Bargaining Frictions in Trading Networks

Arnold Polanski*
University of East Anglia

Fernando Vega-Redondo
Bocconi University & IGIER

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Abstract

In the canonical model of frictionless markets, arbitrage is usually taken to force all trades of homogeneous goods to occur at essentially the same price. In the real world, however, arbitrage possibilities are often severely restricted and this may lead to substantial price heterogeneity. Here we focus on frictions that can be modeled as the bargaining constraints induced by an incomplete trading network. In this context, the *interplay* among the architecture of the trading network, the buyers' valuations, and the sellers' costs shapes the *effective* arbitrage possibilities of the economy. We characterize the configurations that, at an intertemporal bargaining equilibrium, lead to a uniform price. Conceptually, this characterization involves studying how the network positions and valuations/costs of any given set of buyers and sellers affect their *collective* bargaining power relative to a notional or benchmark situation in which the connectivity is complete. Mathematically, the characterizing conditions can be understood as price-based counterparts of those identified by the celebrated Marriage Theorem in matching theory.

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1 Introduction

In this note, we study the phenomenon of price formation in a context where (heterogeneous) buyers and sellers are subject to trading constraints (based on geography, ethnic or language considerations, trust, etc.) that limit their bargaining or/and trading possibilities. In order to highlight the interesting interplay between such constraints and agents' characteristics (valuations and costs), we abstract from any other source of frictions and assume that both buyers and sellers enjoy complete

*Corresponding author: School of Economics, University of East Anglia, Norwich Research Park, NR4 7TJ Norwich, UK. Email: A.Polanski@uea.ac.uk, Tel.: +44-1603-597166

information and are arbitrarily patient. In a nutshell, our aim is to characterize when, under the aforementioned conditions, strategic bargaining leads to an outcome where all trade is conducted at a uniform price.

Why is this question important? Its significance derives from the fact that such a price uniformity is one of the key features associated to transparent and frictionless markets. It is, in a crucial sense, a feature that underlies most of the properties of the market mechanism traditionally emphasized by economists such as, for example, those highlighted by the two so-called Fundamental Welfare Theorems. There is the need, therefore, to understand whether such a uniform-pricing outcome obtains even if trade is carried out in a decentralized manner and possibly subject to substantial frictions. For, as amply documented by empirical evidence, such frictions are substantial in almost all economic domains.[1]

The general mechanism through which markets typically achieve price uniformity is arbitrage. But, of course, a minimal requirement for arbitrage to remove all price disparities is that the trading network be connected i.e. a path must exist between any buyer and every seller. Otherwise, the trading network is effectively divided into separate independent sub-economies and, in general, a uniform price could not be hoped for. At the opposite extreme, complete connectivity – with every buyer being directly linked to every seller – is obviously enough to guarantee a unique price. For, in this case, every price disparity would be readily exploited by some (perfectly informed and infinitely patient) agent. Naturally, the middle ground between complete and minimal connectivity, i.e., an incomplete connected network, represents the truly interesting case to study. This motivates our concrete research question: Given any profile of buyers' valuations and sellers' costs, known to all players, what is the family of networks that lead to a uniform equilibrium price?

To address this question, we model the bargaining setup as follows (a formal description is postponed to Section 2). Bargaining proceeds over discrete periods and, in every one of them, some buyers and sellers are randomly matched in pairs that are consistent with the given trading network. For each such pair, one of the agents is chosen at random to make a proposal, which is immediately implemented if accepted. The matched pairs who strike a deal leave the game and are replaced by agents with identical characteristics, while all other players continue in the game and move into the following period.

In the setup outlined, if buyers and sellers are all homogeneous, i.e. every buyer values the single traded good equally and the (opportunity) cost of each seller is the same, a certain answer to the question posed can be found in Manea (2011, 2016). He proves that bilateral bargaining in a bipartite network leads to a uniform price (i.e., the network is non-discriminatory in his terminology) if and only if for every subset of buyers, the ratio of the number of sellers linked to (at least) one of these buyers to the number of buyers in the subset is greater than or equal to the seller-buyer ratio in the entire network. Our main result generalizes this finding to heterogeneous agents.

[1]For example, Donna, Schenone, and Veramendi (2015) highlight the following cases: labor markets (Mortensen, 2005), eBay (Einav *et al.*, 2015), and automobile markets (Morton *et al.*, 2001).

Naturally, since Manea's model assumes that agents are homogeneous, the conditions he identifies are purely topological, i.e. concern only the architecture of the trading network. For example, with an equal number of buyers and sellers, a uniform trading price is shown to arise at equilibrium if, and only if, the trading network admits a so-called perfect matching, i.e. a network-consistent pairing of agents where every buyer is associated to a single distinct seller.

The situation, however, is in general very different in a market environment that displays some heterogeneity in agents' characteristics. In particular, one expects that individual valuations and costs should interplay in a crucial way with the network architecture to yield insights that go well beyond topological considerations. For example, consider the following network.

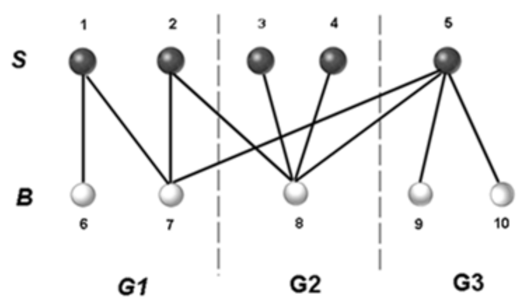

Figure 1: A simple seller-buyer network.

Let us assume first that all buyers in the set B and all sellers in the set S are homogeneous, the former having a unit valuation for the good and the latter a zero cost. Then, as agents become infinitely patient (their discount rate converges to one), it can be easily shown that in any trading component where all trade is conducted at a uniform price, this component-specific price simply reflects the relative numbers of buyers and sellers involved.[2] Furthermore, it can be shown that the overall equilibrium leads to a segmentation of trade into the components labeled $G1$ to $G3$ in Figure 1.

- In $G1$, which is balanced in the sense of including an equal number of buyers and sellers, trade takes place at a common price $p_1 = 1/2$.
- In $G2$, where the single buyer is in the "short side," the trading price is $p_2 = 1/3$.
- In $G3$, where the single seller is in the "short side," the trading price is $p_3 = 2/3$.

Next, we note that trade segmentation can be substantially reinforced or mitigated, depending on sellers' costs and buyers' valuations. Suppose, for example, that under the same trading network

[2] See Section 5 for a detailed explanation of how equilibrium prices are determined when agents become infinitely patient.

as in Figure 1 the valuation of buyer 8 is changed to $v_8 = 10$, all other costs and valuations being kept as in the previous binary case. Then, four trading components arise in equilibrium. One of them consists of the former G_2 enlarged with sellers 2 and 5, i.e. it consists of the agents in the set $\{2, 3, 4, 5, 8\}$. This leaves buyers 9 and 10 as isolated singletons – which means that they do not trade – and induces a residual component consisting of the set $\{1, 6, 7\}$.

Possibly a more interesting situation, somewhat polar to the previous one, obtains if we modify only the costs of sellers 3 and 4, raising them to $c_3 = c_4 = 1/4$, and the valuations of buyers 9 and 10, lowering them to $v_9 = v_{10} = 3/4$. Then, at an equilibrium (with infinitely patient players), all trade occurs at the uniform price $p^* = 1/2$, which is in fact the price that would prevail if buyers and sellers were connected by a complete bipartite network. That is, the resulting outcome is equivalent to one with complete connectivity, even though the network architecture remains unchanged and is therefore quite incomplete.

The previous discussion illustrates the potentially rich interplay between network structure and type profile. In this light, our main contribution is to provide a full and general characterization of the environments – trading networks together with agents' characteristics – that yield a frictionless-like (uniform-price) outcome. As we shall see, this characterization relies on a collection of notional prices – notional in the sense of being purely "conceptual" or "algorithmic," i.e. not effectively implemented. These different prices, which are associated to the various possible subnetworks of the overall original network, serve the purpose of assessing the equilibrium consistency of the uniformly prevailing price. The need to resort to such notional prices in this case derives from the fact that, when agents are heterogeneous, the bargaining options available to the different agents must be evaluated in terms of the characteristics (costs and valuations) of their possible partners and competitors to whom they are connected.

We close this introduction with a brief summary of related literature. The intertemporal bargaining approach to price determination considered here was initiated by the seminal papers by Rubinstein and Wolinsky (1985, 1990) and Gale (1987). Their models presume that, in every period, agents are *randomly matched afresh* to bargain bilaterally with individuals on the other side of the market. Their theoretical frameworks depend, therefore, on a complete social structure. Subsequent literature (e.g. Corominas-Bosch (2004) and Kranton and Minehart (2001)) introduced an incomplete structure into the analysis in the same way as we do here, i.e. by postulating that a given social network restricts bilateral trading possibilities. A limiting feature of their approach is that either the bargaining procedure or the matching mechanism exhibits a degree of coordination that is at odds with the idea of decentralization that we associate to markets. The aforementioned contributions by Manea (2011, 2016) – see also Abreu and Manea (2012) – do not suffer from this drawback but they make the assumption that all agents are completely homogeneous except for their connectivity. It abstracts, therefore, from the core focus of this paper, which is the study of how the interplay between agents characteristics and network structure shapes the effect of trading frictions. Nguyen (2015) generalizes Manea's model by allowing for surplus creation in coalitions larger than pairs and devises a simple method to solve this game in the limit as players become arbitrarily patient. In the proof of our Lemma 1, we show how his convex program computes payoffs and prices in our context.

The rest of the paper is organized as follows. First, in Section 2 we present the model and, in Section 3, the equilibrium notion. Then, the analysis in Section 4 starts with a brief discussion of the extreme case where the trading network is complete. This context is analogous to that studied by Gale (1987) and represents a useful benchmark for the analysis. We proceed in Section 5 to the study of our main case of interest where bargaining takes place in an arbitrary trading network, possibly quite incomplete. We characterize those configurations that lead to a uniform trading price and compare our conditions to the classical ones obtained for a pure matching context. Section 6 concludes the main body of the paper with a summary of the main insights. For the sake smoothness in the presentation, all proofs are included in the Appendix, although their main gist is informally explained in the main text.

2 Model

There is a given set of sellers and a set of buyers. Each agent (buyer or seller) is connected to a certain subset (possibly empty) of agents on the other side of the market (buyers or sellers, respectively). Such connections are formalized through a bipartite *trading network* $G = \{S \cup B, L\}$ where S is the set of sellers, B is the set of buyers, and $L \subseteq \{sb : s \in S, b \in B\}$ stands for the set of undirected links $sb \, (= bs)$ that connect some seller s to some buyer b. It is assumed that every seller $s \in S$ can produce at most one unit of the good being traded and incurs a (production or opportunity) cost c_s in doing so. On the other hand, each buyer $b \in B$ cares for just one unit of the good and has an idiosyncratic valuation v_b for it.

Time is modeled discretely, $t = 1, 2, ...$ At every t, the following two steps take place in sequence:

> *First*, a certain seller-buyer matching $m = \{s_1 b_1 \, , \, s_2 b_2 \, ..., \, s_q b_q\}$ is selected according to some probability distribution φ_G over all feasible matchings. For a matching to be feasible, it must verify two properties: (i) every buyer and seller is included in at most one pair (possibly in none); (ii) every matched pair corresponds to a link in the prevailing trading network G. In particular, m can be empty or it can contain the maximum number of non-intersecting edges (maximum matching). We need not make any assumption on how the particular matching m is selected (i.e. on the probability distribution φ_G) other than supposing that every link sb in G is chosen with some (marginal) probability π_G^{sb} that is positive.
>
> *Second*, for every pair $sb \in m$, one of the two agents is selected at random with equal probability to make a proposal p on the price at which trade can be conducted.
>
> (a) If this proposal is accepted, the good is transferred and the price paid. The buyer b earns

$v_b - p$ and the seller $p - c_s$. These two agents then leave the economy and are replaced by another buyer and seller with the same characteristics who occupy the same network positions next period, $t + 1$.

(b) If the proposal is refused, then agents remain active in the same network position (with the same set of connections) and participate in the new bargaining round taking place at $t + 1$.

The replacement assumption contemplated in (a) was already made by Rubinstein and Wolinsky (1990) and has become common in the recent literature on bargaining and networks (e.g., Manea, 2011). It is particularly useful because, in combination with (b), it allows to model the situation in a stationary manner and hence consider stationary equilibria, as formulated in the next section

3 Trading equilibrium

Given the bipartite network $G = \{S \cup B, L\}$ and a corresponding set of costs $(c_s)_{s \in S}$ and valuations $(v_b)_{b \in B}$, the trading mechanism described above defines a sequential game form governing the bargaining process, i.e. the "rules of the game." Concerning preferences, we make the traditional assumption that, for any $t = 1, 2, ...$, every agent active at t discounts the instantaneous payoffs that might be obtained at some future $t' \geq t$ with the factor $\delta^{t'-t}$, where the discount rate $\delta < 1$ is the same for all players. This intertemporal trading game is played under complete information on all relevant details of the situation (i.e. the payoffs of all agents, the prevailing network, etc.)

As indicated, our analysis of the induced intertemporal game will focus on its *Stationary Subgame Perfect Equilibria* (SSPE), i.e. Subgame Perfect Equilibria where players' strategies are stationary and, hence, the behavior they prescribe within any given period t is independent of what happened at any $t' < t$. More precisely, a stationary strategy σ_i for any given agent $i \in S \cup B$ embodies two distinct components. First, it includes, for every j such that $ij \in L$, a price p_{ij} at which i offers to trade with j when the link ij is chosen by the matching mechanism and i is the proposer. Thus, overall, agent i must have a vector of such proposals $\mathbf{p}_i \equiv (p_{ij})_{ij \in L}$ for all his partners. On the other hand, every agent i must have a function $\psi_{ij} : \mathbb{R} \to \{A, R\}$ that specifies what price proposals from j he will accept (A) or not (R). All these conditional decisions may be gathered in a vectorial function $\boldsymbol{\psi}_i \equiv (\psi_{ij})_{ij \in L}$ that embodies the full range of agent i's behavior as a responder. Combining this function with the aforementioned price offer $\mathbf{p}_i$, we arrive at a (stationary) strategy $\boldsymbol{\sigma}_i = (\mathbf{p}_i, \boldsymbol{\psi}_i)$ for every agent $i \in S \cup B$.[3] The corresponding strategy profile $\boldsymbol{\sigma} \equiv (\boldsymbol{\sigma}_i)_{i \in S \cup B}$ in turn induces a unique vector of expected payoffs that we shall denote by

[3]In principle, players' strategies could also depend on the realized matching. However, equilibrium strategies will not depend on it - as the SSPE condition (1) shows - due to their assumed stationarity and the fact that the market composition is taken to be stationary as well.

$\mathbf{x}(\boldsymbol{\sigma};\, G, \boldsymbol{\theta}, \delta) \equiv (x_i(\boldsymbol{\sigma};\, G, \boldsymbol{\theta}, \delta))_{i\in S\cup B}$, where $\boldsymbol{\theta} \equiv ((c_s)_{s\in S}, (v_b)_{b\in B})$ is simply a shorthand for the combined vector of seller and buyer types. Thus, for the sake of clarity, our notation makes explicit the dependence of payoffs on the strategy profile $\boldsymbol{\sigma}$, the underlying network G, the agents' types, and the discount factor δ.

Given the probability distribution φ_G that formalizes the matching mechanism operating on the network G, recall that π_G^{ij} stands for the marginal probability that the pair $\{i, j\}$ is matched, which we assumed positive if (and only if) $ij \in L$. For any such seller-buyer pair, denote by $v^{ij} \equiv \max\{v_i - c_j, 0\}$ the surplus that can be jointly produced by i and j. Then, the requirement that any SSPE $\tilde{\boldsymbol{\sigma}}$ is *intertemporally self-consistent* implies that the induced payoffs $(\tilde{x}_i)_{i\in S\cup B}$ must satisfy, for every $i \in N \equiv S \cup B$, the following Bellman-like conditions:

$$\tilde{x}_i = \sum_{j:ij\in L} \pi_G^{ij} \left(\frac{1}{2} \max\{v^{ij} - \delta\tilde{x}_j, \delta\tilde{x}_i\} + \frac{1}{2}\delta\tilde{x}_i \right) + \left(1 - \sum_{j:ij\in L} \pi_G^{ij} \right) \delta\tilde{x}_i. \tag{1}$$

These conditions simply state that, at equilibrium, the payoff expectation of every agent i at any general date t must equal the payoff to be expected if he and his matched partner react optimally at that period and the same payoff is anticipated for agent i if he is still active the following period. Nguyen (2015) and Polanski and Lazarova (2014) have shown that the system (1) has a unique solution $\tilde{\mathbf{x}}(G, \boldsymbol{\theta}, \delta)$ for any G and $\boldsymbol{\theta}$, provided $\delta < 1$. In fact, as a generalization of Manea (2011), their results imply that for sufficiently high δ (i.e. $\delta \geq \delta_0$ for some $\delta_0 < 1$) the subset of links that trade with positive probability at equilibrium remains fixed and that the limit equilibrium payoff as $\delta \to 1$:

$$\mathbf{x}^*(G, \boldsymbol{\theta}) \equiv \{x_i^*(G, \boldsymbol{\theta})\}_{i\in S\cup B} \equiv \lim_{\delta\to 1} \tilde{\mathbf{x}}(G, \boldsymbol{\theta}, \delta). \tag{2}$$

is well-defined (and thus unique). We shall refer to $\mathbf{x}^*(G, \boldsymbol{\theta})$ as the *Limit Bargaining Outcome* (LBO). It is worth highlighting that the LBO is independent of the matching procedure (cf., Theorem 2 in Nguyen, 2015). Hence, the LBO is the same for any φ_G such that the implied marginal probabilities satisfy $\pi_G^{ij} > 0$ for all $ij \in L$ and $\pi_G^{ij} = 0$ for all $ij \notin L$. Associated to such LBO, we define the prices $\mathbf{p}^*(G, \boldsymbol{\theta}) \equiv (p_{sb}^*(G, \boldsymbol{\theta}))_{sb\in L}$ at which trade is conducted, if at all, in each of the links of G at *any* SSPE. Specifically, if trade is conducted at some link $sb \in L$ with positive probability, the uniquely associated price is given by,[4]

$$p_{sb}^*(G, \boldsymbol{\theta}) = x_s^*(G, \boldsymbol{\theta}) + c_s = v_b - x_b^*(G, \boldsymbol{\theta}). \tag{3}$$

Instead, if the probability that trade occurs at some link $sb \in L$ is zero, we simply write $p_{sb}^*(G, \boldsymbol{\theta}) = \varnothing$. In what follows, we will often drop the reference to G and $\boldsymbol{\theta}$ and write simply $\mathbf{x}^*$ and $\mathbf{p}^*$ if no confusion arises.

[4]From (3) it obviously follows that each trading pair sb exhausts the whole surplus v^{sb}. That is, we have that $x_s^*(G, \theta) + x_b^*(G, \theta) = v_b - c_s = v^{sb}$.

4 Complete trading network

We start our analysis with the case of complete bipartite networks. Essentially, this case is equivalent to that studied by Gale (1987), where each buyer/seller matches afresh every period with some randomly selected member of the other (continuum) population. Under these circumstances, the equilibrium price p^* at which all trades are conducted is given by the unique price at which the two sides of the market obtain, in aggregate, the same share of the surplus (see Proposition 11 in Gale's paper).

To express formally the former condition, it is convenient to introduce the following notation. Given any price p, let $B(p)$ and $S(p)$ stand for the set of buyers and sellers, respectively, who want to trade at price p. Formally,

$$B(p) \equiv \{b \in B : v_b \geq p\}, \quad S(p) \equiv \{s \in S : c_b \leq p\}. \tag{4}$$

Then, particularized to our finite-population context, the aforementioned condition can be written as follows:

$$\sum_{b \in B(p^*)} (v_b - p^*) = \sum_{s \in S(p^*)} (p^* - c_s), \tag{5}$$

which provides an implicit (unique) determination of the equilibrium price p^*. This price determines via (3) the limit payoffs $\mathbf{x}^*$ of players in $B(p^*) \cup S(p^*)$ (all other agents earn zero). Equivalently, these payoffs obtain as the unique solution to (1) with $\pi_G^{ij} > 0$ for all $ij \in S \times B$ as $\delta \to 1$.

The above condition represents a direct generalization of the well-known result of the bilateral bargaining model studied by Rubinstein (1982) between a single seller and a single buyer that are "infinitely patient". In this case, the surplus is divided equally between the two agents if both have the same ex-ante probability of being the proposer (and therefore being in a position to extract some rents). Similarly, by virtue of the extreme patience of all agents, the two essential considerations in our context can be summarized as follows:

(i) All trades must take place at the same price.

(ii) Given (i), from the point of view of any single agent, the opposite side of the market can be suitably conceived as represented by an "average player."

Hence if (as we have assumed) the probability of being the proposer in any matched pair is the same for the two sides, then the surplus earned in total by each side of the market must be equal – just as in the simple Rubinstein's two-agent context. This is precisely what (5) asserts.

Note that the former reasoning is independent of how often any *particular* agent is selected to be the proposer. As a whole, each side of the market enjoys the same probability and thus, also as a whole, both sides must earn the same share of the surplus.[5] Note, further, that item (i) is of course crucially dependent on the completeness of the network. In the absence of this completeness, the considerations illustrated in Figure 1 come into play, with the interaction between network architecture and the profile of types determining whether effective arbitrage possibilities are in place. A full characterization of this problem is developed in the next section.

5 Price determination in general trading networks

We start with a definition of the key concept involved in our analysis.

Definition *Given a profile of costs and valuations, $\boldsymbol{\theta} = ((c_s)_{s \in S}, (v_b)_{b \in B})$, and a bipartite network $G = \{S \cup B,\, L\}$, the pair $(G, \boldsymbol{\theta})$ is said to be an* Uniform-Price Configuration (UPC) *if the corresponding equilibrium price vector $\mathbf{p}^*(G, \boldsymbol{\theta}) = (p^*_{sb}(G, \boldsymbol{\theta}))_{sb \in L}$ obtained from (3) satisfies for all links $ij, kl \in L$,*

$$p^*_{ij}(G, \boldsymbol{\theta}) \neq \emptyset \neq p^*_{kl}(G, \boldsymbol{\theta}) \Rightarrow p^*_{ij}(G, \boldsymbol{\theta}) = p^*_{kl}(G, \boldsymbol{\theta}).$$

As illustrated in Figure 1, whether price uniformity prevails at equilibrium depends on how the trading network interplays with the type profile of costs and valuations. Two polar cases are straightforward. On the one hand, if the network is complete, the configuration must always induce a uniform price whatever the type profile. In contrast, if the trading network is segmented into several components, it is clear that only exceptionally (i.e. non-generically) one can expect that trade will be conducted at a uniform price. In contrast to these two extreme cases, the most interesting situations lie in the intermediate scenario in which the trading network is connected (i.e. displays a single component) but is well below being complete. In those cases, understanding when a uniform-price outcome obtains is not so clear. To address the problem, a general characterization of uniform-price configurations is provided by the Proposition below.

To state our result formally, the following two pieces of notation are useful.

- First, given any subset of sellers $S' \subseteq S$ and buyers $B' \subseteq B$, let $p' \equiv \mathcal{P}(S', B')$ be the (unique) price that satisfies:

$$\sum_{b \in B(p') \cap B'} (v_b - p') = \sum_{s \in S(p') \cap S'} (p' - c_s). \tag{6}$$

[5] Obviously, this is a particular manifestation of the indicated general independence of the LBO on the matching procedure given by φ_G. In the present case, where the network is complete, the intuitive basis for this conclusion is easier to understand.

A natural interpretation of p' is the uniform price that would notionally prevail in a game in which sellers in S' were *completely* connected to buyers in B' in the network $G' = \{S' \cup B', L'\}$ with $L' = \{sb : s \in S',\ b \in B'\}$.

- Second, for any subset of sellers $S' \subseteq S$ denote by $N_G(S')$ the set of buyers $b \in B$ that are connected to some seller $s \in S'$ in the bipartite network $G = \{S \cup B, L\}$. That is, $N_G(S') \equiv \{b \in B : \exists s \in S' \text{ s.t. } sb \in L\}$. Similarly, we define $N_G(B') \equiv \{s \in S : \exists b \in B' \text{ s.t. } bs \in L\}$.

For the sake of formal simplicity, we assume also that all nodes have a partner with whom they can conceivably trade at some mutually beneficial price – otherwise, the nodes for which this condition does not apply are just "dummies" and can be safely ignored in the analysis. That is, we make the following assumption:

Assumption PL (Profitable Links) For every seller $s \in S$ there is some buyer $b \in B$ such that $sb \in L$ and $v_b > c_s$. Similarly, for every buyer $b \in B$ there is some seller $s \in S$ such that $bs \in L$ and $c_s < v_b$.

We can now state and prove the following characterization result.

Proposition 1. *Consider a bipartite network $G = \{S \cup B,\ L\}$ and a type profile $\boldsymbol{\theta} = ((c_s)_{s \in S}, (v_b)_{b \in B})$ for which Assumption PL holds. Then, the following conditions are equivalent:*

- $(G, \boldsymbol{\theta})$ *is a Uniform-Price Configuration,* (7a)
- $\forall B' \subseteq B, B' \neq \emptyset, \quad \mathcal{P}(N_G(B'), B') \leq \mathcal{P}(S, B),$ (7b)
- $\forall S' \subseteq S, S' \neq \emptyset, \quad \mathcal{P}(S', N_G(S')) \geq \mathcal{P}(S, B).$ (7c)

Proof: See the Appendix.

Informally, our result indicates that any given configuration is an UPC if, and only if, any of the following equivalent (and symmetric) conditions for buyers and sellers hold:

(i) For buyers, each subset of them must be collectively connected to relatively enough low-cost sellers such that they cannot be forced into accepting prices that are higher than the price p^* that all buyers would pay in the absence of trading frictions.

(ii) For sellers, each subset of them has to be collectively connected to relatively enough high-valuation buyers such that they cannot be forced into accepting prices that are lower than p^*, the price that all sellers would receive in the absence of trading frictions.

To understand the intuition for the equivalence of (7a), (7b), and (7c) the essential argument can be explained as follows. First, let us argue that either (7b) or (7c) separately imply (7a) – i.e. uniform pricing. We start by the observation that if trading at equilibrium is not conducted at a uniform price, then there must be two distinct prices, say p_H and p_L, that satisfy $p_H > p^* \equiv \mathcal{P}(S, B) > p_L$. This reflects the fact that whenever the creation of additional links leads to the merger of two trading components[6] into a single one, the induced (uniform) price lies at some intermediate "compromise" between the two original prices (see Lemma 2). The key intuition here is that, whenever any active links are formed across the formerly independent trading components, the relatively weak part in each of those components (buyers in one, sellers in the other) cannot become worse-off. Thus, in the single integrated component that results, the prevailing uniform price must lie in between the two former prices. An analogous conclusion can be readily extended to the general case of any number of trading components, with p_H and p_L now standing for the highest and lowest prices prevailing across all components.

Then, to complete the explanation of this first part of the proposition, consider any configuration $(G, \boldsymbol{\theta})$ where (7a) fails and denote by B' the set of buyers who are in the component trading at the highest price p_H. If, hypothetically, these buyers were connected to the sellers in $N_G(B')$ through a complete and isolated (bipartite) subnetwork, those buyers could not do any better than in the original full network. Heuristically, the reason is that, under those circumstances, we are "artificially" ignoring the additional bargaining options that the sellers in $N_G(B')$ actually enjoy in the network G, i.e. their links to buyers outside B'. This, in effect, contradicts (7b) and thus explains why (7a) implies (7b). A symmetric idea applies to (7c), the focus then turning to the lowest price p_L and the set of sellers trading at that price.

Finally, we explain the reciprocal statement that (7a) implies both (7b) and (7c). Since the argument again applies to buyers and sellers symmetrically and separately, let us focus on the condition (7b) that refers to buyers. Suppose that this latter condition fails and thus we have some set of buyers B' such that, if connected in a complete subnetwork to $N_G(B')$, the (uniform) price p_1 they would attain is higher than $p^* \equiv \mathcal{P}(S, B)$. Let S' stand for the set of sellers not connected to B' – i.e. $S' \equiv S \setminus N_G(B')$ – and denote by p_2 the uniform price that prevails in a trading component where S' is completely connected to $B \setminus B'$. Applying again the reasoning used above, p^* must be conceived as a "compromise" between p_1 and p_2, and therefore $p_1 > p^* > p_2$.

Consider then the network constructed as follows: all buyers in B' are completely connected to the sellers in $N_G(B')$ and all sellers in $S \setminus N_G(B')$ are completely connected to the buyers in $B \setminus B'$. This, in short, is simply the complete bipartite network between the full sets B and S except for *all* the links between buyers in $B \setminus B'$ and the sellers in $N_G(B')$. Let us refer to this network as G', and note that the original G is a subnetwork of it. It is clear that, on the configuration $(G', \boldsymbol{\theta})$, an equilibrium can be constructed where all sellers in $N_G(B')$ are only willing to trade with buyers in B' at price $p_1 > p^*$, while all buyers in $B \setminus B'$ are only willing to trade with sellers in $S \setminus N_G(B')$

[6] A trading component is simply defined as a component of the subnetwork of G consisting of all links that are active at equilibrium, i.e. those that support trade with positive probability.

at price $p_2 < p^*$. Since $G \subset G'$, this contradicts the assertion that all trade at the configuration $(G, \boldsymbol{\theta})$ is conducted at a uniform price.

A quite intuitive application of Proposition 1 arises for the particular case where costs and valuations are homogeneous within each side of the market. In this case, (6) simplifies to,

$$\mathcal{P}(N_G(B'), B') = \frac{\#(B')}{\#(B') + \#(N_G(B'))}, \quad \mathcal{P}(S', N_G(S')) = \frac{\#(N_G(S'))}{\#(S') + \#(N_G(S'))}, \tag{8}$$

where $\#(\cdot)$ represents the cardinality of the set in question. Hence, in particular, for a completely connected component of the trading network (which, of course, also defines a single trading component) the single price prevailing in it simply reflects the buyer-seller ratio in that component. For example, in the network of Figure 1, $p^* \equiv \mathcal{P}(S, B) = \frac{1}{2}$ since there are as many sellers as buyers in that market. In contrast, if we consider for example the subset of buyers given by $B' = \{9, 10\}$, we obtain:

$$\mathcal{P}(N_G(B'), B') = \mathcal{P}(\{5\}, \{9, 10\}) = 2/3 > p^*,$$

which violates Condition (7b). This indicates that the configuration given by the trading network represented in Figure 1 with homogeneous costs and valuations induces price dispersion, i.e. it is *not* a UPC.[7]

Within an homogeneous buyer and seller context, a straightforward observation is that Conditions (7b) and (7c) are formally equivalent to those put forward by the celebrated *Marriage Theorem* (see Hall, 1935, or Chartrand, 1985) to characterize the bipartite networks that admit a so-called perfect matching – i.e. a matching where every node of either part ("man" or "woman") is suitably matched with one (and only one) node of the other part. To see this note that, once the relevant prices have been computed from (8), the aforementioned conditions can be written as follows:

$$\begin{aligned} \forall B' \subseteq B, B' \neq \emptyset, &\quad \frac{\#(B')}{\#(N_G(B'))} \leq \frac{\#(B)}{\#(S)}, \\ \forall S' \subseteq S, S' \neq \emptyset, &\quad \frac{\#(S')}{\#(N_G(S'))} \leq \frac{\#(S)}{\#(B)}. \end{aligned}$$

Therefore, if $\#S = \#B$, they become:

$$\begin{aligned} \forall B' \subseteq B, B' \neq \emptyset, &\quad \#(B') \leq \#(N_G(B')), \\ \forall S' \subseteq S, S' \neq \emptyset, &\quad \#(S') \leq \#(N_G(S')). \end{aligned}$$

[7] As a further illustration of Proposition 1, let us still focus on the trading network displayed in Figure 1 but now consider, as we did in the Introduction, the *heterogeneous* environment obtained from the homogeneous one by changing the valuation of buyer 8 to $v_8 = 10$, while all other costs and valuations remain unchanged. Then (6) implies $p^* \equiv \mathcal{P}(S, B) = 2$ and we find that the subset $S' = \{1\}$ violates the Conditions (7c) as $\mathcal{P}(\{1\}, \{6, 7\}) = 2/3 < p^*$. Instead, it may be easily checked that if the change involves $c_3 = c_4 = 1/4$, and $v_9 = v_{10} = 3/4$ (another case considered in the Introduction), Conditions (7b) and (7c) are satisfied, thus implying that the induced configuration yields a uniform price.

The above conditions simply specify that every subset of agents on one side of the market is connected to a set on the other side that is at least as numerous. This are precisely the conditions established by Hall's Marriage Theorem as necessary and sufficient for a perfect matching.

In fact, the previous observation immediately follows from a result established by Manea (2011). He shows that, for homogeneous buyer-seller networks, a uniform price arises out of bargaining among infinitely patient agents if, and only if, the trading network admits a perfect matching. Thus, in this light, what our above discussion suggests is that a natural interpretation of Proposition 1 can be cast along the following lines. When one moves from a fully homogeneous environment to an heterogeneous one with arbitrary cost and valuation profiles, uniform-price configurations may be characterized by "economic conditions" that generalize those of the Marriage Theorem through the consideration of suitably determined (endogenous) prices. These prices, in essence, reflect the relative "scarcity" of valuable bargaining partners faced by every possible subset of buyers or sellers.

6 Summing up

When the market consists of heterogeneous buyers and sellers, effective arbitrage possibilities derive from a complex interaction of valuations, costs, and the architecture of the network of trading possibilities. In this context, purely topological considerations cannot provide a suitable analysis of the effective frictions impinging on the market. A proper understanding of the problem can only be achieved by an *integration* of topological features and individual characteristics (costs and valuations). This is precisely the approach pursued by our main result, which provides conditions that characterize configurations that are frictionless in the sense of inducing a uniform price across all trades. These conditions formally resemble those highlighted by the graph-theoretic matching literature but introduce the canonical economic mechanism, prices, in assessing the effective bargaining possibilities of agents that can be quite heterogeneous – not only in terms of their network position but also in terms of their individual inherent characteristics.

Appendix

In this Appendix, we provide the formal proof of our main result, Proposition 1. The proof relies on two separate Lemmas, which are stated and proven first.

Lemma 1. *Let the bipartite network $G = \{B \cup S, L\}$ and type profile $\boldsymbol{\theta} = ((c_s)_{s \in S}, (v_b)_{b \in B})$ define an UPC $(G, \boldsymbol{\theta})$ with a trading price p. Consider the network $G' = \{B \cup S, L \cup \{s'b'\}\}$, with $s' \in S$, $b' \in B$, and $s'b' \notin L$. Then, $(G', \boldsymbol{\theta})$ defines an UPC with the same trading price p.*

Proof: Let $\mathbf{x}^*(G, \boldsymbol{\theta})$ be the LBO induced by the UPC $(G, \boldsymbol{\theta})$ where all trades occur at the price

p. By Theorem 2 in Nguyen (2015), this outcome is the unique solution to the following (quadratic) optimization problem:

$$\min_{\mathbf{x}}(\textstyle\sum_{b\in B} x_b^2 + \sum_{s\in S} x_s^2), \quad s.t. \quad \forall sb \in L,\ x_s + x_b \geq \max\{v_b - c_s, 0\}. \tag{9}$$

Consider any seller s and buyer b such that $c_s \leq p \leq v_b$. Their equilibrium payoffs must then be, respectively, given by $x_s^* = p - c_s$ and $x_b^* = v_b - p$. To see this, consider the case of the seller s. Either some of her neighboring buyers in G trade at price p with other buyers or they do not trade at all. In the former case, seller s can also trade at price p. In the latter case, instead, by Assumption 1, there is some price at which s can trade profitably with some of his neighboring buyers. Thus, seller s must be active at equilibrium and his trading price must also be p, since the configuration $(G, \boldsymbol{\theta})$ is an UPC.

Note that the preceding argument applies to any seller-buyer pair (s, b), independently of whether they are connected in G or not. Thus suppose that the link $s'b'$ added to G to obtain G' indeed satisfies $c_{s'} \leq p \leq v_{b'}$. Then, the solution to the optimization problem (9) must satisfy:

$$x_{s'}^* + x_{b'}^* = p - c_{s'} + v_{b'} - p = v_{b'} - c_{s'}, \tag{10}$$

which implies that adding the constraint $x_{s'}^* + x_{b'}^* \geq v_{b'} - c_{s'}$ is redundant, and therefore $\mathbf{x}^*(G, \boldsymbol{\theta})$ is still a solution to the optimization problem obtained after adding this constraint. This means that $\mathbf{x}^*(G', \boldsymbol{\theta}) = \mathbf{x}^*(G, \boldsymbol{\theta})$.

Consider now the alternative case in which the link $s'b'$ added to G does *not* satisfy $c_{s'} \leq p \leq v_{b'}$. For concreteness, suppose that $c_{s'} \leq v_{b'} < p$. Then seller s' trades in G but the buyer b' does not. Hence the corresponding payoffs satisfy $x_{s'}^* = p - c_{s'}$, $x_{b'}^* = 0$ and, therefore,

$$x_{s'}^* + x_{b'}^* = p - c_{s'} > v_{b'} - c_{s'}.$$

Hence, again, if the link $s'b'$ is added to G and the constraint $x_{s'}^* + x_{b'}^* \geq v_{b'} - c_{s'}$ is added to (9) the solution remains unchanged. Thus, as before, we find that $\mathbf{x}^*(G', \boldsymbol{\theta}) = \mathbf{x}^*(G, \boldsymbol{\theta})$, which completes the proof of the Lemma. ■

Lemma 2. *For any disjoint non-empty subsets $S', S'' \subseteq S$ and $B', B'' \subseteq B$ in the bipartite network $G = \{B \cup S, L\}$ with the type profile $\boldsymbol{\theta} = ((c_s)_{s\in S}, (v_b)_{b\in B})$ it holds that,*

$$\mathcal{P}(S', B') < \mathcal{P}(S'', B'') \Rightarrow \mathcal{P}(S', B') < \mathcal{P}(S' \cup S'', B' \cup B'') < \mathcal{P}(S'', B''),$$

where $\mathcal{P}(.)$ is defined in (6).

Proof: By the definition (6) and by the fact that $S' \cap S'' = \emptyset$ and $B' \cap B'' = \emptyset$, it follows for $p = \mathcal{P}(S' \cup S'', B' \cup B'')$ that,

$$\sum_{b\in B(p)\cap B'} (v_b - p) + \sum_{b\in B(p)\cap B''} (v_b - p) = \sum_{s\in S(p)\cap S'} (p - c_s) + \sum_{s\in S(p)\cap S''} (p - c_s).$$

We write the last equality as $f(p; B', S') + f(p; B'', S'') = 0$, where,

$$f(p; \widetilde{B}, \widetilde{S}) \equiv \sum_{b \in B(p) \cap \widetilde{B}} (v_b - p) - \sum_{s \in S(p) \cap \widetilde{S}} (p - c_s), \quad \forall \widetilde{B} \subseteq B, \widetilde{S} \subseteq S.$$

Clearly, the function $f(p; .)$ is strictly decreasing in $p \in R$. Then, we have for $p' = \mathcal{P}(S', B') < p'' = \mathcal{P}(S'', B'')$ the following inequalities,

$$\begin{aligned} f(p'; B', S') &= 0 \Rightarrow f(p'; B', S') + f(p'; B'', S'') = f(p'; B'', S'') > 0, \\ f(p''; B'', S'') &= 0 \Rightarrow f(p''; B', S') + f(p''; B'', S'') = f(p''; B', S') < 0. \end{aligned}$$

As the function $f(p; .)$ is also continuous, it follows that there is a unique $p \in R$ solving $f(p; B', S') + f(p; B'', S'') = 0$ and $p \in (p', p'')$. ■

Proof of Proposition 1: We establish the desired equivalence by proving in turn a sufficient set of different implications.

- (7b)⇒(7a):

For the sake of contradiction, assume that the condition (7b) holds but $(G, \boldsymbol{\theta})$ is not an UPC. Then, there are at least two connected and disjoint subnetworks G' and G'' of G, where trade takes place at the uniform prices p' and p'', respectively, such that $p' \neq p''$. We will call each such subnetwork a *trading component* (TC). In each TC of G, we add all missing links until all buyers and sellers in this component are connected by a complete subnetwork. By Lemma 1, this will not affect the price in this component. Each node that does not belong to any TC (i.e., does not trade in equilibrium) is connected to at least one trading node due to our Assumption PL. For any such non-trading player v, we select one of her trading neighbors in some TC and connect v to all players from the opposite side in this TC. Again, this operation will not change the price in G' as v will be still inactive. Thus, we obtain a collection of completely connected TCs that cover disjoint sets of nodes, whose union is $S \cup B$, and each TC displays a uniform price.

If we now add all missing links between two completely connected TCs with the respective prices p' and p'', then Lemma 2 implies that the price in the merged component lies in the interval (p', p''). If we proceed in this way iteratively merging components, we will arrive at the completely connected bipartite network with the set of nodes $S \cup B$, where all trade takes place at the price $p^* = \mathcal{P}(B, S)$. By the iterative application of Lemma 2, this price must lie strictly between the minimum p_L and the maximum p_H price of the initial completely connected TCs,

$$p_L < p^* = \mathcal{P}(B, S) < p_H.$$

Now, denote by H a trading component with the price p_H. Furthermore, let H_B and H_S be, respectively, the (non-empty) set of active buyers and sellers in H. Note that any active seller

$s \in N_G(H_B)$ (who must of course have her cost $c_s \leq p_H$) will sell at equilibrium only at the highest price p_H, since this price is available to her. Hence, we can write $H_S = N_G(H_B)$ and, therefore,

$$\mathcal{P}(H_B, H_S) = \mathcal{P}(H_B, N_G(H_B)) = p_H > p^*,$$

which contradicts (7b).

- (7a)⇒(7b):

For the sake of contradiction assume that $(G, \boldsymbol{\theta})$ is UPC but (7b) does not hold. Then, there exists a non-empty set $B' \subseteq B$ such that,

$$\mathcal{P}(B', N_G(B')) = p_1 > p^* \equiv \mathcal{P}(B, S). \tag{11}$$

We add all missing links between B' and $N_G(B')$ until these two sets are connected by a complete subnetwork G_1 and we do the same for the sets $S' = S \backslash N_G(B')$ and $N_G(S')$ obtaining the complete subnetwork G_2 (G_1 and G_2 are only connected by links between $N_G(S')$ and $N_G(B')$). We denote by $\widetilde{G}$ the entire network that resulted from this link addition to G and note that $(\widetilde{G}, \boldsymbol{\theta})$ is an UPC by the Lemma 1.

Considering now G_1 and G_2 separately (i.e., ignoring all links between them), they cover disjoint sets of nodes, whose union is $S \cup B$, and each subnetwork displays a uniform price due to their completeness. Then, Lemma 2 and $p_1 > p^*$ imply that p_2 must verify,

$$p_2 = \mathcal{P}(N_G(S'), S') < p^*.$$

Considering now the entire network $\widetilde{G}$, trading at p_k in its subnetwork G_k, $k = 1, 2$, and disagreement (no trade) for any connected pair $(s, b) \in G_1 \times G_2$ forms a limit SSPE with the (unique) expected payoff vector $\mathbf{x}^*$. In particular, it is optimal not to trade for each pair $(s, b) \in N_G(B') \times N_G(S')$, i.e., $s \in G_1$, $b \in G_2$, as the sum of their expected payoffs is higher than their joint surplus,

$$x_s^* + x_b^* = \max\{p_1 - c_s, 0\} + \max\{v_b - p_2, 0\} \geq \max\{p_1 - c_s + v_b - p_2, 0\} > v_b - c_s.$$

As the equilibrium trade in $\widetilde{G}$ occurs at two different prices, $p_1 > p_2$, the configuration $(\widetilde{G}, \boldsymbol{\theta})$ cannot be an UPC.

Given the formal symmetry between buyers and sellers in the model, it is clear that it readily follows that both (7a)⇒(7c) and (7c)⇒(7a). This establishes the equivalence among (7a), (7b), and (7c), thus completing the proof. ■

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