Supplemental Material

for *Stable core symmetries and confined textures for a vortex line in a spinor Bose-Einstein condensate*

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In this Supplemental Material we provide additional discussion of the breaking of axisymmetry in the core of a stable singly quantized vortex in the polar phase of the spin-1 BEC. We also give a brief overview of the Skyrmion textures in different dimensions.

BREAKING OF VORTEX-CORE AXISYMMETRY

In the main text, we show that the singly quantized vortex without internal structure can exhibit different, energetically stable core symmetries as the (spatially uniform) Zeeman shifts are varied. When the level shifts are weak, energy relaxation leads to spontaneous breaking of axial symmetry in the core, splitting the vortex into two half-quantum vortices, as predicted in Ref. [1]. This splitting of the vortex core was recently experimentally observed for a ^{23}Na spin-1 BEC of 3.5×10^6 atoms in an oblate trap with $(\omega_x,\omega_y,\omega_z)=2\pi\times(4.2,5.3,480)$ Hz [2]. In this experiment, singly quantized vortices were created in a condensate initially occupying only the $m=0$ Zeeman level. A spin rotation is then applied by tuning the quadratic Zeeman shift (induced using microwave dressing [3]) to transfer population to the $m=\pm1$ levels. After the spin rotation, the vortices are observed to split into half-quantum vortex pairs with opposite core spin polarization. The resulting half-quantum vortices were identified by *in situ* imaging [2], in which spin-dependent phase-contrast imaging is used to map out the condensate magnetization. The oppositely magnetized FM cores of the half-quantum vortices can then be discerned.

The breaking of axisymmetry and splitting of the singly quantized vortex is made possible by the (uniaxial) *nematic order* exhibited by the polar phase of the spin-1 BEC, which allows the existence of half-quantum vortices. The polar order parameter may be expressed in terms of a condensate phase τ and a unit vector $\hat{\mathbf{d}}$ as [4]

$$\zeta=\frac{e^{i\tau}}{\sqrt{2}}\begin{pmatrix} d_x+id_y\\ \sqrt{2}d_z\\ d_x-id_y\end{pmatrix}=\frac{e^{i\tau}}{\sqrt{2}}\begin{pmatrix} -e^{-i\alpha}\sin\beta\\ \sqrt{2}\cos\beta\\ e^{i\alpha}\sin\beta\end{pmatrix}. \tag{S1}$$

(In the last expression $\hat{\mathbf{d}}$ has been parametrized in terms of azimuthal and polar angles α and β, for later convenience). Note that $\zeta(\tau,\hat{\mathbf{d}})=\zeta(\tau+\pi,-\hat{\mathbf{d}})$. These two states must therefore be identified, and the vector $\hat{\mathbf{d}}$ is understood as an unoriented *nematic axis*. Rotations of $\hat{\mathbf{d}}$ do not contribute to the superfluid flow, making it possible to form a vortex carrying half a quantum of circulation by letting a π winding of τ around the vortex line be accompanied by a $\hat{\mathbf{d}}\to-\hat{\mathbf{d}}$ rotation of the nematic axis, keeping the order parameter single-valued. For example, taking $d_z=0$ for simplicity, a half-quantum vortex can be written as

$$\zeta=\frac{e^{i\varphi/2}}{\sqrt{2}}\begin{pmatrix} -e^{-i\varphi/2}\\ 0\\ e^{i\varphi/2}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} -1\\ 0\\ e^{i\varphi}\end{pmatrix}, \tag{S2}$$

where φ is the azimuthal coordinate around the vortex line. Nematic order also gives rise to half-quantum vortices in, e.g., the A phase of superfluid liquid ^{3}He [5], and to π-disclinations in nematic liquid crystals [6]. The name is also sometimes used in the context of exciton-polariton condensates in reference to a vortex with a π rotation of linear polarization of the photon component [7, 8]. This does not, however, arise from nematic order, but is more reminiscent of a topologically very different coreless vortex in a two-component BEC [9].

We can now understand the splitting of the singly quantized vortex given by Eq. (3) of the main text as

$$\frac{e^{i\varphi}}{\sqrt{2}}\begin{pmatrix} -e^{-i\varphi}\sin\beta\\ \sqrt{2}\cos\beta\\ e^{i\varphi}\sin\beta\end{pmatrix}\to$$
$$\frac{e^{i\varphi_1/2}}{\sqrt{2}}\begin{pmatrix} -e^{-i\varphi_1/2}\sin\beta\\ \sqrt{2}\cos\beta\\ e^{i\varphi_1/2}\sin\beta\end{pmatrix}\oplus\frac{e^{i\varphi_2/2}}{\sqrt{2}}\begin{pmatrix} -e^{-i\varphi_2/2}\sin\beta\\ \sqrt{2}\cos\beta\\ e^{i\varphi_2/2}\sin\beta\end{pmatrix}. \tag{S3}$$

Here the spinors on the right-hand side represent half-quantum vortices [cf. Eq. (S2)]. In these, $\varphi_{1,2}$ are the azimuthal angles relative to each vortex line, and the spinors describe the wave function locally around each vortex core. Away from the core region, the wave function still corresponds to the original singly quantized vortex. (Note that $\oplus$ here indicates the addition of topological defects.)

SKYRMIONS AND BABY SKYRMIONS

Nonsingular textures may be topologically nontrivial by considering maps from a compactified real space to the compact order-parameter space. When the order parameter reaches the same value everywhere sufficiently far away from the (particlelike) texture, the entire boundary enclosing the texture may be identified and the volume in $\mathbb{R}^3$ becomes topologically S^3 (a unit sphere in four dimensions). One may then think of the $S^3\to S^3$ map as distributing (an integer number of copies of) the full order-parameter space over the compactified real space. The corresponding nontrivial textures are the 3D Skyrmions [10]. Analogous structures may be constructed in a two-component BEC [11].

An S^2 order-parameter space, may similarly be distributed over a (2D) real-space surface with fixed, uniform boundary conditions (corresponding to an $S^2 \to S^2$ map). Such a 2D Skyrmion is commonly referred to as a (2D) "baby Skyrmion", being the topologically lower-dimensional analog of the full 3D Skyrmion, and may be realized as a coreless vortex [12–15]. The dimensionality of the baby Skyrmion may be further reduced by considering an S^1 order parameter. For uniform boundary conditions such that 1D space can be compactified to S^1, the resulting $S^1 \to S^1$ map defines a 1D baby Skyrmion. In our system a ferromagnetic spin texture confined inside the core of a vortex line exhibits fixed boundary conditions. As the boundary conditions are twisted (the orientation of the spin vector differs by π in the two ends of the vortex line, the 1D Skyrmion winding number is equal to 1/2. Any further winding of the spin texture would lead to higher Skyrmion winding numbers.

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