g_4)$$

But $(g_1 g_2^k)(g_5^k g_4) = id_{\mathcal{M}}$, as they are semigroup-theoretic inverses of each other. Therefore:

$$g_1 g_2^k g_3 g_5^k g_4 = id_{\mathcal{M}} f_k id_{\mathcal{M}} = f_k$$

and we are done. Moreover, each f_k is written as a product of length $a_k = 2k+3$; this provides a bounding sequence $(a_k)_{k \in \mathbb{N}}$ on the length of product and hence Inv($\mathcal{M}$) is strongly distorted. □

Remarks. (i) If $\mathcal{M}$ satisfies conditions (a)-(c), then by this result and Lemma 3.1.10 both the cofinality and strong cofinality of Inv($\mathcal{M}$), Inj($\mathcal{M}$) and Part($\mathcal{M}$) are uncountable. In this case, all three monoids have the Bergman property by Proposition 3.1.2.

(ii) As any generating set for an inverse semigroup generates using inverses of the elements, we do not need to include g_4 or g_5 when generating Inv($\mathcal{M}$) as an inverse semigroup. Therefore, if $\mathcal{M}$ satisfies conditions (a)-(c), then Inv($\mathcal{M}$) has Sierpiński rank of at most 3 by definition. Furthermore, Inj($\mathcal{M}$) and Part($\mathcal{M}$) have Sierpiński ranks of at most 5 with these assumptions. Whether or not these results are exact is an open question.

Below are a few examples (and a non-example) of structures where this theorem holds.

Example 5.2.2. We show that the random graph R (see Example 2.4.2) satisfies conditions (a)-(c) of Theorem 5.2.1. Let $\Gamma_0 = \bigcup_{i \in \omega} R_i$ such that $R_i \cong R$ for all $i \in \omega$, where $VR_i \cap VR_j = \varnothing$ for $i \neq j$, and for any vertices $a_i \in R_i$ and $a_j \in R_j$ with $i \neq j$, we have that $\{a_i, a_j\} \notin ER$. As a countable union of countable sets is countable, Γ_0 is a countable graph. Since R is universal for countable graphs by Theorem 2.4.6, Γ_0 embeds in R and condition (a) is satisfied. In addition, there exists an isomorphism from Γ_0 to $\Gamma_1 = \bigcup_{i \geq 1} R_i$ and so condition (b) is satisfied.

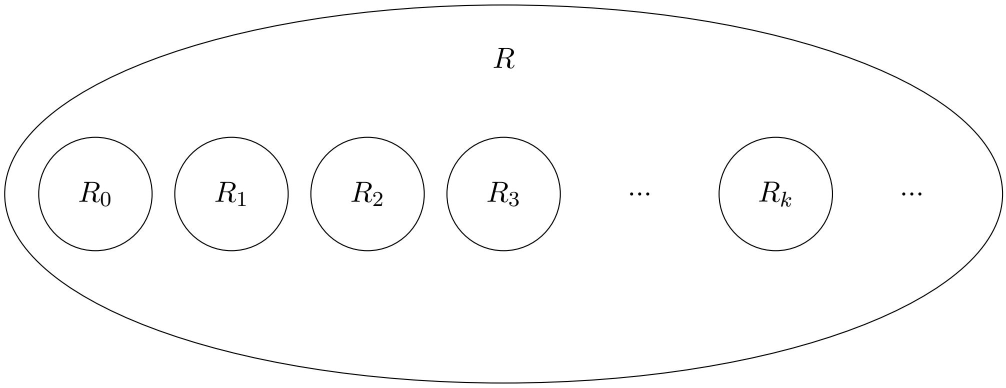


Figure 5.1: Γ in R

Any union of partial isomorphisms f_i, where dom f_i, im $f_i \subseteq R_i$, is also a partial isomorphism due to the independence of R_i and R_j where $i \neq j$. Hence condition (c) is satisfied and thus Inv(R) has uncountable strong cofinality. Similarly, the partial homomorphism (and partial monomorphism) version of condition (c) is satisfied and so Part(R) (and Inj(R)) have uncountable strong cofinality.

Remarks. (i) We can use a similar argument to this to show that the generic oriented graph D, and the generic digraph D^* (see Examples 2.4.8 and 2.4.9), both satisfy conditions (a)-(c). Therefore, the following monoids have uncountable strong cofinality; Inv(D), Inv(D^*), Inj(D), Inj(D^*), Part(D) and Part(D^*).

(ii) Note here that this argument only used the fact that the random graph R was universal for all countable graphs, and does not rely on some of its more distinct properties (like the ARP, or homogeneity). From this, we can use a similar argument to show that if Γ is a countably infinite graph that is universal for all countable graphs, then Inv(Γ), Inj(Γ) and Part(Γ) have uncountable strong cofinality.

Example 5.2.3. The countable dense linear order $(\mathbb{Q}, <)$ (see Example 2.3.12) satisfies the three conditions of Theorem 5.2.1. Note that as every partial homomorphism is a partial isomorphism in $(\mathbb{Q}, <)$, we have that $\mathrm{Inv}(\mathbb{Q}, <) = \mathrm{Inj}(\mathbb{Q}, <) = \mathrm{Part}(\mathbb{Q}, <)$.

Let $\mathcal{Q}_0 = \bigcup_{i\in\omega} Q_i$ be a disjoint union of open intervals $Q_i = (a_i, b_i)$ such that $b_i < a_{i+1}$ for all $i \in \omega$. Since each Q_i is an open interval in $(\mathbb{Q}, <)$, there exists an isomorphism from $(\mathbb{Q}, <)$ to Q_i. Since $\mathcal{Q}_0$ is itself a countable linear order, and the fact that $(\mathbb{Q}, <)$ contains all countable linear orders, $\mathcal{Q}_0$ is an induced substructure of $(\mathbb{Q}, <)$. Therefore, condition (a) is satisfied. We can also find an isomorphism from $\mathcal{Q}_0$ to $\mathcal{Q}_1 = \bigcup_{i\geq 1} Q_i$; this isomorphism satisfies condition (b).

Finally, let $\hat{f}_i$ be a partial isomorphism of Q_i; here $\hat{f}_i$ is order-preserving for every $i \in \omega$. We can see that every element of im $\hat{f}_i$ is less than every element of im $\bar{f}_{i+1}$ as $b_i < a_{i+1}$ for all $i \in \omega$. The union of all these $\hat{f}_i$'s is an order-preserving isomorphism that sends $\bigcup_{i\in\omega}$ dom $\hat{f}_i$ to $\bigcup_{i\in\omega}$ im $\hat{f}_i$, acting like $\hat{f}_i$ on every Q_i. Hence condition (c) is satisfied and we are done; $\mathrm{Inv}(\mathbb{Q}, <)$ has uncountable strong cofinality.

Remark. The same argument also works for the structure $(\mathbb{Q}, \leq)$. Here, as every partial monomorphism is an isomorphism, $\mathrm{Inv}(\mathbb{Q}, \leq) = \mathrm{Inj}(\mathbb{Q}, \leq)$; and $\mathrm{Inv}(\mathbb{Q}, \leq)$ has uncountable strong cofinality. Furthermore, $\mathrm{Part}(\mathbb{Q}, \leq)$ satisfies the partial homomorphism conditions of Theorem 5.2.1 and so this monoid has uncountable strong cofinality as well.

Example 5.2.4. The generic poset P (see Example 2.3.13) also satisfies conditions (a)-(c).

For condition (a), we define $\mathcal{P}_0$ to be an infinite antichain of P_i's such that $P_i \cong P$ for all $i \in \omega$ and each pair P_i and P_j is disjoint for all $i \neq j \in \omega$. Note that $\mathcal{P}_0$ is a countable partial order; so $\mathcal{P}_0$ is an induced substructure of P, satisfying condition (a). Furthermore, there exists an isomorphism taking $\mathcal{P}_0$ to $\mathcal{P}_1 = \bigcup_{i\geq 1} P_i$; so condition (b) is satisfied.

Now for condition (c), note that if $a_i \in P_i$ and $a_j \in P_j$ with $i \neq j$, then a_i and a_j are incomparable elements. Take a countable sequence of partial isomorphisms $(\hat{f}_i)_{i\in\omega}$ where dom $\hat{f}_i$, im $\hat{f}_i \subseteq P_i$. Note that every element of dom $\hat{f}_i$ is incomparable to every element of dom $\hat{f}_j$ with $i \neq j$; this is also the case for im $\hat{f}_i$ and im $\hat{f}_j$. Therefore the union of any set of such functions preserves the incomparability between domains and images of $\hat{f}_i$ and $\hat{f}_j$. As each $\hat{f}_i$ is a partial isomorphism of P, the union $f = \bigcup_{i\in\omega} \hat{f}_i : \bigcup_{i\in\omega} \text{dom } \hat{f}_i \to \bigcup_{i\in\omega} \text{im } \hat{f}_i$ is a partial isomorphism of P. So condition (c) is satisfied and therefore Inv(P) has uncountable strong cofinality. Finally, note that both Part(P) and Inj(P) satisfy the relevant conditions and therefore these also have uncountable strong cofinality.

Remark. As with Example 5.2.2, we can use a similar argument to show that if $\mathcal{P}$ is a countably infinite poset that is universal for all countable posets, then Inv$(\mathcal{P})$, Inj$(\mathcal{P})$ and Part$(\mathcal{P})$ have uncountable strong cofinality.

Example 5.2.5. Contrary to the above three examples, the discrete linear order $(\mathbb{N}, \leq)$ (see Subsection 4.2.2) satisfies conditions (a) and (b) but does not satisfy (c).

To show this, let $\mathcal{N}$ be a disjoint union of countably many isomorphic copies N_i of $(\mathbb{N}, \leq)$, with $\min(N_i) \leq \min(N_{i+1})$ for all $i \in \omega$. Let $\hat{f}_2$ be a partial isomorphism of N_2 and assume without loss of generality that dom $\hat{f}_2$ is finite. As it is so, we have that im $\hat{f}_2$ is finite and hence has a maximal element n. Now, define $\hat{f}_1$ to be a partial isomorphism with singleton domain $\{x\}$ such that x is less than every element of dom $\hat{f}_2$ (such an x exists due to our conditions on $\mathcal{N}$) and singleton image $\{y\}$ such that $y > n$ (such a y exists as A_1 is infinite). Hence the union $\hat{f}_1 \cup \hat{f}_2$ is a function that sends x to y and sends dom $\hat{f}_2$ to a set of elements strictly less than y; but this is not order preserving and is hence not a partial isomorphism of $(\mathbb{N}, \leq)$. So $(\mathbb{N}, \leq)$ does not satisfy condition (c).

5.3 The semilattice of idempotents of Inv$(\mathcal{M})$

Finally in this section, we consider a fourth example of a partial map monoid on a countable first-order structure $\mathcal{M}$; one generated by identity maps on substructures of $\mathcal{M}$. The following definitions are standard, and can be found in [19] or [41].

Let $(X, \leq)$ be a partial order, and let Y be a subset of X. An element $a \in Y$ is called *minimal* if for all $y \in Y$, whenever $y \leq a$ then $y = a$, and it is called *maximal* if for all $y \in Y$, whenever $a \leq y$ then $y = a$ An element $b \in Y$ is called a *minimum* if $b \leq y$ for all $y \in Y$, and a *maximum* if $y \leq b$ for all $y \in Y$. If Y is a non-empty subset of X, say that $c \in X$ is a *lower bound of* Y if $c \leq y$ for all $y \in Y$. If the set of lower bounds of Y is non-empty, and has a maximum element d, then say that d is the *greatest lower bound* or *meet* of Y. If such a d exists, it is unique, and we write $d = \bigwedge\{y \ : \ y \in Y\}$; or $d = a \wedge b$ if $Y = \{a, b\}$.

If $(X, \leq)$ is such that $a \wedge b$ exists for all $a, b \in X$, then we say that $(X, \leq)$ is a *lower semilattice*. If $(X, \leq)$ is a lower semilattice, then for $a, b, c \in X$, both $(a \wedge b) \wedge c$ and $a \wedge (b \wedge c)$ are greatest lower bounds for $\{a, b, c\}$; as greatest lower bounds are unique,

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

and so $(X, \wedge)$ is a semigroup. As $a \wedge a = a$ for all $a \in X$, and $a \wedge b = b \wedge a$ for all $a, b \in X$, the semigroup $(X, \wedge)$ is commutative, and consists entirely of idempotents.

Conversely, if $(E, \cdot)$ is a commutative semigroup consisting entirely of idempotents, then there is a natural partial order on E defined by $a \leq b$ if and only if $ab = a$. Furthermore, the product ab of any two elements a, b of E is the meet of those two elements with respect to this partial order. As the collection of idempotents $E(S)$ of any *inverse* semigroup S is a commutative subsemigroup of S [41], every inverse semigroup contains a *semilattice of idempotents*. Therefore, contained in any symmetric inverse monoid S of an infinite first-order structure $\mathcal{M}$ is a semilattice of idempotents, denoted by $E(S)$.

In this case, the idempotents are the identity maps on subsets of the domain M of a first-order σ-structure $\mathcal{M}$ (by Lemma 5.1.2), with the meet operation on identity maps id_x and id_y for subsets x and y of M given by

$$id_x \wedge id_y = \begin{cases} id_{x\cap y} & \text{if } x \cap y \neq \varnothing \\ \varnothing_E & \text{if otherwise} \end{cases} \tag{5.1}$$

where $\varnothing_E$ is the empty transformation of $E(S)$. This operation corresponds to the composition of partial maps as seen in Equation 2.1 from page 26. By Lemma 5.1.2 and the proof of Corollary 5.1.3, it follows that $|E(S)| = 2^{\aleph_0}$. Furthermore, this construction is independent of the structure $\mathcal{M}$; depending only on the size of the domain. So the symmetric inverse monoid of any countably infinite first-order structure $\mathcal{M}$ has a semilattice of idempotents isomorphic to $E(S)$; justifying the use of the definite article in the section title. From now, we simply refer to the semigroup $E(S)$ by E; this monoid forms the focus of this section.

Since E is an uncountable and hence infinitely generated submonoid of $\mathrm{Inv}(\mathcal{M})$ for any infinite first-order structure $\mathcal{M}$, we can investigate cofinality results for E. To further our investigation into properties of this semigroup, we can split E into constituent parts as shown in Figure 5.2, where B, C and D are subsets of E. The natural partial order on E (see above) is inherited from containment on subsets of $\mathcal{P}(M)$. It is clear that id_M is the maximal element of this partial order, and $\varnothing_E$ is the minimal element. Under this partial order, every element in B is greater than every element in C and every element in C is greater than every element in D. This is reflected in Figure 5.2.

In order to look at results involving this semigroup, we need to study the products of elements of E in more detail. Instead of looking at the images as in Equation 5.1, we can instead look at the identity functions on *complements* of subsets of M. Some set theory tells us that

$$M \smallsetminus (x \cap y) = (M \smallsetminus x) \cup (M \smallsetminus y) \tag{5.2}$$

As $|(M \smallsetminus x) \cap (M \smallsetminus y)| = |M \smallsetminus x|$ if $x \subseteq y$, and a similar result holds if $y \subseteq x$,

$$
\begin{array}{c}
id_M \\
| \\
B = \{id_x \in E \ : \ |x| \text{ is cofinite } \} \\
| \\
C = \{id_x \in E \ : \ |x| \text{ is infinite}, |M \smallsetminus x| \text{ is infinite}\} \\
| \\
D = \{id_x \in E \ : \ |x| \text{ is finite } \} \\
| \\
\varnothing_E
\end{array}
$$

Figure 5.2: The semilattice of idempotents E, decomposed into constituent parts, ordered by the natural partial order on E.

it follows that $|M \smallsetminus (x \cap y)|$ is at least $\max(|M \smallsetminus x|, |M \smallsetminus y|)$. This means that we cannot reduce the size of the complement of the sets when composing two identity functions; proving our next result.

Lemma 5.3.1. *Let $I_k := \{id_x \in E : |M \smallsetminus x| \geq k\} \subseteq E$. Then I_k is an ideal of E. Furthermore, $I_\infty := \{id_y \in E : |M \smallsetminus y| = \aleph_0\}$ is also an ideal of E.* □

Similar to the case of the discrete linear order $(\mathbb{N}, \leq)$ (see Subsection 4.2.2), there is a unique identity function on each subset of M. Therefore, we can proceed along similar lines; motivating our next result.

Lemma 5.3.2. *Let B be as in Figure 5.2. Then B is a countable, infinitely generated submonoid of E.*

Proof. Suppose that id_x and id_y are in B. Then $M \smallsetminus (x \cap y)$ is a union of finite sets by Equation 5.2, and so is finite; hence $id_{x \cap y} = id_x id_y \in B$. As $|M \smallsetminus M|$ is finite, it follows that $id_M \in B$. Since the set B consists of all elements id_x where x has a finite complement, and id_x is unique for every subset x of M, there exists a bijection between B and the set of cofinite subsets of A. Since this set is countable, B is countable.

Assume now that B is generated by some set F. Here, F must generate all the idempotent elements id_x with $|M \smallsetminus x| = 1$. By Lemma 5.3.1 we cannot reduce the size of the complement of one of these idempotents, so F must generate each id_x with $|M \smallsetminus x| = 1$ via elements with complement size 1. As there exist unique functions for each such x, and the only map that has complement size 0 is id_M, it follows that F must contain all id_x such that $|A \setminus x| = 1$. So B is infinitely generated. $\square$

We can sum up cofinality results of E and determine whether or not E satisfies the Bergman property. First, note that $E = B \sqcup (E \setminus B)$, and that $\text{cf}(B) = \aleph_0$ by Lemma 3.1.3.

Proposition 5.3.3. *$cf(E) = scf(E) = \aleph_0$.*

Proof. E has the ideal structure specified in the conditions for Proposition 3.1.11 by Lemma 5.3.1; hence $\text{scf}(E) = \aleph_0$. By Lemma 3.1.3 and Lemma 3.1.4, the cofinality of E is at most $\aleph_0$; but since it cannot be less than this, we are done. $\square$

By using a similar argument to Proposition 4.2.14 and its following deduction, the countable submonoid B in Lemma 5.3.2 can be generated by the set $U = \{id_M\} \cup \{id_x \in E : |M \setminus x| = 1\}$. We now determine whether or not E has the Bergman property.

Proposition 5.3.4. *Let $id_t = id_{t_1} id_{t_2} ... id_{t_k}$ be a product of elements from $U \sqcup (E \setminus B)$. Then $|M \setminus t|$ is either at most k or it is infinite.*

Proof. We prove the statement by induction on the length of product. Note that if any of the elements of the product $id_{t_1} id_{t_2} ... id_{t_k}$ is in C, it follows that $|M \setminus t|$ is immediately infinite by Equation 5.2. So we only consider cases where each element in the product is contained in U.

For the base case, if $k = 1$ then we have that $id_t = id_{t_1}$. By the fact that each identity function is unique we have that $t = t_1$ and so either $t = t_1 = M$ (in which case $|M \setminus t| = 0$) or $|M \setminus t| = 1$ by definition. This proves the base case.

Suppose now that that the inductive hypothesis holds. Multiplying on the right by $id_{t_{k+1}}$ gives $id_t id_{t_{k+1}} = id_{t_1} id_{t_2} ... id_{t_k} id_{t_{k+1}}$. If $id_{t_{k+1}}$ is in C then $|A \setminus t \cap t_{k+1}|$ is immediately infinite; so suppose that $id_{t_{k+1}}$ is in U. If $id_{t_{k+1}} = id_M$ then we are done as $id_t id_{t_{k+1}} = id_t$ and therefore $|M \setminus t \cap M| = |M \setminus t| \leq k$. If $|M \setminus t_{k+1}| = 1$, then by Equation 5.2 $|M \setminus t \cap t_{k+1}|$ takes a maximal value if t and t_{k+1} are disjoint; so this means that $|M \setminus t \cap t_{k+1}| = |M \setminus t| + |M \setminus t_{k+1}| \leq k + 1$ by the inductive hypothesis. $\square$

Therefore $U \sqcup C$ is a generating set of E that is not Cayley bounded; hence E cannot possibly have the Bergman property.

6

Oligomorphic transformation monoids

As mentioned in Section 2.3 and Chapter 3, the automorphism group $\text{Aut}(\mathcal{M})$ of a first-order structure $\mathcal{M}$ is an important concept of understanding the model theory of a structure $\mathcal{M}$. Furthermore, they have an important role to play in the theory of infinite permutation groups. One such example is given by a result of Reyes ([72], reproduced in Theorem 2.3.5) which says that closed subgroups of $\text{Sym}(\mathbb{N})$ under the pointwise convergence topology (see Subsection 2.2.4) are precisely automorphism groups of countable first-order structures. As this happens, we can view $\text{Aut}(\mathcal{M})$ as a *topological group*; since is a closed subset of the Polish space $\text{Sym}(\mathbb{N})$, $\text{Aut}(\mathcal{M})$ is a *Polish group*. There have been extensive studies of automorphisms of first order structures as topological groups; particularly in the field of generic automorphisms [79, 46] and the *small index property* [22, 38, 46].

Cameron and Nešetřil [14] demonstrated that endomorphism monoids play a similar role for infinite transformation monoids; endomorphism (monomorphism) monoids of countable first-order structures are precisely the closed submonoids of $\text{End}(\mathbb{N})$ ($\text{Mon}(\mathbb{N})$) under the product topology. This in turn stimulated studies into topological *monoids*; some results concerning reconstruction problems [6] and generic endomorphisms [52] have been shown.

One of the strongest model-theoretic properties a first-order structure $\mathcal{M}$ can have is $\aleph_0$-categoricity (see Subsection 2.3.3). The celebrated theorem of Engeler, Ryll-Nardzewski and Svenonius (Theorem 2.3.7) proves that a structure $\mathcal{M}$ is $\aleph_0$-categorical if and only if $\text{Aut}(\mathcal{M})$ has finitely many orbits on M^n via the componentwise action on tuples. This means the natural action of $\text{Aut}(\mathcal{M})$ on $\mathcal{M}$ is oligomorphic; if such an action exists, we say that $\mathrm{Aut}(\mathcal{M})$ is an *oligomorphic permutation group*. This equivalence implies that finding $\aleph_0$-categorical structures gives examples of oligomorphic permutation groups, and vice versa. As outlined in Proposition 2.3.9, finding $\aleph_0$-categorical structures is made easier by the connection with homogeneity; in turn, prompting the use of Fraïssé's theorem (Theorem 2.3.11) to find oligomorphic permutation groups.

The notion of homogeneity has been extended to cases where the maps involved are not isomorphisms. The idea of *homomorphism-homogeneity* was developed by Cameron and Nešetřil [14]; in this sense, a first-order structure $\mathcal{M}$ is *HH-homogeneous* if every finite partial homomorphism of $\mathcal{M}$ extends to an endomorphism of $\mathcal{M}$. This idea was subsequently generalised by the two papers of Lockett and Truss [52, 53]; detailing eighteen different varieties of homomorphism-homogeneity based on three types of finite partial maps extending to the six types of endomorphism as outlined in Subsection 2.2.2. An example of one of these is *MB-homogeneity*, a notion previously introduced in Subsection 4.1.4.

Using this work on HH-homogeneity, Mašoluvic and Pech [61] developed the notion of an *oligomorphic transformation monoid*. In this paper, they showed that if $\mathcal{M}$ is a HH-homogeneous structure over a (residually) finite relational language, then $\mathrm{End}(\mathcal{M})$ is an oligomorphic transformation monoid. Following on from this, and the fact endomorphisms of $\mathcal{M}$ preserve *positive* formulas (that is, those well-formed formulas without negation symbols), they went on to demonstrate some model-theoretic results for HH-homogeneous structures concerning positive formulas.

The purpose of this brief chapter is a further generalisation of results mentioned in Subsection 2.3.3, as well as motivating the work of Chapters 7 and 8; which are devoted to developing machinery to find oligomorphic transformation monoids and finding examples of MB-homogeneous graphs respectively.

In Section 4.1, we saw that the bimorphism monoid of a σ-structure $\mathcal{M}$ is an example of a group-embeddable monoid. By definition, any group-embeddable monoid can be viewed as a monoid of *permutations* contained in some symmetric group. If $\mathcal{M}$ is countably infinite, then there is a natural embedding from

Bi($\mathcal{M}$) into Sym($\mathbb{N}$), and so we can view bimorphism monoids as submonoids of Sym($\mathbb{N}$); we call these *infinite permutation monoids*. Section 6.1 establishes a connection between bimorphism monoids of structures and infinite permutation monoids, akin to that of automorphism groups and infinite permutation groups.

Section 6.2 is devoted to extending some of the work of [61] by considering the notions of homomorphism-homogeneity outlined in [53]. We restate the definition of *oligomorphic transformation monoid* from [61], and then present some results determining sufficient conditions for one of the six endomorphism monoids mentioned in Subsection 2.2.2 to be an oligomorphic transformation monoid.

Some of the work in this chapter, and Chapters 7 and 8, forms joint work with David Evans and Robert Gray in [17].

6.1 Infinite permutation monoids

Let $\mathcal{M}$ be a σ-structure. As mentioned in the introduction to Chapter 4, Bi($\mathcal{M}$) is embeddable in Sym(M) via the natural inclusion mapping. As this happens, it is a submonoid of Sym(M) and so we can view Bi($\mathcal{M}$) as a *monoid of permutations*, or simply a *permutation monoid*. Note here that if $\mathcal{M}$ is finite, then Bi($\mathcal{M}$) is finite and hence a group; to rescue this section from triviality, we stipulate that $\mathcal{M}$ is countably infinite. In this case, we say that Bi($\mathcal{M}$) expressed in this fashion is an *infinite permutation monoid*.

Our first result is analogous to those of Reyes ([72], see Theorem 2.3.5) and Cameron and Nešetřil [14]; it characterises closed submonoids of the symmetric group. The proof is along similar lines; we define a *canonical structure* on some infinite permutation monoid T.

Theorem 6.1.1. *Let M be a countable set. A submonoid T of Sym(M) is closed under the pointwise convergence topology if and only if it is the bimorphism monoid of some countable first-order structure $\mathcal{M}$ on M.*

Proof. We begin with the converse direction. Suppose that T = Bi($\mathcal{M}$) is the bimorphism monoid of a structure $\mathcal{M}$ on a countably infinite domain M. Then Bi($\mathcal{M}$) is the intersection of the closed monoids End($\mathcal{M}$) and Sym(M) in the topology on End(M). As Sym(M) is a subspace of End(M), it follows from Theorem 2.2.13 that Bi($\mathcal{M}$) is closed in Sym(M).

For the forward direction, assume that T is a closed submonoid of Sym(M). We define an n-ary relation $R_{\bar{x}}$ by:

$$\bar{y} \in R_{\bar{x}} \text{ if and only if } (\exists s \in T)(\bar{x}s = \bar{y})$$

for each $n \in \mathbb{N}$ and $\bar{x} \in M^n$. Let $\mathcal{M}$ be the relational structure on M with relations $R_{\bar{x}}$ for all $n \in \mathbb{N}$ and all tuples $\bar{x} \in M^n$. The proof that $T = \text{Bi}(\mathcal{M})$ for this structure $\mathcal{M}$ is by containment both ways.

As every element of T is already a permutation of the domain M of $\mathcal{M}$, proving that that T acts as endomorphisms on $\mathcal{M}$ is enough to show that $T \subseteq$ Bi($\mathcal{M}$). Assume then that $s \in T$ and $\bar{y} \in M^n$ such that $R_{\bar{x}}(\bar{y})$ holds. As this happens, there exists $s' \in T$ such that $\bar{x}s' = \bar{y}$; therefore $\bar{x}s's = \bar{y}s$. This means that $R_{\bar{x}}(\bar{y}s)$ holds and so $T \subseteq \text{End}(\mathcal{M})$ by definition; hence $T \subseteq \text{Bi}(\mathcal{M})$.

It remains to show that Bi($\mathcal{M}$) $\subseteq T$; so assume that $\alpha \in \text{Bi}(\mathcal{M})$. Here, it is enough to show that α is a limit point of T; as T is closed, it contains all its limit points. Note that each n-tuple $\bar{x}$ defines a neighbourhood of α, consisting of all functions β such that $\bar{x}\alpha = \bar{x}\beta$. As T is a monoid, it follows that $R_{\bar{x}}(\bar{x})$ holds and so $R_{\bar{x}}(\bar{x}\alpha)$ also holds. By definition of $R_{\bar{x}}$, there exists $t \in T$ such that $\bar{x}\alpha = \bar{x}t$; hence α is a limit point of T. Therefore $\alpha \in T$ and so Bi($\mathcal{M}$) $\subseteq T$. □

Remark. It is a well-known result from descriptive set theory that any closed subset of a Polish space is itself a Polish space (see [44]). As Sym($\mathcal{M}$) is a Polish space, it follows that Bi($\mathcal{M}$) is also a Polish space; so bimorphisms of first order structures provide natural examples of Polish *monoids*. We leave this area of investigation open for now.

Example 6.1.2. In Example 4.1.15, we constructed a graph Γ such that Bi(Γ) $\cong$ $(\mathbb{N}, +)$, the infinite monogenic semigroup with identity. By Theorem 6.1.1, it follows that $T \subseteq \mathrm{Sym}(\mathbb{N})$, where $T \cong (\mathbb{N}, +)$, is a closed submonoid of $\mathrm{Sym}(\mathbb{N})$.

Remark. A positive answer to Question 4.1.16 would confirm that every countable group-embeddable monoid arises as a closed permutation monoid.

Our aim now is to determine a cardinality result for closed submonoids of $\mathrm{Sym}(X)$. For any $\bar{x} \in X^n$, recall that the *pointwise stabilizer* of $\bar{x}$ is the set $\mathrm{St}(\bar{x}) = \{\alpha \in \mathrm{Bi}(\mathcal{M}) : \bar{x}\alpha = \bar{x}\}$.

Theorem 6.1.3. *For any countably infinite first-order structure $\mathcal{M}$, either $|Bi(\mathcal{M})| \leq \aleph_0$ or $|Bi(\mathcal{M})| = 2^{\aleph_0}$, the first alternative holding if and only if the pointwise stabilizer of some tuple is the identity $e \in Bi(\mathcal{M})$.*

Proof. Assume that $\mathrm{St}(\bar{x}) = \{e\}$ for some $\bar{x} \in M^n$. Let $\gamma_1, \gamma_2 \in \mathrm{Bi}(\mathcal{M})$ where $\bar{x}\gamma_1 = \bar{x}\gamma_2 = \bar{y}$. As $\mathrm{Bi}(\mathcal{M})$ embeds in $\mathrm{Sym}(M) = G$, the action of $\mathrm{Bi}(\mathcal{M})$ on $\mathcal{M}$ extends to an action of G on $\mathcal{M}$. As a consequence, there exists a unique $\gamma_1^{-1} \in G$ such that $\gamma_1\gamma_1^{-1} = e$. As $\bar{y}\gamma_1^{-1} = \bar{x}$, it follows that $\bar{x}\gamma_2\gamma_1^{-1} = \bar{x}\gamma_1\gamma_1^{-1} = \bar{x}$. But $\mathrm{St}(\bar{x}) = \{e\}$ and so $\gamma_2\gamma_1^{-1} = e$; hence $\gamma_1 = \gamma_2$. By the fact there are only countably many tuples in $F(\bar{x})$, we are forced to conclude that $\mathrm{Bi}(\mathcal{M})$ is countable in this case.

On the other hand, suppose that $\mathrm{St}(\bar{x}) \neq \{e\}$ for all tuples $\bar{x} \in M^n$. As $\mathcal{M}$ is countably infinite, we can enumerate elements of $\mathcal{M} = \{x_1, x_2, ...\}$. Using this enumeration, we define a sequence of tuples $(\bar{x}_k)_{k\in\mathbb{N}}$ where $\bar{x}_k = (x_1, ..., x_k)$ for all $k \in \mathbb{N}$. Since $\mathrm{St}(\bar{x}) \neq \{e\}$ for all tuples $\bar{x}$ of $\mathcal{M}$, then for each element $\bar{x}_k$ of $(\bar{x}_k)_{k\in\mathbb{N}}$ there exists $t_k \in \mathrm{Bi}(\mathcal{M})$ such that $t_k \neq e$ and $\bar{x}_k t_k = \bar{x}_k$. This in turn induces a sequence $(t_k)_{k\in\mathbb{N}}$ of non-trivial elements of $\mathrm{Bi}(\mathcal{M})$. As the sequence of tuples $(\bar{x}_k)_{k\in\mathbb{N}}$ will eventually encapsulate every element of $\mathcal{M}$, the sequence of bimorphisms $(t_k)_{k\in\mathbb{N}}$ approaches the pointwise stabilizer of $\mathcal{M}$. This is the identity element and so e is a limit point of $\mathrm{Bi}(\mathcal{M})$.

Now, consider the sequence $(t_k\alpha)_{k\in\mathbb{N}}$, where α is some bimorphism of $\mathcal{M}$. Here, $t_k\alpha \neq \alpha$ for any $k \in \mathbb{N}$; for if $t_k\alpha = \alpha$ for some k, then cancellativity of $\mathrm{Bi}(\mathcal{M})$ implies that $t_k = e$, contradicting our earlier observation. It follows that α is a limit point for the sequence $(t_k\alpha)_{k\in\mathbb{N}}$, and so *every* element of $\mathrm{Bi}(\mathcal{M})$ is a limit point. This means that $\mathrm{Bi}(\mathcal{M})$ is a perfect set and thus has cardinality $2^{\aleph_0}$ ([44], see Subsection 2.2.4). $\square$

This section has served as a brief introduction to infinite permutation monoids viewed as topological spaces. We can ask about permutation monoid analogues for established results concerning the topology of infinite permutation groups. As stated above, every closed infinite permutation group is a Polish space; and so automorphism groups of first-order structures provide examples of Polish groups. Since $\mathrm{Bi}(\mathcal{M})$ is closed in $\mathrm{Sym}(\mathbb{N})$, it is an example of a Polish monoid. Following the significant body of literature on Polish groups (see [79, 46] for two instances), we conjecture that there are analogous results for Polish monoids.

Question 6.1.4. *Further develop the theory of Polish monoids.*

6.2 Oligomorphic transformation monoids

Recall that a permutation group $G \subseteq \mathrm{Sym}(X)$ acts *oligomorphically* on X if and only if there are finitely many orbits on X^n for every $n \in \mathbb{N}$ [9]. If the componentwise action of G on tuples of X is oligomorphic, we say that G is an *oligomorphic permutation group*. The next definition, originally of [61], reformulates these concepts in the context of transformation monoids. Recall the notion of a *strong orbit* of a monoid action on a set from Subsection 2.2.3.

Definition 6.2.1 (Definition 2.1, [61])**.** We say that a transformation monoid $T \subseteq \mathrm{End}(X)$ acts *oligomorphically* on X if and only if there are finitely many strong orbits on X^n for every $n \in \mathbb{N}$. If the componentwise action of T on X is oligomorphic, we say that T is an *oligomorphic transformation monoid*.

Remarks. (i) We note that if T is itself a group, then the strong orbits are the group orbits and the definitions coincide; so any oligomorphic permutation group is an oligomorphic transformation monoid.

(ii) If T is a group-embeddable monoid, then $T \subseteq \mathrm{Sym}(X)$ and so T is a permutation monoid. If the componentwise action of T on X^n is oligomorphic, we say that T is an *oligomorphic permutation monoid*.

Our next result, a generalisation of [61, Lemma 2.10], provides more connections between oligomorphic permutation groups and oligomorphic transformation monoids.

Proposition 6.2.2. *Let $T \subseteq End(X)$ be a transformation monoid with group of units U. If U is an oligomorphic permutation group then T is an oligomorphic transformation monoid.*

Proof. As U is an oligomorphic permutation group, there are finitely many group orbits $U(\bar{y})$ with $y \in X^n$ for every $n \in \mathbb{N}$. As every strong orbit $S(\bar{x})$ arises as the union of group orbits $U(\bar{y})$ by Lemma 2.2.12, we conclude that there are at most finitely many strong orbits of T acting on X^n for every natural number n. $\square$

Remark. The Ryll-Nardzewski theorem (Theorem 2.3.7) states that U is an oligomorphic permutation group if and only if it is the automorphism group of some $\aleph_0$-categorical structure $\mathcal{M}$. By Proposition 6.2.2 and the fact that Aut($\mathcal{M}$) acts as the group of units for any endomorphism monoid of $\mathcal{M}$, we conclude that if $\mathcal{M}$ is $\aleph_0$-categorical then $T \in \{\text{End}(\mathcal{M}), \text{Epi}(\mathcal{M}), \text{Mon}(\mathcal{M}), \text{Bi}(\mathcal{M}), \text{Emb}(\mathcal{M})\}$ is an oligomorphic transformation monoid.

Example 6.2.3. There are many examples of homogeneous structures throughout the thesis so far. As posets, graphs and digraphs are first-order structures over a finite relational language, then Aut($\mathcal{M}$) is oligomorphic for each structure $\mathcal{M}$ in the three classification results for posets (Theorem 2.3.14), graphs (Theorem 2.4.10) and digraphs (Theorem 2.4.11) by Proposition 2.3.9. This means that $T \in \{\text{End}(\mathcal{M}), \text{Epi}(\mathcal{M}), \text{Mon}(\mathcal{M}), \text{Bi}(\mathcal{M}), \text{Emb}(\mathcal{M})\}$ is an oligomorphic transformation monoid. In particular, Bi($\mathcal{M}$) in all these cases is an oligomorphic permutation monoid. It may be that these monoids coincide for some structure $\mathcal{M}$, reducing the range of examples. For instance, $\text{Aut}(\mathbb{Q}, <) = \text{End}(\mathbb{Q}, <)$.

These examples of oligomorphic transformation monoids are closely related to $\aleph_0$-categorical structures via their group of units. Our next proposition distances the notion of oligomorphicity in monoids from $\aleph_0$-categoricity by providing a differing source of suitable examples; but first we detail some preliminary conditions. In the same way that homogeneous structures over a finite language provide examples of $\aleph_0$-categorical structures (and hence examples of oligomorphic permutation groups), we turn to *homomorphism-homogeneity* to provide examples of oligomorphic transformation monoids.

We recall the eighteen different notions of homomorphism-homogeneity as developed in the two papers of Lockett and Truss [52, 53]. Following the lead of [53] we denote each type of endomorphism outlined in Subsection 2.2.2 by a symbol: H for endomorphism, E for epimorphism, M for monomorphism, B for bimorphism, I for embedding and A for automorphism. We cannot assert that a finite partial map is surjective; there is no well defined notion of a finite partial epimorphism, for instance. Therefore, there are only three types of finite partial map of a structure: H for homomorphism, M for monomorphism, and I for embedding. Without loss of generality, maps between finite substructures can be taken to be surjective.

Definition 6.2.4. Let $\mathcal{M}$ be a first-order structure, and take X $\in \{\mathrm{H}, \mathrm{M}, \mathrm{I}\}$ and Y $\in \{\mathrm{H}, \mathrm{E}, \mathrm{M}, \mathrm{B}, \mathrm{I}, \mathrm{A}\}$. Say that $\mathcal{M}$ is *XY-homogeneous* if every finite partial map of type X of $\mathcal{M}$ extends to a map of type Y of $\mathcal{M}$. We denote the collection of all notions of homomorphism-homogeneity by $\mathfrak{H}$.

Furthermore, we denote the *class of all XY-homogeneous structures* by XY, and say that $\mathbb{H}$ is the set of all homomorphism-homogeneity classes.

Remark. It is important to make the distinction between a notion of homomorphism-homogeneity and a homomorphism-homogeneity class. For countable structures, it was shown in [53] that a structure is II (MI, HI)-homogeneous if and only if it is IA (MA, HA)-homogeneous; that is, II = IA, MI = MA and HI = HA as classes of homomorphism-homogeneous structures. While all the structures in this thesis are countable (and hence subject to this result), it is crucial that we treat the notions of homomorphism-homogeneity separately. This is apparent in Chapter 7, where we re-prove this result from a Fraïssé-theoretic perspective.

As we have seen, a structure $\mathcal{M}$ is MB-homogeneous if every finite partial monomorphism of $\mathcal{M}$ extends to a bimorphism of $\mathcal{M}$. Regular homogeneity (as in Definition 2.3.8) corresponds to IA-homogeneity using this notation. All possible types of homomorphism-homogeneity given in Definition 6.2.4 are outlined in Figure 6.1.

	isomorphism (I)	monomorphism (M)	homomorphism (H)
End($\mathcal{M}$) (H)	IH	MH	HH
Epi($\mathcal{M}$) (E)	IE	ME	HE
Mon($\mathcal{M}$) (M)	IM	MM	HM
Bi($\mathcal{M}$) (B)	IB	MB	HB
Emb($\mathcal{M}$) (I)	II	MI	HI
Aut($\mathcal{M}$) (A)	IA	MA	HA

Figure 6.1: Table of XY-homogeneity: $\mathcal{M}$ is XY-homogeneous if a finite partial map of type X (column) extends to a map of type Y (row) in the associated monoid.

It follows that some notions of homomorphism-homogeneity are stronger than others. For instance, as every bimorphism is a monomorphism, it follows that every MB-homogeneous structure is also MM-homogeneous. This natural containment induces a partial order on the set $\mathbb{H}$ of homomorphism-homogeneity classes; see Figure 6.2 for a diagram of this order.

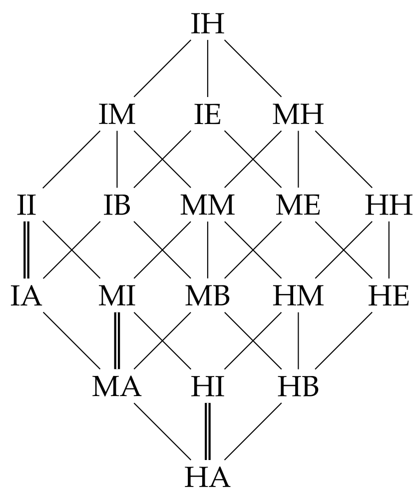


Figure 6.2: The set $\mathbb{H}$ of homomorphism-homogeneity classes partially ordered by inclusion for countable first-order structures. Lines indicate inclusion, double lines indicate equality.

As an isomorphism is both a monomorphism and a homomorphism, it follows that if a structure $\mathcal{M}$ is XY-homogeneous then it is also IY-homogeneous (see Figure 6.2). For shorthand, denote the monoid of maps of type Y by Y$(\mathcal{M})$ for some structure $\mathcal{M}$; for instance, End$(\mathcal{M})$ becomes H$(\mathcal{M})$ in this notation.

Our aim now is to determine the strong orbits of Y$(\mathcal{M})$ on $\mathcal{M}$ where $\mathcal{M}$ is an XY-homogeneous structure, in order to show that Y$(\mathcal{M})$ is an oligomorphic transformation monoid in this case. To do this, we first state and prove two preliminary lemmas. The first lemma is a restatement of (2.3) of [9]. Here, we note that an n-tuple $(a_1, ..., a_n)$ defines a partition of the set $\{1, ..., n\}$, where i, j are in the same part if $a_i = a_j$. Furthermore, for any n-tuple $\bar{a} = (a_1, ..., a_n)$ there exists a k-tuple $\bar{a}' = (a'_1, ..., a'_k)$ formed by the distinct elements of $\bar{a}$ in order of first appearance, where $1 \leq k \leq n$.

Lemma 6.2.5 (2.3, [9])**.** *Let $\bar{a} = (a_1, ..., a_n)$ and $\bar{b} = (b_1, ..., b_n)$ be two n-tuples of M. Then there exists a partial isomorphism f of $\mathcal{M}$ such that $\bar{a}f = \bar{b}$ if and only if the partitions defined by $\bar{a}, \bar{b}$ are equal, and there exists a partial isomorphism f' of $\mathcal{M}$ such that $\bar{a}'f' = \bar{b}'$.* □

The next lemma demonstrates that if there is a bijective homomorphism sending a finite σ-structure $\mathcal{A}$ to $\mathcal{B}$ and vice versa, then these are isomorphisms. The proof of this uses an observation of [53] (see Lemma 2.3.2); stating that an endomorphism of a finite first-order structure is an automorphism if and only if it is a bijection.

Lemma 6.2.6. *If $\mathcal{A}$ and $\mathcal{B}$ are finite σ-structures and $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ are bijective homomorphisms, then $\mathcal{A} \cong \mathcal{B}$ and f, g are isomorphisms.*

Proof. The composition map $fg : \mathcal{A} \to \mathcal{A}$ is a bijective endomorphism of $\mathcal{A}$; by Lemma 2.3.2, fg must be an automorphism of $\mathcal{A}$. For some $\bar{a} \in A^{n_i}$, if $\neg R_i^{\mathcal{A}}(\bar{a})$ and $R_i^{\mathcal{B}}(\bar{a}f)$, then $R_i^{\mathcal{A}}(\bar{a}fg)$ since g is a homomorphism. This is a contradiction as fg is an automorphism of $\mathcal{A}$ and must preserve non-relations. Therefore, f must preserve non-relations and so is an isomorphism. A similar argument shows that g is an isomorphism. □

Remark. As we will see in Chapter 8, this result is far from being true if the structures involved are infinite.

This is enough to determine the strong orbits of $\mathrm{Y}(\mathcal{M})$ for an XY-homogeneous structure $\mathcal{M}$.

Proposition 6.2.7. *Let $\mathcal{M}$ be an XY-homogeneous σ-structure. Then two tuples $\bar{a} = (a_1, ..., a_n)$ and $\bar{b} = (b_1, ..., b_n)$ are in the same strong orbit of $\mathrm{Y}(\mathcal{M})$ if and only if there exists a partial isomorphism f of $\mathcal{M}$ such that $\bar{a}f = \bar{b}$.*

Proof. Suppose that $\alpha, \beta \in \mathrm{Y}(\mathcal{M})$ are maps such that $\bar{a}\alpha = \bar{b}$ and $\bar{b}\beta = \bar{a}$ respectively; so $a_i\alpha = b_i$ and $b_i\beta = a_i$ for $1 \leq i \leq n$. If α sends elements $a_i \neq a_j$ of $\bar{a}$ to elements $a_i\alpha = a_j\alpha$ of $\bar{b}$, then $a_i\alpha\beta = a_j\alpha\beta$ and so $a_i = a_j$; a contradiction. Hence the restrictions $\alpha|_{\bar{a}}$ and $\beta|_{\bar{b}}$ are injective maps and so they are also bijections. Now, we consider the homomorphisms $\alpha|_{\bar{a}} : \mathcal{A} \to \mathcal{B}$ and $\beta|_{\bar{b}} : \mathcal{B} \to \mathcal{A}$, where $\mathcal{A}, \mathcal{B}$ are the structures induced by $\mathcal{M}$ on $\bar{a}, \bar{b}$ respectively. By Lemma 6.2.6, it follows that $\mathcal{A} \cong \mathcal{B}$ and so we can define $f = \alpha|_{\bar{a}}$ to be the required partial isomorphism. Conversely, assume that f is an isomorphism between the structures $\mathcal{A}$ and $\mathcal{B}$. By XY-homogeneity (and hence IY-homogeneity) of $\mathcal{M}$, we extend f to a map $\alpha \in \mathrm{Y}(\mathcal{M})$ such that $\bar{a}\alpha = \bar{b}$. Similarly, we can extend the isomorphism $f^{-1} : \mathcal{B} \to \mathcal{A}$ to a map $\beta \in \mathrm{Y}(\mathcal{M})$ such that $\bar{b}\beta = \bar{a}$. Therefore, $\bar{a}$ and $\bar{b}$ are in the same strong orbit. $\square$

Theorem 6.2.8. *If $\mathcal{M}$ is an XY-homogeneous structure over a finite relational language, then $Y(\mathcal{M})$ is an oligomorphic transformation monoid.*

Proof. By Proposition 6.2.7 and Lemma 6.2.5, the strong orbit $S(\bar{a})$ of $\bar{a}$ consists of all those n-tuples $\bar{b}$ such that the partitions defined by $\bar{a}, \bar{b}$ are equal, and there exists a partial isomorphism f' of $\mathcal{M}$ such that $\bar{a}'f' = b'$. As $\mathcal{M}$ is over a finite relational language, it has finitely many isomorphism types on n-tuples of distinct elements for any $n \in \mathbb{N}$. As the number of partitions of n into k pieces is finite, we conclude that there are finitely many strong orbits of $\bar{a}$ in $\mathrm{Y}(\mathcal{M})$. $\square$

Remark. If $\mathcal{M}$ is an XB-homogeneous structure, then $\mathrm{Bi}(\mathcal{M})$ is an oligomorphic permutation monoid. In Chapter 8, we explore examples of MB-homogeneous graphs and digraphs, producing a range of examples of oligomorphic permutation monoids.

Using this corollary, we can find examples of structures with oligomorphic transformation monoids that are *not* $\aleph_0$-categorical; for instance, $\mathrm{Y}(\mathcal{P})$ for any XY-homogeneous poset $\mathcal{P}$ not in Schmerl's classification (Theorem 2.3.14, see [60] and [53]) is an oligomorphic transformation monoid. Furthermore, Example 2.11 of [61] asserts that the discrete linear order $(\mathbb{N}, \leq)$ (see Subsection 4.2.2) is HH-homogeneous; therefore, $(\mathbb{N}, \leq)$ is an example of a σ-structure where $\mathrm{Aut}(\mathbb{N}, \leq)$ is trivial but $\mathrm{End}(\mathbb{N}, \leq)$ is an oligomorphic endomorphism monoid. This example also provides a converse to Proposition 6.2.2. We present an example of where this occurs for a graph Γ.

Example 6.2.9. Cameron and Nešetřil [14] demonstrated an example of a HH-homogeneous (and MM-homogeneous) graph $\bar{\Gamma}$ with trivial automorphism group, where $\bar{\Gamma}$ is the complement of a rigid, locally finite graph Γ (see Figure 6.3 for a reproduction of this graph).

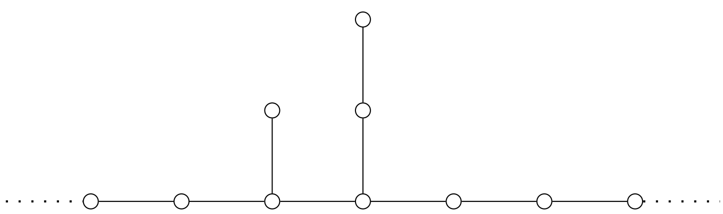


Figure 6.3: Γ, a rigid locally finite graph whose complement $\bar{\Gamma}$ is HH and MM-homogeneous.

Here, $\mathrm{Aut}(\bar{\Gamma}) = \{e\}$, but both $\mathrm{End}(\bar{\Gamma})$ and $\mathrm{Mon}(\bar{\Gamma})$ are oligomorphic transformation monoids by Theorem 6.2.8.

The aim of Chapter 7, and particularly of Chapter 8, is to find more examples of homomorphism-homogeneous structures, widening the range of oligomorphic transformation monoids and oligomorphic permutation monoids. We also investigate several semigroup-theoretic questions related to oligomorphic transformation monoids as well; for instance, for any countable group G, does there exist an oligomorphic transformation monoid T with group of units isomorphic to G? We answer this question in the affirmative for finite groups in Chapter 8 (see Theorem 8.2.11); however, the general countable case remains open.

There are other questions we can ask, particularly in model theory. In part, the initial development of oligomorphic transformation monoids in [61] was to facilitate a discussion on model-theoretic properties of HH-homogeneous structures. As endomorphisms preserve positive formulas, the properties developed were focused on these; for example, an analogous result to Proposition 2.3.9 in [61] asserts that if $\mathcal{M}$ is a first-order structure with oligomorphic endomorphism monoid, then homomorphism-homogeneity is equivalent to Th$(\mathcal{M})$ having quantifier elimination for positive formulas. We can then ask the same questions about the oligomorphic transformation monoids presented here.

Question 6.2.10. *If $\mathcal{M}$ is a structure, let $Y(\mathcal{M}) \in \{Epi(\mathcal{M}), Mon(\mathcal{M}), Bi(\mathcal{M}), Emb(\mathcal{M})\}$ be an oligomorphic transformation monoid. Using this assumption, develop the model theory associated with $\mathcal{M}$. In particular, is XY-homogeneity of $\mathcal{M}$ equivalent to some kind of quantifier elimination in Th$(\mathcal{M})$?*

7

Homomorphism-homogeneous first-order structures

Examples of oligomorphic permutation groups are sought because of their close connection to the theory of $\aleph_0$-categorical structures (Theorem 2.3.7). As outlined in Proposition 2.3.9, oligomorphic permutation groups can arise as automorphism groups of homogeneous structures. Homogeneous structures are completely characterised by Fraïssé's theorem (2.3.11); so finding Fraïssé limits provides examples of oligomorphic permutation groups. Our aim in this chapter is to find a generalisation of Fraïssé's theorem in order to provide more examples of oligomorphic transformation monoids, in line with the connection detailed in Theorem 6.2.8. We begin with a discussion of this aim.

Throughout this chapter, let σ be a relational signature. Suppose that $\mathscr{C}$ is a class of finite σ-structures. The proof of part of Fraïssé's theorem relies on inductively constructing a structure $\mathcal{M}$ whose age is $\mathscr{C}$ and is also homogeneous. This proof relies on properties belonging to $\mathscr{C}$ and the constructed structure $\mathcal{M}$. One is the *joint embedding property* (JEP); this property ensures that we can construct a countable σ-structure $\mathcal{M}$ with age $\mathscr{C}$. The second is the *amalgamation property* (AP), a condition held by a class of finite σ-structures $\mathscr{C}$ that ensures that the countable σ-structure $\mathcal{M}$ with age $\mathscr{C}$ is homogeneous. This is verified by showing that $\mathcal{M}$ has the *extension property*, a necessary and sufficient condition for a countable σ-structure $\mathcal{M}$ to be homogeneous. Fraïssé's theorem also states that any two homogeneous σ-structures with the same age are isomorphic; this is achieved using a back and forth argument constructing the desired isomorphism. The forth part of the argument ensures that the extended map is totally defined; the back part ensures its surjectivity.

In 2006, Cameron and Nešetřil [14] proved an analogue of Fraïssé's theorem for MM-homogeneity (see Figure 6.1). This proof necessitated modification of the amalgamation property to ensure MM-homogeneity; resulting in the *mono-amalgamation property* (MAP). In a slight departure to the technique used to prove Fraïssé's theorem, the proof of the analogous theorem for MM-homogeneity in [14] utilised a forth alone argument; this is because the extended map need not be surjective. It was also realised in [14] that two MM-structures with the same age may be non-isomorphic; instead detailing that two such structures were unique up to a weaker notion called *mono-equivalence*. The proof of this again only required forward steps. Further insights were made by Dolinka [23], who detailed the notion of a *homo-amalgamation property* (HAP). However, the HAP was used in providing examples of oligomorphic endomorphism monoids in order to determine whether or not they had the Bergman property, and not studied from a Fraïssé-theoretical point of view.

In the case of MB-homogeneity, a forth alone approach does not suffice. As the extended map must be surjective, we are required to use a back and forth argument. The fact that monomorphisms are not invertible in general necessitates the use of a second amalgamation property alongside the MAP of [14]; this was defined by Coleman, Evans and Gray [17] using *antimonomorphisms* in the *bi-amalgamation property* (BAP). In a similar situation to [14], two MB-homogeneous structures with the same age may not be isomorphic but instead are unique up to *bi-equivalence*; the proof of this also requires back and forth steps.

In light of these previous generalisations of Fraïssé's theorem, and the multitude of types of homomorphism-homogeneity in $\mathfrak{H}$ (see Definition 6.2.4), the natural aim would be to find an "umbrella" version of Fraïssé's theorem; one that encapsulates all possible notions of homomorphism-homogeneity. This result would supply Fraïssé's theorem, and the versions of [14] and [17], as corollaries. Such a theorem could determine the extent to which a structure is homomorphism-homogeneous with reference to the classes in $\mathbb{H}$. In turn, this will provide a rich source of oligomorphic transformation monoids by Theorem 6.2.8.

However, a compromise must be reached between idealism and practicality for two reasons. First, as discussed above, differing approaches are required if the extended map is surjective; see the contrast between MM-homogeneity and MB-homogeneity for a case in point. In the forth alone case, we can utilise a single modified amalgamation property in order to construct the structure and extend the map. But as monomorphisms and homomorphisms are not "invertible" in general, in the back and forth case we need *two* modified amalgamation properties; one for the forth part and one for the back part. Second, some kinds of homomorphism-homogeneity are easier to deal with than others. There is a distinct dichotomy in $\mathfrak{H}$, split between those whose extended maps are not necessarily the same "type" as the partial map (such as MH-homogeneity, in that a homomorphism is not necessarily an monomorphism), and those whose extended maps are definitely of the same type than the partial map (such as MM, or MI-homogeneity). The former case causes issues in inductively constructing a structure due to the lack of certainty about the extended map; this is discussed in further detail in Section 7.2.

The first of these reasons therefore necessitate *two* similar but markedly different theorems (Theorem 7.0.1 and Theorem 7.0.2) based on whether or not the proof uses forth alone or a back and forth argument. The second allows the two theorems to cover twelve of the eighteen different notions of homomorphism-homogeneity. What constitutes the "relevant" amalgamation property and notion of equivalence will be explained in Section 7.2.

Theorem 7.0.1. *Let $XY \in \{II, MI, MM, HI, HM, HH\}$.*

(1) If $\mathcal{M}$ is an XY-homogeneous σ-structure, then $Age(\mathcal{M})$ has the relevant amalgamation property.

(2) If $\mathscr{C}$ is a class of finite σ-structures with countably many isomorphism types, is closed under isomorphisms and substructures, has the JEP and the relevant amalgamation property, then there exists a XY-homogeneous σ-structure $\mathcal{M}$ with age

$\mathscr{C}$.

(3) *Any two XY-homogeneous σ-structures with the same age are equivalent up to a relevant notion of equivalence.*

Theorem 7.0.2. *Let $XZ \in \{IA, MA, MB, HA, HB, HE\}$.*

(1) *If $\mathcal{M}$ is an XZ-homogeneous σ-structure, then $Age(\mathcal{M})$ has the two relevant amalgamation properties.*

(2) *If $\mathscr{C}$ is a class of finite σ-structures with countably many isomorphism types, is closed under isomorphisms and substructures, has the JEP and the two relevant amalgamation properties, then there exists a XZ-homogeneous σ-structure $\mathcal{M}$ with age $\mathscr{C}$.*

(3) *Any two XZ-homogeneous σ-structures with the same age are equivalent up to a relevant notion of equivalence.*

Whilst not the ideal "umbrella" theorem, these two results are still useful in determining the extent to which a structure is homomorphism-homogeneous; thus providing examples of oligomorphic transformation monoids.

To that end, this chapter is dedicated to the proof of these two theorems; as well as determining a complete picture of homomorphism-homogeneity for some well-known structures. Section 7.1 introduces the concept of an *antihomomorphism* between two σ-structures; this will be important machinery in the back part of the eventual back-and-forth argument in the proof of Theorem 7.0.2. We split Section 7.2 into two pieces proving Theorem 7.0.1 and Theorem 7.0.2 in turn. Finally, Section 7.3 defines the notion of a maximal homomorphism-homogeneity class of a structure, and determines these for some previously seen homogeneous structures.

7.1 Antihomomorphisms

As mentioned in the introduction, homomorphisms are not "invertible" in general. For instance, there could be a homomorphism between two relational σ-structures $\alpha : \mathcal{A} \to \mathcal{B}$ sending a non-relation of $\mathcal{A}$ to a relation in $\mathcal{B}$; that is, such that $\bar{a} \notin R_i^{\mathcal{A}}$ but $\bar{a}\alpha \in R_i^{\mathcal{A}}$. Furthermore, there is no guarantee that the homomorphism is even injective; so α could send two points in $\mathcal{A}$ to the same point in $\mathcal{B}$. In establishing a suitable 'back' amalgamation property for our Fraïssé-style theorem, both of these considerations must be taken into account. To that end, we adapt the concept of a *multifunction* (see Section 2.1) to the setting of relational first-order structures.

Definition 7.1.1. Suppose that $\mathcal{A}, \mathcal{B}$ are two σ-structures and that $f^* : \mathcal{B} \to \mathcal{A}$ is a multifunction. We say that f^* is an *antihomomorphism* if for all $R_i \in \sigma$, we have that if $\neg R_i^{\mathcal{B}}(\bar{b})$ in $\mathcal{B}$ then $\neg R_i^{\mathcal{A}}(\bar{a})$ in $\mathcal{A}$ for all $\bar{a} \in \bar{b}f^*$.

Remark. This definition is equivalent to saying that $f^* : \mathcal{B} \to \mathcal{A}$ is an antihomomorphism if for all $R_i \in \sigma$ and for all $\bar{a} \in \bar{b}f^*$, then $R_i^{\mathcal{A}}(\bar{a})$ implies that $R_i^{\mathcal{B}}(\bar{b})$.

The motivation behind this definition is explained by the following alternate characterisation of antihomomorphisms. We use the notation adopted in Section 2.1: if $f : A \to B$ is a function, denote its converse multifunction by $f^* : B \to A$. It immediately follows that $(f^*)^* = f$.

Lemma 7.1.2. *Let $\mathcal{A}, \mathcal{B}$ be two σ-structures. Then $f^* : \mathcal{B} \to \mathcal{A}$ is a surjective antihomomorphism if and only if $f : \mathcal{A} \to \mathcal{B}$ is a surjective homomorphism.*

Proof. Assume that $f : \mathcal{A} \to \mathcal{B}$ is a surjective homomorphism. As $f : A \to B$ is a surjective function we have that $f^* : B \to A$ is a surjective multifunction. Now suppose that $\neg R_i^{\mathcal{B}}(\bar{b})$. As f must preserve relations, we have $\neg R_i^{\mathcal{A}}(\bar{a})$ whenever $\bar{a}f = \bar{b}$; this is precisely when $\bar{a} \in \bar{b}f^*$. Conversely, suppose that $f^* : \mathcal{B} \to \mathcal{A}$ is a surjective antihomomorphism; therefore $f : A \to B$ is a surjective function. Suppose also that $R_i^{\mathcal{A}}(\bar{a})$ holds. As f^* is an antihomomorphism, it follows that $\bar{a} \notin \bar{b}f^*$ for every $\bar{b}$ such that $\neg R_i^{\mathcal{B}}(\bar{b})$. Since f is a function, it must be that $\bar{a} \in \bar{b}f^*$ for some $\bar{b}$ such that $R_i^{\mathcal{B}}(\bar{b})$; so f is a homomorphism. $\square$

Remark. If $f : \mathcal{A} \to \mathcal{B}$ is any homomorphism, we can restrict the codomain to the image to see that $f : \mathcal{A} \to \mathcal{A}f$ is a surjective homomorphism; and hence, by the above proposition, $f^* : \mathcal{A}f \to \mathcal{A}$ is a surjective antihomomorphism. This technique will be used regularly in Subsection 7.2.2.

This result leads to an immediate corollary; an analogue of Lemma 2.1.3 for σ-structures.

Corollary 7.1.3. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be σ-structures, and $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ are surjective homomorphisms. Then $(fg)* = g^*f^*$ is a surjective antihomomorphism.*

Proof. The multifunctions $(fg)^* : C \to A$ and $g^*f^* : C \to A$ are equal by Lemma 2.1.2. As fg is a surjective homomorphism, it follows that $(fg)^*$ is a surjective antihomomorphism by Lemma 7.1.2. □

Note that if $f : \mathcal{A} \to \mathcal{B}$ is a bijective homomorphism, then $f^* : \mathcal{B} \to \mathcal{A}$ is an bijective function from B to A that preserves non-relations; this is the definition of an *antimonomorphism* $\hat{f} : \mathcal{B} \to \mathcal{A}$ (see [17]). Furthermore, if $f : \mathcal{A} \to \mathcal{B}$ is a isomorphism, then $f^* : \mathcal{B} \to \mathcal{A}$ is exactly f^{-1}, the inverse isomorphism of f. In the style of Lemma 2.1.2, we prove a composition lemma for antihomomorphisms.

Proposition 7.1.4. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be σ-structures. Suppose that $f^* : \mathcal{A} \to \mathcal{B}$ and $g^* : \mathcal{B} \to \mathcal{C}$ are antihomomorphisms. Then their composition $f^*g^* : \mathcal{A} \to \mathcal{C}$ is an antihomomorphism.*

Proof. From Lemma 2.1.2, the relation composition of the underlying multifunctions $f^* : A \to B$ and $g^* : B \to C$ is again a multifunction $f^*g^* : A \to C$. Now suppose that $\neg R_i^{\mathcal{A}}(\bar{a})$ holds. As f^* is an antihomomorphism, $\neg R_i^{\mathcal{B}}(\bar{b})$ holds for all $\bar{b} \in \bar{a}f^*$. As g^* is an antihomomorphism, $\neg R_i^{\mathcal{C}}(\bar{c})$ holds for all $\bar{c} \in \bar{b}g^*$. Therefore $\neg R_i^{\mathcal{C}}(\bar{c})$ holds for all $\bar{c} \in \bigcup_{\bar{b} \in \bar{a}f^*} \bar{b}g^* = \bar{a}f^*g^*$ and so f^*g^* is an antihomomorphism. □

Remarks. We note that as every antimonomorphism and isomorphism is also an antihomomorphism, the product f^*g^* of any antihomomorphism f^* with any antimonomorphism or isomorphism g^* is again an antihomomorphism. This fact turns out to be crucial in the statement of a suitable amalgamation property for the back part of the back-and-forth argument. Furthermore, the product of two antimonomorphisms (or an antihomomorphism and an isomorphism) is again an antimonomorphism.

Finally in this section, we prove a straightforward yet important fact about epimorphisms of an infinite first order structure $\mathcal{M}$.

Lemma 7.1.5. *Let $\mathcal{M}$ be a σ-structure, with $\mathcal{A}$ a finite substructure of $\mathcal{M}$. Then for any $\alpha \in Epi(\mathcal{M})$, there exists a finite structure $\mathcal{B} \subseteq \mathcal{M}$ such that $\mathcal{B}\alpha = \mathcal{A}$.*

Proof. As α is surjective, there exists some set B' such that $B'\alpha = A$, where A is the domain of $\mathcal{A}$. For $A = \{a_1, ..., a_n\}$, B' is partitioned into kernel classes B'_i such that $B'_i\alpha = a_i$ for $1 \leq i \leq n$. By selecting one representative b_i from each B'_i, we induce a structure $\mathcal{B}$ on the finite set $B = \{b_1, ..., b_n\}$ with relations from $\mathcal{M}$. As α is a homomorphism, $\mathcal{B}\alpha = \mathcal{A}$. □

7.2 Two Fraïssé-style theorems

For the rest of the chapter, we will abuse notation slightly and write A, B to mean finite σ-structures on domains A, B respectively.

We recall the collection $\mathfrak{H}$ of notions of homomorphism-homogeneity outlined in Definition 6.2.4. As mentioned in the introduction to this chapter, it is necessary to partition $\mathfrak{H}$ into two pieces based on whether or not the extended map is surjective. This represents the division between cases where a forth alone argument will suffice and the other when we require a back and forth construction. Furthermore, there are some elements of $\mathfrak{H}$ that are weaker notions of homogeneity than others. These are of the form XY where a map of type Y does not necessarily imply that it is a map of type X; for instance, a homomorphism is not necessarily a monomorphism. These phenomena motivate the division of $\mathfrak{H}$ into the following:

- forth alone $\mathfrak{F} = \{\text{XY} \in \mathfrak{H} : \text{X,Y} \in \{\text{H, M, I}\}\}$;
- back and forth $\mathfrak{B} = \{\text{XZ} \in \mathfrak{H} : \text{X} \in \{\text{H, M, I}\}, \text{Z} \in \{\text{E, B, A}\}\}$;
- no implication $\mathfrak{N} = \{\text{IH, IE, IM, IB, MH, ME}\}$;
- implication $\mathfrak{I} = \mathfrak{H} \setminus \mathfrak{N}$.

These choices partition $\mathfrak{H}$ into four parts based on the intersections of $\mathfrak{B}, \mathfrak{F}$ with $\mathfrak{N}, \mathfrak{I}$ (see Figure 7.1, where the boxes represent intersections).

	$\mathfrak{F}$	$\mathfrak{B}$
$\mathfrak{N}$	IH IM MH	IE IB ME
$\mathfrak{I}$	HH HM HI MM MI II	HE HB HA MB MA IA

Figure 7.1: $\mathfrak{F}$, $\mathfrak{B}$, $\mathfrak{N}$, and $\mathfrak{I}$ in $\mathfrak{H}$

Here, it is necessary to note here that II-homogeneity and IA-homogeneity represent two different notions of homomorphism-homogeneity under consideration. As outlined in Figure 6.1, II-homogeneity is where every finite partial isomorphism extends to an *embedding*; IA-homogeneity is where every finite partial isomorphism extends to an automorphism (that is, standard homogeneity from Definition 2.3.8). As stated in Chapter 6, Lockett and Truss proved that the *classes* II and IA coincide [53]. For the purposes of our work in this chapter, we focus on the notions of homomorphism-homogeneity as opposed to the classes of homomorphism-homogeneous structures. This allows us to re-prove these results of Lockett and Truss from a Fraïssé-theoretic standpoint.

7.2.1 Forth alone

This section is devoted to the proof of Theorem 7.0.1. It deals with types of homomorphism-homogeneity in $\mathfrak{F}$ (see Figure 7.1); those that only require a forth construction to prove. Consequently, we have that $X, Y \in \{H, M, I\}$ throughout this subsection. When we say a *map of type X*, we are referring to this instance; so if α is a map of type H, it is a homomorphism. Notice that $I \subseteq M \subseteq H$; as every isomorphism is also a monomorphism, and every monomorphism a homomorphism.

Our eventual aim is to construct a countable structure $\mathcal{M}$ with age $\mathscr{C}$, where $\mathcal{M}$ is XY-homogeneous. The JEP (see Subsection 2.3.3) is required to construct any countable structure $\mathcal{M}$ with age $\mathscr{C}$; it has nothing to do with the homogeneity of the structure $\mathcal{M}$. It is the amalgamation property that is central to homogeneity of a Fraïssé limit; so this must be generalised in order to ensure XY-homogeneity. So to construct such an $\mathcal{M}$ with age $\mathscr{C}$, the class $\mathscr{C}$ must have the JEP and some generalised amalgamation property.

Since different types of homogeneity require different amalgamation properties, it would make sense to define an "umbrella" amalgamation property; one that encompasses all the amalgamation properties required. This is presented as the *XY-amalgamation property*, where X,Y $\in$ {H, M, I}:

> (XYAP) Let $\mathscr{C}$ be a class of finite structures. Then $\mathscr{C}$ has the *XYAP* if for all $A, B_1, B_2 \in \mathscr{C}$, map $f_1 : A \to B_1$ of type X and embedding $f_2 : A \to B_2$, there exists a $D \in \mathscr{C}$, embedding $g_1 : B_1 \to D$ and map $g_2 : B_2 \to D$ of type Y such that $f_1 g_1 = f_2 g_2$ (see Figure 7.2).

$$\begin{tikzcd}
 & \exists D & \\
B_1 \arrow[ur, hook, dotted, "g_1"] & & B_2 \arrow[ul, dotted, "g_2"'] \\
 & A \arrow[ul, "f_1"] \arrow[ur, hook, "f_2"'] &
\end{tikzcd}$$

Figure 7.2: The XY-amalgamation property (XYAP)

Based on choices for X and Y, the XYAP yields nine different amalgamation properties; one for each notion of XY-homogeneity in $\mathfrak{F}$. For instance, the IIAP is the standard amalgamation property, the MMAP is the MAP in [14] and the HHAP is the HAP from [23]. A similar "umbrella" extension property is required for a structure $\mathcal{M}$ with age $\mathscr{C}$; this is defined as the *XY-extension property*:

> (XYEP) A structure $\mathcal{M}$ with age $\mathscr{C}$ has the *XYEP* if for all $A \subseteq B \in \mathscr{C}$ and maps $f : A \to \mathcal{M}$ of type X, there exists a map $g : B \to \mathcal{M}$ of type Y extending f.

Ideally, we would like to have a straight generalisation of Proposition 2.3.10; that a structure $\mathcal{M}$ is XY-homogeneous if and only if $\mathcal{M}$ has the XYEP. However, complications occur in the proof of the converse direction; this is due to the inductive construction of the extended map. For example, suppose that $\mathcal{M}$ has the IMEP and that $f : A \to B$ is an isomorphism. Extending this using the IMEP gives a monomorphism $g : A' \to B'$ where $A \subseteq A'$ and $B \subseteq B'$. However, g is a monomorphism between finite substructures; and so we may not be able to extend g to another monomorphism h between finite substructures. The only way we can continue extending is if the map of type Y is also of type X. This behaviour is the motivating factor in splitting $\mathfrak{H}$ into $\mathfrak{I}$ and $\mathfrak{N}$. In light of this, we show that the XYEP is a necessary condition for XY-homogeneity in general, and that it is also sufficient when the extended map of type Y is also a map of type X.

Proposition 7.2.1. *Let $\mathcal{M}$ be a countable σ-structure with age $\mathscr{C}$.*

(1) Suppose that $XY \in \mathfrak{F}$. If $\mathcal{M}$ is XY-homogeneous, then $\mathcal{M}$ has the XYEP.

(2) Suppose that $XY \in \mathfrak{F} \cap \mathfrak{I}$. If $\mathcal{M}$ has the XYEP, then $\mathcal{M}$ is XY-homogeneous.

Proof. (1) Let $A \subseteq B \in \mathscr{C}$ and $f : A \to \mathcal{M}$ be a map of type X. As $\text{Age}(\mathcal{M}) = \mathscr{C}$, there exists an isomorphism $\theta : B \to B\theta \subseteq \mathcal{M}$. Therefore, $\theta^{-1}f : B\theta \to Af$ is a map of type X between finite substructures of $\mathcal{M}$. As $\mathcal{M}$ is XY-homogeneous, extend $\theta^{-1}f$ to a map $\alpha : \mathcal{M} \to \mathcal{M}$ of type Y. Hence, $\theta\alpha : B \to \mathcal{M}$ is a map of type Y. It remains to show that $\theta\alpha$ extends f; indeed, for any $a \in A$,

$$af = a\theta\theta^{-1}f = a\theta\alpha$$

as α extends $\theta^{-1}f$. Therefore $\mathcal{M}$ has the XYEP.

(2) Suppose that $f : A \to B$ is a map of type X between finite substructures of $\mathcal{M}$. We use a forth argument to extend f to a map α of type Y. As $\mathcal{M}$ is countable, we enumerate the points of $M = \{m_0, m_1, ...\}$. Set $A = A_0, B = B_0$ and $f = f_0$ and assume that we have extended f to a map $f_k : A_k \to B_k$, where $A_i \subseteq A_{i+1}$

and $B_i \subseteq B_{i+1}$ for all $0 \leq i \leq k-1$. At most, we can assume that f_k is a map of type Y. Select $m_i \in \mathcal{M} \setminus A_k$, where i is the least natural number such that $m_i \notin A_k$. We can see that $A_k \cup \{m_i\} \subseteq \mathcal{M}$ belongs to $\mathscr{C}$. As XY $\in \mathfrak{I}$, the f_k of type Y is also of type X; so use the XYEP to find a map $f_{k+1} : A_k \cup \{m_i\} \to \mathcal{M}$ of type Y extending f_k. Repeating this process infinitely many times, ensuring that each m_i appears at some stage, extends f to a map $\alpha : \mathcal{M} \to \mathcal{M}$ of type Y; so $\mathcal{M}$ is XY-homogeneous. □

Our next result demonstrates Theorem 7.0.1 (1).

Proposition 7.2.2. *Suppose that XY $\in \mathfrak{F}$. If a σ-structure $\mathcal{M}$ is XY-homogeneous, then Age($\mathcal{M}$) has the XYAP.*

Proof. Suppose that $A, B_1, B_2 \in$ Age($\mathcal{M}$), $f_1 : A \to B_1$ is a map of type X and $f_2 : A \to B_2$ is an embedding. Without loss of generality, suppose that f_2 is the inclusion map and that $A, B_1, B_2 \subseteq \mathcal{M}$. Using XY-homogeneity of $\mathcal{M}$, extend $f_1 : A \to B$ to a map $\alpha : \mathcal{M} \to \mathcal{M}$ of type Y. Set $D = B_1 \cup B_2\alpha$ and induce the structure on D with relations from $\mathcal{M}$. Finally, take $g_1 : B_1 \to D$ to be the inclusion map and define the map $g_2 = \alpha|_{B_2} : B_2 \to D$ of type Y. We can see that $f_1 g_1 = f_2 g_2$ and so these choices verify the XYAP for Age($\mathcal{M}$). □

Now, we proceed with the proof of Theorem 7.0.1 (2). Recall from Section 2.3 that a class of finite structures $\mathscr{C}$ has the *joint embedding property* (JEP) if for all $A, B \in \mathscr{C}$ there exists a $D \in \mathscr{C}$ such that D jointly embeds A and B.

Proposition 7.2.3. *Suppose that XY $\in \mathfrak{F} \cap \mathfrak{I}$. Let $\mathscr{C}$ be a class of finite σ-structures that is closed under isomorphism and substructures, has countably many isomorphism types, and has the JEP and XYAP. Then there exists a XY-homogeneous σ-structure $\mathcal{M}$ with age $\mathscr{C}$.*

Proof. We build a structure inductively over countably many steps. As $\mathscr{C}$ has countably many isomorphism types, we can enumerate $\mathscr{C} = \{C_0, C_1, ...\}$. Assume that a structure M_k has already been constructed.

If k is even, select $C_i \in \mathscr{C}$ where $k = 2i$. Use the JEP to find a structure $D \in \mathscr{C}$ that jointly embeds M_k and C_i; and define this structure D to be M_{k+1}.

Now suppose that k is odd. Select a triple (A, B, f), where $A \subseteq B \in \mathscr{C}$ and $f : A \to M_k$ is some map of type X. Using the XYAP, we find a structure M_{k+1}, an embedding $e_k : M_k \to M_{k+1}$ and a map $g : B \to M_{k+1}$ of type Y that extends f. As $\mathscr{C}$ has countably many isomorphism types, and there are only finitely many maps f from A into M_k of type X at each stage, we can arrange the steps such that:

- each structure C_i in $\mathscr{C}$ appears at an even step, and:
- each triple (A, B, f) appears at an odd step k, where for every such k, for every $A \subseteq B \in \mathscr{C}$ and every map $f : A \to M_k$ there exists $\ell \geq k$ and embedding $e_{k,\ell} : M_k \to M_\ell$ such that f extends to a map $g : B \to M_\ell$ of type Y.

Arranging the steps this way ensures that every possible amalgamation is performed. As each $M_k \subseteq M_{k+1}$ due to the embedding e_k, we can define the structure $\mathcal{M} = \bigcup_{k\in\mathbb{N}} M_k$. We check that $\mathscr{C} = \text{Age}(\mathcal{M})$ and that $\mathcal{M}$ is XY-homogeneous.

Due to our construction at even steps, $C_i \in \text{Age}(\mathcal{M})$ for every $i \in \mathbb{N}$, and so $\mathscr{C} \subseteq \text{Age}(\mathcal{M})$. We also ensured at every step that each M_k is a member of $\mathscr{C}$; as $\mathscr{C}$ is closed under substructures, it follows that $\text{Age}(\mathcal{M}) \subseteq \mathscr{C}$ and so they are equal. For XY-homogeneity, as XY $\in \mathfrak{F} \cap \mathfrak{I}$ it suffices to show that $\mathcal{M}$ has the XYEP by Proposition 7.2.1. So assume that $A \subseteq B \in \mathscr{C}$ and that $f : A \to \mathcal{M}$ is a map of type X. From the arrangement of steps above, there exists a k such that $Af \subseteq M_k$. From the construction, there exists an $M_\ell \supseteq M_k$ and a map $g : B \to M_\ell$ of type Y extending f. Hence $\mathcal{M}$ has the XYEP and is therefore XY-homogeneous by Proposition 7.2.1. □

All that remains to show is part (3) of Theorem 7.0.1. It was previously mentioned in [14] that two MM-homogeneous structures with the same age need not be isomorphic, but are instead *mono-equivalent*. This inspires a new definition.

Definition 7.2.4. Let $\mathcal{M}, \mathcal{N}$ be σ-structures and suppose that Y $\in$ {H, M, I}. Say that $\mathcal{M}$ and $\mathcal{N}$ are *Y-equivalent* if $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{N})$ and every embedding f :

$A \to \mathcal{N}$ from a finite substructure A of $\mathcal{M}$ can be extended to a map $g : \mathcal{M} \to \mathcal{N}$ of type Y, and vice versa.

Note that if two structures $\mathcal{M}, \mathcal{N}$ are M-equivalent, then they are mono-equivalent in the sense of [14]. If two structures $\mathcal{M}, \mathcal{N}$ are I-equivalent, then they are mutually embeddable.

Proposition 7.2.5. *Let $\mathcal{M}, \mathcal{N}$ be countable σ-structures, and suppose that $XY \in \mathfrak{F} \cap \mathfrak{I}$.*

(1) Suppose that $\mathcal{M}, \mathcal{N}$ are Y-equivalent. Then $\mathcal{M}$ is XY-homogeneous if and only if $\mathcal{N}$ is.

(2) If $\mathcal{M}, \mathcal{N}$ are XY-homogeneous and $Age(\mathcal{M}) = Age(\mathcal{N})$ then $\mathcal{M}, \mathcal{N}$ are Y-equivalent.

Proof. (1) It suffices to show that $\mathcal{N}$ has the XYEP by Proposition 7.2.1 (2). Suppose then that $A \subseteq B \in \text{Age}(\mathcal{N})$ and there exists a map $f : A \to A' \subseteq \mathcal{N}$ of type Y. Note that A need not be isomorphic to A'. As $\text{Age}(\mathcal{M}) = \text{Age}(\mathcal{N})$, there exists a copy A'' of A' in $\mathcal{M}$; fix an embedding $e : A' \to A''$ between the two. Therefore, $e^{-1} : A'' \to A'$ is a isomorphism from a finite substructure of $\mathcal{M}$ into $\mathcal{N}$; as the two are Y-equivalent, we extend this to a map $\alpha : \mathcal{M} \to \mathcal{N}$ of type Y. Now, define a map $h = fe : A \to A''$; this is a map of type Y from A into $\mathcal{M}$. Since $\mathcal{M}$ is XY-homogeneous, it has the XYEP by Proposition 7.2.1 (1) and so we extend h to a map $h' : B \to \mathcal{M}$ of type Y. Now, the map $h'\alpha : B \to \mathcal{N}$ is a map of type Y; we need to show it extends f. So using the facts that α extends e^{-1} and h' extends $h = fe$, we have that for all $a \in A$:

$$af = afee^{-1} = ah'\alpha.$$

Therefore $\mathcal{N}$ has the XYEP.

(2) Let $A \subseteq \mathcal{M}, B \subseteq \mathcal{N}$ and suppose that $f : A \to B$ is an embedding; trivially, f is also a map of type X. We extend f to a map $\alpha : \mathcal{M} \to \mathcal{N}$ of type Y via an inductive argument. As $\mathcal{M}$ is countable, we enumerate its elements $M = \{m_0, m_1, ...\}$. Set $A = A_0, B = B_0$ and $f = f_0$, and suppose that $f_k : A_k \to B_k$ is a map of type Y where $A_i \subseteq A_{i+1}$ and $B_i \subseteq B_{i+1}$ for all $0 \leq i \leq k-1$.

As XY $\in \mathfrak{F} \cap \mathfrak{I}$, f_k is also a map of type X. Select a point $m_i \in \mathcal{M} \setminus A_k$, where i is the least natural number such that $m_i \notin A_k$. We see that $A_k \cup \{m_i\}$ is a substructure of $\mathcal{M}$ and is therefore an element of Age($\mathcal{N}$) by assumption. As $\mathcal{N}$ is XY-homogeneous, by Proposition 7.2.1 (1) it has the XYEP. Using this, extend $f_k : A_k \to \mathcal{N}$ to a map $f_{k+1} : A_k \cup \{m_i\} \to \mathcal{N}$ of type Y. As XY $\in \mathfrak{F} \cap \mathfrak{I}$, we can repeat this process infinitely many times; by ensuring that every $m_i \in \mathcal{M}$ is included at some stage, we can extend the map f to a map $\alpha : \mathcal{M} \to \mathcal{N}$ of type Y as required. We can use a similar argument to construct a map $\beta : \mathcal{N} \to \mathcal{M}$ of type Y; therefore $\mathcal{M}$ and $\mathcal{N}$ are Y-equivalent. □

Remark. Although we have stated that σ is a relational signature in this chapter, this assumption is only used in the definition of an antihomomorphism. These are not used in the proof of Theorem 7.0.1; so it follows that this result should hold for first-order structures in general. With this in mind, Propositions 4.1 and 4.2 of [14] are direct corollaries of Theorem 7.0.1.

7.2.2 Back and forth

We now move on to discussing extension to surjective endomorphisms; this is when XZ $\in \mathfrak{B}$. Due to the lack of symmetry when working with homomorphisms as opposed to isomorphisms, we must provide a backwards condition to achieve the back part of the required back-and-forth argument. Similar to the more conventional amalgamation properties, this backwards condition is defined on finite structures. This will involve using the concept of antihomomorphisms outlined in Section 7.1 in three distinct case; antihomomorphisms ($\overline{\mathrm{H}}$) for homomorphisms (H), antimonomorphisms ($\overline{\mathrm{M}}$) for monomorphisms (M), and inverse isomorphisms ($\bar{\mathrm{I}}$) for isomorphisms (I) (although these notions are the same in this case).

We will write $\bar{f} : B \to A$ to mean some multifunction of type $\overline{\mathrm{X}} \in \{\overline{\mathrm{H}}, \overline{\mathrm{M}}, \bar{\mathrm{I}}\}$ from B to A (see Figure 7.3). This notation is used in another manner: if $f : A \to B$ is a surjective homomorphism of type X, we write $\bar{f} : B \to A$ to be the corresponding surjective antihomomorphism of type $\overline{\mathrm{X}}$: this is uniquely deter-

Type	Map	Converse type (M)	Converse map (H)
H	homomorphism	$\overline{\text{H}}$	antihomomorphism
M	monomorphism	$\overline{\text{M}}$	antimonomorphism
I	isomorphism	$\bar{\text{I}}$	isomorphism

Figure 7.3: Types of finite partial map

mined by Lemma 7.1.2. The context of when we use this will usually be clear. We also recall Proposition 7.1.4 and its following remarks; the composition of two multifunctions of type $\overline{\text{H}}, \overline{\text{M}}, \bar{\text{I}}$ is again a multifunction of type $\overline{\text{H}}, \overline{\text{M}}, \bar{\text{I}}$. Note also that the classes I and $\bar{\text{I}}$ coincide; we use the barred version when applicable throughout for notational simplicity. It can be seen that $\bar{\text{I}} \subseteq \overline{\text{M}} \subseteq \overline{\text{H}}$.

We note that if Z = E then it is a surjective map of type Y = H; likewise, when Z = B we have that Y = M and when Z = A we have that Y = I. This relation is codified by the following set of pairs:

$$\mathscr{S} = \{(\text{E,H}), (\text{B,M}), (\text{A,I})\}. \tag{7.1}$$

It follows that any XZ-homogeneous structure $\mathcal{M}$ is also XY-homogeneous, where the two are related by the relevant pair (Z,Y) $\in \mathscr{S}$. Therefore, we need to ensure that any XZ-homogeneous structure $\mathcal{M}$ we construct is also XY-homogeneous for the appropriate Y; so results in Subsection 7.2.1 should be satisfied by $\mathcal{M}$.

As mentioned previously, new properties are required to take care of extension and amalgamation in the backwards direction to ensure the map is surjective. This is achieved by looking at types of antihomomorphisms; which are denoted by $\overline{\text{X}}, \overline{\text{Y}} \in \{\overline{\text{H}}, \overline{\text{M}}, \bar{\text{I}}\}$. We must pair these together with standard homomorphisms to ensure the correct properties. For instance, if X = H then $\overline{\text{X}} = \overline{\text{H}}$, and so on; see Figure 7.3 for corresponding pairs. Throughout, we let X,Y $\in$ {H, M, I}, $\overline{\text{X}}, \overline{\text{Y}} \in \{\overline{\text{H}}, \overline{\text{M}}, \bar{\text{I}}\}$, and Z $\in$ {E, B, A}. To avoid any potential confusion, whenever we refer to a map of type Z being a surjective map of type Y, the symbol Z is always related to Y in the manner illustrated in $\mathscr{S}$ (see Equation 7.1), in the sense that (Z,Y) $\in \mathscr{S}$.

We proceed by stating our new amalgamation property to accommodate the back portion of a back-and-forth argument; this is the *$\overline{\text{XY}}$-amalgamation property*:

> ($\overline{\text{XY}}$AP) Let $\mathscr{C}$ be a class of finite structures. We say that $\mathscr{C}$ has the $\overline{\text{XY}}$AP if for all $A, B_1, B_2 \in \mathscr{C}$, multifunction $\bar{f}_1 : A \to B_1$ of type $\overline{\text{X}}$ and embedding $f_2 : A \to B_2$, there exists a $D \in \mathscr{C}$, embedding $g_1 : B_1 \to D$ and multifunction $\bar{g}_2 : B_2 \to D$ of type $\overline{\text{Y}}$ such that $\bar{f}_1 g_1 = f_2 \bar{g}_2$ (see Figure 7.4).

$$\begin{tikzcd}
& \exists D & \\
B_1 \arrow[ur, hook, dotted, "g_1"] & & B_2 \arrow[ul, dotted, "\bar{g}_2"'] \\
& A \arrow[ul, "\bar{f}_1"] \arrow[ur, hook, "f_2"'] &
\end{tikzcd}$$

Figure 7.4: The $\overline{\text{XY}}$-amalgamation property ($\overline{\text{XY}}$AP)

Note that this property represents nine different amalgamation conditions. This corresponds to one for each class XZ $\in \mathfrak{B}$, where (Z,Y) $\in \mathscr{S}$ (see Equation 7.1 on the previous page) and X and $\overline{\text{X}}$ are related as in Figure 7.3. For examples, the $\overline{\text{I}}$IAP is the standard amalgamation property, and the $\overline{\text{M}}$MAP is the BAP of [17]. Along similar lines, we can state the *$\overline{\text{XY}}$-extension property*:

> ($\overline{\text{XY}}$EP) Suppose that $\mathcal{M}$ is a structure with age $\mathscr{C}$. For all $A \subseteq B \in \mathscr{C}$ and a multifunction $\bar{f} : A \to \mathcal{M}$ of type $\overline{\text{X}}$, there exists a multifunction $\bar{g} : B \to \mathcal{M}$ of type $\overline{\text{Y}}$ extending $\bar{f}$.

We now turn our attention to necessary and sufficient conditions for XZ-homogeneity, to be used throughout the proof of Theorem 7.0.2. As stated above, we need to ensure that any XZ-homogeneous structure we construct is also XY-homogeneous for the appropriate Y. It follows that such a structure must satisfy all the conditions outlined in Proposition 7.2.1; in particular, XY must be in $\mathfrak{I}$ for part (2). With these restrictions in mind, and a desire to obtain the most general result possible, we show that both the XYEP and $\overline{\text{XY}}$EP are necessary conditions for XZ-homogeneity in general, and that it these are also sufficient when the extended map of type Y is also a map of type X.

Proposition 7.2.6. *Let $\mathcal{M}$ be a σ-structure with age $\mathscr{C}$.*

(1) Suppose that XZ $\in \mathfrak{B}$. If $\mathcal{M}$ is XZ-homogeneous, then $\mathcal{M}$ has both the XYEP and the $\overline{\text{XY}}$EP.

(2) Suppose that XZ $\in \mathfrak{B} \cap \mathfrak{I}$. If $\mathcal{M}$ has the XYEP and the $\overline{\text{XY}}$EP, then $\mathcal{M}$ is XZ-homogeneous.

Proof. (1) As $\mathcal{M}$ is XZ-homogeneous, it is also XY-homogeneous and so it has the XYEP by Proposition 7.2.1 (1). Now, suppose that $A, B \in \mathscr{C}$ and $\bar{f} : A \to \mathcal{M}$ is a multifunction of type $\overline{\text{X}}$. As $\mathscr{C}$ is the age of $\mathcal{M}$, it follows that $\mathcal{M}$ contains copies $A' \subseteq B'$ of A and B and there are isomorphisms $\theta : B \to B'$ and $\theta^{-1} : B' \to B$. Restrict the codomain of $\bar{f}$ to its image to find a map $\bar{f}' : A \to A\bar{f}$; as this is a surjective multifunction of type $\overline{\text{X}}$, we have that $\theta^{-1}\bar{f}' = h^* : A' \to A\bar{f}$ is also a surjective multifunction of type $\overline{\text{X}}$. By Lemma 7.1.2, the converse $h : A\bar{f} \to A'$ of $\theta|_A^{-1}\bar{f}'$ is a surjective map of type X; as $\mathcal{M}$ is XZ-homogeneous, extend h to a map $\beta : \mathcal{M} \to \mathcal{M}$ of type Z. So $\beta\theta^{-1} : \mathcal{M} \to B$ is a surjective map of type Y; by Corollary 7.1.3, define $\bar{g} = \theta\bar{\beta} : B \to \mathcal{M}$ to be the corresponding surjective multifunction of type $\overline{\text{Y}}$. We need to show it extends $\bar{f}$. As β extends h, then $\bar{\beta}$ extends h^*. So for all $a \in A$:

$$a\bar{f} = a\theta\theta^{-1}\bar{f} = a\theta h^* = a\theta\bar{\beta}$$

and hence $\mathcal{M}$ has the $\overline{\text{XY}}$EP.

(2) Now suppose that XZ $\in \mathfrak{B} \cap \mathfrak{I}$; so a multifunction of type $\overline{\text{Y}}$ implies that it is also a multifunction of type $\overline{\text{X}}$. Suppose also that $\mathcal{M}$ has the XYEP and the $\overline{\text{XY}}$EP, and that $f : A \to B$ is a map of type X between substructures of $\mathcal{M}$. We use a back-and-forth argument to show that $\mathcal{M}$ is XZ-homogeneous.

Set $A = A_0$, $B = B_0$ and $f_0 = f$, and assume that we have extended f to a surjective map $f_k : A_k \to B_k$ of type Y (and hence of type X, by assumption), where $A_i \subseteq A_{i+1}$ and $B_i \subseteq B_{i+1}$ for all $0 \leq i \leq k-1$. Furthermore, as $\mathcal{M}$ is countable we can enumerate the elements of $M = \{m_0, m_1, ...\}$.

If k is even, select a point $m_i \in \mathcal{M} \setminus A_k$ where i is the smallest number such that $m_i \notin A_k$, so $A_k \cup \{m_i\} \subseteq \mathcal{M}$. Using the XYEP, extend f_k to a map $f'_{k+1} : A_k \cup \{m_i\} \to B'_k$ of type Y; by restricting the codomain of f'_{k+1} to its image, it follows that $f_{k+1} : A_k \cup \{m_i\} \to B_k \cup \{m_i f'_{k+1}\}$ is a surjective map of type Y extending f_k.

If k is odd, we select a point $m_i \in \mathcal{M} \setminus B_k$ where i is the smallest number such that $m_i \notin B_k$; so $B_k \cup \{m_i\} \subseteq \mathcal{M}$. Note that as f_k is a surjective map of type X, we have that $\bar{f}_k : B_k \to A_k$ is a surjective multifunction of type $\overline{\text{X}}$. Using the $\overline{\text{XY}}$EP, extend $\bar{f}$ to a multifunction $\bar{f}'_{k+1} : B_k \cup \{m_i\} \to \mathcal{M}$ of type $\overline{\text{Y}}$. Restricting the codomain of $\bar{f}_{k+1}$ to its image gives a surjective multifunction $\bar{f}_{k+1} : B_k \cup \{m_i\} \to A_k \cup m_i\bar{f}_{k+1}$ of type $\overline{\text{Y}}$, where $m_i\bar{f}_{k+1} = \{y \in \mathcal{M} : (y, m_i) \in \bar{f}_{k+1}\}$ is a non-empty set. As $\bar{f}_{k+1}$ is a surjective multifunction of type $\overline{\text{Y}}$, we have that $f_{k+1} : A_k \cup m_i\bar{f}_{k+1} \to B_k \cup \{m_i\}$ is a surjective map of type Y extending f_k.

Since XZ $\in \mathfrak{B} \cap \mathfrak{I}$, a map of type Y is also a map of type X; so we can use the XYEP and $\overline{\text{XY}}$EP to repeat this process infinitely many times. By ensuring that each point of $\mathcal{M}$ appears at both an odd and even step, we extend f to a surjective map β of type Y; which is of course a map of type Z and so $\mathcal{M}$ is XZ-homogeneous. □

Remark. Together, Proposition 7.2.1 and Proposition 7.2.6 re-prove [53, Lemma 1.1], which states that a countable structure $\mathcal{M}$ is II (MI, HI)-homogeneous if and only if it is IA (MA, HA)-homogeneous. For if a structure $\mathcal{M}$ is HI-homogeneous, then it has the HIEP by Proposition 7.2.1; this implies that every homomorphism between finite substructures of $\mathcal{M}$ is an isomorphism. Since this happens, it follows that every antihomomorphism between finite substructures of $\mathcal{M}$ is an isomorphism. Finally, as $\mathcal{M}$ has the HIEP it must have the $\overline{\text{HI}}$EP as well and so $\mathcal{M}$ is HA-homogeneous by Proposition 7.2.6. A similar argument works for the equality concerning MI-homogeneous structures. In the II case, the IIEP is the standard extension property (EP) from Proposition 2.3.10, and so any structure $\mathcal{M}$ with the IIEP is homogeneous by the same result.

We now prove Theorem 7.0.2 (1).

Proposition 7.2.7. *Suppose that XZ $\in \mathfrak{B}$. If a structure $\mathcal{M}$ is XZ-homogeneous, then Age($\mathcal{M}$) has the XYAP and the $\overline{\mathrm{XY}}$AP.*

Proof. As $\mathcal{M}$ is XZ-homogeneous then it is XY-homogeneous and so has the XYAP by Proposition 7.2.2. To show that Age($\mathcal{M}$) has the $\overline{\mathrm{XY}}$AP, suppose that $A, B_1, B_2 \in$ Age($\mathcal{M}$), $\bar{f}_1 : A \to B_1$ is a multifunction of type $\overline{\mathrm{X}}$ and $f_2 : A \to B_2$ is an embedding. We can assume without loss of generality that A, B_1, B_2 are actually substructures of $\mathcal{M}$ and that f_2 is the inclusion mapping.

By restricting the codomain of $\bar{f}_1$ to its image, $\bar{f}_1 : A \to A\bar{f}_1$ is a surjective multifunction of type $\overline{\mathrm{X}}$; hence the converse $f_1 : A\bar{f}_1 \to A$ of $\bar{f}_1$ is a surjective map of type X. Use XZ-homogeneity to extend f_1 to a map $\beta : \mathcal{M} \to \mathcal{M}$ of type Z; and so a surjective map of type Y. We see that $B_1\beta$ is a structure containing A, and that $\beta|_{B_1} : B_1 \to B_1\beta$ extends f_1. Define $D = B_1\beta \cup B_2$. As β is surjective, there exists a finite substructure C such that $C\beta = D$ by Lemma 7.1.5. Now, define the map $g_1 : B_1 \to C$ to be the inclusion map. Since β is a surjective map of type Y, $\bar{\beta} : \mathcal{M} \to \mathcal{M}$ is a surjective multifunction of type $\overline{\mathrm{Y}}$ by Lemma 7.1.2. Therefore $\bar{\beta}|_{B_2} : B_2 \to B_2\bar{\beta}$ is a surjective multifunction of type $\overline{\mathrm{Y}}$; furthermore, $B_2\bar{\beta} \subseteq C$ as $B_2 \subseteq D$. Define $\bar{g}_2 : B_2 \to C$ to be the multifunction $\bar{\beta}|_{B_2}$ of type $\overline{\mathrm{Y}}$. It is easy to check that $\bar{f}_1 g_1 = f_2 \bar{g}_2$ and so Age($\mathcal{M}$) has the $\overline{\mathrm{XY}}$AP. □

We now show the existence portion of Theorem 7.0.2. Note that the previously described inductive construction of an infinite structure in Proposition 7.2.3 used even and odd steps to achieve different stages of the construction at different times. Because we have two amalgamation properties, as well as the JEP to ensure a countable structure exists, we proceed using an inductive argument at steps congruent to 0,1,2 mod 3 to accommodate different stages of the construction.

Proposition 7.2.8. *Suppose that XZ $\in \mathfrak{B} \cap \mathfrak{I}$. Let $\mathscr{C}$ be a class of finite σ-structures that is closed under substructures and isomorphism, has countably many isomorphism types and has the JEP, XYAP and the $\overline{\mathrm{XY}}$AP. Then there exists a XZ-homogeneous σ-structure $\mathcal{M}$ with age $\mathscr{C}$.*

Proof. We build the structure iteratively over countably many steps, achieving different goals at each stage of the construction. Assume that M_k has been constructed for some $k \in \mathbb{N}$, and as $\mathscr{C}$ has countably many isomorphism types, we can enumerate $\mathscr{C} = \{C_0, C_1, ...\}$.

If $k \equiv 0$ mod 3, select $A_i \in \mathscr{C}$ such that $k = 3i$. Since $\mathscr{C}$ has the JEP, we can find $D \in \mathscr{C}$ that jointly embeds M_k and A_i; define this structure D to be M_{k+1}. If $k \equiv 1$ mod 3, select a triple (A, B, f) where $A \subseteq B \in \mathscr{C}$ and $f : A \to M_k$. Using the fact that $\mathscr{C}$ has the XYAP, we can find a structure $M_{k+1} \in \mathscr{C}$, embedding $e_k : M_k \to M_{k+1}$ and map $g : B \to M_{k+1}$ of type Y extending f. If $k \equiv 2$ mod 3, then select a triple $(P, Q, \bar{f})$, where $P \subseteq Q \in \mathscr{C}$ and $\bar{f} : P \to M_k$ is a multifunction of type $\overline{\text{X}}$. As $\mathscr{C}$ has the $\overline{\text{XY}}$AP, we can find a structure $M_{k+1} \in \mathscr{C}$, embedding $e_k : M_k \to M_{k+1}$ and a multifunction $\bar{g} : Q \to M_{k+1}$ of type $\overline{\text{Y}}$ extending $\bar{f}$.

As $\mathscr{C}$ has countably many isomorphism types, there are only finitely many maps $f : A \to M_k$ of type X at each stage, and there are finitely many multifunctions $\bar{f} : P \to M_k$ of type $\overline{\text{X}}$ at each stage, we can arrange the steps such that:

- every structure $A_i \in \mathscr{C}$ appears at a 0 mod 3 stage;
- every triple (A, B, f) appears at a $k \equiv 1$ mod 3 stage, where for every such k, for every $A \subseteq B \in \mathscr{C}$ and every map $f : A \to M_k$ of type X, there exists $\ell \geq k$ and embedding $e_{k,\ell} : M_k \to M_\ell$ such that f extends to a map $g : B \to M_\ell$ of type Y.
- every triple $(P, Q, \bar{f})$ appears at a $k \equiv 2$ mod 3 stage, where for every such k, for every $P \subseteq Q \in \mathscr{C}$ and every multifunction $\bar{f} : P \to M_k$ of type $\overline{\text{X}}$ there exists $\ell \geq k$ and embedding $e_{k,\ell} : M_k \to M_\ell$ such that $\bar{f}$ extends to a multifunction $\bar{g} : Q \to M_\ell$ of type $\overline{\text{Y}}$.

Arranging the steps this way ensures that every possible amalgamation is performed. Since $M_k \subseteq M_{k+1}$ for all $k \in \mathbb{N}$ via the embedding e_k, we can define $\mathcal{M} = \bigcup_{k \in \mathbb{N}} M_k$. All that remains to show is that $\mathcal{M}$ has age $\mathscr{C}$ and that $\mathcal{M}$ is XZ-homogeneous.

As we ensured that every $A_i \in \mathscr{C}$ was embedded into M_{k+1} at some 0 mod 3 stage, we have that $\mathscr{C} \subseteq \mathrm{Age}(\mathcal{M})$. At any stage k, our construction ensured that each $M_k \in \mathscr{C}$; as $\mathscr{C}$ is closed under substructures, $\mathrm{Age}(\mathcal{M}) \subseteq \mathscr{C}$ and so they are equal. As XZ $\in \mathfrak{B} \cap \mathfrak{I}$, it is enough to show that $\mathcal{M}$ has the XYEP and the $\overline{\text{XY}}$EP by Proposition 7.2.6 (2). So suppose that $A \subseteq B \in \mathscr{C}$ and that $f : A \to \mathcal{M}$ is a map of type X. From the arrangement of steps above, there exists a k such that $Af \subseteq M_k$. From the construction, there exists an $M_\ell \supseteq M_k$ and a map $g : B \to M_\ell$ of type Y extending f; so $\mathcal{M}$ has the XYEP. Similarly, let $P \subseteq Q \in \mathscr{C}$ and suppose that $\bar{f} : P \to \mathcal{M}$ is a multifunction of type $\overline{\text{X}}$. Then there exists $k \in \mathbb{N}$ such that $P\bar{f} \subseteq M_k$. Therefore, there exists an $\ell \in \mathbb{N}$ with $M_\ell \supseteq M_k$ such that $\bar{f} : P \to M_k$ extends to a multifunction $\bar{g} : Q \to M_\ell$ of type $\overline{\text{Y}}$. So $\mathcal{M}$ has the $\overline{\text{XY}}$EP and therefore $\mathcal{M}$ is XZ-homogeneous. □

Finally, we show part (3) of Theorem 7.0.2. Using the fact that XZ-homogeneous structures have two extension properties, we can ensure that a map between two of them is surjective by using a back-and-forth argument. This motivates a new definition, building on that of Y-equivalence.

Definition 7.2.9. Let $\mathcal{M}, \mathcal{N}$ be σ-structures, and suppose that Z $\in$ {E, B, A} corresponds to the surjective map of type Y $\in$ {H, M, I} by the relation in Equation 7.1 (on page 153). Say that $\mathcal{M}$ and $\mathcal{N}$ are *Z-equivalent* if $\mathrm{Age}(\mathcal{M}) = \mathrm{Age}(\mathcal{N})$ and every embedding $f : A \to \mathcal{N}$ from a finite substructure A of $\mathcal{M}$ into $\mathcal{N}$ extends to a surjective map $g : \mathcal{M} \to \mathcal{N}$ of type Y, and vice versa.

For an example, $\mathcal{M}, \mathcal{N}$ are B-equivalent means that they are bi-equivalent in the sense of [17]. Note that if two structures $\mathcal{M}$ and $\mathcal{N}$ are Z-equivalent, then they are also Y-equivalent where (Z,Y) $\in \mathscr{S}$ (from Equation 7.1).

Proposition 7.2.10. *Suppose that XZ* $\in \mathfrak{B} \cap \mathfrak{I}$*.*

(1) Assume that $\mathcal{M}, \mathcal{N}$ are Z-equivalent. Then $\mathcal{M}$ is XZ-homogeneous if and only if $\mathcal{N}$ is.

(2) If $\mathcal{M}, \mathcal{N}$ are XZ-homogeneous and $\mathrm{Age}(\mathcal{M}) = \mathrm{Age}(\mathcal{N})$, then $\mathcal{M}$ and $\mathcal{N}$ are Z-equivalent.

Proof. (1) As $\mathcal{M}, \mathcal{N}$ are Z-equivalent they are also Y-equivalent and as $\mathcal{M}$ is also XY-homogeneous, so is $\mathcal{N}$ by Proposition 7.2.5; so $\mathcal{N}$ has the XYEP by Proposition 7.2.1. We show now that $\mathcal{N}$ has the $\overline{\text{XY}}$EP. Suppose then that $A \subseteq B \in$ Age($\mathcal{N}$) and there exists a multifunction $\bar{f} : A \to A' \subseteq \mathcal{N}$ of type $\overline{\text{X}}$. Note that A need not be isomorphic to A'. As Age($\mathcal{M}$) = Age($\mathcal{N}$) there exists a copy A'' of A' in $\mathcal{M}$; fix an isomorphism $e : A' \to A''$ between the two. Therefore, e is a isomorphism from a finite structure of $\mathcal{N}$ into $\mathcal{M}$; as the two are Z-equivalent, we extend this to a surjective map $\alpha : \mathcal{N} \to \mathcal{M}$ of type Y. This in turn induces a surjective map $\bar{\alpha} : \mathcal{M} \to \mathcal{N}$ by Lemma 7.1.2. Note that $\bar{\alpha}$ extends the isomorphism $e^{-1} : A'' \to A'$. Now, define a multifunction $\bar{h} = \bar{f}e : A \to A''$ of type $\overline{\text{X}}$; this is a multifunction of type $\overline{\text{X}}$ from A into $\mathcal{M}$. Since $\mathcal{M}$ is XZ-homogeneous, it has the $\overline{\text{XY}}$EP by Proposition 7.2.6 and so we extend $\bar{h}$ to a multifunction $\bar{h}' : B \to \mathcal{M}$ of type $\overline{\text{Y}}$. Here, the multifunction $\bar{h}'\bar{\alpha} : B \to \mathcal{N}$ is also of type $\overline{\text{Y}}$; we need to show it extends $\bar{f}$. As $\bar{h}'$ extends $\bar{h} = \bar{f}e$, it follows that:

$$a\bar{f} = a\bar{f}ee^{-1} = a\bar{f}e\bar{\alpha} = a\bar{h}'\bar{\alpha}$$

for all $a \in A$. Therefore $\mathcal{N}$ has the XZEP.

(2) It is enough to show that $\mathcal{N}$ has the XYEP and the $\overline{\text{XY}}$EP by Proposition 7.2.6. We utilise a back and forth argument constructing the surjective map over infinitely many stages. Let $f : A \to B$ be a bijective embedding from a finite structure $A \subseteq \mathcal{M}$ to a finite substructure $B \subseteq \mathcal{N}$. Set $A = A_0$, $B = B_0$ and $f = f_0$ and assume that $f_k : A_k \to B_k$ is a surjective map of type Y (and so of type X by assumption) extending f_k. Note that as both $\mathcal{M}$ and $\mathcal{N}$ are countable, then there exists enumerations $\mathcal{M} = \{m_0, m_1, ...\}$ and $\mathcal{N} = \{n_0, n_1, ...\}$.

If k is even, select a $m_i \in \mathcal{M} \setminus A_k$, where i is the smallest natural number such that $m_i \notin A_k$. So $A_k \cup \{m_i\} \subseteq \mathcal{M}$, and is also in Age($\mathcal{N}$) by assumption. As $\mathcal{N}$ is XZ-homogeneous it has the XYEP by Proposition 7.2.1 and we use this to extend f_k to a map $f'_{k+1} : A_k \cup \{m_i\} \to \mathcal{N}$ of type Y. Restricting the codomain of f'_{k+1} to its image yields a surjective map $f_{k+1} : A_k \cup \{m_i\} \to B_k \cup \{m_i f_{k+1}\}$ of type Y. If k is odd, select a $n_i \in \mathcal{N} \setminus B_k$ such that i is the smallest natural number such that $n_i \notin B_k$. Hence $B_k \cup \{n_i\} \subseteq \mathcal{N}$ and thus it is an element of Age($\mathcal{M}$) by assumption. As f_k is a surjective map of type Y, its converse $\bar{f}_k : B_k \to A_k$ is a surjective multifunction of type $\overline{\text{Y}}$ by Lemma 7.1.2, and of type $\overline{\text{X}}$ by assumption. As $\mathcal{M}$ is XZ-homogeneous it has the $\overline{\text{XY}}$EP and so we can extend $\bar{f}_k$ to a multifunction $\bar{f}'_{k+1} : B_k \cup \{n_i\} \to \mathcal{M}$ of type $\overline{\text{Y}}$. By restricting the codomain of $\bar{f}'_{k+1}$ to its image, we obtain a surjective multifunction $\bar{f}_{k+1} : B_k \cup \{n_i\} \to A_k \cup n_i\bar{f}'_{k+1}$ of type $\overline{\text{Y}}$, where $n_i\bar{f}'_{k+1} = \{(n_i, y) : y \in \mathcal{M}\}$ is a non-empty set. So by Lemma 7.1.2, there exists a surjective map $f_{k+1} : A_k \cup n_i\bar{f}'_{k+1} \to B_k \cup \{n_i\}$ of type Y extending f_k. By our earlier assumption, as a map of type Y is also a map of type X, we can repeat this process infinitely many times. By ensuring all points in $\mathcal{M}$ appear at even stages and all points in $\mathcal{N}$ appear at odd stages, we construct a surjective map $\alpha : \mathcal{M} \to \mathcal{N}$ of type Y as required. We can use a similar method to show that we can extend any embedding $g : A \to B$ where $A \in \mathcal{N}$ and $B \in \mathcal{M}$ to a surjective map of type Y; proving that $\mathcal{M}$ and $\mathcal{N}$ are Z-equivalent. □

Note that unlike Theorem 7.0.1, the use of antihomomorphisms means that we cannot just drop the assumption that σ is a relational signature; this is because antihomomorphisms are defined on relational structures only. The only time we can do this is if we are dealing with isomorphisms, as the converse $\bar{f}$ of an isomorphism f is the inverse isomorphism f^{-1}. This means that Theorem 7.0.2 works for any first-order structure in the case where XZ = IA; this is Fraïssé's theorem. This remark motivates an open question.

Question 7.2.11. *Can we expand Theorem 7.0.2 to include all first-order structures?*

Of course, the other open problem that arises from this section is:

Question 7.2.12. *Can we expand Theorem 7.0.1 and Theorem 7.0.2 to include those homomorphism-homogeneity classes in $\mathfrak{B}$?*

7.3 Maximal homomorphism-homogeneity classes

This section is devoted to determining the extent to which well known examples of homogeneous structures are also homomorphism-homogeneous. In some cases, verifying that a structure $\mathcal{M}$ is homogeneous involves using a property of $\mathcal{M}$ to determine that $\mathcal{M}$ has the EP, and so is homogeneous by Proposition 2.3.10. Good examples of such properties are the density of $(\mathbb{Q}, <)$, and the ARP characteristic of R (see Figure 2.6 and Proposition 2.4.3). In the homomorphism-homogeneity case, this idea was used by Cameron and Lockett [13] and Lockett and Truss [53] to classify homomorphism-homogeneous posets and determine their position relative to the natural containment order on $\mathbb{H}$. In addition to this, Dolinka [23] used properties of known homogeneous structure to show that they satisfied the *one-point homomorphism extension property* (1PHEP), a necessary and sufficient condition for HH-homogeneity. Our approach in this section is similar to that of Section 3 of [23]; by defining necessary and sufficient conditions for XY, XZ-homogeneity and using properties of structures to show that these are satisfied or not satisfied. As in Subsection 7.2.2, we let X,Y $\in$ {H, M, I}, $\overline{\text{X}}, \overline{\text{Y}} \in \{\overline{\text{H}}, \overline{\text{M}}, \overline{\text{I}}\}$, and Z $\in$ {E, B, A} throughout this section. Furthermore, the pair (Z,Y)$\in \mathscr{S}$ is related as in Equation 7.1.

So to begin this section, we define the *one-point XY-extension property*, and the *one-point $\overline{XY}$-extension property*:

> (1PXYEP) We say that a σ-structure $\mathcal{M}$ with age $\mathscr{C}$ has the 1PXYEP if for all $A \subseteq B \in \mathscr{C}$ with $|B \setminus A| = 1$ and maps $f : A \to \mathcal{M}$ of type X, there exists a map $g : B \to \mathcal{M}$ of type Y extending f.
>
> (1P$\overline{\text{XY}}$EP) Suppose that $\mathcal{M}$ is a σ-structure with age $\mathscr{C}$. Say that $\mathcal{M}$ has the 1P$\overline{\text{XY}}$EP if for all $A \subseteq B \in \mathscr{C}$ with $|B \setminus A| = 1$, and a multifunction $\bar{f} : A \to \mathcal{M}$ of type $\overline{\text{X}}$, there exists a multifunction $\bar{g} : B \to \mathcal{M}$ of type $\overline{\text{Y}}$ extending $\bar{f}$.

For an example, the 1PHHEP is the same thing as the 1PHEP of [23]. These properties, together with the next proposition, provide some of the theoretical basis for the examples that follow.

Proposition 7.3.1. *Suppose that XY $\in \mathfrak{I}$. A countable σ-structure $\mathcal{M}$ has the XYEP / $\overline{\text{XY}}$EP if and only if it has the 1PXYEP / 1P$\overline{\text{XY}}$EP.*

Proof. The forward direction for both the XYEP and $\overline{\text{XY}}$EP cases is clear. We now aim to show that if $\mathcal{M}$ has the 1PXYEP then $\mathcal{M}$ has the XYEP. Assume that $A \subseteq B$ with $|B \smallsetminus A| = n$, and $f : A \to \mathcal{M}$ is a map of type X. We prove the result by induction on the size of this complement; the base case (where n = 1) is true by the assumption that $\mathcal{M}$ has the 1PXYEP.

So suppose that for some $k \in \mathbb{N}$, for any $A \subseteq B \in \mathscr{C}$ where $|B \smallsetminus A| = k$ and any map $f : A \to \mathcal{M}$ of type X can be extended to a map $g : B \to \mathcal{M}$ of type Y. Take $P \subseteq Q \in \mathscr{C}$ where $|Q \smallsetminus P| = k + 1$ and $f' : P \to \mathcal{M}$ to be some map of type X. There exists $S \in \mathscr{C}$ containing P such that $|Q \smallsetminus S| = 1$. By the inductive hypothesis, we can extend f' to a map $h : S \to \mathcal{M}$ of type Y. As XY $\in \mathfrak{I}$, it follows that h is also a map of type X. Now, using the 1PXYEP, extend h to a map $g' : Q \to \mathcal{M}$ of type Y. Since $P \subseteq S \subseteq Q$ and g' extends h which extends f', we have that g' extends f' and so we are done. Using a similar argument, we can show that if $\mathcal{M}$ has the 1P$\overline{\text{XY}}$EP then it has the $\overline{\text{XY}}$EP. □

Remark. Let XY $\in \mathfrak{I}$. Together with Proposition 7.2.1, this lemma states that a countable structure $\mathcal{M}$ has the 1PXYEP if and only if $\mathcal{M}$ is XY-homogeneous. Similarly, by Proposition 7.2.6 a countable structure $\mathcal{M}$ is XZ-homogeneous if and only if it has the 1PXYEP and the 1P$\overline{\text{XY}}$EP, where (Z,Y) are as in $\mathscr{S}$ (Equation 7.1).

By considering properties of partial maps and endomorphisms of structures, our next result places restrictions on certain types of homomorphism-homogeneity. The latter consideration looks at structures known as *cores*; a structure $\mathcal{M}$ is a core if every endomorphism of $\mathcal{M}$ is an embedding. Every $\aleph_0$-categorical structure $\mathcal{M}$ contains a core as the image of an endomorphism of $\mathcal{M}$, and $\mathcal{M}$ is homomorphically equivalent to a model-complete core [4]. Cores play an important role in the theory of constraint satisfaction problems; see [5] for a introduction to the topic. Widely studied examples of cores include the countable dense linear order without endpoints $(\mathbb{Q}, <)$, the complete graph on countably many vertices $K^{\aleph_0}$ and its complement $\bar{K}^{\aleph_0}$. We note that in these three cases every finite partial monomorphism of these structures is an isomorphism, and in the cases of $(\mathbb{Q}, <)$ and $K^{\aleph_0}$ every homomorphism between finite structures is an isomorphism. This straightforward result includes a restatement of Lemma 1.1 of [53].

Lemma 7.3.2. *(1) A structure $\mathcal{M}$ is MI and MA-homogeneous (HI and HA-homogeneous) if and only if $\mathcal{M}$ is IA-homogeneous and every finite partial monomorphism (homomorphism) of $\mathcal{M}$ is an isomorphism.*

(2) If a structure $\mathcal{M}$ is HM or HB-homogeneous, then every finite partial homomorphism of $\mathcal{M}$ is also a monomorphism.

(3) Let $\mathcal{M}$ be a core. If there exists a finite partial monomorphism of $\mathcal{M}$ that is not an isomorphism, then $\mathcal{M}$ is not MH-homogeneous.

Proof. (1) is contained in Lemma 1.1 of [53]; notice that we cannot extend a map that is not a partial isomorphism of $\mathcal{M}$ to an isomorphism of the entire structure $\mathcal{M}$. The converse direction is clear. To show (2), note that if h is a finite partial homomorphism of $\mathcal{M}$ that is not injective, then we cannot possibly extend this to an injective map and so $\mathcal{M}$ does not have the HMEP. For (3), let h be a finite partial monomorphism of a core $\mathcal{M}$ that is not an isomorphism. As any endomorphism of $\mathcal{M}$ is an embedding, we cannot extend h. $\square$

Remark. Note that (1) and (2) also follow from Theorem 7.0.1 and Theorem 7.0.2.

Following the approach of [53] in classifying homomorphism-homogeneous posets, the idea of this section is to look at properties of structures to determine "maximal" homomorphism-homogeneity classes with respect to the containment order on $\mathfrak{H}$. We formally define what we mean by "maximal".

Definition 7.3.3. Let $\mathcal{M}$ be a first order structure. A homomorphism-homogeneity class XY $\in \mathbb{H}$ is *maximal* for $\mathcal{M}$ if $\mathcal{M}$ is XY-homogeneous and $\mathcal{M}$ is not PQ-homogenenous, where PQ $\subseteq$ XY in $\mathbb{H}$. If this happens, we say that XY is a *maximal homomorphism-homogeneity class* (shortened to *maximal hh-class*) for $\mathcal{M}$.

Remark. While this definition describes a *minimal* element in the poset $\mathbb{H}$, it is so named because of the strengths of different notions of homomorphism-homogeneity. For instance, HA-homogeneity is a stronger condition than IA-homogeneity, but HA $\subseteq$ IA in $\mathbb{H}$. In fact, there is an inverse correspondence between the relative strength of notions of homomorphism-homogeneity in $\mathfrak{H}$ and containment of classes in $\mathbb{H}$.

For example, if $\mathcal{M}$ is MB-homogeneous but not MA or HB-homogeneous, then MB is a maximal hh-class for $\mathcal{M}$. A structure $\mathcal{M}$ may have more than one maximal hh-class. Furthermore, the set of maximal hh-classes for $\mathcal{M}$ completely determines the extent of homomorphism-homogeneity satisfied by $\mathcal{M}$; we therefore denote this set by $\mathbb{H}(\mathcal{M})$. As an example $\mathbb{H}((\mathbb{Q}, <)) = \{\text{HA}\}$; this example arose from the classification of homomorphism-homogeneous posets in [53].

If $\mathcal{M}$ is a countable σ-structure where there exists a finite partial monomorphism of $\mathcal{M}$ that is not an isomorphism, and a finite partial homomorphism of $\mathcal{M}$ that is not an monomorphism, then Lemma 7.3.2 implies that the "best possible" maximal hh-classes for $\mathcal{M}$ are IA, MB and HE. As an aside, these classes have important roles to play in the theory of generic endomorphisms [52].

7.3.1 Examples

In this section, we look at a selection of countable homogeneous structures encountered throughout the thesis in order to determine sets of maximal-hh classes for these structures. By restricting ourselves to classes XY $\in \mathfrak{I}$, we can recall Proposition 7.3.1 and the remark that follows it; to show that $\mathcal{M}$ is XY-homogeneous it suffices to show that $\mathcal{M}$ has the 1PXYEP, and to show that $\mathcal{M}$ is XZ-homogeneous it suffices to show that it has the 1PXYEP and the 1P$\overline{\text{XY}}$EP.

Example 7.3.4. It is well-known (see Theorem 2.4.10) that the complete graph on countably many vertices $K^{\aleph_0}$ is homogeneous. Suppose that $h : A \to B$ is a homomorphism between two finite substructures of $K^{\aleph_0}$. Then as h preserves edges, it cannot send two distinct vertices $x_1, x_2 \in VA$ to a single point $v \in VB$;

hence h is injective. As there are no non-edges to preserve, it must preserve non-edges and so h is an embedding. It follows from Lemma 7.3.2 (1) that $K^{\aleph_0}$ is HA-homogeneous and so $\mathbb{H}(K^{\aleph_0}) = \{\text{HA}\}$.

Its complement $\bar{K}^{\aleph_0}$, the infinite null graph, is also homogeneous and as every finite partial monomorphism of $\bar{K}^{\aleph_0}$ preserves non-edges, it is MA-homogeneous by Lemma 7.3.2 (1). We note that there exist non-injective finite partial homomorphisms of $\bar{K}^{\aleph_0}$ and hence it is not HM or HB-homogeneous by Lemma 7.3.2 (3). So if $h : A \to B$ is any finite partial homomorphism, we can define a bijective map $g : \bar{K}^{\aleph_0} \smallsetminus A \to \bar{K}^{\aleph_0} \smallsetminus B$ and note that the map $\alpha : \bar{K}^{\aleph_0} \to \bar{K}^{\aleph_0}$ that acts like h on A and g everywhere else is an epimorphism of $\bar{K}^{\aleph_0}$; so $\bar{K}^{\aleph_0}$ is HE. Hence $\mathbb{H}(\bar{K}^{\aleph_0}) = \{\text{MA, HE}\}$.

Example 7.3.5. Recall from Section 2.4 that a *tournament* is defined to be an oriented, loopless complete graph. By a similar argument to the complete graph in Example 7.3.4, every finite partial homomorphism of a tournament is an embedding. It follows from Lemma 7.3.2 (1) that every countable homogeneous tournament is HA-homogeneous. Therefore, the three countable homogeneous tournaments as classified by Lachlan [48], namely $(\mathbb{Q}, <)$, the random tournament T, and the local order $S(2)$ (see Theorem 2.4.11), are all HA-homogeneous. So HA is the unique maximal hh-class for these three examples.

Example 7.3.6. Let R be the random graph (see Example 2.4.2). Note that there exist finite partial monomorphisms of R that are not isomorphisms and finite partial homomorphisms of R that are not monomorphisms; hence R is not MI or HM-homogeneous by Lemma 7.3.2. We proved that R is MB-homogeneous in Proposition 4.1.20; here, we show that R is HE-homogeneous. To do this, we rely on the ARP characteristic of R (see Proposition 2.4.3).

Let $A \subseteq B \in \text{Age}(R)$ with $B \smallsetminus A = \{b\}$ and suppose that $f : A \to R$ is a homomorphism. Using ARP, we can find a vertex $v \in VR$ such that v is adjacent to everything in $\text{im}\, f$. Let $g : B \to R$ be the function such that $bg = v$ and $g|_A = f$; this is a homomorphism as all edges from A to b are preserved and so R has the 1PHHEP. The proof to show that R has the 1P$\overline{\text{HH}}$EP is similar; in this case, we use ARP to find a vertex $w \in VR$ that is independent of the image of the some antihomomorphism $\bar{f} : A \to R$. The resulting multifunction g is an antihomomorphism as it preserves all non-edges. Therefore, R is HE-homogeneous by Proposition 7.3.1 and Proposition 7.2.6. We conclude that $\mathbb{H}(R) = \{$IA, MB, HE$\}$

Remark. It was shown in [52, Theorem 5.3] that R has a generic endomorphism. As R is HE-homogeneous, it follows from Theorem 2.1 of the same source that this generic endomorphism must be in Epi(R).

Example 7.3.7. As in Example 2.4.8, let D be the generic oriented graph. From that example, D has a characteristic extension property called the oriented Alice's restaurant property (OARP, see Figure 2.9). We show that D is MB-homogeneous using the OARP.

Suppose that $A \subseteq B \in$ Age(D) with $B \setminus A = \{b\}$ and that $f : A \to D$ is a monomorphism. Decompose A into three disjoint sets $b^{\rightarrow} = \{a \in A : b \to a\}$, $b^{\leftarrow} = \{a \in A : b \leftarrow a\}$ and $b^{\|} = \{a \in A : b \parallel a\}$. The injectivity of f means that the sets $b^{\rightarrow}f, b^{\leftarrow}f$ and $b^{\|}f$ are pairwise disjoint subsets of VD. Using the OARP, select a vertex $x \in VD$ such that x has an arc to all elements of $b^{\rightarrow}f$, an arc from all elements of $b^{\leftarrow}f$ and is independent of all elements of $b \parallel f$. Define $g : B \to D$ to be the map such that $bg = x$ and $g|_A = f$; due to our choice of x, this is a monomorphism and so D has the 1PMMEP. The proof that D has the 1P$\overline{\text{MM}}$EP also is analogous to the proof that R has the 1P$\overline{\text{MM}}$EP; hence D is MB-homogeneous.

However, D is not even HH-homogeneous. To see this, consider an endomorphism γ of D, and suppose there exists $v \parallel w \in VD$ such that $v\gamma = w\gamma$. As D is universal and homogeneous, there exists an oriented graph $A = \{v, w, x\}$ such that $x \to v$ and $w \to x$. The image of A under γ is a 2-cycle and this is a contradiction as D does not embed 2-cycles. It follows that every endomorphism of D is a monomorphism. As there exist finite partial homomorphisms of D that are not monomorphisms, by Lemma 7.3.2 (2) D is not HM-homogeneous and hence it is not HH-homogeneous. We conclude that the maximal hh-classes of D are IA and MB.

Example 7.3.8. As in Example 2.4.9, let D^* be the generic digraph. Recall that D^* has a characteristic extension property known as the directed Alice's restaurant property (DARP, see Figure 2.10). Using this, we show that D^* is MB and HE-homogeneous.

Let $A \subseteq B \in \text{Age}(D^*)$ with $B \smallsetminus A = \{b\}$ and suppose that $f : A \to D^*$ is a monomorphism. As im f is finite, we can use DARP to find a vertex $v \in VD^*$ such that there is a 2-cycle between v and every element in im f. Let $g : B \to D^*$ be the injective map such that $bg = v$ and $g|_A = f$; this is a monomorphism as all arcs from A to b are preserved. Therefore D^* has the 1PMMEP. The proof to show that D^* has the 1P$\overline{\text{MM}}$EP is similar; we use DARP to instead find a vertex $w \in VD^*$ that is independent of the finite set im f. The resulting injective map g is an antimonomorphism as it preserves all non-relations. Therefore, D^* is MB-homogeneous by Proposition 7.3.1 and Proposition 7.2.6.

As with Example 7.3.6, we can use a similar argument (by replacing monomorphism, antimonomorphism with homomorphism, antihomomorphism respectively) to show that D^* has the 1PHHEP and the 1P$\overline{\text{HH}}$EP, and so $\mathbb{H}(D^*) = \{$IA, MB, HE$\}$.

Remark. Note the difference between the maximal hh-classes of D, the generic digraph without 2-cycles, and D^*, the generic digraph with 2-cycles.

Example 7.3.9. Let G be the countable homogeneous K_n-free graph for $n \geq 3$. Results of Mudrinski [66] show that G is a core for all such n. Roughly, this is because if an endomorphism adds a single edge or shrinks a single non-edge down to a point, it creates a subgraph isomorphic to K_n. Hence, as there are finite partial monomorphisms of G that are not isomorphisms (sending any non-edge to an edge, for instance) we have that G is not MH-homogeneous by Lemma 7.3.2 (3) and so $\mathbb{H}(G) = \{$IA$\}$.

Example 7.3.10. As introduced in Theorem 2.4.11, the oriented graph analogues of the homogeneous K_n-free graphs are the *Henson digraphs* M_T. These are the Fraïssé limits of the class of all digraphs not embedding elements of some set T of finite tournaments. We show that M_T is a core for any non-empty T containing tournaments of three or more vertices.

Suppose for a contradiction there exists a $\gamma \in \text{End}(M_T)$ such that for some independent pair of vertices v, w we have that $v\gamma = w\gamma$. Select a tournament $Y \in T$ with the least number of vertices, and choose $x, y \in Y$ such that $x \to y$. Create a oriented graph Y' by removing the arc between x and y, adding an extra vertex x' and drawing an arc $x' \to y$; see Figure 7.5 for an example.

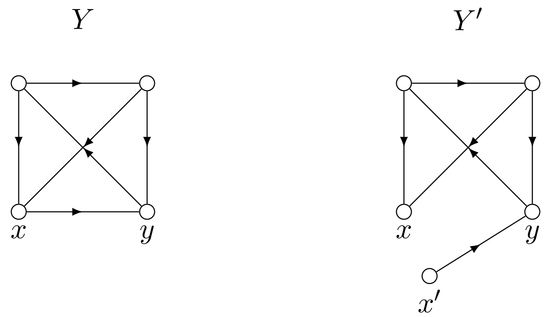


Figure 7.5: Construction of Y' from Y in Example 7.3.10

Note that there is no tournament on any $|Y|$-set of vertices of Y'; so $Y' \in$ Age(M_T). By homogeneity of M_T, we find a copy Y'' of Y' with v, w in place of x, x' respectively, and a vertex u in place of y. The image of Y'' under γ is a oriented graph on $|Y|$ vertices with $v\gamma \to u\gamma$, that preserves all arcs involving v and u. It follows that $Y''\gamma \cong Y$; this is a contradiction as Y does not belong to the age of M_T. So γ must be injective.

Now assume that there exists an independent pair of vertices $v, w \in M_T$ such that $v\gamma \to w\gamma$. Select a $Y \in T$ in the same fashion as before, choose two vertices $x \to y$ of Y and remove this arc to obtain a oriented graph Z that embeds in M_T. Via homogeneity, we find an isomorphic copy of Z with v, w in place of x, y respectively. Hence the image of Z under γ induces a copy of Y in M_T; a contradiction as Y does not belong to the age of M_T. So M_T is a core for any such set of tournaments T; therefore, by Lemma 7.3.2 (3), M_T is not MH-homogeneous. We deduce that the maximal hh-class of M_T is IA for any such T.

Example 7.3.11. Let $S(3)$ be the *myopic local order* introduced in Theorem 2.4.11. Define $S(3)$ as follows. Distribute $\aleph_0$ many points densely around the unit circle such that for every point a there are no points b and c such that $\arg(a, b) = \arg(b, c) = \arg(c, a) = \frac{2\pi}{3}$. Draw an arc $a \to b$ if and only if $\arg(a, b) < 2\pi/3$; note that this means that $S(3)$ embeds no directed 3-cycles. We show that this structure is a core.

Assume that there is an endomorphism γ of $S(3)$ with $a\gamma \to b\gamma$ for some independent pair of points $a, b \in S(3)$. As this occurs, both $\arg(a, b)$ and $\arg(b, a) > 2\pi/3$. As $\arg(a, b) = 2\pi - \arg(b, a)$, it follows that $\arg(b, a) < 4\pi/3$. From this, there exists a point c such that $\arg(b, c) = \arg(c, a) < 2\pi/3$ and so $b \to c$ and $c \to a$. The endomorphism γ then creates a directed 3-cycle (or a loop) and this is a contradiction. Now suppose that for some non-related pair $a, b \in S(3)$, there is an endomorphism γ such that $a\gamma = b\gamma$. As before, we can find a point $c \in S(3)$ such that $b \to c$ and $c \to a$. Since γ is an endomorphism, it must preserve these relations and so $b\gamma \to c\gamma$ and $c\gamma \to a\gamma = b\gamma$. This creates a directed 2-cycle and so is obviously false; therefore, $S(3)$ is a core. Applying Lemma 7.3.2 (3) again implies that $S(3)$ is not MH-homogeneous and so $\mathbb{H}(S(3)) = \{\text{IA}\}$.

Example 7.3.12. Let H be the complement of the homogeneous K_n-free graph for $n \geq 3$. Suppose that $A \subseteq B \in \text{Age}(H)$ with $B \smallsetminus A = \{b\}$ and suppose that $f : A \to H$ is a monomorphism. As $Af \in \text{Age}(H)$ and H is universal for graphs not embedding an independent set of size n, there exists a graph G contained in H that is isomorphic to the graph of Af together with an element x adjacent to all vertices in Af. As there is a partial isomorphism θ between $G \smallsetminus \{x\}$ and Af, using homogeneity we can extend θ to an automorphism of H sending x to some vertex y that is adjacent to everything in Af. Now, let $g : B \to H$ be the map extending f and sending b to y; this is a monomorphism and so H has the 1PMMEP. A result of [23] shows that H has the 1PHHEP, and so is both MM and HH-homogeneous.

We now show that H does not have the 1P$\overline{\text{MM}}$EP and hence cannot be MB-homogeneous. So let A be the complete graph on $n-1$ vertices, and let B be the disjoint union of A with a vertex b. Note that $A \subseteq B \in \text{Age}(H)$. Let $\bar{f} : A \to H$ be an antimonomorphism sending A to an independent set of $n-1$ vertices in H; such a substructure exists by construction. Then as antimonomorphisms preserve non-edges, a potential image point for b in H must be independent of $A\bar{f}$; this cannot happen as then H would induce an independent n-set. So H does not have the 1P$\overline{\text{MM}}$EP. We can also show (using a similar extension argument) that H does not have the 1P$\overline{\text{HH}}$EP is not HE-homogeneous. Therefore, $\mathbb{H}(H) = \{$IA, MM, HH$\}$.

Remark. By using an analogous argument, we can show that the I_n-free digraphs for $n \geq 3$ have the same maximal hh-classes as H.

8

MB-homogeneous graphs and digraphs

All of the examples considered in Section 7.3 are known to be IA-homogeneous structures. Finding homomorphism-homogeneous structures that differ from IA-homogeneous structures has been a point of interest from the advent of the subject; for instance, Cameron and Nešetřil gave an example of a HH-homogeneous graph Γ such that $\mathrm{Aut}(\Gamma) = \{e\}$ (Corollary 2.2 [14], see also Example 6.2.9). Following subsequent work of Mašulović and Pech [61], this is a graph with an oligomorphic endomorphism monoid whose group of units is trivial. Using Theorem 6.2.8, finding examples of structures that are in some way homomorphism-homogeneous whilst not being IA-homogeneous provides examples of oligomorphic transformation monoids without an oligomorphic group of units; for if $\mathrm{Aut}(\mathcal{M})$ is oligomorphic, then $\mathcal{M}$ is $\aleph_0$-categorical. Therefore, following comments made at the end of Chapter 6, one of the two aims of this chapter is to find more examples of oligomorphic transformation monoids that are not based on $\aleph_0$-categorical structures. In particular, we would like to find examples of closed permutation monoids arising as examples of bimorphism monoids of first-order structures; this follows from Theorem 6.1.1.

Typically, finding examples of homogeneous structures relies on strong properties of the structure at hand to demonstrate that isomorphisms between finite substructures extend to an automorphism; a good example would be the ARP characteristic to R (Example 2.4.2). Machinery to find examples of homomorphism-homogeneous structures was the main theme of Chapter 7. Verifying instances of homomorphism-homogeneity in the literature and Section 7.3 usually relied on these characteristic properties of homogeneous structures; however, there are cases that use demonstrably weaker properties. For instance, any countable graph that contains the random graph as a spanning subgraph (see Section 2.4) is MM-homogeneous [14]. This approach is codified by Lockett and Truss in [53]; their wide-ranging classification of homomorphism-homogeneous posets was based on whether or not the posets considered satisfied three structural conditions.

In the case where the considered structure is a graph, such a classification result is extremely ambitious; particularly in the finite case [73]. In addition to this present predicament, there are a lot more examples than in the homogeneous case (see Corollary 8.2.9) and due to [14, Proposition 4.2], two homomorphism-homogeneous graphs with the same age may not be isomorphic. However, there are some positive results for countable graphs. The paper of Rusinov and Schweitzer [73] built on the work of [14]; showing that the two classes MH and HH coincide for graphs and that the only MH-homogeneous graphs that are not MM-homogeneous are disjoint unions of cliques. In the same paper [73], they also provided examples of HH-homogeneous graphs that do not contain R as a spanning subgraph. Following on from the approach of [14], [73] and [53], our second aim is to continue classification work on countable homomorphism-homogeneous graphs, and to develop general theory about homomorphism-homogeneous oriented graphs and digraphs.

Our attention here focuses on classes of homomorphism-homogeneity not yet considered in the case for graphs; MB-homogeneity and HE-homogeneity. As outlined in Chapter 7, MB and HE-homogeneity require satisfaction of 'back' conditions in order to ensure the extended map is surjective; in addition to the structure adhering to the relevant 'forth' conditions. Therefore, as we are able to write down a property of a graph Γ that implies MM (or HH) homogeneity ([14] and [23]), we should be able to deduce a property of Γ to imply the back condition as well; this is considered with Proposition 8.1.6. Furthermore, as we have seen in Theorem 7.0.2, classifying these graphs up to isomorphism is not an option; so an appropriate equivalence condition must be developed and used. This is the notion of *bimorphism equivalence* defined in Definition 8.1.7; every graph with properties outlined in Proposition 8.1.6 is bimorphism equivalent to R. This investigation eventually culminates in the construction of $2^{\aleph_0}$ non-isomorphic MB (and HE)-homogeneous graphs in Corollary 8.2.9; presenting a stark contrast to the countably many IA-homogeneous graphs up to isomorphism detailed in Theorem 2.4.10. Furthermore, each of these examples is bimorphism equivalent to the random graph R (see Section 2.4). We also modify techniques considered in this study to investigate properties of MB-homogeneous oriented graphs and digraphs; this leads to a construction of $2^{\aleph_0}$ non-isomorphic MB-homogeneous oriented graphs in Corollary 8.3.10 and digraphs in Theorem 8.4.3 and Theorem 8.4.6.

In addition, finding examples of MB-homogeneous graphs, oriented graphs and digraphs satisfies our first aim as well. An in-depth study not only provides examples of oligomorphic permutation monoids that do not have a large group of units (as in Proposition 8.2.5) but also provide a range of interesting closed submonoids of the symmetric group on a countably infinite set. Furthermore, we answer a question asked at the end of Chapter 6; we construct a MB-homogeneous structure with an arbitrary finite automorphism group in Theorem 8.2.11, providing an example of an oligomorphic permutation monoid with any finite group of units.

The structure of this chapter is as follows. Section 8.1 details properties of MB-homogeneous graphs, provides a classification of MB-homogeneous graphs that are also IA-homogeneous (Theorem 8.1.4), and defines the two relevant structural properties (Proposition 8.1.6) and notion of equivalence (Definition 8.1.7) used throughout the chapter. In Section 8.2, we construct first one example of a MB-homogeneous graph that is not IA-homogeneous (Example 8.2.1), and then uncountably many non-isomorphic MB-homogeneous graphs (Corollary 8.2.9); all of which are bimorphism equivalent to the random graph R. Also, we prove that for any finite group G, there exists an MB-homogeneous graph Γ such that $\mathrm{Aut}(\Gamma) = G$ (Theorem 8.2.11). In Section 8.3, we define sufficient conditions for an oriented graph to be MB-homogeneous (Example 8.3.6), and then construct uncountably many non-isomorphic MB-homogeneous oriented graphs

(Corollary 8.3.10). Finally, in Section 8.4 we use the previous sections' work for graphs and oriented graphs to present uncountably many non-isomorphic MB-homogeneous digraphs (Theorem 8.4.3 and Theorem 8.4.6).

8.1 Properties of MB-homogeneous graphs

Recall that the examples of MB-homogeneous graphs we have seen so far are: the random graph R (Example 7.3.6), the infinite complete graph $K^{\aleph_0}$, and the infinite null graph $\bar{K}^{\aleph_0}$ (both Example 7.3.4). Here, $K^{\aleph_0}$ and $\bar{K}^{\aleph_0}$ are complements of each other, and R is a self-complementary graph [11]. Our first result shows that this behaviour is true in general.

Proposition 8.1.1. *Let* Γ *be an MB-homogeneous graph. Then its complement* $\bar{\Gamma}$ *is also MB-homogeneous.*

Proof. We note that $A \in \text{Age}(\Gamma)$ if and only if $\bar{A} \in \text{Age}(\bar{\Gamma})$. Furthermore, as Γ is MB-homogeneous, it has the MMEP and $\overline{\text{MM}}$EP by Proposition 7.2.6. Now suppose that $A \subseteq B \in \text{Age}(\Gamma)$ and that $\bar{f} : A \to \Gamma$ is an antimonomorphism. Any such $\bar{f}$ preserves non-edges and may change edges to non-edges; so $\bar{f} : \bar{A} \to \bar{\Gamma}$ is a monomorphism. As Γ has the $\overline{\text{MM}}$EP, $\bar{f}$ can be extended to an antimonomorphism $\bar{g} : B \to \Gamma$; this in turn induces a monomorphism $\bar{g} : \bar{B} \to \bar{\Gamma}$ and hence $\bar{\Gamma}$ has the MMEP. The proof that $\bar{\Gamma}$ has the $\overline{\text{MM}}$EP is similar. □

Following this, we can guarantee that certain subgraphs appear in an MB-homogeneous graph. Cameron and Nešetřil prove that every MM-homogeneous graph must contain an infinite complete subgraph [14, Proposition 2.5]; we extend this result.

Corollary 8.1.2. *Any infinite non-complete, non-null MB-homogeneous graph* Γ *contains both an infinite complete and an infinite null subgraph.*

Proof. Any MB-homogeneous graph is necessarily MM-homogeneous and hence it contains an infinite complete subgraph from the aforementioned result of [14]. By Proposition 8.1.1, $\bar{\Gamma}$ is also MB-homogeneous and so contains an infinite complete subgraph; the result follows from this. □

Remark. A straightforward consequence of this argument is that any countably infinite MB-homogeneous graph Γ is neither a locally finite graph nor the complement of a locally finite graph.

We now examine cases where the graph Γ is disconnected. The case where Γ has no edges was considered in Example 7.3.4. It is shown in [14] that any disconnected, non-null MH-homogeneous graph is a disjoint union of complete graphs, all of which are the same size. By this and Corollary 8.1.2, the only candidates for a disconnected non-null MB-homogeneous graph must be disjoint unions of more than one infinite complete graph. The next result shows that there is only one disconnected, non-null MB-homogeneous graph; re-proving and extending a remark of Rusinov and Schweitzer [73].

Proposition 8.1.3. *Let $\Gamma = \bigsqcup_{i \in I} K_i$, where $K_i \cong K^{\aleph_0}$ for all i in some index set I.*

(1) If I is finite with size $n > 1$, then Γ is MM-homogeneous but not MB-homogeneous.

(2) If I is countably infinite, then Γ is MB-homogeneous.

Proof. In both cases, every $A \in \text{Age}(\Gamma)$ can be decomposed as finite disjoint union of finite complete graphs; so we can write that $A = \bigsqcup_{j=1}^{k} C_j$, where C_j is a complete graph of some finite size.

(1) As $|I| = n$ we note that $k \leq n$ for all $A = \bigsqcup_{j=1}^{k} C_j$ in the age of Γ. Suppose that $A \subseteq B \in \text{Age}(\Gamma)$ with $B \smallsetminus A = \{b\}$. There are two cases for b; either b is completely independent of A or b is related to exactly one C_j for some $1 \leq j \leq k$. If it is the former, we can extend any monomorphism $f : A \to \Gamma$ to a monomorphism $g : B \to \Gamma$ by sending b to any vertex $v \in V\Gamma \smallsetminus A$. If it is the latter, then we can extend any monomorphism $f : A \to \Gamma$ to a monomorphism $g : B \to \Gamma$ by sending b to a vertex $v \in K_i \smallsetminus C_j$, where j is defined as above. Hence Γ has the MMEP by Proposition 7.3.1. However, note that Γ does not embed an independent set of size $n + 1$ and so Γ is not MB-homogeneous by Corollary 8.1.2.

(2) The proof that Γ has the MMEP when I is infinite is as above. Assume then that $A \subseteq B \in \text{Age}(\Gamma)$ with $B \smallsetminus A = \{b\}$, and let $\bar{f} : A \to \Gamma$ be an anti-monomorphism. We note that as A is finite then $A\bar{f}$ is finite; since I is infinite,

there exists $i \in I$ such that $K_i \cap A\bar{f} = \varnothing$. Therefore, regardless of how b is related to A, we can extend $\bar{f}$ to an antimonomorphism $\bar{g} : B \to \Gamma$ by mapping b to some $v \in K_i$, where i is as stated above. Hence Γ has the $\overline{\text{MM}}$EP by Proposition 7.3.1 and so is MB-homogeneous by Proposition 7.2.6. □

Remark. By Proposition 8.1.3 and Proposition 8.1.1, the complement $\bar{\Gamma}$ of $\Gamma = \bigsqcup_{i\in\mathbb{N}} K_i^{\aleph_0}$, where $\bar{\Gamma}$ is the complete multipartite graph with infinitely many partitions each of infinite size, is also MB-homogeneous.

This proposition completes the classification of countable homogeneous graphs that are also MB-homogeneous.

Theorem 8.1.4. *Let Γ be an infinite IA and MB-homogeneous graph. Then Γ is isomorphic to one of the following:*

- *$K^{\aleph_0}$ and its complement $\overline{K}^{\aleph_0}$;*
- *$\bigsqcup_{i\in\mathbb{N}} K_i^{\aleph_0}$ and its complement $\overline{\bigsqcup_{i\in\mathbb{N}} K_i^{\aleph_0}}$;*
- *the random graph R.*

Proof. We check every item in the classification of countably infinite, undirected homogeneous graphs given in Theorem 2.4.10 [49]. We showed that the graphs on the list above are MB-homogeneous in Example 7.3.4, Proposition 8.1.3 and Example 7.3.6 respectively. Any other disconnected homogeneous graph must be the a countable union of finite complete graphs of the same size or a finite union of infinite complete graphs; these are not MB-homogeneous by Proposition 8.1.3. The only other countable homogeneous graphs are the K_n-free graphs and their complements; these are not MB-homogeneous by Example 7.3.9 and Example 7.3.12 respectively. □

The proof that $\Gamma = \bigsqcup_{i\in\mathbb{N}} K_i^{\aleph_0}$ is MB-homogeneous in Proposition 8.1.3 relies on the existence of an independent element to every image of a finite antimonomorphism; more simply, to every finite set of vertices in Γ. Our aim now is to obtain some sufficient conditions for MB-homogeneity along these lines in order to construct some new examples.

Definition 8.1.5. Let Γ be an infinite graph.

- Say that Γ has *property* $(\triangle)$ if for every finite set $U \subseteq V\Gamma$ there exists $u \in V\Gamma$ such that u is adjacent to every member of U.
- Say that Γ has *property* $(\therefore)$ if for every finite set $V \subseteq V\Gamma$ there exists $v \in V\Gamma$ such that v is non-adjacent to every member of V.

(See Figure 8.1 for a diagram of an example.)

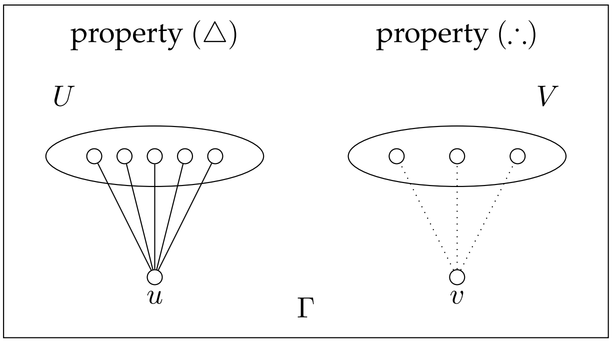


Figure 8.1: A diagram of Definition 8.1.5

Remark. In the language of [73], properties $(\triangle)$ and $(\therefore)$ are the same as a *cone* and *anti-cone* respectively. If a graph Γ has property $(\triangle)$ then it is *algebraically closed* (see [24]). Due to the complementary nature of these properties, it follows that Γ has property $(\therefore)$ if and only if its complement $\bar{\Gamma}$ is algebraically closed.

Note that these properties are what was required to prove that R is both MB and HE-homogeneous in Example 7.3.6. We show that this is true in all cases.

Proposition 8.1.6. *Let Γ be an infinite graph. If Γ has both properties $(\triangle)$ and $(\therefore)$ then Γ is both MB and HE-homogeneous.*

Proof. Suppose that $A \subseteq B \in \text{Age}(\Gamma)$ with $B \setminus A = \{b\}$, and that $f : A \to \Gamma$ is a monomorphism. As A is finite, Af is a finite set of vertices in Γ. By property $(\triangle)$, there exists a vertex v of Γ such that v is adjacent to every element of Af. This means that v is a potential image point of b; so the map $g : B \to \Gamma$ extending f and sending b to v is a monomorphism. Hence, Γ has the MMEP by Proposition 7.3.1. Using property $(\therefore)$ in a similar fashion shows that Γ has the $\overline{\text{MM}}$EP and so is MB-homogeneous by Proposition 7.2.6.

The proof that Γ has the HHEP and $\overline{\text{HH}}$EP is similar. $\square$

Remarks. (i) The converse of this result is not true. The infinite disjoint union of infinite complete graphs in Proposition 8.1.3 (2) is an example of an MB-homogeneous graph with property ($\therefore$) but not property ($\triangle$). Its complement (the complete multipartite graph with infinitely many partitions of infinite size) is an example of an MB-homogeneous graph with property ($\triangle$) but not property ($\therefore$).

(ii) While both properties are not required for MB-homogeneity, we cannot show that a graph with exactly one of property ($\triangle$) or property ($\therefore$) is MB-homogeneous. For example, the infinite disjoint union of *finite* complete graphs (of all the same size) has property ($\therefore$), but is not MB-homogeneous by Proposition 8.1.3 (1). Its complement has property ($\triangle$), but cannot be MB-homogeneous by Proposition 8.1.1.

This shows that if the complement $\bar{\Gamma}$ of an algebraically closed graph Γ is also algebraically closed, then Γ is MB and HE-homogeneous.

We will see below that it is sometimes possible to start with a countable graph Γ, add some edges to obtain a graph $\Delta \ncong \Gamma$, and then add in edges to Δ to obtain a graph Ω such that $\Gamma \cong \Omega$. In this situation, Γ and Δ will be *bimorphism equivalent*. We formalise this notion of equivalence below, extending an idea of [24].

Definition 8.1.7. Let Γ, Δ be two graphs. We say that Γ is *bimorphism equivalent* to Δ if there exist bijective homomorphisms $\alpha : \Gamma \to \Delta$ and $\beta : \Delta \to \Gamma$.

Remarks. This is a weaker version of the idea of B-equivalence introduced in Proposition 7.2.10; every pair of B-equivalent graphs are bimorphism equivalent by definition, but the converse is not true (see Corollary 8.1.10 and Example 8.2.1). Note also that this definition is equivalent to saying that Γ, Δ contain each other as spanning subgraphs.

Justifying the name, this is an equivalence relation of graphs (up to isomorphism); we denote this relation by $\sim_b$. The product $\alpha\beta : \Gamma \to \Gamma$ of the two bimorphisms induces a bimorphism on Γ, and so if $\mathrm{Bi}(\Gamma) = \mathrm{Aut}(\Gamma)$ we have that α and β are necessarily isomorphisms. In this case, $[\Gamma]^{\sim_b}$ is a singleton equivalence class. We now show that bimorphism equivalence preserves properties $(\triangle)$ and $(\therefore)$, and that any two graphs with properties $(\triangle)$ and $(\therefore)$ are bimorphism equivalent.

Proposition 8.1.8. *Let Γ, Δ be bimorphism equivalent graphs via bijective homomorphisms $\alpha : \Gamma \to \Delta$ and $\beta : \Delta \to \Gamma$. Then Γ has properties $(\triangle)$ and $(\therefore)$ if and only if Δ does.*

Proof. Suppose that Γ has property $(\triangle)$ and $X \subseteq V\Gamma$. From this, there exists a vertex $v \in V\Gamma$ adjacent to every element of X. As α is a homomorphism, $X\alpha$ is a finite subset of $V\Delta$ and $v\alpha$ is adjacent to every element of $X\alpha$; since α is bijective, every finite subset Y of $V\Delta$ can be written as $X\alpha$ for some $X \subseteq V\Gamma$. These observations show that Δ has property $(\triangle)$. As α is a bijective homomorphism, there exists a bijective antihomomorphism $\bar{\alpha} : \Delta \to \Gamma$ by Lemma 7.1.2. Since Γ has property $(\therefore)$, for any finite $X \subseteq V\Gamma$ there exists a vertex $w \in \Gamma$ independent of every element in X. Due to the fact that $\bar{\alpha}$ preserves non-edges, any $Y \subseteq V\Delta$ has a vertex $x \in V\Delta$ independent of every element of Y; and as α is bijective this happens for every finite set $X \subseteq V\Gamma$ and so Δ has property $(\therefore)$. The converse direction is symmetric. $\square$

Proposition 8.1.9. *If Γ, Δ are two graphs with properties $(\triangle)$ and $(\therefore)$, then Γ and Δ are bimorphism equivalent.*

Proof. Assume that Γ, Δ are two graphs with properties $(\triangle)$ and $(\therefore)$. We use a back and forth argument to construct a bijective homomorphism $\alpha : \Gamma \to \Delta$ and a bijective antihomomorphism $\bar{\beta} : \Gamma \to \Delta$, which by Lemma 7.1.2 will be the converse of a bijective homomorphism $\beta : \Delta \to \Gamma$. Suppose that $f : \{c\} \to \{d\}$ is a function sending a vertex c of Γ to a vertex d of Δ; this is a bijective homomorphism. Now set $\{c\} = C_0$, $\{d\} = D_0$, $f = f_0$, and assume that we have extended f to a bijective homomorphism $f_k : C_k \to D_k$, where C_i and D_i are finite and $C_i \subseteq C_{i+1}$ and $D \subseteq D_{i+1}$ for all $0 \leq i \leq k-1$. Furthermore, as both

Γ and Δ are countable, we can enumerate their vertices as $V\Gamma = \{c_0, c_1, ...\}$ and $V\Delta = \{d_0, d_1, ...\}$.

If k is even, select a vertex $c_j \in \Gamma$ where j is the smallest number such that $c_j \notin C_k$. As Δ has property $(\triangle)$, there exists a vertex $u \in \Delta$ such that u is adjacent to every element of D_k. Define a map $f_{k+1} : C_k \cup \{c_j\} \to D_k \cup \{u\}$ sending c_j to u and extending f; this map is a bijective homomorphism as any edge from c_j to some element of C_k is preserved (see Figure 8.2 for a diagram of an example).

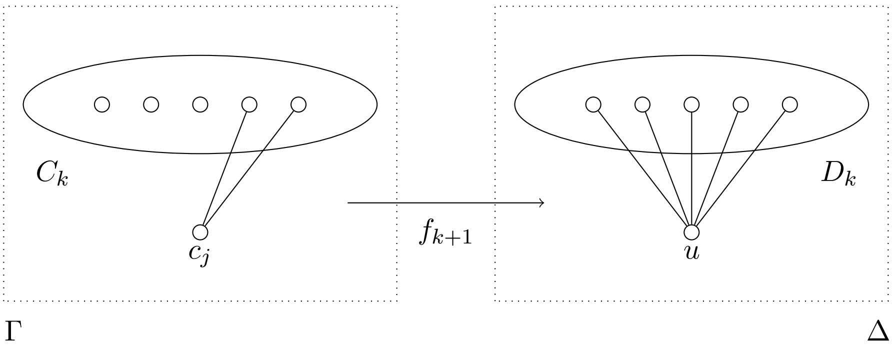


Figure 8.2: k even in proof of Proposition 8.1.9

Now, if k is odd, choose a vertex $d_j \in \Delta$ where j is the smallest number such that $d_j \notin D_k$. As Γ has property $(\therefore)$, there exists a vertex $v \in \Gamma$ such that v is independent of every element of C_k. Define a map $f_{k+1} : C_k \cup \{v\} \to D_k \cup \{d_j\}$ sending c to d_j and extending f_k. Then f_{k+1} is a bijective homomorphism as f_k is and every edge between c and C_k is preserved; because there are none. See Figure 8.3 for a diagram of an example of this stage.

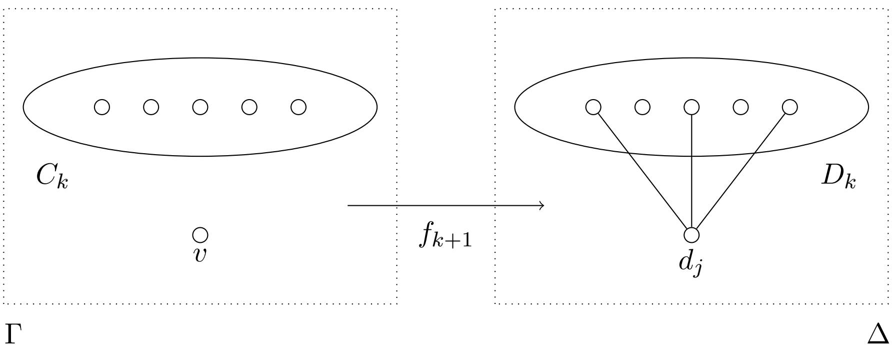


Figure 8.3: k odd in proof of Proposition 8.1.9

Repeating this process infinitely many times, ensuring that each vertex of Γ appears at an even stage and each vertex of Δ appears at an odd stage, defines a bijective homomorphism $\alpha : \Gamma \to \Delta$. We can construct a bijective antihomomorphism $\bar{\beta} : \Gamma \to \Delta$ in a similar fashion; replacing homomorphism with antihomomorphism and using property $(\therefore)$ of Δ at even steps and property $(\triangle)$ of Γ at odd steps. So the converse map $\beta : \Delta \to \Gamma$ is a bijective homomorphism and so Γ and Δ are bimorphism equivalent. $\square$

These two statements mean that any graph with properties $(\triangle)$ and $(\therefore)$ is bimorphism equivalent to any other graph with the same properties. Finally in this section, the next result extends [14, Proposition 2.1 (i)] and establishes a complementary condition to the graph case of [24, Corollary 2.2].

Corollary 8.1.10. *Suppose that Γ is a countable graph. Then Γ has properties $(\triangle)$ and $(\therefore)$ if and only if it is bimorphism equivalent to the random graph R.*

Proof. As R has both properties $(\triangle)$ and $(\therefore)$, the converse direction follows from Proposition 8.1.8, and the forward direction follows from Proposition 8.1.9.

$\square$

Remark. These three results show that the equivalence class $[R]^{\sim_b}$ is precisely the set of all countable graphs Γ with properties $(\triangle)$ and $(\therefore)$.

8.2 Examples of MB-homogeneous graphs

Now, we use properties $(\triangle)$ and $(\therefore)$ to find an MB-homogeneous graph that is not IA-homogeneous.

Example 8.2.1. Let $P = (p_n)_{n \in \mathbb{N}_0}$ be an infinite binary sequence with infinitely many 0's and 1's. Define the graph $\Gamma(P)$ on the infinite vertex set $V\Gamma(P) = \{v_0, v_1, ...\}$ with edge relation $v_i \sim v_j$ if and only if $p_{\max(i,j)} = 0$. From this, we can observe that:

- if $p_i = 0$ then $v_i \sim v_j$ for all natural numbers $j < i$;
- if $p_i = 1$ then $v_i \nsim v_j$ for all $j < i$;

where $<$ is the natural ordering on $\mathbb{N}$. Say that $\Gamma(P)$ is the graph *determined* by the binary sequence P. An example (where $P = (0, 1, 0, 1, ...)$) is given in Figure 8.4.

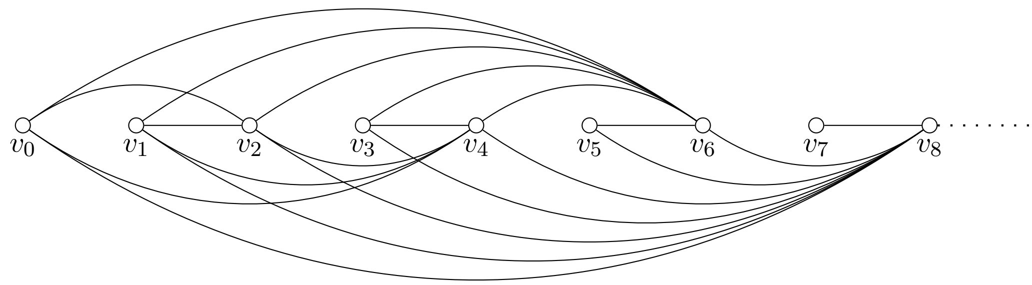


Figure 8.4: $\Gamma(P)$, with $P = (0, 1, 0, 1, ...)$.

As P has infinitely many 0's and 1's, it follows that for every finite subsequence $A = \{a_{i_1}, ..., a_{i_k}\}$ of P there exist natural numbers $c, d > i_k$ such that $p_c = 0$ and $p_d = 1$. This, together with the manner of the construction, ensures that $\Gamma(P)$ has both properties $(\triangle)$ and $(\therefore)$ and is therefore MB-homogeneous by Proposition 8.1.6.

We now show that $\Gamma(P)$ is distinct from the other MB-homogeneous graphs already considered. Note that $\Gamma(P)$ contains both an edge and a non-edge; so it is neither the infinite complete nor the infinite null graph. As $\Gamma(P)$ has both properties $(\triangle)$ and $(\therefore)$ it is neither $\bigsqcup_{i \in \mathbb{N}} K_i^{\aleph_0}$ nor its complement $\overline{\bigsqcup_{i \in \mathbb{N}} K_i^{\aleph_0}}$ by the remarks following Proposition 8.1.6. Finally, as no term p of P can be both 0 and 1 simultaneously, there is no vertex v_p that is adjacent to $\{v_0\}$ and non-adjacent to $\{v_1\}$. Hence $\Gamma(P)$ does not satisfy ARP; so by Proposition 2.4.3, it is not the random graph. So for any sequence P with infinitely many 0's and 1's, $\Gamma(P)$ is an example of an MB-homogeneous graph that is not IA-homogeneous by Theorem 8.1.4.

Remarks. (i) Any infinite binary sequence P with infinitely many 0's and 1's contains every finite binary sequence X as a subsequence. The finite induced subgraphs $\Gamma(X)$ of $\Gamma(P)$ are those induced on $V\Gamma(X)$ by the edge relation of P. As this is true for all such binary sequences P and Q, we conclude that $\Gamma(P)$ and $\Gamma(Q)$ have the same age. It can be shown from here that for any two such sequences P and Q, then $\Gamma(P)$ and $\Gamma(Q)$ are B-equivalent (see Proposition 7.2.10).

(ii) Throughout the rest of this section, any binary sequences P, Q have infinitely many 0's and 1's. This guarantees that any graph $\Gamma(P)$ determined by P has properties ($\triangle$) and ($\therefore$).

Many natural questions arise from this construction. Perhaps the most pressing is: to what extent does the isomorphism type of $\Gamma(P)$ depend on the binary sequence P? Answering this question will help to tell us how many MB-homogeneous graphs there are of this kind. However, there is a notable difficulty in deciding this question in that two graphs $\Gamma(P), \Gamma(Q)$ determined by binary sequences P, Q respectively have the same age; so deciding whether the two graphs are isomorphic or not requires some thought. Our method of achieving a partial solution to this problem is by investigating the automorphism group as an invariant of $\Gamma(P)$. Our first lemma establishes a convention for the zeroth place of such a sequence.

Lemma 8.2.2. *Suppose that $\Gamma(P)$ and $\Gamma(Q)$ are the graphs on the vertex sets $V = \{v_0, v_1, ...\}$ and $W = \{w_0, w_1, ...\}$ determined by the binary sequences $P = (p_n)_{n\in\mathbb{N}_0}$ and $Q = (q_n)_{n\in\mathbb{N}_0}$ respectively. Furthermore assume that $p_i = q_i$ for all $i > 0$. Then $\Gamma(P) \cong \Gamma(Q)$.*

Proof. If $p_0 = q_0$ the binary sequences are the same and we are done. If $p_0 \neq q_0$, note that the graph induced on $\{v_1, v_2, ...\}$ by $\Gamma(P)$ is isomorphic to the graph induced on $\{w_1, w_2, ...\}$ by $\Gamma(Q)$. Then the graph induced on $N(v_0) = \{v_j \in V\Gamma(P) : p_j = 0, j \geq 1\}$ by $\Gamma(P)$ and the graph induced on $N(w_0) = \{w_j \in V\Gamma(P) : q_j = 0, j \geq 1\}$ by $\Gamma(Q)$ are isomorphic. As $p_j = q_j$ for all $j \geq 1$, the map $\theta : \Gamma(P) \to \Gamma(Q)$ sending v_i to w_i for all $i \in \mathbb{N}$ is an isomorphism. $\square$

Following this, we can take $p_0 = p_1$ for any binary sequence P without loss of generality; we adopt this as convention for the rest of the section. Now, if P is a binary sequence, denote the k^{th} consecutive string of 0′s and 1′s by O_k and I_k respectively, and denote the vertex sets corresponding to these subsequences by VO_k and VI_k (see Figure 8.5). Furthermore, denote the graph induced by $\Gamma(P)$ on any subset X of $V\Gamma$ by $\Gamma(X)$. The next pair of lemmas deal with what graphs induced on neighbourhoods of vertices in some $\Gamma(P)$ look like.

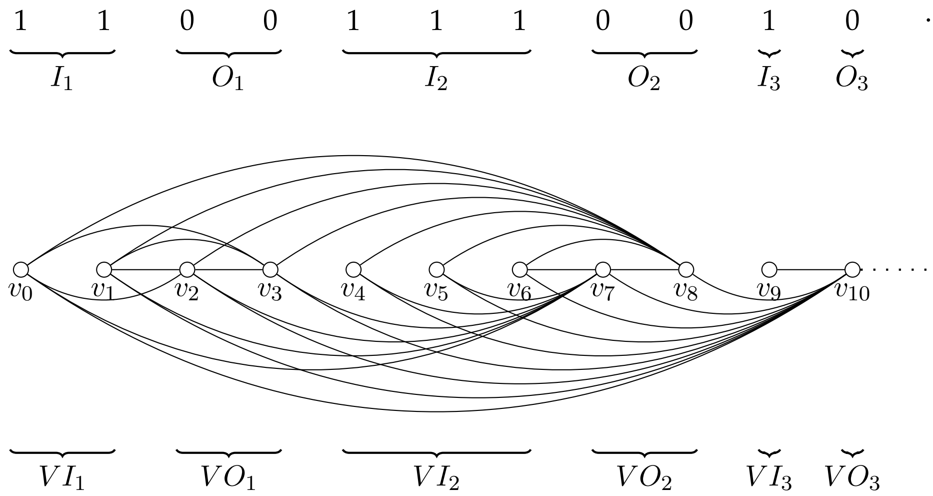


Figure 8.5: $\Gamma(P)$, with $P = (1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, ...)$, illustrating the definition of O_k, I_k, and VO_k, VI_k.

Lemma 8.2.3. *Let $P = (p_n)_{n\in\mathbb{N}_0}$ be a binary sequence, and let $\Gamma(P)$ be the graph determined by P. Suppose that v_j is a vertex in VI_n for some $n \in \mathbb{N}$. Then $\Gamma(N(v_j)) \cong K^{\aleph_0}$.*

Proof. From the assumption that $p_j = 1$, the observation of Example 8.2.1, and the definition of an edge in $\Gamma(P)$, we have that $v_j \sim v_k$ if and only if $j < k$ and $p_k = 0$; so $N(v_j) = \{v_k \in V\Gamma(P) : k > j, p_k = 0\}$. As there are infinitely many 0′s in P and j is finite, it follows that $N(v_j)$ is infinite. Now, take $v_a, v_b \in N(v_j)$. Here, $v_a \sim v_b \in \Gamma(N(v_j))$ if and only if $p_{\max(a,b)} = 0$; but $p_a = p_b = 0$ and so any two vertices of $\Gamma(N(v_j))$ are adjacent. $\square$

Lemma 8.2.4. *Let $P = (p_n)_{n\in\mathbb{N}_0}$ be a binary sequence, and let $\Gamma(P)$ be the graph determined by P. Suppose that v, w are vertices in some $VO_k = \{v_{k_1}, ..., v_{k_n}\}$ corresponding to some $O_k = \{p_{k_1}, ..., p_{k_n}\}$. Then $N(v) \cup \{v\} = N(w) \cup \{w\}$ and $\Gamma(N(v)) \cong \Gamma(N(w))$.*

Proof. Define the sets $X = \{v_j \in V\Gamma(P) : j < i_1\}$ and $Y = \{v_k \in V\Gamma(P) : k \geq i_1, a_k = 0\}$. Due to the construction of Γ in Example 8.2.1, $N(v) \cup \{v\} = X \cup Y = N(w) \cup \{w\}$. For all $u \in V\Gamma(VO_k)$, it is easy to see that $N(u) = X \cup (Y \setminus \{u\})$. Using a similar argument to the proof of Lemma 8.2.3, it follows that $\Gamma(Y \setminus \{u\})$ is an infinite complete graph for any $u \in \Gamma(VO_k)$, and every element in Y is adjacent to every element of X. For any $v, w \in VO_k$ define a map $f : X \cup (Y \setminus \{v\}) \to X \cup (Y \setminus \{w\})$ fixing X pointwise and sending $(Y \setminus \{v\})$ to $(Y \setminus \{w\})$ in any fashion; this is an isomorphism from $\Gamma(N(v))$ to $\Gamma(N(w))$.. □

Before our next proposition, recall that if there exists $\gamma \in \mathrm{Aut}(\Gamma)$ such that $v\gamma = w$, then the graphs induced on $N(v)$ and $N(w)$ are isomorphic.

Proposition 8.2.5. *Let $P = (p_n)_{n \in \mathbb{N}_0}$ be a binary sequence, and let $\Gamma(P)$ be the graph determined by the binary sequence P. Then*

$$Aut(\Gamma(P)) = \prod_{k \in \mathbb{N}} (Aut(\Gamma(O_k)) \times Aut(\Gamma(I_k))),$$

the infinite direct product of automorphism groups on each $\Gamma(O_k)$ and $\Gamma(I_k)$.

Proof. Using Lemma 8.2.2, we set the convention that $p_0 = p_1$ throughout; hence either O_1 or I_1 (depending on whether P starts with a 0 or 1) has size at least 2. For some i, write $O_i = \{p_{i_1}, ..., p_{i_n}\}$ and $I_i = \{q_{i_1}, ..., q_{i_m}\}$, and we write $VO_i = \{v_{i_1}, ..., v_{i_n}\}$ and $VI_i = \{w_{i_1}, ..., w_{i_m}\}$.

We first show that any automorphism of $\Gamma(P)$ fixes both VO_k and VI_k setwise for all $k \in \mathbb{N}$, using a series of claims. Then, we prove that any bijective map from $\Gamma(P)$ to itself fixing every point except those in a single VO_k (or VI_k), and acting as an automorphism on that VO_k (or VI_k), is an automorphism of $\Gamma(P)$. We begin with our first claim.

Claim 1. If $v \in VO_i$ and $w \in VO_j$ with $i \neq j$, then $\Gamma(N(v)) \not\cong \Gamma(N(w))$.

Proof of Claim 1. We write $VO_i = \{v_{i_1}, ..., v_{i_n}\}$ and $VO_j = \{v_{j_1}, ..., v_{j_m}\}$. Without loss of generality, suppose that $i < j$; so $i_1 < i_2 < ... < i_n < j_1 < j_2 < ... < j_m$.

We define the following sets:

$$
\begin{aligned}
X_i &= \{v_k \in V\Gamma(P) : k < i_1\} \\
X_j &= \{v_k \in V\Gamma(P) : k < j_1\} \\
Y_i &= \{v_k \in V\Gamma(P) : k \geq i_1, p_k = 0\} \\
Y_j &= \{v_k \in V\Gamma(P) : k \geq j_1, p_k = 0\}.
\end{aligned}
$$

Lemma 8.2.4 asserts that $N(v) = X_i \cup (Y_i \smallsetminus \{v\})$ and $N(w) = X_j \cup (Y_j \smallsetminus \{w\})$. By reasoning outlined in the proof of Lemma 8.2.4, if $v_a \nsim v_b \in \Gamma(N(v))$ then $v_a, v_b \in X_i$. Note also that both X_i and X_j are both finite sets, and so any maximum independent set of X_i, X_j (and hence of $N(v), N(w)$) must be finite. Consider the sets $A = \{v_a : a < i_1, p_a = 1\} \cup \{v_0\}$ and $B = \{v_b : b < j_1, p_b = 1\} \cup \{v_0\}$, contained in X_i and X_j respectively. So if $v_a, v_d \in A$, then $p_{\max(a,d)} = 1$ and so $v_a \nsim v_d$. Now, if $v_c \in X_i \smallsetminus A$ then $p_c = 0$, and as $v_c \neq v_0$ it follows that $p_{\max(c,0)} = 0$ and so $v_c \sim v_0$. Hence A is the maximum independent set in X_i; using a similar argument, we can show that B is the maximum independent set in X_j. Since $i < j$ there exists an I_k between O_i and O_j; so there is a $e \in \mathbb{N}$ where $i_n < e < j_1$ and $p_e = 1$. Hence $v_e \in B \smallsetminus A$ and so $|B| > |A|$. Therefore, as $\Gamma(N(v))$ and $\Gamma(N(w))$ have maximum independent sets of different sizes, they are not isomorphic and this concludes the proof of Claim 1.

This shows that there is no automorphism α of $\Gamma(P)$ sending a vertex $v \in VO_i$ to a vertex $w \in VO_j$ where $i \neq j$, as they have non-isomorphic neighbourhoods. We move on to our next claim.

Claim 2. Suppose that $v \in VO_i$ (with $i \geq 2$) and $w \in VI_j$. Then there is no automorphism of $\Gamma(P)$ sending v to w.

Proof of Claim 2. Set $VO_i = \{v_{i_1}, ..., v_{i_n}\}$, and define the set $A = \{v_a : a < i_1, p_a = 1\} \cup \{v_0\}$ as in the previous claim. As $i \geq 2$, there exists a c such that $0 < c < i_1$ with $p_c = 1$. Hence we have that $|A| \geq 2$ and so $\Gamma(N(v))$ has at least one non-edge. However, $\Gamma(N(w)) \cong K^{\aleph_0}$ by Lemma 8.2.3 and so there cannot possibly be an automorphism of $\Gamma(P)$ sending v to w. This proves the second claim.

From these claims, any automorphism α of $\Gamma(P)$ fixes VO_i setwise for $i \geq 2$. Now, for $k \geq 2$, any VI_k (or VI_{k+1}) sandwiched between VO_k and VO_{k+1}, as VI_k (or VI_{k+1}) are the only vertices not adjacent to any vertex in VO_k and adjacent to every vertex in VO_{k+1}. As VO_k and VO_{k+1} are fixed setwise, we conclude that VI_k (or VI_{k+1}) is fixed setwise for $k \geq 2$.

Claim 3. VO_1 and VI_1 are fixed setwise by any automorphism of $\Gamma(P)$.

Proof of Claim 3: There are two cases to consider; where P either begins with a 0 or a 1.

Case 1 ($p_0 = 0$)**.** Suppose for a contradiction that there exists an $\alpha \in \text{Aut}(\Gamma)$ sending some $v \in VO_1$ to some $w \in VI_1$. The assumption in this case implies that $|VO_1| \geq 2$ and so $\Gamma(VO_1)$ has an edge from v to some $u \in VO_1$. But $w \in VI_1$ and thus is independent of everything in $VO_1 \cup VI_1$; so the edge between v and u is not preserved by α and so VO_1 is fixed setwise. Here, VI_1 are the only vertices not adjacent to any vertex in VO_1 and adjacent to every vertex in VO_2. As VO_1 and VO_2 are fixed setwise, then VI_1 is also fixed setwise.

Case 2 ($p_0 = 1$)**.** From this assumption, we have that $|VI_1| \geq 2$ and so $\Gamma(VI_1)$ contains a non-edge. Hence for some $v \in VO_1$, the graph $\Gamma(N(v))$ contains a non-edge. Therefore, from Lemma 8.2.3 and Claim 1, $\Gamma(N(v)) \ncong \Gamma(N(w))$ for some $w \in V\Gamma(P) \smallsetminus VO_1$ and so VO_1 is fixed setwise. As this happens, both VI_1 and VI_2 are fixed setwise. This concludes the proof of Claim 3.

So for any $k \in \mathbb{N}$, we have that VO_k and VI_k are fixed setwise by any automorphism of $\Gamma(P)$. It remains to prove that any automorphism of $\Gamma(VO_k)$ or $\Gamma(VI_k)$ whilst fixing every other point in $\Gamma(P)$ is an automorphism of $\Gamma(P)$. Here, as $\Gamma(VO_k)$ is a complete graph and $\Gamma(VI_k)$ is a null graph, it follows that $\text{Aut}(\Gamma(VO_k)) \cong \text{Sym}(|VO_k|)$ and $\text{Aut}(\Gamma(VI_k)) \cong \text{Sym}(|VI_k|)$ for all $k \in \mathbb{N}$.

So suppose that $f : \Gamma(P) \to \Gamma(P)$ is a bijective map that acts as an automorphism on some $\Gamma(VO_k)$ (that is, some permutation of the set VO_k) and fixes

$V\Gamma \setminus VO_k$. By Lemma 8.2.4, any two elements $v, w \in VO_k$ have the same extended neighbourhood $N(v) \cup \{v\} = N(w) \cup \{w\}$ (and $\Gamma(N(v)) \cong \Gamma(N(w))$). Here, f preserves all edges and non-edges between VO_k and $V\Gamma(P) \setminus VO_k$ and so it is an automorphism of $\Gamma(P)$. The proof that the bijective map $g : \Gamma(P) \to \Gamma(P)$ that acts as an automorphism of some $\Gamma(VI_k)$ and fixes every other point is similar. $\square$

Remarks. (i) This proposition guarantees the existence of at least countably many non-isomorphic MB-homogeneous graphs. For instance, define a binary sequence $P^n = (p_i)_{i\in\mathbb{N}}$ by the following:

$$p_i = \begin{cases} 1 \text{ if } i = 0, 1, 2, ..., n-1, n+1, n+3, ... \\ 0 \text{ if } i = n, n+2, ... \end{cases}$$

So P^n is a sequence of n many 1's followed by alternating 0's and 1's. It follows that Aut($\Gamma(P^n)$) is the infinite direct product of one copy of Sym(n) together with infinitely many trivial groups; so Aut($\Gamma(P^n)$) $\cong$ Sym(n). This is true for all $n \geq 2$. So if $m \neq n$, then Aut($\Gamma(P^m)$) $\ncong$ Aut($\Gamma(P^n)$) and so $\Gamma(P^m) \ncong \Gamma(P^n)$.

(ii) If $\Gamma(P)$ is the graph determined by the binary sequence $P = (0, 1, 0, 1, ...)$ (as exhibited in Example 8.2.1), then this result implies that Aut($\Gamma(P)$) $\cong$ C_2. Using this and MB-homogeneity of $\Gamma(P)$, one can show that $\Gamma(P)$ is an example of a structure such that $|\text{Aut}(\Gamma(P))| \leq \aleph_0$ but $|\text{Bi}(\Gamma(P))| = 2^{\aleph_0}$; demonstrating that Theorem 6.1.3 is not a natural consequence of Theorem 2.3.6.

Furthermore, it is straightforward to see that if two monoids S, S' have non-isomorphic groups of units U, U' respectively, then $S \ncong S'$. Hence we have constructed several examples of non-isomorphic oligomorphic permutation monoids on B-equivalent structures. By Proposition 6.2.7, this means that each of these oligomorphic permutation monoids have the same strong orbits. Finally, we note from a remark made earlier that any $\Gamma(P)$ constructed in this manner is also HE-homogeneous; therefore, we have constructed countably many non-isomorphic HE-homogeneous graphs.

8.2.1 From countably many to uncountably many

We now aim to use the theory established here to construct $2^{\aleph_0}$ non-isomorphic examples of MB-homogeneous graphs. The idea now is to add in pairwise non-embeddable finite structures into the age of some $\Gamma(P)$ to ensure uniqueness up to isomorphism, without adversely affecting properties $(\triangle)$ and $(\therefore)$.

To this end, let $A = (a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers. We use A to recursively define a binary sequence $PA = (p_i)_{i\in\mathbb{N}}$ as follows:

Base: 0 followed by a_1 many 1's.

Inductive: Assuming that the n^{th} stage of the sequence has been constructed, add a 0 followed by a_{n+1} many 1's to the right hand side of the sequence.

For instance, if $A = (2, 3, 5, ...)$ then $PA = (0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, ...)$. As PA has infinitely many 0's and 1's then $\Gamma(PA)$, the graph determined by PA, is MB-homogeneous.

The eventual plan is to draw finite graphs onto the independent sets induced on $\Gamma(PA)$ by strings of consecutive 1's in PA. By selecting a suitable countable family of pairwise non-embeddable graphs that do not appear in the age of $\Gamma(PA)$, we ensure a collection of graphs with different ages. The family of graphs we use are the *cycle graphs*. These are graphs C_n on n vertices with n edges such that the degree of each vertex $v \in VC_n$ is 2 (see Figure 8.6 for an example on four vertices). The next pair of lemmas demonstrate that the family of cycle graphs have the properties we require.

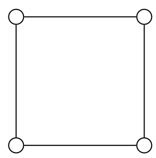


Figure 8.6: The cycle graph C_4

Lemma 8.2.6. *Let C_m and C_n be two cycle graphs with $m, n \geq 3$. Then C_m embeds in C_n (and vice versa) if and only if $m = n$; in which case they are isomorphic.*

Proof. Without loss of generality suppose that that $m < n$; it is clear that C_n does not embed in C_m. Now suppose for a contradiction there is an embedding $\theta : C_m \to C_n$. As $m < n$, we select a vertex $v_i \in VC_n \setminus \operatorname{im}\theta$ such that $v_i \sim v_j$, where v_j is in the image of θ. However, as θ is an embedding, v_j is adjacent to two separate members of im θ and so v_j has degree 3 in C_n. This is a contradiction as every vertex of C_n has degree 2. The converse direction is trivial. $\square$

Lemma 8.2.7. *Let P be any binary sequence, and let $\Gamma(P)$ be the graph determined by P. Then $\Gamma(P)$ does not embed any cycle graph of size $m \geq 4$.*

Proof. Let $P' = (p_{i_1}, p_{i_2}, ..., p_{i_m})$ and $VX = \{v_{i_1}, ..., v_{i_m}\} \subseteq V\Gamma(P)$, with $m \geq 4$. Let X be the graph on VX with edges induced by the subsequence P' of $\Gamma(P)$, and suppose X is an m-cycle. As each vertex in a cycle graph has degree 2, it follows that $p_{i_j} \neq 0$ for $i_j > i_3$. But this means that $p_{i_4} = ... = p_{i_m} = 1$ and so v_{i_4} has degree 0 in X. This is a contradiction and so X is not an m-cycle. $\square$

Hence $\mathcal{C} = \{C_n : n \geq 4\}$ is a countable family of pairwise non-embeddable graphs such that Age$(\Gamma(PA)) \cap \mathcal{C} = \varnothing$. So for a strictly increasing sequence of natural numbers $A = (a_n)_{n\in\mathbb{N}}$ with $a_i \geq 4$, construct $\Gamma(PA)$ in the usual fashion. Here, the size of each I_k in PA is $a_k = m$. For each independent set of vertices $VI_k = \{v_{i_1}, ..., v_{i_m}\}$, draw an m-cycle on its vertices by $v_{i_1} \sim v_{i_m}$ and $v_{i_j} \sim v_{i_{j+1}}$ for $1 \leq j \leq m - 1$, thus creating a new graph $\Gamma(PA)'$ (see Figure 8.7). When we construct a graph $\Gamma(PA)'$ as we have done here, we say that $\Gamma(PA)'$ is the graph *determined by PA with cycles induced.*

In particular, note that even with these additional structures, $\Gamma(PA)'$ is still MB-homogeneous as it has properties $(\triangle)$ and $(\therefore)$. This is because adding in edges does nothing to alter property $(\triangle)$, and the cycles remain independent of each other so we can still find an independent vertex for every finite set, preserving property $(\therefore)$. However, since we have added so many structures to the

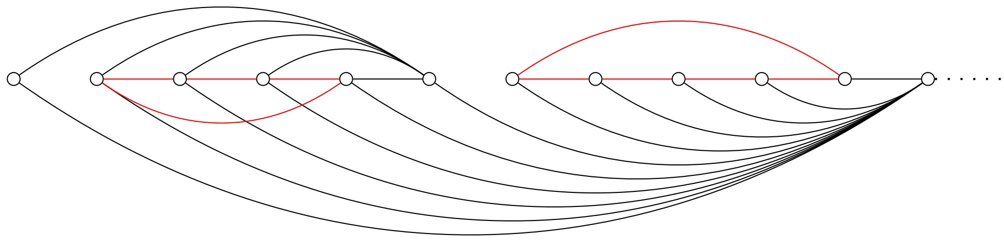


Figure 8.7: $\Gamma(PA)'$ corresponding to the sequence $A = (4, 5, 6, ...)$, with added cycles highlighted in red.

age, we must be careful that there are no extra cycles of sizes not expressed in the sequence A. The next proposition alleviates this concern.

Proposition 8.2.8. *Suppose that $A = (a_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers with $a_1 \geq 4$, and suppose that $m \geq 4$ is a natural number such that $m \neq a_n$ for all $n \in \mathbb{N}$. Then the graph $\Gamma(PA)'$ determined by PA with cycles induced does not contain an m-cycle as an induced subgraph.*

Proof. Suppose that $M \subseteq \Gamma(PA)'$ is an m-cycle. Then the edge set of M is a combination of the edges of the graph on $VM = \{v_{i_1}, ..., v_{i_m}\}$ determined by the finite subsequence $Q = (q_{i_1}, ..., q_{i_m})$ of PA (where $i_1 < i_2 < ... < i_m$), and the edges from cycles added to $\Gamma(PA)$ to make $\Gamma(PA)'$. We aim to show that $M = \Gamma(I_n)$ for some $n \in \mathbb{N}$; that is, the only cycles of size ≥ 4 in $\Gamma(PA)'$ are precisely those we added. As $m \geq 4$, it follows that M cannot embed a 3-cycle by Lemma 8.2.6.

First, assume that Q contains a 0. If $q_{i_n} = 0$ for some $3 < n \leq m$, then by construction of $\Gamma(PA)'$, $d_M(v_{i_n}) \geq n - 1 \geq 3$. As M is an m-cycle, $d_M(v) = 2$ for all $v \in M$; so this is a contradiction and $q_{i_n} \neq 0$ for some $3 < n \leq m$. Therefore $q_{i_4} = ... = q_{i_m} = 1$ and the only elements of Q that can be 0 are q_{i_1}, q_{i_2} and q_{i_3}. We now split our consideration into cases.

Case 1 ($q_{i_3} = 0$)**.** As we assume this, v_{i_3} is adjacent to both v_{i_1} and v_{i_2}. If $q_{i_2} = 0$, then $v_{i_2} \sim v_{i_1}$, creating a 3-cycle; this is a contradiction as M does not embed a 3-cycle. Therefore, $q_{i_2} = 1$ and so v_{i_1} is adjacent to some v_{i_j} where $3 < j \leq m$. Since $q_{i_j} = 1$ for all $i_j > i_3$, it follows that the edge between v_{i_1} and v_{i_j} was induced by an added cycle. This implies that $v_{i_1}, v_{i_j} \in V\Gamma(I_k)$ for some k. Therefore, $(b_{i_1}, ..., b_{i_j})$ is a sequence of 1's where $i_1 < i_2 < ... < i_j$ are consecutive natural numbers; a contradiction as $i_1 < i_3 < i_j$ and so $1 = q_{i_3} = 0$. Hence, $q_{i_3} \neq 0$.

Case 2 ($q_{i_2} = 0$)**.** Since this happens, v_{i_2} is adjacent to v_{i_1} and some vertex $v_{i_j} \neq v_{i_1}$; so $i_j > i_2$. If q_{i_j} is 1, then v_{i_j} is non-adjacent to any vertex v_{i_k} with $i_k < i_j$ and $q_{i_k} = 0$, as the construction of $\Gamma(PA)'$ ensures that no edges are drawn in this case. But this is a contradiction as $q_{i_2} = 0$ and $v_{i_j} \sim v_{i_2}$. So q_{i_j} must be 0 in this case; but $i_j > i_2$ and so $q_{i_j} = 1$ by Case 1 and the argument preceding Case 1. This is a contradiction and so $q_{i_2} = 1$.

A similar argument to that of Case 2 holds for when $q_{i_1} = 0$ and so no element of Q is 0. Hence, Q is made up of 1's; however, these may be from different I_k's. As no element of Q is 0, we conclude that two vertices in M have an edge between them only if they are contained in the same VI_k and have an edge between them in $\Gamma(I_k)$. As M is connected, it follows that $M \subseteq \Gamma(I_k)$ for some k. Finally, as $\Gamma(I_k)$ is an a_k-cycle embedding an m-cycle, we are forced to conclude that $m = a_k$ by Lemma 8.2.6 and so we are done. □

We can now draw our main conclusions.

Corollary 8.2.9. *Suppose that $A = (a_n)_{n\in\mathbb{N}}$ and $B = (b_n)_{n\in\mathbb{N}}$ are two different strictly increasing sequences of natural numbers with $a_1, b_1 \geq 4$. Then $\Gamma(PA)' \not\cong \Gamma(PB)'$.*

Proof. As A and B are different sequences of natural numbers, there exists a $j \in \mathbb{N}$ such that $a_j \neq b_j$; without loss of generality assume that $a_j < b_j$. Hence $\Gamma(PA)'$ embeds an a_j-cycle; but as $a_j \notin B$, by Proposition 8.2.8 $\Gamma(PB)'$ does not embed an a_j-cycle. Hence Age($\Gamma(PA)'$) $\neq$ Age($\Gamma(PB)'$) and so they are not isomorphic. □

This result proves the following:

Theorem 8.2.10. *There exists $2^{\aleph_0}$ many non-isomorphic, non-bi-equivalent MB and HE-homogeneous graphs, each of which is bimorphism equivalent to the random graph R.*

Proof. As there are $2^{\aleph_0}$ strictly increasing sequences of natural numbers, we have continuum many non-isomorphism examples of $\Gamma(PA)'$ by Corollary 8.2.9. Since these graphs have different ages, this means we have constructed $2^{\aleph_0}$ many non-B-equivalent graphs. Furthermore, as each these examples has property $(\triangle)$ and $(\therefore)$, they are MB and HE-homogeneous by Proposition 8.1.6 and bimorphism equivalent to R by Corollary 8.1.10. $\square$

Remark. As a consequence of this, $|[R]^{\sim_b}| = 2^{\aleph_0}$.

Finally in this subsection, we can utilise this technique of overlaying finite graphs in order to prove a striking result. Recall that Frucht's theorem (see [10]) states that any finite group H arises as the automorphism group of some graph Γ. This has been extended (by Frucht himself, [33]) to state that there are countably many 3-regular graphs G such that $\text{Aut}(G) \cong H$.

Theorem 8.2.11. *Any finite group H arises as the automorphism group of an MB-homogeneous graph Γ.*

Proof. By Theorem 2.4.1 there exist countably many graphs G such that $\text{Aut}(G) \cong H$ where $d_G(v) = 3$ for all $v \in VG$. As there are only finitely many graphs of size less than or equal to 5, there exists a graph Δ, where $|V\Delta| = n \geq 6$ and $d_\Delta(v) = 3$ for all $v \in V\Delta$, such that $\text{Aut}(\Delta) \cong H$. By the handshake lemma (see Section 2.4), such a graph must have a total of $3n/2$ edges out of a total of $(n^2 - n)/2$ possible edges; as $n \geq 6$, this means that Δ must induce at least 6 nonedges.

Define a binary sequence $P = (p_i)_{i\in\mathbb{N}}$ by the following:

$$p_i = \begin{cases} 1 \text{ if } i = 0, 1, 2, ..., n-1, n+1, n+3, ... \\ 0 \text{ if } i = n, n+2, ... \end{cases}$$

So P is a sequence of n many 1's followed by alternating 0's and 1's; and so has infinitely many of each. Using the notation established above for Lemma 8.2.2, it follows that $|I_1| = n$ and $|O_k| = |I_m| = 1$ for $k \geq 1$ and $m \geq 2$. Let $\Gamma(P)$ be the graph determined by P on $V\Gamma(P) = \{v_0, v_1, ...\}$, and draw in edges on

VI_1 in any fashion such that $\Gamma(VI_1) \cong \Delta$. We then obtain a graph $\Gamma(P)'$ (see Figure 8.8).

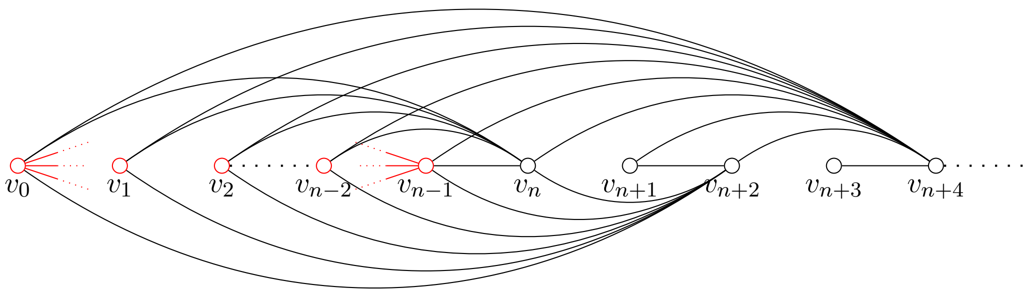


Figure 8.8: $\Gamma(P)'$, with Δ highlighted in red

In a similar fashion to Proposition 8.2.5, we aim to show that VO_i and VI_i are fixed setwise for any automorphism α of $\Gamma(P)'$ through a series of claims.

Claim 1. If $v_a \in VO_i$ and $v_b \in VO_j$ with $i \neq j$, then $\Gamma(N(v)) \ncong \Gamma(N(w))$.

Proof of Claim 1. As $|VO_k| = 1$ for all $k \in \mathbb{N}$, we have that $VO_i = \{v_a\}$ and $VO_j = \{v_b\}$. Assume without loss of generality that $a < b$. We define the following sets:

$$\begin{aligned} X_a &= \{v_k \in V\Gamma(P)' : k < a\} \\ X_b &= \{v_k \in V\Gamma(P)' : k < b\} \\ Y_a &= \{v_k \in V\Gamma(P)' : k \geq a, p_k = 0\} \\ Y_b &= \{v_k \in V\Gamma(P)' : k \geq b, p_k = 0\}. \end{aligned}$$

Lemma 8.2.4 applies in this situation; so we have that $N(v_a) = X_a \cup Y_a$ and $N(v_b) = X_b \cup Y_b$. As in the proof of Proposition 8.2.5, any maximum independent set of $\Gamma(N(v_a)), \Gamma(N(v_b))$ is contained in X_a, X_b respectively. Now, as $\Gamma(VI_1) \cong \Delta$ is a 3-regular graph on more than six vertices, there exists some maximum independent set $M \subseteq VI_1$ of $\Gamma(VI_1)$ with size greater than or equal to 2.

Now we consider the sets

$$A = \{v_c \in V\Gamma(P)' : n-1 < c < a, p_c = 1\} \cup M$$

and

$$B = \{v_c \in V\Gamma(P)' : n-1 < c < b, p_c = 1\} \cup M,$$

the maximum independent sets of X_a and X_b respectively. As $i < j$, there exists d such that $a < d < b$ with $p_d = 1$. Hence $v_d \in B \setminus A$ and so $|B| > |A|$. Since A, B are maximum independent sets of $\Gamma(N(v_a))$ and $\Gamma(N(v_b))$ respectively with different sizes, we conclude that $\Gamma(N(v_a)) \not\cong \Gamma(N(v_b))$. This ends the proof of Claim 1.

This shows that there exists no automorphism γ of $\Gamma(P)'$ sending any $v \in VO_i$ to $w \in VO_j$ with $i \neq j$.

Claim 2. There exists no automorphism sending $v \in VO_k$ to $w \in VI_m$ for all $k, m \in \mathbb{N}$.

Proof of Claim 2. We split the proof into two cases; where $m = 1$ and where $m \geq 2$. For the latter, $\Gamma(N(w)) \cong K^{\aleph_0}$ for any $w \in VI_m$ with $m \geq 2$ by Lemma 8.2.3. But as $\Gamma(VI_1)$ is not a complete graph, we have that $\Gamma(VO_k)$ contains a non-edge for all $k \in \mathbb{N}$ and so $\Gamma(N(v)) \not\cong \Gamma(N(w))$ in this case. It remains to show that there is no automorphism sending $v \in VO_k$ to $w \in VI_1$. In this case $\Gamma(N(w))$ is the union of an infinite complete graph K and $G = \Gamma(N_{\Gamma(VI_1)}(w))$, with every vertex of K connected to every vertex of G. This means any non-edge of $\Gamma(N(w))$ must be induced by G; as $|N_{\Gamma(VI_1)}(w)| = 3$, there are at most 3 of them for any $w \in VI_1$. However, as X_a contains VI_1, we have that $\Gamma(N(v))$ contains Δ as an induced subgraph. By the reasoning above, Δ has at least 6 non-edges and therefore so does $\Gamma(N(v))$. This means that $\Gamma(N(v)) \not\cong G(N(w))$ for any $v \in VO_k$ and any $w \in VI_1$ and so we are done.

Here, VO_k is fixed setwise for all $k \in \mathbb{N}$; as $|VO_k| = 1$ for all such k we have that they are also fixed pointwise. As in the proof of Proposition 8.2.5, it follows that any VI_k sandwiched between VO_{k-1} and VO_k are the only vertices not adjacent to every vertex in VO_k and adjacent to every vertex in VO_{k+1}. As VO_k and VO_{k+1} are fixed setwise, we deduce that VI_k is fixed setwise (and hence pointwise) for $k \geq 2$. We conclude that VI_1 is fixed setwise under automorphisms of $\Gamma(P)'$.

Finally, we show that any bijective map $\gamma : \Gamma(P)' \to \Gamma(P)'$ acting as an automorphism on VI_1 and fixing everything else is an automorphism of $\Gamma(P)'$. As every $v \in VO_k$ for all k is connected to each $u \in VI_1$, and every $w \in VI_m$ for all m is independent of each $u \in VI_1$, it follows that γ preserves all edges and non-edges of $\Gamma(P)'$ and so $\gamma \in \mathrm{Aut}(\Gamma(P)')$. $\square$

Remark. Using this together with Theorem 6.2.8, it follows that for any finite group U there exists an oligomorphic permutation monoid with group of units isomorphic to U.

We finish this section with a question concerning bimorphism equivalence of MB-homogeneous graphs in general. A positive answer to this would constitute the best classification result possible for MB-homogeneous graphs, given the amount and range of examples above.

Question 8.2.12. *Is every countable MB-homogeneous graph bimorphism equivalent to one of the five graphs in Theorem 8.1.4?*

Related to this question is the following, slightly more general question:

Question 8.2.13. *Are there only countably many MB-homogeneous graphs up to bimorphism equivalence?*

Finally, we proved in Proposition 8.1.6 that every graph with properties $(\triangle)$ and $(\because)$ is both MB and HE-homogeneous. As every MB-homogeneous graph we have constructed here has properties $(\triangle)$ and $(\because)$ is also HE-homogeneous graph, we can ask:

Question 8.2.14. *Is there a countably infinite MB-homogeneous graph that is not*

HE-homogeneous? Conversely, is there a HE-homogeneous graph that is not MB-homogeneous?

8.3 MB-homogeneous oriented graphs

8.3.1 Properties of MB-homogeneous oriented graphs

Recall from Section 2.4 that an oriented graph is a loopless digraph that does not contain a 2-cycle. Following on from the work in Section 8.1 and Section 8.2, our aim in this section is to provide a range of examples of MB-homogeneous oriented graphs. As before, we begin searching for a new pair of properties implying MB-homogeneity that an oriented graph may have; these properties must ensure that both the 1PMMEP and the 1P$\overline{\text{MM}}$EP hold for such an oriented graph. In this case, the 1P$\overline{\text{MM}}$EP is the easier of the two; we can take a version of property ($\therefore$); for the 1PMMEP, some more thought is required.

Definition 8.3.1. Let $\mathcal{G}$ be an oriented graph. Say that $\mathcal{G}$ has *property* ($\leftrightarrows$) if for all finite, disjoint subsets U, V of $V\mathcal{G}$ there exists $x \in V\mathcal{G}$ such that $x \to u$ for all u in U and $v \to x$ for all $v \in V$. Say that $\mathcal{G}$ has *property* ($\therefore$) if for every finite subset $W \subseteq V\mathcal{G}$ there exists a $y \in V\mathcal{G}$ such that $w \| y$ for all w in W. (See Figure 8.9 for a diagram illustrating property ($\leftrightarrows$).)

Remark. Note that a straight generalisation of property ($\triangle$) into two digraph conditions (one with arrows to the set, and one with arrows from) would not suffice. The idea behind defining this property is to find suitable image points to extend functions; with two digraph conditions, this task is not achievable due to both in and out relations that need to be preserved, and mapped to a single vertex (see proof of Lemma 8.3.2 for more details).

These two properties together are sufficient conditions for MB-homogeneity.

Lemma 8.3.2. *Let $\mathcal{G}$ be an oriented graph with both properties ($\leftrightarrows$) and ($\therefore$). Then $\mathcal{G}$ is MB-homogeneous.*

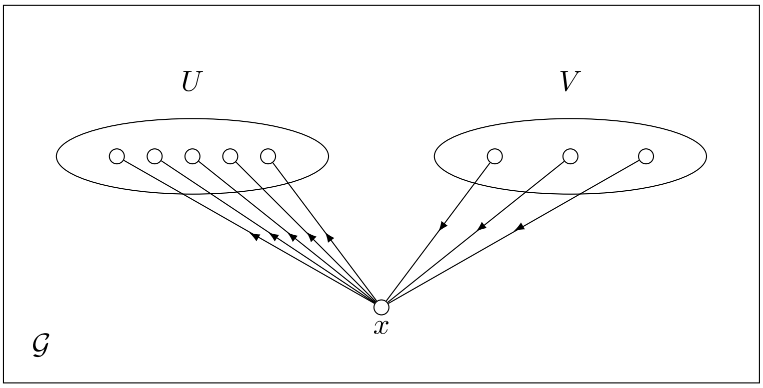


Figure 8.9: Property ($\leftrightarrows$)

Proof. It suffices to show that $\mathcal{G}$ has both the 1PMMEP and 1P$\overline{\text{MM}}$EP from Proposition 7.3.1. So suppose that $A \subseteq B \in \text{Age}(\mathcal{G})$ where $B \smallsetminus A = \{b\}$ and that $f : A \to \mathcal{G}$ is a monomorphism. As outlined in Section 2.4, we can decompose A into three sets of vertices based on the relation with b in B: $A^{\rightarrow}(b)$, $A^{\leftarrow}(b)$ and $A^{\|}(b)$. Take the union of $A^{\rightarrow}(b)$ with $A^{\|}(b)$. As f is injective, the subsets $(A^{\rightarrow}(b) \cup A^{\|}(b))f$ and $A^{\leftarrow}(b)f$ of $V\mathcal{G}$ are both finite and disjoint. Using property ($\leftrightarrows$), find a vertex x such that $x \to af$ for all $af \in (A^{\rightarrow}(b) \cup A^{\|}(b))f$ and $a'f \to x$ for all $a'f \in A^{\leftarrow}(b)$. We can define a monomorphism $g : B \to \mathcal{G}$ extending f and sending $b \in B$ to the vertex $x \in V\mathcal{G}$; hence $\mathcal{G}$ has the 1PMMEP. Now suppose that $\bar{f} : A \to \mathcal{G}$ is an antimonomorphism. As $\mathcal{G}$ has property ($\therefore$), there exists a vertex $w \in V\mathcal{G}$ such that $w\|af$ for all $af \in Af$. Define an antimonomorphism $\bar{g} : B \to \mathcal{G}$ extending f and sending $b \in B$ to the vertex $w \in V\mathcal{G}$; so $\mathcal{G}$ has the 1P$\overline{\text{MM}}$EP and we are done. □

Remark. Unlike the analogous case for undirected graphs (Proposition 8.1.6), an oriented graph $\mathcal{G}$ having properties ($\leftrightarrows$) and ($\therefore$) does *not* imply that $\mathcal{G}$ is HE-homogeneous. For example, the generic oriented graph D (see Example 2.4.8) has properties ($\leftrightarrows$) and ($\therefore$) but is not HE-homogeneous by Example 7.3.7.

In a similar fashion to Definition 8.1.7, we can say that two countably infinite oriented graphs $\mathcal{G}$ and $\mathcal{H}$ are *bimorphism equivalent* if there exists bijective homomorphisms $\alpha : \mathcal{G} \to \mathcal{H}$ and $\beta : \mathcal{H} \to \mathcal{G}$.

Proposition 8.3.3. *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are bimorphism equivalent oriented graphs via the bijective homomorphisms $\alpha : \mathcal{G} \to \mathcal{H}$ and $\beta : \mathcal{H} \to \mathcal{G}$. Then $\mathcal{G}$ has properties*

$(\therefore)$ *and* $(\leftrightarrows)$ *if and only if* $\mathcal{H}$ *does.*

Proof. Suppose that $\mathcal{G}$ has property $(\leftrightarrows)$, and that X, Y are two finite, disjoint subsets of $V\mathcal{G}$. By property $(\leftrightarrows)$, there exists a vertex v such that $v \to x$ for all $x \in U$ and $y \to v$ for all $y \in Y$. Since α is a function, $X\alpha$ and $Y\alpha$ are finite subsets of $\mathcal{H}$ and as α is injective, these two sets are also disjoint. As α is a homomorphism, the vertex $v\alpha \in V\mathcal{H}$ is a vertex such that $v\alpha \to x\alpha$ for all $x\alpha \in X\alpha$ and $y\alpha \to v\alpha$ for all $y \in Y\alpha$. Finally, because α is surjective, every finite subset $Z \subseteq V\mathcal{H}$ can be written as $W\alpha$ for some finite $W \subseteq V\mathcal{G}$ by Lemma 7.1.5. These observations prove that $\mathcal{H}$ has property $(\leftrightarrows)$.

Now suppose that $\mathcal{G}$ has property $(\therefore)$. As $\beta : \mathcal{H} \to \mathcal{G}$ is a bijective homomorphism, the converse map $\bar{\beta} : \mathcal{G} \to \mathcal{H}$ is a bijective antihomomorphism by Lemma 7.1.2. Select a finite subset $W \subseteq V\mathcal{G}$. As $\mathcal{G}$ has property $(\therefore)$, there exists a vertex y that is independent of W. As $\bar{\beta}$ is a bijective antihomomorphism (and therefore a function), we have that;

- $W\bar{\beta}$ is a finite subset of $V\mathcal{H}$;
- $y\bar{\beta}$ is a single vertex that is independent of $W\bar{\beta}$ in $\mathcal{H}$, and;
- Every finite $Y \subseteq V\mathcal{H}$ can be written as some finite $X\bar{\beta}$, where W is a finite subset of $V\mathcal{G}$.

These observations show that $\mathcal{H}$ has property $(\therefore)$. The proof of the converse direction is symmetric. □

Proposition 8.3.4. *If* $\mathcal{G}, \mathcal{H}$ *are two oriented graphs with properties* $(\leftrightarrows)$ *and* $(\therefore)$*, then* $\mathcal{G}$ *and* $\mathcal{H}$ *are bimorphism equivalent.*

Proof. Suppose that $\mathcal{G}, \mathcal{H}$ are two graphs with properties $(\leftrightarrows)$ and $(\therefore)$. Similar to Proposition 8.1.9, we use a back-and-forth argument to construct a bijective homomorphism $\alpha : \mathcal{G} \to \mathcal{H}$ and a bijective antihomomorphism $\bar{\beta} : \mathcal{G} \to \mathcal{H}$, which by Lemma 7.1.2 will be the converse of a bijective homomorphism $\beta : \mathcal{H} \to \mathcal{G}$. Assume that $f : \{x\} \to \{y\}$ is some function sending a vertex x of $\mathcal{G}$ to a vertex y of $\mathcal{H}$; this is a bijective homomorphism. Now set $\{x\} = X_0$, $\{y\} = Y_0$,

$f = f_0$, and suppose that we have extended f to a bijective homomorphism $f_k : X_k \to Y_k$, where X_i, Y_i are finite and $X_i \subseteq X_{i+1}$ and $Y \subseteq Y_{i+1}$ for all $0 \leq i \leq k-1$. In addition to this, since both $\mathcal{G}$ and $\mathcal{H}$ are countable, we can enumerate their vertices as $V\mathcal{G} = \{x_0, x_1, ...\}$ and $V\mathcal{H} = \{y_0, y_1, ...\}$.

If k is even, select a vertex $x_j \in \mathcal{G}$ where j is the smallest number such that $x_j \notin X_k$. As this happends, we can decompose X_k into the three neighbourhood sets $X^{\to}(x_j)$, $X^{\leftarrow}(x_j)$ and $X^{\parallel}(x_j)$. As f_k is bijective, the subsets $(X^{\to}(x_j) \cup X^{\parallel}(x_j))f_k$ and $X^{\leftarrow}(x_j)f_k$ of VY_k are both finite, disjoint, and their union is VY_k. As $\mathcal{H}$ has property $(\leftrightarrows)$, there exists a vertex $u \in \mathcal{H}$ such that $af_k \to u$ for all $af_k \in (X^{\to}(x_j) \cup X^{\parallel}(x_j))f_k$ and $u \to af_k$ for all $X^{\leftarrow}(x_j)f_k$. Define a map $f_{k+1} : X_k \cup \{x_j\} \to Y_k \cup \{u\}$ sending x_j to u and extending f; this map is a bijective homomorphism as any relation between x_k and some element of Y_k is preserved (see Figure 8.10 for a diagram of an example).

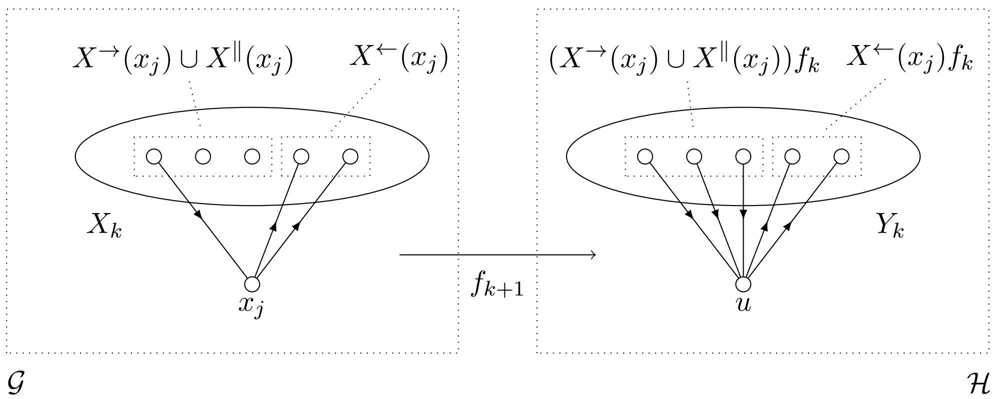


Figure 8.10: k even in proof of Proposition 8.3.4

Now, if k is odd, choose a vertex $y_j \in \mathcal{H}$ where j is the smallest number such that $y_j \notin Y_k$. As $\mathcal{G}$ has property $(\therefore)$, there exists a vertex $v \in \mathcal{G}$ such that v is independent of every element of X_k. Define a map $f_{k+1} : X_k \cup \{v\} \to Y_k \cup \{y_j\}$ sending v to y_j and extending f_k. Then f_{k+1} is a bijective homomorphism as f_k is and every relation between v and X_k is preserved by the fact that there are none (see Figure 8.11 for a diagram of an example of this stage).

We can repeat this process infinitely many times, ensuring that each vertex of

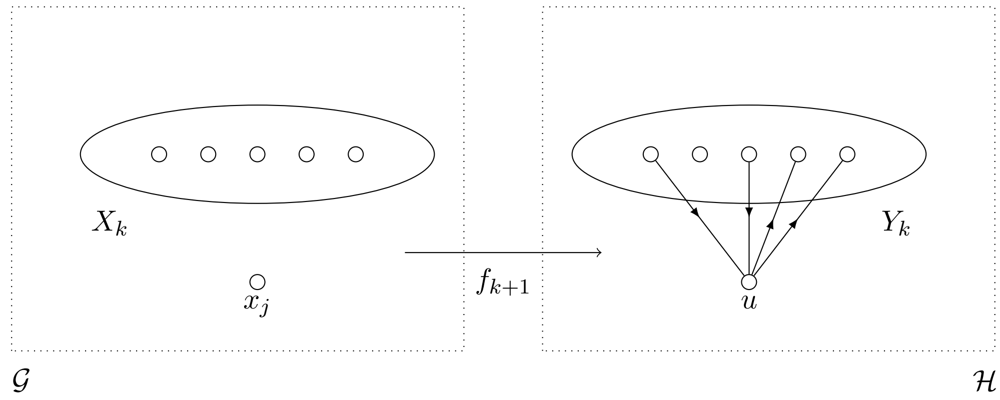


Figure 8.11: k odd in proof of Proposition 8.3.4

$\mathcal{G}$ appears at an even stage and each vertex of $\mathcal{H}$ appears at an odd stage. Doing this defines a bijective homomorphism $\alpha : \mathcal{G} \to \mathcal{H}$. We can construct a bijective antihomomorphism $\bar{\beta} : \mathcal{G} \to \mathcal{H}$ in a similar fashion; replacing homomorphism with antihomomorphism and using property $(\therefore)$ of $\mathcal{H}$ at even steps and property $(\leftrightarrows)$ of $\mathcal{G}$ as above at odd steps. So the converse map $\beta : \mathcal{H} \to \mathcal{G}$ is a bijective homomorphism; proving that $\mathcal{G}$ and $\mathcal{H}$ are bimorphism equivalent. $\square$

Corollary 8.3.5. *An oriented graph $\mathcal{G}$ has properties $(\leftrightarrows)$ and $(\therefore)$ if and only if $\mathcal{G}$ is bimorphism equivalent to the generic oriented graph D (see Example 2.4.9).*

Proof. As D has both properties $(\leftrightarrows)$ and $(\therefore)$, the converse direction follows from Proposition 8.3.3 and the forward direction is a consequence of Proposition 8.3.4. $\square$

Inspired by the construction of the random graph R outlined in Péresse [69, Lemma 3.10.2], we use this machinery to demonstrate an example of a countable MB-homogeneous oriented graph that is not isomorphic to the generic oriented graph D (see Example 2.4.8).

Example 8.3.6. Let $\mathcal{G}$ be a finite oriented graph with at least one vertex. We define an oriented graph $\mathcal{H}(\mathcal{G})$ inductively over countably many steps H_i where $i \in \mathbb{N}$. To begin with, let H_0 be the oriented graph $\mathcal{G}$, and assume that H_n has been constructed. For every finite subset $A \subseteq VH_n$, add a vertex v_A and draw arcs from v_A to every $v \in A$ and draw arcs to v_A from every $w \in VH_n \setminus A$. Additionally, add a vertex u_{n+1} such that $u_{n+1} \| u$ for all $u \in VH_n$, and $u_{n+1} \| v_A$

for all finite subsets $A \subseteq VH_n$. Say that the resulting digraph is H_{n+1}; this is countable as each H_n is finite. As H_{n+1} contains H_n for every $n \in \mathbb{N}$, define $\mathcal{H}(\mathcal{G}) = \bigcup_{i \in \mathbb{N}} H_i$. Then $\mathcal{H}(\mathcal{G})$ is a countably infinite oriented graph with properties $(\therefore)$ and $(\leftrightarrows)$; so it is MB-homogeneous by Lemma 8.3.2.

A diagram illustrating the construction of $\mathcal{H}(\mathcal{G})$ at the H_2 stage, when $\mathcal{G}$ is the oriented graph on a single vertex $\{v\}$, is given in Figure 8.12.

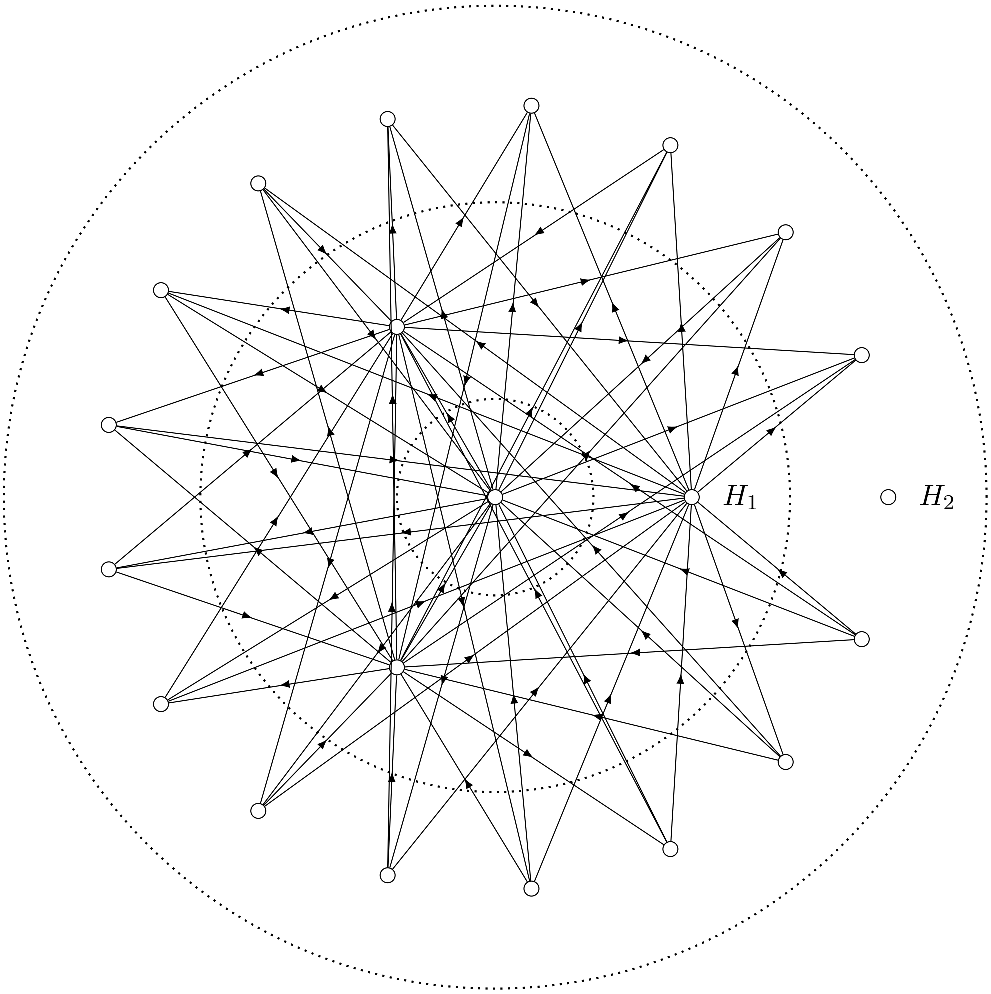


Figure 8.12: H_2 in the construction of $\mathcal{H}(\mathcal{G})$, where $\mathcal{G}$ is a single vertex

We show that $\mathcal{H}(\mathcal{G})$ does not have the extension property characteristic of the countable generic oriented graph (see Example 2.4.9). So as $|VH_0| \geq 1$, it follows that $|VH_1 \smallsetminus VH_0| \geq 3$ via a counting argument. Now, take $X = VH_0$, $Y = \{y\}$ (where $y \in VH_1 \smallsetminus VH_0$) and $Z = VH_1 \smallsetminus (VH_0 \cup \{y\})$; therefore, X, Y, Z

are a trio of non-empty, finite and pairwise disjoint sets. As $\mathcal{H}(\mathcal{G})$ has property $(\leftrightarrows)$, we can find a vertex $v \in VH_n$ for some $n \geq 2$ such that $v \to u$ for all $u \in U$ and $z \to v$ for all $z \in Z$. As $y \in VH_1 \smallsetminus VH_0$, then either $v \to y$ or $y \to v$ due to our construction. Therefore, there is no vertex $v \in VH_n$ satisfying the OARP characteristic of D for these sets X, Y, Z; hence $\mathcal{H}(\mathcal{G}) \ncong D$.

Remark. Note that the construction of $\mathcal{H}(\mathcal{G})$ did not depend on the size of the initial oriented graph $\mathcal{G}$. As we add a vertex for each *finite* subset A of VH_n at each H_{n+1} step, it follows that we add at most countably many vertices. So if $\mathcal{G}$ were countably infinite, then $\mathcal{H}(\mathcal{G})$ when constructed in this fashion is also countably infinite. This means we can build a countable MB-homogeneous oriented graph $\mathcal{H}'$ with any countable oriented graph $\mathcal{G}$ as an induced subgraph. Whether or not this $\mathcal{H}(\mathcal{G})$ is isomorphic to D or not is unclear, given the amount of freedom in the choice of $\mathcal{G}$; for instance, taking $\mathcal{G} = D$ may present issues.

8.3.2 Uncountably many MB-homogeneous oriented graphs

As in the undirected case, the idea is to construct uncountably many examples of MB-homogeneous oriented graphs by using a countable family of pairwise non-embeddable oriented graphs and incorporating them into some MB-homogeneous construction based on strictly increasing sequence of natural numbers. Similar to before, cycle graphs are the family we utilise; however, in the oriented case there are two distinct notions (see Figure 8.13):

- an *oriented cycle graph* on n vertices, an orientation of some cycle graph C_n;
- the *cycle digraph* D_n on n vertices, the unique orientation of the cycle graph C_n where every $v \in D_n$ has indegree and outdegree 1.

In our consideration, we use the cycle digraphs. The first result is the oriented graph analogue of Lemma 8.2.6.

Lemma 8.3.7. *Let D_m be the cycle digraph on m vertices, and let $\mathcal{G}$ be some oriented cycle graph on n vertices, where $m, n \geq 3$. Then $\mathcal{G}$ embeds in D_m if and only if $m = n$ and $\mathcal{G}$ is a cycle digraph; in which case they are isomorphic.*

D_4 $\mathcal{G}$

Figure 8.13: The cycle digraph D_4 and an oriented cycle graph $\mathcal{G}$ on 4 vertices

Proof. Assume first that $m < n$; it is clear in this case that $\mathcal{G}$ does not embed in D_m. Now suppose that $n < m$ and assume for a contradiction there is an embedding $\theta : \mathcal{G} \to D_m$. As $n < m$, we select a vertex $v_i \in VD_m \setminus \operatorname{im}\theta$ such that $v_i \sim v_j$, where v_j is in the image of θ. However, as θ is an embedding, v_j has arcs to two separate members of im θ and so the sum of indegree and outdegree of v_j in $\mathcal{G}$ is 3. This is a contradiction as this sum is at most 2 for every vertex v_k of $\mathcal{G}$; so $m = n$. Now, if $\mathcal{G}$ is an oriented cycle graph that is not a cycle digraph, then there exists at least one vertex $v \in V\mathcal{G}$ such that either the indegree or the outdegree of v in $\mathcal{G}$ is 2. As there are no such vertices in a cycle digraph, $\mathcal{G}$ cannot embed in D_m. The converse direction is trivial. □

As any cycle digraph is an oriented cycle graph, it follows from this lemma that the collection of cycle digraphs $(D_n)_{n\geq 3}$ are a countable family of pairwise non-embeddable digraphs. Our next lemma investigates the intersection of this collection with the age of $\mathcal{H}(\mathcal{G})$ as described in Example 8.3.6.

Lemma 8.3.8. *Let $\mathcal{G}$ be an oriented graph on one vertex, and suppose that $\mathcal{H}(\mathcal{G})$ is constructed as in Example 8.3.6. Then $\mathcal{H}(\mathcal{G})$ does not embed any odd cycle digraphs of size ≥ 5.*

Proof. Suppose that $\mathcal{C}$ is an cycle digraph of size ≥ 5. We show that $\mathcal{C}$ can only contain vertices from only two H_i's, and then deduce that this is only possible when $\mathcal{C}$ is a cycle digraph of even size. So let $x_1, x_2, x_3 \in \mathcal{C}$, where $x_1 \in H_i$, $x_2 \in H_j \setminus H_i$ and $x_3 \in H_k \setminus H_j$, where $i < j < k$. There are two cases to consider; where x_3 is independent of x_1 and x_2, and where there is some arc in either direction between x_3 and one of x_1, x_2.

Case 1. If x_3 is independent of x_1 and x_2, it is independent of every other vertex in H_k by construction of $\mathcal{H}(\mathcal{G})$. As $|V\mathcal{C}| \geq 5$, there exists a vertex $y \in \mathcal{C}$ where either $y \to x_3$ or $x_3 \to y$. As this happens, $y \in H_n$ for some $n > k$ by construction. From this, and in either of these scenarios, there must be an arc in either direction between y and x_1, and also an arc in either direction between y and x_2. So the sum of the indegree and outdegree of y in $\mathcal{C}$ is at least 3; a contradiction and so x_3 is not independent of x_1 and x_2.

Case 2. Now suppose that there is some arc in either direction between x_3 and one of x_1, x_2. By construction, this means that there is an arc in either direction between x_3 and both x_1 and x_2. There are then two further cases as $x_1, x_2, x_3 \in \mathcal{C}$; either $x_1 \to x_3$ and $x_3 \to x_2$, or $x_2 \to x_3$ and $x_3 \to x_1$. Assume the former. Now, it must be true that x_2 is independent of x_1; as if there was an arc in either direction between x_1 and x_2, this would create an oriented cycle graph of size 3 contained in $\mathcal{C}$ which is impossible by Lemma 8.3.7. As this happens, there exists a $y \in \mathcal{C}$ such that $x_2 \to y$; as $y \in \mathcal{H}(\mathcal{G})$, it follows that $y \in H_n$ and $n > j$ by construction. But in this case, there is an arc in either direction between y and x_1; this then creates an oriented cycle graph M on 4 vertices. However, this cannot happen as M would then be an induced subgraph of $\mathcal{C}$; this is impossible by Lemma 8.3.7.

This completes the proof that any cycle digraph $\mathcal{C}$ of size ≥ 5 contains only vertices from two H_i's. By construction, any pair of vertices $u, v \in H_i$ for some i are independent of each other; we can then deduce that $\mathcal{C}$ can be represented as a bipartite digraph. By a result of graph theory (see [21]), a cycle graph is bipartite if and only if the order is even; this result is applicable in the cycle digraph case. So $\mathcal{C}$ must be of even order and hence no odd cycle digraphs of size ≥ 5 embed in $\mathcal{H}(\mathcal{G})$. □

We can now describe a modified construction of Example 8.3.6. Let $S = (s_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence of odd natural numbers where $s_1 \geq 5$, and let $\mathcal{G}$ be some finite oriented graph. Define an oriented graph $\mathcal{H}(\mathcal{G}, S)$ inductively as follows. Take $\mathcal{G} = H_0$, and assume that H_n has been constructed for some $n \in \mathbb{N}_0$. As in Example 8.3.6, for every finite subset A of H_n add a vertex

v_A, and add arcs where $v_A \to a$ for all $a \in A$ and $b \to v_A$ for all $b \in VH_n \setminus A$. In addition to this, when $i = n+1$, add s_i many vertices independent of every element in H_n *and* every v_A, and draw a cycle digraph of size s_i on these vertices. Call the resulting oriented graph H_{n+1} and note that as $H_n \subseteq H_{n+1}$ for every $n \in \mathbb{N}_0$, we can define $\mathcal{H}(\mathcal{G}, S) = \bigcup_{j\in\mathbb{N}} H_j$. A diagram outlining a portion of this construction when $\mathcal{G}$ is a single vertex and $S = (5, 7, ...)$ is given in Figure 8.14. Note that $\mathcal{H}(\mathcal{G}, S)$ has properties ($\leftrightarrows$) and ($\therefore$) for any $\mathcal{G}$ and any S and so is MB-homogeneous by Figure 8.9.

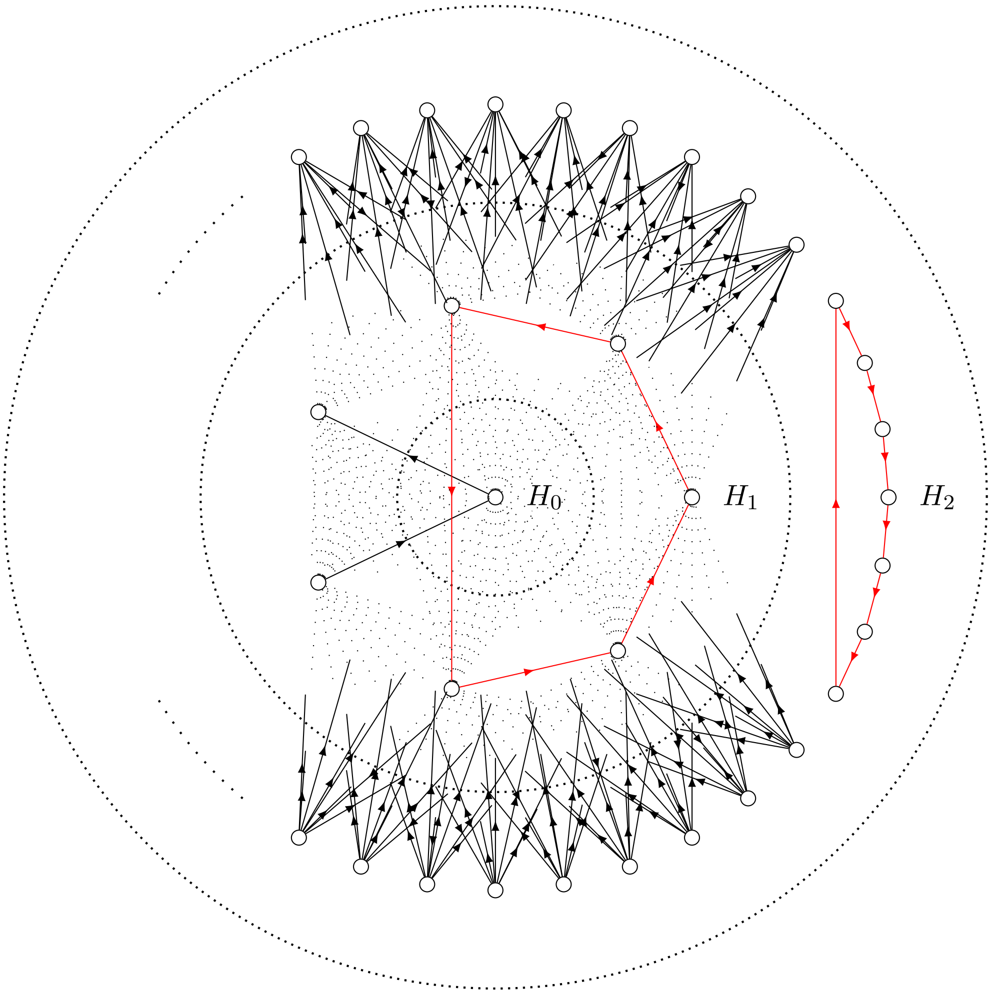


Figure 8.14: H_2 in the construction of $\mathcal{H}(\mathcal{G}, S)$, where $\mathcal{G}$ is a single vertex and S is the sequence $(5, 7, ...)$, with added cycles coloured in red

Lemma 8.3.9. *Let $\mathcal{G}$ be the oriented graph on one vertex. Suppose that $S = (s_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of odd natural numbers with $a_1 \geq 5$ and that $m \geq 5$ is an* *odd natural number such that $m \neq s_i$ for all $i \in \mathbb{N}$. Then the oriented graph $\mathcal{H}(\mathcal{G}, S)$ does not embed a cycle digraph of size m.*

Proof. Let M_i denote the cycle digraph added at the ith stage of construction, and suppose that $\mathcal{C}$ is a cycle digraph of size m, where $m \geq 5$ is an odd natural number. It suffices to show that that $\mathcal{C} = M_i$ for some $i \in \mathbb{N}$; so assume for a contradiction that $\mathcal{C} \neq M_i$ for any i. It follows from Lemma 8.3.7 that any such cycle digraph $\mathcal{C}$ must contain at least one arc from some M_i. As this happens, there must be at least two vertices in $VC \cap VM_i$. Furthermore, as $\mathcal{C} \neq M_i$, there exists an arc in either direction between some v in M_i and $w \in VC \setminus VM_i$. Without loss of generality, we can also assume that there is an arc in either direction between this $v \in M_i$ and some $u \in M_i$. By construction, this means that $w \in H_n$, where $n > i$. From this, there is an arc between w and every element in $\mathcal{C} \cap M_i$; in particular, there is an arc in either direction between w and u. So then there exists at least one oriented cycle graph of size 3 contained in $\mathcal{C}$; a contradiction. Therefore, $\mathcal{C} = M_i$ for some $i \in \mathbb{N}$. □

Corollary 8.3.10. *Let $\mathcal{G}$ be the oriented graph on one vertex. Suppose that $S = (s_i)_{i\in\mathbb{N}}$ and $T = (t_i)_{i\in\mathbb{N}}$ are two different strictly increasing sequences of odd natural numbers with $s_1, t_1 \geq 5$. Then $\mathcal{H}(\mathcal{G}, S) \not\cong \mathcal{H}(\mathcal{G}, T)$.*

Proof. The proof is almost exactly as in Corollary 8.2.9, with cycle graphs replaced by cycle digraphs. □

Theorem 8.3.11. *There are $2^{\aleph_0}$ many pairwise non-isomorphic, non-B-equivalent oriented graphs, each of which is bimorphism equivalent to the generic oriented graph D.*

Proof. As there are $2^{\aleph_0}$ many strictly increasing sequences of odd natural numbers, there are $2^{\aleph_0}$ pairwise non-isomorphic oriented graphs $\mathcal{H}(\mathcal{G}, S)$ by Corollary 8.3.10. For $S \neq T$, as each $\mathcal{H}(\mathcal{G}, S)$ has a different age from $\mathcal{H}(\mathcal{G}, T)$, they are non-B-equivalent. As each example $\mathcal{H}(\mathcal{G}, S)$ has properties ($\leftrightarrows$) and ($\therefore$), it follows from Corollary 8.3.5 that it is bimorphism equivalent to D. □

This section, along with those oriented graphs studied in Section 7.3, barely scratch the surface of the study of homomorphism-homogeneous oriented graphs.

We therefore ask the following two questions, the oriented graph analogues of those posed at the end of Section 8.2:

Question 8.3.12. *Analogously to Theorem 8.1.4, classify the IA-homogeneous oriented graphs that are also MB-homogeneous.*

Question 8.3.13. *Is every MB-homogeneous oriented graph bimorphism equivalent to an MB and IA-homogeneous oriented graph? How many MB-homogeneous oriented graphs are there up to bimorphism equivalence?*

8.4 MB-homogeneous digraphs

Finally in this chapter, we modify techniques from previous sections in order to show similar results for digraphs. Unlike the oriented graph case, we can directly transfer some of the theory in Section 8.1 and Section 8.2 to the case for digraphs; this is because of the existence of 2-cycles.

Definition 8.4.1. Let $\mathcal{D}$ be an infinite digraph.

- Say that $\mathcal{D}$ has *property* $(\uparrow\downarrow)$ if for every finite set $U \subseteq V\mathcal{D}$ there exists $u \in V\mathcal{D}$ such that there is a 2-cycle between u and every member of U.
- Say that $\mathcal{D}$ has *property* $(\therefore)$ if for every finite set $W \subseteq V\mathcal{D}$ there exists $w \in V\mathcal{D}$ such that w is independent of every member of V.

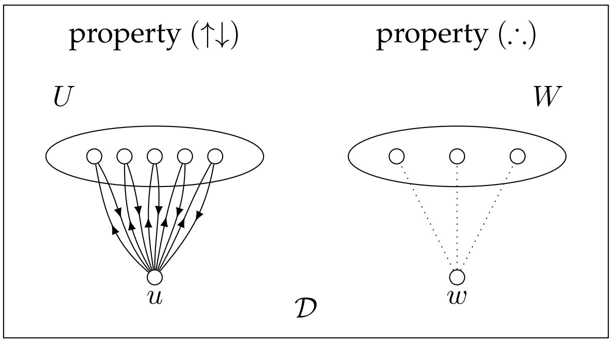


Figure 8.15: A diagram of Definition 8.4.1

Remark. Property $(\uparrow\downarrow)$ in Definition 8.4.1 is a restatement for what it means for a digraph to be *algebraically closed* [62]. In the same source, McPhee noted that this notion of algebraic closure for digraphs is the same notion as algebraic closure for graphs; hence, any graph that is algebraically closed in the class of graphs is also algebraically closed in the class of digraphs.

We now state a lemma analogous to Proposition 8.1.6 and Lemma 8.3.2.

Lemma 8.4.2. *Let $\mathcal{D}$ be a digraph with properties $(\uparrow\downarrow)$ and $(\therefore)$. Then $\mathcal{D}$ is MB and HE-homogeneous.*

Proof. The proof of this is a similar argument to Proposition 8.1.6, with digraph in place of graph, property $(\uparrow\downarrow)$ instead of property $(\triangle)$, and 2-cycle in place of adjacency. $\square$

We now utilise a technique of [24, Section 4]. Consider an undirected graph Γ. By Example 2.3.1, Γ interprets a binary relation $E \subseteq V\Gamma \times V\Gamma$, and models the first-order formulae expressing irreflexivity and symmetry of that relation E. Hence for any two vertices a and b in an undirected graph Γ, it follows that $\{a, b\}$ is an edge of Γ if and only if $(a, b), (b, a) \in E$. Therefore, we can view any simple, undirected graph $\Gamma = (V\Gamma, E\Gamma)$ as a loopless digraph $\mathcal{D}(\Gamma) = (V\Gamma, A\mathcal{D}(\Gamma))$ where

$$\{a, b\} \in E\Gamma \Leftrightarrow (a, b), (b, a) \in A\mathcal{D}(G).$$

So $\mathcal{D}(\Gamma)$, consists solely of non-arcs and 2-cycles, where there is a 2-cycle between vertices a and b in $\mathcal{D}(\Gamma)$ if and only if there is an edge between a and b in Γ. See Figure 8.16 for a diagram of this idea.

Figure 8.16: The cycle graph C_4 and its corresponding digraph $\mathcal{D}(C_4)$

It is easy to see that any infinite undirected graph Γ has properties $(\triangle)$ and $(\therefore)$ if and only if the digraph $\mathcal{D}(\Gamma)$ has properties $(\uparrow\downarrow)$ and $(\therefore)$. Furthermore, we can check to see that Lemma 8.2.6, Lemma 8.2.7 and Proposition 8.2.8 hold when we view graphs as digraphs. Thus, we can prove the following:

Theorem 8.4.3. *There exists $2^{\aleph_0}$ many non-isomorphic, non-B-equivalent MB and HE-homogeneous digraphs.*

Proof. It follows from Corollary 8.2.9 that we can construct $2^{\aleph_0}$ many non-isomorphic, non-B-equivalent digraphs. As each example of an undirected graph Γ described in Corollary 8.2.9 has properties $(\triangle)$ and $(\therefore)$, its corresponding digraph $\mathcal{D}(\Gamma)$ has properties $(\uparrow\downarrow)$ and $(\therefore)$ and so is MB and HE-homogeneous by Lemma 8.4.2. □

Whilst we have uncountably many examples of MB-homogeneous digraphs here, they are simply a restatement of what we have already found. Ideally, we would like some examples of MB-homogeneous digraphs that cannot be viewed as undirected graphs. We can do this by using the technique of overlaying *directed* cycles on graphs constructed in the manner of Figure 8.7.

Example 8.4.4. Let $A = (a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with $a_1 \geq 4$, and let PA be the associated binary sequence as defined at the start of Subsection 8.2.1. Construct the undirected graph $\Gamma(PA)$ in the fashion of Example 8.2.1, and take $\mathcal{E}(PA) = \mathcal{D}(\Gamma(PA))$ to be its corresponding digraph. As before, there are independent sets induced on vertices $VI_k = \{v_{i_1}, ..., v_{i_n}\}$ corresponding to the k^{th} string of 1's in the binary sequence PA. On each VI_k, induce a *directed* cycle of length k by $v_{i_m} \to v_{i_1}$ and $v_{i_j} \to v_{i_{j+1}}$ for $1 \leq j \leq m-1$ to obtain the digraph $\mathcal{E}(PA)'$ (see Figure 8.17 for a diagram).

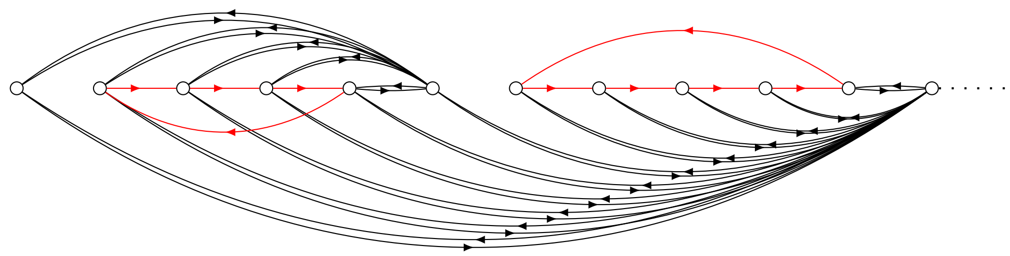


Figure 8.17: $\mathcal{E}(PA)'$ corresponding to the sequence $A = (4, 5, 6, ...)$, with added cycles highlighted in red.

Here, $\mathcal{E}(PA)'$ has both properties $(\uparrow\downarrow)$ and $(.\!\because)$ and so is an MB-homogeneous digraph by Lemma 8.4.2. However, because of the presence of directed cycles on independent sets. $\mathcal{E}(PA)'$ cannot be viewed as an undirected graph Γ.

Analogously to Proposition 8.2.8, it follows (using Lemma 8.3.7) that any digraph $\mathcal{E}(PA)'$ defined in this fashion does not embed a directed cycle graph other than those added in the construction. Using this observation, we can prove that:

Corollary 8.4.5. *Suppose that $A = (a_n)_{n\in\mathbb{N}}$ and $B = (b_n)_{n\in\mathbb{N}}$ are two different strictly increasing sequences of natural numbers with $a_1, b_1 \geq 4$. Then $\mathcal{E}(PA)' \ncong \mathcal{E}(PB)'$.*

Proof. As in Corollary 8.2.9, with cycles replaced by directed cycles. □

The next result follows as an immediate consequence of this.

Theorem 8.4.6. *There exists $2^{\aleph_0}$ many non-isomorphic, non-bi-equivalent MB and HE-homogeneous digraphs that cannot be viewed as an undirected graph Γ.* □

Finally in this section, we extend the notion of bimorphism equivalence to digraphs, using this to prove a digraph version of to Corollary 8.1.10 and Corollary 8.3.5. In a similar way to Definition 8.1.7, say that two digraphs $\mathcal{D}$ and $\mathcal{E}$ are *bimorphism equivalent* if there exist bijective homomorphisms $\alpha : \mathcal{D} \to \mathcal{E}$ and $\beta : \mathcal{E} \to \mathcal{D}$.

Proposition 8.4.7. *Let $\mathcal{D}, \mathcal{E}$ be bimorphism equivalent digraphs via bijective homomorphisms $\alpha : \mathcal{D} \to \mathcal{E}$ and $\beta : \mathcal{E} \to \mathcal{D}$. Then $\mathcal{D}$ has properties $(\uparrow\downarrow)$ and $(.\!\because)$ if and only if Δ does.*

Proof. Follows from a similar argument to Proposition 8.1.8, replacing property $(\triangle)$ by property $(.\!\because)$, adjacency with 2-cycle, and non-edge with non-arc. □

Proposition 8.4.8. *If $\mathcal{D}, \mathcal{E}$ are two digraphs with properties $(\uparrow\downarrow)$ and $(.\!\because)$, then $\mathcal{D}$ and $\mathcal{E}$ are bimorphism equivalent.*

Proof. Assume that $\mathcal{D}, \mathcal{E}$ are two digraphs with properties $(\uparrow\downarrow)$ and $(.\!\because)$. As in the proof of Proposition 8.1.9, we will use a back and forth argument to construct a bijective homomorphism $\alpha : \mathcal{D} \to \mathcal{E}$ and a bijective antihomomorphism $\bar{\beta} : \mathcal{D} \to \mathcal{E}$, which by Lemma 7.1.2 will be the converse of a bijective homomorphism $\beta : \mathcal{E} \to \mathcal{D}$. So assume that $f : \{c\} \to \{d\}$ is a function sending a vertex c of $\mathcal{D}$ to a vertex d of $\mathcal{E}$; this is a bijective homomorphism. Now set $\{c\} = C_0$, $\{d\} = D_0$, $f = f_0$, and assume that we have extended f to a bijective homomorphism $f_k : C_k \to D_k$, where C_i and D_i are finite and $C_i \subseteq C_{i+1}$ and $D \subseteq D_{i+1}$ for all $0 \leq i \leq k-1$. Furthermore, as both $\mathcal{D}$ and $\mathcal{E}$ are countable, we can enumerate their vertices as $V\mathcal{D} = \{c_0, c_1, ...\}$ and $V\mathcal{E} = \{d_0, d_1, ...\}$.

If k is even, select a vertex $c_j \in \mathcal{D}$ where j is the smallest number such that $c_j \notin C_k$. As $\mathcal{E}$ has property $(\uparrow\downarrow)$, there exists a vertex $u \in \mathcal{E}$ such that there is a 2-cycle between u and every element of D_k. Define a map $f_{k+1} : C_k \cup \{c_j\} \to D_k \cup \{u\}$ sending c_j to u and extending f; this map is a bijective homomorphism as any arc between c_j and any element of C_k in either direction is preserved (see Figure 8.18 for a diagram of an example).

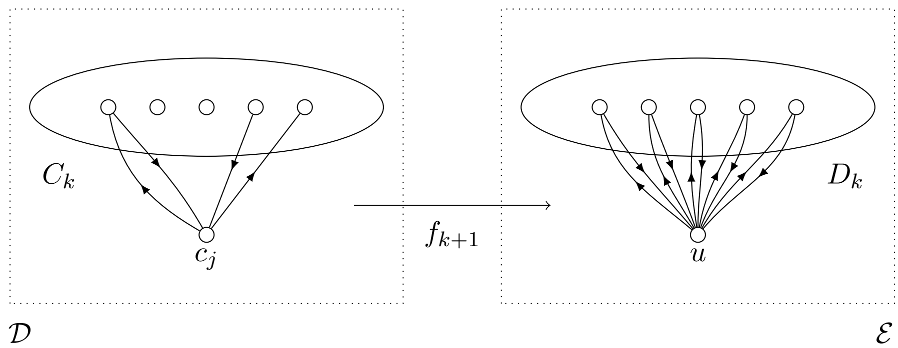


Figure 8.18: k even in proof of Proposition 8.4.8

Now, if k is odd, choose a vertex $d_j \in \mathcal{E}$ where j is the smallest number such that $d_j \notin D_k$. As $\mathcal{D}$ has property $(\therefore)$, there exists a vertex $v \in \mathcal{D}$ such that v is independent of every element of C_k. Define a map $f_{k+1} : C_k \cup \{v\} \to D_k \cup \{d_j\}$ sending c to d_j and extending f_k. Then f_{k+1} is a bijective homomorphism as f_k is and every arc between c and C_k is preserved; because there are no arcs to preserve. See Figure 8.19 for a diagram of an example of this stage.

Repeating this process infinitely many times, ensuring that each vertex of $\mathcal{D}$

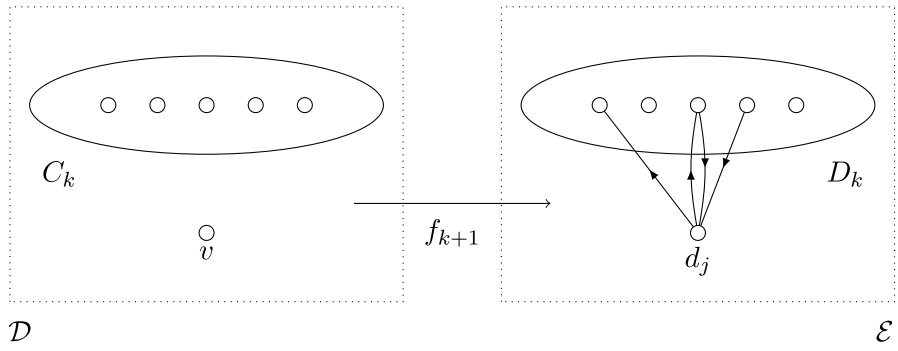


Figure 8.19: k odd in proof of Proposition 8.4.8

appears at an even stage and each vertex of $\mathcal{E}$ appears at an odd stage, defines a bijective homomorphism $\alpha : \mathcal{D} \to \mathcal{E}$. We can construct a bijective antihomomorphism $\bar{\beta} : \mathcal{D} \to \mathcal{E}$ in a similar fashion; replacing homomorphism with antihomomorphism and using property $(\therefore)$ of $\mathcal{E}$ at even steps and property $(\uparrow\downarrow)$ of $\mathcal{D}$ at odd steps. So the converse map $\beta : \mathcal{E} \to \mathcal{D}$ is a bijective homomorphism and so $\mathcal{D}$ and $\mathcal{E}$ are bimorphism equivalent. $\square$

Finally, recall from Example 2.4.9 that D^* is the generic digraph with characteristic extension property DARP. Note that the DARP of D^* implies that D^* has both properties $(\uparrow\downarrow)$ and $(\therefore)$. We therefore have the digraph version of Corollary 8.1.10 and Corollary 8.3.5:

Corollary 8.4.9. *Suppose that $\mathcal{D}$ is a countable digraph. Then $\mathcal{D}$ has properties $(\uparrow\downarrow)$ and $(\therefore)$ if and only if $\mathcal{D}$ is bimorphism equivalent to the generic digraph D^*.*

This means that any countable digraph $\mathcal{D}$ that arises as an example in either Theorem 8.4.3 or Theorem 8.4.6 is bimorphism equivalent to the generic digraph D^*.

We end on natural generalisations of the conjectures given in Question 8.3.12 and Question 8.3.13.

Question 8.4.10. *(1) Classify the IA-homogeneous digraphs that are also MB-homogeneous.*

(2) Is every MB-homogeneous digraph bimorphism equivalent to an MB and IA-homogeneous digraph? How many MB-homogeneous digraphs are there up to bimorphism equivalence?

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