ON FINITE COMPLETE REWRITING SYSTEMS, FINITE DERIVATION TYPE, AND AUTOMATICITY FOR HOMOGENEOUS MONOIDS

ALAN J. CAIN, ROBERT D. GRAY, AND ANTÓNIO MALHEIRO

Abstract. This paper investigates the class of finitely presented monoids defined by homogeneous (length-preserving) relations from a computational perspective. The properties of admitting a finite complete rewriting system, having finite derivation type, being automatic, and being biautomatic are investigated for this class of monoids. The first main result shows that for any consistent combination of these properties and their negations, there is a homogeneous monoid with exactly this combination of properties. We then introduce the new concept of abstract Rees-commensurability (an analogue of the notion of abstract commensurability for groups) in order to extend this result to show that the same statement holds even if one restricts attention to the class of n-ary homogeneous monoids (where every side of every relation has fixed length n). We then introduce a new encoding technique that allows us to extend the result partially to the class of n-ary multihomogenous monoids.

1. Introduction

Numerous interesting algebras arise as semigroup algebras $K[S]$, where K is a field and S is a homogeneous semigroup (that is, a semigroup that is defined by a presentation where all relations are length-preserving); examples include algebras yielding set-theoretic solutions to the Yang–Baxter equation and quadratic algebras of skew type (see for example [1, 2, 3] and [4, 5, 6]), algebras related to Young diagrams, representation theory and algebraic combinatorics such as the plactic and Chinese algebras (see [7, Ch. 5], [8, 9] and [10, 11, 12]), and algebras defined by permutation relations (see [13, 4, 14]). In these examples, there are strong connections between the structure of the algebra $K[S]$ and that of the underlying semigroup S. Further motivation for studying this class comes from other important semigroups in the literature that admit homogeneous presentations, such as the hypoplactic monoid [15], shifted plactic monoid [16], monoids with the same multihomogeneous growth as the plactic monoid [17], trace monoids [18],

The first author was supported by an FCT Ciência 2008 fellowship and later by an Investigador FCT fellowship (IF/01622/2013/CP1161/CT0001).

The second author was partially supported by the EPSRC grant EP/N033353/1 'Special inverse monoids: subgroups, structure, geometry, rewriting systems and the word problem'.

This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

divisibility monoids [19], queue monoids [20], and positive braid monoids [21, 22].

When investigating a semigroup S defined by homogeneous relations, and its associated semigroup algebra $K[S]$, a useful first step is to find a good set of normal forms (canonical representatives over the generating set) for the elements of the monoid, and thus for elements of the algebra. (See the list of open problems in [13, Section 3] for more on the importance of this problem in the context of semigroups defined by permutation relations.) Specifically we would like a set of normal forms that is a regular language, and we want to be able to compute effectively with these normal forms. Two situations where such a good set of normal forms does exist are for monoids that admit presentations by finite complete rewriting systems (see [23]), and for monoids and semigroups that are automatic (see [24, 25]). Each of these properties also has implications for properties of the corresponding semigroup algebra. Indeed, if the semigroup admits a finite complete rewriting system, then the semigroup algebra admits a finite Gröbner–Shirshov basis (see [26] for an explanation of the connection between Gröbner–Shirshov bases and complete rewriting systems), while the automaticity of the semigroup implies that the algebra is an automaton algebra in the sense of Ufnarovskij; see [27] and [4, Section 1].

Many of the examples of homogeneous semigroups mentioned above have been shown to admit presentations by finite complete rewriting systems, and have been shown to be biautomatic; see for example [28, 29, 30, 31, 32, 33]. It is natural to ask to what extent these results generalise to arbitrary homogeneous semigroups. One can ask: Does every homogeneous semigroup admit a presentation by a finite complete rewriting system? Is every such semigroup biautomatic? Within the class of homogeneous semigroups, what is the relationship between admitting a finite complete rewriting system and being biautomatic? (For general semigroups, these properties are independent; see [34]). The aim of this paper is to make a comprehensive investigation of these questions. In fact, we shall consider two different strengths of automaticity, called automaticity and biautomaticity, and we shall also investigate the homotopical finiteness property of finite derivation type (FDT) in the sense of Squier [35], which is a finiteness property that is satisfied by monoids that admit presentations by finite complete rewriting systems (full definitions of all of these concepts will be given in Section 2).

There are various degrees of homogeneity that one can impose on a semigroup presentation. We shall consider finite presentations $\langle A \mid \mathcal{R} \rangle$ which are:

- homogeneous: relations are length-preserving;
- multihomogeneous: for each letter a in the alphabet A, and for every relation $u = v$ in $\mathcal{R}$, the number of occurrences of the letter a in u equals the number of occurrences of the letter a in v;
- n-ary homogeneous: there is a fixed global constant n such that for every relation $u = v$ in $\mathcal{R}$ the lengths of the words u and v are both n;
- n-ary multihomogeneous: simultaneously n-ary homogeneous and multihomogeneous.

FCRS $\implies$ FDT		BIAUTO $\implies$ AUTO		Example	See
Y	Y	Y	Y	Plactic monoid	[28]
Y	Y	N	Y	$\mathsf{M}^{\text{FCRS}}_{\text{AUTO}}$	Example 3.1
Y	Y	N	N	$\mathsf{M}^{\text{FCRS}}_{\text{NONAUTO}}$	Example 3.6
N	Y	Y	Y	$\mathsf{M}^{\text{FDT}}_{\text{BIAUTO}}$	Example 3.8
N	N	Y	Y	$\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$	Example 3.10
N	Y	N	Y	$\mathsf{M}^{\text{FCRS}}_{\text{AUTO}} * \mathsf{M}^{\text{FDT}}_{\text{BIAUTO}}$	Section 4
N	Y	N	N	$\mathsf{M}^{\text{FDT}}_{\text{BIAUTO}} * \mathsf{M}^{\text{FCRS}}_{\text{NONAUTO}}$	Section 4
N	N	N	Y	$\mathsf{M}^{\text{FCRS}}_{\text{AUTO}} * \mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$	Section 4
N	N	N	N	$\mathsf{M}^{\text{FCRS}}_{\text{NONAUTO}} * \mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$	Section 4

TABLE 1. Summary of examples of homogeneous monoids exhibiting all consistent combinations of the properties FCRS, FDT, BIAUTO, and AUTO. Examples with the same combinations of properties also exist in the class of n-ary homogeneous monoids (see Section 5).

FCRS $\implies$ FDT		BIAUTO $\implies$ AUTO		Exists?	See
Y	Y	Y	Y	Y	[28]
Y	Y	N	Y	Y	Theorem 5.2
Y	Y	N	N	Y	Theorem 5.2
N	Y	Y	Y	?	Question 5.4
N	N	Y	Y	Y	Theorem 5.3
N	Y	N	Y	?	Question 5.4
N	Y	N	N	?	Question 5.4
N	N	N	Y	Y	Theorem 5.3
N	N	N	N	Y	Theorem 5.3

TABLE 2. Summary of the existence of examples of multihomogeneous monoids with consistent combinations of the properties FCRS, FDT, BIAUTO, and AUTO. Examples with the same combinations of properties also exist in the class of n-ary multihomogeneous monoids.

Of course, the most restricted class listed here is the class of n-ary multihomogeneous presentations.

For brevity, we introduce the following terminology for the four properties we are interested in: a monoid is

- FCRS if it admits a presentation via a finite complete rewriting system (with respect to some finite generating set);
- FDT if it has finite derivation type;
- BIAUTO if it is biautomatic;
- AUTO if it is automatic.

We will also use the natural negated terms: non-FCRS, non-FDT, non-BIAUTO, and non-AUTO.

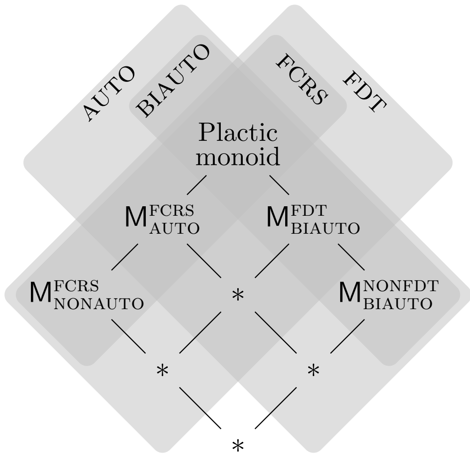

FIGURE 1. The semilattice showing the relationship between examples. By taking the free product of two examples, one obtains a new monoid whose properties are given by taking the logical conjunction (that is, the 'and' operation) of corresponding properties of the original example monoids. This corresponds to the meet operation in this semilattice.

We are interested in which combinations of these properties a homogeneous monoid can have. Since in general FCRS implies FDT, and BIAUTO implies AUTO, not all combinations will be possible. We refer to any combination of properties that satisfies these restrictions, and that does not contain a property and its negation, as *consistent*. Our first main result shows that any consistent combination is possible within the class of homogeneous monoids. We show this by constructing examples of homogeneous monoids with each consistent combination of properties. We adopt the following naming scheme: in the example monoid $\mathsf{M}^{\text{A}}_{\text{B}}$, the superscript A will be one of FCRS, FDT, or NONFDT, indicating that the monoid is respectively FCRS (and thus also FDT), FDT but not FCRS, or non-FDT (and thus also non-FCRS); while the subscript B will be one of BIAUTO, AUTO, or NONAUTO, indicating that the monoid is respectively BIAUTO (and thus also AUTO), AUTO but not BIAUTO, or non-AUTO (and thus also non-BIAUTO). In Section 3, we presents the fundamental examples $\mathsf{M}^{\text{FCRS}}_{\text{AUTO}}$, $\mathsf{M}^{\text{FDT}}_{\text{BIAUTO}}$, $\mathsf{M}^{\text{FCRS}}_{\text{NONAUTO}}$, and $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$. Section 4 contains general results about the behaviour of the various properties under free products of monoids, which we then use to construct the remaining examples. These results are summarised in Table 1 and the relationship between the various examples is illustrated in Figure 1.

In Sections 6 and 7 we introduce new concepts and prove new results, in order to study the combinations of properties can occur in even more restricted classes. We first introduce and investigate the notion of abstract Rees-commensurability (an analogue of abstract commensurability for groups [36, §§ iv.27ff.]), which allows us to show that every consistent combination can arise within the class of n-ary homogeneous monoids. (Thus Table 1 and Figure 1 could also describe the situation for n-ary homogeneous monoids.) We then develop a new encoding technique that embeds a homogeneous monoid into a 2-generated multihomogeneous monoid. This encoding technique allows us to obtain most of the consistent combinations of properties in the class of multihomogeneous or n-ary multihomogenous monoids. Specifically, it allows us to construct n-ary multihomogeneous monoids with any possile combination of the properties FCRS, BIAUTO, and AUTO, or any combination of the properties FDT, BIAUTO, and AUTO. However, it does not allow us to construct examples to separate the properties FCRS and FDT within the class of multihomogeneous or n-ary multihomogeneous monoids. Table 2 summarises the known consistent combinations of properties in the class of multihomogeneous monoids. Using the results for abstract Rees commensurability, the same table also describes the situation for the class of n-ary multihomogeneous monoids.

2. Preliminaries

The subsection below on derivation graphs, homotopy bases and finite derivation type is self-contained, but it can be complemented with [37, 38]. There is an alternative formulation of the same concepts in terms of strict monoidal categories/groupoids and higher-dimensional variations of them, and homotopical algebra in higher categories [39, 40]. However, our approach, using Squier complexes, is the same one used in papers by Otto [41, 42], Wang [43], Pride and the second and third authors [44], and the third author [45]; we will require methods and results from these papers in Sections 4, 6, and 7.

For further information on automatic semigroups, see [25]. We assume familiarity with basic notions of automata and regular languages (see, for example, [46]) and transducers and rational relations (see, for example, [47]), although we will recall some key results that we use frequently. For background on string rewriting systems we refer the reader to [48, 23].

2.1. **Words, rewriting systems, and presentations.** We denote the empty word (over any alphabet) by ε. For an alphabet A, we denote by A^* the set of all words over A. When A is a generating set for a monoid M, every element of A^* can be interpreted either as a word or as an element of M. For words $u, v \in A^*$, we write $u = v$ to indicate that u and v are equal as words and $u =_M v$ to denote that u and v represent the same element of the monoid M. The length of $u \in A^*$ is denoted $|u|$, and, for any $a \in A$, the number of symbols a in u is denoted $|u|_a$. We denote by u^{rev} the reversal of a word u; that is, if $u = a_1 \cdots a_{n-1}a_n$ then $u^{\mathrm{rev}} = a_n a_{n-1} \cdots a_1$, with $a_i \in A$. If $\mathcal{R}$ is a relation on A^*, then $\mathcal{R}^{\#}$ denotes the smallest monoid congruence generated by $\mathcal{R}$.

We use standard terminology and notation from the theory of string rewriting systems; see [23] or [48] for background reading.

If M is a monoid, a presentation of M is a pair $\langle A \mid \mathcal{R} \rangle$ such that M is isomorphic to the quotient $A^*/\mathcal{R}^{\#}$, in which case, the elements of $\mathcal{R}$ are called the defining relations. We write $[u]_M$ for the set of words in A^* equal to u in M. The presentation $\langle A \mid \mathcal{R} \rangle$ is *homogeneous* (respectively, *multihomogeneous*) if for every $(u, v) \in \mathcal{R}$ and $a \in A$, we have $|u| = |v|$ (respectively, $|u|_a = |v|_a$). That is, in a homogeneous presentation, defining relations preserve length; in a multihomogenous presentation, defining relations preserve the number of each symbol. A monoid is *homogeneous* (respectively, *multihomogeneous*) if it admits a homogeneous (respectively, multihomogeneous) presentation. Note that homogeneous and multihomogeneous presentations are not required to be finite presentations.

A *string rewriting system*, or simply a *rewriting system*, is a pair $(A, \mathcal{R})$, where A is a finite alphabet and $\mathcal{R}$ is a set of pairs (ℓ, r), usually written $\ell \to r$, known as *rewriting rules* or simply *rules*, drawn from $A^* \times A^*$. The single reduction relation $\to_{\mathcal{R}}$ is defined as follows: $u \to_{\mathcal{R}} v$ (where $u, v \in A^*$) if there exists a rewriting rule $(\ell, r) \in \mathcal{R}$ and words $x, y \in A^*$ such that $u = x\ell y$ and $v = xry$. That is, $u \to_{\mathcal{R}} v$ if one can obtain v from u by substituting the word r for a subword ℓ of u, where $\ell \to r$ is a rewriting rule. The reduction relation $\to_{\mathcal{R}}^*$ is the reflexive and transitive closure of $\to_{\mathcal{R}}$. The subscript $\mathcal{R}$ is omitted when it is clear from context. The process of replacing a subword ℓ by a word r, where $\ell \to r$ is a rule, is called *reduction* by application of the rule $\ell \to r$; the iteration of this process is also called reduction. A word $w \in A^*$ is *reducible* if it contains a subword ℓ that forms the left-hand side of a rewriting rule in $\mathcal{R}$; it is otherwise called *irreducible*.

The rewriting system $(A, \mathcal{R})$ is *finite* if both A and $\mathcal{R}$ are finite. The rewriting system $(A, \mathcal{R})$ is *noetherian* if there is no infinite sequence $u_1, u_2, \ldots$ of words from A^* such that $u_i \to u_{i+1}$ for all $i \in \mathbb{N}$. That is, $(A, \mathcal{R})$ is noetherian if any process of reduction must eventually terminate with an irreducible word. The rewriting system $(A, \mathcal{R})$ is *confluent* if, for any words $u, u', u'' \in A^*$ with $u \to^* u'$ and $u \to^* u''$, the pair u' and u'' *resolves*, that is, there exists a word $v \in A^*$ such that $u' \to^* v$ and $u'' \to^* v$. It is well known that a noetherian system is confluent if and only if all critical pairs resolve, where critical pairs are obtained by considering overlaps of left-hand sides of the rewrite rules in $\mathcal{R}$; see [23] for more details. A rewriting system that is both confluent and noetherian is *complete*. If a monoid admits a presentation with respect to some generating set A that forms a finite complete rewriting system $\mathcal{R}$, the monoid is FCRS. In that case, the irreducible elements form a set of unique normal forms, over A, for the elements of the monoid.

The *Thue congruence* $\leftrightarrow_{\mathcal{R}}^*$ is the equivalence relation generated by $\to_{\mathcal{R}}$. The elements of the monoid presented by $\langle A \mid \mathcal{R} \rangle$ are the $\leftrightarrow_{\mathcal{R}}^*$-equivalence classes. The relations $\leftrightarrow_{\mathcal{R}}^*$ and $\mathcal{R}^\#$ coincide.

Let M be a homogeneous monoid. Let $\langle A \mid \mathcal{R} \rangle$ be a homogeneous presentation of M. Without lost of generality, we can assume that $\mathcal{R}$ has no trivial relations of the form $a = a'$, for letters a, a' in A. Since none of the generators represented by A can be non-trivially decomposed, the alphabet A represents a unique minimal generating set for M, and any generating set must contain this minimal generating set. Any two words over A representing the same element of M must be of the same length. So there is a well-defined function $\lambda : M \to \mathbb{N}$ where $x\lambda$ is defined to be the length of any word over A representing x. (Here and elsewhere we shall write the function symbol on the right.) It is easy to see that λ is a homomorphism. Following [49, Definition 2.1 in §4] the function λ is called a *grading* of M, and so homogeneous monoids are *graded monoids*.

2.2. **Derivation graphs, homotopy bases, and finite derivation type.** Associated with any monoid presentation $\langle A \mid \mathcal{R} \rangle$ is a 2-complex $\mathcal{D}$, called the *Squier complex*, whose 1-skeleton has vertex set A^* and edges corresponding to applications of relations from $\mathcal{R}$, and that has 2-cells adjoined for each instance of "non-overlapping" applications of relations from $\mathcal{R}$ (see below for a formal definition of non-overlapping relations). The free monoid A^* acts on $\mathcal{D}$ in a natural way via left and right multiplication. A collection of closed paths in $\mathcal{D}$ is called a *homotopy base* if the complex obtained by adjoining cells for each of these paths, and those that they generate under the action of the free monoid on the Squier complex, has trivial fundamental groups. A monoid defined by a presentation is said to have *finite derivation type* (or FDT for short) if the corresponding Squier complex admits a finite homotopy base. It was shown by Squier [35] that the property FDT is independent of the choice of a finite presentation, so we may speak of FDT monoids. The original motivation for studying this notion is Squier's result [35] which says that if a monoid admits a presentation by a finite complete rewriting system then the monoid must have finite derivation type. The study of these concepts is motivated further by the fact that the fundamental groups of connected components of Squier complexes, which are called *diagram groups*, have turned out to be a very interesting class of groups; see [50]. In recent important work [51], acyclic polygraphs have been used to define a higher-dimensional homotopical finiteness condition for higher categories. In particular, this work gives rise to a definition of FDT_n that extends the notion of finite derivation type to arbitrary dimensions.

In more detail, with any monoid presentation $\mathcal{P} = \langle A \mid \mathcal{R} \rangle$ we associate a graph (in the sense of Serre [52]) as follows. The *derivation graph* of $\mathcal{P}$ is an infinite graph $\Gamma = \Gamma(\mathcal{P}) = (V, E, \iota, \tau, ^{-1})$ with *vertex set* $V = A^*$, and *edge set* E consisting of the collection of 4-tuples

$$\{(w_1, r, \epsilon, w_2) : \; w_1, w_2 \in A^*, r \in \mathcal{R}, \text{ and } \epsilon \in \{+1, -1\}\}.$$

The functions $\iota, \tau : E \to V$ associate with each edge $\mathbb{E} = (w_1, r, \epsilon, w_2)$ (with $r = (r_{+1}, r_{-1}) \in \mathcal{R}$) its initial and terminal vertices $\iota\mathbb{E} = w_1 r_\epsilon w_2$ and $\tau\mathbb{E} = w_1 r_{-\epsilon} w_2$, respectively. The mapping $^{-1} : E \to E$ associates with each edge $\mathbb{E} = (w_1, r, \epsilon, w_2)$ an inverse edge $\mathbb{E}^{-1} = (w_1, r, -\epsilon, w_2)$.

A non-empty path in Γ is a sequence of edges $\mathbb{P} = \mathbb{E}_1 \circ \mathbb{E}_2 \circ \ldots \circ \mathbb{E}_n$, written in the diagrammatic order, where $\tau\mathbb{E}_i = \iota\mathbb{E}_{i+1}$ for $i = 1, \ldots, n-1$. Here $\mathbb{P}$ is a path from $\iota\mathbb{E}_1$ to $\tau\mathbb{E}_n$ and we extend the mappings ι and τ to paths by defining $\iota\mathbb{P} = \iota\mathbb{E}_1$ and $\tau\mathbb{P} = \tau\mathbb{E}_n$. The inverse of a path $\mathbb{P} = \mathbb{E}_1 \circ \mathbb{E}_2 \circ \ldots \circ \mathbb{E}_n$ is the path $\mathbb{P}^{-1} = \mathbb{E}_n^{-1} \circ \mathbb{E}_{n-1}^{-1} \circ \ldots \circ \mathbb{E}_1^{-1}$, which is a path from $\tau\mathbb{P}$ to $\iota\mathbb{P}$. A *closed path* is a path $\mathbb{P}$ satisfying $\iota\mathbb{P} = \tau\mathbb{P}$. For two paths $\mathbb{P}$ and $\mathbb{Q}$ with $\tau\mathbb{P} = \iota\mathbb{Q}$ the composition $\mathbb{P} \circ \mathbb{Q}$ is defined.

We denote the set of paths in Γ by $P(\Gamma)$, where for each vertex $w \in V$ we include a path 1_w with no edges, called the *empty path* at w. The free monoid A^* acts on both sides of the set of edges E of Γ by

$$x \cdot \mathbb{E} \cdot y = (xw_1, r, \epsilon, w_2 y)$$

where $\mathbb{E} = (w_1, r, \epsilon, w_2)$ and $x, y \in A^*$. This extends naturally to a two-sided action of A^* on $P(\Gamma)$ where for a path $\mathbb{P} = \mathbb{E}_1 \circ \mathbb{E}_2 \circ \ldots \circ \mathbb{E}_n$ we define

$$x \cdot \mathbb{P} \cdot y = (x \cdot \mathbb{E}_1 \cdot y) \circ (x \cdot \mathbb{E}_2 \cdot y) \circ \ldots \circ (x \cdot \mathbb{E}_n \cdot y).$$

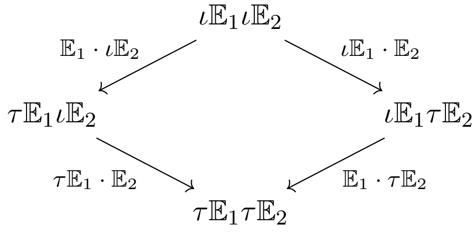

FIGURE 2. Disjoint derivations in Γ.

If $\mathbb{P}$ and $\mathbb{Q}$ are paths such that $\iota\mathbb{P} = \iota\mathbb{Q}$ and $\tau\mathbb{P} = \tau\mathbb{Q}$ then we say that $\mathbb{P}$ and $\mathbb{Q}$ are *parallel*, and write $\mathbb{P} \parallel \mathbb{Q}$. We use $\parallel$ to denote the subset of $P(\Gamma) \times P(\Gamma)$ of all pairs of parallel paths.

An equivalence relation $\sim$ on $P(\Gamma)$ is called a *homotopy relation* if it is contained in $\parallel$ and satisfies the following four conditions.

(1) If $\mathbb{E}_1$ and $\mathbb{E}_2$ are edges of Γ, then
$$(\mathbb{E}_1 \cdot \iota\mathbb{E}_2) \circ (\tau\mathbb{E}_1 \cdot \mathbb{E}_2) \sim (\iota\mathbb{E}_1 \cdot \mathbb{E}_2) \circ (\mathbb{E}_1 \cdot \tau\mathbb{E}_2).$$

(2) For any $\mathbb{P}, \mathbb{Q} \in P(\Gamma)$ and $x, y \in A^*$
$$\mathbb{P} \sim \mathbb{Q} \ \text{ implies } \ x \cdot \mathbb{P} \cdot y \sim x \cdot \mathbb{Q} \cdot y.$$

(3) For any $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S} \in P(\Gamma)$ with $\tau\mathbb{R} = \iota\mathbb{P} = \iota\mathbb{Q}$ and $\iota\mathbb{S} = \tau\mathbb{P} = \tau\mathbb{Q}$
$$\mathbb{P} \sim \mathbb{Q} \ \text{ implies } \ \mathbb{R} \circ \mathbb{P} \circ \mathbb{S} \sim \mathbb{R} \circ \mathbb{Q} \circ \mathbb{S}.$$

(4) If $\mathbb{P} \in P(\Gamma)$ then $\mathbb{P}\mathbb{P}^{-1} \sim 1_{\iota\mathbb{P}}$, where $1_{\iota\mathbb{P}}$ denotes the empty path at the vertex $\iota\mathbb{P}$.

The idea behind condition 1 is the following. Suppose that a word w has two disjoint occurrences of rewriting rules in the sense that $w = \alpha r_\epsilon \beta \alpha' r'_{\epsilon'} \beta'$ where $\alpha, \beta, \alpha', \beta' \in A^*$, $r, r' \in R$ and $\epsilon, \epsilon' \in \{-1, +1\}$. Let $\mathbb{E}_1 = (\alpha, r, \epsilon, \beta)$ and $\mathbb{E}_2 = (\alpha', r', \epsilon', \beta')$. Then the paths
$$\mathbb{P} = (\mathbb{E}_1 \cdot \iota\mathbb{E}_2) \circ (\tau\mathbb{E}_1 \cdot \mathbb{E}_2), \quad \mathbb{P}' = (\iota\mathbb{E}_1 \cdot \mathbb{E}_2) \circ (\mathbb{E}_1 \cdot \tau\mathbb{E}_2)$$
give two different ways of rewriting the word $w = \alpha r_\epsilon \beta \alpha' r'_{\epsilon'} \beta'$ to the word $w = \alpha r_{-\epsilon} \beta \alpha' r'_{-\epsilon'} \beta'$, where in $\mathbb{P}$ we first apply the left-hand relation and then the right-hand, while in $\mathbb{P}'$ the relations are applied in the opposite order; see Figure 2. We want to regard these two paths as being essentially the same, and this is achieved by condition 1. Equivalent paths under this condition are said to be homotopic by *disjoint derivations*. This relation is also often refereed to as the exchange relation or the interchange law in the literature; see [40].

For a subset C of $\parallel$, the homotopy relation $\sim_C$ *generated by* C is the smallest (with respect to inclusion) homotopy relation containing C. The homotopy relation generated by the empty set $\varnothing$ is denoted by $\sim_0$. If $\sim_C$ coincides with $\parallel$, then C is called a *homotopy base* for Γ. The presentation $\langle A \mid \mathcal{R} \rangle$ is said to have *finite derivation type* if the derivation graph Γ of $\langle A \mid \mathcal{R} \rangle$ admits a finite homotopy base. A finitely presented monoid M is said to have *finite derivation type*, or to be FDT, if some (and hence any by [35, Theorem 4.3]) finite presentation for M has finite derivation type.

It is not difficult to see that a subset C of $\|$ is a homotopy base of Γ if and only if the set

$$\{(\mathbb{P} \circ \mathbb{Q}^{-1}, 1_{\iota\mathbb{P}}) : (\mathbb{P}, \mathbb{Q}) \in C\}$$

is a homotopy base for Γ. Thus we say that a set D of closed paths is a homotopy base if the corresponding set $\{(\mathbb{P}, 1_{\iota\mathbb{P}}) : \mathbb{P} \in D\}$ is a homotopy base.

2.3. **Rational relations.** For references purposes, we briefly recall here some basic definitions and results regarding rational relations; we only consider relations of the form $R \subseteq A^* \times B^*$. The set of rational relations between A^* and B^* is the smallest subset of $\mathbb{P}A^* \times B^*$ that contains the empty set $\emptyset$, all singleton sets $\{(u, v)\}$, and is closed under the operations of union, product, and the Kleene star

$$X \mapsto X^* = \bigcup_{i=0}^{\infty} X^i.$$

Note that the set of rational relations is also closed under the Kleene plus operation $X \mapsto X^+ = X^*X$.

Proposition 2.1 ([47, Examples 5.1 & 5.5])**.** *For any regular language $L \subseteq A^*$, the relation*

$$\{\, (u, u) : u \in L \,\} \subseteq A^* \times A^*$$

is rational.

Proposition 2.2. *Let $K \subseteq A^*$ and $L \subseteq B^*$ be regular languages. If $R \subseteq A^* \times B^*$ is a rational relation, then $R \cap (K \times L)$ is a rational relation. In particular, $K \times L$ is a rational relation.*

Proof. Let $\mathcal{T}$ be a transducer recognizing R and let $\mathcal{M}$ and $\mathcal{N}$ be finite automata recognizing K and L respectively. Adapt $\mathcal{T}$ to simulate $\mathcal{M}$ and $\mathcal{N}$ on the inputs from its first and second tapes, respectively, and to accept only if the simulated copies of $\mathcal{M}$ and $\mathcal{N}$ are in accept states. This adapted transducer recognizes $R \cap (K \times L)$. ☐

2.4. **Automaticity and biautomaticity.**

Definition 2.3. Let A be an alphabet and let \$ be a new symbol not in A. Define the mapping ${}_\sqcup{}^\$: A^* \times A^* \to ((A \cup \{\$\}) \times (A \cup \{\$\}))^*$ by

$$(u_1 \cdots u_m, v_1 \cdots v_n) \mapsto \begin{cases} (u_1, v_1) \cdots (u_m, v_n) & \text{if } m = n, \\ (u_1, v_1) \cdots (u_n, v_n)(u_{n+1}, \$) \cdots (u_m, \$) & \text{if } m > n, \\ (u_1, v_1) \cdots (u_m, v_m)(\$, v_{m+1}) \cdots (\$, v_n) & \text{if } m < n, \end{cases}$$

and the mapping ${}^\${}_\sqcup : A^* \times A^* \to ((A \cup \{\$\}) \times (A \cup \{\$\}))^*$ by

$$(u_1 \cdots u_m, v_1 \cdots v_n) \mapsto \begin{cases} (u_1, v_1) \cdots (u_m, v_n) & \text{if } m = n, \\ (u_1, \$) \cdots (u_{m-n}, \$)(u_{m-n+1}, v_1) \cdots (u_m, v_n) & \text{if } m > n, \\ (\$, v_1) \cdots (\$, v_{n-m})(u_1, v_{n-m+1}) \cdots (u_m, v_n) & \text{if } m < n, \end{cases}$$

where $u_i, v_i \in A$.

Definition 2.4. Let M be a finitely generated monoid. Let A be a finite set of generators for M and let $L \subseteq A^*$ be a regular language such that every element of M has at least one representative in L. For each $a \in A \cup \{\varepsilon\}$, define the relations

$$L_a = \{(u, v) : u, v \in L, ua =_M v\}$$
$$_aL = \{(u, v) : u, v \in L, au =_M v\}.$$

The pair (A, L) is an *automatic structure* for M if $L_a^\$$ is a regular language over $(A \cup \{\$\}) \times (A \cup \{\$\})$ for all $a \in A \cup \{\varepsilon\}$. A monoid M is *automatic*, or AUTO, if it admits an automatic structure with respect to some finite generating set.

The pair (A, L) is a *biautomatic structure* for M if $L_a^\$$, $_aL^\$$, $^\$L_a$, and $^\$_aL$ are regular languages over $(A \cup \{\$\}) \times (A \cup \{\$\})$ for all $a \in A \cup \{\varepsilon\}$. A monoid M is *biautomatic*, or BIAUTO, if it admits a biautomatic structure with respect to some finite generating set. [Note that BIAUTO implies AUTO.]

Hoffmann & Thomas have made a careful study of biautomaticity for semigroups [53]. They distinguish four notions of biautomaticity for semigroups that require at least one of $L_a^\$$ and $^\$L_a$ and at least one of $_aL^\$$ and $^\$_aL$ to be regular. These notions are all equivalent for groups and more generally for cancellative semigroups [53, Theorem 1] but distinct for semigroups [53, Remark 1 & § 4]. BIAUTO clearly implies all four Hoffmann–Thomas notions of biautomaticity. However, we shall shortly prove that, within the class of homogeneous monoids, any of the Hoffmann–Thomas notions of biautomaticity implies BIAUTO (see Proposition 2.7).

In proving that $R^\$$ or $^\$R$ is regular, where R is a relation on A^*, a useful strategy is to prove that R is a rational relation (that is, a relation recognized by a finite transducer [47, Theorem 6.1]) and then apply the following result, which is a combination of [54, Corollary 2.5] and [53, Proposition 4]:

Proposition 2.5. *If $R \subseteq A^* \times A^*$ is rational relation and there is a constant k such that $\big||u| - |v|\big| \leq k$ for all $(u, v) \in R$, then $R^\$$ and $^\$R$ are regular.*

Now we shall prove some results on automaticity and biautomaticity for the class of homogeneous monoids.

Unlike the situation for groups, both automaticity and biautomaticity for monoids and semigroups are dependent on the choice of generating set [25, Example 4.5]. However, for monoids, biautomaticity and automaticity are independent of the choice of *semigroup* generating sets [55, Theorem 1.1]. In our particular case of homogeneous monoids, we do have independence of the choice of generating set:

Proposition 2.6. *Let M be a homogeneous monoid that is* AUTO *(respectively,* BIAUTO*). Then for any finite generating set C of M there is a language K over C such that (C, K) is an automatic (respectively, biautomatic) structure for M.*

Proof. We first consider the case for AUTO. Suppose (B, L) is an automatic structure for M.

Notice that both the alphabet B and the alphabet C must contain a subalphabet representing the unique minimal generating set of M. Without loss of generality, assume that they both contain the alphabet A representing this minimal generating set. For each $b \in B$, let $w_b \in A^*$ be such that $w_b =_M b$. Let $\mathcal{Q} \subseteq B^* \times A^*$ be the relation

$$\{ (b, w) : b \in B \}^*.$$

Since $\mathcal{Q}$ is simply the subset of $B^* \times A^*$ obtained by taking the Kleene star of the finite set of elements of the form (b, w_b), it is by definition a rational relation. Let

$$K = L \circ \mathcal{Q} = \{v \in A^* : (\exists u \in L)\big((u, v) \in \mathcal{Q}\big)\}.$$

Let $a \in A \cup \{\varepsilon\}$. Then

$$\begin{aligned}(u, v) \in K_a &\iff u \in K \wedge v \in K \wedge ua =_M v \\ &\iff (\exists u', v' \in L)((u', u) \in \mathcal{Q} \wedge (v', v) \in \mathcal{Q} \wedge u'a =_M v') \\ &\iff (\exists u', v' \in L)((u', u) \in \mathcal{Q} \wedge (v', v) \in \mathcal{Q} \wedge (u', v') \in L_a) \\ &\iff (u, v) \in \mathcal{Q}^{-1} \circ L_a \circ \mathcal{Q}.\end{aligned}$$

Since a composition of rational relations is rational [47, Theorem 4.4], it follows that $K_a = \mathcal{Q}^{-1} \circ L_a \circ \mathcal{Q}$ is a rational relation. Furthermore

$$\begin{aligned}(u, v) \in K_a &\implies ua =_M v \\ &\implies (ua)\lambda = v\lambda \\ &\implies u\lambda + a\lambda = v\lambda \\ &\implies |u| + 1 = |v|;\end{aligned}$$

thus $K_a^{\$}$ is a regular language by Proposition 2.5.

For $c \in C - A$, let $u = u_1 \cdots u_m \in A^*$ be such that $u = c$; then $K_c^{\$} = (K_{u_1} \circ K_{u_2} \circ \cdots \circ K_{u_m})^{\$}$ is regular by [25, Proposition 2.3] and similarly ${}^{\$}K_c$ is regular. Hence (C, K) is an automatic structure for M.

For BIAUTO, assume (B, L) is a biautomatic structure for M and follow the above reasoning to show that each of the languages $K_a^{\$}$, ${}_aK^{\$}$, ${}^{\$}K_a$, and ${}^{\$}_aK$ are regular. □

Proposition 2.7. *Let M be a homogeneous monoid, let B be a finite generating set for M, and let L be a regular language over B such that every element of M has at least one representative in L and such that, for each $b \in B \cup \{\varepsilon\}$, at least one of $L_b^{\$}$ and ${}^{\$}L_b$ and at least one of ${}_bL^{\$}$ and ${}^{\$}_bL$ is regular. Then M is* BIAUTO.

Proof. Suppose $L_b^{\$}$ and ${}_bL^{\$}$ are regular; the other cases are similar.

As in the proof of Proposition 2.6, the alphabet B must contain the unique minimal generating set A of M. Construct the relation $\mathcal{Q} \subseteq B^* \times A^*$ as in the proof of Proposition 2.6. Let $K = L \circ \mathcal{Q}$.

Let $a \in A \subseteq B$. Then at least one of $L_a^{\$}$ and ${}^{\$}L_a$ and at least one of ${}_aL^{\$}$ and ${}^{\$}_aL$ is regular. In particular, L_a and ${}_aL$ are rational relations. So $K_a = \mathcal{Q}^{-1} \circ L_a \circ \mathcal{Q}$ and ${}_aK = \mathcal{Q}^{-1} \circ {}_aL \circ \mathcal{Q}$ are rational relations. If (u, v) is in K_a or ${}_aK$, then $|u| + 1 = |v|$. Hence $K_a^{\$}$, ${}^{\$}K_a$, ${}_aK^{\$}$, and ${}^{\$}_aK$ are all regular by Proposition 2.5. Since $a \in A \cup \{\varepsilon\}$ was arbitrary, this proves that (A, K) is a biautomatic structure for M. □

Despite the positive results obtained so far, note that AUTO does not imply BIAUTO in the class of homogeneous monoids, as we shall see below in Example 3.1.

3. Fundamental examples

3.1. An FCRS, FDT, non-BIAUTO, AUTO homogeneous monoid.

In this subsection we present a homogeneous monoid that is FCRS and thus FDT, is AUTO, but is not BIAUTO. By considering the reversal semigroup of this example we will get a homogeneous monoid that admits a finite complete rewriting system but is not automatic.

Example 3.1. Let $\mathsf{M}^{\text{FCRS}}_{\text{AUTO}}$ be the monoid defined by the presentation $\langle A \mid \mathcal{R}\rangle$, where $A = \{a, b, c\}$ and $\mathcal{R}$ consists of the rewriting rules

$$\begin{aligned} cxyz &\to cxcz, \qquad\qquad (x, y, z \in \{a, b\}) \\ cbca &\to cacb. \end{aligned}$$

Proposition 3.2. *The monoid* $\mathsf{M}^{\text{FCRS}}_{\text{AUTO}}$ *is* FCRS*.*

Proof. The rewriting system $(A, \mathcal{R})$ is noetherian because every rewriting rule either decreases the number of non-c symbols, or it stays the same and decreases the number of symbols b to the left of symbols a. To see that it is confluent, notice that the only overlaps are those between the left-hand side of $cbca \to cacb$ and the left-hand side of a rule of the form $cayz \to cacz$, where $y, z \in \{a, b\}$. However, they resolve since we have

$$cbcayz \to \begin{cases} cacbyz \to cacbcz \\ cbcacz \to cacbcz \end{cases}, \text{ for any } y, z \in \{a, b\}.$$

Therefore $(A, \mathcal{R})$ is confluent. □

Proposition 3.3. *The monoid* $\mathsf{M}^{\text{FCRS}}_{\text{AUTO}}$ *is* AUTO*, but non-*BIAUTO*.*

Proof. Let L be the language of normal forms of $(A, \mathcal{R})$. Since $(A, \mathcal{R})$ is finite, L is regular [23, Lemma 2.1.3]. Let $u \in L$. Consider the following cases separately:

(1) uc must also be in normal form, since no left-hand side of a rewriting rule ends in c. Hence

$$L_c = \{\, (u, u) : u \in L \,\}(\varepsilon, c),$$

and so L_c is rational by Proposition 2.1.

(2) If ub is not in normal form, then ub must end with the left-hand side of a rewriting rule. Hence $u = u'cxy$ for some $x, y \in \{a, b\}$, then $ub = u'cxyb \to u'cxcb$. This word $u'cxcb$ is in normal form since $u'cx$ (which is a prefix of u) is in normal form and no rewriting rule has left-hand side $cxcb$ for any $x \in \{a, b\}$. Thus

$$\begin{aligned} L_b &= \{\, (u, u)(\varepsilon, b) : u \in L, ub \in L \,\} \\ &\quad \cup \{\, (u'cxy, u'cxcb) : x, y \in \{a, b\}, u'cxy \in L \,\} \\ &= \{\, (u, u)(\varepsilon, b) : u \in L, ub \in L \,\} \\ &\quad \cup \big(\{\, (u', u') : u' \in L \,\}\{\, (cxy, cxcb) : x, y \in \{a, b\} \,\} \cap (L \times L)\big), \end{aligned}$$

and so L_b is rational by Proposition 2.2.

(3) If ua is not in normal form, then ua must end with the left-hand side of a rewriting rule and so either $u = u'cbc$ or $u = u'cxy$ for some $x, y \in \{a, b\}$.
   (a) If $u = u''(cb)^\alpha y$, with $y \in \{a, b, c\}$ and where $\alpha \geq 1$ is maximal, then $ua = u''(cb)^\alpha ya \to^* u''ca(cb)^\alpha$, since $ua = u''(cb)^\alpha ya \to u''(cb)^\alpha ca$ when $y \in \{a, b\}$, and then $cbca \to cacb$. Now, $u''ca(cb)^\alpha$ is in normal form since u'' and $ca(ab)^\alpha$ are in normal form and the only left-hand side of a rewriting rule of that ends in ca is $cbca$, and α is maximal.
   (b) If $u = u'cay$, with $y \in \{a, b\}$, then $ua = u'caya \to u'caca$ and this word is in normal form since $u'ca$ is in normal form.

Therefore

$$\begin{aligned}
L_a &= \{\,(u, ua) : u \in L, ua \in L\,\} \\
&\quad \cup \{\,(u''(cb)^\alpha y, u''ca(cb)^\alpha) : \alpha \in \mathbb{N}, y \in \{a, b, c\}, u''(cb)^\alpha y \in L, u'' \notin A^*cb\,\} \\
&\quad \cup \{\,(u'cay, u'caca) : y \in \{a, b\}, u'cay \in L\,\} \\
&= \big(\{\,(u, u)) : u \in L\,\}(\varepsilon, a) \cap (L \times L)\big) \\
&\quad \cup \big(\{\,(u''u'') : u'' \in L \setminus (A^*cb)\,\}(\varepsilon, ca)(cb, cb)^+\{\,(y, \varepsilon) : y \in \{a, b, c\}\,\} \cap L \times L\big) \\
&\quad \cup \big(\{\,(u', u') : u' \in L\,\}\{\,(cay, caca) : y \in \{a, b\}\,\} \cap L \times L\big),
\end{aligned}$$

and so L_a is a union of relations, each of which is rational by Proposition 2.1 and 2.2.

Note also that $L_\varepsilon = \{(u, u) : u \in L\}$ is rational. Hence L_x is a rational relation for any $x \in A \cup \{\varepsilon\}$. Moreover, if (u, v) lies in one of these relations, then $\big||u| - |v|\big| \leq 1$ and so $L_x^\$$ is regular for all $x \in A \cup \{\varepsilon\}$ by Propositions 2.5 and 2.2. Hence $\mathsf{M}_{\text{AUTO}}^{\text{FCRS}}$ is AUTO.

Suppose, with the aim of obtaining a contradiction, that $\mathsf{M}_{\text{AUTO}}^{\text{FCRS}}$ is BIAUTO. Then by Proposition 2.6 it admits a biautomatic structure (A, L). Thus L is a regular language mapping onto $\mathsf{M}_{\text{AUTO}}^{\text{FCRS}}$ and ${}_cL^\$$ is regular. This contradicts Lemma 3.4 below. ☐

Lemma 3.4. *There is no regular language $L \subseteq A^*$ such that L maps onto $\mathsf{M}_{\text{AUTO}}^{\text{FCRS}}$ and ${}_cL^\$$ is regular.*

Proof. Suppose, with the aim of obtaining a contradiction, that such a language L exists. Then $({}_cL \circ {}_cL^{-1})^\$$ is regular. Let n be an even number exceeding the number of states in an automaton recognizing $({}_cL \circ {}_cL^{-1})^\$$. Observe that

$${}_cL \circ {}_cL^{-1} = \{(u, v) \in L : cu =_{\mathsf{M}} cv\}.$$

Notice that $a^n b^{n+1}$ is not represented by any other word over A and similarly $b^n a^n b$ is not represented by any other word over A. So $a^n b^{n+1}, b^n a^n b \in L$. Furthermore,

$$ca^n b^{n+1} \to^* (ca)^{n/2}(cb)^{(n/2)+1}$$

and

$$\begin{aligned}
cb^n a^n b &\to^* (cb)^{n/2}(ca)^{n/2}cb \\
&\to^* (ca)^{n/2}(cb)^{(n/2)+1}
\end{aligned}$$

and so $(a^n b^{n+1}, b^n a^n b) \in {}_cL \circ {}_cL^{-1}$. Since n exceeds the number of states in an automaton recognizing $({}_cL \circ {}_cL^{-1})^{\$}$, we can apply the pumping lemma to a segment of the word that lies within the first n letters of $(a^n b^{n+1}, b^n a^n b)^{\$}$ (that is, to a subword of the form $(a^k, b^k)^{\$}$ for some $k \geq 1$) to see that $(a^{n+ik} b^{n+1}, b^{n+ik} a^n b) \in {}_cL \circ {}_cL^{-1}$ for some $k \geq 1$ and for all $i \in \mathbb{N}$. Hence, by definition of the relation ${}_cL$, we have $ca^{n+ik} b^{n+1} =_{\mathsf{M}} cb^{n+ik} a^n b$. But

$$ca^{n+2k} b^{n+1} \to^* (ca)^{n/2+k} (cb)^{(n/2)+1}$$

and

$$\begin{aligned} cb^{n+2k} a^n b &\to^* (cb)^{n/2+k} (ca)^{n/2} cb \\ &\to^* (ca)^{n/2} (cb)^{(n/2)+k+1}, \end{aligned}$$

and so the normal forms of $ca^{n+ik} b^{n+1}$ and $cb^{n+ik} a^n b$ are unequal; this contradicts the previous equality. Thus M is not BIAUTO. ☐

3.2. An FCRS, FDT, non-BIAUTO, non-AUTO homogeneous monoid.

Definition 3.5. Let S be a monoid defined by a presentation $\langle A \mid \mathcal{R} \rangle$. Denote by M^{rev} the monoid defined by the presentation $\langle A \mid \mathcal{R}^{\mathrm{rev}} \rangle$, where $\mathcal{R}^{\mathrm{rev}} = \{(l^{\mathrm{rev}}, r^{\mathrm{rev}}) : (l, r) \in \mathcal{R}\}$, that is called the *reversal monoid* of M. [Note that $(M^{\mathrm{rev}})^{\mathrm{rev}} \simeq M$.]

Example 3.6. Let $\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}} = (\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{AUTO}})^{\mathrm{rev}}$, which is defined by the presentation $\langle A \mid \mathcal{R}^{\mathrm{rev}} \rangle$, where $\langle A \mid \mathcal{R} \rangle$ is the presentation defining Example 3.1.

Since $\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}}$ is presented by $\langle A \mid \mathcal{R}^{\mathrm{rev}} \rangle$ we can argue as in the proof of Proposition 3.2 that the rewriting system is noetherian and that the overlaps, that result in critical pairs, resolve in a similar way. Thus $\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}}$ is also FCRS and thus FDT.

Proposition 3.7. $\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}}$ *is non-*AUTO.

Proof. Suppose, with the aim of obtaining a contradiction, that $\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}}$ is AUTO. Let (A, L) be an automatic structure for $\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}}$. Then $L_a^{\$}$ is regular for all $a \in A \cup \{\varepsilon\}$. Since $\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}}$ is homogeneous, if $(u, v) \in L_a$, then $\big||u| - |v|\big| \leq 1$ and so ${}^{\$}L_a$ is regular by [56, Corollary 4.2]. Notice that $({}^{\$}L_a)^{\mathrm{rev}} = \big({}_a(L^{\mathrm{rev}})\big)^{\$}$. Hence L^{rev} is a regular language mapping onto $\big(\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}}\big)^{\mathrm{rev}}$ such that $\big({}_a(L^{\mathrm{rev}})\big)^{\$}$ is regular. Since $\big(\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}}\big)^{\mathrm{rev}} \simeq \mathsf{M}^{\mathrm{FCRS}}_{\mathrm{AUTO}}$, this contradicts Lemma 3.4 and so $\mathsf{M}^{\mathrm{FCRS}}_{\mathrm{NONAUTO}}$ is indeed non-AUTO. ☐

3.3. A non-FCRS, FDT, BIAUTO, AUTO homogeneous monoid.

The following homogeneous monoid was introduced by Katsura and Kobayashi [57, Example 3], who showed that it is non-FCRS, but is FDT. We shall prove that it is BIAUTO and thus AUTO.

Example 3.8. Let $A = \{a, b_i, c_i, d_i : i = 1, 2, 3\}$ and let $\mathcal{R}$ consist of the rewriting rules

$$b_i a \to a b_i \qquad i \in \{1, 2, 3\}, \tag{3.1}$$

$$c_j b_j \to c_1 b_1 \qquad j \in \{2, 3\}, \tag{3.2}$$

$$b_j d_j \to b_1 d_1 \qquad j \in \{2, 3\}. \tag{3.3}$$

Let $\mathsf{M}^{\text{FDT}}_{\text{BIAUTO}} = \langle A \mid \mathcal{R} \rangle$. Then $\mathsf{M}^{\text{FDT}}_{\text{BIAUTO}}$ is FDT [57, § 4] but is non-FCRS [57, Proposition 3].

Proposition 3.9. *The monoid* $\mathsf{M}^{\text{FDT}}_{\text{BIAUTO}}$ *of Example 3.8 is* BIAUTO *and thus* AUTO.

Proof. Let $\mathcal{S}$ consist of the following rewriting rules:

$$b_i a \rightarrow a b_i \qquad i \in \{1, 2, 3\}, \tag{3.4}$$

$$c_j a^k b_j \rightarrow c_1 a^k b_1 \qquad j \in \{2, 3\}, k \in \mathbb{N} \cup \{0\}, \tag{3.5}$$

$$c_1 a^k b_1 d_j \rightarrow c_j a^k b_1 d_1 \qquad j \in \{2, 3\}, k \in \mathbb{N} \cup \{0\}, \tag{3.6}$$

$$b_j d_j \rightarrow b_1 d_1 \qquad j \in \{2, 3\}. \tag{3.7}$$

Notice that every rule in $\mathcal{S}$ is a consequence of those in $\mathcal{R}$. Indeed, using rules in $\mathcal{R}$, we have

$$c_j a^k b_j \leftrightarrow c_j a^{k-1} b_j a \leftrightarrow \ldots \leftrightarrow c_j b_j a^k \leftrightarrow c_1 b_1 a^k \leftrightarrow c_1 a b_1 a^{k-1} \leftrightarrow \ldots \leftrightarrow c_1 a^k b_1;$$

and

$$c_1 a^k b_1 d_j \leftrightarrow \ldots \leftrightarrow c_1 b_1 a^k d_j \leftrightarrow c_j b_j a^k d_j \leftrightarrow \ldots \leftrightarrow c_j a^k b_j d_j \leftrightarrow c_j a^k b_1 d_1.$$

Consider any ordering $<$ of A satisfying $b_1 < b_j < a < d_1 < d_j$ (for $j \in \{2, 3\}$). By [48, Lemma 2.4.3] the right-to-left length-plus-lexicographic order induced by $<$ is noetherian. Moreover, any rewriting using a rule in $\mathcal{S}$ decreases a word with respect to this ordering. Thus the rewriting system $(A, \mathcal{S})$ is noetherian.

To see that $(A, \mathcal{S})$ is confluent, notice that there are two possible overlaps of left-hand sides of rewriting rules: an overlap of (3.4) and (3.5), and an overlap of (3.5) and (3.7). However, critical pairs resolve, since

$$c_j a^k b_j a \rightarrow \begin{cases} c_j a^{k+1} b_j \rightarrow c_1 a^{k+1} b_1 \\ c_1 a^k b_1 a \rightarrow c_1 a^{k+1} b_1 \end{cases}$$

and

$$c_j a^k b_j d_j \rightarrow \begin{cases} c_j a^k b_1 d_1 \\ c_1 a^k b_1 d_j \rightarrow c_j a^k b_1 d_1 \end{cases} .$$

Let L be the language of $\mathcal{S}$-irreducible words of A^*. That is,

$$L = A^* - A^* \Big(\{b_1 a, b_2 a, b_3 a\} \cup \{c_2, c_3\} a^* \{b_2, b_3\} \cup c_1 a^* b_1 \{d_2, d_3\} \cup \{b_2 d_2, b_3 d_3\} \Big) A^*;$$

thus L is regular. To prove that (A, L) is an automatic structure for $\mathsf{M}^{\text{FDT}}_{\text{BIAUTO}}$, we first show that L_x and ${}_xL$ are rational relations for all $x \in A \cup \{\varepsilon\}$. Since L is a cross-section for $\mathsf{M}^{\text{FDT}}_{\text{BIAUTO}}$, the relations L_ε and ${}_\varepsilon L$ are both equal to the equality relation and hence are trivially rational.

So let $x \in A$ and $w \in L$. Suppose first that $x \in \{b_2, b_3\}$, then there may be a left-hand side of a rewriting rule of type (3.5) at the rightmost end of the word wx. In this case, w must be of the form $w' c_j a^k$ for some $w' \in L$ (since a prefix of a irreducible word is irreducible), $j \in \{2, 3\}$ and $k \in \mathbb{N} \cup \{0\}$; applying the rewriting rule yields $w' c_1 a^k b_1$, which is in normal form since no left-hand side of a rewrite rule contains c_1 except for (3.6), which clearly cannot be applied. So when $x \in \{b_2, b_3\}$, at most one application of a rewrite rule at the rightmost end turns wx into a normal form word. Hence

$$\begin{aligned} L_{b_j} &= \{\, (w, wx) : w \in L \setminus Lc_j a^* \,\} \\ &\qquad \cup \{\, (w'c_j a^k, w'c_1 a^k b_1) : w' \in L, k \in \mathbb{N} \cup \{0\} \,\} \\ &= \{\, (w, w) : w \in L \setminus Lc_j a^* \,\}(\varepsilon, x) \\ &\qquad \cup \{\, (w', w') : w' \in L \,\}(c_j, c_1)(a, a)^*(\varepsilon, b_1). \end{aligned}$$

Hence each L_{b_j} is a rational relation by Proposition 2.2.

Suppose now that $x \in \{d_2, d_3\}$. Reasoning similar to the previous paragraph shows that wx is in normal form, or one application of a rewrite rule of type (3.6) or (3.7) turns it into normal form. (Note that an application of a rule of type (3.7) might be followed by one of type (3.5), but these can be replaced by one of type (3.6).) Hence

$$\begin{aligned} L_{d_j} &= \{\, (w, wx) : w \in L \setminus (Lc_1 a^* b_1 \cup Ld_2 \cup Ld_3) \,\} \\ &\qquad \cup \{\, (w'c_1 a^k b_1, w'c_j a^k b_1 d_1) : w' \in L, k \in \mathbb{N} \cup \{0\} \,\} \\ &\qquad \cup \{\, (w'b_j, w'b_1 d_1) : w' \in L \setminus Lc_1 a^* \,\} \\ &= \{\, (w, w) : w \in L \setminus (Lc_1 a^* b_1 \cup Ld_2 \cup Ld_3) \,\}(\varepsilon, x) \\ &\qquad \cup \{\, (w', w') : w' \in L \,\}(c_1, c_j)(a, a)^*(b_1, b_1)(\varepsilon, d_1) \\ &\qquad \cup \{\, (w', w') : w' \in L \setminus Lc_1 a^* \,\}(b_j, b_1)(\varepsilon, d_1). \end{aligned}$$

Hence each L_{d_j} is a rational relation by Proposition 2.2.

Suppose that $x = a$. Then the only rewriting rules that can apply to wx are a sequence of rule of type (3.4), rewriting $w'yb_{i_1} \cdots b_{i_k} a$ (where $y \notin \{b_1, b_2, b_3\}$) to $w'yab_{i_1} \cdots b_{i_k}$. This resulting word is in normal form, since the only way a rewriting rule could apply was if $y = c_{i_1}$, but this means the word w would contain $c_{i_1} b_{i_1}$, which contradicts $w \in L$. Hence

$$\begin{aligned} L_a &= \{\, (w'yb_{j_1} \cdots b_{j_k}, w'yab_{j_1} \cdots b_{j_k}) : w'y \in L, b_{j_1}, \ldots, b_{j_k} \in \{1, 2, 3\}, k \in \mathbb{N} \cup \{0\} \,\} \\ &= \{\, (w'y, w'y) : w'y \in L \,\}(\varepsilon, a)\{\, (b_j, b_j) : j \in \{1, 2, 3\} \,\}^*. \end{aligned}$$

Finally, if $x \in \{b_1, c_1, c_2, c_3, d_1\}$, then wx is already in normal form: hence, in this case,

$$L_x = \{\, (w, w)(\varepsilon, x) : w \in L \,\}.$$

In each case, L_x is a rational relation. Since $\mathsf{M}^{\mathrm{FDT}}_{\mathrm{BIAUTO}}$ is homogeneous, if $(u, v) \in L_x$ for $x \in A$, then $|v| = |ua| = |u| + 1$. Furthermore, if $(u, v) \in L_\varepsilon$, then $|u| = |v|$. Hence $L_x^{\$}$ and ${}^{\$}L_x$ are regular for all $x \in A \cup \{\varepsilon\}$ by Proposition 2.5.

Similar reasoning shows that ${}_xL$ is a rational relation: if $x \in \{b_1, b_2, b_3\}$ and $w \in L$, then rewriting xw to normal form can consist of a sequence of applications of rules of type (3.4) followed possibly by one of type (3.3); for all other x, at most one rewriting rule is required. Proposition 2.5 then applies to show that ${}_xL^{\$}$ and ${}^{\$}_{x}L$ are regular.

Hence (A, L) is a biautomatic structure for $\mathsf{M}^{\mathrm{FDT}}_{\mathrm{BIAUTO}}$. ☐

3.4. **A non-FCRS, non-FDT, BIAUTO, AUTO homogeneous monoid.** In this section we give an example of a homogeneous monoid that is non-FDT and thus non-FCRS, but which is BIAUTO and thus AUTO.

Example 3.10. Let $A = \{a, b\}$ and let $\mathcal{R}$ be the rewriting system on $A \cup \{c\}$ consisting of the three rules:

$$\begin{array}{lcl} K_a & : & ac \rightarrow ca \\ K_b & : & bc \rightarrow cb \\ C & : & cab \rightarrow cbb. \end{array}$$

Let $\mathcal{P}$ be the presentation $\langle A \cup \{c\} \mid \mathcal{R}\rangle$, and let $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ be the monoid presented by $\mathcal{P}$.

Theorem 3.11. *The monoid* $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$*:*

(1) *has* $A^* \cup c^+b^*a^*$ *as set of unique normal forms (that is, it is a set, over the generating set* A*, in one-to-one correspondence with the elements of* $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$*);*
(2) *is* BIAUTO *and thus* AUTO*;*
(3) *is non-*FDT *and thus non-*FCRS.

Part 1 of Theorem 3.11 will follow from Lemma 3.13 below, and part 2 is proved in Lemma 3.14. Then, the rest of the subsection will be devoted to proving that $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ is non-FDT, thus establishing part 3.

Remark 3.12. The methods we use here to prove that Example 3.10 is not-FDT are similar to those used in the proof of [44, Theorem 1]. In particular we will use the notion of critical peaks, and resolution of critical peaks, in our proof. We refer the reader to [44, Section 2] for the definitions of these concepts, and their connection with complete rewriting systems and FDT.

Let us begin by fixing some of the notation. We start by adding to $\mathcal{P}$ infinitely many rules of the form

$$\overline{C}_u \ : \ cuab \rightarrow cubb \ (u \in A^*)$$

and denote by $\mathcal{R}'$ the set of all these rules. Notice first that $\overline{C}_\epsilon$ is precisely the rule C defined above and that, for any word $u \in A^*$ the words $cuab$ and $cubb$ represent the same element of the monoid $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$, since in the word $cuab$ we can use relations of the form K_x to pass the letter c through the word u from left to right, then replace cab by cbb using the relation C, and finally move the c back through u again from right to left using the relations K_x. It follows that the presentations $\mathcal{P} = \langle A \cup \{c\} \mid \mathcal{R}\rangle$ and $\overline{\mathcal{P}} = \langle A \cup \{c\} \mid \mathcal{R} \cup \mathcal{R}'\rangle$ are equivalent presentations, in the sense that two words $u, v \in (A \cup \{c\})^*$ are equivalent modulo the relations $\mathcal{R}$ if and only if they are equivalent modulo the relations $\mathcal{R} \cup \mathcal{R}'$. In particular, the monoid $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ is also defined by the infinite presentation

$$\overline{\mathcal{P}} = \left\langle A \cup \{c\} \mid \mathcal{R} \cup \mathcal{R}'\right\rangle.$$

Lemma 3.13. *The infinite presentation* $\overline{\mathcal{P}}$ *is a complete presentation of* $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$*. The set of irreducible words with respect to this complete rewriting system is*

$$A^* \cup c^+b^*a^* = A^* \cup \{\, c^ib^jc^l : i \in \mathbb{N}, j, k \in \mathbb{N} \cup \{0\}\,\}.$$

Proof. The fact that $\overline{\mathcal{P}}$ is a presentation for $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ follows from the comments made before the statement of the lemma. By considering the (left-to-right) length-plus-lexicographic ordering on $\{a, b, c\}^*$ induced by $a > b > c$ one sees that the rewriting system $\overline{\mathcal{P}}$ is noetherian.

The set of irreducible words under this rewriting system is the set $A^* \cup c^+b^*a^*$. Indeed, if a irreducible word contains a symbol c, it cannot be to the left of a symbol from A, otherwise we could apply a relation K_a or K_b. Moreover, if the word also contains a symbol b, then all symbols a must be to the right of the rightmost symbol b, since otherwise we could use a relation of the form $\overline{C}_u$.

Finally, to prove that $\overline{\mathcal{P}}$ is confluent it suffices to consider all possible overlaps between left-hand sides of the rewriting rules K_x ($x \in A$) and $\overline{C}_u$ ($u \in A^*$), showing that all critical peaks arising from these overlaps resolve (see [44, Section 2]). There are three different ways in which these rewrite rules can overlap, giving rise to three types of critical peaks, all of which can be resolved; see Figure 3. This proves that $\overline{\mathcal{P}}$ is confluent and thus completes the proof of the lemma. □

Lemma 3.14. *The monoid* $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ *is* BIAUTO*.*

Proof. Let $L = A^* \cup c^+b^*a^*$. We will prove that $(A \cup \{c\}, L)$ is a biautomatic structure for $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$. By the previous lemma, L is a regular language such that every element of $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ has a unique representative in L. Hence

$$L_\varepsilon = \{\, (w, w) : w \in L \,\};$$

thus L_ε is a rational relation.

Now let $u \in L$. Regardless of whether $u \in A^*$ or $u \in c^+b^*a^*$, the word ua also lies in L. Hence

$$L_a = \{\, (u, ua) : u \in L \,\} = \{\, (u, u) : u \in L \,\}(\varepsilon, a)$$

is a rational relation by Proposition 2.1.

If $u \in A^*$, then ub also lies in L. On the other hand, if $u = c^ib^ja^k$, then $ub = c^ib^ja^kb \to^* c^ib^{j+k+1}$ via a sequence of applications of rules $\overline{C}_u$. Hence

$$\begin{aligned} L_b &= \{\, (u, ub) : u \in A^* \,\} \cup \{\, (c^ib^ja^k, c^ib^{j+k+1}) : i \geq 1, j, k \geq 0 \,\} \\ &= \{\, (u, u) : u \in A^* \,\}(\varepsilon, b) \cup (c, c)^+(b, b)^*(a, b)^*(\varepsilon, a) \end{aligned}$$

is a rational relation by Proposition 2.1.

If $u = a^k$, then $uc = a^kc \to^* ca^k$ using a sequence of applications of rules K_a. If $u \in A^*$ and u contains at least one symbol b, then $u = u'ba^k$ for some $u' \in A^*$ and $k \in \mathbb{N} \cup \{0\}$ and so $uc = u'ba^kc \to^* cu'ba^k \to^* cb^{|u'|+1}a^k$ by a sequences of applications of rules of K_a and K_b and a sequence of applications of rules $\overline{C}_u$. Hence

$$\begin{aligned} L_c = &\{\, (a^k, ca^k) : k \geq 0 \,\} \\ &\cup \{\, (uba^k, cb^{|u|+1}a^k) : u \in A^*, k \geq 0 \,\} \\ &\cup \{\, (c^ib^ja^k, c^{i+1}b^ja^k) : i \geq 1, j, k \geq 0 \,\} \\ = &\,(\varepsilon, c)(a, a)^* \\ &\cup (\varepsilon, c)\{(a, b), (b, b)\}^*(b, b)(a, a)^* \\ &\cup (\varepsilon, c)(c, c)^+(b, b)^*(a, a)^* \end{aligned}$$

is a rational relation.

Similar reasoning shows that

$$\begin{aligned}
{}_aL &= \{\, (u, au) : u \in A^* \,\} \\
&\quad \cup \{\, (c^i a^k, c^i a^{k+1}) : i \geq 1, k \geq 0 \,\} \\
&\quad \cup \{\, (c^i b^j a^k, c^i b^{j+1} a^k) : i, j \geq 1, k \geq 0 \,\} \\
&= (\varepsilon, a)\{(a, a), (b, b)\}^* \\
&\quad \cup (c, c)^+ (a, a)^* (\varepsilon, a) \\
&\quad \cup (c, c)^+ (b, b)^+ (\varepsilon, b)(a, a)^*; \\
{}_bL &= \{\, (u, bu) : u \in A^* \,\} \\
&\quad \cup \{\, (c^i b^j a^k, c^i b^{j+1} a^k) : i \geq 1, j, k \geq 0 \,\} \\
&= (\varepsilon, b)\{(a, a), (b, b)\}^* \\
&\quad \cup (c, c)^+ (\varepsilon, b)(b, b)^+ (aa, a)^* \\
{}_cL &= \{\, (a^k, ca^k) : k \geq 0 \,\}; \\
&\quad \cup \{\, (uba^k, cb^{|u|+1} a^k) : u \in A^*, k \geq 0 \,\} \\
&\quad \cup \{\, (c^i b^j a^k, c^{i+1} b^j a^k) : i \geq 1, j, k \geq 0 \,\} \\
&= (\varepsilon, c)(a, a)^* \\
&\quad \cup (\varepsilon, c)\{(a, b), (b, b)\}^* (b, b)(a, a)^* \\
&\quad \cup (\varepsilon, c)(c, c)^+ (b, b)^* (a, a)^*;
\end{aligned}$$

thus ${}_aL$, ${}_bL$, and ${}_cL$ are all rational.

Since $(u, v) \in {}_xL \cup L_x$ for any $x \in A \cup \varepsilon$ implies $\big||u| - |v| \leq 1\big|$, Proposition 2.5 shows that their images under ${}_\lrcorner{}^{\$}$ and ${}^{\$}{}_\lrcorner$ are regular. Hence (A, L) is a biautomatic structure for $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$. ☐

Let Γ denote the derivation graph of $\mathcal{P}$, and $\overline{\Gamma}$ the derivation graph of $\overline{\mathcal{P}}$. Let Γ_Z denote the connected components of Γ with vertex set the set of all words in $A \cup \{c\}$ with at least two occurrences of the letter c. Likewise let $\overline{\Gamma}_Z$ be the connected component of $\overline{\Gamma}$ with the same vertex set as Γ_Z.

There are three infinite families $(\overline{CT1})$, $(\overline{CT2})$ and $(\overline{CT3})$ of closed paths in $\overline{\Gamma}$, as displayed in Figure 3, that correspond to resolutions of all critical peaks. Each such closed path we obtain we shall call a *critical circuit*. Let us denote by $\overline{\mathcal{C}}$ the critical circuits of the form $(\overline{CT1})$ and $(\overline{CT3})$ and denote by $\overline{\mathcal{Z}}$ the critical circuits of the form $(\overline{CT2})$. Observe that the critical circuits in $\overline{\mathcal{Z}}$ are in $\overline{\Gamma}_Z$ since the words labelling vertices in $(\overline{CT2})$ all contain two occurrences of the letter c. The set of critical circuits $\overline{\mathcal{C}} \cup \overline{\mathcal{Z}}$ forms an infinite homotopy base for $\overline{\Gamma}$ (see [44, Lemma 3]).

We now want to use the infinite homotopy base $\overline{\mathcal{C}} \cup \overline{\mathcal{Z}}$ for $\overline{\Gamma}$ to obtain an infinite homotopy base for Γ. In order to do this we need to take the critical circuits $(\overline{CT1})$–$(\overline{CT3})$ and transform them into circuits in the derivation graph Γ by replacing each occurrence of an edge $\overline{C}_u$ by a corresponding path in Γ.

As mentioned above when proving that $\mathcal{P}$ and $\overline{\mathcal{P}}$ are equivalent presentations, the edges $\overline{C}_u$ can be realized in Γ by paths C_u which are defined

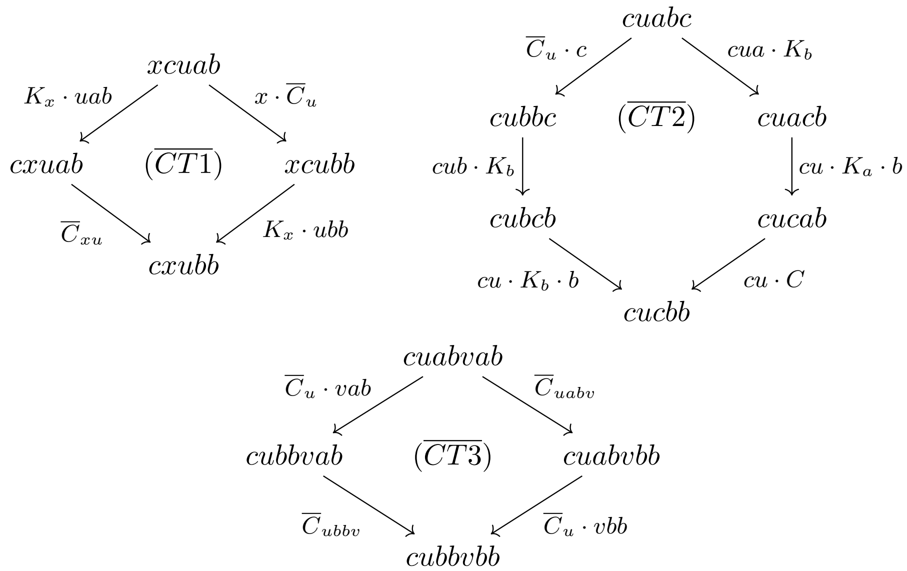

FIGURE 3. Resolutions of the critical peaks in the derivation graph $\overline{\Gamma}$ of the presentation $\overline{\mathcal{P}}$. Here $x, y \in A$ and $u, v \in A^*$.

inductively as follows: we first set C_ϵ to be the rule C, and then for $u = xu'$, with $x \in A$ and $u' \in A^*$ we set C_u to be the path

$$(3.8) \qquad C_u: \quad cxu'ab \xrightarrow{K_x^{-1} \cdot u'ab} xcu'ab \xrightarrow{x \cdot C_{u'}} xcu'bb \xrightarrow{K_x \cdot u'bb} cxu'bb.$$

So, C_u is the path in Γ from $cuab$ to $cubb$ given by commuting c through u using the relations K_x, applying the relation C to transform $ucab$ into $ucbb$, and then commuting c back through u again using the relations K_x, ending at the vertex $cubb$.

Now let us define a mapping φ from the set of paths $P(\overline{\Gamma})$ in $\overline{\Gamma}$ to the set of paths $P(\Gamma)$ in Γ. Let $\varphi : P(\overline{\Gamma}) \to P(\Gamma)$ be the map given by $(\alpha \cdot \overline{C}_u \cdot \beta)\varphi = \alpha \cdot C_u \cdot \beta$, for all $\alpha, \beta \in (A \cup \{c\})^*$ and $u \in A^*$, and defined to be the identity on every other edge of $\overline{\Gamma}$. Let $\mathcal{C} = (\overline{\mathcal{C}})\varphi$ and $\mathcal{Z} = (\overline{\mathcal{Z}})\varphi$. Since $\overline{\mathcal{C}} \cup \overline{\mathcal{Z}}$ forms a homotopy base for $\overline{\Gamma}$ it follows that $\mathcal{C} \cup \mathcal{Z}$ is an infinite homotopy base for Γ (see [35, Corollary 3.7]).

Observe that the infinite homotopy base $\mathcal{C} \cup \mathcal{Z}$ for Γ is nothing more than the set of circuits in Γ obtained by taking the set of circuits $(\overline{CT1})$–$(\overline{CT3})$ and replacing each occurrence of the edge $\overline{C}_u$ by the path C_u defined in (3.8). Let us denote this corresponding set of circuits $\mathcal{C} \cup \mathcal{Z}$ in Γ by $(CT1)$–$(CT3)$.

The monoid $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ is presented by the finite presentation $\mathcal{P}$ and Γ is the derivation graph of $\mathcal{P}$ which has an infinite homotopy base $\mathcal{C} \cup \mathcal{Z}$.

Lemma 3.15 ([35, Discussion before Definition 3.3])**.** *If* $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ *were* FDT *then there would be a finite subset* $\mathcal{C}_0 \cup \mathcal{Z}_0$ *of* $\mathcal{C} \cup \mathcal{Z}$ *which would be a finite homotopy base for* Γ*.*

Sketch. Since $\mathcal{D}$ is a finite homotopy base, each path in $\mathcal{D}$ is homotopic to an empty path using finitely many paths from $\mathcal{C} \cup \mathcal{Z}$, and thus the finite subset of $\mathcal{C} \cup \mathcal{Z}$ consisting of all paths arising this way is a homotopy base. □

Our aim now is to show that this leads to a contradiction, and thus conclude that $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ is not FDT. In order to do this we shall now define a mapping from the set of paths $P(\Gamma)$ in Γ into the integral monoid ring $\mathbb{Z}\mathsf{M}$. Define $\Phi : P(\Gamma) \to \mathbb{Z}M$ to be the unique map which extends the mapping:

- $(\alpha \cdot K_a \cdot \beta)\Phi = \overline{\alpha}$, $\quad (\alpha \cdot K_b \cdot \beta)\Phi = \overline{\alpha}$, $\quad$ and
- $(\alpha \cdot C \cdot \beta)\Phi = \overline{\alpha}$,

where $\alpha, \beta \in (A \cup \{c\})^*$ and $\overline{\alpha} \in \mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ denotes the element of $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ represented by the word α, to paths in such a way that

$$(\mathbb{P} \circ \mathbb{Q})\Phi = (\mathbb{P})\Phi + (\mathbb{Q})\Phi \;\text{ and }\; (\mathbb{P}^{-1})\Phi = -(\mathbb{P})\Phi.$$

The following basic properties of Φ are then easily verified for all paths $\mathbb{P}, \mathbb{Q} \in P(\Gamma)$ and words $\alpha, \beta \in (A \cup \{c\})^*$:

(1) $(\alpha \cdot \mathbb{P} \cdot \beta)\Phi = \overline{\alpha} \cdot (\mathbb{P})\Phi$
(2) $(\mathbb{P} \circ \mathbb{P}^{-1})\Phi = 0$
(3) $([\mathbb{P}, \mathbb{Q}])\Phi = 0$ where
$$[\mathbb{P}, \mathbb{Q}] = (\mathbb{P} \cdot \iota\mathbb{Q}) \circ (\tau\mathbb{P} \cdot \mathbb{Q}) \circ (\mathbb{P}^{-1} \cdot \tau\mathbb{Q}) \circ (\iota\mathbb{P} \cdot \mathbb{Q}^{-1}).$$
(4) If $\mathbb{P} \sim_0 \mathbb{Q}$ then $(\mathbb{P})\Phi = (\mathbb{Q})\Phi$.

Here, property 4 follows from properties 1, 2 and 3. Note that property 4 implies that Φ induces a well-defined map on the homotopy classes of paths of Γ.

In what follows we shall often omit bars from the top of words in the images under Φ and simply write words from $(A \cup \{c\})^*$ with the obvious intended meaning. Recall that $\overline{\mathcal{C}}$ is the set of critical circuits of the form $(\overline{CT1})$ and $(\overline{CT3})$, and that $\mathcal{C} = (\overline{\mathcal{C}})\varphi$ is the corresponding set of circuits in Γ. Let $F = A^*$ denote the free monoid on the alphabet A.

Lemma 3.16. *If* $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ *is* FDT *then the submodule* $\langle(\mathcal{C})\Phi\rangle_{\mathbb{Z}F}$*, of the left* $\mathbb{Z}F$*-module* $\mathbb{Z}F$*, generated by* $(\mathcal{C})\Phi$ *is a finitely generated left* $\mathbb{Z}F$*-module.*

Proof. Assume that $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ is FDT and therefore $\mathcal{P}$ is FDT. Since $\mathcal{C} \cup \mathcal{Z}$ is a homotopy base for its derivation graph Γ, by Lemma 3.15 there are finite subsets $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{Z}_0 \subseteq \mathcal{Z}$ such that $\mathcal{C}_0 \cup \mathcal{Z}_0$ is a finite homotopy base for Γ. Let $\mathbb{P} \in \mathcal{C}$ be arbitrary. We claim that $(\mathbb{P})\Phi \in \langle(\mathcal{C}_0)\Phi\rangle_{\mathbb{Z}F}$. Once established, this will prove the lemma, since $(\mathcal{C}_0)\Phi$ is a finite subset of $\langle(\mathcal{C})\Phi\rangle_{\mathbb{Z}F}$.

By [44, Lemma 2], since $\mathbb{P}$ is a closed path in Γ and $\mathcal{C}_0 \cup \mathcal{Z}_0$ is a homotopy base for Γ, we can write

$$\mathbb{P} \sim_0 \mathbb{P}_1^{-1} \circ (\alpha_1 \cdot \mathbb{Q}_1 \cdot \beta_1) \circ \mathbb{P}_1 \circ \cdots \circ \mathbb{P}_n^{-1} \circ (\alpha_n \cdot \mathbb{Q}_n \cdot \beta_n) \circ \mathbb{P}_n, \tag{3.9}$$

where each $\mathbb{P}_i \in P(\Gamma)$, $\alpha_i, \beta_i \in (A \cup \{c\})^*$ and $\mathbb{Q}_i \in (\mathcal{C}_0 \cup \mathcal{Z}_0)^{\pm 1}$.

Since the vertices of $\mathbb{P}$ have exactly one c, and all the relations in the presentation $\mathcal{P}$ involve the letter c, it follows that $\alpha_i, \beta_i \in A^*$ and $\mathbb{Q}_i \in \mathcal{C}_0$, for all $i \in \{1, \ldots, n\}$. Applying Φ gives

$$(\mathbb{P})\Phi = \alpha_1(\mathbb{Q}_1)\Phi + \cdots + \alpha_n(\mathbb{Q}_n)\Phi \in \langle(\mathcal{C}_0)\Phi\rangle_{\mathbb{Z}F}$$

as claimed. $\square$

To complete our proof, it remains to compute the subset $(\mathcal{C})\Phi$ of $\mathbb{Z}F$ and then prove that the submodule $\langle(\mathcal{C})\Phi\rangle_{\mathbb{Z}F}$ of $\mathbb{Z}F$ is not finitely generated as a left $\mathbb{Z}F$-module. Recall that $\overline{\mathcal{C}}$ is the set of critical circuits of the form

$(\overline{CT1})$ and $(\overline{CT3})$, and that $\mathcal{C} = (\overline{\mathcal{C}})\varphi$. So $\mathcal{C}$ is the set of closed paths $(CT1)$ and $(CT3)$ obtained by applying the mapping φ to the closed paths $(\overline{CT1})$ and $(\overline{CT3})$, that is, obtained by taking each occurrence of $\overline{C}_u$ and replacing it by the path C_u.

From equation (3.8) we have, for any word $u \in A^*$, if $u = xu'$, with $x \in A$, the equality $(C_u)\Phi = -1_{\mathsf{M}} + \overline{x} \cdot (C_{u'})\Phi + 1_{\mathsf{M}} = \overline{x} \cdot (C_{u'})\Phi$, and thus we deduce that $(C_u)\Phi = u$. Using this fact, the result of computing a critical circuit in $\mathcal{C}$ under the map Φ is given by:

- $(\mathbb{P})\Phi = 0_{\mathbb{Z}\mathsf{M}}$, for a critical circuit $\mathbb{P}$ of the family $(CT1)$;
- $(\mathbb{P})\Phi = u(ab - bb)v$, for a critical circuit $\mathbb{P}$ of the family $(CT3)$.

The next lemma completes the proof of Theorem 3.11.

Lemma 3.17. *The submodule $\langle(\mathcal{C})\Phi\rangle_{\mathbb{Z}F}$ of $\mathbb{Z}F$, where*

$$(\mathcal{C})\Phi = \{u(ab - bb)v \mid u, v \in A^*\} \cup \{0_{\mathbb{Z}\mathsf{M}}\},$$

is not finitely generated as a left $\mathbb{Z}F$-module, and therefore $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ is not FDT.

Proof. Suppose, with the aim of obtaining a contradiction, that $\langle(\mathcal{C})\Phi\rangle_{\mathbb{Z}F}$ is finitely generated as a left $\mathbb{Z}F$-module. Then there exists a finite subset X of $(\mathcal{C})\Phi$ such that $\langle X\rangle_{\mathbb{Z}F} \supseteq (\mathcal{C})\Phi$. Let $n \in \mathbb{N}$ be the maximal length of a word v where $u(ab-bb)v \in X$. We shall show that $(ab-bb)a^{n+1}$ belongs to $(\mathcal{C})\Phi$ but not to $\langle X\rangle_{\mathbb{Z}F}$. Suppose that in $\mathbb{Z}F$ we have

$$(ab - bb)a^{n+1} = \sum_{i=1}^{k} \alpha_i(ab - bb)v_i,$$

where $\alpha_i, v_i \in A^*$ and $|v_i| \leq n$, for $i = 1, \ldots, k$. Then for some $j \in \{1, \ldots, k\}$ we have either $bba^{n+1} = \alpha_j abv_j$ or $bba^{n+1} = \alpha_j bbv_j$ in the free monoid $F = \{a, b\}^*$, which clearly contradicts the fact that $|v_i| \leq n$. We conclude that $(\mathcal{C})\Phi$ is not finitely generated as a left $\mathbb{Z}F$-module, and it then follows from Lemma 3.16 that $\mathsf{M}^{\text{NONFDT}}_{\text{BIAUTO}}$ is not FDT. ☐

4. Free products of homogeneous monoids

In Section 3, we gave four examples of homogeneous monoids that possess certain combinations of the properties FCRS, FDT, BIAUTO, and AUTO. In this section, we use free products to construct examples with the remaining consistent combinations of these properties. Note that if monoids M_1 and M_2 have homogeneous presentations $\langle A_1 \mid \mathcal{R}_1\rangle$ and $\langle A_2 \mid \mathcal{R}_2\rangle$, then their free product $M_1 * M_2$ is defined by the presentation $\langle A_1 \cup A_2 \mid \mathcal{R}_1 \cup \mathcal{R}_2\rangle$ and is thus also homogeneous.

First we consider the interaction of the free product with BIAUTO and AUTO. It is known that the free product of two monoids is AUTO if and only if each of the monoids is AUTO ([25, Theorem 6.2], [55, Theorem 1.2]). While it would be possible to extend this result to biautomaticity for general monoids, this generalization does not appear in the literature. This paper required the biautomaticity result only for homogeneous monoids, and the proofs are simpler in this case:

Proposition 4.1. *Let M_1 and M_2 be homogeneous monoids. Then $M_1 * M_2$ is* AUTO *(respectively,* BIAUTO*) if and only if M_1 and M_2 are* AUTO *(respectively,* BIAUTO*).*

Proof. Let $\langle A_1 \mid \mathcal{R}_1 \rangle$ and $\langle A_2 \mid \mathcal{R}_2 \rangle$ be homogeneous presentations for M_1 and M_2, respectively.

Suppose $M_1 * M_2$ is AUTO; the proof for BIAUTO is similar. By Proposition 2.6, $M_1 * M_2$ admits an automatic structure $(A_1 \cup A_2, L)$. Since M_1 is homogeneous, there is no non-empty word over A_1 representing the identity of M_1; hence every word in $(A_1 \cup A_2)^*$ representing an element of M_2 must lie in A_2^*. Thus $(A_2, L \cap A_2^*)$ is an automatic structure for M_2; similarly, $(A_1, L \cap A_1^*)$ is an automatic structure for M_1.

On the other hand, suppose M_1 and M_2 are AUTO; again, the proof for BIAUTO is similar. Then there are automatic structures (A_1, L_1) and (A_2, L_2) for M_1 and M_2 respectively. By [25, Corollary 5.5], assume without loss of generality that every element of M_1 and M_2 has a unique representative in L_1 and L_2, respectively. Let

$$L = \{\varepsilon\} \tag{4.1}$$

$$\cup \Big(\big(L_1 \cap (A_1^+ \times A_1^+)\big)\big(L_2 \cap (A_2^+ \times A_2^+)\big)\Big)^+ \tag{4.2}$$

$$\cup \Big(\big(L_1 \cap (A_1^+ \times A_1^+)\big)\big(L_2 \cap (A_2^+ \times A_2^+)\big)\Big)^* \big(L_1 \cap (A_1^+ \times A_1^+)\big) \tag{4.3}$$

$$\cup \Big(\big(L_2 \cap (A_2^+ \times A_2^+)\big)\big(L_1 \cap (A_1^+ \times A_1^+)\big)\Big)^+ \tag{4.4}$$

$$\cup \Big(\big(L_2 \cap (A_2^+ \times A_2^+)\big)\big(L_1 \cap (A_1^+ \times A_1^+)\big)\Big)^* \big(L_2 \cap (A_2^+ \times A_2^+)\big); \tag{4.5}$$

note that L is a disjoint union of the languages (4.1)–(4.5). (Note that the language L is not equal to $(L_1 L_2)^*$: every word in L is a product of words strictly alternating between non-empty words from L_1 and words from L_2.)

Since every element of $M_1 * M_2$ is represented by a unique element of L,

$$L_\varepsilon = \{\, (u, u) : u \in L \,\}$$

is a rational relation.

Now let $a \in A_1$, and let v_a be the unique word in L_1 that is equal to a (note that v_a will either be a itself or another generator that is equal to a). Suppose that $u \in L$ lies in (4.2) or (4.5). Then uv_a is the unique word in L that is equal to ua. On the other hand, if $u \in L$ lies in (4.3) or (4.4), then it is of the form $u's$, where $s \in L_2$ and s is the maximal suffix of u lying in A_1^*. In this case, the unique word in L equal to ua is $u't$, where t is the unique word in L_1 equal to sa: that is, where $(s, t) \in (L_1)_a$. Hence

$$\begin{aligned}
L_a = {} & \{(\varepsilon, v_a)\} \\
& \cup \big((L_1)_\varepsilon \cap (A_1^+ \times A_1^+)\big)\big((L_2)_\varepsilon \cap (A_2^+ \times A_2^+)\big)(\varepsilon, v_a) \\
& \cup \Big(\big((L_1)_\varepsilon \cap (A_1^+ \times A_1^+)\big)\big((L_2)_\varepsilon \cap (A_2^+ \times A_2^+)\big)\Big)^* \big((L_1)_a \cap (A_1^+ \times A_1^+)\big) \\
& \cup \big((L_2)_\varepsilon \cap (A_2^+ \times A_2^+)\big)\Big(\big((L_1)_\varepsilon \cap (A_1^+ \times A_1^+)\big)\big((L_2)_\varepsilon \cap (A_2^+ \times A_2^+)\big)\Big)^* \big((L_1)_a \cap (A_1^+ \times A_1^+)\big) \\
& \cup \Big(\big((L_2)_\varepsilon \cap (A_2^+ \times A_2^+)\big)\big((L_1)_\varepsilon \cap (A_1^+ \times A_1^+)\big)\Big)^* \big((L_2)_\varepsilon \cap (A_2^+ \times A_2^+)\big)\big((L_1)_a \cap (A_1^+ \times A_1^+)\big).
\end{aligned}$$

Thus L_a is a rational relation.

Since $M_1 * M_2$ is homogeneous, if $(u, v) \in L_x$ for $x \in A$, then $|v| = |ua| = |u| + 1$. Furthermore, if $(u, v) \in L_\varepsilon$, then $|u| = |v|$. Hence $L_x^{\$}$ and ${}^{\$}L_x$ are regular for all $x \in A \cup \{\varepsilon\}$ by Proposition 2.5. Thus $M_1 * M_2$ is automatic.

(This reasoning is essentially [25, Proof of Theorem 6.2], but simplified because we consider only homogeneous semigroups.) ☐

Now we consider the interaction of free product with FCRS within the class of homogeneous monoids.

Theorem 4.2 ([58, Theorem D])**.** *Let M_1 and M_2 be monoids. Suppose that M_2 has no non-trivial left- or right-invertible elements. If $M_1 * M_2$ is* FCRS*, then M_1 and M_2 are* FCRS*.*

Proposition 4.3. *Let M_1 and M_2 be homogeneous monoids. Then $M_1 * M_2$ is* FCRS *if and only if M_1 and M_2 are* FCRS*.*

Proof. Suppose $M_1 * M_2$ is FCRS. Since M_1 and M_2 are homogeneous, neither contains any non-trivial left- or right-invertible elements. So M_1 and M_2 are both FCRS by Theorem 4.2.

The converse part follows by [42, Proposition 4.3]. ☐

Finally, we recall the following result on the interaction of the free product and FDT:

Theorem 4.4 ([41, 42])**.** *Let M_1 and M_2 be monoids. Then $M_1 * M_2$ is* FDT *if and only if M_1 and M_2 are* FDT*.*

Now, the examples in Section 3 suffice to construct the remaining examples in Figure 1 by taking their free products. Regarding Figure 1 as a semilattice, examples in Section 3 correspond to the elements of this semilattice that have no non-trivial decomposition. Proposition 4.3, Theorem 4.4, and Proposition 4.1 together show that by taking a free product, one obtains a new monoid whose properties are given by taking the logical conjunction (that is, the 'and' operation) of the corresponding properties. That is, given two monoids $\mathsf{M}_{\text{B}}^{\text{A}}$ and $\mathsf{M}_{\text{D}}^{\text{C}}$, their free product $\mathsf{M}_{\text{B}}^{\text{A}} * \mathsf{M}_{\text{D}}^{\text{C}}$ will have the weaker of the two properties A and B (which lie in $\{$FCRS, FDT, NONFDT$\}$) and the weaker of the two properties C and D (which lie in $\{$BIAUTO, AUTO, NONAUTO$\}$). Thus, to summarize:

- $\mathsf{M}_{\text{AUTO}}^{\text{FCRS}} * \mathsf{M}_{\text{BIAUTO}}^{\text{FDT}}$ is non-FCRS, FDT, non-BIAUTO, AUTO;
- $\mathsf{M}_{\text{BIAUTO}}^{\text{FDT}} * \mathsf{M}_{\text{NONAUTO}}^{\text{FCRS}}$ is non-FCRS, FDT, non-BIAUTO, non-AUTO;
- $\mathsf{M}_{\text{AUTO}}^{\text{FCRS}} * \mathsf{M}_{\text{BIAUTO}}^{\text{NONFDT}}$ is non-FCRS, non-FDT, non-BIAUTO, AUTO;
- $\mathsf{M}_{\text{NONAUTO}}^{\text{FCRS}} * \mathsf{M}_{\text{BIAUTO}}^{\text{NONFDT}}$ is non-FCRS, non-FDT, non-BIAUTO, non-AUTO.

Theorem 4.5. *For each consistent combination of the properties* FCRS*,* FDT*,* BIAUTO*,* AUTO*, and their negations, there exists a homogeneous monoid with exactly that combination of properties.*

5. From homogeneous to n-ary multihomogeneous monoids

Thus far we have proved that for every consistent combination of the properties FCRS, FDT, BIAUTO, AUTO, and their negations, there exists a homogeneous monoid with exactly those properties. In the remainder of the paper, we show that such monoids exist in the more restricted class of n-ary homogenous monoids, and show that monoids with some consistent combinations of these properties exist in the even more restricted class of n-ary multihomogeneous monoids. The current section describes the overall strategy and results; the following two sections then develop the necessary concepts and techniques. First, we introduce and investigate the theory of abstractly Rees-commensuable semigroups in Section 6, ultimately proving Corollary 6.5, which implies that from each homogeneous monoid listed in Table 1, we can obtain an n-ary homogeneous monoid with the same combination of properties. Thus we have the following analogy of Theorem 4.5 for n-ary homogeneous monoids:

Theorem 5.1. *For each consistent combination of the properties* FCRS*,* FDT*,* BIAUTO*,* AUTO*, and their negations, there exists an n-ary homogeneous monoid with exactly that combination of properties.*

Since we have examples of n-ary homogeneous monoids with all consistent combinations of the properties FCRS, FDT, BIAUTO, AUTO, and their negations, the next step is to extend these examples to n-ary multihomogeneous monoids. With that aim, in Section 7 we define and investigate an embedding of a (n-ary) homogeneous monoid into a 2-generated (n-ary) multihomogeneous monoid. As stated below in Corollary 7.13, passing to and from the (n-ary) multihomogeneous monoid preserves FDT, AUTO, and BIAUTO. Furthermore, passing *to* the (n-ary) multihomogeneous monoid preserves FCRS. It is unknown whether FCRS is preserved passing back to the original (n-ary) homogeneous monoid, or, equivalently, whether non-FCRS is preserved on passing to the (n-ary) multihomogeneous monoid. Applying this embedding technique to the list of n-ary homogeneous monoids we discussed above, we get the following results:

Theorem 5.2. *For each consistent combination of the properties* FCRS*,* BIAUTO*,* AUTO*, and their negations, there exists an n-ary multihomogeneous monoid with exactly that combination of properties.*

Theorem 5.3. *For each consistent combination of the properties* FDT*,* BIAUTO*,* AUTO*, and their negations, there exists an n-ary multihomogeneous monoid with exactly that combination of properties.*

To obtain an analogue of Theorem 5.1 for n-ary multihomogeneous monoids it would be sufficient to find a multihomogeneous monoid that is non-FCRS, FDT, and BIAUTO. Indeed, in that case, combining Theorems 5.2 and 5.3 with the results of Section 4 and noting that a free product of multihomogeneous monoids is again multihomogeneous, we would get, for any consistent combination of BIAUTO, AUTO, and their negations, an example of an FDT, non-FCRS multihomogeneous monoid with exactly that combination of properties. Therefore, from Corollary 6.5 we would get examples of n-ary multihomogeneous monoids with exactly the same discussed properties. Joining these examples with the examples from Theorems 5.2 and 5.3 we would get the intended result. Thus we have the following question:

Question 5.4. Does there exist a multihomogeneous monoid that is FDT, but non-FCRS?

6. Abstractly Rees-commensurable semigroups

We introduce a new definition which is inspired by the notions of *abstractly commensurable groups* [36, §§ iv.27ff.] and *Rees index* for semigroups [59]. A subsemigroup T of a given semigroup S has finite Rees index if $S \setminus T$ is finite. In that case the semigroup S is said to be a *small extension* of T, and T a *large subsemigroup* of S. The main interest is that large subsemigroups and small extensions of a given semigroup share many important properties of that semigroup (see [60] for a survey).

Definition 6.1. Two semigroups S_1 and S_2 are said to be *abstractly Rees-commensurable* if there are finite Rees index subsemigroups $T_i \subseteq S_i$ (for $i = 1, 2$) with $T_1 \cong T_2$.

Is is easy to verify that abstract Rees-commensurability is an equivalence relation on semigroups.

This notion can be naturally extended to ideals.

Definition 6.2. Two semigroups S_1 and S_2 are said to be *abstract Rees-ideal-commensurable* if there are finite Rees index ideals $U_i \subseteq S_i$ (for $i = 1, 2$) with $U_1 \cong U_2$.

The idea behind these notions is that abstract Rees-ideal-commensurable semigroups share many important properties, such as FCRS, FDT, BIAUTO, and AUTO.

Proposition 6.3. FCRS*,* FDT*,* BIAUTO*, and* AUTO *are preserved under abstract Rees-ideal-commensurability:*

Proof.

- It is known that FCRS is inherited by small extensions [43, Theorem 1] and by large subsemigroups [61, Theorem 1.1]. Although the result for small extensions was stated in the context of monoids, it can be naturally extended to semigroups. These two results imply that FCRS is preserved under abstract Rees(-ideal)-commensurability.
- It is known that FDT is inherited by small extensions [43, Theorem 2] of monoids, and by large semigroup *ideals* [45, Theorem 1]. We recall that the notion of finite derivation type was first introduced for monoids, but it was naturally extended to the semigroup case [62, Section 2].

  The result on small extensions can be easily adapted for the semigroup case. Indeed, let T be a semigroup and consider the monoid T^1 obtained from T by adding an identity. If T is FDT the derivation graph of T^1 can be obtained from the derivation graph of T by adding an extra connected component with a single vertex corresponding to the empty word. Thus T^1 is FDT. Now, if S is a small extension of the semigroup T, it turns out that the monoid S^1 is a small extension of the monoid T^1. Therefore, by [43, Theorem 2] the monoid S^1 is FDT. But S is a large ideal of S^1, and by [45, Theorem 1] we conclude that S is FDT.

  The two results on small extensions and large ideals of semigroups show that FDT is preserved under abstract Rees-ideal-commensurability.

- By [63, Theorem 1.1] and its natural analogue for BIAUTO, both AUTO and BIAUTO are inherited by small extensions and by large subsemigroups, and therefore AUTO and BIAUTO are preserved under abstract Rees(-ideal)-commensurability.

□

The preceding result is important because, in the case of (multi)homogeneous and n-ary (multi)homogeneous monoids, the following result holds:

Proposition 6.4. *Every finitely presented (multi)homogeneous monoid is Rees-ideal-commensurable to an n-ary (multi)homogeneous monoid, where n can be chosen arbitrarily as long as it is greater than or equal to the length of the longest relation in $\mathcal{R}$.*

Proof. Let $\langle A \mid \mathcal{R}\rangle$ be a finite homogeneous presentation of a monoid M. Let n be chosen arbitrarily, provided it is greater than or equal to the maximum length of a relation in $\mathcal{R}$. We are going to construct an n-ary (multi)homogeneous monoid M' and show it is abstract Rees-ideal-commensurable to M by finding isomorphic ideals I and I' of M and M' respectively.

Let $I = \{[u]_M \in M : |u| \geq n\}$. Note that I is an ideal of finite Rees index in M, since its complement in M is the finite set $\bigcup_{0\leq i<n}\{[a_1]_M \cdots [a_i]_M : a_1, \ldots, a_i \in A\}$.

Let $\mathcal{R}' = \{(u\ell v, urv) : (\ell, r) \in \mathcal{R}, u, v \in A^*, |u\ell v| = n\}$. Consider the n-ary (multi)homogeneous presentation $\langle A \mid \mathcal{R}'\rangle$, and let M' be the monoid defined by it.

The set $I' = \{[u]_{M'} \in M' : |u| \geq n\}$ is an ideal of M'. Moreover, I' has finite Rees index in M'.

Since $\mathcal{R}'$ is contained in the Thue congruence generated by $\mathcal{R}$, we can define a map $\varphi : I' \to I$, with $([u]_{M'})\,\varphi = [u]_M$. Given $[u]_M$ in I, thus with $u \in A^*$ and $|u| \geq n$, we have $[u]_{M'} \in I'$ by definition of I'. Therefore φ is surjective. It is also injective since, for any $[u]_{M'}$, $[v]_{M'} \in I'$ such that $([u]_{M'})\,\varphi = ([v]_{M'})\,\varphi$, we get $[u]_M = [v]_M$, with $|u| = |v| \geq n$, and therefore $[u]_{M'} = [v]_{M'}$. It is routine to check that φ is a homomorphism. Thus I and I' are isomorphic finite Rees index ideals of M and M'. So M and M' are abstract Rees-ideal-commensurable. □

Corollary 6.5. *Let P be a set of properties preserved under abstract Rees-ideal-commensurability. Then there exists a (multi)homogeneous monoid satisfying every property in P if and only if there exists an n-ary (multi)homogeneous monoid satisfying every property in P.*

Note that the set of properties P can contain 'negative' properties like 'not finitely generated'. As an immediate consequence, we obtain Theorem 5.1

7. From n-ary homogeneous to n-ary multihomogeneous monoids

We now develop the embedding technique that allows us to construct multihomogeneous examples. The technique embeds a homogeneous monoid into a multihomogenous monoid that shares several of the properties under study. If the original monoid is n-ary homogeneous, then the monoid it is embedded into is n-ary multihomogenous, so we use the formulations '(n-ary) homogeneous' and '(n-ary) multihomogeneous' to indicate that statements apply to both the general and n-ary cases.

Let us start by fixing some notation that will be maintained throughout this section. Let M be a finitely generated (n-ary) homogeneous monoid, and let $\langle A \mid \mathcal{R} \rangle$ be a (n-ary) homogeneous presentation defining M. Recall that $A = \{a_1, \ldots, a_m\}$ is minimal, so that any other generating set of M contains A (see discussion at the end of Subsection 2.1).

We define a homomorphism

$$\phi : A^* \to \{x, y\}^*, \qquad a_i \mapsto x^2 y^i x y^{m+1-i}.$$

Denote by N the finitely generated monoid presented by $\langle x, y \mid \mathcal{R}\phi \rangle$, where $\mathcal{R}\phi$ denotes the set $\{(\ell\phi, r\phi) : (\ell, r) \in \mathcal{R}\}$.

Proposition 7.1. *The monoid N, defined by the presentation $\langle x, y \mid \mathcal{R}\phi \rangle$, is ($n(m+4)$-ary) multihomogeneous.*

Proof. The presentation $\langle A \mid \mathcal{R} \rangle$ is homogeneous hence, for any $(u, v) \in \mathcal{R}$, we have $|u| = |v|$. Since $a_i\phi$ contains 3 symbols x and $m + 1$ symbols y for all $i \in \{1, \ldots, m\}$, it follows that $|u\phi|_x = 3|u| = 3|v| = |v\phi|_x$ and $|u\phi|_y = (m + 1)|u| = (m + 1)|v| = |v\phi|_y$. Hence $\langle x, y \mid \mathcal{R}\phi \rangle$ is multihomogeneous.

If the presentation $\langle A \mid \mathcal{R} \rangle$ is n-ary homogeneous, then also for any relation $(u, v) \in \mathcal{R}$, we have $|u| = |v| = n$. But in that case we get $|u\phi| = |u\phi|_x + |u\phi|_y = 3|u| + (m+1)|u| = 3n + (m+1)n = n(m+4)$, and similarly for $|v\phi|$. ☐

A subset C of $\{x, y\}^*$ is a *code* if it is a set of free generators for the submonoid of $\{x, y\}^*$ generated by C [64, Subsection 7.2]. That is the case for the set $A\phi = \{x^2 y^i x y^{m+1-i} : i = 1, \ldots, m\}$. Furthermore, $A\phi$ is what is called a *prefix code*, since no word of $A\phi$ is a proper prefix of another word in $A\phi$ [7, Chapter 6].

By [7, Proposition 6.1.3], since ϕ induces a bijection from A to the code $A\phi$, we conclude that ϕ is injective, and for that reason ϕ is called a *coding morphism for $A\phi$*.

Proposition 7.2. *The monoid M presented by $\langle A \mid \mathcal{R} \rangle$ embeds into the monoid N presented by $\langle x, y \mid \mathcal{R}\phi \rangle$ via the map $[u]_M \mapsto [u\phi]_N$, and the words over $\{x, y\}$ representing elements of (the image of) M are precisely the words in $(A^*)\phi$.*

Proof. Consider the natural projection ρ_N of $\{x, y\}^*$ onto N, whose kernel is generated by $\mathcal{R}\phi$. Since ϕ is injective, the kernel of the composition $\phi \circ \rho_N : A^* \to N$ is the congruence generated by $\mathcal{R}$.

The natural projection of ρ_M of A^* onto M has the same kernel has the map $\phi \circ \rho_N$. By [64, Theorem 1.5.2] there is a monomorphism $\overline{\phi} : M \to N$ such that $(M)\overline{\phi} = (A^*)\phi \circ \rho_N$ and $([u]_M)\overline{\phi} = (u)\phi \circ \rho_N = [u\phi]_N$. ☐

We shall now turn our attention to investigating the relationship between FDT holding in M and FDT holding in N. For this we shall prove some results which relate the Squier graphs of M and N.

Let Δ denote the subgraph of $\Gamma(\langle x, y \mid R\phi\rangle)$ induced on the set of vertices $(A\phi)^*$.

Let us extend the mapping $\phi : A^* \to \{x, y\}^*$ to a mapping from the derivation graph $\Gamma(\langle A \mid \mathcal{R}\rangle)$ to the derivation graph $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$ mapping an edge $\mathbb{E} = (w_1, (\ell, r), \epsilon, w_2)$ to an edge $\mathbb{E}\phi = (w_1\phi, (\ell\phi, r\phi), \epsilon, w_2\phi)$. The mapping ϕ extends to a mapping between paths by putting

$$(\mathbb{E}_1\mathbb{E}_2 \ldots \mathbb{E}_k)\phi = (\mathbb{E}_1\phi)(\mathbb{E}_2\phi) \ldots (\mathbb{E}_k\phi),$$

for any edges $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_k$ of $\Gamma(\langle A \mid \mathcal{R}\rangle)$.

Lemma 7.3. *The mapping $\phi : \Gamma(\langle A \mid \mathcal{R}\rangle) \to \Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$ has the following properties.*

(1) *ϕ maps A^* bijectively to $(A\phi)^*$;*
(2) *For every edge $\mathbb{E}$ in $\Gamma(\langle A \mid \mathcal{R}\rangle)$ we have*

$$\iota(\mathbb{E}\phi) = (\iota\mathbb{E})\phi, \quad \textit{and} \quad \tau(\mathbb{E}\phi) = (\tau\mathbb{E})\phi;$$

(3) *For every edge $\mathbb{E}$ in $\Gamma(\langle A \mid \mathcal{R}\rangle)$ and every pair of words $u, v \in A^*$ we have*

$$(u \cdot \mathbb{E} \cdot v)\phi = u\phi \cdot \mathbb{E}\phi \cdot v\phi;$$

(4) *ϕ maps the edge-set of $\Gamma(\langle A \mid \mathcal{R}\rangle)$ bijectively to the edge-set of Δ.*

Proof. (i) It follows from the fact that ϕ is an injective homomorphism from A^* into $\{x, y\}^*$, and thus ϕ maps bijectively A^* to its image $(A^*)\phi = (A\phi)^*$.
(ii) & (iii) The identities follow from the definition of the extended map ϕ on edges and from the fact that ϕ is a homomorphism.
(iv) The injectivity of the edge-set of $\Gamma(\langle A \mid \mathcal{R}\rangle)$ to the edge-set of Δ, follows from the injectivity of ϕ. Now, consider an edge $\mathbb{F} = (z_1, (\ell\phi, r\phi), \epsilon, z_2)$ of Δ, and suppose without lost of generality, that $\epsilon = +1$.

The word $\iota\mathbb{F} = z_1(\ell\phi)z_2$ decomposes as a concatenation of words in $A\phi$; that is, words of the form $x^2y^jxy^{m+1-j}$ for various $j \in \{1, \ldots, m\}$. Since subwords x^2 only occur at the start of such words in $A\phi$, the x^2 at the start of $\ell\phi$ lies at the start of a word from $A\phi$ in the decomposition of $\iota\mathbb{F}$. Since all words in $A\phi$ have the same length, $\ell\phi$ must also finish at the end of some word from $A\phi$ in the decomposition of $\iota\mathbb{F}$. Hence z_1 and z_2 are (possibly empty) concatenations of words in $A\phi$, and so $z_1 = w_1\phi$ and $z_2 = w_2\phi$ for some $w_1, w_2 \in A^*$. Thus $\mathbb{F} = \mathbb{E}\phi$, for the edge $\mathbb{E} = (w_1, (\ell, r), +1, w_2)$ of $\Gamma(\langle A \mid \mathcal{R}\rangle)$. $\square$

Let $\psi : (A\phi)^* \to A^*$ be the right inverse of the injective mapping ϕ. Note that $(w_1w_2)\psi = (w_1\psi)(w_2\psi)$, for any elements $w_1, w_2 \in (A\phi)^*$. By part (iv) of the above lemma we can extend ψ to a mapping of edges by simply setting $(\mathbb{E}\phi)\psi = \mathbb{E}$ and using the fact that every edge of Δ has the form $\mathbb{E}\phi$ for a unique edge $\mathbb{E}$ from $\Gamma(\langle A \mid R\rangle)$.

Lemma 7.4. *The mapping $\psi : \Delta \to \Gamma(\langle A \mid R\rangle)$ has the following properties.*

(1) *ψ maps $(A\phi)^*$ bijectively to A^*;*
(2) *For every edge $\mathbb{E}$ in Δ we have*

$$\iota(\mathbb{E}\psi) = (\iota\mathbb{E})\psi, \quad \textit{and} \quad \tau(\mathbb{E}\psi) = (\tau\mathbb{E})\psi;$$

(3) *For every edge* $\mathbb{E}$ *in* Δ *and every pair of words* $u, v \in \{x, y\}^*$ *we have that* $u \cdot \mathbb{E} \cdot v$ *is an edge in* Δ *if and only if* $u, v \in (A\phi)^*$ *in which case*

$$(u \cdot \mathbb{E} \cdot v)\psi = u\psi \cdot \mathbb{E}\psi \cdot v\psi.$$

(4) ψ *maps the edge-set of* Δ *bijectively to the edge-set of* $\Gamma(\langle A \mid R\rangle)$.

Proof. (i) Follows immediatly from the definition of ψ.
(ii) From Lemma 7.3 each edge of Δ has the form $\mathbb{E}\phi = (w_1\phi, (\ell\phi, r\phi), \epsilon, w_2\phi)$, with $w_1, w_2 \in A^*$, $(\ell, r) \in \mathcal{R}$ and $\epsilon = \pm 1$. The image of the edge under ψ is $\mathbb{E}$. Since ϕ is an homomorphism and from the definition of ψ we get $((w_1\phi)(\ell\phi)(w_2\phi))\psi = w_1\ell w_2$ and $((w_1\phi)(r\phi)(w_2\phi))\psi = w_1 r w_2$ as required.
(iii) Suppose that $\mathbb{E}$ is an edge in Δ and that for some $u, v \in \{x, y\}^*$ we have $u \cdot \mathbb{E} \cdot v$ an edge in Δ. So the vertex $u(\iota\mathbb{E})v$ of $\mathbb{E}$ is a word in $(A\phi)^*$. Arguing as in Lemma 7.3 (iv) we can conclude that $u, v \in (A\phi)^*$.

The converse part of the equivalence follows trivially.

If $\mathbb{E}$ in Δ and $u, v \in (A\phi)^*$, there exists $\mathbb{E}'$ in $\Gamma(\langle A \mid R\rangle)$ and $u', v' \in A^*$, such that $(u \cdot \mathbb{E} \cdot v) = u'\phi \cdot \mathbb{E}\phi \cdot v'\phi$. From the definition of ψ we get $(u \cdot \mathbb{E} \cdot v)\psi = u\psi \cdot \mathbb{E}\psi \cdot v\psi$.
(iv) It follows from the definition of ψ and Lemma 7.3. □

From the above lemmas, we can regard, up to the given encoding, the graph $\Gamma(\langle A \mid \mathcal{R}\rangle)$ as a subgraph of $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$, identifying $\Gamma(\langle A \mid \mathcal{R}\rangle)$ with Δ.

Lemma 7.5. *Let* $\mathcal{C}$ *be a set of closed paths in* $\Gamma(\langle A \mid R\rangle)$. *Then* $\mathcal{C}\phi$ *is a set of closed paths in* Δ. *Moreover, for every closed path* $\mathbb{P}$ *in* $\Gamma(\langle A \mid R\rangle)$, *if* $\mathbb{P} \sim_{\mathcal{C}} 1_{\iota\mathbb{P}}$ *in* $\Gamma(\langle A \mid R\rangle)$ *then* $\mathbb{P}\phi \sim_{\mathcal{C}\phi} 1_{(\iota\mathbb{P})\phi}$ *in* $\Gamma(\langle x, y \mid R\phi\rangle)$.

Proof. The result follows by [35, Theorem 3.6]. □

At this point, our strategy starts to emerge. The identification between Δ with the derivation graph $\Gamma(\langle A \mid \mathcal{R}\rangle)$, will allow us to get a finite homotopy base of $\Gamma(\langle A \mid \mathcal{R}\rangle)$, from a finite homotopy bases of $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$.

Lemma 7.6. *Let* $\mathcal{D}$ *be a set of closed paths in* Δ. *Then* $\mathcal{D}\psi$ *is a set of closed paths in* $\Gamma(\langle A \mid R\rangle)$. *Moreover, for every closed path* $\mathbb{P}$ *in* Δ, *if* $\mathbb{P} \sim_{\mathcal{D}} 1_{\iota\mathbb{P}}$ *in* $\Gamma(\langle x, y \mid R\phi\rangle)$ *then* $\mathbb{P}\psi \sim_{\mathcal{D}\psi} 1_{(\iota\mathbb{P})\psi}$ *in* $\Gamma(\langle A \mid R\rangle)$.

Proof. The result follows from [39, Lemma 1], where the author uses strict monoidal categories to prove an analogue of [35, Theorem 3.6].

Indeed, in that context, both $\Gamma(\langle A \mid R\rangle)$ and Δ are strict monoidal categories (notice that Δ is not a derivation graph, and so we can not refer to [35, Theorem 3.6] to prove the lemma). The mapping ψ is a functor that preserves the multiplicative structure, by Lemma 7.4. The homotopy relation $\sim_{\mathcal{D}\psi}$ is what is called a 2-congruence on Δ, satisfying $\mathbb{P}\psi \sim_{\mathcal{D}\psi} 1_{(\iota\mathbb{P})\psi}$, for any closed path $\mathbb{P}$ in $\mathcal{D}$. From [39, Lemma 1], it follows that $\mathbb{P}\psi \sim_{\mathcal{D}\psi} 1_{(\iota\mathbb{P})\psi}$, for any closed path $\mathbb{P}$ of Δ such that $\mathbb{P} \sim_{\mathcal{D}} 1_{\iota\mathbb{P}}$. □

Next, we describe the remaining connected componnets of $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$.

Lemma 7.7. *The derivation graph* $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$ *has the following properties:*

(1) *If a vertex z has maximal factors (with respect to length) u and v in $(A\phi)^*$, then either $z = z_0uz_2 = z_0vz_2$, or $z = z_0uz_1vz_2$, or $z = z_0vz_1uz_2$, with $z_0, z_1, z_2 \notin (A\phi)^+$ and z_1 non-empty;*
(2) *Any vertex has a unique decomposition of the form*

$$w_0u_1w_1 \cdots w_{k-1}u_kw_k,$$

where each u_i is a maximal factor in $(A\phi)^*$, and each w_i has no factor in $(A\phi)^*$;
(3) *Any edge has the form $z_1 \cdot (\mathbb{E}\phi) \cdot z_2$, for some edge $\mathbb{E}$ in $\Gamma(\langle A \mid \mathcal{R}\rangle)$, where $\iota\mathbb{E}\phi$ and $\tau\mathbb{E}\phi$ are maximal factors in $(A\phi)^*$ of the corresponding initial and terminal vertices of $\mathbb{E}\phi$, and z_1, z_2 are not in $(A\phi)^+$.*

Proof. (i) Let u and v be maximal factors in $(A\phi)^*$ of z. Suppose u and v overlap in the word z. Without lost of generality, suppose that $z = u_0uu_1 = v_0vv_1$ with $|u_0| \leq |v_0| < |u_0u|$. Since $v \in (A\phi)^*$ is a sequence of words of the form $x^2y^jxy^{m+1-j}$ for various $j \in \{1, \ldots, m\}$, and subwords x^2 only occur at the start of such words in $A\phi$, the x^2 at the start of v lies at the start of a word from $A\phi$ in the decomposition of u. By the maximality of v and since $z = u_0uu_1 = v_0vv_1$ we get $u_0 = v_0$. Since all words in $A\phi$ have the same length, u and v must also finish at the end of some word from $A\phi$. By the maximality of u and v we get $u = v$.
(ii) The result follows from (i).
(iii) From the definition of $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$, an edge has for initial and terminal vertices words t and s in $\{x, y\}^*$ where one is obtained from the other by a single application of a defining relation in $\mathcal{R}\phi$, that is, $s = w_0(\ell\phi)w_1$ and $t = w_0(r\phi)w_1$, with $(\ell, r) \in \mathcal{R}$. Notice that $\ell\phi, r\phi \in (A\phi)^*$. So, we can consider maximal factors p and q in $(A\phi)^*$ having as factors $\ell\phi$ and $r\phi$, respectively, in such a way that

$$s = z_1(p\phi)z_2,$$
$$t = z_1(q\phi)z_2.$$

The words p and q are the initial and terminal vertices of an edge $\mathbb{E}$ in $\Gamma(\langle A \mid \mathcal{R}\rangle)$. The result follows as in the statement. □

Lemma 7.8. *A length-two path in $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$ has one of the forms:*
(1) *$(z_0 \cdot \mathbb{E}_1\phi \cdot z_2) \circ (z_0 \cdot \mathbb{E}_2\phi \cdot z_2)$, and $\mathbb{E}_1 \circ \mathbb{E}_2$ is a path in $\Gamma(\langle A \mid \mathcal{R}\rangle)$; or*
(2) *$(z_0 \cdot \mathbb{E}_1\phi \cdot z_1(\iota\mathbb{E}_2\phi)z_2) \circ (z_0(\tau\mathbb{E}_1\phi)z_1 \cdot \mathbb{E}_2\phi \cdot z_2)$; or*
(3) *$(z_0(\iota\mathbb{E}_2\phi)z_1 \cdot \mathbb{E}_1\phi \cdot z_2) \circ (z_0 \cdot \mathbb{E}_2\phi \cdot z_1(\tau\mathbb{E}_1\phi)z_2)$,*

for $z_0, z_1, z_2 \notin (A\phi)^+$ and z_1 non-empty, and $\mathbb{E}_1, \mathbb{E}_2$ edges in $\Gamma(\langle A \mid \mathcal{R}\rangle)$.

Proof. By Lemma 7.7 (iii), the path of consecutive edges has the form $(w_1 \cdot (\mathbb{E}_1\phi) \cdot w_2) \circ (z_1 \cdot (\mathbb{E}_2\phi) \cdot z_2)$. In the conditions of that lemma the words $\tau\mathbb{E}_1\phi$ and $\iota\mathbb{E}_2\phi$ are maximal factors in $(A\phi)^*$ of the same word $w_1(\tau\mathbb{E}_1\phi)w_2 = z_1(\iota\mathbb{E}_2\phi)z_2$. From Lemma 7.7 (i) we have three possible cases. Each of the cases corresponds to one the possible cases (i), (ii) and (iii) of the statement. □

Lemma 7.9. *Let $\mathbb{P}$ be a non-empty (closed) path in $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$. Then*

$$\mathbb{P} \sim_0 \mathbb{P}_1 \circ \mathbb{P}_2 \circ \cdots \circ \mathbb{P}_k, \tag{7.1}$$

for some $k \in \mathbb{N}$*, and where each* $\mathbb{P}_i$ *has the form*

$$w_0(\tau\mathbb{Q}_1\phi)w_1 \cdots (\tau\mathbb{Q}_{i-1}\phi)w_{i-1} \cdot (\mathbb{Q}_i\phi) \cdot w_i(\iota\mathbb{Q}_{i+1}\phi) \cdots w_{k-1}(\iota\mathbb{Q}_k\phi)w_k,$$

for some (closed) path $\mathbb{Q}_i$ *in* $\Gamma(\langle A \mid \mathcal{R}\rangle)$*, with* $w_0, \ldots, w_k \notin (A\phi)^*$ *and* $w_1, \ldots, w_{k-1}$ *non-empty.*

Proof. The key observation to prove the lemma is to note that in Lemma 7.8 the paths on items (ii) and (iii) are $\sim_0$-homotopic. Indeed, this is a consequence of disjoint derivations.

By Lemma 7.7 (ii) the initial vertex of $\mathbb{P}$ can be factorized in the form

$$w_0u_1w_1 \cdots w_{k-1}u_kw_k,$$

where each u_i is a maximal factor in $(A\phi)^*$ and each w_i has no factor in $(A\phi)^*$. Notice, also by the same lemma, that any edge in $\mathbb{P}$ has a factorization of the form $z_1 \cdot (\mathbb{E}\phi) \cdot z_2$, for some edge $\mathbb{E}$ in $\Gamma(\langle A \mid \mathcal{R}\rangle)$, where $\iota\mathbb{E}\phi$ and $\tau\mathbb{E}\phi$ are maximal factors in $(A\phi)^*$ of the corresponding initial and terminal vertices of $\mathbb{E}\phi$.

Since w_i's have no factor in $(A\phi)^*$, no relation from $\mathcal{R}\phi$ is going to be applied to them, and hence they are fixed in this path. Also, for each $i \in \{1, \ldots, k\}$ we can identify B_i as the set of edges in $\mathbb{P}$ where the relation is applied to a word between w_{i-1} and w_i.

From Lemma 7.8 any two consecutive edges $\mathbb{E}_i \circ \mathbb{E}_j$, with $\mathbb{E}_i \in B_i$ and $\mathbb{E}_j \in B_j$, and $i > j$, we get $\mathbb{E}_i \circ \mathbb{E}_j \sim_0 \mathbb{E}_j \circ \mathbb{E}_i$ by disjoint derivations.

Consequently, taking $i = 1$ we can group together all edges in B_1 by finding a $\sim_0$-homotopic path in which all those edges are at the beginning of the path. Proceeding in this way with the remaining edges we get the intended result. ☐

The following result can be viewed as a generalization of Lemma 2.3 in [65].

Proposition 7.10. *The monoid* M *is* FDT *if and only if the monoid* N *is* FDT*.*

Proof. Suppose that M is FDT and let $\mathcal{C}$ be a finite homotopy base (of closed paths) for $\Gamma(\langle A \mid \mathcal{R}\rangle)$. We shall see that the finite set $\mathcal{C}\phi = \{\mathbb{P}\phi : \mathbb{P} \in \mathcal{C}\}$ of closed paths is a homotopy base for $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$.

Let $\mathbb{P}$ be a non-empty closed path in $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$. By Lemma 7.9 there are closed paths $\mathbb{Q}_i$, for $i = 1, \ldots, k$, in $\Gamma(\langle A \mid \mathcal{R}\rangle)$ such that

$$\mathbb{P} \sim_0 \mathbb{P}_1 \circ \mathbb{P}_2 \circ \cdots \circ \mathbb{P}_k,$$

and each $\mathbb{P}_i = w_i \cdot \mathbb{Q}_i\phi \cdot w'_i$, for some $w_i, w'_i \in \{x, y\}^*$.

Since $\mathcal{C}$ is a homotopy base, for each $i = 1, \ldots, k$, we have $\mathbb{Q}_i \sim_{\mathcal{C}} 1_{\iota(\mathbb{Q}_i)}$. Thus $\mathbb{P}_i \sim_{\mathcal{C}\phi} 1_{\iota(\mathbb{P}_i\phi)}$ by Lemma 7.5, for all $i = 1, \ldots, k$, which in turn implies that $\mathbb{P} \sim_{\mathcal{C}\phi} 1_{\iota(\mathbb{P}\phi)}$. Consequently, $\mathcal{C}\phi$ is a finite homotopy base for $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$, and so N is FDT.

Conversely, suppose that N is FDT, and let $\mathcal{E}$ be a finite homotopy base (of closed paths) for $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$. Again by Lemma 7.9, for each $\mathbb{P}$ in $\mathcal{E}$, there are closed paths $\mathbb{Q}_1, \ldots, \mathbb{Q}_k$ in $\Gamma(\langle A \mid \mathcal{R}\rangle)$, such that

$$\mathbb{P} \sim_0 \mathbb{P}_1 \circ \mathbb{P}_2 \circ \cdots \circ \mathbb{P}_k,$$

and $\mathbb{P}_i = w_i \cdot \mathbb{Q}_i\phi \cdot w_i'$, for some $w_i, w_i' \in \{x,y\}^*$. Denote by $\mathcal{C}$ the finite set of all $\mathbb{Q}_i$'s, for all $\mathbb{P} \in \mathcal{E}$, and let $\mathcal{C}\phi$ denote the set $\{\mathbb{P}\phi : \mathbb{P} \in \mathcal{C}\}$. From the definition of $\mathcal{C}\phi$, the paths $\mathbb{P}_i$ satisfy $\mathbb{P}_i \sim_{\mathcal{C}\phi} 1_{\iota(\mathbb{P}_i)}$, and so each closed path $\mathbb{P}$ in $\mathcal{E}$ satisfies $\mathbb{P} \sim_{\mathcal{C}\phi} 1_{\iota(\mathbb{P})}$. Thus, the set $\mathcal{C}\phi$ generates the homotopy relation $\sim_{\mathcal{E}}$, and therefore also $\mathcal{C}\phi$ is a finite homotopy base of $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$. Note that $\mathcal{C}\phi$ is a set of closed paths in Δ and that $\mathcal{C}\phi\psi = \mathcal{C}$.

To conclude the proof, let $\mathbb{P}$ be a closed path in $\Gamma(\langle A \mid \mathcal{R}\rangle)$. Since $\mathcal{C}\phi$ is a homotopy base of $\Gamma(\langle x, y \mid \mathcal{R}\phi\rangle)$, we have $\mathbb{P}\phi \sim_{\mathcal{C}\phi} 1_{\iota(\mathbb{P}\phi)}$. By Lemma 7.6, we get $\mathbb{P}\phi\psi \sim_{\mathcal{C}\phi\psi} 1_{\iota(\mathbb{P}\phi\psi)}$. Since $\mathbb{P}\phi\psi = \mathbb{P}$ we get $\mathbb{P} \sim_{\mathcal{C}} 1_{\iota(\mathbb{P})}$ as required. Therefore, $\mathcal{C}$ is a finite homotopy base of $\Gamma(\langle A \mid \mathcal{R}\rangle)$, and so M is FDT. ☐

Proposition 7.11. *The monoid M is* AUTO *(respectively,* BIAUTO*) if and only if the monoid N is* AUTO *(respectively,* BIAUTO*).*

Proof. We will prove the result for BIAUTO; the result for AUTO follows by considering multiplication only on one side.

Suppose that N is BIAUTO. By Proposition 2.6, there is a biautomatic structure $(\{x,y\}, K)$ for N. By Proposition 7.2, words over $\{x,y\}$ representing elements of the image under ϕ of M are precisely those in $(A\phi)^*$. So $K \cap (A\phi)^*$ must map onto the image of M.

The map ϕ is a rational relation, since

$$\phi = \{\, (a_i, x^2y^ixy^{m+1i}) : a_i \in A \,\}^*.$$

Thus its converse ϕ^{-1} is also a rational relation. Let $L = K\phi^{-1}$; note that $L \subseteq A^*$. Since ϕ^{-1} is a rational relation and K is regular, it follows that L is also regular. Since $K \cap (A\phi)^*$ must map onto the image of M, the language L maps onto M. For any $a \in A \cup \{\varepsilon\}$,

$$\begin{aligned}
(u,v) \in L_a &\iff u \in L \wedge v \in L \wedge ua =_M v \\
&\iff (\exists u', v' \in K)\big((u,u') \in \phi \wedge (v,v') \in \phi \wedge u'(a\phi) =_N v'\big) \\
&\iff (\exists u', v' \in K)\big((u,u') \in \phi \wedge (v,v') \in \phi \wedge (u',v') \in K_{a\phi}\big) \\
&\iff (u,v) \in \phi \circ K_{a\phi} \circ \phi^{-1};
\end{aligned}$$

thus $L_a = \phi \circ K_{a\phi} \circ \phi^{-1}$. Since $(\{x,y\}, K)$ is an automatic structure for N, each relation $K_{a\phi}$ is rational. Since ϕ and ϕ^{-1} are rational relations, it follows that L_a is a rational relation. Since M is homogeneous, $(u,v) \in L_a \implies \big||u| - |v|\big| \leq 1$, and so $L_a^{\$}$ and ${}^{\$}L_a$ are regular by Proposition 2.5. Symmetrical reasoning shows that ${}_aL^{\$}$ and ${}^{\$}_aL$ are regular. Hence (A, L) is a biautomatic structure for M.

Now suppose that M is BIAUTO. By Proposition 2.6, there is a biautomatic structure (A, L) for M. The aim is to construct a biautomatic structure for N.

Let $J = \{x,y\}^+ - \{x,y\}^*A^+\phi\{x,y\}^*$; so J consists of all non-empty words over $\{x,y\}^*$ that do not contain subwords equal to elements $M\phi$. Notice in particular that $\varepsilon \notin J$ and that J is closed under taking non-empty subwords. Note further that J is regular. For future use, let $\Delta = \{\, (z,z) : z \in J \,\}$; note that Δ is a rational relation by Proposition 2.1.

Consider a word $w \in \{x, y\}^+$. By Lemma 7.7, w can be uniquely factored into an alternating product of words from J and words from $A^+\phi$. That is,

$$w = z_0 u_1 z_1 \cdots z_{k-1} u_k z_k, w \tag{7.2}$$

where each u_i lies in $A^+\phi$ and each z_i lies in J, except that z_0 and z_k may also be ε. Note that each u_i is equal to some word in $L\phi - \{\varepsilon\}$. The idea is to build an automatic structure where the language of representatives consists of alternating products of words from J and words from $L\phi$. That is, the language will consist of words (7.2) where each u_i lies in $L\phi - \{\varepsilon\}$.

More formally, let

$$K = (J \cup \{\varepsilon\})\big((L\phi \cap \{x, y\}^+)J\big)^*(J \cup \{\varepsilon\});$$

note that K is regular since J is regular and ϕ is a homomorphism, and that every element of N is equal to some word in K by the reasoning in the previous paragraph.

Now let $w \in K$ and factor w as (7.2) where each u_i lies in $L\phi - \{\varepsilon\}$ and each z_i lies in J, except that z_0 and z_k may also be ε.

Consider right-multiplication of w by a generator x. Since $z_k x$ cannot contain a subword from $A^+\phi$ (since z_k contains no such subword, and such subwords do not end with symbols x), the words in L to which wx is equal are precisely words of the form $z_0 u_1' z_1 \cdots z_{k-1} u_k' z_k x$, where each u_i' is any word in $L\phi$ that is equal to u_i. (Recally that defining relations in $\mathcal{R}\phi$ only apply to words in $A^+\phi$; thus the subwords z_i are fixed.) Hence

$$K_x = (\Delta \cup \{(\varepsilon, \varepsilon)\})\big((\phi^{-1} \circ L_\varepsilon \circ \phi)\Delta\big)^*\big(L_\varepsilon\phi(\varepsilon, x) \cup (\varepsilon, x)\big)$$

is a rational relation.

Now consider right-multiplication of w by a generator y. The $z_k y$ may end with a subword $x^2 y^i x y^{m-i+1} \in A\phi$, and thus it is necessary to distinguish three cases:

- $z_k = t x^2 y^i x y^{m-i}$, where $t \neq \varepsilon$. In this case, the factorization of wy into an alternating product of subwords from J and $A^+\phi$ is

  $$w = z_0 u_1 z_1 \cdots z_{k-1} u_k t x^2 y^i x y^{m-i};$$

  note that $t \in J$ since J is closed under taking subwords. The words in L to which wy is equal are precisely words of the form $z_0 u_1' z_1 \cdots z_{k-1} u_k' z_k v$, where each u_i' is any word in $L\phi$ that is equal to u_i, and v is any word in $L\phi$ that is equal to $x^2 y^i x y^{m-i} = a_i\phi$. Let $H_i = \{\, p \in L : p = a_i \,\}$; then H_i is regular by [25, Proposition 3.1]. (In fact, since M is homogeneous, H can contain only generators from A and so must be finite.) Let

  $$K_y^{(1)} = (\Delta \cup m\{(\varepsilon, \varepsilon)\})\big((\phi^{-1} \circ L_\varepsilon \circ \phi)\Delta\big)^* \bigcup_{i=1}^{m}\big(\{x^2 y^i x y^{m-i}\} \times H_i\phi\big).$$

  Since $H_i\phi$ is regular, $\{x^2 y^i x y^{m-i}\} \times H_i\phi$ is a rational relation by Proposition 2.2 and so $K_y^{(1)}$ is a rational relation since it is a concatenation of rational relations.

  Then $K_y^{(1)}$ is a rational relation that describes right-multiplication by a generator y in this case.

- $z_k = x^2y^ixy^{m-i}$. In this case, the factorization of wy into an alternating product of subwords from J and $A^+\phi$ is
$$w = z_0u_1z_1 \cdots z_{k-1}u_kx^2y^ixy^{m-i};$$
note that the last factor is $u_kx^2y^ixy^{m-i} \in A^+\phi$. The words in L to which wy is equal are precisely words of the form $z_0u_1'z_1 \cdots z_{k-1}v$, where each u_i' is any word in $L\phi$ that is equal to u_i, and v is any word in $L\phi$ that is equal to $u_kx^2y^ixy^{m-i}$. That is, v can be any word such that $(u_k, v) \in L_{a_i}\phi$. Let
$$\begin{aligned} K_y^{(2)} = (\Delta \cup \{(\varepsilon,\varepsilon)\})\big((\phi^{-1} \circ L_\varepsilon \circ \phi)\Delta\big)^* \\ \cdot \bigcup_{i=1}^{n} \big((\phi^{-1} \circ L_{a_i} \circ \phi)(x^2y^ixy^{m-i}, \varepsilon)\big). \end{aligned}$$
Then $K_y^{(2)}$ is a rational relation that describes right-multiplication by a generator y in this case.
- z_k does not end with a suffix of the form $x^2y^ixy^{m-i}$. The words in L to which wy is equal are precisely words of the form $z_0u_1'z_1 \cdots z_{k-1}u_k'z_ky$, where each u_i' is any word in $L\phi$ that is equal to u_i. Let
$$\begin{aligned} K_y^{(3)} = (\Delta \cup \{(\varepsilon,\varepsilon)\})\big((\phi^{-1} \circ L_\varepsilon \circ \phi)\Delta\big)^*(\phi^{-1} \circ L_\varepsilon \circ \phi) \\ \cdot \Big\{ (v, v) : v \in J - \{x, y\} \bigcup_{i=1}^{m} x^2y^ixy^{m-i} \Big\}(\varepsilon, y). \end{aligned}$$
Then $K_y^{(2)}$ is a rational relation that describes right-multiplication by a generator y in this case.

Thus $K_y = K_y^{(1)} \cup K_y^{(2)} \cup K_y^{(3)}$ is a rational relation.

Since N is homogeneous, $(u, v) \in K_t \implies \big||u| - |v|\big| \leq 1$, and so $K_t^{\$}$ and ${}^{\$}K_t$ are regular by Proposition 2.5.

Similar reasoning shows that ${}_tK^{\$}$ and ${}^{\$}_tK$. Hence $(\{x, y\}, K)$ is a biautomatic structure for N. ☐

Proposition 7.12. *If $\langle B \mid \mathcal{Q}\rangle$ is a finite presentation for M, and so $A \subseteq B$, then*
$$\mathcal{P} = \langle x, y, B \mid \mathcal{Q},\ (a\phi, a)\ (\forall a \in A)\rangle$$
is a finite presentation defining N. Moreover, the presentation $\langle B \mid \mathcal{Q}\rangle$ is complete if and only if the presentation $\mathcal{P}$ is complete. Thus, if M is FCRS *then N is* FCRS.

Proof. Using Tietze transformations we obtain from the presentation $\langle x, y \mid \mathcal{R}\phi\rangle$ a new presentation for N as follows: for each $a \in A$, insert a generator a and a relation $(a\phi, a)$, thus obtaining a Tietze equivalent presentation
$$\langle x, y, A \mid \mathcal{R}\phi,\ (a\phi, a)\ (\forall a \in A)\rangle\,.$$
Since ϕ is a homomorphism, identifying each symbol a with the word $a\phi$, performing these substitutions on the words from $\mathcal{R}\phi$, we obtain another Tietze equivalent presentation defining the same monoid:
$$\langle x, y, A \mid \mathcal{R},\ (a\phi, a)\ (\forall a \in A)\rangle\,.$$

As it is possible to obtain from the presentation $\langle A \mid \mathcal{R} \rangle$ the presentation $\langle B \mid \mathcal{Q} \rangle$ using finitely many Tietze transformations, we can obtain from the presentation $\langle x, y, A \mid \mathcal{R},\ (a\phi, a)\ (\forall a \in A) \rangle$ the presentation $\mathcal{P}$ by using the same Tietze transformations.

Suppose that $\langle B \mid \mathcal{Q} \rangle$ is also complete. Observe that $\mathcal{Q}$ relates words from the alphabet B, and that a relation from the set $\mathcal{E} = \{(a\phi, a) : a \in A\}$ has left-hand side in $\{x, y\}^*$ and right-hand side in A. Thus, if $w \rightarrow_{\mathcal{Q}} w' \rightarrow_{\mathcal{E}} w''$, we can find $\overline{w}$ such that $w \rightarrow_{\mathcal{E}} \overline{w} \rightarrow_{\mathcal{Q}} w''$. Indeed, since $w' \rightarrow_{\mathcal{E}} w''$ the word w' has a factor from the alphabet $\{x, y\}^*$. Since $w \rightarrow_{\mathcal{Q}} w'$ the word w' is obtained from w by changing some factor in B^*. Thus, the word w also contains the left-hand side of the relation from $\mathcal{E}$ applied to w'. This means that w has both left-hand sides of the relations being applied, and those left-hand sides do not overlap. So, we can alternatively apply first the relation from $\mathcal{E}$, obtaining a word $\overline{w}$, and then apply the relation from $\mathcal{Q}$, obtaining the word w''.

Hence, $\rightarrow_{\mathcal{E}}$ quasi-commutes over $\rightarrow_{\mathcal{Q}}$, that is, $\rightarrow_{\mathcal{Q}} \circ \rightarrow_{\mathcal{E}} \subseteq \rightarrow_{\mathcal{E}} \circ \rightarrow^*_{\mathcal{Q} \cup \mathcal{E}}$. By [66, Theorem 1], the rewriting system $(\{x, y\} \cup A, \mathcal{Q} \cup \mathcal{E})$ is terminating if, and only if, both $\mathcal{Q}$ and $\mathcal{E}$ are terminating. By assumption $\mathcal{Q}$ is terminating and from where $\mathcal{E}$ is also terminating by length-reduction.

As before, since $\mathcal{Q}$ relates words from the alphabet B, and that relations from $\mathcal{E}$ have left-hand side in $\{x, y\}^*$ and right-hand side in A, we can also deduce that whenever $w' \leftarrow^*_{\mathcal{Q}} w \rightarrow^*_{\mathcal{E}} w''$, there exists $\overline{w}$ such that $w' \rightarrow^*_{\mathcal{E}} \overline{w} \leftarrow^*_{\mathcal{Q}} w''$. Therefore, the relations $\rightarrow^*_{\mathcal{E}}$ and $\leftarrow^*_{\mathcal{Q}}$ commute, that is, $\leftarrow^*_{\mathcal{Q}} \circ \rightarrow^*_{\mathcal{E}} \subseteq \rightarrow^*_{\mathcal{E}} \circ \leftarrow^*_{\mathcal{Q}}$. By [48, Lemma 2.7.10], if $\mathcal{E}$ and $\mathcal{Q}$ are confluent and $\rightarrow^*_{\mathcal{E}}$ and $\leftarrow^*_{\mathcal{Q}}$ commute, then $\mathcal{Q} \cup \mathcal{E}$ is also confluent. Since by assumption $\mathcal{Q}$ is confluent it remains to show that $\mathcal{E}$ is confluent.

Observe that two left-hand sides $a_i\phi$ and $a_j\phi$ of rules in $\mathcal{E}$ can overlap if and only if $a_i\phi = a_j\phi$. Since ϕ is injective we get $a_i = a_j$. Therefore, $\mathcal{E}$ has no critical pairs and thus is also confluent.

Conversely, suppose that $\mathcal{P}$ is a complete presentation. Since $\langle B \mid \mathcal{Q} \rangle$ is contained in $\mathcal{P}$, we deduce that $\langle B \mid \mathcal{Q} \rangle$ is terminating. Confluence also holds from the fact that in $\mathcal{P}$ all critical pairs are resolved, in particular, those arising from relations in $\mathcal{Q}$. Hence, each resolution associated to relations from $\mathcal{Q}$, can only involve relations from $\mathcal{Q}$, since left-hand sides of rules in $\mathcal{E}$ belong to $\{x, y\}^*$. Therefore, $\langle B \mid \mathcal{Q} \rangle$ is complete. □

It is unknown if the property FCRS is preserved from N to M. If this were true, we would have an example satisfying the conditions of Question 5.4.

Combining Propositions 7.10, 7.11, and 7.12, we conclude the following:

Corollary 7.13. *For each of the properties* FCRS*, (non-)*FDT*, (non-)*AUTO *and (non-)*BIAUTO*, the (n-ary) homogeneous monoid M has that property if and only if the (n-ary) multihomogeneous monoid N has that property.*

References

[1] P. Etingof, T. Schedler, A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, Duke Math. J. 100 (2) (1999) 169–209. doi:10.1215/S0012-7094-99-10007-X.

[2] E. Jespers, J. Okniński, Noetherian semigroup algebras, Vol. 7 of Algebras and Applications, Springer, Dordrecht, 2007.

[3] F. Cedó, E. Jespers, J. Okniński, Retractability of set theoretic solutions of the Yang-Baxter equation, Adv. Math. 224 (6) (2010) 2472–2484. doi:10.1016/j.aim.2010.02.001.

[4] F. Cedó, E. Jespers, J. Okniński, Algebras and groups defined by permutation relations of alternating type, J. Algebra 324 (6) (2010) 1290–1313. doi:10.1016/j.jalgebra.2010.07.014.

[5] T. Gateva-Ivanova, E. Jespers, J. Okniński, Quadratic algebras of skew type and the underlying monoids, J. Algebra 270 (2) (2003) 635–659. doi:10.1016/j.jalgebra.2003.06.005.

[6] E. Jespers, J. Okniński, Quadratic algebras of skew type, in: Algebras, rings and their representations, World Sci. Publ., Hackensack, NJ, 2006, pp. 93–112. doi:10.1142/9789812774552_0009.

[7] M. Lothaire, Algebraic Combinatorics on Words, no. 90 in Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2002.

[8] F. Cedó, J. Okniński, Plactic algebras, J. Algebra 274 (1) (2004) 97–117. doi:10.1016/j.jalgebra.2003.12.004.

[9] A. Lascoux, M.-P. Schützenberger, Le monoïde plaxique, in: Noncommutative structures in algebra and geometric combinatorics, no. 109 in Quaderni de "La Ricerca Scientifica", CNR, Rome, 1981, pp. 129–156.
URL http://igm.univ-mlv.fr/~berstel/Mps/Travaux/A/1981-1PlaxiqueNaples.pdf

[10] J. Cassaigne, M. Espie, D. Krob, J.-C. Novelli, F. Hivert, The Chinese Monoid, Int. J. Alg. Comput. 11 (3) (2001) 301–334. doi:10.1142/S0218196701000425.

[11] J. Jaszuńska, J. Okniński, Structure of Chinese algebras, J. Algebra 346 (1) (2011) 31–81. doi:10.1016/j.jalgebra.2011.08.020.

[12] F. Cedó, J. Okniński, Minimal spectrum and the radical of chinese algebras, Algebr. Represent. Theor. 16 (4) (2013) 905–930. doi:10.1007/s10468-012-9339-1.

[13] F. Cedó, E. Jespers, J. Okniński, Finitely presented algebras and groups defined by permutation relations, Journal of Pure and Applied Algebra 214 (7) (2010) 1095–1102. doi:10.1016/j.jpaa.2009.09.015.

[14] F. Cedó, E. Jespers, G. Klein, Finitely presented monoids and algebras defined by permutation relations of abelian type, Journal of Pure and Applied Algebra 216 (5) (2012) 1033–1039. doi:10.1016/j.jpaa.2011.10.021.

[15] J.-C. Novelli, On the hypoplactic monoid, Discrete Mathematics 217 (1-3) (2000) 315–336. doi:10.1016/S0012-365X(99)00270-8.

[16] L. Serrano, The shifted plactic monoid, Mathematische Zeitschrift 266 (2) (2009) 363–392. doi:10.1007/s00209-009-0573-0.

[17] G. Duchamp, D. Krob, Plactic-growth-like monoids, in: M. Ito, H. Jürgensen (Eds.), Words, Languages and Combinatorics, II, World Scientific, River Edge, NJ, 1994, p. 124–142.

[18] V. Diekert, Y. Métivier, Partial commutation and traces, in: Handbook of formal languages, Vol. 3, Springer, Berlin, 1997, pp. 457–533.

[19] D. Kuske, Divisibility monoids: presentation, word problem, and rational languages, in: Fundamentals of computation theory (Riga, 2001), Vol. 2138 of Lecture Notes in Comput. Sci., Springer, Berlin, 2001, pp. 227–239. doi:10.1007/3-540-44669-9_23.

[20] M. Huschenbett, D. Kuske, G. Zetzsche, The monoid of queue actions, in: E. Csuhaj-Varjú, M. Dietzfelbinger, Z. Ésik (Eds.), Mathematical Foundations of Computer Science 2014, no. 8634 in Lecture Notes in Comput. Sci., Springer, pp. 340–351. doi:10.1007/978-3-662-44522-8_29.

[21] J. Birman, K. H. Ko, S. J. Lee, A new approach to the word and conjugacy problems in the braid groups, Adv. Math. 139 (2) (1998) 322–353. doi:10.1006/aima.1998.1761.

[22] P. Dehornoy, L. Paris, Gaussian groups and garside groups, two generalisations of artin groups, Proceedings of the London Mathematical Society 79 (3) (1999) 569–604. doi:10.1112/S0024611599012071.

[23] R. V. Book, F. Otto, String Rewriting Systems, Texts and Monographs in Computer Science, Springer, 1993.

[24] D. B. Epstein, J. W. Cannon, D. F. Holt, S. V. Levy, M. S. Paterson, W. P. Thurston, Word Processing in Groups, Jones & Bartlett, Boston, MA, 1992.

[25] C. M. Campbell, E. F. Robertson, N. Ruškuc, R. M. Thomas, Automatic semigroups, Theoret. Comput. Sci. 250 (1-2) (2001) 365–391. doi:10.1016/S0304-3975(99)00151-6.

[26] A. Heyworth, Rewriting as a special case of non-commutative Gröbner basis theory, in: M. Atkinson, N. Gilbert, J. Howie, S. Linton, E. Robertson (Eds.), Computational and Geometric Aspects of Modern Algebra, no. 275 in London Mathematical Society Lecture Note Series, Cambridge University Press, 2000, pp. 101–105. doi:10.1017/CBO9780511600609.009.

[27] V. A. Ufnarovskij, Combinatorial and asymptotic methods in algebra, in: Algebra, VI, Vol. 57 of Encyclopaedia Math. Sci., Springer, Berlin, 1995, pp. 1–196. doi:10.1007/978-3-662-06292-0_1.

[28] A. J. Cain, R. D. Gray, A. Malheiro, Finite Gröbner–Shirshov bases for Plactic algebras and biautomatic structures for Plactic monoids, J. Algebra 423 (2015) 37–53. arXiv:1205.4885, doi:10.1016/j.jalgebra.2014.09.037.

[29] A. J. Cain, R. D. Gray, A. Malheiro, Rewriting systems and biautomatic structures for Chinese, hypoplactic, and sylvester monoids, Internat. J. Algebra Comput. 25 (1-2). arXiv:1310.6572, doi:10.1142/S0218196715400044.

[30] F. Cedó, J. Okniński, Gröbner bases for quadratic algebras of skew type, Proceedings of the Edinburgh Mathematical Society 55 (2) (2012) 387–401. doi:10.1017/S0013091511000447.

[31] E. Güzel Karpuz, Complete rewriting system for the Chinese monoid, Appl. Math. Sci (Ruse) 4 (22) (2010) 1081–1087.
URL http://www.m-hikari.com/ams/ams-2010/ams-21-24-2010/karpuzAMS21-24-2010-2.pdf

[32] Ł. Kubat, J. Okniński, Gröbner-Shirshov bases for plactic algebras, Algebra Colloquium 21 (04) (2014) 591–596. arXiv:1010.3338v1, doi:10.1142/S1005386714000534.

[33] Y. Chen, J. Qiu, Gröbner–Shirshov basis for the Chinese monoid, Journal of Algebra and Its Applications 07 (05) 623–628. arXiv:0804.0972v2, doi:10.1142/S0219498808003028.

[34] F. Otto, A. Sattler-Klein, K. Madlener, Automatic monoids versus monoids with finite convergent presentations, in: T. Nipkow (Ed.), Rewriting Techniques and Applications, no. 1379 in Lecture Notes in Comput. Sci., Springer, 1998, pp. 32–46. doi:10.1007/BFb0052359.

[35] C. C. Squier, F. Otto, Y. Kobayashi, A finiteness condition for rewriting systems, Theoret. Comput. Sci. 131 (2) (1994) 271–294. doi:10.1016/0304-3975(94)90175-9.

[36] P. de la Harpe, Topics in Geometric Group Theory, University of Chicago Press, 2000.

[37] F. Otto, Y. Kobayashi, Properties of monoids that are presented by finite convergent string-rewriting systems: a survey, in: D.-Z. Du, K.-I. Ko (Eds.), Advances in algorithms, languages, and complexity, Kluwer Academic Publishers, Dordrecht, 1997, p. 225–266. doi:10.1007/978-1-4613-3394-4_12.

[38] Y. Kobayashi, F. Otto, On homotopical and homological finiteness conditions for finitely presented monoids, International Journal of Algebra and Computation 11 (03) (2001) 391–403. doi:10.1142/S0218196701000577.

[39] Y. Lafont, A new finiteness condition for monoids presented by complete rewriting systems (after Craig C. Squier), Journal of Pure and Applied Algebra 98 (3) (1995) 229 – 244. doi:10.1016/0022-4049(94)00043-I.

[40] Y. Guiraud, P. Malbos, Higher-dimensional categories with finite derivation type, Theory Appl. Categ. 22 (2009) No. 18, 420–478.

[41] F. Otto, Modular properties of monoids and string-rewriting systems., Nehaniv, Chrystopher L. (ed.) et al., Algebraic engineering. Proceedings of the international workshop on formal languages and computer systems, Kyoto, Japan, March 18–21, 1997 and proceedings of the first international conference on semigroups and algebraic

engineering held in Aizu, Japan, March 24–28, 1997. Singapore: World Scientific. 538-554 (1999). (1999).
URL http://www.theory.informatik.uni-kassel.de/techreps/TR-97-6.ps

[42] F. Otto, On properties of monoids that are modular for free products and for certain free products with amalgamated submonoids.
URL http://www.theory.informatik.uni-kassel.de/techreps/TR-97-6.ps

[43] J. Wang, Finite Complete Rewriting Systems and Finite Derivation Type for Small Extensions of Monoids, J. Algebra 204 (2) (1998) 493–503. doi:10.1006/jabr.1997.7388.

[44] R. Gray, A. Malheiro, S. Pride, On properties not inherited by monoids from their Schützenberger groups, Information and Computation 209 (7) (2011) 1120–1134. doi:10.1016/j.ic.2011.03.004.

[45] A. Malheiro, Finite derivation type for large ideals, Semigroup Forum 78 (3) (2008) 450–485. doi:10.1007/s00233-008-9109-x.

[46] J. E. Hopcroft, J. D. Ullman, Introduction to Automata Theory, Languages, and Computation, 1st Edition, Addison–Wesley, Reading, MA, 1979.

[47] J. Berstel, Transductions and Context-Free Languages, no. 38 in Leitfäden der Angewandten Mathematik und Mechanik, B. G. Teubner, Stuttgart, 1979.

[48] F. Baader, T. Nipkow, Term Rewriting and All That, Cambridge University Press, 1999.

[49] M. Droste, W. Kuich, H. Vogler, Handbook of Weighted Automata, 1st Edition, Springer Publishing Company, Incorporated, 2009.

[50] V. Guba, M. Sapir, Diagram groups, Mem. Amer. Math. Soc. 130 (620) (1997) viii+117. doi:10.1090/memo/0620.

[51] Y. Guiraud, P. Malbos, Higher-dimensional normalisation strategies for acyclicity, Advances in Mathematics 231 (3-4) (2012) 2294–2351. doi:10.1016/j.aim.2012.05.010.

[52] J.-P. Serre, Trees, Springer-Verlag, 1980.

[53] M. Hoffmann, R. M. Thomas, Biautomatic semigroups, in: M. Liskiewicz, R. Reischuk (Eds.), Fundamentals of Computation Theory, no. 3623 in Lecture Notes in Computer Sciene, Springer, 2005, pp. 56–67. doi:10.1007/11537311_6.

[54] C. Frougny, J. Sakarovitch, Synchronized rational relations of finite and infinite words, Theoret. Comput. Sci. 108 (1) (1993) 45–82. doi:10.1016/0304-3975(93)90230-Q.

[55] A. J. Duncan, E. F. Robertson, N. Ruškuc, Automatic monoids and change of generators, Mathematical Proceedings of the Cambridge Philosophical Society 127 (3) (1999) 403–409. doi:10.1017/S0305004199003722.

[56] M. Hoffmann, R. M. Thomas, Notions of automaticity in semigroups, Semigroup Forum 66 (3) (2002) 337–367. doi:10.1007/s002330010161.

[57] M. Katsura, Y. Kobayashi, Constructing finitely presented monoids which have no finite complete presentation, Semigroup Forum 54 (1) (1997) 292–302. doi:10.1007/BF02676612.

[58] S. J. Pride, J. Wang, Rewriting systems, finiteness conditions, and associated functions, in: Algorithmic problems in groups and semigroups (Lincoln, NE, 1998), Trends Math., Birkhäuser Boston, Boston, MA, 2000, pp. 195–216.

[59] A. Jura, Determining ideals of a given finite index in a finitely presented semigroup, Demonstratio Math. 11 (3) (1978) 813–827.

[60] A. J. Cain, V. Maltcev, For a few elements more: A survey of finite Rees index, Tech. rep. (7 2013). arXiv:1307.8259.

[61] K. Wong, P. Wong, On finite complete rewriting systems and large subsemigroups, J. Algebra 345 (1) (2011) 242–256. doi:10.1016/j.jalgebra.2011.08.022.

[62] A. Malheiro, Finite derivation type for Rees matrix semigroups, Theoret. Comput. Sci. 355 (3) (2006) 274–290. doi:10.1016/j.tcs.2005.12.011.

[63] M. Hoffmann, R. M. Thomas, N. Ruškuc, Automatic semigroups with subsemigroups of finite ress index, International Journal of Algebra and Computation 12 (03) (2002) 463–476. doi:10.1142/S0218196702000833.

[64] J. M. Howie, Fundamentals of Semigroup Theory, no. 12 in London Mathematical Society Monographs: New Series, Clarendon Press, Oxford University Press, New York, 1995.

[65] Y. Kobayashi, Finite Homotopy Bases of One-Relator Monoids, J. Algebra 229 (2) (2000) 547–569. `doi:10.1006/jabr.1999.8251`.

[66] L. Bachmair, N. Dershowitz, Commutation, transformation, and termination, in: J. H. Siekmann (Ed.), 8th International Conference on Automated Deduction, Vol. 230 of Lecture Notes in Computer Science, Springer Berlin Heidelberg, 1986, pp. 5–20. `doi:10.1007/3-540-16780-3\_76`.

Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, 2829–516 Caparica, Portugal

E-mail address: `a.cain@fct.unl.pt`

School of Mathematics, University of East Anglia, Norwich NR4 7TJ, United Kingdom

E-mail address: `Robert.D.Gray@uea.ac.uk`

Departamento de Matemática & Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, 2829–516 Caparica, Portugal

E-mail address: `ajm@fct.unl.pt`