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Inducing stability in hedonic games

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Abstract

In many applications of coalition formation games, a key issue is that some desirable coalition structures are not elements of the core of these games. In these cases, it would be useful for an authority which aims to implement a certain outcome to know how far from the original game is the nearest game where the desirable outcome is part of the core. This question is at the center of this study. Focusing on hedonic games, we uncover previously unexplored links between such games and transferrable utility games, and develop a tailor-made solution concept for the transferrable utility game, the implementation core, to provide an answer to our question.

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Keywords

hedonic game, implementation core, Kemeny distance, stability.

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Abstract

In many applications of coalition formation games, a key issue is that some desirable coalition structures are not elements of the core of these games. In these cases, it would be useful for an authority which aims to implement a certain outcome to know how far from the original game is the nearest game where the desirable outcome is part of the core. This question is at the center of this study. Focusing on hedonic games, we uncover previously unexplored links between such games and transferrable utility games, and develop a tailor-made solution concept for the transferrable utility game, the implementation core, to provide an answer to our question.

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1 Introduction

Given a situation in which a certain desirable outcome is unachievable, one may ask the following two questions:

- How far off from the desirable outcome is the achievable?

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- How far off is the situation where the desirable outcome is achievable?

The former question underlied the theory of the second best. The latter – and the one at the center of our work – contributes to the theory of economic design.

An economic designer can be viewed as an authority external to an economic situation who is endowed with the knowledge of the range of possible outcomes and a clear judgement of their desirability. Traditionally, its role has been to exercise authority to impose an outcome on the society, as in the case of a social planner, or to design specific and detailed procedures as an auctioneer would. The designer we aim to inform with our study targets fundamental preferences, instead, and operates in the context of coalition formation. In particular, our focus is on hedonic games (cf. Drèze and Greenberg, 1980) - a class of non-transferrable utility games where the individual decision about which group to join depends (at least to an extent) on the identity of the other group members.

A key issue and a very natural desirable property of any outcome in coalition formation is the coalitional stability, i.e., the immunity of coalitions to deviations by individuals or groups of players. In the analysis of the stability of coalitions the notion of the core takes a central stage. In many real-life applications of coalition formation problems, however, the core is either empty or the most desirable coalition is not an element of it. Take as an example a market with avoidable fixed costs, extensively studied both theoretically and empirically in Sjostrom (1989). In such a market firms may form agreements to adhere to common price lists and output schedules. In any possible agreement structure, however, there is a firm that faces incentives to renege on the agreement. International environmental agreements present us with another silent example where the most desirable form of agreement, the grand coalition, is not stable and a stable cooperation structure of binding agreements is evasive.

Much research in cooperative game theory has focused on studying the underlying properties of the game that ensures non-emptiness of the core. In the context of hedonic games, Banerjee et al. (2001) and Bogomolnaia and Jackson (2002) focus on properties of preference profiles, while Pápai (2004) identifies properties of permissible coalition structures.

Our research agenda is complementary to that: we identify games as near as possible to the original where the core is non-empty, and moreover, a pre-specified coalitional structure is an element of it. The distance between the two games is defined on the differences between the two corresponding preference profiles. Here we employ a standard measure of a swap distance in orders - the Kemeny distance (cf. Kemeny, 1959) - for its natural characteristics.[1]

For the purposes of this exercise we translate the hedonic game into a transferrable utility game (TU-game) using the rankings of players and the specified desirable coalition structure, π. The value of each coalition in the TU-game can be interpreted as the claim this coalition makes against the formation of π. To analyze this game we develop a tailored solution concept for the TU-game that assumes the presence of an external authority that has a budget B to satisfy all claims by the coalition. We name this concept the *implementation core* and show that the implementation core when the budget is zero is non-empty if and only if π is an element of the core of the hedonic game. In case the implementation core is empty, we consider a minimal increase in the authority's budget to render the implementation core at the new budget level non-empty. We show that this minimal increase in the budget equals the Kemeny distance between the original game and a hedonic game where π is a core element. We further show how any implementation core element can be used to construct a new hedonic game with the desirable property that π is core stable for the new game. Thus, one can interpret the changes in the players' preference profiles to the new games as compensations measured in coalitional ranking that these players receive when π is implemented.

In this last respect our analysis is in the spirit of an early work of van Gellekom et al. (1999) and more recent ones by Bejan and Gómez (2009) and Bachrach et al. (2013), who study in the context of TU-games how much one needs to increase the value of the grand coalition in a cooperative game for a certain solution concept applied to this game to be non-empty. van Gellekom et al. (1999) take the increase in the grand coalition's value as an increase in prosperity while the latter works interpret it is as the necessary subsidy given out by an exogenous regulator. A similar question to ours is also studied by Nguyen and Vohra (2016) in the context of many-to-one matching problems with couples. In their study the distance between a matching problem with an empty core and a nearby problem where the core is non-empty is measured

[1] We refer to a recent work by Can and Storcken (2013) for a characterization of the Kemeny distance. In addition, Bossert et al. (2016) argue that such distance-based measures can successfully be applied to assessing social status changes and social mobility.

in terms of the capacity of the firm-side of the market.

Next we present the set-up of the problem and in section 3 we discuss illustrative examples. In section 4 we present our main results.

2 A preliminary discussion

Given a hedonic game $(N, \succeq)$ and a partition π of the player set N, we start by constructing an associated TU-game (N, v_π). In this TU-game, v_π is a standard characteristic function representing the *claim* a coalition may have with respect to the partition π. More precisely, for each nonempty coalition $S \subseteq N$, we define $v_\pi(S)$ to be the minimal amount (defined in terms of rank differences) one needs in order to please the "*cheapest*" player in S and thus, to make S a non-blocking coalition for π in the hedonic game $(N, \succeq)$. For a given budget $B \geq 0$ that can be used by an authority to implement π, we define then in a suitable way the *implementation core* of (N, v_π) denoted by I-CORE$(N, v_\pi; B)$ such that

- π is in the core of the hedonic game if and only if I-CORE$(N, v_\pi; 0)$ is non-empty;
- when π is in the core of the hedonic game, then I-CORE$(N, v_\pi; 0)$ is a singleton in which the unique allocation assigns zero payoff to each player.

When π is not core stable in the hedonic game (and thus, the corresponding TU-game has an empty implementation core when the implementation budget B is 0), we then precede as follows. We calculate the smallest $B > 0$ necessary for the implementation core of the TU-game to be non-empty. Each allocation in such an implementation core provides information on a smallest modification of the preference profile $\succeq$ for the hedonic game $(N, \succeq)$ such that the partition π becomes core stable in the modified hedonic game $(N, \succeq')$. More precisely, the payoff to each player in such an allocation indicates the number of positions his coalition in π is to be raised in order to make π core stable, and hence, the budget B indicates the Kemeny distance between the modified hedonic game $(N, \succeq')$ and the original one $(N, \succeq)$.

We present below the construction of the TU-game. Next, we provide a few exploratory examples. We finish by analyzing the relation between the hedonic game, its associated TU-game, and their respective cores.

Consider the TU-game constructed in the following way.

1. Take a hedonic game $(N, \succeq)$ in which players' preferences over coalitions are strict (i.e., for each $i \in N$, the preference $\succeq_i$ of player i is a complete, transitive and antisymmetric binary relation over the collection $\mathcal{N}_i = \{S \subseteq N \mid i \in S\}$ of coalitions containing i), and take a partition π of N to be implemented.

2. For each player $i \in N$, we record his ranking $\text{RANK}_i(S, \succeq)$ of coalitions in $\mathcal{N}_i$ based on his preferences $\succeq_i$. The coalition at the bottom of player i's preference list is given the lowest rank 1, and the one at the top is given the highest rank $2^{|N|-1}$ by that player[2], i.e.,

   - $\text{RANK}_i(S, \succeq) = \big|\{T \in \mathcal{N}_i \mid S \succeq_i T\}\big|$ for each $S \in \mathcal{N}_i$.

   For example, in a game of three players, each player can be a member of only four coalitions (i.e., when $|N| = 3$, $|\mathcal{N}_i| = 2^{|N|-1} = 4$ for each $i \in N$) and thus, the coalition most preferred by a player is ranked fourth by this player.

3. We then use players' rankings to construct (N, v_π) as follows. For each nonempty coalition $S \subseteq N$,

   $$v_\pi(S) = \min\{\text{DIFF}_i(S, \pi, \succeq) \mid i \in S\},$$

   where for each $i \in S$,

   $$\begin{aligned} \text{DIFF}_i(S, \pi, \succeq) &= \big|\{T \in \mathcal{N}_i \mid S \succeq_i T \succ_i \pi(i)\}\big| \\ &= \big|\{T \in \mathcal{N}_i \mid S \succeq_i T\} \setminus \{T \in \mathcal{N}_i \mid \pi(i) \succeq_i T\}\big| \\ &= \max\big\{0, \text{RANK}_i(S, \succeq) - \text{RANK}_i(\pi(i), \succeq)\big\}, \end{aligned}$$

   and $\pi(i)$ is the unique member of π containing i, i.e. $\{\pi(i)\} = \pi \cap \mathcal{N}(i)$.

By this definition, the following connection between the *individual rationality* of partition π, i.e., $\pi(i) \succeq_i \{i\}$ holding for each $i \in N$, and 0-*normalization* of the corresponding TU-game (N, v_π), i.e., $v_\pi(\{i\}) = 0$ holding for each $i \in N$, is obtained.

[2] Notice that our results hold true for any affine transformation of the RANK function.

Fact 1 *π is individually rational if and only if (N, v_π) is 0-normalized.* ■

Notice that v_π is constructed to capture whether a non-empty coalition $S \subseteq N$ is blocking π in the hedonic game $(N, \succeq)$. By definition, $v_\pi(S) = 0$ implies that there exists a player $i \in S$ such that $\text{DIFF}_i(S, \pi, \succeq) = 0$, and moreover, $\text{DIFF}_i(S, \pi, \succeq) = 0$ implies $S \in \{T \in \mathcal{N}_i \mid \pi(i) \succeq_i T\}$, i.e., $\pi(i) \succeq_i S$. In other words, $v_\pi(S) = 0$ means that there exists a member of S who (weakly) prefers her coalition in π to S, and therefore, has no incentive to block π via S. On the other hand, $v_\pi(S) > 0$ implies that $\text{DIFF}_i(S, \pi, \succeq) > 0$ for each $i \in S$, i.e., $S \notin \{T \in \mathcal{N}_i \mid \pi(i) \succeq_i T\}$ for each $i \in S$, and equivalently, $S \succ_i \pi(i)$ for each $i \in S$. In other words, $v_\pi(S) > 0$ means that π can be blocked by S. Hence, the following fact is obtained.

Fact 2 *π is core stable in $(N, \succeq)$ if and only if $v_\pi(S) = 0$ for each coalition $S \subseteq N$.*

Here we interpret the value of $v_\pi(S)$ as the minimum compensation measured in units of ranks that the "*cheapest*" member of S must be given to make him prefer his coalition in π to S. For all other members of S the compensation needed to induce the same change in preferences is at least as high.

Last, we assume that the central authority that seeks to implement the partition π as a core outcome has a budget $B \geq 0$ that it can use discretionary to compensate players for their membership in π. Given B we define the notion of an *implementation core* of the TU-game (N, v_π) as

$$\text{I-CORE}(N, v_\pi; B) = \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{l} x(N) = B \text{ and } \forall S \subseteq N, \\ \max\{x_i - \text{DIFF}_i(S, \pi, \succeq) \mid i \in S\} \geq 0 \end{array} \right\}.$$

Notice that for $B = v_\pi(N) \geq 0$ there is no difference between the efficiency requirement for the standard core notion and the one defined above. Moreover, as we show in Fact 6 below, in the I-CORE the no-blocking conditions with respect to each coalition are stronger than those in the standard core definition. We refer to these condition as *implementation* conditions. We further illustrate this point through the discussion of some examples in the next section.

Fact 3 *Each $x \in$ I-CORE$(N, v_\pi; B)$ is non-negative.*

Proof. Each $x \in \text{I-CORE}(N, v_\pi; B)$ satisfies, for each $i \in N$,

$$x_i - \text{DIFF}_i(\{i\}, \pi, \succeq) = \max\{x_j - \text{DIFF}_j(\{i\}, \pi, \succeq) \mid j \in \{i\}\} \geq 0.$$

From non-negativity of each $\text{DIFF}_i(\{i\}, \pi, \succeq)$, each x_i is non-negative. ■

Fact 4 $\left|\text{I-CORE}(N, v_\pi; B)\right| \leq \left|\text{I-CORE}(N, v_\pi; B')\right|$ *if* $B \leq B'$.

Proof. It is obvious by observing that, for each $x \in \text{I-CORE}(N, v_\pi; B)$, an element of $\text{I-CORE}(N, v_\pi; B')$ can be obtained by increasing a component of x, say x_1, by $B' - B$. ■

Fact 5 *π is core stable in $(N, \succeq)$ if and only if* $\text{I-CORE}(N, v_\pi; 0) \neq \emptyset$.

Proof. By definition, we have $x(N) = 0$ for each $x \in \text{I-CORE}(N, v_\pi; 0)$. Then, from Fact 3, $\text{I-CORE}(N, v_\pi; 0)$ is either empty or equals to $\{0^N\}$. Moreover, $0^N \in \text{I-CORE}(N, v_\pi; 0)$ if and only if $\max\{-\text{DIFF}_i(S, \pi, \succeq) \mid i \in S\} \geq 0$ for each $S \subseteq N$, which is equivalent to $v_\pi(S) \leq 0$ for each $S \subseteq N$. By definition, each $\text{DIFF}_i(S, \pi, \succeq)$ is non-negative, and thus, each $v_\pi(S)$ is non-negative as well. Therefore, from Fact 2, π is core stable in $(N, \succeq)$ if and only if $\text{I-CORE}(N, v_\pi; 0) \neq \emptyset$. ■

Fact 6 $\text{I-CORE}(N, v_\pi; v_\pi(N)) \subseteq Core(N, v_\pi)$.

Proof. It suffices to show that $x(S) \geq v_\pi(S)$ for each $S \subseteq N$ under the assumption that $\max\{x_i - \text{DIFF}_i(S, \pi, \succeq) \mid i \in S\} \geq 0$ for each $S \subseteq N$.

Let $S \subseteq N$ be an arbitrary non-empty coalition. From Fact 3, we have

$$x(S) - v_\pi(S) \geq \max\{x_i \mid i \in S\} - v_\pi(S) = \max\{x_i - v_\pi(S) \mid i \in S\}.$$

By definition of v_π, we have, for each $i \in S$,

$$x_i - v_\pi(S) = x_i - \min\{\text{DIFF}_j(S, \pi, \succeq) \mid j \in S\} \geq x_i - \text{DIFF}_i(S, \pi, \succeq).$$

Hence, again by assumption, we have

$$x(S) - v_\pi(S) \geq \max\{x_i - \text{DIFF}_i(S, \pi, \succeq) \mid i \in S\} \geq 0,$$

and the proof is completed. ■

3 How to use the implementation core

With the help of the following examples we illustrate how the elements of the implementation core can be used to minimally modify the preference profile in the original hedonic game so that the desirable partition from the central authority's point of view becomes an element of core of the modified game.

Example 1 Take the following hedonic game.

$\succeq_1$	$\succeq_2$	$\succeq_3$	$\text{RANK}_i(S, \succeq)$
12	23	13	4
13	12	123	3
1	2	23	2
123	123	3	1

The core of this hedonic game is empty.

Let's first consider partition $\pi = \{12, 3\}$ and notice that it is individually rational. The corresponding TU-game is constructed as follows.

$$\begin{aligned} v_\pi(1) &= \text{DIFF}_1(1, \pi, \succeq) = 0 \\ v_\pi(2) &= \text{DIFF}_2(2, \pi, \succeq) = 0 \\ v_\pi(3) &= \text{DIFF}_3(3, \pi, \succeq) = 0 \\ v_\pi(12) &= \min\{\text{DIFF}_1(12, \pi, \succeq), \text{DIFF}_2(12, \pi, \succeq)\} = \min\{0, 0\} = 0 \\ v_\pi(13) &= \min\{\text{DIFF}_1(13, \pi, \succeq), \text{DIFF}_3(13, \pi, \succeq)\} = \min\{0, 3\} = 0 \\ v_\pi(23) &= \min\{\text{DIFF}_2(23, \pi, \succeq), \text{DIFF}_3(23, \pi, \succeq)\} = \min\{1, 1\} = 1 \\ v_\pi(123) &= \min\{\text{DIFF}_1(123, \pi, \succeq), \text{DIFF}_2(123, \pi, \succeq), \text{DIFF}_3(123, \pi, \succeq)\} \\ &= \min\{0, 0, 2\} = 0. \end{aligned}$$

From $v_\pi(1) = v_\pi(2) = v_\pi(3) = v_\pi(123) = 0$, $x \in Core(N, v_\pi)$ only if $x = 0^N$, but $x = 0^N$ implies $x(12) = 0 < v_\pi(12)$. Therefore, $Core(N, v_\pi)$ is empty, and by Fact 6, the implementation core $\text{I-CORE}(N, v_\pi; B)$ with respect to $B = 0$ of this game is empty.

Let us now increase B so that we obtain a non-empty $\text{I-CORE}(N, v_\pi; B)$: in this case we need $B \geq 1$. We focus on minimal increases to the budget and thus consider $\text{I-CORE}(N, v_\pi; 1)$. The two elements of the I-CORE are $(0, 1, 0)$ and $(0, 0, 1)$. To verify that these are indeed I-CORE elements we need to check that $\max\{x_2 - 1, x_3 - 1\} \geq 0$ which is the case in both payoff vectors $(0, 1, 0)$ and $(0, 0, 1)$.

Now we are going to use the above information to modify the preference profile of hedonic game $(N, \succeq)$ such that π is in the core of the modified game. We will interpret a zero payoff in the I-CORE element of the TU-game as an indication that the corresponding player's preference should remain unchanged and a positive payoff as an indication that in the corresponding player's preference the ranking of his coalition in π should go up by the number of places indicated by his payoff. Thus, to render $\{12, 3\}$ core stable,

- the I-CORE element $(0, 1, 0)$ indicates to modify the preference profile from $\succeq$ to $\succeq'$ in such a way that $\succeq_1 = \succeq'_1$ and $\succeq_3 = \succeq'_3$, and then modify the preference of player 2, $\succeq_2$ such that in the new preference $\succeq'_2$ one has $\text{RANK}_2(12, \succeq') = 4$ and $\text{RANK}_2(23, \succeq') = 3$, i.e., $\succeq_2 \setminus \succeq'_2 = \{(23, 12)\}$ and $\succeq'_2 \setminus \succeq_2 = \{(12, 23)\}$.

- the I-CORE element $(0, 0, 1)$ indicates to modify the preference profile from $\succeq$ to $\succeq''$ in such a way that $\succeq_1 = \succeq''_1$ and $\succeq_2 = \succeq''_2$, and then modify the preference of player 3, $\succeq_3$ such that in the new preference $\succeq''_3$, $\text{RANK}_3(3, \succeq'') = 2$ and $\text{RANK}_3(23, \succeq'') = 1$, i.e., $\succeq_3 \setminus \succeq''_3 = \{(23, 3)\}$ and $\succeq''_3 \setminus \succeq_3 = \{(3, 23)\}$.

Notice that in both cases the modification of the original preference profile renders $\{12, 3\}$ core stable in $(N, \succeq')$ and $(N, \succeq'')$, and that in both cases the Kemeny distance between the original and modified profiles is 1.

$\succeq'_1$	$\succeq'_2$	$\succeq'_3$	$\text{RANK}_i(S, \succeq')$
12	$\boxed{12}$	13	4
13	$\boxed{23}$	123	3
1	2	23	2
123	123	3	1

and

$\succeq''_1$	$\succeq''_2$	$\succeq''_3$	$\text{RANK}_i(S, \succeq'')$
12	23	13	4
13	12	123	3
1	2	$\boxed{3}$	2
123	123	$\boxed{23}$	1

Next, consider partition $\varphi = \{1, 2, 3\}$. Obviously, it is individually rational. The corresponding TU-game is constructed as follows.

$$
\begin{aligned}
v_\varphi(1) &= \textsc{diff}_1(1, \varphi, \succeq) = 0 \\
v_\varphi(2) &= \textsc{diff}_2(2, \varphi, \succeq) = 0 \\
v_\varphi(3) &= \textsc{diff}_3(3, \varphi, \succeq) = 0 \\
v_\varphi(12) &= \min\{\textsc{diff}_1(12, \varphi, \succeq), \textsc{diff}_2(12, \varphi, \succeq)\} = \min\{2, 1\} = 1 \\
v_\varphi(13) &= \min\{\textsc{diff}_1(13, \varphi, \succeq), (\textsc{diff}_3(13, \varphi, \succeq)\} = \min\{1, 3\} = 1 \\
v_\varphi(23) &= \min\{\textsc{diff}_2(23, \varphi, \succeq), \textsc{diff}_3(23, \varphi, \succeq)\} = \min\{2, 1\} = 1 \\
v_\varphi(123) &= \min\{\textsc{diff}_1(123, \varphi, \succeq), \textsc{diff}_2(123, \varphi, \succeq), \textsc{rank}_3(123, \varphi, \succeq)\} \\
&= \min\{0, 0, 2\} = 0.
\end{aligned}
$$

Again from $v_\varphi(1) = v_\varphi(2) = v_\varphi(3) = v_\varphi(123) = 0$, the implementation core $\textsc{i-core}(N, v_\varphi; B)$ with $B = 0$ of this game is empty. Let us now consider an increase in the discretionary budget to $B = 3$. Then $\textsc{i-core}(N, v_\varphi; 3)$ is a singleton and consists of the payoff vector $(1, 1, 1)$. To see that this is indeed an I-CORE element we focus on the coalitions with strictly positive claims: 12, 13, and 23. We have then $\max\{x_1 - 2, x_2 - 1\} = \max\{-1, 0\} = 0 \geq 0$, $\max\{x_1 - 1, x_3 - 3\} = \max\{0, -2\} = 0 \geq 0$, and $\max\{x_2 - 2, x_3 - 1\} = \max\{-1, 0\} = 0 \geq 0$, i.e., the implementation conditions are satisfied.

This unique core element indicates that we need to modify the preference lists of all three players in the hedonic game by moving the respective coalition up one rank. The modified preference profile is given below:

$\succeq_1'''$	$\succeq_2'''$	$\succeq_3'''$	$\textsc{rank}_i(S, \succeq''')$
12	23	13	4
$\boxed{1}$	$\boxed{2}$	123	3
$\boxed{13}$	$\boxed{12}$	$\boxed{3}$	2
123	123	$\boxed{23}$	1

Clearly, partition $\varphi = \{1, 2, 3\}$ is in the core of the modified hedonic game $(N, \succeq''')$. The Kemeny distance between the $\succeq_i$ and $\succeq_i'''$ is 1 for each player $i \in N$, and so the Kemeny distance between the original hedonic game $(N, \succeq)$ and the modified one $(N, \succeq''')$ is 3.

Example 2 Take the following hedonic game

$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$	$\text{RANK}_i(S, \succeq)$
12	23	13	14	8
1234	1234	1234	24	7
13	12	23	1234	6
14	2	3	4	5
1	24	34	34	4
123	123	123	234	3
124	124	234	124	2
134	234	134	134	1

and consider partition $\pi_1 = \{1234\}$ which is the only core element for $(N, \succeq)$. The characteristic function in the corresponding TU-game (N, v_{π_1}) is

S	1	2	3	4	12	13	14	23	24	34	123	124	134	1234
$v_{\pi_1}(S)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The unique I-CORE element with $B = 0$ is $(0, 0, 0, 0)$ confirming the core stability of partition π_1.

Next consider partition $\pi_2 = \{14, 23\}$. This partition is not an element of the core of the hedonic game $(N, \succeq)$ as it is blocked by coalition $\{13\}$ as $13 \succ_1 14$ and $13 \succ_3 23$. The corresponding TU-game is as follows.

S	1	2	3	4	12	13	14	23	24	34	123	124	134	1234
$v_{\pi_2}(S)$	0	0	0	0	0	1	0	0	0	0	0	0	0	0

Clearly, the implementation core with $B = 0$ of this TU-game is empty. To obtain a non-empty I-CORE the minimal increase in B must be such that players 1 and 3 can jointly obtain at least 1, thus we first try $B = 1$. Let's first consider allocation $(0, 0, 1, 0)$, i.e. where the whole of the budget $B = 1$ is allocated to satisfy the claim of player 3. Notice, however, that in this case the implementation condition is not satisfied:

$$\max \left\{ \begin{array}{c} x_1 - \text{DIFF}_1(13, \pi_2, \succeq) \\ x_3 - \text{DIFF}_3(13, \pi_2, \succeq) \end{array} \right\} = \max\{0 - 1, 1 - 2\} = -1 < 0.$$

The unique I-CORE element with $B = 1$ thus is $(1, 0, 0, 0)$. To verify this notice that

$$\max \left\{ \begin{array}{c} x_1 - \text{DIFF}_1(13, \pi_2, \succeq) \\ x_3 - \text{DIFF}_3(13, \pi_2, \succeq) \end{array} \right\} = \max\{1 - 1, 0 - 2\} = 0 \geq 0.$$

Thus, we can consider a modification of hedonic game such that the rank of coalition 14 is increased by 1 (from rank 5 to rank 6) and the rank of coalition 13 is decreased by 1 (from rank 6 to rank 5) in the preference list of player 1. We let all other players' preference lists remain the same, in this way we obtain the modified preference profile:

$\succeq'_1$	$\succeq'_2$	$\succeq'_3$	$\succeq'_4$	$\text{RANK}_i(S, \succeq')$
12	23	13	14	8
1234	1234	1234	24	7
$\boxed{14}$	12	23	1234	6
$\boxed{13}$	2	3	4	5
1	24	34	34	4
123	123	123	234	3
124	124	234	124	2
134	234	134	134	1

Notice that the Kemeny distance between preference profile $\succeq$ and $\succeq'$ is one. It is easy to verify that partition π_2 is an element of the core of the modified hedonic game $(N, \succeq')$.

Last, consider partition $\pi_3 = \{1, 2, 3, 4\}$. Clearly it is not a core element of the hedonic game $(N, \succeq)$ as it is blocked, for example, by coalition 12 as $12 \succ_1 1$ and $12 \succ_2 2$. The corresponding TU-game is as follows.

S	1	2	3	4	12	13	14	23	24	34	123	124	134	1234
$v_{\pi_3}(S)$	0	0	0	0	1	2	1	1	0	0	0	0	0	1

The implementation core with $B = 0$ of the TU game is empty as $B = 0$ is insufficient to satisfy the claim of coalition 13, for example. Furthermore notice, that increasing B to 2 is not sufficient to obtain a non-empty implementation: consider the allocation $(1, 0, 1, 0)$ where the claims of all coalitions are satisfied and notice that this allocation is not implementable as, for example,

$$\max \left\{ \begin{array}{c} 1 - \text{DIFF}_1(12, \pi_3, \succeq) \\ 0 - \text{DIFF}_2(12, \pi_3, \succeq) \end{array} \right\} = \max\{1 - 4, 0 - 1\} = -1 < 0.$$

Here the minimum increase in B to make $\text{I-CORE}(N, v_{\pi_3}; B)$ non-empty is by 5, and the unique element of $\text{I-CORE}(N, v_{\pi_3}; 5)$ is $(2, 1, 1, 1)$. To see that this is indeed an element of the implementation core, notice that the allocation satisfied efficiency, $x_1+x_2+x_3+x_4 = 5$, and as for the implementability conditions we have:

$$\max\left\{\begin{array}{l} 2-\textsc{diff}_1(12,\pi_3,\succeq) \\ 1-\textsc{diff}_2(12,\pi_3,\succeq) \end{array}\right\} = \max\{2-4, 1-1\} = 0;$$

$$\max\left\{\begin{array}{l} 2-\textsc{diff}_1(13,\pi_3,\succeq) \\ 1-\textsc{diff}_3(13,\pi_3,\succeq) \end{array}\right\} = \max\{2-2, 1-3\} = 0;$$

$$\max\left\{\begin{array}{l} 2-\textsc{diff}_1(14,\pi_3,\succeq) \\ 1-\textsc{diff}_4(14,\pi_3,\succeq) \end{array}\right\} = \max\{2-1, 1-3\} = 1;$$

$$\max\left\{\begin{array}{l} 1-\textsc{diff}_2(23,\pi_3,\succeq) \\ 1-\textsc{diff}_3(23,\pi_3,\succeq) \end{array}\right\} = \max\{1-3, 1-1\} = 0;$$

$$\max\left\{\begin{array}{l} 2-\textsc{diff}_1(1234,\pi_3,\succeq) \\ 1-\textsc{diff}_2(1234,\pi_3,\succeq) \\ 1-\textsc{diff}_3(1234,\pi_3,\succeq) \\ 1-\textsc{diff}_4(1234,\pi_3,\succeq) \end{array}\right\} = \max\{2-3, 1-2, 1-2, 1-1\} = 0.$$

The allocation dictates the following modifications in the preference profile of the hedonic game:

- Player 1's preference list: increase the rank of coalition 1 by two –from rank 4 to rank 6; and
- Players 2, 3, and 4's preference lists: increase the rank of the singleton coalition by one.

The new revised preference profile is given below.

$\succeq''_1$	$\succeq''_2$	$\succeq''_3$	$\succeq''_4$	$\textsc{rank}_i(S,\succeq'')$
12	23	13	14	8
1234	1234	1234	24	7
$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	6
$\boxed{13}$	$\boxed{12}$	$\boxed{23}$	$\boxed{1234}$	5
$\boxed{14}$	24	34	34	4
123	123	123	234	3
124	124	234	124	2
134	234	134	134	1

Notice that the Kemeny distance between preference profiles $\succeq$ and $\succeq''$, which is the sum of Kemeny distances between the individual players' preference lists is 5 which equals to the budget level B in the implementation core.

4 Implementation core and nearest hedonic games

The discussion above clearly points out that given a hedonic game, a desirable partition, and its corresponding TU-game there are links between the minimum budget level in the implementation core for which it is non-empty and the Kemeny distance between the preference profile of the original hedonic game and the modified game which is "*nearest*" to it and in which the desirable partition is core stable. Next, we explore these relations more formally.

Let $\succeq$ and $\succeq'$ be two arbitrary profiles. The *Kemeny distance* $\delta_N(\succeq, \succeq')$ between $\succeq$ and $\succeq'$ is defined as follows.

$$\delta_N(\succeq, \succeq') = \sum_{i \in N} \delta_i(\succeq, \succeq'),$$

where, for each $i \in N$,

$$\begin{aligned} \delta_i(\succeq, \succeq') &= \frac{1}{2}\Big|\big(\succeq_i \setminus \succeq'_i \big) \cup \big(\succeq'_i \setminus \succeq_i \big)\Big| \\ &= \frac{1}{2}\Big(\big| \succeq_i \setminus \succeq'_i \big| + \big| \succeq'_i \setminus \succeq_i \big|\Big). \end{aligned}$$

Lemma 1

$$\delta_N(\succeq, \succeq') \geq \sum_{i \in N} \max\big\{\textsc{diff}_i(T, \pi, \succeq) \mid T \in \mathcal{N}_i, \pi(i) \succeq'_i T\big\}.$$

Proof. Let $i \in N$, let $T \in \mathcal{N}_i$ satisfy $\pi(i) \succeq'_i T$. It suffices to show that $\delta_i(\succeq, \succeq') \geq \textsc{diff}_i(T, \pi, \succeq)$. When $\textsc{diff}_i(T, \pi, \succeq) = 0$, the inequality obviously holds. Suppose $\textsc{diff}_i(T, \pi, \succeq) > 0$, and let $k = \textsc{diff}_i(T, \pi, \succeq)$. By definition, there exist k coalitions $S_1, S_2, \ldots, S_k \in \mathcal{N}_i$ such that

$$T = S_k \succ_i S_{k-1} \succ_i \cdots \succ_i S_1 \succ_i \pi(i).$$

Then, it follows from transitivity that $S_k \succ_i \pi(i)$ and $S_k \succ_i S_j \succ_i \pi(i)$ for each $1 \leq j < k$. From antisymmetry and $\pi(i) \succ'_i S = T_k$, we have $S_k \not\succ'_i \pi(i)$, and moreover, by combining with transitivity, we have $S_k \not\succ'_i S_j$ or $S_j \not\succ'_i \pi(i)$ for each $1 \leq j < k$. Hence, $| \succeq_i \setminus \succeq'_i |$ is at least k. From completeness, $a \not\succ'_i b$ implies $b \succ'_i a$, and hence $| \succeq'_i \setminus \succeq_i |$ is at least k as well. Therefore, $\delta_i(\succeq, \succeq')$ is at least k. ■

A profile $\succeq'$ is called $(\pi, \succeq)$-*normalized* if for each $i \in N$,

- $\textsc{rank}_i(\pi(i), \succeq') \geq \textsc{rank}_i(\pi(i), \succeq)$, and
- for all $S, T \in \mathcal{N}_i \setminus \{\pi(i)\}$, $S \succeq_i T$ if and only if $S \succeq'_i T$.

Lemma 2 *Each $(\pi, \succeq)$-normalized profile $\succeq'$ satisfies*

$$\delta_N(\succeq, \succeq') = \sum_{i \in N} \max \left\{ \textsc{diff}_i(T, \pi, \succeq) \mid T \in \mathcal{N}_i, \pi(i) \succeq'_i T \right\}.$$

Proof. Let $\succeq'$ be a $(\pi, \succeq)$-normalized profile, let $i \in N$, and let $S \in \mathcal{N}_i$ be the coalition satisfying $\pi(i) \succeq'_i S$ and

$$\textsc{diff}_i(S, \pi, \succeq) = \max \left\{ \textsc{diff}_i(T, \pi, \succeq) \mid T \in \mathcal{N}_i, \pi(i) \succeq'_i T \right\}.$$

By definition of $(\pi, \succeq)$-normalized profile, $\pi(i) \succ'_i T'$ and $T' \succ_i \pi(i)$ if and only if $T' \in \{T \in \mathcal{N}_i \mid S \succeq_i T \succ_i \pi(i)\}$. Since $\pi(i) \succ'_i T'$ implies $T' \not\succ'_i \pi(i)$, we have

$$\begin{aligned} | \succeq_i \setminus \succeq'_i | = | \succeq'_i \setminus \succeq_i | &= |\{T \in \mathcal{N}_i \mid S \succeq_i T \succ_i \pi(i)\}| \\ &= \textsc{diff}_i(S, \pi, \succeq) \end{aligned}$$

Therefore, $\delta_i(\succeq, \succeq') = \max \left\{ \textsc{diff}_i(T, \pi, \succeq) \mid T \in \mathcal{N}_i, \pi(i) \succeq'_i T \right\}$ for each $i \in N$, which implies the claim. ■

Observe that each $(\pi, \succeq)$-normalized profile $\succeq'$ can be uniquely determined (and constructed) by specifying the value of $\delta_i(\succeq, \succeq')$ for each $i \in N$ in such a way that, for each $i \in N$,

- $\textsc{rank}_i(\pi(i), \succeq') = \textsc{rank}_i(\pi(i), \succeq) + \delta_i(\succeq, \succeq')$,
- $\textsc{rank}_i(S, \succeq') = \textsc{rank}_i(S, \succeq) - 1$ if $0 < \textsc{diff}_i(S, \pi, \succeq) \leq \delta_i(\succeq, \succeq')$, and

- $\text{RANK}_i(S, \succeq') = \text{RANK}_i(S, \succeq)$ otherwise.

In other words, for each $i \in N$, $\pi(i)$ is raised $\delta_i(\succeq, \succeq')$ positions in player i's preference.

Let $(N, \succeq)$ be the original hedonic game, and π be a partition of N.

Lemma 3 *If π is core stable in $(N, \succeq')$ for some profile $\succeq'$, then there exists a $(\pi, \succeq)$-normalized profile $\succeq''$ such that $\delta(\succeq, \succeq'') \leq \delta(\succeq, \succeq')$ and π is core stable in $(N, \succeq'')$ as well.*

Proof. First, assume that π is core stable in $(N, \succeq')$. It follows that each non-empty coalition $S \subseteq N$ satisfies $\pi(i) \succeq'_i S$ for some $i \in S$, which is equivalent to

$$\bigcup_{i \in N} \{S \in \mathcal{N}_i \mid \pi(i) \succeq'_i S\} = 2^N \setminus \{\emptyset\}.$$

Let $d_i = \max\left\{\text{DIFF}_i(S, \pi, \succeq) \mid S \in \mathcal{N}_i, \pi(i) \succeq'_i S\right\}$ for each $i \in N$. It follows that

$$\left\{S \in \mathcal{N}_i \mid \pi(i) \succeq'_i S\right\} \subseteq \left\{S \in \mathcal{N}_i \mid \text{RANK}_i(S, \succeq) \leq \text{RANK}_i(\pi(i), \succeq) + d_i\right\}.$$

Let $\succeq''$ be the $(\pi, \succeq)$-normalized profile satisfying $\delta_i(\succeq, \succeq'') = d_i$ for each $i \in N$. From Lemma 1 and 2, we have $\delta_N(\succeq, \succeq'') \leq \delta_N(\succeq, \succeq')$. Moreover, by definition of $\succeq''$, we have

$$\left\{S \in \mathcal{N}_i \mid \pi(i) \succeq''_i S\right\} = \left\{S \in \mathcal{N}_i \mid \text{RANK}_i(S, \succeq) \leq \text{RANK}_i(\pi(i), \succeq) + d_i\right\}.$$

It implies that $\bigcup_{i \in N}\{S \in \mathcal{N}_i \mid \pi(i) \succeq''_i S\} = 2^N \setminus \{\emptyset\}$, and therefore, π is core stable in $(N, \succeq'')$ as well. ∎

Lemma 4 *Let $\succeq'$ be an arbitrary $(\pi, \succeq)$-normalized profile. If π is core stable in $(N, \succeq')$, then $\text{I-CORE}\big(N, \succeq; \delta_N(\succeq, \succeq')\big) \neq \emptyset$.*

Proof. Suppose $\succeq'$ is a $(\pi, \succeq)$-normalized profile such that π is core stable in $(N, \succeq')$. In the following, we show that, for each $S \subseteq N$,

$$\max\left\{\delta_i(\succeq, \succeq') - \text{DIFF}_i(S, \pi, \succeq) \mid i \in S\right\} \geq 0.$$

Then, it follows that $x = (x_1, x_2, \ldots, x_n)$ satisfying $x_i = \delta_i(\succeq, \succeq')$ for each $i \in N$ is an element of $\text{I-CORE}\big(N, \succeq; \delta_N(\succeq, \succeq')\big)$, and we are done.

Suppose $S \subseteq N$ is such that $\min\{\textsc{diff}_i(S,\pi,\succeq) \mid i \in S\} = 0$. Since $\delta_i(\succeq,\succeq')$ is non-negative, we have

$$\begin{aligned}\max\{\delta_i(\succeq,\succeq') - \textsc{diff}_i(S,\pi,\succeq) \mid i \in S\} &\geq \max\{-\textsc{diff}_i(S,\pi,\succeq) \mid i \in S\} \\ &= \min\{\textsc{diff}_i(S,\pi,\succeq) \mid i \in S\} \\ &= 0.\end{aligned}$$

Now suppose $S \subseteq N$ is such that $\textsc{diff}_i(S,\pi,\succeq) > 0$ for each $i \in S$, which implies $\textsc{diff}_i(S,\pi,\succeq) = \textsc{rank}_i(S,\succeq) - \textsc{rank}_i(\pi(i),\succeq)$ for each $i \in S$. It implies that $S \neq \pi(i)$ for each $i \in S$, and moreover, from the fact that π is core stable in $(N,\succeq')$, we have $\textsc{rank}_i(\pi(i),\succeq') > \textsc{rank}_i(S,\succeq')$ for some $i \in S$. Since $\succeq'$ is a $(\pi,\succeq)$-normalized profile, we have $\textsc{rank}_i(S,\succeq') = \textsc{rank}_i(S,\succeq) - 1$ and $\textsc{rank}_i(\pi(i),\succeq') = \textsc{rank}_i(\pi(i),\succeq) + \delta_i(\succeq,\succeq')$. Therefore,

$$\begin{aligned}\delta_i(\succeq,\succeq') - \textsc{diff}_i(S,\pi,\succeq) &= \delta_i(\succeq,\succeq') - \big(\textsc{rank}_i(S,\succeq) - \textsc{rank}_i(\pi(i),\succeq)\big) \\ &= \textsc{rank}_i(\pi(i),\succeq') - (\textsc{rank}_i(S,\succeq') + 1) \\ &\geq 0,\end{aligned}$$

which implies $\max\{\delta_i(\succeq,\succeq') - \textsc{diff}_i(S,\pi,\succeq) \mid i \in S\} \geq 0$. ■

Lemma 5 *If* $\textsc{i-core}\big(N,\succeq;B\big) \neq \emptyset$*, then there exists a* $(\pi,\succeq)$*-normalized profile* $\succeq'$ *such that* π *is core stable in* $(N,\succeq')$ *and* $\delta_N(\succeq,\succeq') \leq B$*.*

Proof. Let $x \in \textsc{i-core}\big(N,\succeq;B\big)$, and let $\succeq^x$ be the $(\pi,\succeq)$-normalized profile such that, $\delta_i(\succeq,\succeq^x) = \min\{x_i, 2^n - \textsc{rank}_i(\pi(i),\succeq)\}$ for each $i \in N$. Hence, we have $\delta_N(\succeq,\succeq^x) \leq x(N) = B$. Observe that, for each $i \in N$, we have $\textsc{rank}_i(\pi(i),\succeq^x) = \textsc{rank}_i(\pi(i),\succeq) + x_i$ or $\textsc{rank}_i(\pi(i),\succeq^x) = 2^{n-1}$, i.e., $\pi(i)$ is raised x_i positions or put at the top of player i's preference list. In the following, we show that π is core stable in $(N,\succeq^x)$.

By definition of $\textsc{i-core}$, we have, for each $S \subseteq N$, there exists $i \in S$ such that $x_i - \textsc{diff}_i(S,\pi,\succeq) \geq 0$. If $\textsc{diff}_i(S,\pi,\succeq) = 0$, then we have $\pi(i) \succeq_i S$; otherwise, i.e., $\textsc{diff}_i(S,\pi,\succeq) > 0$, we have

$$x_i - \big(\textsc{rank}_i(S,\succeq) - \textsc{rank}_i(\pi(i),\succeq)\big) \geq 0.$$

When $\delta_i(\succeq,\succeq^x) = x_i$, we have $\textsc{rank}_i(\pi(i),\succeq^x) = \textsc{rank}_i(\pi(i),\succeq) + x_i$, and thus,

$$\textsc{rank}_i(\pi(i),\succeq^x) \geq \textsc{rank}_i(S,\succeq) \geq \textsc{rank}_i(S,\succeq^x),$$

which implies $\pi(i) \succeq_i^x S$. When $\delta_i(\succeq, \succeq^x) \neq x_i$, we have $\text{RANK}_i(\pi(i), \succeq^x) = 2^{n-1}$, i.e., $\pi(i)$ at the top of player i's preference list with respect to $\succeq^x$, and obviously, $\pi(i) \succeq_i S$. Therefore, π is core stable in $(N, \succeq^x)$. ∎

From Lemma 4 and 5, our main result is obtained.

Proposition 6 *Let $B_{\min}$ be the smallest non-negative budget such that* $\text{I-CORE}(N, v_\pi; B_{\min}) \neq \emptyset$. *Then, $\delta_N(\succeq, \succeq') \geq B_{\min}$ holds for each profile $\succeq'$ such that π is core stable in $(N, \succeq')$, and moreover, this lower bound $B_{\min}$ is tight.*

5 Further research

Our approach allows to completely order all partitions of a player set N with respect to the extent to which they are stable in a pre-specified hedonic game $(N, \succeq)$. More precisely, for each partition π of N, one can uniquely assign the minimal budget $B_{\min}^{\pi}$ such that $\text{I-CORE}(N, v_\pi; B_{\min}^{\pi}) \neq \emptyset$. Then, naturally, the complete order $\trianglerighteq$ over the set of all partitions of N is defined as follows: for any two partitions π and π', $\pi \trianglerighteq \pi'$ if and only if $B_{\min}^{\pi} \leq B_{\min}^{\pi'}$. Clearly, if the hedonic game has a non-empty core, then the top indifference class of $\trianglerighteq$ contains all core stable partitions in $(N, \succeq)$.

Noticing that various matching problems can be seen as particular specifications of a hedonic game, our idea of ordering the partitions with respect to their stability could be useful for instance as to compare mechanisms (assigning partitions to each profile of preferences) in these specific matching problems with respect to their stability properties. For instance, one could say that a mechanism f is more stable than the mechanism g if, at each preference profile, $\pi_f \trianglerighteq \pi_g$ with π_f and π_g being the partitions induced by f and g, respectively. It may clearly happen that two mechanisms are not comparable with respect to their stability and so, looking for comparable mechanisms seems for us to be a valuable direction for further research.

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