On the unicity of types for representations of reductive p-adic groups

Peter Latham

August, 2016

A thesis submitted to the School of Mathematics of the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy (PhD)

Abstract

We consider the unicity of types for various classes of supercuspidal representations of reductive p-adic groups, with a view towards establishing instances of the inertial Langlands correspondence. We introduce the notion of an archetype, which we define to be a conjugacy class of typical representations of maximal compact subgroups.

In the case of supercuspidal representations of a special linear group, we generalize the functorial results of Bushnell and Kutzko relating simple types in $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$ to cover all archetypes; from this we deduce that any archetype for a supercuspidal representation of $\mathbf{SL}_N(F)$ is induced from a maximal simple type. We then provide an explicit description of the number of archetypes contained in a given supercuspidal representation of $\mathbf{SL}_N(F)$. We next consider depth zero supercuspidal representations of an arbitrary group, where we are able to show that the only archetypes are the depth zero types constructed by Morris.

We end by showing that there exists a unique inertial Langlands correspondence from the set of archetypes contained in regular supercuspidal representations to the set of regular inertial types. In the cases of $\mathbf{SL}_N(F)$ and depth zero supercuspidals of arbitrary groups, we describe completely the fibres of this inertial correspondence; in general, we formulate a conjecture on how these fibres should look for all regular inertial types.

Contents

Acknowledgements

First and foremost, I would like to thank Shaun Stevens for his excellent supervision over the last three years. I have learned an incredible amount from him very quickly, and truly could not have asked for a better supervisor.

I have benefited, at various points from helpful discussions with and and advice from Jeffrey Adler, Colin Bushnell, David Helm, Guy Henniart, Joshua Lansky, Vanessa Miemietz, Santosh Nadimpalli, Tom Oliver, Jack Shotton, Daniel Skodlerack and Maarten Solleveld.

My time in Norwich has been particularly enjoyable; I thank all of my friends and colleagues for helping to make it so.

Finally, I wish to thank my parents for their constant support and encouragement.

Chapter 1

Introduction

1.1 Overview

Let F be a non-archimedean local field of residual characteristic p. Given a prime $\ell \neq p$, the local Langlands conjectures – which are now known to hold in many significant cases – predict a natural parametrization of various classes of continuous ℓ-adic representations of the absolute Weil group of F in terms of the smooth ℓ-adic irreducible representations of the locally profinite groups of rational points of connected reductive algebraic groups defined over F. We call such groups *p-adic groups* for short. The theory of Galois representations has a well-deserved reputation for difficulty – in particular, there are few general constructions of such representations (and those which are known, such as via ℓ-adic étale cohomology, are difficult to work with), and these constructions tend to be rather non-uniform as p varies. In contrast, however, the representation theory of p-adic groups admits a number of explicit constructions. One motivating hope for the work in this thesis is that a sufficiently detailed understanding of explicit constructions in the representation theory of p-adic groups should allow one to, via the local Langlands correspondence, obtain new, explicit, information about Galois representations.

The representation theory of p-adic groups is closely related to the structure of the underlying algebraic group. In particular, one obtains a natural notion of parabolic subgroups of a p-adic group $G = \mathbf{G}(F)$ and their Levi factors, which allows one to consider the process of *parabolic induction*. Since the proper Levi subgroups of G are of strictly smaller semisimple rank, their representation theory should be, in some sense, simpler. Parabolic induction provides an extremely well-behaved collection of functors from the categories of finite-length representations of proper Levi subgroups of G, to the category of finite-length representations of G. However, not every irreducible representation of G arises as a subquotient of some parabolically induced representation. We say that the representations which do not come from parabolic induction are *supercuspidal*. This leads to a two-part strategy for classifying the irreducible representations of G: firstly, one constructs the supercuspidal representations of all of the Levi subgroups of G (including G itself), and then one decomposes the resulting parabolically induced representations into irreducibles.

While neither of these steps is easy to deal with, the construction of supercuspidal representations presents the more immediate problem of the two. Due to ideas originating with Howe, it is expected that, given a supercuspidal representation π of G, there should exist an open, compact-modulo-centre subgroup $\tilde{J}$ of G and an irreducible representation Λ of $\tilde{J}$ such that π is isomorphic to the representation of G compactly induced from Λ. However, this conjecture has a well-deserved reputation for difficulty, and remains wide-open in general. Despite this, there has been substantial progress in many cases: the conjecture is now known when G is a general or special linear group due to Bushnell and Kutzko [BK93a, BK93b, BK94], a classical group (i.e. special orthogonal, symplectic or unitary) when p is odd due to Stevens [Ste08], an inner form of a general linear group due to Sécherre and Stevens [Séc05, SS08], for the "depth zero" supercuspidal representations of an arbitrary group due to Moy–Prasad and Morris [MP94, MP96, Mor99], and for the "tamely ramified" supercuspidal representations of an arbitrary group due to Yu and Kim [Yu01, Kim07]. In this latter case, we only know that all such supercuspidals arise via these constructions provided the field F satisfies certain assumptions, which in particular are satisfied when p is sufficiently large relative to the semisimple rank of $\mathbf{G}$. Recently, a construction has been given for the "epipelagic" supercuspidal representations by Reeder and Yu, which is valid for any group provided that p is sufficiently large [RY14]; the condition on p has since been removed by Fintzen [Fin15].

In each of these cases, the construction proceeds by first constructing a *type* for each of the supercuspidal representations. Given a supercuspidal representation π of a p-adic group G, a type for π is a pair (J, λ) consisting of a compact open subgroup J of G and an irreducible representation λ of J such that any other irreducible (not necessarily supercuspidal) representation of G with a non-zero λ-isotypic component must be isomorphic to $\pi \otimes \omega$, for some unramified character ω of G. This may be rephrased by saying that the irreducible representations with non-zero λ-isotypic component are precisely those in the same subcategory as π in the block decomposition of the category of smooth representations of G due to Bernstein and Deligne [Ber84]; in this way one also obtains a definition of types for non-cuspidal representations. Continuing to focus on the supercuspidal case, there is a close relationship between types and the compact inducing data that we expect to exist for supercuspidals. Given such a datum $(\tilde{J}, \Lambda)$ and a supercuspidal representation $\pi = \text{c-}\operatorname{Ind}_{\tilde{J}}^{G} \Lambda$, an irreducible subrepresentation λ of the restriction of Λ to its maximal compact subgroup will be typical for π. However, from the abstract definition of a type it is not possible to show that a type (J, λ) extends to a compact inducing datum for π – in order for this to be the case, one needs further, explicit information on the structure of (J, λ).

In particular, no such problems arise for the known constructions of types. We expect that there should not exist any types which do not arise from such compact inducing data, and that these compact inducing data should be essentially unique (which is to say, unique up to conjugacy and induction to intermediate open, compact-modulo-centre subgroups). These expectations are the basic idea behind the central idea of this thesis: what we call the *unicity of types*.

Before elaborating on this, let us consider the implications of these ideas in terms of the local Langlands correspondence, which (when restricted to the supercuspidal representations of G) takes the form of a surjective, finite-to-one map from the set $\mathrm{Cusp}(G)$ of isomorphism classes of supercuspidal representations of G to a certain set $\mathcal{L}(G)$ of L-parameters for G (i.e. certain homomorphisms from the Weil group of G to the Langlands dual of G). Suppose that, to each $\pi \in \mathrm{Cusp}(G)$, we can associate at least one type (J_π, λ_π) for π; the idea is that this should correspond to something on the Galois side of the picture. The obvious guess is that this should correspond to the restriction to the inertia group I_F of F of the L-parameter associated to π by the local Langlands correspondence. Indeed, this is essentially the case: given a type (J, λ), we may pick an arbitrary supercuspidal π in which (J, λ) is contained, apply the local Langlands correspondence, and restrict the resulting L-parameter to the inertia group. It is easy to see that the resulting map is independent of the choice of π, and so we obtain a map from the set of types contained in supercuspidal representations to a certain set of homomorphisms from I_F to the Langlands dual of G; we call the image of this map the set of *cuspidal inertial types*.

There is, however, an immediate problem with this. In order for this to be a useful procedure, it should behave in much the same way as the local Langlands correspondence. In particular, it is desirable that this map is surjective and finite-to-one; but it is easy to see that, in the above formulation, this cannot be the case – given a type (J, λ), one may conjugate (J, λ) by an arbitrary element of G, or induce λ to a compact subgroup of G containing J and take an irreducible component. Both of these processes result in a new type which will be mapped to the same inertial type as (J, λ). In particular, the fibres of this map will almost always be infinite.

The solution is to instead form the quotient by the equivalence relation which identifies such closely related types, and consider the resulting correspondence. This is equivalent to considering what we call *archetypes*: G-conjugacy classes of types which are defined on maximal compact subgroups of G. We then get a map from the set of archetypes to the set of cuspidal inertial types, which we hope to be surjective and finite-to-one. Recalling that we expect that all types for supercuspidal representations arise from an essentially unique collection of compact inducing data, we are naturally led to conjecture the following:

Conjecture (The unicity of types for supercuspidals)**.** *Let π be a supercuspidal representation of a p-adic group G. There exists an archetype for π and, given a fixed maximal compact subgroup K of G, there exists at most one archetype for π defined on K.*

The point of this conjecture is that this gives us extremely strong control over the fibres of the above map from archetypes to cuspidal inertial types which, moreover, is seen to be surjective; if we have a sufficiently well-behaved such map, we refer to it as an *inertial Langlands correspondence* for G.

This is a generalization of the unicity of types for $\mathbf{GL}_N(F)$ as considered by Henniart and Paškūnas [BM02, Pas05]. Since $\mathbf{GL}_N(F)$ contains a unique conjugacy class of maximal compact subgroups, it is not necessary for them to take as complex a setup as the one we use. However, it is easy to see that they prove precisely the above conjecture for all supercuspidal representations of $\mathbf{GL}_N(F)$, and deduce an inertial Langlands correspondence – which, in this case, takes the form of a bijection.

While the inertial Langlands correspondence is of interest in its own right, it has also found many applications around the Langlands programme, particularly when one wishes to consider modular representations. While this thesis will only be concerned with representations taking characteristic zero coefficients, there are analogous Langlands correspondences for positive characteristic coefficient fields; it is natural to ask whether these correspondences are related. Through the Breuil–Mézard conjecture [BM02], the inertial Langlands correspondence has been seen to be intimately related with such questions, allowing one to describe congruence between Galois representations in terms of the theory of types. One motivation of the work in this thesis is that it should allow for the possibility of obtaining such relations for groups other than $\mathbf{GL}_N(F)$.

While we will, for the most part, only consider supercuspidal representations during this thesis, much of the above can be made to work for arbitrary irreducible representations. Since parabolic induction is the natural means of constructing non-cuspidal representations, it is natural to ask whether there is a related process for, from types for supercuspidal representations of Levi subgroups of G, producing types for the resulting parabolically induced representations of G. Indeed, there is such a procedure, which is given by the Bushnell–Kutzko theory of covers [BK98]: this reduces the problem to constructing a cover of each type for a supercuspidal representation of a Levi subgroup of G. In many cases, this has been completed: for general linear groups by Bushnell and Kutzko [BK99], for special linear groups by Goldberg and Roche [GR02], for inner forms of general linear groups by Sécherre and Stevens [SS12], for classical groups when p is odd by Stevens and Miyauchi [MS14], and for depth zero representations of arbitrary groups by Morris [Mor99].

However, there are a number of complications which currently prevent us from being able to formulate such general conjectures as those above for non-cuspidal representations. Indeed, if one drops the adjective "supercuspidal" from our conjecture, then it is easily seen to fail even for $\mathbf{GL}_2(F)$, where the Steinberg representation admits as types both the trivial representation of $\mathbf{GL}_2(\mathcal{O})$ and the representation of $\mathbf{GL}_2(\mathcal{O})$ inflated from the Steinberg representation of its reductive quotient. It seems that such conjectures would be heavily reliant on completely general, uniform constructions of types which satisfy certain additional properties; at the time of writing such constructions are beyond reach. Nonetheless, in the case of $\mathbf{GL}_N(F)$, the unicity of types has been completely described by Henniart when $N = 2$ [BM02], and recently by Nadimpalli in most other cases for arbitrary N [Nad14, Nad15]. We will make some brief comments on how it should be possible to generalize this to special linear groups.

1.2 Summary of results

The major results of this thesis are a combination of the results of three papers: [Lat16c], [Lat16a] and [Lat16b]. Of these, the second is a generalization of the first, the results of which correspond to those in Chapter 4, and the third considers a different question, corresponding to Chapter 5. Chapter 6 consists of an interpretation of the results of the previous two chapters in terms of Galois theory.

1.2.1 Chapter 4

In this chapter, we consider the unicity of types for supercuspidal representations of special linear groups. Since this problem is well-understood for general linear groups, the approach is to exploit the close connection between the representation theories of these groups in order to establish a form of functoriality for types, which allows us to transfer over the unicity result from $\mathbf{GL}_N(F)$.

The starting point is the construction of maximal simple types due to Bushnell and Kutzko: these are a collection of types, providing precisely one conjugacy class of such contained in each supercuspidal representation of $\mathbf{GL}_N(F)$ which, essentially by Clifford theory, then leads to a similar collection of types for the supercuspidal representations of $\mathbf{SL}_N(F)$. In particular, there is a natural "functorial" relationship between these two collections of types, corresponding to the inclusion $\mathbf{SL}_N \hookrightarrow \mathbf{GL}_N$. On the other hand, the unicity of types is known for $\mathbf{GL}_N(F)$: for each supercuspidal representation π of $\mathbf{GL}_N(F)$, there exists a unique π-archetype, which is represented by a representation of the form $(\mathbf{GL}_N(\mathcal{O}), \tau)$, where τ is induced from a maximal simple type contained in π. We therefore ask whether the functoriality for maximal simple types extends to archetypes.

So one considers the following setup: given a supercuspidal representation π of $\mathbf{GL}_N(F)$ and an irreducible subquotient $\bar{\pi}$ of $\pi\downarrow_{\mathbf{SL}_N(F)}$, suppose that we are also given an archetype for $\bar{\pi}$, which for the sake of simplicity we assume in this discussion to be of the form $(\mathbf{SL}_N(\mathcal{O}), \bar{\tau})$ (although there are N conjugacy classes of maximal compact subgroups of $\mathbf{SL}_N(F)$ which must be considered, these are all conjugate under the action of $\mathbf{GL}_N(F)$). The functorial relation which we would expect to hold is that, among the irreducible subrepresentations of the infinite-length representation of $\mathbf{GL}_N(\mathcal{O})$ induced from $\bar{\tau}$, there exists an archetype for π. Certainly, it is possible to identify among these representations a canonical candidate Ψ for such an archetype, and it is a trivial matter to see that, at the very least, Ψ cannot be too far away from being a π-archetype. Indeed, one sees that Ψ may only be contained in irreducible representations of the form $\pi \otimes (\chi \circ \det)$, where χ is a multiplicative character of F. Moreover, it is also easy to see that χ^N must be unramified (Lemma 4.1.2).

So it remains to us to argue by contradiction and show that χ itself must be unramified. This is the major technical step in the proof, and is one which we are currently only able to complete under the additional assumption that π is *essentially tamely ramified*; in the general case one encounters significant problems of an arithmetic nature, due to the possibility that there can exist wildly ramified order N characters which leave the simple type contained in π invariant under twisting. The approach is to make use of Paškūnas' approach for $\mathbf{GL}_N(F)$: since the representation Ψ is contained in $\pi\downharpoonright_{\mathbf{GL}_N(\mathcal{O})}$, we follow Paškūnas by splitting the irreducible subrepresentations of $\pi\downharpoonright_{\mathbf{GL}_N(\mathcal{O})}$ into three classes: the unique typical representation, the "type A" subrepresentations, and the "type B" subrepresentations. We show that if Ψ were of type A or B then we would obtain a contradiction, and come to our first important result:

Theorem (4.6.1)**.** *The representation Ψ is a π-archetype.*

This allows us to easily transfer the unicity results for $\mathbf{GL}_N(F)$ over to $\mathbf{SL}_N(F)$: since the unique π-archetype is of the form $\mathrm{Ind}_J^{\mathbf{GL}_N(\mathcal{O})}\ \lambda$ for some maximal simple type (J, λ), we obtain a non-zero map $\bar{\tau} \to \mathrm{Res}_{\mathbf{SL}_N(\mathcal{O})}^{\mathbf{GL}_N(\mathcal{O})} \mathrm{Ind}_J^{\mathbf{GL}_N(\mathcal{O})}\ \lambda$. Applying Bushnell and Kutzko's results on simple types for $\mathbf{SL}_N(F)$, we obtain the unicity of types for $\mathbf{SL}_N(F)$:

Theorem (4.7.2)**.** *Any archetype for an essentially tame supercuspidal representation $\bar{\pi}$ of $\mathbf{SL}_N(F)$ is induced from a maximal simple type. In particular, there exists at most one $\bar{\pi}$-archetype on each conjugacy class of maximal compact subgroups of $\mathbf{SL}_N(F)$, which is contained in $\bar{\pi}$ with multiplicity one.*

With this in place, an obvious follow-up question is to ask precisely how many $\bar{\pi}$-archetypes there are. From the above, it is clear that this is equivalent to counting the number of inclusions of the group on which the maximal simple type for $\bar{\pi}$ is defined into the various maximal compact subgroups of $\mathbf{SL}_N(F)$. This turns out to have an extremely satisfying answer. Given a supercuspidal representation $\bar{\pi}$ of $\mathbf{SL}_N(F)$ and a supercuspidal representation π of $\mathbf{GL}_N(F)$ such that $\bar{\pi}$ is contained in $\pi \downharpoonright_{\mathbf{SL}_N(F)}$, one defines the *ramification degree* $e_{\bar{\pi}}$ of $\bar{\pi}$ by requiring that there are $N/e_{\bar{\pi}}$ unramified characters χ of $F^\times$ such that $\pi \simeq \pi \otimes (\chi \circ \det)$. This leads to the following:

Theorem (4.7.5)**.** *Let $\bar{\pi}$ be an essentially tame supercuspidal representation of $\mathbf{SL}_N(F)$. Then there exist precisely $e_{\bar{\pi}}$ archetypes for $\bar{\pi}$. Moreover, any two such archetypes are conjugate under the action of $\mathbf{GL}_N(F)$.*

This gives us a rather complete and explicit understanding of the theory of types for supercuspidal representations of $\mathbf{SL}_N(F)$. While we are currently unable to make quite as much progress for non-cuspidal representations, we end by making some remarks on the extent to which these methods generalize.

1.2.2 Chapter 5

This chapter establishes the unicity of types for depth zero supercuspidal representations of an arbitrary p-adic group $G = \mathbf{G}(F)$, i.e. those supercuspidal representations with a non-zero vector fixed by the maximal pro-p open normal subgroup of a parahoric subgroup of G. The quotient of a parahoric subgroup by this pro-p radical is then naturally a finite reductive group over the residue field of F; through this quotient, depth zero representations admit a natural description in terms of the representation theory of finite reductive groups, things are sufficiently simple for us to make progress.

The starting point here is the construction of the *unrefined depth zero types* for these constructions due to Moy–Prasad and Morris: these are pairs (G_x, σ), where G_x is a maximal parahoric subgroup of G, and σ is inflated from a cuspidal irreducible representation of a certain quotient of σ which identifies with a reductive group over the residue field of F. We know that every depth zero supercuspidal representation of G contains a unique conjugacy class of such types. However, we should point out that if G_x is not maximal as a compact subgroup of G, then these may only be types in a slightly more general sense than that discussed thus far. In practice, however, this distinction is of little importance.

We approach the problem of unicity in two steps. We first establish the analogue of unicity on the level of these unrefined depth zero types, before performing an extension step to the corresponding maximal compact subgroup – which will be the maximal compact subgroup of the normalizer of G_x, which leads to unicity in the usual sense.

For the first of these steps, we make use of connections with the representation theory of finite reductive groups and the Bushnell–Kutzko theory of covers. We proceed in the usual way: given a depth zero supercuspidal representation π of G and an unrefined depth zero type (G_x, σ) for π, let G_y denote an arbitrary maximal parahoric subgroup of G. Then we may embed $\pi \downharpoonright_{G_y}$ into the representation $\mathrm{Res}_{G_y}^G \operatorname{c-Ind}_{G_x}^G \ \sigma$, and perform a Mackey decomposition on this latter representation. In this decomposition, we identify a family of conjugates of σ (which will be empty unless G_y is conjugate to G_x), along with a family of irreducible representations Ξ which are induced from the inverse image in G_y of *proper* parabolic subgroups of the reductive quotient of G_y. This eventually leads us to, for each such Ξ, the construction of finitely many unrefined depth zero types for proper Levi subgroups of G. We show that Ξ must intertwine with the Bushnell–Kutzko cover of one of these types, which leads to our preliminary form of unicity:

Theorem (5.3.1)**.** *Let π be a depth zero supercuspidal representation of G, and let (G_x, σ) be an unrefined depth zero type contained in π. Any representation of a parahoric subgroup of G which is typical for the same Bernstein components as (G_x, σ) must be conjugate to σ.*

While the basic idea behind this result is a rather simple one, the proof ends up being quite technical, requiring a lot of work involving Bruhat–Tits theory, which in essence reduces the problem to choosing a compatible system of embeddings of buildings of various Levi subgroups of G into that of G.

It then remains for us to extend this result to cover archetypes in the usual sense. Given the above result, this is quite straightforward, and leads to the following:

Theorem (5.4.2)**.** *Let π be a depth zero supercuspidal representation of G. Then there exists a unique π-archetype, which is induced from an unrefined depth zero type contained in π.*

1.2.3 Chapter 6

This chapter consists of a reformulation of the results of the previous two in terms of the inertial Langlands correspondence. Recall that we expect this to take the form of a surjective, finite-to-one map from the set of archetypes contained in supercuspidal representations of $G = \mathbf{G}(F)$ to the set of cuspidal inertial types for G.

We first consider the case of $\mathbf{SL}_N(F)$. We begin by establishing a form of converse to Theorem 4.6.1, which is considerably easier to do:

Theorem (6.2.2)**.** *Let π be a supercuspidal representation of $\mathbf{GL}_N(F)$, and let $(\mathbf{GL}_N(\mathcal{O}), \tau)$ be the unique π-archetype. For any irreducible subquotient $\bar{\pi}$ of $\pi \downharpoonright_{\mathbf{SL}_N(F)}$, there exist a $g \in G$ and an irreducible subrepresentation $\bar{\tau}$ of ${}^g\tau \downharpoonright_{{}^g\mathbf{SL}_N(\mathcal{O})}$ such that $({}^g\mathbf{SL}_N(\mathcal{O}), \bar{\tau})$ is a $\bar{\pi}$-archetype.*

This allows us to justify our claim that our approach has been via functoriality between $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$: when combined with Theorem 4.6.1, it gives a relationship between archetypes and L-packets for $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$ (Theorem 6.2.4). From this, it is a simple matter to obtain a complete description of the inertial Langlands correspondence for $\mathbf{SL}_N(F)$:

Theorem (6.2.5)**.** *For $G = \mathbf{SL}_N(F)$, the inertial Langlands correspondence is a surjective map with finite fibres. Each of its fibres consists of a single orbit of archetypes under $\mathbf{GL}_N(F)$-conjugacy. If $\bar{\pi}$ is a supercuspidal representation of $\mathbf{SL}_N(F)$ contained in $\pi \downharpoonright_{\mathbf{SL}_N(F)}$ for some supercuspidal representation π of $\mathbf{GL}_N(F)$, and φ is the inertial type obtained by restricting the L-parameter associated to $\bar{\pi}$ to the inertia group, then the fibre* *above φ is of cardinality* $e_{\bar{\pi}} \cdot \mathrm{length}(\pi \downharpoonright_{\mathbf{SL}_N(F)})$.

Moreover, there is a natural functorial diagram relating the inertial Langlands correspondences for $\mathbf{GL}_N(F)$ *and* $\mathbf{SL}_N(F)$.

Next, we describe what we call the *tame* inertial Langlands correspondence, by which we mean the inertial correspondence for depth zero supercuspidal representations of arbitrary p-adic groups $G = \mathbf{G}(F)$. Here, we obtain a similar statement, expressing the order of the fibres of the inertial Langlands correspondence in terms of those of the local Langlands correspondence, although we see that it is possible for a new form of complication to arise, due to disconnectefdness phenomena on the level of parahoric subgroups. To each depth zero supercuspidal representation π, we associate a finite set $\mathfrak{S}_\pi$ of inertia classes of depth zero supercuspidal representations – the set of those classes, the elements of which contain the same unrefined depth zero type as π. We then have the following result:

Theorem (6.3.7)**.** *Suppose that the local Langlands correspondence exists and is unique for the depth zero supercuspidal representations of a p-adic group G. Let φ be an L-parameter for G, and let π be in the L-packet corresponding to φ. Then the fibre of above the restriction to the inertia group of φ is of cardinality $\sum \#\mathfrak{S}_\pi$, as π ranges through the elements of the L-packet above φ.*

Chapter 2

Preliminaries

2.1 Local fields

We fix, once and for all, a non-archimedean local field F of residual characteristic p, by which we mean a locally compact field, complete with respect to a non-archimedean discrete valuation, and such that the residue field is finite of characteristic p. We write $\mathcal{O}_F$ for the ring of integers of F (i.e. its discrete valuation ring), $\mathfrak{p}_F$ for the maximal ideal of $\mathcal{O}_F$, and $\mathbf{k}_F = \mathcal{O}_F/\mathfrak{p}_F$ for the residue field, which we take to be of cardinality q_F. Whenever there is no danger of ambiguity, we will drop the subscript F from the notation. We also write $\|\cdot\|_F$ for the (normalized) multiplicative valuation on F, and $v_F = \log_{q_F} \circ \|\cdot\|_F$ for the additive valuation.

2.2 Reductive p-adic groups

We will be concerned with what we will refer to as *p-adic groups*. Strictly speaking, this is an abuse of terminology: by a p-adic group, we will mean the group $G = \mathbf{G}(F)$ of F-rational points of a connected reductive algebraic group $\mathbf{G}$ defined over F, for F a non-archimedean local field of residual characteristic p. Such a group comes equipped with a natural locally profinite (which is to say locally compact, Hausdorff and totally disconnected) topology: $\mathbf{G}$ admits a faithful algebraic representation $\mathbf{G} \to \mathbf{GL}_N$ for some N, which induces an embedding $G \hookrightarrow \mathbf{GL}_N(F)$. The group $\mathbf{GL}_N(F)$, viewed as a dense subset of the locally profinite space F^{N^2} comes equipped with a locally profinite topology; one then puts the subspace topology on G. This process is independent of the choice of representation $\mathbf{G} \hookrightarrow \mathbf{GL}_N$.

2.3 Smooth representations

Our goal will be to study the representation theory of these p-adic groups. There are three natural families of coefficient rings to consider representations over, indexed by a prime ℓ: the ℓ-adic representations over $\bar{\mathbb{Q}}_\ell$, the corresponding integral representations over $\bar{\mathbb{Z}}_\ell$, and the ℓ-modular representations over $\bar{\mathbb{F}}_\ell$. In each of these cases, there is a further division depending on whether or not ℓ is equal to p. Our focus will be entirely on the case of $\bar{\mathbb{Q}}_\ell$-representations with $\ell \neq p$. More specifically, we will consider smooth representations:

Definition 2.3.1. Let R be an algebraically closed field of characteristic zero, and let G be a locally profinite group. We say that a representation $\pi : G \to \mathrm{Aut}_R(V)$ of G is *smooth* if, for every vector $v \in V$, the stabilizer of v in G is open. We denote by $\mathrm{Rep}_R(G)$ the category of smooth R-representations of G.

Equivalently, this says that a smooth representation must be continuous with respect to the discrete topology on V. Since this is independent of the usual ℓ-adic topology, we may choose an isomorphism $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$ (and hence an equivalence of categories $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(G) \simeq \mathrm{Rep}_{\mathbb{C}}(G)$) and instead consider smooth representations of G over $\mathbb{C}$. This will be our point of view for the majority of the thesis – we will often simply write $\mathrm{Rep}(G)$ rather than $\mathrm{Rep}_{\mathbb{C}}(G)$.

We now briefly record some of the fundamental properties of smooth representations of G for later use.

Theorem 2.3.2 (Schur's lemma, [Ren10, Lemma B.II])**.** *Suppose that* π, π' *are irreducible representations of a second countable locally profinite group* G*. Then* $\mathrm{Hom}_G(\pi, \pi') \neq 0$ *if and only if* $\pi \simeq \pi'$*. If* $\pi \simeq \pi'$*, then* $\mathrm{End}_G(\pi) = \mathbb{C}$*.*

In practice, the secound countability hypothesis is unimportant: every p-adic group is second countable, as are all of its closed subgroups.

Definition 2.3.3. Let $(\pi, V) \in \mathrm{Rep}(G)$. We say that (π, V) is *admissible* if, for every compact open subgroup K of G, the space V^K of K-stable vectors in V is finite-dimensional.

These admissible representations satisfy a number of desirable properties. Again, this will not be an issue for us:

Theorem 2.3.4 ([Ren10, Théorème VI.2.2])**.** *Let* π *be a smooth irreducible representation of* G*. Then* π *is admissible.*

It then follows by induction and the exactness of the functor $V \mapsto V^K$ that any finite length representation of G is admissible. In particular, almost all of the representations which we consider will be admissible.

2.4 Hecke algebras

We now describe the analogue for p-adic groups of the usual group algebra, which is given by the Hecke algebra of G. We begin by recalling the notion of Haar measure:

Definition 2.4.1. A (left) *Haar measure* μ on G is a non-zero Radon measure μ on G which is left-translation invariant, i.e. a measure $\mu : G \to \mathbb{R} \cup \{\infty\}$ which is finite on compact sets, outer-regular on Borel sets and inner-regular on open sets, and such that, for all measurable sets E and all $g \in G$, one has $\mu(gE) = \mu(E)$.

Then any locally compact Hausdorff group admits a left Haar measure, which is unique up to a multiplicative constant. One may define a right Haar measure in a similar way, at which point it is natural to ask whether these measures coincide.

Definition 2.4.2. Let μ be a left Haar measure on G. We define the *modular function* $\delta_G : G \to \mathbb{R}_+^\times$ by requiring that, for all $g \in G$ and all measurable sets E, one has $\mu(Eg)\delta_G(g) = \mu(E)$.

The modular function then turns out to be a character of G.

Definition 2.4.3. We say that G is *unimodular* if δ_G is trivial, i.e. if every right Haar measure is a left Haar measure.

Theorem 2.4.4 ([Ren10, Proposition V.5.4])**.** *Let G be a reductive p-adic group. Then G is unimodular.*

So we are free to fix, once and for all, a choice μ of left Haar measure on G, with respect to which we may perform Lebesgue integration.

Definition 2.4.5. We denote by $\mathcal{C}_c^\infty(G)$ the set of locally constant compactly supported functions $G \to \mathbb{C}$.

Through Lebesgue integration with respect to μ, we may define a convergent convolution product on $\mathcal{C}_c^\infty(G)$ by

$$(f_1 * f_2)(x) = \int_G f_1(g) f_2(xg^{-1}) \, d\mu(g).$$

This convolution product then gives $\mathcal{C}_c^\infty(G)$ the structure of an associative (non-unital) $\mathbb{C}$-algebra.

Definition 2.4.6. We call the algebra $(\mathcal{C}_c^\infty(G), *)$ the *Hecke algebra* of G, and denote it by $\mathcal{H}(G)$.

Then $\mathcal{H}(G)$ fills the role of the group algebra for smooth representations:

Theorem 2.4.7 ([Ren10, Théorème III.1.4])**.** *There is an equivalence of categories* $\mathrm{Rep}(G) \simeq \mathcal{H}(G)\text{-Mod}$.

Remark 2.4.8. There is a slight issue with this approach, in that the above equivalence is *not* natural – it is dependant on the choice of Haar measure. However, by being slightly more careful and viewing the Hecke algebra as an algebra of distributions rather than of functions, one may remove this problem.

2.5 Induction

If H is a closed subgroup of G, then there is a natural containment $\mathcal{H}(H) \subset \mathcal{H}(G)$, and so we may define induction and restriction functors, respectively, by

$$\begin{aligned} \operatorname{Ind}_H^G &: \mathcal{H}(H)\text{-Mod} \to \mathcal{H}(G)\text{-Mod}, \quad M \mapsto \mathcal{H}(G) \otimes_{\mathcal{H}(H)} M \\ \operatorname{Res}_H^G &: \mathcal{H}(G)\text{-Mod} \to \mathcal{H}(H)\text{-Mod}, \quad M \mapsto \operatorname{Hom}_{\mathcal{H}(H)}(M, \mathcal{H}(H)). \end{aligned}$$

By the usual tensor-hom adjunction, these functors $(\operatorname{Res}_H^G, \operatorname{Ind}_H^G)$ form an adjoint pair. In the case that H is open, the restriction functor also admits a right-adjoint coinduction functor $\text{c-}\operatorname{Ind}_H^G : \mathcal{H}(H)\text{-Mod} \to \mathcal{H}(G)\text{-Mod}$.

Fixing our choice μ of right Haar measure on G, and hence an equivalence of categories $\operatorname{Rep}(G) \simeq \mathcal{H}(G)\text{-Mod}$, we may realise these as functors between representations as follows.

For a smooth representation $(\pi, V) \in \operatorname{Rep}(G)$, restriction is simply the set-theoretic restriction functor. Now let $(\sigma, W) \in \operatorname{Rep}(H)$. Then let $\operatorname{Ind}_H^G (\sigma, W)$ denote the set of functions $f : G \to W$ which transform on the left according to σ, i.e. $f(hg) = \sigma(h)f(g)$ for $h \in H$ and $g \in G$, and such that there is a compact open subgroup K of G such that $f(gk) = f(g)$ for $g \in G$ and $k \in K$. Then $\operatorname{Ind}_H^G (\sigma, W)$ is naturally a representation of G by left translation

Altenatively, we may define $\text{c-}\operatorname{Ind}_H^G (\sigma, W)$ to be the subrepresentation of $\operatorname{Ind}_H^G (\sigma, W)$ consisting of those functions which are compactly supported modulo H.

These functors realise the induction and coinduction functors on the Hecke algebra side of the equivalence $\operatorname{Rep}(G) \simeq \mathcal{H}(G)\text{-Mod}$, and thus we have functors

$$\begin{aligned} \operatorname{Ind}_H^G &: \operatorname{Rep}(H) \to \operatorname{Rep}(G), \quad \text{and} \\ \text{c-}\operatorname{Ind}_H^G &: \operatorname{Rep}(H) \to \operatorname{Rep}(G). \end{aligned}$$

Then, of course, these functors will still form adjoint pairs:

Theorem 2.5.1 (Frobenius reciprocity, [Ren10, Théorème III.2.5])**.** *The functor* Res_H^G *is left-adjoint to* Ind_H^G*. If* H *is also open in* G*, then the functor* Res_H^G *is right-adjoint to* $\text{c-}\mathrm{Ind}_H^G$*.*

Remark 2.5.2. There are, of course, cases where Ind_H^G coincides with $\text{c-}\mathrm{Ind}_H^G$, most obviously when G is compact. Using the Iwasawa decomposition of G, one sees that parabolic induction is another example of such, which we will discuss shortly.

2.6 Intertwining

Often, the notion of *intertwining* of representations will allow us to conveniently state results.

Definition 2.6.1. Let J, J' be open subgroups of G, and let λ, λ' be representations of J, J', respectively. We say that λ *intertwines with* λ' if there exists a $g \in G$ such that $\mathrm{Hom}_{J \cap {}^g J'}(\lambda, {}^g\lambda') \neq 0$. In the case that $\lambda = \lambda'$, we say that g *intertwines* λ. Write $\mathbf{I}_G(\lambda)$ for the set of $g \in G$ which intertwine λ.

We will often need to know that two representations intertwine in order to apply certain results. To see this, one often uses the following:

Lemma 2.6.2. *Let* J, J' *be open subgroups of* G *and* λ, λ' *be irreducible representations of* J, J'*, respectively. Suppose that there exists an irreducible representation* π' *of* G *which contains both* λ *and* λ'*. Then* λ *intertwines with* λ'*.*

Proof. Both $\mathrm{Hom}_J(\lambda, \mathrm{Res}_J^G\ \pi)$ and $\mathrm{Hom}_{J'}(\mathrm{Res}_{J'}^G\ \pi, \lambda')$ are non-zero, so we may apply Frobenius reciprocity and see that $\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_J^G\ \lambda, \pi)$ and $\mathrm{Hom}_G(\pi, \mathrm{Ind}_{J'}^G\ \lambda')$ are nonzero; hence $\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_J^G\ \lambda, \mathrm{Ind}_{J'}^G\ \lambda') \neq 0$. Then we compute

$$\begin{aligned} 0 &\neq \mathrm{Hom}_J(\lambda, \mathrm{Res}_J^G \mathrm{Ind}_{J'}^G\ \lambda') \\ &= \mathrm{Hom}_J(\lambda, \bigoplus_{J' \backslash G / J} \mathrm{Ind}_{{}^g J' \cap J}^J \mathrm{Res}_{{}^g J' \cap J}^{{}^g J'}\ {}^g\lambda'). \end{aligned}$$

Hence λ is a subrepresentation of $\mathrm{Ind}_{{}^g J' \cap J}^J \mathrm{Res}_{{}^g J' \cap J}^{{}^g J'}\ {}^g\lambda$ for some g. Applying Frobenius reciprocity again, we see that g intertwines λ with λ'. □

2.7 Parabolic induction and supercuspidals

Our reductive p-adic group $G = \mathbf{G}(F)$ comes equipped with a collection of special subgroups. Recall that an algebraic subgroup $\mathbf{P}$ of the algebraic group $\mathbf{G}$ is F-parabolic if $\mathbf{G}/\mathbf{P}$ is a projective variety and $\mathbf{P}$ is defined over F. We will usually drop the F from our notation, and only speak of parabolic subgroups defined over F. Then a parabolic subgroup of G is a group $P = \mathbf{P}(F)$ of the F-rational points of an F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$.

Parabolic subgroups of $\mathbf{G}$ will in general no longer be reductive; forming the quotient by the unipotent radical $\mathbf{N}$ of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ will always give a reductive group $\mathbf{M}$, and there will exist subgroups of $\mathbf{P}$ which are naturally isomorphic to $\mathbf{M}$; such subgroups are the Levi factors of $\mathbf{P}$. We say that a Levi subgroup of G is a group of the form $M = \mathbf{M}(F)$ where $\mathbf{M}$ is the Levi factor of some parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$.

Thus, given a Levi subgroup M of G and a parabolic P with Levi factor M, there is a natural *inflation* functor $\mathrm{Inf}_M^P : \mathrm{Rep}(M) \to \mathrm{Rep}(P)$, where $\mathrm{Inf}_M^P\ (\sigma, W)$ is given by the composition

$$P = MN \twoheadrightarrow M \xrightarrow{\sigma} \mathrm{Aut}_{\mathbb{C}}(W).$$

This gives a procedure for constructing representations of a reductive p-adic group G from smaller reductive subgroups M: we inflate to a parabolic P with Levi factor M, and then induce to G. To simplify certain statements, it is convenient to normalize this process:

Definition 2.7.1. Let $P = MN$ be a parabolic subgroup of G with Levi factor M. The functor of *normalized parabolic induction* $\mathrm{Ind}_{M,P}^G : \mathrm{Rep}(M) \to \mathrm{Rep}(G)$ is the composition

$$\mathrm{Ind}_{M,P}^G : \zeta \mapsto \mathrm{Ind}_P^G\ (\delta_P^{1/2} \otimes \mathrm{Inf}_M^P\ \zeta).$$

By the Iwasawa decomposition of G, the functor obtained by performing the above process, but with compact induction rather than induction, is identical to $\mathrm{Ind}_{M,P}^G$. Often, we will simply drop the adjective "normalized" and speak of parabolic induction. One important property of parabolic induction is that the resulting representations are never too large:

Theorem 2.7.2 ([Ren10, Lemme VI.6.2])**.** *Let $P = MN$ be a parabolic subgroup of G, and let $\zeta \in \mathrm{Rep}(M)$ be admissible. Then $\mathrm{Ind}_{M,P}^G\ \zeta$ is admissible. If ζ is irreducible, then $\mathrm{Ind}_{M,P}^G\ \zeta$ is of finite length.*

If ζ is irreducible, then as $\mathrm{Ind}_{M,P}^G\ \zeta$ is finite length it will always have an irreducible subrepresentation. In fact, allowing P to range over all parabolic subgroups Q of G with Levi factor M, any irreducible subquotient of $\mathrm{Ind}_{M,P}^G\ \zeta$ will be realizable as an irreducible subrepresentation of $\mathrm{Ind}_{M,Q}^G\ \zeta$ for some M.

The functor of parabolic induction admits a left-adjoint functor given by *Jacquet restriction.* For a parabolic subgroup $P = MN$ of G and a smooth representation (π, V) of G, denote by $V(N)$ the space of N-coinvariants, i.e. $V(N) = \langle v - \pi(n)v \mid v \in V,\ n \in N\rangle$. Restricting π to P, $V/V(N)$ naturally gives a representation of M. This defines a functor $r_N : \mathrm{Rep}(G) \to \mathrm{Rep}(M)$ by $V \mapsto V/V(N)$.

Definition 2.7.3. Let $P = MN$ be a parabolic subgroup of G. The (normalized) Jacquet restriction functor is the functor $r_{M,P}^G : \mathrm{Rep}(G) \to \mathrm{Rep}(M)$ given by $r_{M,P}^G\ (\pi, V) = r_N\ (\pi \otimes \delta_P^{-1/2}, V)$.

Theorem 2.7.4 ([Ren10, Théorème VI.1.1])**.** *The functor $r_{M,P}^G$ is left-adjoint to $\mathrm{Ind}_{M,P}^G$.*

As before, one sees that $\mathrm{Ind}_{M,P}^G$ should also have a right-adjoint. However, it is rather more difficult to identify what this functor should be, despite the question having a rather simple answer in the end:

Theorem 2.7.5 (Bernstein's second adjunction theorem, [Ren10, Théorème VI.9.7])**.** *Let $P = MN$ be a parabolic subgroup of G, and let $P^{\mathrm{op}} = MN^{\mathrm{op}}$ be the opposite parabolic, i.e. the unique parabolic with Levi factor M and unipotent radical N^{op} such that $N \cap N^{\mathrm{op}} = \{1\}$. Then the functor $r_{M,P^{\mathrm{op}}}^G$ is right-adjoint to $\mathrm{Ind}_{M,P}^G$.*

One natural question is whether parabolic induction, allowing P to range over all *proper* parabolic subgroups of G, leads to a construction of all representations of G as irreducible subquotients of parabolically induced representations. This turns out not to be the case.

Definition 2.7.6. Let $\pi \in \mathrm{Irr}(G)$.

(i) We say that π is *cuspidal* if it is not a quotient of $\mathrm{Ind}_{M,P}^{G}\ \zeta$, for any parabolic subgroup $P = MN$ of G and any $\zeta \in \mathrm{Irr}(M)$.

(ii) We say that π is *supercuspidal* if it is not a *sub*quotient of $\mathrm{Ind}_{M,P}^{G}\ \zeta$, for any parabolic subgroup $P = MN$ of G and any $\zeta \in \mathrm{Irr}(M)$.

We write $\mathrm{Irr}_{\mathrm{sc}}(G)$ for the set of isomorphism classes of supercuspidal representations of G, and $\mathrm{Rep}_{\mathrm{sc}}(G)$ for the full subcategory of $\mathrm{Rep}(G)$ consisting of those representations all of whose irreducible subquotients lie in $\mathrm{Irr}_{\mathrm{sc}}(G)$.

While these notions are *a priori* distinct, it turns out that when $R = \mathbb{C}$ or $\bar{\mathbb{Q}}_\ell$ a representation is cuspidal if and only if it is supercuspidal. This is not the case more generally.

Theorem 2.7.7. *Let $\pi \in \mathrm{Irr}(G)$. The following are equivalent:*

(i) The representation π is supercuspidal.

(ii) For any parabolic subgroup $P = MN$ of G, one has $r_{M,P}^{G}\ \pi = 0$.

One of the central problems regarding the representation theory of G is the following long-standing folklore conjecture:

Conjecture 2.7.8. *Let π be a supercuspidal representation of G. Then there exists an open, compact-modulo-centre subgroup $\tilde{J}$ of G and an irreducible representation Λ of $\tilde{J}$ such that $\pi \simeq \text{c-}\mathrm{Ind}_{\tilde{J}}^{G}\ \Lambda$.*

This is now known in many cases, as we discussed in the introduction. A kind of converse result which we will often make use of (and which is considerably easier to prove) is the following:

Theorem 2.7.9 ([Car84, Proposition 1.5])**.** *Let $\tilde{J}$ be an open, compact-modulo centre subgroup of G, and let Λ be an irreducible representation of $\tilde{J}$. If $\mathbf{I}_G(\Lambda) = \tilde{J}$, then* $\text{c-}\mathrm{Ind}_{\tilde{J}}^{G}\ \Lambda$ *is irreducible and supercuspidal.*

2.8 Bernstein decomposition

Definition 2.8.1. A *cuspidal datum* in G is a pair (M, ζ) consisting of a Levi subgroup M of G and an irreducible representation ζ of M.

It turns out that any irreducible representation of G comes from an essentially unique cuspidal datum:

Theorem 2.8.2 ([Ren10, Lemme VI.7.1])**.** *Let $\pi \in \mathrm{Irr}(G)$, and suppose that the two cuspidal data (M, ζ) and (M', ζ') are such that, for some parabolics $P = MN$ and $P' = M'N'$, the representation π is a subquotient of both $\mathrm{Ind}_{M,P}^{G}\ \zeta$ and $\mathrm{Ind}_{M',P'}^{G}\ \zeta'$. Then there exists a $g \in G$ such that $M' = {}^{g}M$ and $\zeta' = {}^{g}\zeta$.*

Definition 2.8.3. We call the conjugacy class of cuspidal data associated to π by the above the *supercuspidal support* of π.

We put an equivalence relation on the collection of cuspidal data by saying that two cuspidal data (M, ζ) and (M', ζ') are *inertially equivalent* if there exists an unramified character ω of M' such that (M, ζ) is G-conjugate to $(M', \zeta' \otimes \omega)$. Denote the inertial equivalence class of (M, ζ) by $[M, \zeta]_G$.

Definition 2.8.4. Let $\pi \in \mathrm{Irr}(G)$. The *inertial support* of π is the inertial equivalence class of its supercuspidal support.

Denote by $\mathfrak{B}(G)$ the set of inertial equivalence classes of cuspidal data. For each $\mathfrak{s} \in \mathfrak{B}(G)$, we may define a full subcategory $\mathrm{Rep}^{\mathfrak{s}}(G)$ of $\mathrm{Rep}(G)$ consisting of those representations such that all irreducible subquotients have inertial support $\mathfrak{s}$. Then we have the following result:

Theorem 2.8.5 (Bernstein decomposition, [Ber84])**.** *The categories $\mathrm{Rep}^{\mathfrak{s}}(G)$ are indecomposable. The category $\mathrm{Rep}(G)$ factors as a product:*

$$\mathrm{Rep}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Rep}^{\mathfrak{s}}(G).$$

Similarly, for any subset $\mathfrak{S}$ of $\mathfrak{B}(G)$, we write $\mathrm{Rep}^{\mathfrak{S}}(G) = \prod_{\mathfrak{s} \in \mathfrak{S}} \mathrm{Rep}^{\mathfrak{s}}(G)$, and $\mathrm{Irr}^{\mathfrak{S}}(G) = \coprod_{\mathfrak{s} \in \mathfrak{S}} \mathrm{Irr}^{\mathfrak{s}}(G)$.

Example 2.8.6. If π is a supercuspidal representation of G, then the set $\mathrm{Irr}^{\mathfrak{s}}(G)$ is precisely the set of unramified twists of π, i.e. the set of representations $\pi \otimes \omega$, where ω is an unramified character of G.

2.9 Types and archetypes

The Bernstein decomposition suggests that it may be profitable to study each subcategory $\mathrm{Rep}^{\mathfrak{s}}(G)$ by restriction to compact open subgroups – for example, two supercuspidal representations are in the same inertia class if and only if they agree upon restriction to any compact open subgroup. This leads to the notion of a *type*:

Definition 2.9.1. Let $\mathfrak{S} \subset \mathfrak{B}(G)$, and let (J, λ) be a pair consisting of a compact open subgroup J of G and a smooth irreducible representation λ of J.

(i) We say that (J, λ) is $\mathfrak{S}$-typical if, for any $\pi \in \mathrm{Irr}(G)$, one has that $\mathrm{Hom}_J(\pi \downharpoonright_J, \lambda) \neq 0 \Rightarrow \pi \in \mathrm{Irr}^{\mathfrak{S}}(G)$.

(ii) We say that (J, λ) is an $\mathfrak{S}$-type if it is $\mathfrak{S}$-typical and one has that $\pi \in \mathrm{Irr}^{\mathfrak{S}}(G) \Rightarrow \mathrm{Hom}_J(\pi \downharpoonright_J, \lambda) \neq 0$.

In the case that $\mathfrak{S} = \{\mathfrak{s}\}$ is a singleton, we simply speak of $\mathfrak{s}$-types rather than $\{\mathfrak{s}\}$-types.

Thus, the idea of a type is that a construction of an $\mathfrak{s}$-type for each subcategory $\mathrm{Rep}^{\mathfrak{s}}(G)$ of $\mathrm{Rep}(G)$ would allow for a description of $\mathrm{Rep}(G)$ entirely in terms of the identification of the presence of certain representations of compact subgroups.

Example 2.9.2. The original example of a type is the trivial representation of the Iwahori subgroup in $\mathbf{GL}_2(F)$, where an irreducible representation has an Iwahori-fixed vector if and only if it is an unramified twist of the Steinberg representation $\mathrm{St} = (\mathrm{Ind}_B^{\mathbf{GL}_2(F)} \mathbb{1}_B)/\mathbb{1}_{\mathbf{GL}_2(F)}$, where B is the standard Borel subgroup of upper-triangular matrices.

Our main question throughout this thesis will be: in the cases where types are constructed, to what extent are these types unique? If one has a unique $\mathfrak{s}$-type for each $\mathfrak{s}$, it will be genuinely possible to completely describe the representation theory of G by looking for a single $\mathfrak{s}$-type (J, λ) for each $\mathfrak{s}$ in the restrictions to J of representations π of G. True uniqueness isn't possible due to the following simple observation:

Lemma 2.9.3. *Let $\mathfrak{s} \in \mathfrak{B}(G)$ and let (J, λ) be a $\mathfrak{s}$-type. If K is a compact open subgroup of G containing J, then any irreducible subrepresentation of $\mathrm{Ind}_J^K\ \lambda$ is $\mathfrak{s}$-typical. If $\mathrm{Ind}_J^K\ \lambda$ is itself irreducible, then it is an $\mathfrak{s}$-type.*

Proof. This is immediate by Frobenius reciprocity. □

However, one might hope that $\mathfrak{s}$-types (K, τ) with K maximal compact in G might be unique. Again, there is a slight problem in that there exists inertia classes $\mathfrak{s}$ which have an $\mathfrak{s}$-type (J, λ) with J non-maximal, but which admit no $\mathfrak{s}$-type (K, τ) with τ maximal. Consider, for example, the following:

Example 2.9.4. Let $G = \mathbf{GL}_2(F)$. Then, as noted above, the Steinberg representation $\mathrm{St} = (\mathrm{Ind}_B^G\ \mathbb{1}_B)/\mathbb{1}_G$ contains the $[T, \mathbb{1}_T]_G$-type $(U_{\mathfrak{I}}, \mathbb{1}_{U_{\mathfrak{I}}})$, where $U_{\mathfrak{I}}$ is the standard Iwahori subgroup of G and T is the split maximal torus in G. Let $K = \mathbf{GL}_2(\mathcal{O})$ denote the standard maximal compact subgroup of G. Then $U_{\mathfrak{I}} \subset K$, but we claim that there does not exist a $[T, \mathbb{1}_T]_G$-type of the form (K, τ). Indeed, by definition of St we have a short exact sequence

$$0 \longrightarrow \mathbb{1}_G \longrightarrow \mathrm{Ind}_B^G\ \mathbb{1}_B \longrightarrow \mathrm{St} \longrightarrow 0.$$

As the functor $V \mapsto V^K$ is exact, this gives a short exact sequence

$$0 \longrightarrow \mathbb{1}_K \longrightarrow (\mathrm{Ind}_B^G\ \mathbb{1}_B)^K \longrightarrow \mathrm{St}^K \longrightarrow 0.$$

By the Iwasawa decomposition $G = BK$, we have $(\mathrm{Ind}_B^G\ \mathbb{1}_B)^K = \mathbb{1}_K$, and so $\mathrm{St}^K = 0$. But now, $\mathrm{Irr}^{[T,\mathbb{1}_T]_G}(G)$ clearly contains the trivial representation of G, and so the only possible $[T, \mathbb{1}_T]_G$-type defined on K is the trivial representation of K. As $\mathrm{St}^K = 0$, this cannot be a type.

So the best we can hope for is that there is a unique $\mathfrak{s}$-typical representation (K, τ) with K maximal compact for each $\mathfrak{s}$. Again, this will turn out to not quite be possible – although it seems likely that when $\mathfrak{s} = [G, \pi]_G$ is a supercuspidal class and K is fixed, there will always be at most one $\mathfrak{s}$-typical representation defined on K. As we are free to conjugate typical representations within G, it makes sense to consider conjugacy classes of typical representations one at a time. This leads to the following modification of the definition of a type:

Definition 2.9.5. Let $\mathfrak{s} \in \mathfrak{B}(G)$. An $\mathfrak{s}$-*archetype* is a conjugacy class of $\mathfrak{s}$-typical representations (K, τ) with K maximal compact in G.

We will usually abuse notation when discussing archetypes, referring to a single $\mathfrak{s}$-typical representation of some maximal compact subgroup as an $\mathfrak{s}$-archetype; it should always be understood that the action of G by conjugacy is implicit in such statements.

Remark 2.9.6. If $\mathfrak{s} = [G, \pi]_G$ is a supercuspidal inertia class, then the distinction between $\mathfrak{s}$-types and $\mathfrak{s}$-typical representations is irrelevant. Indeed, suppose that (J, λ) is $\mathfrak{s}$-typical. Then the compactly induced representation $\text{c-Ind}_J^G\ \lambda$ has some irreducible subquotient π' which is necessarily an unramified twist of π, say $\pi' = \pi \otimes \omega$. Now π will be an irreducible subquotient of $(\text{c-Ind}_J^G\ \lambda) \otimes \omega^{-1} = \text{c-Ind}_J^G\ (\lambda \otimes \omega^1)$. But, as ω is unramified and J is compact, this is simply equal to $\text{c-Ind}_J^G\ \lambda$, and so (J, λ) is contained in every element of $\text{Irr}^{\mathfrak{s}}(G)$.

Our main results will be focused on counting the number of archetypes contained in representations in various situations. In particular, we will focus on the case of $[G, \pi]_G$-archetypes, where π is a supercuspidal representation of G. Building on the expectation that such types correspond precisely to the compact inducing data for π, we conjecture the following:

Conjecture 2.9.7 (The unicity of types for supercuspidals)**.** *Let π be a supercuspidal representation of G. Then there exists a $[G, \pi]_G$-archetype, and there exists at most one $[G, \pi]_G$-archetype defined on each conjugacy class of maximal compact subgroups of G.*

The main technical results of this thesis will be the verification of this conjecture in a number of cases.

2.10 The Hecke algebra of a type

Let us discuss the role of types on the Hecke algebra side of the equivalence of categories $\mathbf{M} : \mathrm{Rep}(G) \simeq \mathcal{H}(G)\text{-Mod}$. Recall that we have a fixed right Haar measure μ on G.

Let (J, λ) be a pair consisting of an irreducible representation λ of a compact open subgroup J of G. We associate to λ an idempotent element e_λ of $\mathcal{H}(G)$ by setting

$$e_\lambda(g) = \frac{\dim\ \lambda}{\mu(J)} \mathrm{tr}\ \lambda(g^{-1})\chi_J(g),$$

where χ_J is the characteristic function on J. Then $e_\lambda * \mathcal{H}(G) * e_\lambda$ is a unital subalgebra of $\mathcal{H}(G)$ with identity e_λ.

We may also associate a different algebra to λ by setting $\mathcal{H}(G, \lambda) = \mathrm{End}_G(\text{c-}\mathrm{Ind}_J^G\ \lambda)$.

Let $\mathrm{Rep}^\lambda(G)$ denote the full subcategory of $\mathrm{Rep}(G)$ consisting of those representations generated by their λ-isotypic vectors. We describe a pair of functors between the categories $\mathrm{Rep}^\lambda(G)$, $\mathcal{H}(G, \lambda)$-Mod and $e_\lambda * \mathcal{H}(G) * e_\lambda$-Mod.

Let $\mathbf{M}^\lambda : \mathrm{Rep}(G) \to e_\lambda * \mathcal{H}(G) * e_\lambda\text{-Mod}$ be the functor mapping a smooth representation to its λ-isotypic subspace, and let $\mathbf{M}_\lambda : \mathrm{Rep}(G) \to \mathcal{H}(G, \lambda)\text{-Mod}$ be the functor $\mathrm{Hom}_J(\lambda, -)$.

Theorem 2.10.1 ([BK98]). *With the notation above, the following are equivalent:*

(i) The subcategory $\mathrm{Rep}^\lambda(G)$ *of* $\mathrm{Rep}(G)$ *is closed under subquotients.*

(ii) The functor $\mathbf{M}^\lambda$ *induces an equivalence of categories* $\mathrm{Rep}^\lambda(G) \simeq e_\lambda * \mathcal{H}(G) * e_\lambda\text{-Mod}$.

(iii) The functor $\mathbf{M}_\lambda$ *induces an equivalence of categories* $\mathrm{Rep}^\lambda(G) \simeq \mathcal{H}(G, \lambda)\text{-Mod}$.

(iv) There exists a finite set $\mathfrak{S} \subset \mathfrak{B}(G)$ *such that*

$$\mathrm{Rep}^\lambda(G) = \prod_{\mathfrak{s} \in \mathfrak{S}} \mathrm{Rep}^{\mathfrak{s}}(G).$$

(v) There exists a finite set $\mathfrak{S} \subset \mathfrak{B}(G)$ *such that* (J, λ) *is an* $\mathfrak{S}$*-type.*

This allows us to easily translate the properties of types between the viewpoints of representations of G and of $\mathcal{H}(G)$-modules. In particular, this will be useful when we come to discuss G-covers of types.

Chapter 3

Simple and semisimple types

In this chapter, we give a terse overview of the Bushnell–Kutzko theory of simple and semisimple types for general and special linear groups, focusing on a goal of making sufficiently explicit their construction of types for each of the Bernstein components of these groups in order for application to the subsequent chapters. We make no attempt to be comprehensive, and will avoid going into the technical details wherever possible; the full theory is developed in [BK93a, BK93b, BK94, BK98, BK99, GR02].

3.1 Hereditary orders

The starting point for the construction of types in $\mathbf{GL}_N(F)$ is the observation that $\mathbf{GL}_N(F)$ is naturally the group of units of an algebra. Let V denote a fixed N-dimensional F-vector space, and let $A = \mathrm{End}_F(V)$. Then $G \simeq \mathrm{Aut}_F(V) = A^\times$. Inside of A, there live a particular class of $\mathcal{O}$-modules, which themselves give a natural means of working with the parahoric subgroups of G (which will be introduced in complete generality in Chapter 5).

Definition 3.1.1. A *(left) hereditary $\mathcal{O}$-order* in A is a subring $\mathfrak{A}$ of A which is a $\mathcal{O}$-lattice in A, such that every left $\mathfrak{A}$-lattice is $\mathfrak{A}$-projective.

When there is no danger of ambiguity, we will often drop the mention of $\mathcal{O}$, and simple speak of hereditary orders. Up to change of basis, every hereditary order $\mathfrak{A}$ in A consists of a ring of block matrices, with matrices on and above the diagonal taking entries in $\mathcal{O}$, and matrices below the diagonal taking entries in $\mathfrak{p}$. Working with such a basis, the Jacobson radical $\mathfrak{P} = \mathrm{rad}\,\mathfrak{A}$ of a hereditary order $\mathfrak{A}$ is the two-sided invertible fractional ideal of $\mathfrak{A}$ consisting of block matrices, with matrices above the diagonal taking values in $\mathcal{O}$, and matrices on and below the diagonal taking values in $\mathfrak{p}$. Define the *lattice period* of $\mathfrak{A}$ to be the least integer $e_{\mathfrak{A}} = e_{\mathfrak{A}/\mathcal{O}}$ such that $\mathfrak{P} = \varpi^{e_{\mathfrak{A}}}\mathfrak{A}$. Viewing a hereditary order $\mathfrak{A}$ in the block form described above, we say that $\mathfrak{A}$ is *principal* if each of its blocks is of the same size.

The relationship between hereditary orders and parahoric subgroups is then that, given a hereditary order $\mathfrak{A}$, the unit group $U_{\mathfrak{A}} = \mathfrak{A}^\times$ is a parahoric subgroup of G, and every such subgroup arises in this way. Moreover, one recovers the Moy–Prasad filtration of $U_{\mathfrak{A}}$ by setting $U_{\mathfrak{A}}^k = 1 + \mathfrak{P}^k$ for $k > 0$. If one defines the *normalizer* of $\mathfrak{A}$ to be $\mathfrak{K}_{\mathfrak{A}} = \{g \in G \mid {}^g\mathfrak{A} = \mathfrak{A}\}$, then $\mathfrak{K}_{\mathfrak{A}}$ is an open, compact-modulo-centre subgroup of G normalizing $U_{\mathfrak{A}}$ and containing $U_{\mathfrak{A}}$ as its maximal compact subgroup.

Finally, we note that a hereditary order $\mathfrak{A}$ defines a valuation $v_{\mathfrak{A}}$ on A by $v_{\mathfrak{A}}(x) = \max\{n \in \mathbb{Z} \mid x \in \mathfrak{P}^n\}$.

3.2 Strata

We fix, once and for all, a level 1 additive character ψ of F, i.e. a character of F trivial on $\mathfrak{p}$ but non-trivial on $\mathcal{O}$.

Definition 3.2.1. A *stratum* in A is a quadruple $[\mathfrak{A}, n, r, \beta]$ consisting of a hereditary $\mathcal{O}$-order $\mathfrak{A}$, integers $n > r \geq 0$, and an element β of $\mathfrak{P}^{-n}$.

We say that two strata $[\mathfrak{A}, n, r, \beta]$ and $[\mathfrak{A}, n', r, \beta']$ are *equivalent* if $\beta = \beta' \bmod \mathfrak{P}^{-r}$; this fits in with the point of view that a stratum may simply be viewed as a coset $\beta + \mathfrak{P}^{-r}$.

Given a stratum $[\mathfrak{A}, n, r, \beta]$ in A, if $0 \leq \lfloor \frac{n}{2} \rfloor \leq r < n$, then one obtains a character ψ_beta of $U_{\mathfrak{A}}^{r+1}/U_{\mathfrak{A}}^{n+1}$ by setting $\psi_\beta = \psi \circ \mathrm{tr}_{E/F}(\beta(x-1))$. Every such character arises in this way, and strata $[\mathfrak{A}, n, r, \beta]$ and $[\mathfrak{A}, n, r, \beta']$ satisfying $0 \leq \lfloor \frac{n}{2} \rfloor \leq r < n$ define the same character if and only if they are equivalent.

We will be particularly interested in certain classes of strata:

Definition 3.2.2. Let $[\mathfrak{A}, n, r, \beta]$ be a stratum. We say that $[\mathfrak{A}, n, r, \beta]$ is *pure* if:

(i) $E = F[\beta]$ is a field;

(ii) $E^\times \subset \mathfrak{K}_{\mathfrak{A}}$; and

(iii) $n = -v_{\mathfrak{A}}(\beta)$.

If $F[\beta] = F$, then we say that $[\mathfrak{A}, n, r, \beta]$ is *scalar*.

Given such a pure stratum $[\mathfrak{A}, n, r, \beta]$, one may view V as an $E = F[\beta]$-vector space, which naturally leads one to consider the subalgebra $B = B_\beta = \mathrm{End}_E(V)$ of A. One then obtains a hereditary $\mathfrak{o}_E$-order $\mathfrak{B} = \mathfrak{B}_\beta = \mathfrak{A} \cap B$ in B with Jacobson radical $\mathfrak{Q} = \mathfrak{Q}_\beta = \mathfrak{P} \cap B$. For $k \in \mathbb{Z}$, set

$$\mathfrak{N}_k(\beta, \mathfrak{A}) = \{x \in \mathfrak{A} \mid \beta x - x\beta \in \mathfrak{P}^k\}.$$

We then define a constant $k_0(\beta, \mathfrak{A})$ by setting

$$k_0(\beta, \mathfrak{A}) = \begin{cases} \max\{k \in \mathbb{Z} \mid \mathfrak{N}_k(\beta, \mathfrak{A}) \not\subset \mathfrak{B} + \mathfrak{P} & \text{if } F[\beta] \neq F \\ -\infty & \text{if } F[\beta] = F \end{cases}$$

In particular, if $[\mathfrak{A}, n, 0, \beta]$ is non-scalar, then $k_0(\beta, \mathfrak{A}) \geq v_{\mathfrak{A}}(\beta)$. Moreover, one may show that equivalence of strata preserves k_0.

Definition 3.2.3. We say that a pure stratum $[\mathfrak{A}, n, r, \beta]$ is *simple* if $r < -k_0(\beta, \mathfrak{A})$.

Remark 3.2.4. While the definition of $k_0(\beta, \mathfrak{A})$ seems unclear at first, it plays a rather simple – although fundamental – role in the theory. As we will see in the next section, one approximates strata in terms of a sequence of strata over finite field extensions E/F. The constant $k_0(\beta, \mathfrak{A})$ essentially measures how far along this sequence one is able to go before being required to move on to the next field extension in the sequence.

3.3 Approximation of strata

There is an initial class of simple strata which are rather easy to describe; these are those given by minimal elements. Given a field extension $E = F[\beta]/F$, we say that β is *minimal* over F if either $E = F$ or, whenever $\mathfrak{A}$ is such that $E^\times \subset \mathfrak{K}_{\mathfrak{A}}$, one has that $v_{\mathfrak{A}}(\beta) = k_0(\beta, \mathfrak{A})$. If $[\mathfrak{A}, -v_{\mathfrak{A}}(\beta), r, \beta]$ is a stratum with β minimal over F, then $[\mathfrak{A}, -v_{\mathfrak{A}}(\beta), r, \beta]$ will always be simple. Sometimes these minimal simple strata are called *alfalfa strata*.

For the non-minimal simple strata, one must pass to larger extensions of F within A in order to completely describe the stratum. The main tool in order to do this is the notion of a tame corestriction:

Definition 3.3.1. Let $\beta \in A$ be such that $E = F[\beta]$ is a field, and let $B = B_\beta$. A *tame corestriction on A relative to E/F* is a linear (B, B)-bimodule homomorphism $s : A \to B$ such that, for any hereditary $\mathcal{O}$-order $\mathfrak{A}$ with $E^\times \subset \mathfrak{K}_{\mathfrak{A}}$, one has $s(\mathfrak{A}) = \mathfrak{A} \cap B$.

There will always exist such a tame corestriction, which will be unique up to a scalar in $\mathcal{O}_E^\times$. Given a pure stratum $[\mathfrak{A}, n, r, \beta]$, together with a tame corestriction $s : A \to B_\beta$, then $[\mathfrak{B}_\beta, r, r-1, s(\beta)]$ will be a stratum in B.

Proposition 3.3.2. *Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum in A.*

(i) There exists a simple stratum $[\mathfrak{A}, n, r, \gamma]$ equivalent to $[\mathfrak{A}, n, r, \beta]$. Moreover, if $[\mathfrak{A}, n, r, \beta']$ is a pure stratum equivalent to $[\mathfrak{A}, n, r, \beta]$, then $[\mathfrak{A}, n, r, \beta']$ is simple if and only if $F[\beta']/F$ is of minimal degree among all such extensions of F.

(ii) Suppose that $r = -k_0(\beta, \mathfrak{A})$ *and let* $[\mathfrak{A}, n, r, \gamma]$ *be a simple stratum equivalent to* $[\mathfrak{A}, n, r, \beta]$*. If* $s_\gamma : A \to B_\gamma$ *is a tame corestriction relative to* $F[\gamma]/F$*, then* $[\mathfrak{B}_\gamma, r, r-1, s_\gamma(\beta-\gamma)]$ *is equivalent to a simple stratum in* B_γ*.*

3.4 Simple characters

With this in place, we come to the technical heart of the theory – the theory of simple characters. These are approximations to the types which we will eventually construct, from which the large majority of the desirable properties of these types will be inherited. the first step is to assign to a simple stratum some rings, the unit groups of which will be pro-p subgroups of G.

Definition 3.4.1. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, and set $r = -k_0(\beta, \mathfrak{A})$.

(i) If β is minimal over F, set

$$\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B}_\beta + \mathfrak{P}^{\lfloor \frac{n}{2} \rfloor + 1}.$$

(ii) If β is not minimal, let $[\mathfrak{A}, n, r, \gamma]$ be a simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$. Pure

$$\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B}_\beta + \mathfrak{H}(\gamma, \mathfrak{A}) \cap \mathfrak{P}^{\lfloor \frac{r}{2} \rfloor + 1}.$$

There is also a slightly larger ring $\mathfrak{J}(\beta, \mathfrak{A})$ defined in a similar manner:

Definition 3.4.2. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, and set $r = -k_0(\beta, \mathfrak{A})$.

(i) If β is minimal over F, set

$$\mathfrak{J}(\beta, \mathfrak{A}) = \mathfrak{B}_\beta + \mathfrak{P}^{\lfloor \frac{n+1}{2} \rfloor}.$$

(ii) If β is not minimal, let $[\mathfrak{A}, n, r, \gamma]$ be a simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$. Pure

$$\mathfrak{J}(\beta, \mathfrak{A}) = \mathfrak{B}_\beta + \mathfrak{J}(\gamma, \mathfrak{A}) \cap \mathfrak{P}^{\lfloor \frac{r+1}{2} \rfloor}.$$

Each of these sets comes equipped with a natural filtration, by setting, for $k \geq 0$, $\mathfrak{H}^k(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap \mathfrak{P}^K$ and $\mathfrak{J}^k(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap \mathfrak{P}^k$.

Proposition 3.4.3. *(i) The sets $\mathfrak{H}(\beta, \mathfrak{A})$ and $\mathfrak{J}(\beta, \mathfrak{A})$ are well-defined $\mathfrak{o}$-orders in A (and so, in particular, rings).*

(ii) For each $k \geq 0$, the sets $\mathfrak{H}^k(\beta, \mathfrak{A})$ and $\mathfrak{J}^k(\beta, \mathfrak{A})$ are $\mathfrak{B}_\beta$-bimodules satisfying $\mathfrak{Q}_\beta \mathfrak{H}^k(\beta, \mathfrak{A}) = \mathfrak{H}^k(\beta, \mathfrak{A}) \mathfrak{Q}_\beta$ and $\mathfrak{Q}_\beta \mathfrak{J}^k(\beta, \mathfrak{A}) = \mathfrak{J}^k(\beta, \mathfrak{A}) \mathfrak{Q}_\beta$, respectively.

(iii) For $t \leq r - 1$, if $[\mathfrak{A}, n, t, \beta']$ is a simple stratum equivalent to $[\mathfrak{A}, n, t, \beta]$, then $\mathfrak{H}^k(\beta', \mathfrak{A}) = \mathfrak{H}^k(\beta, \mathfrak{A})$ for $k \geq \max\{0, t + 1 - \lfloor \frac{r+1}{2} \rfloor\}$ and $\mathfrak{J}^k(\beta', \mathfrak{A}) = \mathfrak{J}^k(\beta, \mathfrak{A})$ for $k \geq \max\{0, t - \lfloor \frac{r}{2} \rfloor\}$.

Definition 3.4.4. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, and let $m \geq 0$. we set $H^m(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap U_{\mathfrak{A}}^m$ and $J^m(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap U_{\mathfrak{A}}^m$.

In particular, one has $H^0(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A})^\times$ and $J^0(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A})^\times$. We will often denote this latter group simple by $J(\beta, \mathfrak{A})$.

Proposition 3.4.5. *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum.*

(i) The groups $H^0(\beta, \mathfrak{A})$ and $J^0(\beta, \mathfrak{A})$ are compact open subgroups of G with decreasing filtrations by open normal subgroups given by $H^m(\beta, \mathfrak{A})$ and $J^m(\beta, \mathfrak{A})$, respectively. Moreover, $H^m(\beta, \mathfrak{A})$ is open and normal in $J^0(\beta, \mathfrak{A})$ for any m.

(ii) For any m, the groups $H^m(\beta, \mathfrak{A})$ and $J^m(\beta, \mathfrak{A})$ are normalized by $\mathfrak{K}_{\mathfrak{B}_\beta}$.

(iii) The group $\tilde{J}(\beta, \mathfrak{A}) = \mathfrak{K}_{\mathfrak{B}_\beta} J^1(\beta, \mathfrak{A})$ is compact modulo its centre, and contains $J(\beta, \mathfrak{A})$ as its unique maximal compact subgroup. The group $J(\beta, \mathfrak{A})$ contains as a unique maximal pro-p subgroup the group $J^1(\beta, \mathfrak{A})$.

The point of introducing these groups is that the groups $H^k(\beta, \mathfrak{A})$ will come equipped with an important class of characters – the simple characters – satisfying a number of functorial properties. Eventually, this will lead to the definition of simple types defined on the groups $J(\beta, \mathfrak{A})$.

Definition 3.4.6. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum with $n = -v_{\mathfrak{A}}(\beta)$, and let $0 \leq m < n$.

(i) Suppose that β is minimal over F. If $\lfloor \frac{n}{2} \rfloor \leq m < n$, set $\mathcal{C}(\mathfrak{A}, m, \beta) = \{\psi_\beta\}$. If $0 \leq m < \lfloor \frac{n}{2} \rfloor$, let $\mathcal{C}(\mathfrak{A}, m, \beta)$ be the set of characters θ of $H^{m+1}(\beta, \mathfrak{A})$ such that:

(a) the restriction of θ to $H^{\lfloor \frac{n}{2} \rfloor + 1}(\beta, \mathfrak{A})$ is equal to ψ_β; and

(b) the restriction of θ to $H^{m+1}(\beta, \mathfrak{A}) \cap B_\beta^\times$ factors through $\det_{B_\beta}$.

(ii) Now suppose that β is not minimal over F. Let $r = -k_0(\beta, \mathfrak{A})$ and let $[\mathfrak{A}, n, r, \gamma]$ be a simple stratum equivalent to the pure stratum $[\mathfrak{A}, n, r, \beta]$. Suppose that $\mathcal{C}(\mathfrak{A}, m', \gamma)$ has already been defined for all m'. If $r \leq m < n$, we let $\mathcal{C}(\mathfrak{A}, m, \beta) = \mathcal{C}(\mathfrak{A}, m, \beta')$. Otherwise, we let $\mathcal{C}(\mathfrak{A}, m, \beta)$ be the set of characters θ of $H^{m+1}(\beta, \mathfrak{A})$ such that:

(a) if $\lfloor \frac{r}{2} \rfloor \leq m < r$, then $\theta = \theta_0 \psi_{\beta - \gamma}$ for some $\theta_0 \in \mathcal{C}(\mathfrak{A}, m, \gamma)$;

(b) if $0 \leq m < \lfloor \frac{r}{2} \rfloor$, then the restriction of θ to $H^{\lfloor \frac{r}{2} \rfloor + 1}(\beta, \mathfrak{A})$ is equal to $\theta_0 \psi_{\beta - \gamma}$ for some $\theta_0 \in \mathcal{C}(\mathfrak{A}, \lfloor \frac{r}{2} \rfloor, \gamma)$;

(c) the restriction of θ to $H^{m+1}(\beta, \mathfrak{A}) \cap B_\beta^\times$ factors through $\det_{B_\beta}$; and

(d) θ is normalized by $\mathfrak{K}_{\mathfrak{B}_\beta}$.

We call any character θ contained in one of the sets $\mathcal{C}(\mathfrak{A}, m, \beta)$ for some simple stratum $[\mathfrak{A}, n, r, \beta]$ (with $n = -v_{\mathfrak{A}}(\beta)$) a *simple character.*

Among the sets of simple characters, those sets $\mathcal{C}(\mathfrak{A}, 0, \beta)$ will be of the most significance:

Theorem 3.4.7. *Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. There exists a finite set $\mathfrak{S}_\theta \subset \mathfrak{B}(G)$ such that θ is an $\mathfrak{S}_\theta$-type. Moreover, for every irreducible representation π of G, there exists a simple stratum $[\mathfrak{A}, n, 0, \beta]$ such that π contains some simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$.*

Remark 3.4.8. It is important to point out that the sets $\mathfrak{S}_\theta$ do *not* partition $\mathfrak{B}(G)$. This is true if one restricts to supercuspidal representations, but in general a non-cuspidal irreducible representation of G will contain multiple simple characters.

While these simple characters satisfy many important functorial properties, most of these will not be relevant to our purposes. We only note the most significant of these properties, the *intertwining implies conjugacy property*:

Theorem 3.4.9. *Let* θ *and* θ' *be simple characters contained in the sets* $\mathcal{C}(\mathfrak{A}, 0, \beta)$ *and* $\mathcal{C}(\mathfrak{A}', 0, \beta')$, *respectively. If* $\mathbf{I}_G(\theta, \theta') \neq \emptyset$, *then there exists a* $g \in G$ *such that* ${}^gH^1(\beta, \mathfrak{A}) = H^1(\beta', \mathfrak{A}')$ *and* ${}^g\theta = \theta'$.

Moreover, a supercuspidal representation π *of* G *contains precisely one* G*-conjugacy class of simple characters.*

The importance of simple characters is then made clear by the following:

Proposition 3.4.10. *Let* $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. *The group* $\tilde{J}(\beta, \mathfrak{A}) = \mathfrak{K}_{\mathfrak{B}_\beta} J^1(\beta, \mathfrak{A})$ *coincides with the* G*-normalizer of* θ.

This suggests that it should be possible to find, among the irreducible subrepresentations of $\mathrm{Ind}_{H^1(\beta,\mathfrak{A})}^{\tilde{J}(\beta,\mathfrak{A})} \theta$, a collection of irreducible representations Λ_π for each supercuspidal representation π in $\mathrm{Irr}_{\mathfrak{S}_\theta}(G)$ such that $\pi \simeq \text{c-}\mathrm{Ind}_{\tilde{J}(\beta,\mathfrak{A})}^{G} \Lambda_\pi$. Indeed, this will turn out to be the case.

3.5 Simple types in $\mathbf{GL}_N(F)$

So it remains to construct these representations Λ_π of $\tilde{J}(\beta, \mathfrak{A})$, which will be done by first constructing $[G, \pi]_G$-types of the form $(J(\beta, \mathfrak{A}), \lambda)$. We take advantage of the well-behaved ascending chain of compact open subgroups of G given by

$$H^1(\beta, \mathfrak{A}) \subset J^1(\beta, \mathfrak{A}) \subset J(\beta, \mathfrak{A}).$$

Thus the first step is to understand the representation $\mathrm{Ind}_{H^1(\beta,\mathfrak{A})}^{J^1(\beta,\mathfrak{A})} \theta$. This turns out to be rather simple:

Proposition 3.5.1. *Let* $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. *There exists a unique irreducible representation* η *of* $J^1(\beta, \mathfrak{A})$ *which contains* θ *upon restriction to* $H^1(\beta, \mathfrak{A})$. *Moreover, upon restriction to* $H^1(\beta, \mathfrak{A})$, *the representation* η *becomes isomorphic to a direct sum of* $(J^1(\beta, \mathfrak{A}) : H^1(\beta, \mathfrak{A}))^{1/2}$ *copies of* θ.

We call η the *Heisenberg extension* of θ.

The next step is then to describe $\mathrm{Ind}_{J^1(\beta,\mathfrak{A})}^{J(\beta,\mathfrak{A})}\ \eta$. Since $J^1(\beta,\mathfrak{A})$ is normal in $J(\beta,\mathfrak{A})$, it suffices to construct an extension κ of η to $J(\beta,\mathfrak{A})$, and then form the representation $\kappa \otimes \mathrm{Ind}_{J^1(\beta,\mathfrak{A})}^{J(\beta,\mathfrak{A})}\ \mathbb{1}_{J^1(\beta,\mathfrak{A})}$. Due to our goal of describing supercuspidal representations as being compactly induced from the groups $\tilde{J}(\beta,\mathfrak{A})$, we also require that this extension κ satisfies an additional intertwining property:

Definition 3.5.2. Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ have Heisenberg extension η. A β-*extension* of η is an irreducible representation κ of $J(\beta,\mathfrak{A})$ extending η which is intertwined by $B_\beta^\times$.

Proposition 3.5.3. *Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ have Heisenberg extension η.*

(i) There exists a β-extension of η.

(ii) If κ is a β-extension of η, then all other β-extensions of η are of the form $\kappa \otimes (\chi \circ \det_B)$, for some character χ of $\mathbf{k}_E^\times$, viewed as a character of $\mathcal{O}_E^\times$ via inflation.

(iii) Distinct characters χ of $\mathbf{k}_E^\times$ yield distinct β-extensions. Thus there are precisely $q_E - 1$ distinct β-extensions of η.

While it may seem troublesome that β-extensions are not unique, this is not an issue. It remains for us to form an irreducible subrepresentation of $\kappa \otimes \mathrm{Ind}_{J^1(\beta,\mathfrak{A})}^{J(\beta,\mathfrak{A})}\ \mathbb{1}_{J^1(\beta,\mathfrak{A})}$, which is to say that it remains for us to consider the representations $\kappa\otimes\sigma$, where σ is an irreducible representation of $J(\beta,\mathfrak{A})/J^1(\beta,\mathfrak{A})$. There are two cases: if σ is non-cuspidal, then $\kappa \otimes \sigma$ may only be contained in non-cuspidal irreducible representations of G (as we will see during the proof of Lemma 4.4.2). On the other hand, if σ is cuspidal then so is any twist $\sigma \otimes (\chi \circ \det)$, which means that we may as well just fix a single choice of β-extension.

Remark 3.5.4. In the case that θ is a *maximal* simple character, i.e. $\mathfrak{B}_\beta \simeq \mathrm{Mat}_{N/[E:F]}(\mathcal{O}_E)$ is a maximal hereditary order, β-extensions become rather simple: unless $U_{\mathfrak{B}_\beta}/U_{\mathfrak{B}_\beta}^1 \simeq \mathbf{GL}(2,2)$, every extension of η to $J(\beta,\mathfrak{A})$ is a β-extension.

We are then able to give the main definition:

Definition 3.5.5. A *simple type* in G is a pair (J, λ) consisting of a compact open subgroup J of G and an irreducible representation λ of J, of one of the two following forms.

(i) $J = J(\beta, \mathfrak{A})$ and $\lambda = \kappa \otimes \sigma$, where:

(a) $\mathfrak{A}$ is a principal hereditary $\mathcal{O}$-order in A and $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum with $n = -v_{\mathfrak{A}}(\beta)$;

(b) κ is a β-extension of some $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$; and

(c) making the identification

$$J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \simeq U_{\mathfrak{B}_\beta}/U^1_{\mathfrak{B}_\beta} \simeq \prod_{i=1}^{e_{\mathfrak{B}_\beta}} \mathbf{GL}_{N/[E:F]}(\mathbf{k}_E),$$

the representation σ is the inflation to $J(\beta, \mathfrak{A})$ of an $e_{\mathfrak{B}_\beta}$-fold tensor power $\sigma_0^{e_{\mathfrak{B}_\beta}}$ of some cuspidal representation σ_0 of $\mathbf{GL}_{N/[E:F]}(\mathbf{k}_E)$.

(ii) $J = U_{\mathfrak{A}}$ for some principal hereditary $\mathcal{O}$-order $\mathfrak{A}$, so that $U_{\mathfrak{A}}/U^1_{\mathfrak{A}} \simeq \prod_{i=1}^{e_{\mathfrak{A}}} \mathbf{GL}_{N/e_{\mathfrak{A}}}(\mathbf{k}_F)$, and λ is the inflation to $U_{\mathfrak{A}}$ of the $e_{\mathfrak{A}}$-fold tensor power $\sigma_0^{e_{\mathfrak{A}}}$ of some cuspidal representation σ_0 of $\mathbf{GL}_{N/e_{\mathfrak{A}}}(\mathbf{k}_F)$.

The distinction between these two cases is really just a case of formality, since β-extensions only make sense in the first case. In practice, the second case can be viewed as a degenerate case of the first.

The etymology of these simple types is then justified by the following:

Proposition 3.5.6. *Let (J, λ) be a simple type in G. Then there exists an $\mathfrak{s} \in \mathfrak{B}(G)$ such that (J, λ) is an $\mathfrak{s}$-type. Moreover, if π is an irreducible representation of G containing some simple type, then π lies in the discrete series of G, and every discrete series representation of G contains some simple type.*

(We recall that by a discrete series representation, we mean either a supercuspidal representation, or a generalized Steinberg representation. This means that π lies in the discrete series of G if it is an irreducible representation of inertial support of the form $[M, \zeta]_G$, with $M = G_M^{N/M}$ and $\zeta \simeq \zeta_0^{\otimes N/M}$ for some $M|N$ and some supercuspidal representation ζ_0

of G_M).

In the case of the simple types contained in supercuspidal representations, one obtains a particularly rich theory.

Definition 3.5.7. Let (J, λ) be a simple type in G. We say that (J, λ) is a *maximal* simple type if the hereditary $\mathcal{O}_E$-order $\mathfrak{B}_\beta$ is maximal, i.e. if $e_{\mathfrak{B}_\beta/\mathcal{O}_E} = 1$.

Note that in the second case of the definition of a simple type, this simply means that $\mathfrak{A}$ is a maximal $\mathcal{O}$-order in A.

Theorem 3.5.8. *(i) Let π be a supercuspidal representation of G. Then π contains some maximal simple type (J, λ), which is a $[G, \pi]_G$-type.*

(ii) Conversely, if (J, λ) is a maximal simple type in G, then there exists a supercuspidal representation π of G containing (J, λ).

(iii) Let (J, λ) be a maximal simple type associated to the simple stratum $[\mathfrak{A}, n, 0, \beta]$, and let $E = F[\beta]$. Then one has $\mathbf{I}_G(\lambda) = E^\times J$. Moreover, if π is a supercuspidal representation of G containing (J, λ), then there exists a unique extension Λ of λ to $E^\times J$ such that $\pi \simeq \text{c-Ind}_{E^\times J}^G\ \Lambda$.

From the fact that $\mathbf{I}_G(\lambda) = E^\times J$, one obtains as an immediate corollary the following fact which will be fundamental to our work in the next chapter:

Corollary 3.5.9. *Let (J, λ) be a maximal simple type in G, and suppose that $K \supset J$ is compact open. Then $\text{Ind}_J^K\ \lambda$ is irreducible, and $(K, \text{Ind}_J^K\ \lambda)$ is a type for the same Bernstein component as (J, λ).*

Finally, we note that the maximal simple types inherit an intertwining implies conjugacy property from their underlying simple characters:

Theorem 3.5.10. *Suppose that (J, λ) and (J', λ') are maximal simple types in G such that $\mathbf{I}_G(\lambda, \lambda') \neq \emptyset$. Then there exists a $g \in G$ such that ${}^gJ = J'$ and ${}^g\lambda \simeq \lambda'$.*

3.6 Simple types in $\mathbf{SL}_N(F)$

With the construction of maximal simple types for $G = \mathbf{GL}_N(F)$ having been completed, we wish to obtain a similar construction for $\bar{G} = \mathbf{SL}_N(F)$, where one is essentially able to transfer the results over from G by Clifford theory. Throughout this section, given a closed subgroup H of G, we will write $\bar{H} = H \cap \bar{G}$.

Given a simple type (J, λ) in G, the restriction to $\bar{J}$ of λ will split as a direct sum of irreducible representations, however these are not quite the right representations to consider: it is possible that one must consider representations on a slightly larger group than $\bar{J}$, due to the fact that $\lambda \otimes (\chi \circ \det) \downharpoonright_{\bar{J}} \simeq \lambda \downharpoonright_{\bar{J}}$ for any character χ of F.

Definition 3.6.1. Let (J, λ) be a simple type in G arising from the simple stratum $[\mathfrak{A}, n, 0, \beta]$. The *projective normalizer* of (J, λ) is the group $J^+ = J^+(\lambda)$ consisting of those $x \in U_{\mathfrak{A}}$ such that ${}^x\lambda \simeq \lambda \otimes (\chi \circ \det)$ for some $\chi \in \mathbf{X}(F)$.

Proposition 3.6.2. *Let (J, λ) be a simple type in G.*

(i) The projective normalizer J^+ contains J as a closed normal subgroup.

(ii) The quotient J^+/J is a finite abelian p-group of exponent a divisor of N.

(iii) One has $J\bar{J}^+ = J^+$.

In particular, one has $\mathrm{Res}^{J^+}_{\bar{J}^+} \mathrm{Ind}^{J^+}_{J} \ \lambda \simeq \mathrm{Ind}^{\bar{J}^+}_{\bar{J}} \mathrm{Res}^{J}_{\bar{J}} \ \lambda$.

Definition 3.6.3. A simple type in $\bar{G}$ is a pair of the form $(\bar{J}^+, \mu)$, where μ is an irreducible subrepresentation of $\mathrm{Res}^{J^+}_{\bar{J}^+} \mathrm{Ind}^{J^+}_{J} \ \lambda$ for some simple type (J, λ) in G. We say that $(\bar{J}^+, \mu)$ is maximal if (J, λ) is maximal.

One then has results analogous to those for the maximal simple types in G:

Theorem 3.6.4. *(i) Let $\bar{\pi}$ be a supercuspidal representation of $\bar{G}$. Then $\bar{\pi}$ contains some maximal simple type $(\bar{J}^+, \mu)$, which is a $[\bar{G}, \bar{\pi}]_{\bar{G}}$-type.*

(ii) Conversely, if $(\bar{J}^+, \mu)$ is a maximal simple type in $\bar{G}$, then the representation $\text{c-}\mathrm{Ind}^{\bar{G}}_{\bar{J}^+} \ \mu$ *is irreducible and supercuspidal.*

Here, part *(ii)* is simpler than the analogous result for G, since $\bar{G}$ has a compact centre.

One also has an intertwining implies conjugacy property for the maximal simple types in $\bar{G}$:

Theorem 3.6.5. *Let $(\bar{J}^{+}, \mu)$ and $(\bar{J}'^{+}, \mu')$ be two maximal simple types in $\bar{G}$. If $\mathbf{I}_{\bar{G}}(\mu, \mu') \neq \emptyset$, then there exists a $g \in \bar{G}$ such that $\bar{J}'^{+} = {}^{g}\bar{J}^{+}$ and $\mu' \simeq {}^{g}\mu$.*

3.7 Covers and semisimple types

We have seen that the theory of simple types provides a construction of types for the discrete series of $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$. The obvious next question is whether one may extend these constructions to all of the Bernstein components of these groups. This is indeed possible, with the approach being the theory of types and covers, which provides an abstract approach to extending types via parabolic induction. For this section, we return to the general setting and let $G = \mathbf{G}(F)$ denote an arbitrary connected reductive p-adic group.

Let $P = MN$ be an F-parabolic subgroup of G with Levi factor M, and let $P^{\mathrm{op}} = MN^{\mathrm{op}}$ denote the opposite parabolic, i.e. the unique F-parabolic subgroup of G with Levi factor M, the unipotent radical of which intersects trivially with N. Let ζ be a supercuspidal representation of M, and let (J_M, λ_M) be an $[M, \zeta]_M$-type. We wish to construct an $[M, \zeta]_G$-type (J, λ) satisfying certain compatibility properties with respect to (J_M, λ_M).

Definition 3.7.1. Let (J_M, λ_M) be an $[M, \zeta]_M$-type. A *G-cover* of (J_M, λ_M) is a pair (J, λ) consisting of a compact open subgroup J of G satisfying $J \cap M = J_M$ and an irreducible representation λ of J extending λ_M such that:

(i) (J, λ) is decomposed with respect to (M, P) in the sense that $J = (J \cap N^{\mathrm{op}})(J \cap M)(J \cap N)$, and both $J \cap N^{\mathrm{op}}$ and $J \cap N$ are contained in $\ker \lambda$; and

(ii) for every parabolic subgroup Q of G with Levi factor M¡ there exists an invertible element of $\mathcal{H}(G,\lambda)$ supported on a double coset Jz_QJ for some strongly (Q,J)-positive element z_Q of $Z(M)$ (in the sense of [BK98, Def. 6.15]).

The point of this definition is the following:

Theorem 3.7.2. *Let $\mathfrak{s}_M = [M,\zeta]_M$ and let $\mathfrak{s} = [M,\zeta]_G$. Let (J,λ) be a G-cover of some $\mathfrak{s}_M$-type (J_M,λ_M). Then (J,λ) is an $\mathfrak{s}$-type. Moreover, there exists a family $t_Q : \mathcal{H}(G,\lambda) \to \mathcal{H}(M,\lambda_M)$ of ring homomorphisms as Q ranges over the parabolic subgroups $Q = MU$ of G with Levi factor M, such that the following diagrams commute:*

$$\begin{array}{ccc}
\mathrm{Rep}^{\mathfrak{s}}(G) & \xrightarrow[\mathbf{M}_\lambda]{\sim} & \mathcal{H}(G,\lambda)\text{-Mod} \\
{\scriptstyle r_U}\downarrow & & \downarrow{\scriptstyle t_Q^*} \\
\mathrm{Rep}^{\mathfrak{s}_M}(M) & \xrightarrow[\mathbf{M}_{\lambda_M}]{\sim} & \mathcal{H}(M,\lambda_M)\text{-Mod}
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Rep}^{\mathfrak{s}}(G) & \xrightarrow[\mathbf{M}_\lambda]{\sim} & \mathcal{H}(G,\lambda)\text{-Mod} \\
{\scriptstyle \mathrm{Ind}_{M,Q}^G}\uparrow & & \uparrow{\scriptstyle \mathrm{Hom}_{\mathcal{H}(M,\lambda_M)}(\mathcal{H}(G,\lambda),-)} \\
\mathrm{Rep}^{\mathfrak{s}_M}(M) & \xrightarrow[\mathbf{M}_{\lambda_M}]{\sim} & \mathcal{H}(M,\lambda_M)\text{-Mod}
\end{array}$$

Here, t_Q^* denotes the restriction functor.

In the situations of interest to us, one will always be able to construct a cover of a maximal simple type.

Theorem 3.7.3 ([BK99, GR02])**.** *Let $G = \mathbf{GL}_N(F)$ or $\mathbf{SL}_N(F)$ and let (J_M,λ_M) be a maximal simple $[M,\zeta]_M$-type. Then there exists a G-cover (J,λ) of (J_M,λ_M).*

We call such a cover a *semisimple type.*

Remark 3.7.4. We emphasize that, while we have only defined semisimple types for $\mathbf{GL}_N(F)$ and $\mathbf{SL}_N(F)$ (more generally, they have been defined for representations of classical groups and of inner forms of $\mathbf{GL}_N(F)$), the notion of a cover is completely general. In particular, in Chapter 5 we will encounter covers of another class of types. However, we restrict the nomenclature of semisimple types to covers of simple types.

3.8 The unicity of types for $\mathbf{GL}_N(F)$

We end by discussing the resolution of Conjecture 2.9.7 in the case of $F = \mathbf{GL}_N(F)$, together with its generalization to non-cuspidals.

Since G contains a unique conjugacy class of maximal compact subgroups – that of $K = \mathbf{GL}_N(\mathcal{O})$ – for each irreducible representation π of G (of inertial support $[M,\zeta]_G$, say), there is a canonical identification between the set of $[M,\zeta]_G$-archetypes and the set of $[M,\zeta]_G$-typical representations of K. This allows us to easily translate existing results on unicity for G into our language of archetypes. We first discuss the supercuspidal case, where the results are more complete:

Theorem 3.8.1 ([BM02, Pas05])**.** *Let π be a supercuspidal representation of G. Then there exists a unique $[G,\pi]_G$-type (K,τ), which is obtained by taking a maximal simple $[G,\pi]_G$-type (J,λ) with $J \subset K$, and letting τ be the representation $\mathrm{Ind}_J^K\ \lambda$.*

Of course, the representation τ must be irreducible, since the K-intertwining of λ is the intersection with K of its G-intertwining, which is to say $\mathbf{I}_K(\lambda) = K \cap \mathbf{I}_G(\lambda) = K \cap E^\times J = J$.

In the case that π is an irreducible non-cuspidal representation of G, the situation is slightly more complicated. We have the following:

Theorem 3.8.2. *Let π be an irreducible representation of G of inertial support $[M,\zeta]_G$, and let (K,τ) be an $[M,\zeta]_G$-type. Then there exists a semisimple $[M,\zeta]_G$-type (J,λ) such that $\tau \simeq \mathrm{Ind}_J^K\ \lambda$. Moreover, unless $q = 2$, there exists a single semisimple $[M,\zeta]_G$-type such that the irreducible subrepresentations of $\mathrm{Ind}_J^K\ \lambda$ exhaust the $[M,\zeta]_G$-archetypes.*

In the case of $\mathbf{GL}_2(F)$, this is due to Henniart in [BM02]. The general case is due to Nadimpalli: the cases of $\mathbf{GL}_3(F)$ and a large number of other inertia classes in $\mathbf{GL}_N(F)$ are dealt with in [Nad14] and [Nad15], while the remaining cases are completed but currently not written up.

Chapter 4

The unicity of types for $\mathbf{SL}_N(F)$

In this chapter, we investigate the question of unicity for special linear groups, giving an expanded exposition of the results of [Lat16c] and [Lat16a]. In order to simplify notation, for the remainder of the chapter we introduce the convention that $G = \mathbf{GL}_N(F)$ and $\bar{G} = \mathbf{SL}_N(F)$. Given a closed subgroup H of G, we let $\bar{H} = H \cap \bar{G}$. We also denote by $K = \mathbf{GL}_N(\mathcal{O})$ the standard maximal compact subgroup of G, and $\bar{K} = \mathbf{SL}_N(\mathcal{O})$. When required to consider groups of various orders, we make use of the notation $G_{N_i} = \mathbf{GL}_{N_i}(F)$, etc. We also write $\mathbf{X}(F)$ for the group of characters $F^\times \to \mathbb{C}^\times$, $\mathbf{X}_{\mathrm{ur}}(F)$ for the subgroup of unramified characters, and $X_N(F)$ for the subgroup consisting of characters χ such that χ^N is unramified.

4.1 The main lemma

Our approach is based on the following observation that there is a close relationship between the representation theory of G and that of $\bar{G}$:

Lemma 4.1.1. *Let $M = G_{N_1} \times \cdots \times G_{N_r}$ be a standard Levi subgroup of G, and let*

$\zeta = \zeta_1 \otimes \cdots \otimes \zeta_r$ be a supercuspidal representation of M. If $\zeta' = \zeta_1' \otimes \cdots \otimes \cdots \zeta_r'$ is another supercuspidal representation of M which becomes isomorphic to ζ upon restriction to $\bar{M}$, then there exists a $\chi \in \mathbf{X}(F)$ such that $\zeta' \simeq \zeta(\chi \circ \det)$.

Proof. Restricting further to $\bar{\bar{M}} = \bar{G}_{N_1} \times \cdots \times \bar{G}_{N_r}$, the representations ζ and ζ' become tensor products of representations, all of which are finite-length representations with all irreducible subquotients supercuspidal. Hence, by [BK93b, Proposition 1.17], for each i there must exist a $\chi_i \in \mathbf{X}(F)$ such that $\zeta_i' \simeq \zeta_i(\chi_i \circ \det)$.

Our claim is that one may find a $\chi \in \mathbf{X}(F)$ such that $(\chi_1 \circ \det) \cdots (\chi_r \circ \det) = \chi \circ \det$. We show this for $r = 2$; the claim then follows in general by induction. Let $x = (x_1, x_2) \in M = G_{N_1} \times G_{N_2}$. As $(\chi_1 \circ \det) \cdot (\chi_2 \circ \det)$ is trivial on $\bar{M}$ and for $(x_1, x_2) \in \bar{M}$ one has $\det x_2 = \det x_1^{-1}$, it must be the case that $\chi_1(\det x_1) = \chi_2(\det x_1)$ for all $x = (x_1, x_2) \in \bar{M}$. As det is surjective, it follows that $\chi_1 = \chi_2$, so that $(\chi_1 \circ \det) \cdot (\chi_2 \circ \det) = \chi_1 \circ \det$, as required. □

The idea is then the following. If $\bar{\pi}$ is a supercuspidal representation of $\bar{G}$, we may arbitrarily choose a (necessarily supercuspidal) irreducible representation π of G such that $\bar{\pi}$ is a subquotient of $\pi \downharpoonright_{\bar{G}}$. Let (K, τ) be the standard representative of the unique $[G, \pi]_G$-archetype. Then as π is a subquotient of $\text{c-Ind}_K^G\ \tau$, it is certainly the case that $\bar{\pi}$ is a subquotient of $\bigoplus_{K \backslash G / \bar{G}} \text{Ind}_{^g\bar{K}}^{\bar{G}} \text{Res}_{^g\bar{K}}^{^gK}\ {}^g\tau$, and one might reasonably hope that, among the irreducible subrepresentations of ${}^g\tau \downharpoonright_{^g\bar{K}}$ one finds a complete set of representatives of the $[\bar{G}, \bar{\pi}]_{\bar{G}}$-archetypes.

Indeed, this will turn out to be the case, but is a rather deep fact. If, however, one is willing to ask for slightly less, it is possible to get an immediate result in this direction which, moreover, holds true even for the case of non-cuspidal representations, where the above justification for our approach is not quite as clear:

Lemma 4.1.2. *Let π be an irreducible representation of G of inertial support $[M, \zeta]_G$, let $\bar{\pi}$ be an irreducible subquotient of $\pi \downharpoonright_{\bar{G}}$, and let $\bar{\mathfrak{s}}$ be the inertial support of $\bar{\pi}$. Suppose that, for some $g \in G$, there exists an $\bar{\mathfrak{s}}$-archetype $({}^g\bar{K}, \bar{\tau})$. Then there exists an irreducible* *subrepresentation* Ψ *of* $\pi\downharpoonright_{{}^gK}$ *which contains* $\bar\tau$*. Moreover,* $({}^gK,\Psi)$ *is* $\mathfrak{S}$*-typical for* $\mathfrak{S}=\{[M,\zeta\otimes(\chi\circ\det)]_G\mid\chi\in\mathbf{X}_N(F)\}$.

Proof. Without loss of generality, we may as well just consider the case ${}^g\bar K=\bar K$. First, note that such a Ψ clearly exists – by Frobenius reciprocity, one has

$$\mathrm{Hom}_K(\mathrm{Ind}_{\bar K}^K\ \bar\tau,\mathrm{Res}_K^G\ \pi)=\mathrm{Hom}_{\bar K}(\bar\tau,\mathrm{Res}_{\bar K}^{\bar G}\,\mathrm{Res}_{\bar G}^G\ \pi)\neq 0,$$

and so some irreducible subrepresentation of $\mathrm{Ind}_{\bar K}^K\ \bar\tau$ is contained in π. Fix such a choice of representation Ψ.

Now suppose that π' is an irreducible representation of G such that $\mathrm{Hom}_K(\pi'\downharpoonright_K,\Psi)\neq 0$. Then

$$0\neq\mathrm{Hom}_K(\mathrm{Ind}_{\bar K}^K\ \bar\tau,\mathrm{Res}_K^G\ \pi')=\mathrm{Hom}_{\bar K}(\bar\tau,\mathrm{Res}_{\bar K}^{\bar G}\,\mathrm{Res}_{\bar G}^G\ \pi').$$

Hence π' must contain $\bar\pi$ upon restriction to $\bar G$. Suppose that π' is of inertial support $[M',\zeta']_G$. Then, as $\bar\pi$ is a subrepresentation of both $Res_{\bar G}^G\,\mathrm{Ind}_{M,P}^G\ \zeta=\mathrm{Ind}_{\bar M,\bar P}^{\bar G}\,\mathrm{Res}_{\bar M}^M\ \zeta$ and $\mathrm{Res}_{\bar G}^G\,\mathrm{Ind}_{M',P'}^G\ \zeta'=\mathrm{Ind}_{\bar M',\bar P'}^{\bar G}\,\mathrm{Res}_{\bar M'}^{M'}\ \zeta'$, for some parabolic subgroups P and P', it follows by the uniqueness of supercuspidal supports that $\mathrm{Res}_{\bar M}^M\ \zeta$ and $\mathrm{Res}_{\bar M'}^{M'}\ \zeta'$ must contain some irreducible subquotients which are isomorphic up to $\bar G$-conjugacy. In particular, we may as well take $M=M'$. By Clifford theory, the representations $\mathrm{Res}_{\bar M}^M\ \zeta$ and $\mathrm{Res}_{\bar M}^M\ \zeta'$ are direct sums over a full orbit under M-conjugacy of irreducible representations of $\bar M$, and so these two representations have a subquotient in common if and only if they are equal. Hence, it follows from Lemma 4.1.1 that $\zeta'\simeq\zeta\otimes(\chi\circ\det)$ for some $\chi\in\mathbf{X}(F)$.

So the representation π' is of inertial support $[M,\zeta\otimes(\chi\circ\det)]_G$ and contains Ψ. As π also contains Ψ, we may compare central characters to see that $\omega_{\pi'}\downharpoonright_{\mathcal{O}^\times}=\omega_\pi\downharpoonright_{\mathcal{O}^\times}$. As $F^\times\subset M$, one must have $\omega_{\pi'}=\omega_\pi\otimes(\chi\circ\det)$, so that $\chi\circ\det$ is trivial on $\mathcal{O}^\times$, which is to say that χ^N is unramified. □

Remark 4.1.3. As the set $\{{}^g\bar K\mid g\in G\}$ exhausts the maximal compact subgroups of $\bar G$, we can now apply Frobenius reciprocity to see that, at the very least, every archetype in $\bar G$ is restricted from some appropriate $\mathfrak{S}$-type.

So we fix such an $\mathfrak{S}$-typical representation Ψ in the hopes that we may see that Ψ is actually $[G,\pi]_G$-typical. This would be sufficient to determine Ψ and show that any archetype in $\bar{G}$ is contained in the restriction of an archetype in G.

4.2 Some results on simple characters

Before we move on to the proof that the representation Ψ constructed in Lemma 4.1.2 is $[G,\pi]_G$-typical, we establish some preparatory results on simple characters. These will essentially form the observations which a large part of our proof boils down to, in that they allow us to resolve the fundamental difficulty that it can, *a priori*, be difficult to distinguish between a simple character θ and its twist $\theta(\chi\circ\det)$ for some $\chi\in\mathbf{X}(F)$. Obviously, the first necessary observation is that this twist is itself a simple character. This is the main point of the appendix to [BK94], and our proof is essentially a replication of their proof, rephrased slightly in places in order to be more convenient for us. In particular, the argument is completely unoriginal. We freely quote facts established by Bushnell and Kutzko during their proof.

Lemma 4.2.1. *Let $\theta\in\mathcal{C}(\mathfrak{A},0,\beta)$ be a simple character, and let $\chi\in\mathbf{X}(F)$. Then $\theta(\chi\circ\det)$ is a simple character contained in $\mathcal{C}(\mathfrak{A},0,\beta')$ for some β'. Moreover, one has $\mathcal{C}(\mathfrak{A},0,\beta')=\{\Theta(\chi\circ\det)\mid\Theta\in\mathcal{C}(\mathfrak{A},0,\beta)\}$.*

Proof. Let $[\mathfrak{A},n,0,\beta]$ be the simple stratum giving rise to the set $\mathcal{C}(\mathfrak{A},0,\beta)$. If $\beta\in F$ then $H^1(\beta,\mathfrak{A})=U_{\mathfrak{A}}$ and $\mathcal{C}(\mathfrak{A},0,\beta)$ consists of characters of the form $\omega\circ\det$, from which the claim is clear.

So assume that $\beta\notin F$. Then, for $c\in F$, the stratum $[\mathfrak{A},-v_{\mathfrak{A}}(\beta+c),0,\beta+c]$ is simple, and one has $\mathfrak{H}(\beta,\mathfrak{A})=\mathfrak{H}(\beta+c,\mathfrak{A})$ and $\mathfrak{J}(\beta,\mathfrak{A})=\mathfrak{J}(\beta+c,\mathfrak{A})$. Let $\chi\in\mathbf{X}(F)$ be such that $\chi\circ\det$ agrees with ψ_c on $U_{\mathfrak{A}}^{\lfloor\frac{-v_{\mathfrak{A}}(c)}{2}\rfloor+1}$. Then $\mathcal{C}(\mathfrak{A},0,\beta+c)=\mathcal{C}(\mathfrak{A},0,\beta)\otimes(\chi\circ\det)$. So we need only observe that, for any $\chi\in\mathbf{X}(F)$, one may find a $c\in F$ such that $\chi\circ\det=\psi_c$ on $U_{\mathfrak{A}}^{\lfloor\frac{-v_{\mathfrak{A}}(c)}{2}\rfloor+1}$. □

The next complication is that χ is viewed as a character of G through $\det_F$, while the determinant with which the groups $H^m(\beta, \mathfrak{A})$ are most naturally compatible is $\det_E$, where $E = F[\beta]$. This results in the presence of blocks of length $e_{\mathfrak{A}}$ in the filtration $H^m(\beta, \mathfrak{A})$ of $H^1(\beta, \mathfrak{A})$ on which $\chi \circ \det$ is essentially the same.

Lemma 4.2.2. *Suppose that $k \in \mathbb{N}$ is such that $\chi \circ \det$ is non-trivial on $H^{ke_{\mathfrak{A}}+1}(\beta, \mathfrak{A})$. Then $\chi \circ \det$ is non-trivial on $H^{(k+1)e_{\mathfrak{A}}}(\beta, \mathfrak{A})$.*

Proof. As $\det_F = \mathbf{N}_{E/F} \circ \det_E$, this follows from the observation that, for any m, one has $\det_F(H^m) = \det_F(H^m \cap B^\times) = \det_F(U^m_{\mathfrak{B}_\beta})$, and that one has $\mathbf{N}_{E/F}(1 + \mathfrak{p}_E^{m+1}) = \mathbf{N}_{E/F}(1 + \mathfrak{p}_E^m)$ for $ke + 1 \leq m \leq (k+1)e - 1$, while $\mathbf{N}_{E/F}(1 + \mathfrak{p}_E^m)$ is strictly larger than $\mathbf{N}_{E/F}(1 + \mathfrak{p}_E^{m+1})$ otherwise. □

Finally, at the root of most of our arguments will be a use of Lemma 4.1.2 in order to see that some simple character θ is conjugate to $\theta(\chi \circ \det)$, for some $\chi \in \mathbf{X}_N(F)$. In general, it is possible for such a situation to arise, although this may only occur for certain wildly ramified characters χ.

Definition 4.2.3. Let π be a supercuspidal representation of G containing the simple stratum $[\mathfrak{A}, n, 0, \beta]$. We say that π is *essentially tame* if p is coprime to $e_{\mathfrak{A}}$.

Similarly, say that a supercuspidal representation $\bar{\pi}$ of $\bar{G}$ is essentially tame if it is contained in an essentially tame supercuspidal representation of G. The reason for distinguishing these supercuspidals is the following [BK93a, Remark 3.5.14]:

Lemma 4.2.4. *Let $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$ and $\theta \in \mathcal{C}(\mathfrak{A}', m, \beta')$ be such that $H^{m+1}(\beta, \mathfrak{A}) = H^{m+1}(\beta', \mathfrak{A}')$, and suppose that p is coprime to $e_{\mathfrak{A}}$ (and hence also to $e_{\mathfrak{A}'}$). If there exists a $g \in G$ such that ${}^g\theta = \theta'$, then $\theta = \theta'$.*

We will only complete the proof of our main result – that Ψ is a $[G, \pi]_G$-type – in the case that π is essentially tame. However, there are parts of the proof do not rely on this assumption; we therefore prove some of the preliminary results in greater generality.

Remark 4.2.5. The possibility that simple characters defined on the same group $H^{m+1}(\beta, \mathfrak{A})$ can intertwine without being equal leads to significant technical difficulties, as evidenced in [BK94]. While it should be expected that the results in this chapter remain true even when π is not essentially tame, their proofs should become considerably more difficult. In particular, the Bushnell–Kutzko theory alone appears to be insufficient to rule out the possibility that Ψ is contained in $\pi \otimes (\chi \circ \det)$ for some wildly ramified character χ of sufficiently large level. Indeed, approaching the problem directly via Bushnell–Kutzko theory only allows one to rule out the possibility that Ψ is contained in $\pi \otimes (\chi \circ \det)$ for and ramified character χ of level no greater than $-k_0(\beta, \mathfrak{A})$ (via a simple application of [BK93a, Lemma 3.5.10]).

4.3 Decompositions of $\pi\downharpoonright_K$

Let us begin by fixing, once and for all, some notation. Let π be a supercuspidal representation of G, and let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum such that π contains some (necessarily maximal) simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. Let κ be a β-extension of θ to $J = J(\beta, \mathfrak{A})$, and let σ be a cuspidal irreducible representation of J/J^1 such that π contains the simple type $(J, \lambda) = (J, \kappa \otimes \sigma)$. We write $E = F[\beta]$.

In order to see that the representation Ψ constructed in 4.1.2 is actually $[G, \pi]_G$-typical, we proceed by contradiction. As $\pi\downharpoonright_K$ contains a unique $[G, \pi]_G$-typical subrepresentation, if Ψ were not $[G, \pi]_G$-typical then it would be contained in the complement of this representation in $\pi\downharpoonright_K$. In each of a number of cases, we then show that this implies the existence of a representation π' of G which contains Ψ but cannot possibly be of the form $\pi' \simeq \pi \otimes (\chi \circ \det)$ for some $\chi \in \mathbf{X}_N(F)$.

Writing $\tilde{J}$ for the G-normalizer of J, there exists a unique extension Λ of λ to $\tilde{J}$ such that $\pi \simeq \text{c-}\operatorname{Ind}_{\tilde{J}}^G \Lambda$. This allows us to perform a Mackey decomposition of $\pi\downharpoonright_K$:

$$\pi\downharpoonright_K = \bigoplus_{\tilde{J}\backslash G/K} \operatorname{Ind}_{{}^gJ\cap K}^K \operatorname{Res}_{{}^gJ\cap K}^{{}^gJ} {}^g\lambda.$$

While this will be the main decomposition with which we work, it has the serious disadvantage that the coset space $\tilde{J}\backslash G/K$ is *extremely* difficult to describe. This forces us to also work with a slightly coarser decomposition. Let $\rho = \operatorname{Ind}_J^{U_{\mathfrak{A}}} \lambda$, and let $\tilde{\rho} = \text{c-}\operatorname{Ind}_{\tilde{J}}^{\mathfrak{K}_{\mathfrak{A}}} \Lambda$, so that $\pi \simeq \text{c-}\operatorname{Ind}_{\mathfrak{K}_{\mathfrak{A}}}^G \tilde{\rho}$. This then gives us

$$\pi \downharpoonright_K = \bigoplus_{\mathfrak{K}_{\mathfrak{A}} \backslash G/K} \operatorname{Ind}_{{}^gU_{\mathfrak{A}} \cap K}^K \operatorname{Res}_{{}^gU_{\mathfrak{A}} \cap K}^{{}^gU_{\mathfrak{A}}} {}^g\rho.$$

This decomposition has the advantage that there is a natural set of distinguished coset representatives for $\mathfrak{K}_{\mathfrak{A}} \backslash G/K$, which are described explicitly by Paskunas in [Pas05, Lemma 5.3]. In particular, these representatives consist of diagonal matrices with entries in $\varpi^{\mathbb{Z}}$. Whenever we discuss a coset in $\mathfrak{K}_{\mathfrak{A}} \backslash G/K$, we will implicitly assume that it is represented in this form. We now recall Paskunas' definition of type A and B cosets:

Definition 4.3.1. Let $\mathfrak{K}_{\mathfrak{A}} gK \neq \mathfrak{K}_{\mathfrak{A}} K$ be a non-trivial coset.

(i) We say that $\mathfrak{K}_{\mathfrak{A}} gK$ is *of type A* if the map $U_{\mathfrak{A}} \cap {}^{g^{-1}}K \to U_{\mathfrak{A}}/U_{\mathfrak{A}}^1$ is not surjective; or

(ii) we say that $\mathfrak{K}_{\mathfrak{A}} gK$ is *of type B* if the map $U_{\mathfrak{A}} \cap {}^{g^{-1}}K \to U_{\mathfrak{A}}/U_{\mathfrak{A}}^1$ is surjective.

Often, we will simply refer to g being of type A or B, with the obvious meaning. Similarly, we will say that an irreducible subrepresentation ξ of $\pi \downharpoonright_K$ is of type A or B if ξ is a subrepresentation of $\operatorname{Ind}_{{}^gU_{\mathfrak{A}} \cap K}^K \operatorname{Res}_{{}^gU_{\mathfrak{A}} \cap K}^{{}^gU_{\mathfrak{A}}} {}^g\rho$, for g of type A or B, respectively.

This leads to three classes of irreducible subrepresentation ξ of $\pi \downharpoonright_K$: the trivial summand $\operatorname{Ind}_{U_{\mathfrak{A}}}^K \rho$, which represents the unique $[G, \pi]_G$-archetype, the type A subrepresentations, and the type B subrepresentations. We will show that Ψ cannot be of type A or B. In each case, we are forced to use rather different approaches.

Remark 4.3.2. The notions of type A and B cosets are closely related to the lattice period $e_{\mathfrak{A}}$ of the hereditary order $\mathfrak{A}$ or, equivalently, the ramification degree of E/F. If $e_{\mathfrak{A}} = 1$, then every coset is of type A, while if $e_{\mathfrak{A}} = N$ then every coset is of type B. In particular, for N prime the two cases correspond precisely to those considered by Henniart in the appendix to [BM02]. In general though, a representation π will contain representations of types both A and B in its restriction to K.

4.4 Representations of type A

We first consider the type A subrepresentations of $\pi\downharpoonright_K$. If $\mathfrak{K}_{\mathfrak{A}}gK$ is of type A, then the map $J\cap{}^{g^{-1}}K\to J/J^1$ given by mod-J^1 reduction is also not surjective. Let H denote its image. The crucial observation is that H is a *sufficiently small* subgroup of J/J^1 in the sense of [Pas05, Definition 6.2]. This has a technical definition, but for us it suffices to say that this means that H is unable to distinguish between the irreducible representations of J/J^1. The key result of Paskunas of which we will make use is the following:

Lemma 4.4.1 ([Pas05, Proposition 6.8])**.** *For every irreducible subrepresentation ξ of $\sigma\downharpoonright_H$ there exists an irreducible representation $\sigma'\not\simeq\sigma$ of J/J^1 such that $\mathrm{Hom}_H(\sigma'\downharpoonright_H,\xi)\neq 0$.*

So let Ψ be the representation constructed in Lemma 4.1.2, and suppose that Ψ is of type A, corresponding to the coset $\mathfrak{K}_{\mathfrak{A}}gK$. By the above lemma, there exists an irreducible representation $\sigma'\not\simeq\sigma$ of J/J^1 such that Ψ is also a subrepresentation of $\mathrm{Ind}_{{}^gJ\cap K}^K\mathrm{Res}_{{}^gJ\cap K}^{{}^gJ}\ {}^g(\kappa\otimes\sigma')$. In the case that σ' may be taken to be non-cuspidal, this leads to an obvious contradiction:

Lemma 4.4.2. *Suppose that there exists a non-cuspidal irreducible representation σ' of J/J^1 such that $\Psi\hookrightarrow\mathrm{Ind}_{{}^gJ\cap K}^K\mathrm{Res}_{{}^gJ\cap K}^{{}^gJ}\ {}^g(\kappa\otimes\sigma')$. Then there exists a non-cuspidal irreducible representation π' of G which contains Ψ upon restriction to K.*

Proof. Let σ'' be any non-cuspidal irreducible representation of J/J^1. Upon restriction to H^1, the representation $\kappa\otimes\sigma''$ becomes isomorphic to a sum of copies of θ, so any irreducible representation π' of G containing $\kappa\otimes\sigma''$ must lie in $\mathrm{Irr}^{\mathfrak{S}_\theta}(G)$. If such a representation π' were supercuspidal, then it would contain some maximal simple type (J,λ'). As supercuspidal may contain only a single conjugacy class of simple characters, it must be the case that $\lambda'=\kappa\otimes\Sigma$ for some irreducible cuspidal representation Σ of J/J^1. Then we may perform a Mackey decomposition to obtain

$$\pi'\downharpoonright_J=\bigoplus_{J\backslash G/J}\mathrm{Ind}_{{}^hJ\cap J}^J\mathrm{Res}_{{}^gJ\cap J}^{{}^hJ}\ {}^h\lambda'.$$

Let h be a coset representative such that $\kappa\otimes\sigma''\hookrightarrow\mathrm{Ind}_{{}^hJ\cap J}^J\mathrm{Res}_{{}^hJ\cap J}^{{}^hJ}\ {}^h\lambda'$. By Frobenius reciprocity, it follows that $h\in\mathbf{I}_J(\kappa\otimes\sigma'',\kappa\otimes\Sigma)$. By [BK93a, Proposition 5.3.2], this intertwining set is contained in $J\mathbf{I}_{B^\times}(\sigma'' \downharpoonright_{U_{\mathfrak{B}}}, \Sigma \downharpoonright U_{\mathfrak{B}})J$. By [Pas05, Proposition 6.15], σ'' and Σ do not intertwine, so that this set is empty, giving a contradiction. Hence π' must be non-cuspidal.

As we have

$$\Psi \hookrightarrow \mathrm{Ind}_{{}^gJ\cap K}^{K} \mathrm{Res}_{{}^gJ\cap K}^{{}^gJ} \ {}^g(\kappa \otimes \sigma) \simeq \mathrm{Ind}_{{}^gJ\cap K}^{K} \mathrm{Res}_{{}^gJ\cap K}^{{}^gJ} \ {}^g(\kappa \otimes \sigma') \hookrightarrow \mathrm{Res}_K^G \operatorname{c-Ind}_J^G (\kappa \otimes \sigma'),$$

we may form the representation $\langle\Psi\rangle$ generated by Ψ in $\operatorname{c-Ind}_J^G (\kappa\otimes\sigma')$. This representation will certainly admit some irreducible quotient π' which contains Ψ. Hence, it suffices to show that such an irreducible quotient of $\langle\Psi\rangle$ contains a representation of the form $\kappa\otimes\sigma''$ with σ'' a non-cuspidal irreducible representation of J/J^1. As $\langle\Psi\rangle \hookrightarrow \operatorname{c-Ind}_J^G (\kappa\otimes\sigma)$, we have

$$\begin{aligned}\mathrm{Res}_J^G \langle\Psi\rangle &\hookrightarrow \mathrm{Res}_J^K \operatorname{c-Ind}_J^G (\kappa\otimes\sigma') \\ &= \bigoplus_{J\backslash G/J} \mathrm{Ind}_{{}^hJ\cap J}^{J} \mathrm{Res}_{{}^hJ\cap J}^{{}^hJ} \ {}^h(\kappa\otimes\sigma').\end{aligned}$$

Now, $\langle\Psi\rangle$ contains θ upon restriction to H^1, which implies that $\langle\Psi\rangle \in \mathrm{Rep}^{\mathfrak{S}_\theta}(G)$, as $\langle\Psi\rangle$ is generated by a single vector. Then any irreducible quotient of $\langle\Psi\rangle$ must contain $\kappa\otimes\Sigma'$ for some irreducible representation Σ' of J/J^1, and it remains to show that Σ' cannot be cuspidal. If Σ' were cuspidal then we would obtain an inclusion $\kappa\otimes\Sigma' \hookrightarrow \mathrm{Ind}_{{}^hJ\cap J}^{J} \mathrm{Res}_{{}^hJ\cap J}^{{}^hJ} \ {}^g(\kappa\otimes\sigma')$, which is to say that $h \in \mathbf{I}_J(\kappa\otimes\Sigma', \kappa\otimes\sigma')$. Applying [Pas05, Proposition 6.16] again, we see that this intertwining set may only be non-empty if Σ' is non-cuspidal. □

Thus it remains to consider the case that any such σ' must be cuspidal. In this case, we are able to see that σ' must be of a rather specific form:

Lemma 4.4.3. *Suppose that π is essentially tame, and suppose that there exists a cuspidal representation $\sigma \not\simeq \sigma'$ of J/J^1 such that $\Psi \hookrightarrow \mathrm{Ind}_{{}^gJ\cap K}^{K} \mathrm{Res}_{{}^gJ\cap K}^{{}^gJ} \ {}^g(\kappa\otimes\sigma')$. Then there exist a $\chi \in \mathbf{X}_N(F)$ with χ trivial on $\det J^1$ such that $\sigma' \simeq \sigma\otimes(\chi\circ\det)$.*

Proof. Let λ' be the maximal simple type $\kappa\otimes\sigma'$, which must necessarily be contained in some supercuspidal representation π' of G which is contained in some inertia class other than that of π. Then Ψ must be a subrepresentation of $\pi' \downharpoonright_K$, and so, by Lemma 4.1.2, we know that $\pi' \simeq \pi \otimes (\chi \circ \det)$ for some $\chi \in \mathbf{X}_N(F)$. Hence the archetypes for π' and $\pi \otimes (\chi \circ \det)$ coincide.

Respectively, these archetypes are represented by $\mathrm{Ind}_J^K\ \lambda'$ and $\mathrm{Ind}_J^K\ (\lambda \otimes (\chi \circ \det))$. As $\pi' \simeq \pi \otimes (\chi \circ \det)$, it follows from Lemma 4.2.1 that π' contains the simple character $\theta(\chi \circ \det)$. Hence, as supercuspidal representations may contain only a single conjugacy class of simple characters, the simple characters θ and $\theta(\chi \circ \det)$ must be G-conjugate. Since π is essentially tame, simple characters contained in π may only be conjugate if they are equal, hence χ is trivial on $\det H^1 = \det J^1$.

We now return to the isomorphism $\mathrm{Ind}_J^K\ \lambda' \simeq \mathrm{Ind}_J^K\ (\lambda \otimes (\chi \circ \det))$. Then we have $\mathbf{I}_K(\lambda', \lambda \otimes (\chi \circ \det)) \neq \emptyset$. As $\det H^1 = \det J^1$, both λ' and $\lambda \otimes (\chi \circ \det)$ contain κ upon restriction to J^1. Thus, by [BK93a, Proposition 5.3.2], we have

$$\mathbf{I}_K(\lambda', \lambda \otimes (\chi \circ \det)) \subset K \cap J\mathbf{I}_{B^\times}(\lambda' \downharpoonright_{U_{\mathfrak{B}_\beta}}, \lambda \otimes (\chi \circ \det) \downharpoonright_{U_{\mathfrak{B}_\beta}})J \subset JB^\times J = J,$$

from which we see that $\lambda' \simeq \lambda \otimes (\chi \circ \det)$. Now we apply the functor $\mathbf{K}_\kappa = \mathrm{Hom}_{J^1}(\kappa, -)$, which gives $\sigma' \simeq \sigma \otimes (\chi \circ \det)$. □

4.5 Representations of type B

We now consider the case that the representation Ψ constructed in Lemma 4.1.2 is a type B subrepresentation of $\pi \downharpoonright_K$, corresponding to the coset $\mathfrak{K}_{\mathfrak{A}} gK$. There are then two further subcases which, despite allowing for essentially the same argument, require a slightly different setup.

So let us suppose first that $k_0(\beta, \mathfrak{A}) \neq -1$. In this case one has $H^1(\beta, \mathfrak{A}) = U^1_{\mathfrak{B}_\beta} H^2(\beta, \mathfrak{A})$, and so we may view a non-trivial character μ of $(1+\mathfrak{p}_E)/(1+\mathfrak{p}_E^2)$ as a character of H^1/H^2 via

$$H^1/H^2 \xrightarrow{\sim} U^1_{\mathfrak{B}_\beta}/U^2_{\mathfrak{B}_\beta} \xrightarrow{\det_{\mathfrak{B}_\beta}} (1+\mathfrak{p}_E)/(1+\mathfrak{p}_E^2) \xrightarrow{\mu} \mathbb{C}^\times.$$

On the other hand, if $k_0(\beta,\mathfrak{A}) = -1$ then the above no longer works. Instead, let $[\mathfrak{A}, n, 1, \gamma]$ be a simple stratum equivalent to the pure stratum $[\mathfrak{A}, n, 1, \beta]$. Then $\theta\psi_{\beta-\gamma}^{-1}$ is a simple character in $\mathcal{C}(\mathfrak{A}, 0, \gamma)$.

To combine these two cases, we let μ be as above if $k_0(\beta,\mathfrak{A}) \neq -1$, and let $\mu = \psi_{\beta-\gamma}^{-1}$ otherwise. As noted by Paskunas during the proofs of [Pas05, Propositions 7.3, 7.16], in each of these two cases we then have $\theta\mu = \theta$ on $H^1 \cap {}^{g^{-1}}K$. Moreover, in each case the character μ is trivial on H^2.

Lemma 4.5.1. *Suppose that π is essentially tame. Then the representation Ψ cannot be of type B.*

Proof. We have, for some integer n, a chain of inclusions

$$\begin{aligned}\Psi &\hookrightarrow \mathrm{Ind}_{{}^gJ\cap K}^{K} \mathrm{Res}_{{}^gJ\cap K}^{{}^gJ}\ {}^g\lambda \\ &\hookrightarrow \mathrm{Ind}_{{}^gJ\cap K}^{K} \mathrm{Ind}_{{}^gH^1\cap K}^{{}^gJ\cap K} \mathrm{Res}_{{}^gH^1\cap K}^{{}^gJ}\ {}^g\lambda \\ &= \mathrm{Ind}_{{}^gH^1\cap K}^{K}\ {}^g\theta \downharpoonright_{{}^gH^1\cap K}^{\oplus n} \\ &= \mathrm{Ind}_{{}^gH^1\cap K}^{K}\ {}^g(\theta\mu) \downharpoonright_{{}^gH^1\cap K}^{\oplus n} \\ &\hookrightarrow \mathrm{Res}_K^G\, \text{c-}\mathrm{Ind}_{H^1}^G\ \theta\mu.\end{aligned}$$

Now, by Lemma 4.1.2, any irreducible subquotient of $\text{c-}\mathrm{Ind}_K^G\ \Psi$ is a supercuspidal representation of the form $\pi \otimes (\chi \circ \det)$ for some $\chi \in \mathbf{X}_N(F)$. Hence there exists a supercuspidal representation of this form which contains the simple character $\theta\mu$. But now $\pi \otimes (\chi \circ \det)$ clearly also contains the simple character $\theta(\chi \circ \det)$, and so, since a supercuspidal representation contains a unique conjugacy class of simple characters, there exists a $\chi \in \mathbf{X}_N(F)$ such that $\theta\mu$ is conjugate to $\theta(\chi \circ \det)$.

Suppose that χ is trivial on $\det H^1$. Then $\theta\mu$ is conjugate to θ, but Paskunas establishes during the proofs of [Pas05, Propositions 7.3, 7.16] that, in either case this is not possible. So χ is non-trivial on $\det H^1$. As we are assuming that Ψ is a representation of type B, it must be the case that $e_{\mathfrak{A}} > 1$; hence χ is non-trivial on $\det H^2$. But now, in either case,

μ is trivial on H^2 and so we see that θ is conjugate to $\theta(\chi \circ \det)$ on H^2, which, since π is essentially tame, implies that χ is trivial on $\det H^2$, a contradiction. $\square$

4.6 Refinement of the main lemma for supercuspidals

The results of the previous two sections complete all of the preparation needed to deduce that the representation Ψ is $[G,\pi]_G$-typical, for π supercuspidal. We retain our notation from before, so that there exists a simple stratum $[\mathfrak{A}, n, 0, \beta]$, a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ with β-extension κ and a cuspidal representation σ of J/J^1 such that π contains the simple type $(J, \lambda) = (J, \kappa \otimes \sigma)$.

Theorem 4.6.1. *Suppose that π is an essentially tame supercuspidal representation of G. Then the representation Ψ constructed in Lemma 4.1.2 is a $[G,\pi]_G$-type.*

Proof. Suppose not. By Lemma 4.5.1 we see that Ψ must be a type A representation. By Lemma 4.4.1, there exists an irreducible representation σ' of J/J^1 such that Ψ is a subrepresentation of $\mathrm{Ind}_{^gJ\cap K}^{K} \mathrm{Res}_{^gJ\cap K}^{^gJ}\ {}^g(\kappa \otimes \sigma')$. If σ' may be taken to be non-cuspidal then, by Lemma 4.4.2, there exists a non-cuspidal irreducible representation π' of G containing Ψ, which is impossible by Lemma 4.1.2.

So suppose that the only such irreducible representations σ' of J/J^1 are cuspidal, so that we are assuming that $\mathrm{Ind}_H^{J/J^1}\ \xi$ splits as a direct sum of cuspidal irreducible representations of J/J^1. If the extension E/F is totally ramified then by [Pas05, Corollary 6.6] the group H is contained in some proper parabolic subgroup of J/J^1. But now, if the N is the unipotent radical of a parabolic subgroup of J/J^1 opposite to one containing H, then the restriction to N of $\mathrm{Ind}_H^{J/J^1}\ \xi$ surjects onto $\mathrm{Ind}_{H\cap N}^{N} \mathrm{Res}_{H\cap N}^{H}\ \xi$. As H intersects trivially with N, this is simply the regular representation of N, which certainly contains the trivial representation of N, contradicting the cuspidality of $\mathrm{Ind}_H^{J/J^1}\ \xi$.

Hence we need only consider the case that E/F is not totally ramified. If the only irreducible representations σ' of J/J^1 such that $\Psi \hookrightarrow \mathrm{Ind}_{{}^gJ\cap K}^{K} \mathrm{Res}_{{}^gJ\cap K}^{{}^gJ}\ {}^g(\kappa\otimes\sigma')$ are cuspidal, then all such representations σ' are of the form $\sigma' \simeq \sigma\otimes(\chi\circ\det)$ for some character $\chi\in\mathbf{X}_N(F)$ trivial on $\det J^1$, by Lemma 4.4.3. Hence we see that all such representations σ' actually identify upon restriction to H. Moreover, these representations will identify upon restriction to the product $H\cdot\mathbf{SL}_{N/[E:F]}(\mathbf{k}_E)\subset\mathbf{GL}_{N/[E:F]}(\mathbf{k}_E)\simeq J/J^1$.

Now, there are precisely $\gcd(N,q_F-1)$ characters $\chi\in\mathbf{X}_N(F)$ which are trivial on $\det J^1$. Thus, writing $\mathscr{H} = H\cdot\mathbf{SL}_{N/[E:F]}(\mathbf{k}_E)\subset J/J^1$ and $\Xi=\sigma\downharpoonright_{\mathscr{H}}$, if we can show that $\mathrm{Ind}_{\mathscr{H}}^{J/J^1}\ \Xi$ contains at least q_F distinct irreducible subrepresentations then we will have a contradiction, as every irreducible subrepresentation of $\mathrm{Ind}_H^{J/J^1}\ \sigma\downharpoonright_H\twoheadrightarrow\mathrm{Ind}_{\mathscr{H}}^{J/J^1}\ \Xi$ is a twist of σ by such a χ.

Since $\mathscr{H}$ contains the commutator subgroup $\mathbf{SL}_{N/[E:F]}(\mathbf{k}_E)$ of J/J^1, the representation $\mathrm{Ind}_{\mathscr{H}}^{J/J^1}\ \mathbb{1}$ will split as a multiplicity-free direct sum of $[J/J^1:\mathscr{H}]$ distinct characters of J/J^1; hence the number of distinct irreducible subrepresentations of $\mathrm{Ind}_{\mathscr{H}}^{J/J^1}\ \Xi$ is at least the index of $\mathscr{H}$ in J/J^1. Thus, it suffices for us to show that this index is no smaller than q_F.

As E/F is not totally ramified, $\mathbf{k}_E/\mathbf{k}_F$ is a non-trivial extension. Then there exists a proper subextension $\mathbf{k}$ of $\mathbf{k}_E$ which contains $\mathbf{k}_F$ and is of maximal degree among such extensions of $\mathbf{k}_F$ such that H contains only $\mathbf{k}$-rational points of J/J^1 (by combining [Pas05, Lemma 6.5] and [Pas05, Corollary 6.6]). Thus, if $f=f(E/F)$ is the residue class degree of E/F then $\mathbf{k}\simeq\mathbb{F}_{q_F^{f-1}}$, and so we may certainly take as a lower bound for $[J/J^1:\mathscr{H}]$ the number

$$\frac{|\mathbf{GL}_{N/[E:F]}(\mathbf{k}_E)|}{|\mathbf{GL}_{N/[E:F]}(\mathbf{k})\cdot\mathbf{SL}_{N/[E:F]}(\mathbf{k}_E)|}=\frac{|\mathbf{GL}_{N/[E:F]}(\mathbf{k}_E)|}{|\mathbf{SL}_{N/[E:F]}(\mathbf{k}_E)|}\cdot\left(\frac{\mathbf{GL}_{N/[E:F]}(\mathbf{k})}{\mathbf{SL}_{N/[E:F]}(\mathbf{k})}\right)^{-1}=\frac{q_F^f-1}{q_F^{f-1}-1}.$$

This is then no less than q_F, as required. □

Remark 4.6.2. One actually obtains a slight strengthening of this result for free: by arguing precisely as in Lemma 4.1.2, one sees that any irreducible subrepresentation of

$\mathrm{Ind}_{\bar{K}}^{K}\ \bar{\tau}$ is contained in some supercuspidal representation π containing $\bar{\pi}$ upon restriction to $\bar{G}$. Then *every* irreducible subrepresentation Ψ of $\mathrm{Ind}_{\bar{K}}^{K}\ \bar{\tau}$ is a $[G,\pi]_G$-type for some such π. This also means that an analogue of Lemma 4.1.1 holds true for $\bar{\tau}$: if τ, τ' are two irreducible representations of K which contain $\bar{\tau}$ upon restriction, then $\tau' \simeq \tau \otimes (\chi \circ \det)$ for some character $\chi \in \mathbf{X}(F)$.

4.7 The main unicity results

With the refinement of Lemma 4.1.2 having been completed, we are able to immediately obtain a number of results on the unicity of types in $\bar{G}$. We first discuss our results for the supercuspidal representations, where we are able to easily deduce that any archetype is represented by a representation induced from a simple type:

Lemma 4.7.1. *Let π be an essentially tame supercuspidal representation of G, and let (K,τ) be the unique $[G,\pi]_G$-archetype. Then every irreducible component of $\tau\downharpoonright_{\bar{K}}$ is induced from a maximal simple type in $\bar{G}$.*

Proof. The representation τ must be of the form $\tau = \mathrm{Ind}_J^K\ \lambda$ for some maximal simple type (J,λ) in G. Then we may perform a Mackey decomposition to obtain

$$\begin{aligned}\mathrm{Res}_{\bar{K}}^{K}\mathrm{Ind}_J^K\ \lambda &= \bigoplus_{J\backslash K/\bar{K}} \mathrm{Ind}_{{}^gJ\cap\bar{K}}^{\bar{K}}\mathrm{Res}_{{}^gJ\cap\bar{K}}^{{}^gJ}\ {}^g\lambda \\ &= \bigoplus_{J\backslash K/\bar{K}} \mathrm{Ind}_{{}^g\bar{J}^+}^{\bar{K}}\mathrm{Ind}_{{}^g\bar{J}}^{{}^g\bar{J}^+}\mathrm{Res}_{{}^g\bar{J}}^{{}^gJ}\ {}^g\lambda \\ &= \bigoplus_{J\backslash K/\bar{K}} \mathrm{Ind}_{{}^g\bar{J}^+}^{\bar{K}}\ {}^g(\mathrm{Ind}_{\bar{J}}^{\bar{J}^+}\mathrm{Res}_{\bar{J}}^{J}\ \lambda).\end{aligned}$$

But now, as $J\bar{J}^+ = J^+$, one has $\mathrm{Ind}_{\bar{J}}^{\bar{J}^+}\mathrm{Res}_{\bar{J}}^{J}\ \lambda = \mathrm{Res}_{\bar{J}^+}^{J^+}\mathrm{Ind}_J^{J^+}\ \lambda$. By definition, the irreducible components of this representation are maximal simple types in $\bar{G}$ and the result follows. $\square$

Theorem 4.7.2. *Let $\bar{\pi}$ be an essentially tame supercuspidal representation of $\bar{G}$.*

(i) If $(\mathscr{K},\bar{\tau})$ is a $[\bar{G},\bar{\pi}]_{\bar{G}}$-archetype, then there exists a maximal simple type $(\bar{J}^+,\mu)$ with $\bar{J}^+ \subset \mathscr{K}$ such that $\bar{\tau} \simeq \mathrm{Ind}_{\bar{J}^+}^{\mathscr{K}}\ \mu$.

(ii) If $(\bar{J}^+, \mu)$ is a maximal simple type contained in $\bar{\pi}$ and $\mathscr{K}$ is a maximal compact subgroup of $\bar{G}$ containing $\bar{J}^+$, then the representation $\operatorname{Ind}_{\bar{J}^+}^{\mathscr{K}} \mu$ is the unique $[\bar{G}, \bar{\pi}]_{\bar{G}}$-typical representation of $\mathscr{K}$.

Proof. Without loss of generality, assume that $\mathscr{K} = \bar{K}$. By Theorem 4.6.1, if π is a supercuspidal representation of G which contains $\bar{\pi}$ upon restriction, then $\bar{\tau}$ is an irreducible component of the restriction to $\bar{K}$ of (K, τ), where (K, τ) is the unique $[G, \pi]_G$-archetype. Then *(i)* follows immediately from Lemma 4.7.1.

For *(ii)*, it remains to check that, given two distinct maximal simple types $(\bar{J}^+, \mu)$ and $(\bar{J}'^+, \mu)$ contained in $\bar{\pi}$ which are, moreover, contained in the same conjugacy class of maximal compact subgroups, these simple types provide the same archetypes through induction. Thus, we may as well assume that $\bar{J}^+, \bar{J}'^+ \subset \bar{K}$. As $(\bar{J}^+, \mu)$ and $(\bar{J}'^+, \mu')$ are $[\bar{G}, \bar{\pi}]_{\bar{G}}$-types, $\bar{\pi}$ will appear as an irreducible subquotient of the induced representations $\operatorname{c-Ind}_{\bar{J}^+}^{\bar{G}} \mu$ and $\operatorname{c-Ind}_{\bar{J}'^+}^{\bar{G}} \mu'$; hence we will have

$$\begin{aligned} 0 &\neq \operatorname{Hom}_{\bar{G}}(\operatorname{c-Ind}_{\bar{J}^+}^{\bar{G}} \mu, \operatorname{c-Ind}_{\bar{J}'^+}^{\bar{G}} \mu') \\ &= \operatorname{Hom}_{\bar{J}^+}(\mu, \operatorname{Res}_{\bar{J}^+}^{\bar{G}} \operatorname{c-Ind}_{\bar{J}'^+}^{\bar{G}} \mu') \\ &= \bigoplus_{\bar{J}'^+ \backslash \bar{G} / \bar{J}^+} \operatorname{Hom}_{\bar{J}^+}(\mu, \operatorname{Ind}_{{}^g\bar{J}^+ \cap \bar{J}'^+}^{\bar{J}^+} \operatorname{Res}_{{}^g\bar{J}^+ \cap \bar{J}'^+}^{{}^g\bar{J}^+} {}^g\mu') \\ &= \bigoplus_{\bar{J}'^+ \backslash \bar{G} / \bar{J}^+} \operatorname{Hom}_{{}^g\bar{J}'^+ \cap \bar{J}^+}(\operatorname{Res}_{{}^g\bar{J}'^+ \cap \bar{J}^+}^{\bar{J}^+} \mu, \operatorname{Res}_{{}^g\bar{J}'^+ \cap \bar{J}^+}^{{}^g\bar{J}'^+} {}^g\mu'), \end{aligned}$$

and so $\mathbf{I}_{\bar{G}}(\mu, \mu') \neq \emptyset$. But then, by the intertwining implies conjugacy property, there will exist a $g \in \bar{G}$ such that ${}^g(\operatorname{Ind}_{\bar{J}^+}^{\bar{K}} \mu) \simeq \operatorname{Ind}_{\bar{J}'^+}^{{}^g\bar{K}} \mu'$. As $\bar{J}'^+$ is contained in at most one maximal compact subgroup in each $\bar{G}$-conjugacy class, we must actually have ${}^g\bar{K}' = \bar{K}$, and so $(\bar{J}^+, \mu)$ and $(\bar{J}'^+, \mu')$ induce to the same archetype. □

Given a fixed choice of maximal compact subgroup $\mathscr{K}$, this gives a complete description of the theory of $[\bar{G}, \bar{\pi}]_{\bar{G}}$-typical representations of $\mathscr{K}$. However, it is natural to ask whether it is possible to give a uniform description of the number of conjugacy classes of $\mathscr{K}$ for which there exists a $[\bar{G}, \bar{\pi}]_{\bar{G}}$-typical representation of $\mathscr{K}$. This turns out to have a rather satisfying answer.

Definition 4.7.3. Let $\bar{\pi}$ be a supercuspidal representation of $\bar{G}$, and let π be a supercuspidal representation of G which contain $\bar{\pi}$ upon restriction. Suppose that π contains some simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. Then we define the *ramification degree* of $\bar{\pi}$ to be $e_{\bar{\pi}} = e_{\mathfrak{A}}$.

Remark 4.7.4. This is indeed well-defined – any two such representations π containing $\bar{\pi}$ are related by a twist by some character $\chi \circ \det$, which does not affect the lattice period of the hereditary order from which the (pairwise G-conjugate) simple characters contained in π are constructed.

Theorem 4.7.5. *Let $\bar{\pi}$ be an essentially tame supercuspidal representation of $\bar{G}$. Then $\bar{\pi}$ contains precisely $e_{\bar{\pi}}$ archetypes. Moreover, each of these $e_{\bar{\pi}}$ archetypes contained in $\bar{\pi}$ are pairwise G-conjugate.*

Proof. After Theorem 4.7.2, it remains only for us to show that, given a maximal simple type $(\bar{J}^+, \mu)$ in $\bar{G}$, the group $\bar{J}^+$ embeds into precisely $e_{\bar{\pi}}$ conjugacy classes of maximal compact subgroups of $\bar{G}$.

Let $(\bar{J}^+, \mu)$ be a maximal simple type contained in $\bar{\pi}$, of the form of an irreducible subrepresentation of $\mathrm{Ind}_{\bar{J}}^{\bar{J}^+} \mathrm{Res}_{\bar{J}}^{J}\ \lambda$ for some maximal simple type (J, λ) in G, and suppose that (J, λ) is constructed via the simple stratum $[\mathfrak{A}, n, 0, \beta]$. Let $E = F[\beta]$. Let $\mathscr{K}$ be a maximal compact subgroup of $\bar{G}$ containing $\bar{J}^+$, and let ϖ_E be a uniformizer of E. Then the groups ${}^{\varpi_E^j}\mathscr{K}$, $0 \leq j \leq e_{\mathfrak{A}} - 1$ are each contained in a different G-conjugacy class of maximal compact subgroups of $\bar{G}$ and, since the G-normalizer of (J, λ) is precisely $E^\times J$, one sees that $\bar{J}^+$ is contained in each of the ${}^{\varpi_E^j}\mathscr{K}$. Write $\mathscr{K}_i = {}^{\varpi_E^j}\mathscr{K}$ for $0 \leq j \leq e_{\mathfrak{A}} - 1$. Then we have seen that the $e_{\mathfrak{A}}$ representations $\mathrm{Ind}_{\bar{J}^+}^{\mathscr{K}_i}\ \mu$ are irreducible and represent $[\bar{G}, \bar{\pi}]_{\bar{G}}$-archetypes. Moreover, since the powers of ϖ_E are contained in $N_G(J) = E^\times J$, these representations are pairwise G-conjugate.

It remains to show that $\bar{J}^+$ does not admit a containment into a member of any other conjugacy class of maximal compact subgroups of $\bar{G}$. Writing

$$\nu = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \varpi & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

i.e. letting ν be the matrix with $\nu_{i,i+1} = 1$ for $1 \leq i \leq N-1$, $\nu_{N,1} = \varpi$ and $\nu_{i,j} = 0$ otherwise (with the point being that ν is then a uniformizer of the totally ramified degree N extension $F[\nu]/F$), the N compact open subgroups ${}^{\nu^j}\bar{K}$, $0 \leq j \leq N-1$ form a system of representatives of the N conjugacy classes of maximal compact subgroups in $\bar{G}$. Without loss of generality, assume that $\bar{J}^+ \subset \bar{K}$; then there is a choice ϖ_E of uniformizer of E such that ${}^{\varpi_E^j}\bar{K} \subset {}^{\nu^{Nj/e_{\mathfrak{A}}}}\bar{K}$ for each $0 \leq j \leq e_{\mathfrak{A}} - 1$. The group $\bar{J}/\bar{J}^1 \simeq \mathbf{SL}_{N/[E:F]}(\mathbf{k}_E)$ contains the kernel of the norm map $\mathbf{N}_{\mathbf{k}_L/\mathbf{k}_F}$ on some degree $N/[E:F]$ extension $\mathbf{k}_L/\mathbf{k}_F$. This kernel is a cyclic group of order $\frac{q^{N/e_{\mathfrak{A}}}-1}{q-1}$. Suppose that $\bar{J}^+$ were contained in ${}^{\nu^k}\bar{K}$ for some value of k other than the $e_{\mathfrak{A}}$ values constructed above. Then one would have $\bar{J}^+ \subset \left(\bigcap_{i=0}^{e_{\mathfrak{A}}-1} {}^{\nu^{jN/e_{\mathfrak{A}}}}\bar{K}\right) \cap {}^{\nu^k}\bar{K}$. This group is equal to $\bar{U}_{\mathfrak{C}}$ for some hereditary order $\mathfrak{C}$ of lattice period $e_{\mathfrak{A}} + 1$ (note that no issue arises if $e_{\mathfrak{A}} = N$; one has already constructed all possible archetypes).

By Zsigmondy's theorem, unless $N/e_{\mathfrak{A}} = 2$ and $q = 2^i - 1$ or $N/e_{\mathfrak{A}} = 6$ and $q = 2$, there exists a prime r dividing $q^{N/e_{\mathfrak{A}}} - 1$ but not dividing $q^s - 1$ for any $1 \leq s < N/e_{\mathfrak{A}}$. If $N/e_{\mathfrak{A}} = 6$ and $q = 2$, let $r = 63$, and if $N/e_{\mathfrak{A}} = 2$ and $q = 2^i - 1$, let $r = 4$. While in the latter two cases r is composite, it will be coprime to $q^s - 1$ for every $1 \leq s < N/e_{\mathfrak{A}}$, which suffices for our purposes. Thus, via the embedding $\ker \mathbf{N}_{\mathbf{k}_L/\mathbf{k}_F} \hookrightarrow \bar{J}/\bar{J}^1$ one obtains in each case an order r element of $\bar{J}/\bar{J}^1$, which, lifts to give an order r element of $\bar{J}$. The inclusion

$$\bar{J}/\bar{J}^1 \hookrightarrow J/J^1 \simeq \mathbf{GL}_{N/[E:F]}(\mathbf{k}_E) \hookrightarrow \mathbf{GL}_{N/e_{\mathfrak{A}}}(\mathbf{k}_F) \hookrightarrow \prod_{i=1}^{e_{\mathfrak{A}}} \mathbf{GL}_{N/e_{\mathfrak{A}}}(\mathbf{k}_F)$$

maps $\bar{J}/\bar{J}^1$ to a block-diagonal embedding, with the blocks being pairwise Galois conjugate. Hence, each of the blocks $\mathbf{GL}_{N/e_{\mathfrak{A}}}(\mathbf{k}_F)$ contains an order r element. However, as one also has $\bar{J} \subset \bar{U}_{\mathfrak{C}}$, one again obtains an order r element of $U_{\mathfrak{C}}/U_{\mathfrak{C}}^1 \simeq \prod_{i=1}^{e_{\mathfrak{A}}+1} \mathbf{GL}_{N_i}(\mathbf{k}_F)$ for some partition $N = N_1 + \cdots + N_{e_{\mathfrak{A}}+1}$ of N. Among these N_i, there will be $e_{\mathfrak{A}} - 1$ which are equal to $N/e_{\mathfrak{A}}$, and the remaining two will be distinct from $N/e_{\mathfrak{A}}$. Hence, in the image of $\ker \mathbf{N}_{E/F} \hookrightarrow U_{\mathfrak{C}}/U_{\mathfrak{C}}^1$, one obtains an order r element in a block, which is actually contained in the standard parabolic subgroup of $\mathbf{GL}_{N/e_{\mathfrak{A}}}(\mathbf{k}_F)$ corresponding to the Levi subgroup $\mathbf{GL}_{N_l}(\mathbf{k}_F) \times \mathbf{GL}_{N_k}(\mathbf{k}_F)$, for some $l + k = N/e_{\mathfrak{A}}$. But the order of this group is $\left(\prod_{i=0}^{N_l-1} q^i(q^{N_l-i} - 1)\right) \cdot \left(\prod_{j=0}^{N_j-1} q^i(q^{N_k-1} - 1)\right)$. Thus, r must divide one of these factors. Clearly r cannot divide q^t for any t; otherwise r could not divide $q^{N/e_{\mathfrak{A}}} - 1$. Also, as $N_l - i, N_k - i < N/e_{\mathfrak{A}}$ for all relevant i, it cannot be the case that r divides $|\mathbf{GL}_{N_l}(\mathbf{k}_F) \times \mathbf{GL}_{N_k}(\mathbf{k}_F)|$, giving the desired contradiction.

Hence, the only way in which one might obtain more than $e_{\bar{\pi}}$ archetypes is if $\bar{\pi}$ contained simple types which are G-conjugate but not $\bar{G}$-conjugate. By the intertwining implies conjugacy property, this cannot be the case. □

Finally, a somewhat trivial observation is that archetypes necessarily appear with multiplicity 1, just as is clearly the case in G:

Proposition 4.7.6. *Let $\bar{\pi}$ be an essentially tame supercuspidal representation of $\bar{G}$, and let $(\mathscr{K}, \bar{\tau})$ be a $[\bar{G}, \bar{\pi}]_{\bar{G}}$-archetype. Then $\bar{\tau}$ occurs in $\bar{\pi}$ with multiplicity 1.*

Proof. There exists a maximal simple type $(\bar{J}^+, \mu)$ contained in $\bar{\pi}$ such that $\bar{J}^+ \subset \mathscr{K}$ and $\bar{\tau} \simeq \mathrm{Ind}_{\bar{J}^+}^{\mathscr{K}} \ \mu$, and one also has $\bar{\pi} \simeq \text{c-}\mathrm{Ind}_{\bar{J}^+}^{\bar{G}} \ \mu$. Then we calculate

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{K}}(\bar{\pi} \downarrow_{\mathscr{K}}, \bar{\tau}) &= \mathrm{Hom}_{\mathscr{K}}(\mathrm{Res}_{\mathscr{K}}^{\bar{G}} \text{c-}\mathrm{Ind}_{\bar{J}^+}^{\bar{G}} \ \mu, \mathrm{Ind}_{\bar{J}^+}^{\mathscr{K}} \ \mu) \\
&= \bigoplus_{\bar{J}^+ \backslash \bar{G} / \bar{J}^+} \mathrm{Hom}_{\bar{J}^+}(\mathrm{Ind}_{{}^g\bar{J}^+ \cap \bar{J}^+}^{\bar{J}^+} \mathrm{Res}_{{}^g\bar{J}^+ \cap \bar{J}^+}^{{}^g\bar{J}^+} \ {}^g\mu, \mu) \\
&= \bigoplus_{\bar{J}^+ \backslash \bar{G} / \bar{J}^+} \mathrm{Hom}_{{}^g\bar{J}^+ \cap \bar{J}^+}({}^g\mu, \mu).
\end{aligned}$$

The $\bar{G}$-intertwining of μ is precisely $\bar{J}^+$ (as $\text{c-Ind}_{\bar{J}^+}^{\bar{G}}\ \mu$ is irreducible and supercuspidal), and thus one has $\text{Hom}_{{}^g\bar{J}^+\cap\bar{J}^+}({}^g\mu,\mu) \neq 0$ if and only if $g \in \bar{J}^+$, and so this space is one-dimensional as required. □

Remark 4.7.7. The reader should note that the results of this section only require that the supercuspidal representations in question are essentially tame in order to apply Theorem 4.6.1; if one assumes that Theorem **??** is true in general, then the results of this section follow in complete generality, with identical proofs.

4.8 Remarks on the non-cuspidal case

Since Lemma 4.1.2 makes no assumptions on the cuspidality of π, one would expect that our strategy of proof extends to cover the non-cuspidal representations. Indeed, this turns out to very nearly be the case, although there are some complications due to the types under consideration being induced from semisimple types, rather than from maximal simple types. This means that we are unable to rule out a small family of additional possibilities other than that the representation Ψ is typical.

We begin by setting up some notation. Let π be an irreducible representation of G of supercuspidal support (M,ζ), where $M = G_{N_1} \times \cdots \times G_{N_r}$ is a standard Levi subgroup of G and $\zeta = \zeta_1 \otimes \cdots \otimes \zeta_r$ is a supercuspidal representation of M. Then there exist simple strata $[\mathfrak{A}_i, n_i, 0, \beta_i]$ in $\text{Aut}_F(F^{N_i})$, respectively, for $1 \leq i \leq r$, giving rise to simple characters $\theta_i \in \mathcal{C}(\mathfrak{A}_i, 0, \beta_i)$ and β_i-extensions of θ_i to $J_i = J(\beta_i, \mathfrak{A}_i)$, together with cuspidal representations σ_i of J_i/J_i^1 such that, writing $\lambda_i = \kappa_i \otimes \sigma_i$, for each i the pair (J_i, λ_i) is a maximal simple $[G_{N_i}, \zeta_i]_{G_{N_i}}$-type . Thus, writing $J_M = J_1 \times \cdots \times J_r$ and $\lambda_M = \lambda_1 \otimes \cdots \otimes \lambda_r$, the pair (J_M, λ_M) is a maximal simple $[M,\zeta]_M$-type.

With this in place, we assume that an appropriate form of unicity is satisfied by π:

Hypothesis 4.8.1. *Let π be an irreducible representation of G of inertial support $[M,\zeta]_G$, and let (K,τ) represent a $[M,\zeta]_G$-archetype. Then there exists a maximal simple $[M,\zeta]_M$-*

type (J_M, λ_M), together with a G-cover (J, λ) of (J_M, λ_M) with $J \subset K$, such that τ is isomorphic to an irreducible subrepresentation of $\mathrm{Ind}_J^K\ \lambda$.

Remarks 4.8.2. (i) This hypothesis is certainly expected to be satisfied, and is known in many cases: for $\mathbf{GL}_2(F)$ due to Henniart [BM02], and for many inertia classes of representations in $\mathbf{GL}_N(F)$ due to Nadimpalli [Nad14, Nad15].

(ii) Moreover, when $q \neq 2$, it is expected that a single choice (J, λ) of semisimple $[M, \zeta]_G$-type will suffice in order to construct a representative of all of the $[M, \zeta]_G$-archetypes as above.

We begin by combining our results on supercuspidal representations with parabolic induction in order to rule out most of the alternatives to our desired unicity result. So, let π be an irreducible representation of G as above. By choosing ζ and P appropriately, we may realize π as a subrepresentation of $\mathrm{Ind}_{M,P}^G \operatorname{c-Ind}_{\tilde{J_M}}^M\ \Lambda_M$, where $\tilde{J_M} = E_M^\times J_M$ and Λ_M is an appropriate extension of λ_M to $\tilde{J}_M$. Here, E_M is the étale F-algebra $E_1 \times \cdots \times E_r$, with $E_i = F[\beta_i]$ for each i.

For H a closed subgroup of G, write $K_H = K \cap H$. Then the restriction to K of π embeds into the Mackey decomposition of a representation as follows:

$$\begin{aligned}
\pi\downharpoonright_K &\hookrightarrow \mathrm{Res}_K^G \mathrm{Ind}_{M,P}^G \operatorname{c-Ind}_{\tilde{J}_M}^M\ \Lambda_M \\
&= \mathrm{Ind}_{K_M,K_P}^K \mathrm{Res}_{K_M}^M \operatorname{c-Ind}_{\tilde{J}_M}^M\ \Lambda_M \\
&= \bigoplus_{\tilde{J}_M \backslash M / K_M} \mathrm{Ind}_{K_M,K_P}^K \mathrm{Ind}_{{}^gJ_M \cap K_M}^{K_M} \mathrm{Res}_{{}^gJ_M \cap K_M}^{{}^gJ_M}\ {}^g\lambda_M.
\end{aligned}$$

Here, the first equality follows from the Iwasawa decomposition, where we are abusing notation slightly by writing Ind_{K_M,K_P}^K for the composition of $\mathrm{Ind}_{K_P}^K$ with the inflation functor $\mathrm{Rep}(K_M) \to \mathrm{Rep}(K_P)$.

Proposition 4.8.3. *The representation Ψ constructed in Lemma 4.1.2 must be isomorphic to a subrepresentation of* $\mathrm{Ind}_{K_M,K_P}^K \mathrm{Ind}_{J_M}^{K_M}\ \lambda_M$.

Proof. We proceed much as in the supercuspidal case. Since many of the arguments will be exactly the same, we will be rather more brief. Since Ψ is a subrepresentation of $\pi\downharpoonright_K$, there exists a $g\in G$ such that Ψ is a subrepresentation of $\mathrm{Ind}_{K_M,K_P}^{K}\mathrm{Ind}_{{}^gJ_M\cap K_M}^{K_M}\mathrm{Res}_{{}^gJ_M\cap K_M}^{{}^gJ_M}\ {}^g\lambda_M$. We say that an element $g\in G$ such that $\tilde{J}_M gK_M\neq\tilde{J}_M GK_M$ is of type B if, for some $1\leq i\leq r$, the map ${}^gU_{\mathfrak{A}_i}\cap K_i\to U_{\mathfrak{A}_i}/U_{\mathfrak{A}_i}^1$ is not surjective; if not, say that g is of type A.

Arguing precisely as in Lemma 4.5.1, we see that Ψ may not be contained in the representation $\mathrm{Ind}_{K_M,K_P}^{K}\mathrm{Ind}_{{}^gJ_M\cap K_M}^{K_M}\mathrm{Res}_{{}^gJ_M\cap K_M}^{{}^gJ_M}\ {}^g\lambda_M$ for g of type B. So suppose for contradiction that g is of type A, and suppose first that, for some $1\leq j\leq r$, there exists an irreducible non-cuspidal representation σ_j' of J_j/J_j^1 such that $\Psi\hookrightarrow\mathrm{Ind}_{K_M,K_P}^{K}\mathrm{Ind}_{{}^gJ_M\cap K_M}^{K_M}\mathrm{Res}_{{}^gJ_M\cap K_M}^{{}^gJ_M}\ {}^g\lambda_M'$, where λ_M' is the representation $\bigotimes_{i=1}^r\kappa_i\otimes\sigma_i'$, where $\sigma_i'=\sigma_i$ for $i\neq j$. Then, just as in Lemma 4.4.2, Ψ is contained in an irreducible representation of G, the supercuspidal support of which is defined on a *proper* Levi subgroup of M, which cannot be the case.

So it remains to treat the case that there is no such σ_j'. Then, arguing as in Lemma 4.4.3, we see that, for each i, and any r irreducible representations σ_i' of J_i/J_i^1, $1\leq i\leq r$ such that $\Psi\hookrightarrow\mathrm{Ind}_{K_M,K_P}^{K}\mathrm{Ind}_{{}^gJ_M\cap K_M}^{K_M}\mathrm{Res}_{{}^gJ_M\cap K_M}^{{}^gJ_M}\ {}^g\lambda_M'$, where $\lambda_M'=\bigotimes_{i=1}^r\kappa_i\otimes\sigma_i'$, there must exist characters $\chi_i\in\mathbf{X}_N(F)$ with each χ_i trivial on $\det J_i^1$ such that $\sigma_i'\simeq\sigma\otimes(\chi\circ\det)$. In particular, fix a choice of $1\leq i\leq r$. Then we may use the argument from the proof of Theorem 4.6.1 to see that there are more non-isomorphic such σ_i' than there are choices for χ_i, giving the desired contradiction. □

Note that this is precisely the statement of Theorem 4.6.1 if we specify $M=G$. However, in the case that M is a proper Levi subgroup of G, this result doesn't immediately show that Ψ must be an $[M,\zeta]_G$-type: the representation $\mathrm{Ind}_{K_M,K_P}^{K}\mathrm{Ind}_{J_M}^{K_M}\ \lambda_M$ certainly contains the representation of K induced from the semisimple type (J,λ) covering (J_M,λ_M), but it is of infinite length, and also contains many subrepresentations which are not induced from a cover of (J_M,λ_M). It seems likely that in order to rule out the possibility that Ψ is an atypical subrepresentation of $\mathrm{Ind}_{K_M,K_P}^{K}\mathrm{Ind}_{J_M}^{K_M}\ \lambda_M$, one must make use of the results in [Nad15].

Chapter 5

The unicity of types for depth zero supercuspidals

The focus of this chapter is to establish the unicity of types for depth zero supercuspidal representations of an arbitrary semisimple, simply connected group. We begin by recalling some of the necessary background, before presenting the main results of [Lat16b].

5.1 Background on Bruhat–Tits theory

We begin by recalling some of the basic constructions from Bruhat–Tits theory which we will need in later sections. Unless specified otherwise, we will follow the constructions from [SS97, Section 1].

5.1.1 Affine apartments

We begin by fixing some notation, which will remain in place for the rest of the paper. Let $\mathbf{G}$ be a connected reductive group defined over F with Lie algebra $\mathfrak{g}$, and denote by

$G = \mathbf{G}(F)$ the F-rational points of $\mathbf{G}$, taken with its natural locally profinite topology. Let $\mathbf{S}$ be a maximal F-split torus in $\mathbf{G}$, which will contain the split component $\mathbf{Z}_{\mathbf{G}}^0$ of $\mathbf{G}$. Let $\mathbf{T}$ denote the centralizer in $\mathbf{G}$ of $\mathbf{S}$; then $\mathbf{T}$ is a minimal F-Levi subgroup of $\mathbf{G}$. Let $\Phi \subset \mathbf{X}^*(\mathbf{S})$ be the complete root system for $\mathbf{G}$ relative to $\mathbf{S}$, and let Φ^{aff} denote the corresponding set of affine roots: the elements of Φ^{aff} are then of the form $\psi = \phi + n$, for some $n \in \mathbb{Z}$, and there is a gradient function $\psi \mapsto \dot{\psi}$ from Φ^{aff} to Φ, given by $\phi + n \mapsto \phi$. Let $\mathbf{W}_{\mathbf{G}}$ be the Weyl group of $\mathbf{G}$, i.e. $\mathbf{W}_{\mathbf{G}} = N_G(\mathbf{S}(F))/\mathbf{T}(F)$, and let $\mathbf{W}_{\mathbf{G}}^{\text{aff}} = N_G(\mathbf{S}(F))/\mathbf{T}(\mathscr{O})$ denote the affine Weyl group. Let $\check{\Phi} \subset \mathbf{X}_*(\mathbf{S})$ be the set of coroots dual to Φ, and denote by $\phi \mapsto \check{\phi}$ the natural duality between Φ and $\check{\Phi}$.

Since $\mathbf{Z}_{\mathbf{G}}^0$ is a subtorus of $\mathbf{S}$, its cocharacter lattice identifies with a sublattice of the cocharacter lattice of $\mathbf{S}$.

Definition 5.1.1. The *(affine) apartment* associated to the maximal F-split torus $\mathbf{S}$ of $\mathbf{G}$ is the Euclidean space $\mathscr{A}(G, \mathbf{S}) = (\mathbf{X}_*(\mathbf{S})/\mathbf{X}_*(\mathbf{Z}_{\mathbf{G}}^0)) \otimes_{\mathbb{Z}} \mathbb{R}$.

Note that $\mathscr{A}(G, \mathbf{S})$ naturally inherits the structure of a simplicial complex from the underlying quotient lattice.

5.1.2 Root subgroups

To each root $\phi \in \Phi$, we may associate a root subgroup: we let $\mathbf{U}_\phi$ denote the unique smooth connected F-group subscheme of $\mathbf{G}$ such that $\mathbf{U}_\phi$ is normalized by $\mathbf{S}$ and such that $\text{Lie}(\mathbf{U}_\phi)$, together with its natural $\mathbf{S}$-action, identifies with the weight space $\mathfrak{g}_\phi + \mathfrak{g}_{2\phi} \subset \mathfrak{g}$. We then write $U_\phi = \mathbf{U}_\phi(F)$.

More generally, to each point $x \in \mathscr{A}(G, \mathbf{S})$, we may associate a subgroup of U_ϕ. A root $\phi \in \Phi$ naturally induces a linear form $\mathscr{A}(G, \mathbf{S}) \to \mathbb{R}$, as well as an involution $s_\phi \in \mathbf{W}_{\mathbf{G}}$, which acts on $\mathscr{A}(G, \mathbf{S})$ as $s_\phi \cdot x = x - \phi(x)\check{\phi}$. For each $u \in U_\phi \backslash \{1\}$, the set $U_{-\phi} u U_\phi \cap N_G(\mathbf{S}(F))$ then contains a single element $m(u)$ whose image in $\mathbf{W}_{\mathbf{G}}$ is the reflection corresponding to ϕ. There exists a real number $l(u)$ such that, for all $x \in \mathscr{A}(G, \mathbf{S})$, one has $m(u)x = s_\phi \cdot x - l(u)\check{\phi}$. This defines a discrete filtration of U_ϕ by

$U_{\phi,r} = \{u \in U_\phi(F)\backslash\{1\} \mid l(u) \geq r\} \cup \{1\}$, where it is to be understood that $U_{\phi,\infty}$ is trivial. Each $x \in \mathscr{A}(G, \mathbf{S})$ then induces a function $f_x : \Phi \to \mathbb{R} \cup \{\infty\}$ by $\phi \mapsto -\phi(x)$; we let U_x denote the subgroup of G generated by the $U_{\phi,f_x(\phi)}$, as ϕ runs through Φ.

5.1.3 The affine and enlarged buildings

We now impose the temporary assumption that $\mathbf{G}$ is F-quasi-split. We define an equivalence relation $\sim$ on $G \times \mathscr{A}(G, \mathbf{S})$ by saying that $(g, x) \sim (h, y)$ if and only if there exists an $n \in N_G(\mathbf{S}(F))$ such that $nx = y$ and $g^{-1}hn \in U_x$. Let $\mathscr{B}(G)$ denote the quotient $(G \times \mathscr{A}(G, \mathbf{S}))/\sim$.

As $\mathbf{G}$ quasi-splits over some finite unramified extension, we define the building in general by Galois descent. Let E/F be a finite unramified extension over which $\mathbf{G}$ becomes quasi-split. Then the natural action of $\mathrm{Gal}(E/F)$ on $\mathbf{G}(E)$ extends to an action on $\mathscr{B}(\mathbf{G}(E))$.

Definition 5.1.2. Let E/F be a finite unramified extension over which $\mathbf{G}$ quasi-splits. The *(affine) Bruhat–Tits building* of the p-adic group $G = \mathbf{G}(F)$ is the space $\mathscr{B}(G) = \mathscr{B}(\mathbf{G}(E))^{\mathrm{Gal}(E/F)}$ of $\mathrm{Gal}(E/F)$-fixed points in $\mathscr{B}(\mathbf{G}(E))$.

This space then carries a natural action of G via left translation.

Theorem 5.1.3 ([BT72, BT84])**.** *The space $\mathscr{B}(G)$ is a contractible, finite-dimensional, locally finite Euclidean simplicial complex on which G acts properly by simplicial automorphisms.*

Here, the simplices in $\mathscr{B}(G)$ are inherited from those in the apartments $\mathscr{A}(G, \mathbf{S})$. We call such a simplex a *facet*.

In the case that $\mathbf{Z_G}(F)$ is not compact, it will be more convenient for us to work with a slight modification of this building. Denote by $\mathcal{V}^1$ the dual of $\mathbf{X}^*(\mathbf{G}) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 5.1.4. The *enlarged Bruhat–Tits building* of the p-adic group $G = \mathbf{G}(F)$ is $\mathscr{B}^1(G) = \mathscr{B}(G) \times \mathcal{V}^1$. The action of G on $\mathscr{B}(G)$, viewed as the subspace $\mathscr{B}(G) \times \{1\}$ of

$\mathscr{B}^1(G)$, extends to an action on $\mathscr{B}^1(G)$ via $g \cdot (x, v) = (g \cdot x, v + \theta(g))$, where $\theta(g)(\chi) = -v_F(\chi(g))$.

Say that a *facet* of $\mathscr{B}^1(G)$ is a subspace of the form $V \times \mathcal{V}^1$, where $V \subset \mathscr{B}(G)$ is a facet in the above sense. We adopt the convention that facets in $\mathscr{B}(G)$ are closed – the reader should note that some other authors take the facets to be the interior of these simplices (with the interior of a vertex being the vertex itself). We will always make it clear when we wish only to consider the interior. While a point $x \in \mathscr{B}(G)$ may be contained in multiple facets, there will always exist a unique facet of minimal dimension which contains x; we denote the facet in $\mathscr{B}^1(G)$ associated to this facet by $\bar{x}$.

Given a vertex $x \in \mathscr{B}(G)$, we say that the *link* of x is the union of the facets in which x is contained.

5.1.4 Parahoric group schemes

Proposition 5.1.5 ([BT72, BT84])**.** *Let $x \in \mathscr{B}(G)$.*

(i) The G-isotropy subgroup $\tilde{G}_x$ of $\bar{x} \subset \mathscr{B}^1(G)$ is a compact open subgroup of G.

(ii) There exists a unique smooth affine $\mathscr{O}$-group subscheme $\tilde{\mathbf{G}}_x$ of $\mathbf{G}$ with generic fibre $\mathbf{G}$ such that $\tilde{\mathbf{G}}_x(\mathscr{O}) = \tilde{G}_x$.

In general, the schemes $\tilde{\mathbf{G}}_x$ will *not* be connected. The special fibre of $\tilde{\mathbf{G}}_x$ will identify with a reductive $\mathbf{k}$-group scheme; the inverse image under the projection onto the special fibre of the connected component of this $\mathbf{k}$-group scheme is then a smooth connected $\mathscr{O}$-group subscheme of $\tilde{\mathbf{G}}_x$, which we denote by $\mathbf{G}_x$. We call the connected group scheme $\mathbf{G}_x$ the *parahoric group scheme* associated to x. Its group $G_x = \mathbf{G}_x(\mathscr{O})$ of $\mathscr{O}$-rational points is then a compact open subgroup of G, which we call the *parahoric subgroup* of G associated to x.

Proposition 5.1.6 ([BT72, BT84])**.** *The group G_x contains a unique maximal pro-p normal subgroup G_x^+, which identifies with the group of $\mathscr{O}$-rational points of the pro-unipotent* *radical of $\mathbf{G}_x$. The quotient G_x/G_x^+ is isomorphic to the group of $\mathbf{k}$-rational points of the connected component of the special fibre of $\tilde{\mathbf{G}}_x$, which is a finite reductive group over $\mathbf{k}$.*

Moreover, we may *explicitly* describe the groups G_x and G_x^+ in terms of generators and relations, see, for example, [Fin15, 2.3]. In order to do so, we need to generalize the notion of a root subgroup, associating to each *affine* root a similar root subgroup: for $\psi \in \Phi^{\mathrm{aff}}$, let $U_\psi = \{u \in \mathbf{U}_{\dot{\psi}}(F) \mid u = 1 \text{ or } \alpha_{\dot{\psi}}(u) \geq \psi\rangle$; here, $\alpha_{\dot{\psi}}$ is the unique affine function on $\mathscr{A}(G, \mathbf{T})$ with gradient $\dot{\psi}$ which vanishes on the hyperplane of points fixed by $v(m(u))$, where $v : \mathbf{Z}_{\mathbf{G}}(F) \to \mathbf{X}_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ is the function defined by $\chi(v(z)) = -v_F(\chi(z))$ for $z \in \mathbf{Z}_{\mathbf{G}}(F)$ and $\chi \in \mathbf{X}^*(\mathbf{Z}_G)$.

Proposition 5.1.7. *Let $x \in \mathscr{B}(G)$ and, for $r \geq 0$, denote by T_r the subgroup of $\mathbf{T}(F) \cap G_x$ generated by those $t \in \mathbf{T}(F) \cap G_x$ such that, for all $\chi \in \mathbf{X}^*(\mathbf{S})$, one has $v_F(\chi(t) - 1) \geq r$. Then one has:*

(i) $G_x = \langle \mathbf{T}(F) \cap G_x, U_\psi \mid \psi \in \Phi^{\mathrm{aff}},\ \psi(x) \geq 0\rangle$; and

(ii) $G_x^+ = \langle \mathbf{T}(F) \cap G_x, U_\psi \mid \psi \in \Phi^{\mathrm{aff}},\ \psi(x) > 0\rangle$.

One also has an order-reversing bijection between the set of parahoric subgroups of G and the simplicial skeleton of $\mathscr{B}(G)$:

Proposition 5.1.8 ([BT72,BT84])**.** *Let $x, y \in \mathscr{B}(G)$. Then one has $G_x \subset G_y$ if and only if $\bar{y} \subset \bar{x}$.*

In particular, the Iwahori subgroups of G correspond to the facets of maximal dimension (the *chambers*), while the maximal parahoric subgroups of G correspond to the vertices.

In a closely related manner, we may also associate to the interior of each facet in $\mathscr{B}(G)$ an F-Levi subgroup of G. The intuition behind this is as follows: if $G_y \subset G_x$ is a strict containment of parahoric subgroups of G, then the image in G_x/G_x^+ of G_y identifies with a proper parabolic subgroup of G_x/G_x^+. The Levi subgroup M we construct will then be precisely the one such that the image in G_x/G_x^+ of $M \cap G_x$ identifies with the standard Levi factor of the image of G_y.

Proposition 5.1.9 ([MP96, Proposition 6.4]). *Let $x \in \mathscr{B}(G)$. The algebraic subgroup $\mathbf{M}$ of $\mathbf{G}$ generated by $\mathbf{T}$, together with the root subgroups $\mathbf{U}_\phi$ for those $\phi \in \Phi$ such that some affine root $\phi + n$, $n \in \mathbb{Z}$, vanishes on $\bar{x}$, is an F-Levi subgroup of G. The group $\mathbf{M}(F) \cap G_x$ is a maximal parahoric subgroup of $\mathbf{M}(F)$, and there is a natural identification $(\mathbf{M}(F) \cap G_x)/(\mathbf{M}(F) \cap G_x)^+ = G_x/G_x^+$.*

5.1.5 An example

We end our discussion of Bruhat–Tits theory by looking at the explicit example of $\mathbf{SL}_N(F)$, where the above may all be described explicitly.

Theorem 5.1.10 ([Gar97, Chapter 19]). *Let $G = \mathbf{SL}_N(F)$, and fix a frame $\mathscr{F} = \{\mu_1, \dots, \mu_n\}$ of an N-dimensional F-vector space V. Let $\mathscr{V}(\mathscr{F})$ denote the set of homothety classes of $\mathfrak{o}$-lattices L in V which may be written as $L = L_1 + \cdots + L_n$, where each L_i is in the span of μ_i. Define an incidence relation $\sim$ on $\mathscr{V}(\mathscr{F})$ by $L \sim L'$ if there exist representatives of L and L' with $L' \subset L$ and such that $\mathfrak{p}$ annihilates the quotient $\mathfrak{o}$-module L/L'.*

Let G act on the resulting simplicial complex generated by the vertices $\mathscr{V}(\mathscr{F})$ and the incidence relation $\sim$ by left translation. There exists a G-equivariant simplicial isomorphism between this complex and some affine apartment of $\mathscr{B}(G)$. The G-stabilizer of a point in this apartment is equal to the intersection of the stabilizers of the vertices in the closure of the point.

In particular, one may see that $\mathscr{B}(G)$ is a regular simplicial complex, the apartments of which are $n-1$ dimensional regular simplicial complexes.

Given this formulation, it is simple to see that the parahoric subgroups of G are then simply the intersections with G of the unit groups of the hereditary orders considered in Chapter 3. Moreover, given a parahoric subgroup $G \cap U_{\mathfrak{A}}$ of G, its pro-unipotent radical is $G \cap U_{\mathfrak{A}}^1$.

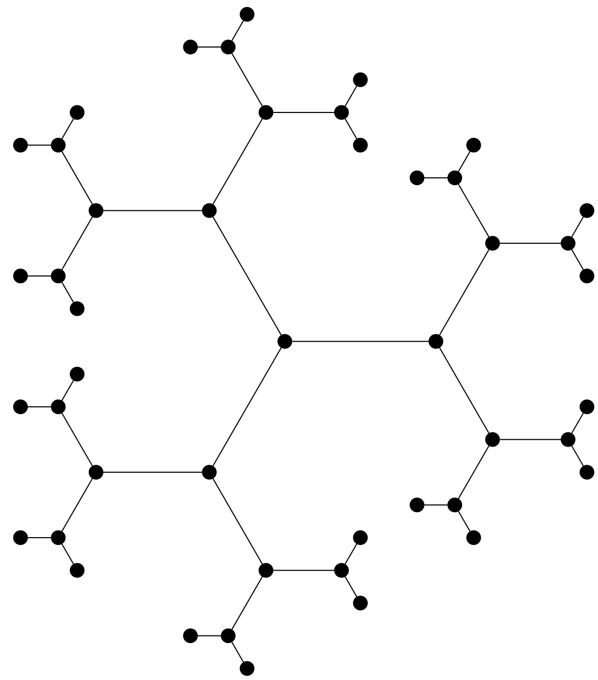


Figure 5.1: The affine building of $\mathbf{SL}_2(\mathbb{Q}_2)$.

If one has a non-maximal parahoric subgroup G_z and a maximal parahoric subgroup G_x with $G_z \subset G_x$, then the image in G_x/G_x^+ of G_z is easily seen to be a proper Levi subgroup of $G_x/G_x^+ \simeq \mathbf{SL}_N(\mathbf{k}_F)$ which is naturally isomorphic to G_z/G_z^+. It is also simple to see what the Levi subgroup of G associated to the point $z \in \mathscr{B}(G)$ should be: the Levi subgroup of G_x/G_x^+ corresponding to G_z is the $\mathbf{k}_F$-rational points of a $\mathbf{k}_F$-group scheme $\mathbf{M}$; the Levi subgroup of G associated to z is then the group of F-rational points of the generic fibre of the $\mathscr{O}$-group scheme $\mathbf{M} \times_{\mathrm{Spec}\ \mathbf{k}_F} \mathrm{Spec}\ \mathscr{O}$ – in other words, it is the Levi subgroup which is "of the same shape" as $\mathbf{M}(\mathbf{k}_F)$, but with coefficients in F.

5.2 Depth zero types

In [MP94, MP96, Mor99], Moy–Prasad and Morris independently construct natural conjugacy classes of $[G,\pi]_G$-types for depth zero supercuspidals π. We briefly recall these constructions.

Definition 5.2.1. An *(unrefined) depth zero type* in G is a pair (G_x, σ) consisting of a parahoric subgroup G_x of G and an irreducible cuspidal representation σ of G_x/G_x^+.

The etymology of these depth zero types is due to the following:

Theorem 5.2.2 ([Mor99, Theorem 4.5])**.** *Let (G_x, σ) be an unrefined depth zero type in G. Then there exists a finite set $\mathfrak{S}_\sigma \subset \mathfrak{B}(G)$ such that (G_x, σ) is an $\mathfrak{S}_\sigma$-type. Any depth zero irreducible representation π of G contains a unique G-conjugacy class of unrefined depth zero types. Moreover, if G_x is a maximal parahoric subgroup of G, then* $\mathrm{Irr}^{\mathfrak{S}_\sigma}(G)$ *consists only of supercuspidal representations.*

Morris also shows that there are natural relations between the unrefined depth zero types in G and those in its Levi subgroups: certain of the unrefined depth zero types in G are covers of those defined on a *maximal* parahoric subgroup of M, in the sense of [BK98]. We will require a slightly different formulation to that given by Morris in the non-cuspidal case. Unravelling Morris' approach, he takes an F-Levi subgroup M of G and an unrefined depth zero type (M_x, σ) in M, to which one associates a finite set $\mathfrak{S}_\sigma \subset \mathfrak{B}(M)$ as usual. This set then defines a finite subset $\mathfrak{S}'_\sigma$ of $\mathfrak{B}(G)$: the set $\mathfrak{S}'_\sigma$ consists of the inertia classes $[M, \zeta]_G$, for those $[M, \zeta]_M \in \mathfrak{S}_\sigma$. Morris then chooses an embedding $\mathscr{B}(M) \hookrightarrow \mathscr{B}(G)$ which maps the vertex $x \in \mathscr{B}(M)$ into some facet of positive dimension in $\mathscr{B}(G)$ which is associated to M by Proposition 5.1.9. To this facet corresponds a non-maximal parahoric subgroup J of G, and the representation σ of $M_x \subset J$ extends by the trivial character to a representation of J; the resulting pair (J, σ) is then a G-cover of (M_x, σ). since one may freely conjugate the resulting cover by elements of G, we may use Morris' result in the following form:

Theorem 5.2.3. *[[Mor99, Theorem 4.8]] Let $x \in \mathscr{B}(G)$, and let $M = \mathbf{M}(F)$ be the F-Levi subgroup of G associated to $\bar{x}$. Let $x_M \in \mathscr{B}(M)$ be such that $M_{z_M} = M \cap G_x$, and let (M_{x_M}, σ) be a depth zero type in M. Choose an embedding $j_M : \mathscr{B}(M) \hookrightarrow \mathscr{B}(G)$ such that $\overline{j_M(x_M)} = \bar{x}$. Then there exists a G-cover of (M_{x_M}, σ) which is of the form $(G_{j_M(x_M)}, \lambda)$, where $\lambda \downharpoonright_{M_{x_M}} = \sigma$. Moreover, the group $G_{j_M(x_M)}$ has an Iwahori decomposition with respect to the standard parabolic subgroup of G with Levi factor M, and λ is trivial on the upper- and lower-unipotent parts of M in this decomposition.*

The significance of this cover is that it will be an $\mathfrak{S}'_\sigma$-type. In particular, any irreducible subquotient of $\text{c-}\operatorname{Ind}_{G_{j_M(x_M)}}^G \lambda$ must be non-cuspidal.

Given a depth zero type (G_x, σ), we wish to "refine" this type in order to obtain an $\mathfrak{s}$-type for each $\mathfrak{s} \in \mathfrak{S}_\sigma$. Let K denote the maximal compact subgroup of the G-normalizer of G_x; if G_x is a maximal parahoric subgroup of G then K is a maximal compact subgroup of G which contains G_x as a normal subgroup of finite index.

Theorem 5.2.4 ([Mor99, Theorem 4.7]). *Suppose that G_x is a maximal parahoric subgroup of G. For each irreducible subrepresentation τ of $\operatorname{Ind}_{G_x}^K \sigma$, there exists an $\mathfrak{s} \in \mathfrak{S}_\sigma$ such that (K, τ) is an $\mathfrak{s}$-type. Conversely, for every $\mathfrak{s} \in \mathfrak{S}_\sigma$, there exists an $\mathfrak{s}$-type of this form.*

Thus, for each depth zero supercuspidal representation π of G, we have a construction of a $[G, \pi]_G$-archetype. Our goal is to show that these exhaust all such archetypes.

5.3 Unicity on the level of parahoric subgroups

We first establish a partial result, which may be viewed as a unicity result on the level of unrefined types. This is our main technical result, and the proof will occupy the remainder of this section.

Theorem 5.3.1. *Let π be a depth zero supercuspidal representation of G, and let (G_x, σ) be an unrefined depth zero type contained in π. Suppose that y is a vertex in $\mathscr{B}(G)$ such that there exists some $\mathfrak{S}_\sigma$-typical representation σ' of G_y. Then (G_y, σ') is G-conjugate to (G_x, σ).*

Since our approach will rely on identifying explicit relations among the parahoric subgroups of G, we begin by fixing a notion of a standard parahoric subgroup. Fix, once and for all, a chamber $X \subset \mathscr{B}(G)$, and let i be an element of the interior of X; thus $I = G_i$ is an Iwahori subgroup of G. We refer to X as the *standard chamber* of $\mathscr{B}(G)$. Let $P_\emptyset$ denote a parabolic subgroup of G with Levi factor the Levi subgroup associated to i and such that $G_i = (P \cap G_i)G_i^+$, so that $P_\emptyset$ is a minimal F-parabolic subgroup of G. We say that a parabolic subgroup P of G is standard if $P \supset P_\emptyset$, and that a parahoric subgroup G_x of G is standard if $x \in X$.

Proof of Theorem 5.3.1. Since every parahoric subgroup of G is conjugacy to a standard parahoric subgroup, π contains a depth zero type (G_x, σ) such that G_x is standard. Our claim is then that, given any maximal parahoric subgroup G_y of G, the irreducible subrepresentations τ of $\pi \downharpoonright_{G_y}$ are atypical, unless G_x is conjugate to G_y and τ is conjugate to σ. Alternatively, it suffices to check this for the *standard* maximal parahoric subgroups G_y, so that we don't need to worry about conjugacy.

Certainly, π appears as a subquotient of $\text{c-Ind}_{G_x}^G\ \sigma$, and so we may embed $\pi \downharpoonright_{G_y}$ into the Mackey decomposition of $\text{Res}_{G_y}^G \text{c-Ind}_{G_x}^G\ \sigma$:

$$\pi \downharpoonright_{G_y} \hookrightarrow \bigoplus_{G_x \backslash G / G_y} \text{Ind}_{K_g}^{G_y} \text{Res}_{K_g}^{G_y}\ {}^g\sigma,$$

where we write $K_g = {}^gG_x \cap G_y$. Any double coset in the space $G_x \backslash G / G_y$ admits a representative in $\mathbf{W}_{\mathbf{G}}^{\text{aff}}$; we will always assume that the representative g is such a representative *which is of shortest length.* In particular, this guarantees the following:

Lemma 5.3.2 ([Mor93, Lemma 3.19, Corollary 3.20])**.** *If g is a shortest length coset representative for $G_x \backslash G / G_y$ in $\mathbf{W}_{\mathbf{G}}^{\text{aff}}$ and either $G_x \neq G_y$ or $g \notin N_G(G_x)$, then the image in G_y/G_y^+ of K_g is a proper parabolic subgroup $\mathscr{P}$ of G_y/G_y^+. The preimage in G_y of $\mathscr{P}$ is $G_y^+ K_g$, and there exists a point z in the link of y such that $G_z = G_y^+ K_g$.*

Let τ be an irreducible subrepresentation of $\pi \downharpoonright_{G_y}$, and suppose that τ is a subrepresentation of $\text{Ind}_{K_g}^{G_y} \text{Res}_{K_g}^{{}^gG_x}\ {}^g\sigma$ for some g satisfying the above hypotheses (if $G_x = G_y$ and $g \in N_G(G_x)$), then τ is necessarily a conjugate of σ). Then there exists an irreducible subrepresentation Ξ of $\text{Res}_{K_g}^{{}^gG_x}\ {}^g\sigma$ such that $\tau \hookrightarrow \text{Ind}_{K_g}^{G_y}\ \Xi$. Note that Ξ is clearly trivial on ${}^gG_x \cap G_y$.

Let $M = \mathbf{M}(F)$ denote the F-Levi subgroup of G associated to z by Proposition 5.1.9, and let $P = \mathbf{P}(F)$ denote the standard parabolic subgroup of G with Levi factor M. From Proposition 5.1.9, we obtain the following:

Lemma 5.3.3. *There exists a vertex $z_M \in \mathscr{B}(M)$ such that $M_{z_M} = M \cap G_z$. One has $M_{z_M}^+ = M \cap G_z^+ = M \cap G_y^+$, which induces a natural identification $M_{z_M}/M_{z_M}^+ = \mathscr{M}$.*

We now need to choose an embedding of $\mathscr{B}(M)$ into $\mathscr{B}(G)$, which comes down to choosing the image of z_M. We let $j_M : \mathscr{B}(M) \hookrightarrow \mathscr{B}(G)$ be an embedding which maps z_M into the link of $g \cdot x$ in a way that $g \cdot x$ lies on the unique geodesic γ from $j_M(z_M)$ to y, and such that $M \cap G_{j_M(z_M)} = M_{z_M}$. Note that, while such an embedding exists (since the open facets containing z_M and $j_M(z_M)$ are of the same dimension; by a density argument it an be assumed that $j_M(z_M)$ lies on the geodesic), it is clearly not unique, it is essentially unique for our purposes: its restriction to the link of z_M in $\mathscr{B}(M)$ is unique up to the image in $\gamma \cap \overline{j_M(z_M)}$ of z_M, which has no effect on the parahoric subgroups which will be defined via this embedding.

With this in place, we are ready to begin our examination of the representation τ. The crucial observation is the following, which is a simple consequence of Proposition 5.1.7, once one recalls that M is the Levi subgroup associated to the point z:

Lemma 5.3.4. *We have $M_{z_M}^+ \subset {}^g G_x^+$. In particular, the representation Ξ is trivial on $M_{z_M}^+$.*

Proof. By Propositions 5.1.7 and 5.1.9, it suffices to check that if $\psi \in \Phi^{\mathrm{aff}}$ is an affine root such that $\dot{\psi} + n$ vanishes on $\overline{j_M(z_M)}$ for some n and the restriction ψ_M of ψ to an affine root for M satisfies $\psi(z_M) > 0$, then one has $\psi(gx) > 0$. Since $\dot{\psi} + n$ vanishes on $\overline{j_M(z_M)}$, we see that $\dot{\psi}_M$ is constant on $\bar{z}_M$, hence $\dot{\psi}$ is constant and takes the same value on $\overline{j_M(z_M)}$, and hence on the closure of $\overline{j_M(z_M)}$. Since gx lies in the closure of $\overline{j_M(z_M)}$, the claim follows. □

So, in particular, the representation $\Xi \downharpoonright_{M_{z_M}}$ identifies with a representation of $M_{z_M}/M_{z_M}^+ = \mathscr{M}$. Given an irreducible subrepresentation ξ of $\Xi \downharpoonright_{M_{z_M}}$, we therefore have a notion of the cuspidal support of ξ, as the unique $\mathscr{M}$-conjugacy class of pairs $(\mathscr{L}_\xi, \zeta_\xi)$ of cuspidal representations of Levi subgroups of $\mathscr{M}$ such that ξ is contained in the representation parabolically induced from $(\mathscr{L}_\xi, \zeta_\xi)$. Let $\mathscr{Q}_\xi$ be a parabolic subgroup of $\mathscr{M}$ with Levi factor $\mathscr{L}_\xi$, standard in the sense that it contains the image in $\mathscr{M}$ of our fixed Iwahori subgroup I of G (which, upon intersection with M, gives an Iwahori subgroup of M). The following is then a simple observation:

Lemma 5.3.5. *The inverse image in G_y of $\mathscr{Q}_\xi$ is a standard parahoric subgroup of G, corresponding to some point w in the standard chamber of $\mathscr{B}(G)$, and one has a containment $G_w \subset G_z$.*

Indeed, this inverse image must certainly be a parahoric subgroup of G contained in G_z. Moreover, since it contains the inverse image of the minimal Levi in G_y/G_y^+ corresponding to our fixed Iwahori subgroup of G, it contains this Iwahori subgroup, and so corresponds to a point in the standard chamber.

As before, let $L = \mathbf{L}(F)$ denote the F-Levi subgroup associated to the point $w \in \mathscr{B}(G)$, and let $w_L \in \mathscr{B}(L)$ be a vertex such that $L_{w_L} = L \cap M_{z_M}$. Choose some embedding $\iota : \mathscr{B}(L) \hookrightarrow \mathscr{B}(M)$ such that $L_{w_L} = L \cap M_{\iota(w_L)}$. The embedding $j_M : \mathscr{B}(M) \hookrightarrow \mathscr{B}(G)$ then gives us an embedding $j_L = j_M \downharpoonright_{\mathscr{B}(L)}: \mathscr{B}(L) \hookrightarrow \mathscr{B}(G)$. With this in place, by Theorem 5.2.3, we are able to construct a G-cover (J_ξ, λ_ξ) of the depth zero type (L_{w_L}, ζ_ξ) such that $J_\xi = G_{j_L(w_L)}$.

We begin by considering, for each fixed choice of irreducible representation ξ of $M_{z_M}/M_{z_M}^+$ as above, the space

$$\begin{aligned}\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_{J_\xi}^G\ \lambda_\xi, \text{c-}\mathrm{Ind}_{K_g}^G\ \Xi) &= \mathrm{Hom}_{J_\xi}(\lambda_\xi, \mathrm{Res}_{J_\xi}^G \text{c-}\mathrm{Ind}_{K_g}^G\ \Xi) \\ &= \bigoplus_{K_g\backslash G/J_\xi} \mathrm{Hom}_{J_\xi}(\lambda_\xi, \mathrm{Ind}_{J_\xi\cap {}^hK_g}^{J_\xi} \mathrm{Res}_{J_\xi\cap {}^hK_g}^{{}^hK_g}\ {}^h\Xi) \\ &= \bigoplus_{K_g\backslash G/J_\xi} \mathrm{Hom}_{J_\xi\cap {}^hK_g}(\lambda_\xi, {}^h\Xi).\end{aligned}$$

This space then surjects onto the summand corresponding to $h = 1$, namely onto the space$\mathrm{Hom}_{J_\xi\cap K_g}(\lambda_\xi, \Xi)$.

Since λ_ξ is an extension of (L_{w_L}, ζ_ξ) by the trivial character of the unipotent subgroups in the Iwahori decomposition of $G_{j_L(w_L)}$ with respect to L, it must certainly be the case that $\lambda_\xi \downharpoonright_{J_\xi \cap K_g}$ is trivial on the upper and lower unipotent parts of the Iwahori decomposition of $G_{j_M(z_M)}$ with respect to the standard parabolic subgroup of G with Levi factor M_{z_M}. This means that, if we knew that $J_\xi \cap K_g \subset M_{z_M}{}^g G_x^+$, then we would be able to make the identification

$$\mathrm{Hom}_{J_\xi \cap K_g}(\lambda_\xi, \Xi) = \mathrm{Hom}_{J_\xi \cap M_{z_M}}(\lambda_\xi, \Xi),$$

where the latter space is clearly non-zero due to the construction of λ_ξ as a cover of the cuspidal support of an irreducible subrepresentation of $\Xi \downharpoonright_{M_{z_M}}$. So it remains to check that $J_\xi \cap K_g \subset M_{z_M}{}^g G_x^+$. Since $J_\xi = G_{j_L(w_L)} \subset G_{j_M(z_M)}$ and ${}^g G_x = G_{gx}$, it suffices to check that any of the generators obtained from Proposition 5.1.7 for the group $G_{j_M(z_M)} \cap K_g = G_{j_M(z_M)} \cap G_{gx} \cap G_y$ are contained in $M_{z_M} G_{gx}^+$. That is to say, if $\psi \in \Phi^{\mathrm{aff}}$ is an affine root such that $\psi(j_M(z_M)), \psi(gx)$ and$\psi(y)$ are all non-negative, then we must check that either $\psi(gx)$ is strictly positive, or the restriction ψ_M of ψ to an affine root of M satisfies $\psi_M(z_M) \geq 0$. Suppose that $\psi(gx) = 0$. Since $\psi(j_M(z_M))$ is non-negative by assumption, it follows that $\psi_M(z_M)$ is non-negative.

Now we return to the representation τ. Since, as ξ ranges over the irreducible subrepresentation the image in ξ of $\mathrm{Hom}_{K_g}(\bigoplus_\xi \mathrm{Ind}_{J_\xi}^{K_g}\ \lambda_\xi, \Xi)$ generate Ξ, composing the non-zero maps $\text{c-}\mathrm{Ind}_{J_\xi}^G\ \lambda_\xi \to \text{c-}\mathrm{Ind}_{K_g}^G\ \Xi$ and $\text{c-}\mathrm{Ind}_{K_g}^G\ \Xi \to \text{c-}\mathrm{Ind}_{G_y}^G\ \tau$ results in, for some ξ, a non-zero map $\text{c-}\mathrm{Ind}_{J_\xi}^G\ \lambda_\xi \to \text{c-}\mathrm{Ind}_{G_y}^G\ \tau$.

So the representation τ is contained in some irreducible subquotient of $\text{c-}\mathrm{Ind}_{J_\xi}^G\ \lambda_\xi$, for some irreducible subrepresentation ξ of $\Xi \downharpoonright_{M_{z_M}}$. Since λ_ξ is a G-cover of (L_ξ, ζ_ξ), any such irreducible subquotient must be non-cuspidal, and so τ is contained in some non-cuspidal irreducible representation of G, and hence cannot be $\mathfrak{S}_\sigma$-typical. □

In the next section, it will be convenient for us to have a slight generalization of this result, showing that there do not exist any $\mathfrak{S}_\sigma$-types defined on non-maximal parahoric subgroups of G:

Proposition 5.3.6. *Let $y \in \mathscr{B}(G)$, and let π be a depth zero supercuspidal representation of G. If y is not a vertex, then no irreducible subrepresentation of $\pi \downharpoonright_{G_y}$ may be $\mathfrak{S}_\sigma$-typical, where (G_x, σ) is the unrefined depth zero type contained in π.*

Proof. Suppose that y is not a vertex. Since y is contained in the interior of a facet of positive dimension, there exists a vertex $z \in \mathscr{B}(G)$ such that $G_y \subset G_z$. Let Ξ be an irreducible subrepresentation of $\pi \downharpoonright_{G_y} = \pi \downharpoonright_{G_z} \downharpoonright_{G_y}$. By Theorem 5.3.1, unless z is conjugate to x under the action of G, we may find a non-cuspidal irreducible representation π' of G in which Ξ is contained. Similarly, we conclude that if z is conjugate to x, then Ξ must be isomorphic to an irreducible subrepresentation of ${}^g\sigma \downharpoonright_{G_y}$, for some $g \in N_G(G_x)$. So without loss of generality, let us assume that $G_y \subset G_z$ and that Ξ is an irreducible subrepresentation of $\sigma \downharpoonright_{G_y}$.

Projecting onto the reductive quotient G_x/G_x^+, we see that Ξ is an irreducible subrepresentation of the restriction of σ to the proper parabolic subgroup $\mathscr{P} = G_y/(G_y \cap G_x^+)$ of G_x/G_x^+. Let $\mathscr{P}$ have a standard Levi decomposition $\mathscr{P} = \mathscr{M}\mathscr{N}$, and let ξ be an irreducible subrepresentation of the restriction to $\mathscr{M}$ of Ξ, so that ξ has cuspidal support $(\mathscr{L}, \zeta)$, say. Let $\mathscr{Q}$ denote the standard parabolic subgroup of G_x/G_x^+ with Levi factor $\mathscr{L}$, and let $\mathscr{Q}^{\mathrm{op}}$ denote the parabolic subgroup opposite to $\mathscr{Q}$. Forming the inverse image in G_x of $\mathscr{Q}^{\mathrm{op}}$, we obtain a standard parahoric subgroup G_w corresponding to some point w in the standard chamber of $\mathscr{B}(G)$. One obtains an identification $G_w/G_w^+ = \mathscr{L}$, and the pair (G_w, ζ) is an unrefined depth zero type. Now, just as in the proof of Theorem 5.3.1, the space $\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_{G_w}^G\ \zeta, \text{c-}\mathrm{Ind}_{G_y}^G\ \Xi)$ surjects onto the space $\mathrm{Hom}_{G_w \cap G_y}(\zeta, \Xi)$. The group $G_w \cap G_y$ is the inverse image in G_x of $\mathscr{Q}^{\mathrm{op}} \cap \mathscr{P}$, which is precisely $\mathscr{L}$. Since ζ is the cuspidal support of the irreducible subrepresentation ξ of $\Xi \downharpoonright_{\mathscr{M}}$, this latter space is certainly non-zero. So we see that $\text{c-}\mathrm{Ind}_{G_y}^G\ \Xi$ contains a non-cuspidal irreducible subquotient, which is to say that Ξ must be atypical. □

5.4 Extension to archetypes

So it remains for us to consider the impact of Theorem 5.3.1 once one performs an extension to refined depth zero types. Any maximal compact subgroup of G contains finitely many parahoric subgroups of G, but *not every maximal compact subgroup must contain a maximal parahoric subgroup* – for example, given a ramified quadratic extension E/F, the group $U(1,1)(E/F)$ contains a maximal compact subgroup isomorphic to $O_2(F)$, the only parahoric subgroup in which is an Iwahori subgroup. Given a maximal compact subgroup K of G, the maximal compact subgroup of the G-normalizer of the largest parahoric subgroup contained in K coincides with K. We wish to see that if a $[G,\pi]_G$-type τ is defined on K, then K must contain G_x and τ must be isomorphic to a subrepresentation of $\mathrm{Ind}_{G_x}^K\ \sigma$ – that is to say, we wish to see that the refined depth zero types are precisely the archetypes for depth zero supercuspidals.

In order to avoid worrying about conjugacy, let us adopt the convention that, when speaking of an archetype (K,τ), any parahoric subgroup contained in K is standard (as is clearly possible).

Lemma 5.4.1. *Let π be a depth zero supercuspidal representation of G, and let (K,τ) be a $[G,\pi]_G$-archetype. Then K contains the maximal parahoric subgroup G_x of G on which the unrefined depth zero type for π is defined, and $\tau\downharpoonright_{G_x}$ is isomorphic to a sum of unrefined depth zero types, which are pairwise K-conjugate.*

Proof. Let y be such that G_y is the largest parahoric subgroup contained in K, so that K is the maximal compact subgroup of the normalizer of G_y (note that y will be a vertex if and only if K normalizes the parahoric subgroup corresponding to some vertex). Since $\tau\hookrightarrow\pi\downharpoonright_K$, we have that $\tau\downharpoonright_{G_y}\hookrightarrow\pi\downharpoonright_{G_y}$, and we have seen by Proposition 5.3.6 that any irreducible subrepresentation of this latter representation which is not an unrefined depth zero type (as may only happen when y is a conjugate of x) must be contained in an irreducible depth zero non-cuspidal representation of G. So pick an irreducible subrepresentation ρ of $\tau\downharpoonright_{G_y}$, and suppose for contradiction that ρ is contained in such a non-cuspidal representation π'. The representation π' contains an unrefined depth zero type (G_w,σ') with G_w a non-maximal standard parahoric subgroup of G; let R denote the subrepresentation of $\text{c-}\mathrm{Ind}_{G_w}^G\ \sigma'$ generated by ρ. Since ρ is irreducible, it is generated by a single vector v, say, and so R coincides with the subrepresentation of $\text{c-}\mathrm{Ind}_{G_w}^G\ \sigma$

generated by v.

As this representation is finitely generated, it admits an irreducible quotient Ψ, say. Since Ψ contains σ, it has a vector fixed by G_w^+, and so ρ has a non-zero vector fixed by $G_y \cap G_w^+$. In particular, τ must also have a vector fixed by $G_y \cap G_w^+$. We claim that, given any point $w \in \mathscr{B}(G)$ and any point $y \in \mathscr{B}(G)$ contained in the interior of a facet of higher dimension than y, there exists a $g \in G$ such that $G_w^+ \subset G_{gy}$. Indeed, G_w^+ is contained in the pro-unipotent radical of an Iwahori subgroup, and hence in the Iwahori subgroup itself. There is an element of the orbit of y in the chamber corresponding to this Iwahori subgroup, and the claim follows. So we may conjugate our choice of depth zero type (G_w, σ') and assume without loss of generality that $G_w^+ \subset G_y$. Hence τ has a non-zero vector fixed by G_w^+, and so there exists an irreducible subquotient of $\text{c-Ind}_K^G\ \tau$ with such a vector. On the other hand, (K, τ) is a $[G, \pi]_G$-type, and so any such subquotient must be supercuspidal. But a supercuspidal may not possess a non-zero vector fixed by the pro-unipotent radical of any non-maximal parahoric subgroup of G.

In order to avoid this contradiction, we conclude that the restriction to G_y of τ is a sum of unrefined depth zero types. So by Proposition 5.3.6 y must be a vertex and, moreover, by Theorem 5.3.1, y must be conjugate to x under the action of G. The result follows. $\square$

With this in place, we come to our main result:

Theorem 5.4.2 (The unicity of types for depth zero supercuspidals)**.** *Let π be a depth zero supercuspidal representation of G. Then there exists a unique $[G, \pi]_G$-archetype (K, τ). The group K contains a maximal parahoric subgroup G_x of G, and the restriction to G_x of τ is isomorphic to a sum of unrefined depth zero types contained in π.*

Proof. Let (K, τ) be such an archetype, and let (G_x, σ) be an unrefined depth zero type contained in π, with G_x standard. We have seen that, without loss of generality, we may assume that K contains G_x and $\tau\downharpoonright_{G_x}$ is isomorphic to a sum of conjugates of σ (possibly with multiplicity). So the $[G, \pi]_G$-archetypes are exhausted by the $[G, \pi]_G$-typical subrepresentations of $\mathrm{Ind}_{G_x}^{K}\ \sigma$. There is a unique conjugacy class of such representations.

Indeed, let τ' be another such representation. Since both τ and τ' are $[G,\pi]_G$-types, π arises as a subquotient of both $\text{c-}\mathrm{Ind}_K^G\ \tau$ and $\text{c-}\mathrm{Ind}_K^G\ \tau'$, so that τ and τ' intertwine in G. Certainly $\tau\restriction_{G_x}$ must then also intertwine with $\tau'\restriction_{G_x}$, which is to say that there exists a $g\in G$ such that

$$0\neq \mathrm{Hom}_{{}^gG_x\cap G_x}(\tau,{}^g\tau')=\mathrm{Hom}_{G_x}(\mathrm{Res}_{G_x}^{K}\ \tau,\mathrm{Ind}_{{}^gG_x\cap G_x}^{G_x}\mathrm{Res}_{{}^gG_x\cap G_x}^{{}^gG_x}\ {}^g\tau').$$

We have seen that the restriction to G_x of τ is a sum of unrefined depth zero types, say $\tau\restriction_{G_x}=\bigoplus_h {}^g\sigma^{\oplus m(\sigma)}$. On the other hand, we have seen that any subrepresentation of

$$\mathrm{Ind}_{{}^gG_x\cap G_x}^{G_x}\mathrm{Res}_{{}^gG_x\cap G_x}^{{}^gG_x}\ {}^g\tau'=\mathrm{Ind}_{{}^gG_x\cap G_x}^{G_x}\mathrm{Res}_{{}^gG_x\cap G_x}^{{}^gG_x}\ {}^g\left(\bigoplus_{K/N_K(\sigma)}{}^h\sigma^{\oplus m(\sigma)}\right)$$

such that $g\notin N_G(G_x)$ must be atypical. So $g\in N_G(G_x)$. Since ${}^gK$ contains ${}^gG_x=G_x$ and K is the unique maximal compact subgroup in which G_x is contained, we must have ${}^gK=K$, so that $\tau'\simeq{}^g\tau$ for some $g\in N_G(K)$. $\square$

In particular, we have established Conjecture 2.9.7 in the case of a depth zero supercuspidal representation of an arbitrary group.

Chapter 6

Some instances of the inertial Langlands correspondence

6.1 Background on the local Langlands correspondence

6.1.1 The Weil group

We fix, once and for all, a separable algebraic closure $\bar{F}/F$. We wish to study the representation theory of the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. It turns out that this is not quite the right group to consider; instead we make a slight modification. Recall that one has a canonical surjection $\mathrm{Gal}(\bar{F}/F) \twoheadrightarrow \mathrm{Gal}(\bar{\mathbf{k}}_F/\mathbf{k}_F) \simeq \hat{\mathbb{Z}}$; the *inertia group* of F is $I_F = \ker(\mathrm{Gal}(\bar{F}/F) \twoheadrightarrow \mathrm{Gal}(\bar{\mathbf{k}}_F/\mathbf{k}_F))$. Let $\mathrm{Frob} \in \mathrm{Gal}(\bar{\mathbf{k}}_F/\mathbf{k}_F)$ denote the q-th power Frobenius automorphism. Say that an element $\sigma \in \mathrm{Gal}(\bar{F}/F)$ is a *geometric Frobenius automorphism* if its image in $\mathrm{Gal}(\bar{\mathbf{k}}_F/\mathbf{k}_F)$ is Frob^{-1}.

Definition 6.1.1. The *Weil group* of F is the subgroup W_F of $\mathrm{Gal}(\bar{F}/F)$ generated by

I_F and the geometric Frobenius elements, equipped with the product topology, where the topology on $I_F \simeq \mathrm{Gal}(\bar{F}/F^{\mathrm{ur}})$ is that inherited from the Krull topology on $\mathrm{Gal}(\bar{F}/F)$.

Recalling that local class field theory gives rise to a continuous injective homomorphism $F^\times \hookrightarrow \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$, we note that the image of $F^\times$ is precisely W_F^{ab}. That is to say, the Weil group arises naturally, and really is the correct object to work with.

6.1.2 Representations of the Weil group

While $\mathrm{Gal}(\bar{F}/F)$ is a profinite group, W_F is a locally profinite dense subgroup of $\mathrm{Gal}(\bar{F}/F)$. In particular, W_F can be understood to have more representations than $\mathrm{Gal}(\bar{F}/F)$: via the embedding $W_F \hookrightarrow \mathrm{Gal}(\bar{F}/F)$, every representation of $\mathrm{Gal}(\bar{F}/F)$ determines a representation of W_F. Since W_F is dense in $\mathrm{Gal}(\bar{F}/F)$, no two distinct Galois representations determine the same representation of W_F. On the other hand, not every representation of W_F arises in this manner.

We have been deliberately vague about the coefficient fields of Galois representations so far. We will be, at least initially, concerned with *continuous* representations $W_F \to \mathrm{Aut}_{\bar{\mathbb{Q}}_\ell}(V)$, for some finite-dimensional $\bar{\mathbb{Q}}_\ell$-vector space V, and some prime $\ell \neq p$. Denote the category of such representations by $\mathrm{Rep}^{\mathrm{fin}}_{\bar{\mathbb{Q}}_\ell}(W_F)$. Since these representations are topological in nature, it is convenient to give a different, purely algebraic way of working with them. To this end, we recall the following:

Definition 6.1.2. The *Weil–Deligne group* of F is the group $W_F' = W_F \times \mathbf{SL}_2(\mathbb{C})$.

Theorem 6.1.3 ([GR10, Section 2.10])**.** *There is an equivalence of categories between* $\mathrm{Rep}^{\mathrm{fin}}_{\bar{\mathbb{Q}}_\ell}(W_F)$ *and the category of finite-dimensional complex representations of* W_F' *which are smooth on* W_F *and algebraic on* $\mathbf{SL}_2(\mathbb{C})$*, under which the resulting action of* Frob *is semisimple.*

In a slight abuse of notation, let us denote this latter category by $\mathrm{Rep}(W_F')$. It is well known that $\mathrm{Rep}^{\mathrm{fin}}_{\bar{\mathbb{Q}}_\ell}(W_F)$ is equivalent to the category of finite-dimensional Deligne representations of W_F; see for example [BH06, Theorem 32.6]. The above equivalence then comes from transporting the nilpotent operator $\mathfrak{n}$ occurring in the Deligne representation to a representation of $\mathbf{SL}_2(\mathbb{C})$; we call the representation of $\mathbf{SL}_2(\mathbb{C})$ occurring in a representation in $\mathrm{Rep}(W_F')$ the *monodromy operator*.

6.1.3 Langlands dual groups

Given a connected reductive group $\mathbf{G}$ defined over F, we may uniquely determine $\mathbf{G}$ by its root datum $[\mathbf{X}^*, \Phi, \mathbf{X}_*, \check{\Phi}]$. Here, $\mathbf{X}^*$ is the group of F-rational characters of $\mathbf{G}$ and $\Phi \subset \mathbf{X}^*$ is a complete reduced set of roots. Dually, $\mathbf{X}_*$ is the set of F-rational cocharacters of $\mathbf{G}$ and $\check{\Phi} \subset \mathbf{X}_*$ is the set of coroots dual to Φ. This induces a natural duality on the set of all such root data by $[\mathbf{X}^*, \Phi, \mathbf{X}_*, \check{\Phi}] \mapsto [\mathbf{X}_*, \check{\Phi}, \mathbf{X}^*, \Phi]$. The root datum dual to that of $\mathbf{G}$ then defines a unique connected reductive algebraic group over $\mathbb{C}$, which we denote by ${}^L\mathbf{G}^0$.

As suggested by the notation, this will be the connected component of some other algebraic group. Since W_F naturally acts on the root datum of $\mathbf{G}$, we obtain an induced action on that of ${}^L\mathbf{G}^0$, and hence an action of W_F on ${}^L\mathbf{G}^0$ itself.

Definition 6.1.4. The *Langlands dual group* of $G = \mathbf{G}(F)$ is the complex reductive group ${}^L\mathbf{G} = {}^L\mathbf{G}^0 \rtimes W_F$.

Examples 6.1.5. When one takes $\mathbf{G}$ to be a classical group, the Langlands dual group will clearly have a classical group as its connected component, corresponding to the usual dualities between the classical groups. In particular, we have the following:

(i) ${}^L\mathbf{GL}_N = \mathbf{GL}_N$;

(ii) ${}^L\mathbf{SL}_N = \mathbf{PGL}_N$;

(iii) ${}^L\mathbf{SO}_{2N+1} = \mathbf{Sp}_{2N}$;

(iv) ${}^L\mathbf{SO}_{2n} = \mathbf{SO}_{2N}$.

6.1.4 The local Langlands conjecture

Definition 6.1.6. An *L-parameter* for G is a homomorphism $\varphi : W_F' \to {}^L\mathbf{G}$ which is smooth upon restriction to W_F and algebraic upon restriction to $\mathbf{SL}_2(\mathbb{C})$, under which the action of Frob is semisimple, and such that the composition $\varphi \downharpoonright_{W_F}$ with the projection $W_F' \to W_F$ is the identity. We say that two L-parameters are equivalent if they are ${}^L\mathbf{G}^0$-conjugate, and denote by $\mathcal{L}(G)$ the set of equivalence classes of L-parameters for G.

Among the L-parameters for G, there are certain subsets of parameters which will be of particular interest for us. We call the algebraic action of $\mathbf{SL}_2(\mathbb{C})$ encoded in an L-parameter its associated *monodromy operator*, and say that an L-parameter φ is *regular* if it has a trivial monodromy operator. We say that φ is *discrete* if its image is not contained in any proper Levi subgroup of ${}^L\mathbf{G}$. We say that φ is *tame* if it is trivial upon restriction to the wild inertia group $I_F^+ \subset I_F$, i.e. the maximal normal open pro-p subgroup of I_F.

Conjecture 6.1.7 (The local Langlands conjecture)**.** *There exists a unique surjective finite-to-one map* $\mathrm{rec} : \mathrm{Irr}(G) \to \mathcal{L}(G)$ *satisfying a number of properties, among which are the following:*

(i) $\mathrm{rec}(\pi)$ *is discrete if and only if* π *is discrete series.*

(ii) $\varphi \in \mathcal{L}(G)$ *is regular if and only if every representation in the fibre of* rec *above* φ *is supercuspidal.*

(iii) $\varphi \in \mathcal{L}(G)$ *is tame if and only if every representation in the fibre of* rec *above* φ *is of depth zero.*

(iv) If π, π' *are supercuspidal representations of* G *which are unramified twists of one another, than there exists an unramified character* ω *of* W_F *such that* $\mathrm{rec}(\pi') \simeq \mathrm{rec}(\pi) \otimes \omega$.

We emphasize that the above four properties are *not* strong enough to uniquely characterize rec: one needs to impose a number of additional, much stronger conditions. However, these require a large amount of work to state, and will not be of direct relevance to the application we have in mind.

We call the finite fibres of rec the *L-packets* in $\mathrm{Irr}(G)$.

Remarks 6.1.8. While the local Langlands conjecture is still unknown in full generality, there are now many situations in which the conjecture is known to be true. In [HT01, Hen00], Harris–Taylor and Henniart proved the conjecture in the case of $G = \mathbf{GL}_N(F)$ (in which case rec is a bijection $\mathrm{Irr}(G) \to \mathcal{L}(G)$). From this, Labesse–Langlands and Gelbart–Knapp obtain a proof of the conjecture for $\mathbf{SL}_N(F)$ [LL79, GK82]. More recently, Arthur has used endoscopic transfer to deduce from the work of Harris and Taylor the local Langlands conjecture for quasi-split classical groups [Art13]. In [GT11], Gan and Takeda prove the conjecture for $\mathbf{GSp}_4(F)$. Finally, in [DR09], DeBacker and Reeder construct a map rec satisfying the above properties, as well as a number of additional desirable properties between a large number of depth zero supercuspidal representations and the tame regular discrete L-parameters; in Section 6.3.1, we will give an overview of this construction.

6.1.5 Existence of the inertial Langlands correspondence

We now construct what we call the *inertial Langlands correspondence*, under assumption of the local Langlands conjecture, for a certain class of irreducible representations. While this construction is straightforward, in order for it to be useful one needs to show that it satisfies additional properties – this will require the unicity of types.

So suppose that we have a surjective, finite-to-one map $\mathrm{rec} : \mathrm{Irr}(G) \to \mathcal{L}(G)$. Let $\mathcal{L}_{\mathrm{reg}}(G)$ denote the subset of $\mathcal{L}(G)$ consisting of equivalence classes of regular L-parameters, and let $\mathrm{Irr}_{\mathrm{reg}}(G)$ denote its inverse image under rec; thus $\mathrm{Irr}_{\mathrm{reg}}(G)$ is precisely the union of those L-packets in $\mathrm{Irr}(G)$ which consist only of supercuspidal representations.

Definition 6.1.9. An *inertial type* for G is a homomorphism $I_F \times \mathbf{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$ which is smooth on I_F, algebraic on $\mathbf{SL}_2(\mathbb{C})$, and extends to an L-parameter for G. We say that two inertial types for G are equivalent if they admit equivalent extensions, and denote by $\mathcal{I}(G)$ the set of equivalence classes of inertial types for G.

Thus, by definition, the map $\mathrm{Res}^{W_F'}_{I_F\times\mathbf{SL}_2(\mathbb{C})}$ maps $\mathcal{L}(G)$ surjectively onto $\mathcal{I}(G)$; denote by $\mathcal{I}_{\mathrm{reg}}(G)$ the image of $\mathcal{L}_{\mathrm{reg}}(G)$. Since $\mathcal{L}_{\mathrm{reg}}(G)$ and $\mathcal{I}_{\mathrm{reg}}(G)$ then consist of L-parameters and inertial types with trivial monodromy operator, we can – and will – identify elements of these sets with homomorphisms $W_F \to {}^L\mathbf{G}$ and $I_F \to {}^L\mathbf{G}$, respectively.

Definition 6.1.10. Suppose that, for each $\mathfrak{s} \in \mathfrak{B}(G)$ such that $\mathrm{Irr}^{\mathfrak{s}}(G) \cap \mathrm{Irr}_{\mathrm{reg}}(G) \neq \emptyset$, there exists an $\mathfrak{s}$-archetype, and denote by $\mathcal{A}^{\mathfrak{s}}(G)$ the set of equivalence classes of $\mathfrak{s}$-archetypes. Let $\mathcal{A}_{\mathrm{reg}}(G)$ denote the union of the $\mathcal{A}^{\mathfrak{s}}(G)$ over all such $\mathfrak{s}$.

Proposition 6.1.11 (Existence of the inertial Langlands correspondence)**.** *Suppose that the local Langlands conjecture is true for G. Then there exists a unique surjective map* $\mathrm{iner} : \mathcal{A}_{\mathrm{reg}}(G) \to \mathcal{I}_{\mathrm{reg}}(G)$ *such that, for any map $T : \mathrm{Irr}_{\mathrm{reg}}(G) \to \mathcal{A}_{\mathrm{reg}}(G)$ which assigns to π a $[G,\pi]_G$-archetype, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{reg}}(G) & \xrightarrow{\mathrm{rec}} & \mathcal{L}_{\mathrm{reg}}(G) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^{W_F}_{I_F}} \\ \mathcal{A}_{\mathrm{reg}}(G) & \xrightarrow[\mathrm{iner}]{} & \mathcal{I}_{\mathrm{reg}}(G) \end{array}$$

Proof. Pick a map $R : \mathcal{A}_{\mathrm{reg}}(G) \to \mathrm{Irr}_{\mathrm{reg}}(G)$ which, to each $\mathfrak{s}$-archetype (K,τ), assigns an element of $\mathrm{Irr}^{\mathfrak{s}}(G)$, and set $\mathrm{iner} = \mathrm{Res}^{W_F}_{I_F} \circ \mathrm{rec} \circ R$. We claim that iner is well-defined. Indeed, take two such maps R, R', and pick an archetype (K,τ) such that $R(K,\tau) \neq R'(K,\tau)$. Then $R(K,\tau)$ and $R'(K,\tau)$ must be unramified twists of one another. Since rec respects unramified twisting, we see that $\mathrm{Res}^{W_F}_{I_F} \circ \mathrm{rec} \circ R(K,\tau) = \mathrm{Res}^{W_F}_{I_F} \circ \mathrm{rec} \circ R'(K,\tau)$. The claimed properties of iner then follow immediately. □

However, in order for the map iner to be genuinely interesting, one would need to have a more explicit understanding of its fibres. In particular, these should be finite – we note that this follows from Conjecture 2.9.7 (indeed, the statement that the fibres of iner are finite can be viewed as a weak form of the unicity of types). In the remainder of this chapter, we will describe what is already known about iner in the case of $G = \mathbf{GL}_N(F)$, and establish the properties of its fibres in two new cases: that of $\mathbf{SL}_N(F)$, and that of depth zero supercuspidal representations of an arbitrary group.

6.2 The inertial Langlands correspondence for $\mathbf{SL}_N(F)$

The aim of this section is to completely describe the map iner for $G = \mathbf{SL}_N(F)$, making use of the unicity results contained in Chapter 4. We resume the notation of Chapter 4. In particular, we let $G = \mathbf{GL}_N(F)$ and $\bar{G} = \mathbf{SL}_N(F)$.

6.2.1 The correspondence for $\mathbf{GL}_N(F)$

Although not stated in the language of the preceding section, the inertial correspondence for $\mathbf{GL}_N(F)$ is already known, due to Henniart and Paškūnas [BM02, Pas05]. Since $\mathbf{GL}_N(F)$ contains a unique conjugacy class of maximal compact subgroups – that of $\mathbf{GL}_N(\mathcal{O})$, they use the equivalent formulation that any supercuspidal representation π of $\mathbf{GL}_N(F)$ contains a unique $[G,\pi]_G$-typical representation of $\mathbf{GL}_N(\mathcal{O})$. This leads to the following result:

Theorem 6.2.1 ([BM02, Pas05])**.** *The inertial Langlands correspondence for $\mathbf{GL}_N(F)$ is a bijection $\mathcal{A}_{\mathrm{reg}}(\mathbf{GL}_N(F)) \to \mathcal{I}_{\mathrm{reg}}(GLN(F))$.*

6.2.2 Restriction of archetypes from $\mathbf{GL}_N(F)$ to $\mathbf{SL}_N(F)$

Theorem 4.6.1 describes how one may "lift" supercuspidal archetypes from $\bar{G}$ to G; a natural question to ask after this is whether there is an analogous result for the restriction of archetypes from G to $\bar{G}$. This turns out to be rather simpler than going from $\bar{G}$ to G.

Proposition 6.2.2. *Let π be a supercuspidal representation of G, and let ${}^{G}(K,\tau)$ be the unique $[G,\pi]_G$-archetype. Let $\bar{\pi}$ be an irreducible component of $\pi\downharpoonright_{\bar{G}}$. Then there exists a $g \in G$ and an irreducible component $\bar{\tau}$ of ${}^{g}\tau\downharpoonright_{{}^{g}\bar{K}}$ such that ${}^{\bar{G}}({}^{g}\bar{K},\bar{\tau})$ is an archetype for $\bar{\pi}$.*

Proof. We may assume without loss of generality, by conjugating if necessary, that $\bar{\pi} = \text{c-Ind}_{\bar{K}}^{\bar{G}}\ \tilde{\mu}$, where $\tilde{\mu} = \text{c-Ind}_{\bar{J}^+}^{\bar{K}}\ \mu$ is the induction to $\bar{K}$ of a maximal simple type. Let $\{\bar{\tau}_j\}$ be the finite set of irreducible components of $\tau \downharpoonright_{\bar{K}}$. We first show that any $\pi' \in \text{Irr}(\bar{G})$ containing one of the $\bar{\tau}_j$ upon restriction must appear in the restriction to $\bar{G}$ of π. We have

$$\begin{aligned} 0 &\neq \bigoplus_j \text{Hom}_{\bar{K}}(\bar{\tau}_j, \pi') \\ &= \text{Hom}_{\bar{K}}(\text{Res}_{\bar{K}}^{K}\ \tau, \text{Res}_{\bar{K}}^{\bar{G}}\ \pi') \\ &= \text{Hom}_{\bar{G}}(\text{c-Ind}_{\bar{K}}^{\bar{G}} \text{Res}_{\bar{K}}^{K}\ \tau, \pi'), \end{aligned}$$

and so we obtain $\pi' \twoheadleftarrow \text{c-Ind}_{\bar{K}}^{\bar{G}} \text{Res}_{\bar{K}}^{K}\ \tau \hookrightarrow \text{Res}_{\bar{G}}^{G} \text{c-Ind}_{K}^{G}\ \tau$. Every irreducible subquotient of the representation $\text{c-Ind}_{K}^{G}\ \tau$ is a twist of π, and hence coincides with π upon restriction to $\bar{G}$, so that any such representation π' must be of the required form. Hence the possible representations π' all lie in a single G-conjugacy class of irreducible representations of $\bar{G}$. Let $g \in \bar{G}$ be such that ${}^g\pi' \simeq \bar{\pi}$, so that $\pi' \simeq \text{c-Ind}_{{}^g\bar{K}}^{\bar{G}}\ {}^g\tilde{\mu}$, and choose j so that π' contains $\bar{\tau}_j$. We claim that $({}^g\bar{K}, {}^g\bar{\tau}_j)$ is the required type.

It suffices to show that any G-conjugate of $\bar{\pi}$ containing $({}^g\bar{K}, {}^g\bar{\tau}_j)$ is isomorphic to $\bar{\pi}$. Suppose that, for some $h \in G$, we have $\text{Hom}_{{}^g\bar{K}}({}^h\bar{\pi}, {}^g\bar{\tau}_j) \neq 0$. The representation ${}^h\bar{\pi}$ is of the form ${}^h\bar{\pi} = \text{c-Ind}_{{}^h\bar{J}^+}^{\bar{G}}\ {}^h\mu$, and so ${}^g\bar{\tau}_j$ must be induced from some maximal simple type $(\bar{J}'^+, \mu')$, say. Then

$$\begin{aligned} 0 &\neq \text{Hom}_{{}^g\bar{K}}(\text{Res}_{{}^g\bar{K}}^{\bar{G}}\ \bar{\pi}, {}^g\bar{\tau}_j) \\ &= \text{Hom}_{\bar{J}'^+}(\text{Res}_{\bar{J}'^+}^{\bar{G}} \text{c-Ind}_{{}^h\bar{J}^+}^{\bar{G}}\ {}^h\mu, \mu') \\ &= \bigoplus_{{}^h\bar{J}^+\backslash\bar{G}/\bar{J}'^+} \text{Hom}_{\bar{J}'^+}(\text{c-Ind}_{{}^{xh}\bar{J}^+\cap\bar{J}'^+}^{\bar{J}'^+} \text{Res}_{{}^{xh}\bar{J}^+\cap\bar{J}'^+}^{{}^{xh}\bar{J}^+}\ {}^{xh}\mu, \mu') \\ &= \bigoplus_{{}^h\bar{J}^+\backslash\bar{G}/\bar{J}'^+} \text{Hom}_{{}^{xh}\bar{J}^+\cap\bar{J}'^+}(\text{Res}_{{}^{xh}\bar{J}^+\cap\bar{J}'^+}^{{}^{xh}\bar{J}^+}\ {}^{xh}\mu, \text{Res}_{{}^{xh}\bar{J}^+\cap\bar{J}'^+}^{\bar{J}'^+}\ \mu'). \end{aligned}$$

Then ${}^h\mu$ and μ' must intertwine in $\bar{G}$, and the intertwining implies conjugacy property shows that the types ${}^h\mu$ and μ' must actually be $\bar{G}$-conjugate, hence π' is $\bar{G}$-conjugate to $\bar{\pi}$. Then $\pi' \simeq \pi$, and the result follows. □

6.2.3 The correspondence

We start by recalling how the local Langlands correspondence for G leads to the correspondence for $\bar{G}$, following [LL79] and [GK82]. Given an irreducible representation π of G, by Clifford theory it restricts to $\bar{G}$ as a semisimple representation, the irreducible components of which consist of a single orbit under G-conjugacy of irreducible representations. Moreover, it may be seen that $\pi \downharpoonright_{\bar{G}}$ is multiplicity-free. These conjugacy classes of representations will be the L-packets in $\mathrm{Irr}(\bar{G})$. On the other hand, there is a canonical map ${}^L\mathbf{GL}_N = \mathbf{GL}_N(\mathbb{C}) \to \mathbf{PGL}_N(\mathbb{C}) = {}^L\mathbf{SL}_N$ given by composition with the projection $P : \mathbf{GL}_N(\mathbb{C}) \to \mathbf{PGL}_N(\mathbb{C})$. We then obtain a diagram

$$\begin{array}{ccc} \mathrm{Irr}(G) & \xrightarrow{\mathrm{rec}} & \mathcal{L}(G) \\ {\scriptstyle I}\big\uparrow & & \big\downarrow{\scriptstyle P} \\ \mathrm{Irr}(\bar{G}) & & \mathcal{L}(\bar{G}) \end{array}$$

where I is *any* map $\mathrm{Irr}(\bar{G}) \to \mathrm{Irr}(G)$ which maps an irreducible representation $\bar{\pi}$ of $\bar{G}$ to an irreducible subquotient of $\mathrm{Ind}_{\bar{G}}^{G}\ \bar{\pi}$.

Theorem 6.2.3 ([LL79, GK82])**.** *The composition $P \circ \mathrm{rec} \circ I : \mathrm{Irr}(\bar{G}) \to \mathcal{L}(G)$ is independent of the choice of I, and is the local Langlands correspondence for $\bar{G}$. The L-packets in $\bar{G}$ are precisely the orbits under G-conjugacy of irreducible representations of $\bar{G}$.*

With this, we may combine Theorem 4.6.1 with Proposition 6.2.2 in order to give a functorial description of the relationship between types in G and types in $\bar{G}$:

Theorem 6.2.4. *Let π be a supercuspidal representation of G, and let (K, τ) be the unique $[G, \pi]_G$-archetype. Let Π be the L-packet of irreducible components of $\pi \downharpoonright_{\bar{G}}$. Then the set of archetypes for the representations in Π is precisely the set of the $(\mathscr{K}, \bar{\tau})$, for $(\mathscr{K}, \bar{\tau})$ an irreducible component of ${}^g\tau \downharpoonright_{{}^g\bar{K}}$, for some $g \in G$.*

Proof. We show that the set of typical representations of $\bar{K}$ for some $\bar{\pi} \in \Pi$ is equal to the set of irreducible components of $\tau \downharpoonright_K$; the general result then follows immediately. Let $(\bar{K}, \bar{\tau})$ be an archetype for some $\bar{\pi} \in \Pi$. By Theorem 4.6.1, $\bar{\tau}$ is of the required form. Conversely, the irreducible components of $\tau \downharpoonright_K$ are all K-conjugate by Clifford theory, and so if one of them is a type for some element of Π then they all must be. Applying Proposition 6.2.2, at least one of these irreducible components must be a type for some $\bar{\pi} \in \Pi$, as required. $\square$

With this in place, we come to the main result of this section:

Theorem 6.2.5 (Inertial local Langlands correspondence for $\mathbf{SL}_N(F)$)**.** *There exists a unique surjective map* iner : $\mathcal{A}_{\mathrm{sc}}(\bar{G}) \twoheadrightarrow \mathcal{I}_{\mathrm{sc}}(\bar{G})$ *with finite fibres such that, for* any *map T assigning to a supercuspidal representation $\bar{\pi}$ of $\bar{G}$ one of the $e_{\bar{\pi}}$ archetypes contained in $\bar{\pi}$, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{sc}}(\bar{G}) & \xrightarrow{\mathrm{rec}} & \mathcal{L}_{\mathrm{sc}}(\bar{G}) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^{W_F}_{I_F}} \\ \mathcal{A}_{\mathrm{sc}}(\bar{G}) & \xrightarrow[\mathrm{iner}]{} & \mathcal{I}_{\mathrm{sc}}(\bar{G}) \end{array}$$

Each of the fibres of iner *consists of the full orbit under G-conjugacy of an archetype, with the fibre above an inertial type φ being of cardinality $e_\varphi \ell_\varphi$.*

Moreover, for any map R assigning to each $[G,\pi]_G$-archetype a $[\bar{G},\bar{\pi}]_{\bar{G}}$-archetype, for $\bar{\pi}$ an irreducible subquotient of $\pi \downharpoonright_{\bar{G}}$, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{\mathrm{sc}}(G) & \xrightarrow{\mathrm{iner}} & \mathcal{I}_{\mathrm{sc}}(G) \\ {\scriptstyle R}\downarrow & & \downarrow \\ \mathcal{A}_{\mathrm{sc}}(\bar{G}) & \xrightarrow[\mathrm{iner}]{} & \mathcal{I}_{\mathrm{sc}}(\bar{G}) \end{array}$$

where the map $\mathcal{I}_{\mathrm{sc}}(G) \to \mathcal{I}_{\mathrm{sc}}(\bar{G})$ is given by composition with the projection $\mathbf{GL}_N(\mathbb{C}) \to \mathbf{PGL}_N(\mathbb{C})$.

Proof. Let S be any map which assigns to an archetype a (necessarily supercuspidal) irreducible representation of $\bar{G}$ in which the archetype is contained. Then we let iner denote the composition $\mathrm{Res}^{W_F}_{I_F} \circ \mathrm{rec} \circ S$. Let $\varphi \in \mathcal{I}_{\mathrm{sc}}(\bar{G})$, and let $\tilde{\varphi}$ be an irreducible extension of φ to W_F. Let $\Pi = \mathrm{rec}^{-1}(\tilde{\varphi})$. Then $\Pi = \{\bar{\pi}_i\}$ is an L-packet of supercuspidal representations of $\bar{G}$, which consists precisely of the set of irreducible subquotients of some supercuspidal representation π of G. Let $\psi = \mathrm{rec}(\pi)$. Then, by [Pas05, Corollary 8.2], there exists a unique smooth irreducible representation τ of K such that, for all irreducible representations ρ of G, we have that ρ contains τ upon restriction to K if and only if $\mathrm{rec}(\rho)\downharpoonright_{I_F}\simeq \psi\downharpoonright_{I_F}$. Then (K,τ) is the unique $[G,\pi]_G$-archetype, and by Theorem 6.2.4 we know that the (finite) set $\{(\mathscr{K}_i,\bar{\tau}_i)\}$ of $[\bar{G},\bar{\pi}_i]_{\bar{G}}$-archetypes is precisely the set of archetypes representated by the irreducible subrepresentations of $({}^g\bar{K},{}^g\tau\downharpoonright_{{}^g\bar{K}})$, as g ranges over G. Let $\mathfrak{S}$ be the set of inertial equivalence classes of representations in Π. As each of the $(\mathscr{K}_i,\bar{\tau}_i)$ is an archetype, it follows that, for all irreducible representations $\bar{\pi}$ of $\bar{G}$, we have that $\bar{\pi}$ contains one of the $\bar{\tau}_i$ upon restriction to $\mathscr{K}_i$ if and only if $[\bar{G},\bar{\pi}]_{\bar{G}}\in\mathfrak{S}$, if and only if $\bar{\pi}_i\in\Pi$, if and only if $\mathrm{rec}(\bar{\pi})\downharpoonright_{I_F}\simeq\varphi$. Thus the map $\mathrm{iner}=\mathrm{Res}_{I_F}^{W_F}\circ\mathrm{rec}\circ S$ defines the unique map which can make the necessary diagram commute.

We now consider the fibres of iner. Let $\varphi\in\mathcal{I}_{\mathrm{sc}}(\bar{G})$. Then each of the archetypes in $\mathrm{iner}^{-1}(\varphi)$ is represented by a representation of the form $\bar{\tau}=\mathrm{Ind}_{\bar{J}^+}^{\mathscr{K}}\ \mu$, for some maximal simple type $(\bar{J}^+,\mu)$ and some maximal compact subgroup $\mathscr{K}$ of $\bar{G}$. Moreover, any G-conjugate of $(\mathscr{K},\bar{\tau})$ is also in the fibre above φ. Indeed, given $g\in G$, one has ${}^g\tau=\mathrm{Ind}_{{}^g\bar{J}^+}^{{}^g\mathscr{K}}\ {}^g\mu$. If $\bar{\pi}$ is a supercuspidal representation of $\bar{G}$ containing $(\bar{J}^+,\mu)$, then ${}^g\bar{\pi}$ contains $({}^g\bar{J}^+,{}^g\mu)$, and hence contains $({}^g\mathscr{K},{}^g\bar{\tau})$. Since $\bar{\pi}$ and ${}^g\bar{\pi}$ are contained in some common supercuspidal representation π of G upon restriction, they lie in the same L-packet and are therefore in the same fibre of iner. Conversely, we have already seen that each of the archetypes in $\mathrm{iner}^{-1}(\varphi)$ are pairwise G-conjugate.

So it remains only to calculate the cardinality of $\mathrm{iner}^{-1}\varphi$. Let $\tilde{\varphi}\in\mathcal{L}(\bar{G})$ be such that $\mathrm{Res}_{I_F}^{W_F}\ \tilde{\varphi}=\varphi$, and let $\bar{\pi}$ be contained in the L-packet $\mathrm{rec}^{-1}(\tilde{\varphi})$ corresponding to $\tilde{\varphi}$. Then the cardinality of this L-packet is precisely the length of $\pi\downharpoonright_{\bar{G}}$, where π is some supercuspidal representation of G such that $\bar{\pi}$ is a subquotient of $\pi\downharpoonright_{\bar{G}}$, i.e. $\#\mathrm{rec}^{-1}(\tilde{\varphi})=\ell_\varphi$. The fibre $\mathrm{iner}^{-1}(\varphi)$ is then equal to the disjoint union of the sets of archetypes contained in each of the ℓ_φ elements of the L-packet $\mathrm{rec}^{-1}(\tilde{\varphi})$. As these elements are precisely an orbit under G-conjugacy of supercuspidal representations of $\bar{G}$, they each contain the same number of archetypes, which is $e_{\bar{\pi}}$ by Theorem 4.7.5. We therefore see that $\#\mathrm{iner}^{-1}(\varphi)=e_{\bar{\pi}}\ell_{\bar{\pi}}$.

The commutativity of the second diagram is then simply a reinterpretation of Theorem 6.2.4 in terms of the inertial correspondences. □

Remark 6.2.6. It is possible to give a more explicit description of the fibres of the inertial correspondence, by describing more explicitly the length ℓ_φ. We freely make use of the results of section 1 of [BK94] in order to do so. Given a supercuspidal representation π of G and an irreducible subrepresentation $\bar{\pi}$ of $\pi\downharpoonright_{\bar{G}}$, the length of $\pi\downharpoonright_{\bar{G}}$ is equal to the cardinality of the finite group $\mathfrak{G}(\pi) = G/N_G(\bar{\pi})$, which is isomorphic to the dual group of $\mathscr{S}(\pi) = \{\chi \in \mathbf{X}(F) \mid \pi \simeq \pi \otimes (\chi \circ \det)\}$. Since this group is finite abelian, it is of the same cardinality as its dual; so we see that $\mathrm{length}(\pi\downharpoonright_{\bar{G}}) = \#\{\chi \in \mathbf{X}(F) \mid \pi \simeq \pi \otimes (\chi \circ \det)\}$. This group partitions according to the level of characters into a disjoint union of finitely many sets of the same cardinality; thus it must be of cardinality a multiple (k, say) of the cardinality of the group of *unramified* characters χ of F which preserve π via twisting, which we know to be $N/e_{\bar{\pi}}$. So, in particular, there exists a positive integer k such that $\mathrm{iner}^{-1}(\varphi) = e_{\bar{\pi}}kN/e_{\bar{\pi}} = kN$.

6.3 The tame inertial Langlands corespondence

The focus of this section is to describe completely the fibres of the restriction of the inertial correspondence the the depth zero supercuspidal representations *of an arbitrary group*, using the results of Chapter 5.

6.3.1 The DeBacker-Reeder construction

We begin by briefly recalling some of the relevant details of the construction of a Langlands correspondence for (a certain subset of) the depth zero supercuspidal representations of (certain) reductive p-adic groups due to DeBacker and Reeder [DR09].

Recall that an *inner form* of a p-adic group $G = \mathbf{G}(F)$ is a group $H = \mathbf{H}(F)$ such that $\mathbf{G}(\bar{F}) = \mathbf{H}(\bar{F})$. The inner forms of G are naturally parametrized by the Galois cohomology group $H^1(\mathrm{Gal}(\bar{F}/F), \mathbf{G}^{\mathrm{ad}})$, where $\mathbf{G}^{\mathrm{ad}} = \mathbf{G}/\mathbf{Z}_{\mathbf{G}}$ denotes the adjoint form of $\mathbf{G}$. There is then a natural map $H^1(\mathrm{Gal}(\bar{F}/F), \mathbf{G}) \to H^1(\mathrm{Gal}(\bar{F}/F), \mathbf{G}^{\mathrm{ad}})$ which is neither injective nor surjective in general; an inner form in the image of this map is said to be *pure*. For the remainder of this section, we impose the following hypothesis:

Hypothesis 6.3.1. *We suppose that G is a pure inner form of an unramified group, i.e. there exists an F-quasi split pure inner form of G which is E-split for some finite unramified extension E/F.*

The DeBacker–Reeder construction associates a finite L-packet to each of the L-parameters in a certain subset of the set of equivalence classes of tame regular L-parameters for such a group G. Denote by I_F^{tame} the quotient I_F/I_F^+.

Definition 6.3.2. A *tame regular semisimple elliptic L-parameter* (TRSELP) is a pair $\varphi = (s, f)$ consisting of:

(i) a continuous homomorphism $s : I_F^{\mathrm{tame}} \to \hat{\mathbf{T}}$ for some maximal torus $\hat{\mathbf{T}}$ in ${}^L\mathbf{G}$ satisfying $C_{{}^L\mathbf{G}}(\hat{\mathbf{T}}) = \hat{\mathbf{T}}$; and

(ii) an element $f \in \hat{N} = N_{{}^L\mathbf{G}}(\hat{\mathbf{T}})$ satisfying certain conditions which will not be important for our purposes; see [DR09, Section 4.1].

We denote by $\mathcal{L}_{\mathrm{tame,reg}}(G)$ the set of equivalence classes of TRSELPs.

Since we will need to define a rather large number of objects via TRSELPs, we will abuse terminology slightly and drop the adjective "semisimple elliptic" from our notation. In the case that the centre of $\mathbf{G}$ is connected, the TRSELPs should be precisely those tame regular parameters corresponding to L-packets which consist *only* of depth zero supercuspidal representations. In the case that $\mathbf{G}$ does not have a connected centre, then the TRSELPs should be the tame regular parameters corresponding to L-packets consisting of supercuspidal representations which are generic in some sense – for example, when F is of odd residual characteristic, then the L-packets in $\mathrm{Irr}(\mathbf{SL}_2(F))$ which contain depth zero supercuspidals generically contain two elements; there exists a unique such packet containing four elements (corresponding to the unique irreducible supercuspidal representation of $\mathbf{GL}_2(\mathbf{k}_F)$ which restricts reducibly to $\mathbf{SL}_2(\mathbf{k}_F)$), the parameter of which is *not* a TRSELP.

Definition 6.3.3. A *tame regular inertial type* for G is the restriction to I_F of an element of $\mathcal{L}_{\text{tame,reg}}(G)$. Note that this is equivalent to defining a tame regular inertial type to be a homomorphism $s : I_F^{\text{tame}} \to {}^L\mathbf{G}$ as in Definition 6.3.2. Say that two tame regular inertial types s, s' are equivalent if there exist choices f, f' such that the TRSELPs (s, f) and (s', f') are equivalent, and denote by $\mathcal{I}_{\text{tame,reg}}(G)$ the set of equivalence classes of tame regular inertial types.

In particular, a TRSELP consists precisely of the data of an underlying tame regular inertial type together with a compatible action of Frob; thus we get a well-defined surjective map $\text{Res}_{I_F}^{W_F} : \mathcal{L}_{\text{tame,reg}}(G) \to \mathcal{I}_{\text{tame,reg}}(G)$. With this in place, we are ready to sketch out the construction of the L-packet associated to a TRSELP. Fix a choice $\varphi = (s, f) \in \mathcal{L}_{\text{tame,reg}}(G)$ of TRSELP. By duality on the root datum of $\mathbf{G}$, the maximal torus $\hat{\mathbf{T}} \subset {}^L\mathbf{G}$ determines a unique maximal split torus $\mathbf{T} \subset \mathbf{G}$; let $X = X_*(\mathbf{T})$. Let $\hat{\vartheta}$ denote the automorphism of ${}^L\mathbf{G}$ arising from the action of Frob (via the action of W_F on ${}^L\mathbf{G}$ inherited from the action of W_F on the root datum of $\mathbf{G}$). This automorphism $\hat{\vartheta}$ then gives a dual automorphism ϑ of X. Moreover, upon restriction to $\hat{\mathbf{T}}$, the element $\varphi(\text{Frob})$ of ${}^L\mathbf{G}$ normalizes $\hat{\mathbf{T}}$ and acts by an element of the form $\hat{\vartheta}\hat{w}$, for some $\hat{w}$ in the Weyl group of $\hat{\mathbf{T}}$. This element $\hat{w}$ then also gives a dual automorphism w of X. We denote by X_w the pre-image in X of the torsion subgroup $[X/(1-w\vartheta)X]_{\text{tors}}$ of $X/(1-w\vartheta)X$.

Now fix a choice of $\lambda \in X_w$. To λ, one may associate a certain 1-cocycle u_λ [DR09, Section 2.7]; the twisted Frobenius $F_\lambda = \text{Ad}(u_\lambda) \circ \text{Frob}$ acts on the apartment $\mathscr{A}(G, \mathbf{T})$ and stabilizes a unique vertex x_λ; we therefore obtain for each $\lambda \in X_w$ a maximal parahoric subgroup G_λ of G.

Moreover, by the local Langlands correspondence for tori (which is well-known, but reproved in [DR09, Section 4.3]), the homomorphism $s : I_F^{\text{tame}} \to \hat{\mathbf{T}}$ corresponds to a character of $\mathbf{T}(F)$. DeBacker and Reeder associate to (φ, λ) a certain conjugate T_λ of $\mathbf{T}(F)$, and hence a character θ_λ of T_λ. This character will be of depth zero in the sense that it will be trivial on $T_\lambda \cap G_\lambda^+$, but non-trivial on $T_\lambda \cap G_\lambda$. In particular, we may identify θ_λ with a character of the torus in the reductive quotient G_λ / G_λ^+ obtained as the image of $T_\lambda \cap G_\lambda$. Deligne–Lusztig theory then gives a virtual representation $R_{T_\lambda}^{\theta_\lambda}$ of G_λ/G_λ^+. At this point, the "genericity" property of a TRSELP (as opposed to an arbitrary tame regular L-parameter) guarantees that the character θ_λ is in general position, so that one of $\pm R_{T_\lambda}^{\theta_\lambda}$ will be an irreducible cuspidal representation σ_λ of G_λ/G_λ^+. We therefore obtain an unrefined depth zero type $(G_\lambda, \sigma_\lambda)$.

At this point, the elements of $\mathrm{Irr}(G)$ which arise as subquotients of $\text{c-}\mathrm{Ind}_{G_\lambda}^G \ \sigma_\lambda$ for some TRSELP φ and some $\lambda \in X_w$ are grouped into L-packets $\Pi(\varphi)$ according to, in particular, the following rules:

(i) As λ ranges through a set of elements of X_w such that no two representations σ_λ are conjugate in G, and all possible conjugacy classes of unrefined depth zero types $(G_\lambda, \sigma_\lambda)$ arise, there exists for each λ a unique element π_λ of $\Pi(\varphi)$ containing the unrefined depth zero type $(G_\lambda, \sigma_\lambda)$. In other words, the elements of $\Pi(\varphi)$ are precisely a choice of irreducible subquotient of $\text{c-}\mathrm{Ind}_{G_\lambda}^G \ \sigma_\lambda$ for each λ; this choice is determined by f, and will turn out to be irrelevant to our purposes.

(ii) As f varies through all possible TRSELPs (s, f) with the equivalent inertial type as that of φ, the union of the packets $\Pi(\varphi)$ is equal to the set of irreducible subquotients of the representations $\text{c-}\mathrm{Ind}_{G_\lambda}^G \ \sigma_\lambda$, as λ varies. Moreover, if f, f' are such that $\Pi(s, f) \cap \Pi(s, f') \neq \emptyset$, then the TRSELPs (s, f) and (s, f') are equivalent.

Moreover, DeBacker and Reeder show that this process results in (at least once one views TRSELPs as simultaneously being L-parameters for the class of pure inner forms of G) L-packets which are *stable* in a technical, character-theoretic sense, and that these are the smallest such stable L-packets. Thus, while they do not show that the resulting correspondence satisfies all of the conditions which are expected of the local Langlands correspondence, it is extremely likely that it is the correct such correspondence.

Hypothesis 6.3.4. *We assume that the assignment of an L-packet of depth zero supercuspidals to each TRSELP given by DeBacker and Reeder is the such assignment satisfying all of the expected properties of the local Langlands correspondence.*

We note that, in particular, properties *(i) – (ii)* in Conjecture 6.1.7 *are* known to hold for the DeBacker–Reeder correspondence.

Definition 6.3.5. Say that an representation of G is *tame regular (semisimple elliptic)* if it is contained in $\Pi(\varphi)$ for some TRSELP φ. Denote by $\mathrm{Irr}_{\mathrm{tame,reg}}(G)$ the set of isomorphism classes of tame regular representations of G.

Note that a tame regular representation is necessarily an irreducible depth zero supercuspidal representation of G.

We realize the DeBacker–Reeder construction as a surjective, finite-to-one map $\mathrm{rec} : \mathrm{Irr}_{\mathrm{tame,reg}}(G) \to \mathcal{L}_{\mathrm{tame,reg}}(G)$ which assigns to each $\pi \in \mathrm{Irr}_{\mathrm{tame,reg}}(G)$ the unique $\varphi \in \mathcal{L}_{\mathrm{tame,reg}}(G)$ such that π is contained in $\Pi(\varphi)$. As an immediate consequence of properties (i) and (ii) above of the L-packets $\Pi(\varphi)$, we immediately obtain the following:

Lemma 6.3.6. *Let $\pi, \pi' \in \mathrm{Irr}_{\mathrm{tame,reg}}(G)$. Then $\mathrm{rec}(\pi)\downharpoonright_{I_F} \simeq \mathrm{rec}(\pi')\downharpoonright_{I_F}$ if and only if π and π' contain a common unrefined depth zero type.*

6.3.2 The fibres of iner

Having described the local Langlands correspondence for $\mathrm{Irr}_{\mathrm{tame,reg}}(G)$, we describe a similar correspondence on the level of types and inertial types.

Denote by $\mathcal{D}_{\mathrm{tame,reg}}(G)$ the set of conjugacy classes of unrefined depth zero types which are contained in some element of $\mathrm{Irr}_{\mathrm{tame,reg}}(G)$, and by $\mathcal{A}_{\mathrm{tame,reg}}(G)$ the set of $[G,\pi]_G$-archetypes, as π ranges through $\mathrm{Irr}_{\mathrm{tame,reg}}(G)$. Since, by Theorem 5.3.1, each such $[G,\pi]_G$-archetype (K,τ) restricts to the unique conjugacy class of maximal parahoric subgroups contained in the conjugacy class of K as a direct sum of pairwise G-conjugate unrefined depth zero types, there is a canonical surjective map $\mathcal{A}_{\mathrm{tame,reg}}(G) \to \mathcal{D}_{\mathrm{tame,reg}}(G)$.

Moreover, since $\pi \in \mathrm{Irr}_{\mathrm{tame,reg}}(G)$ contains a unique element of $\mathcal{D}_{\mathrm{tame,reg}}(G)$ and a unique element of $\mathcal{A}_{\mathrm{tame,reg}}(G)$, there are also canonical surjective maps $T : \mathrm{Irr}_{\mathrm{tame,reg}}(G) \to \mathcal{A}_{\mathrm{tame,reg}}(G)$ and $D : \mathrm{Irr}_{\mathrm{tame,reg}}(G) \to \mathcal{D}_{\mathrm{tame,reg}}(G)$; it is clear that D factors through T via canonical map $\mathcal{A}_{\mathrm{tame,reg}}(G) \to \mathcal{D}_{\mathrm{tame,reg}}(G)$.

Theorem 6.3.7 (The tame inertial Langlands correspondence)**.** *Let G be a pure inner form of an unramified p-adic group.*

(i) There exists a unique surjective, finite-to-one map $\mathrm{iner}_{\mathcal{D}} : \mathcal{D}_{\mathrm{tame,reg}}(G) \to \mathcal{I}_{\mathrm{tame,reg}}(G)$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{tame,reg}}(G) & \xrightarrow{\mathrm{rec}} & \mathcal{L}_{\mathrm{tame,reg}}(G) \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_{I_F}^{W_F}} \\ \mathcal{D}_{\mathrm{tame,reg}}(G) & \xrightarrow[\mathrm{iner}_{\mathcal{D}}]{} & \mathcal{I}_{\mathrm{tame,reg}}(G) \end{array}$$

Given $\varphi \in \mathcal{L}_{\mathrm{tame,reg}}(G)$, one has $\#\mathrm{iner}_{\mathcal{D}}^{-1}(\varphi \downharpoonleft_{I_F}) = \#\mathrm{rec}^{-1}(\varphi)$.

(ii) There exists a unique surjective, finite-to-one map $\mathrm{iner} : \mathcal{A}_{\mathrm{tame,reg}}(G) \to \mathcal{I}_{\mathrm{tame,reg}}(G)$ *such that the following diagram commutes;*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{tame,reg}}(G) & \xrightarrow{\mathrm{rec}} & \mathcal{L}_{\mathrm{tame,reg}}(G) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_{I_F}^{W_F}} \\ \mathcal{A}_{\mathrm{tame,reg}}(G) & \xrightarrow[\mathrm{iner}]{} & \mathcal{I}_{\mathrm{tame,reg}}(G) \end{array}$$

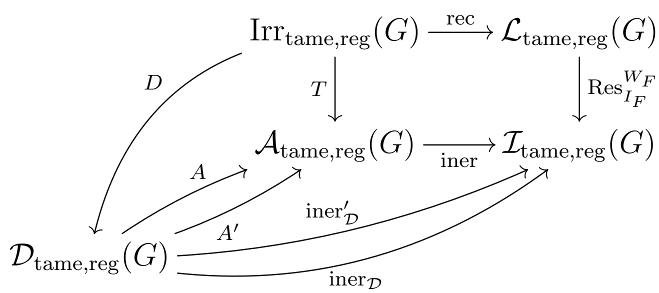
The map iner *factors uniquely through* $\mathrm{iner}_{\mathcal{D}}$ *via the canonical map* $\mathcal{A}_{\mathrm{tame,reg}}(G) \to \mathcal{D}_{\mathrm{tame,reg}}(G)$ *and, given $\varphi \in \mathcal{L}_{\mathrm{tame,reg}}(G)$, one has*

$$\#\mathrm{iner}^{-1}(\varphi \downharpoonleft_{I_F}) = \sum_{(G_x,\sigma)\in \mathrm{iner}_{\mathcal{D}}^{-1}(\varphi\downharpoonleft_{I_F})} \#\mathfrak{S}_\sigma.$$

Proof. Let $R : \mathcal{A}_{\mathrm{tame,reg}}(G) \to \mathrm{Irr}_{\mathrm{tame,reg}}(G)$ be *any* map which, to an archetype (K, τ), assigns an irreducible subquotient of $\text{c-}\mathrm{Ind}_K^G\ \tau$; we then define $\mathrm{iner} = \mathrm{Res}_{I_F}^{W_F} \circ \mathrm{rec} \circ R$. This is well-defined: since (K, τ) is a $[G, \pi]_G$-archetype for some $\pi \in \mathrm{Irr}_{\mathrm{tame,reg}}(G)$, any two subquotients of $\text{c-}\mathrm{Ind}_K^G\ \tau$ are unramified twists of one another; however, given an unramified character ω of G, restricting to I_F induces an isomorphism between $\mathrm{rec}(\pi) \downharpoonleft_{I_F}$

and $\mathrm{rec}(\pi \otimes \omega) \downharpoonright_{I_F}$. It follows immediately that iner is the unique map $\mathcal{A}_{\text{tame,reg}}(G) \to \mathcal{I}_{\text{tame,reg}}(G)$ such that the diagram in *(ii)* commutes. Since rec and $\mathrm{Res}_{I_F}^{W_F}$ are surjective, it follows that iner is also surjective.

We now establish *(i)*. Fix a map $A : \mathcal{D}_{\text{tame,reg}}(G) \to \mathcal{A}_{\text{tame,reg}}(G)$ which, to each unrefined depth zero type (G_x, σ), assigns an irreducible subrepresentation of $\mathrm{Ind}_{G_x}^{K}\ \sigma$, where K denotes the maximal compact subgroup of $N_G(G_x)$. Then we define $\mathrm{iner}_{\mathcal{D}} : \mathcal{D}_{\text{tame,reg}}(G) \to \mathcal{I}_{\text{tame,reg}}(G)$ by setting $\mathrm{iner}_{\mathcal{D}} = \mathrm{iner} \circ A$. We must first check that $\mathrm{iner}_{\mathcal{D}}$ is well-defined. Let $A' : \mathcal{D}_{\text{tame,reg}}(G) \to \mathcal{A}_{\text{tame,reg}}(G)$ be another such map, and let $\mathrm{iner}'_{\mathcal{D}} = \mathrm{iner} \circ A'$. Then we have a diagram



We have seen that the top square and commutes and that D factors through T, while in the bottom triangle A commutes with $\mathrm{iner}_{\mathcal{D}}$ and A' commutes with $\mathrm{iner}'_{\mathcal{D}}$ by definition. Suppose that $\mathrm{iner}_{\mathcal{D}} \neq \mathrm{iner}'_{\mathcal{D}}$, and pick $(G_x, \sigma) \in \mathcal{D}_{\text{tame,reg}}(G)$ such that $\mathrm{iner}_{\mathcal{D}}(G_x, \sigma) \neq \mathrm{iner}'_{\mathcal{D}}(G_x, \sigma)$. Hence $A(G_x, \sigma)$ and $A'(G_x, \sigma)$ must lie in different fibres of iner, which is to say that they must lie in different fibres of $\mathrm{Res}_{I_F}^{W_F} \circ \mathrm{rec} \circ R$. This means that $\text{c-}\mathrm{Ind}_K^G\ A(G_x, \sigma)$ and $\text{c-}\mathrm{Ind}_K^G\ A'(G_x, \sigma)$ admit irreducible subquotients lying in different fibres of $\mathrm{Res}_{I_F}^{W_F} \circ \mathrm{rec}$. Hence there exist irreducible subquotients ρ, ρ', say, of $\text{c-}\mathrm{Ind}_{G_x}^G\ \sigma$ which lie in different fibres of $\mathrm{Res}_{I_F}^{W_F} \circ \mathrm{rec}$. By Lemma 6.3.6 this is not the case, and so $\mathrm{iner}_{\mathcal{D}}$ is well-defined.

Since $\mathrm{iner}_{\mathcal{D}}$ is well-defined, iner factors uniquely through $\mathrm{iner}_{\mathcal{D}}$ via the canonical map $\mathcal{A}_{\text{tame,reg}}(G) \to \mathcal{D}_{\text{tame,reg}}(G)$. Let $\varphi \in \mathcal{L}_{\text{tame,reg}}(G)$. We claim that the fibres $\mathrm{iner}_{\mathcal{D}}^{-1}(\varphi \downharpoonright_{I_F})$ and $\mathrm{rec}^{-1}(\varphi)$ are in canonical bijection. There is a canonical injective map between these two sets. Indeed, given $(G_x, \sigma) \in \mathrm{iner}_{\mathcal{D}}^{-1}(\varphi \downharpoonright_{I_F})$, there exists a unique element of $\mathrm{rec}^{-1}(\varphi)$ containing (G_x, σ). This gives a map $\mathrm{iner}_{\mathcal{D}}^{-1}(\varphi\restriction_{I_F}) \to \mathrm{rec}^{-1}(\varphi)$, which is injective since a depth zero irreducible representation of G contains a unique unrefined depth zero type. Moreover, this map is clearly seen to be surjective by the DeBacker–Reeder construction: the fibre corresponds to a choice of f in the pair $(\varphi\restriction_{I_F}, f)$. It follows that $\mathrm{iner}_{\mathcal{D}}^{-1}(\varphi\restriction_{I_F}) = \mathrm{iner}^{-1}(\varphi)$.

Finally, since iner factors uniquely through $\mathrm{iner}_{\mathcal{D}}$, it follows that the elements of the fibre $\mathrm{iner}^{-1}(\varphi\restriction_{I_F})$ are precisely the irreducible subrepresentations of the representations $\mathrm{Ind}_{G_x}^{K_x}\ \sigma$ (modulo G-conjugacy), where (G_x, σ) ranges through the elements of $\mathrm{iner}_{\mathcal{D}}^{-1}(\varphi\restriction_{I_F}$) and K_x denotes the maximal compact subgroup of $N_G(G_x)$. Each subrepresentation τ of some $\mathrm{Ind}_{G_x}^{K_x}\ \sigma$ is an $\mathfrak{s}$-type, for some $\mathfrak{s} \in \mathfrak{S}_\sigma$. We have already seen that there is a unique $\mathfrak{s}$-type (up to conjugacy) for each $\mathfrak{s}$; it follows that

$$\#\mathrm{iner}^{-1}(\varphi\restriction_{I_F}) = \sum_{(G_x,\sigma)\in\mathrm{iner}_{\mathcal{D}}^{-1}(\varphi\restriction_{I_F})} \#\mathfrak{S}_\sigma,$$

as desired. □

Remarks 6.3.8. (i) In [Mac80], Macdonald constructs a "Langlands correspondence" for certain finite reductive groups, via p-adic methods. In particular, this provides – for certain groups – a correspondence between cuspidal irreducible representations of finite reductive groups and certain homomorphisms from the inertia group to complex tori. One interpretation of the above result is that it shows that the DeBacker–Reeder construction subsumes Macdonald's result, providing via the map $\mathrm{iner}_{\mathcal{D}}$ a similar correspondence (this correspondence also allows one to consider families of representations of possibly distinct finite reductive groups which are simultaneously associated to the same inertial type; this gives a very simple instance of "functoriality" for cuspidal representations of finite reductive groups).

(ii) In the case that the maximal parahoric subgroups of G are all maximal as compact subgroups of G (which is equivalent to requiring that each of the group schemes $\tilde{\mathbf{G}}_x$, for $x \in \mathscr{B}(G)$ a vertex, is connected) – as happens, for example, when $\mathbf{G}$ is semisimple and simply connected – the sets $\mathcal{A}_{\mathrm{tame,reg}}(G)$ and $\mathcal{D}_{\mathrm{tame,reg}}(G)$ and the maps iner and $\text{iner}_{\mathcal{D}}$ coincide; in particular, in this case the description of the inertial correspondence consists simply of the statement *(i)* above.

(iii) The reader should note that the definitions of iner and $\text{iner}_{\mathcal{D}}$ do *not* rely on our unicity results: one can see that these are the unique well-defined surjective maps making the appropriate diagrams commute without knowing anything about unicity. However, the real strength of the above result is in the description of the fibres of these maps, for which our unicity results are crucial.

(iv) An undesirable aspect of our result is that we only obtain a description for $\text{Irr}_{\text{tame,reg}}(G)$. There should exist a more general correspondence, but there are a number of obstacles preventing the proof of such a result. Firstly, and most significantly, we do not yet have a construction of the local Langlands correspondence for arbitrary depth zero irreducible representations of G; in particular, this prevents the observation that any map analogous to $\text{iner}_{\mathcal{D}}$ is well-defined from being made. On top of this, there are two further serious complications. Firstly, the relationship between depth zero types and archetypes becomes far more complicated, meaning that describing the fibres of iner via such a simple formula is unlikely to be possible. For example, in $\mathbf{GL}_2(F)$, one already sees that while the Steinberg representation contains a unique unrefined depth zero type (the trivial representation of the Iwahori subgroup), it contains two archetypes: the trivial representation of $\mathbf{GL}_2(\mathcal{O})$, and the inflation to $\mathbf{GL}_2(\mathcal{O})$ of the Steinberg representation of $\mathbf{GL}_2(\mathbf{k})$. Finally, there are additional complications in the proof of unicity which would require methods distinct to those employed in this paper – it is likely that our methods would suffice only to show that any typical representation of a maximal parahoric subgroup of G is contained in a certain infinite length representation (corresponding to the trivial Mackey summand here; once one considers representations with supercuspidal support defined on a proper parabolic subgroup, this summand is no longer irreducible).

6.4 An inertial Langlands conjecture

We end by stating a conjecture on how we expect the fibres of the inertial Langlands correspondence to look in general. We have, beyond the case of $\mathbf{GL}_N(F)$, understood two additional instances of the correspondence: that of $\mathbf{SL}_N(F)$, where we allow representations to be arbitrarily complicated but specify that the group is rather simple to understand; and that of depth zero supercuspidals, where we allow the group to be arbitrarily complicated, but specify that the representations are as simple as possible. In each of these cases, we see that one obtains a multiplicity in the fibres of iner with respect to those of rec, although these multiplicities arise for completely different reasons – one arises from the representation admitting a type on a sufficiently small parahoric subgroup, while the other arises from the group admitting parahoric subgroups which aren't their own compact normalizer.

Recall that we have, under no assumptions on G, the diagram

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{reg}}(G) & \xrightarrow{\mathrm{rec}} & \mathcal{L}_{\mathrm{reg}}(G) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^{W_F}_{I_F}} \\ \mathcal{A}_{\mathrm{reg}}(G) & \xrightarrow[\mathrm{iner}]{} & \mathcal{I}_{\mathrm{reg}}(G) \end{array}$$

and that, moreover, the map iner is always surjective. We assume Conjecture 2.9.7. Since there are finitely many conjugacy classes of parahoric subgroups of G, this implies that the map iner is finite-to-one.

Now let $\varphi \in \mathcal{L}_{\mathrm{reg}}(G)$, and let $\pi \in \mathrm{rec}^{-1}(\varphi)$. Denote by $\mathrm{Type}(\pi)$ the set of $[G,\pi]_G$-types (J,λ). We associate two constants to π:

- For each $(J,\lambda) \in \mathrm{Type}(\pi)$, let $e_\pi(J,\lambda)$ denote the number of conjugacy classes of maximal compact subgroups of G into which J admits a containment. We then define e_π to be the maximum of the $e_\pi(J,\lambda)$, for $(J,\lambda) \in \mathrm{Type}(\pi)$. Note that e_π is the natural analogue of the ramification degree which appears in the correspondence for $\mathbf{SL}_N(F)$.

- For each $(J, \lambda) \in \mathrm{Type}(\pi)$, pick a maximal compact subgroup K with $J \subset K$, and let G_x be the unique maximal parahoric subgroup contained in K. Since $G_x \cap J$ is open, it will be of finite index in J, so that the restriction to $G_x \cap J$ of λ will be $\mathfrak{S}_\lambda$-typical for some finite subset $\mathfrak{S}_\lambda$ of $\mathscr{B}(G)$. We expect that the set $\mathfrak{S}_\lambda$ should consist only of supercuspidal inertia classes. We then let f_π denote the maximum of the cardinalities $\#\mathfrak{S}_\lambda$, for $(J, \lambda) \in \mathrm{Type}(\pi)$. This constant is the the natural analogue of the multiplicity appearing in the tame inertial correspondence.

Conjecture 6.4.1. *There exists a unique surjective, finite-to-one map* $\mathrm{iner} : \mathcal{A}_{\mathrm{reg}}(G) \to \mathcal{I}_{\mathrm{reg}}(G)$ *such that following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Irr}_{\mathrm{reg}}(G) & \xrightarrow{\mathrm{rec}} & \mathcal{L}_{\mathrm{reg}}(G) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_{I_F}^{W_F}} \\ \mathcal{A}_{\mathrm{reg}}(G) & \xrightarrow[\mathrm{iner}]{} & \mathcal{I}_{\mathrm{reg}}(G) \end{array}$$

Given $\varphi \in \mathcal{L}_{\mathrm{reg}}(G)$, *one should have*

$$\#\mathrm{iner}^{-1}(\varphi \downharpoonright_{I_F}) = \sum_{\pi \in \mathrm{rec}^{-1}(\varphi)} e_\pi f_\pi.$$

We suggest that this statement is the form that the strongest possible result on the unicity of types for regular supercuspidal representations should take in complete generality. Of course, a proof of this result is completely out of reach given the current state of knowledge – it would rely on a completely general construction of types satisfying additional properties such as intertwining implies conjugacy, as well as a proof of the local Langlands conjectures for arbitrary groups.

Bibliography

[Art13] James Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013, Orthogonal and symplectic groups. MR 3135650

[Ber84] J. N. Bernstein, *Le "centre" de Bernstein*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, pp. 1–32. MR 771671 (86e:22028)

[BH06] Colin J. Bushnell and Guy Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006. MR 2234120 (2007m:22013)

[BK93a] Colin J. Bushnell and Philip C. Kutzko, *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ, 1993. MR 1204652 (94h:22007)

[BK93b] ———, *The admissible dual of* $\mathrm{SL}(N)$. *I*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 2, 261–280. MR 1209709 (94a:22033)

[BK94] ———, *The admissible dual of* $\mathrm{SL}(N)$. *II*, Proc. London Math. Soc. (3) **68** (1994), no. 2, 317–379. MR 1253507 (94k:22035)

[BK98] ______, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 582–634. MR 1643417 (2000c:22014)

[BK99] ______, *Semisimple types in* GL_n, Compositio Math. **119** (1999), no. 1, 53–97. MR 1711578 (2000i:20072)

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart. MR 1944572 (2004i:11052)

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251. MR 0327923

[BT84] ______, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 60, 197–376. MR 756316

[Car84] H. Carayol, *Représentations cuspidales du groupe linéaire*, Ann. Sci. École Norm. Sup. (4) **17** (1984), no. 2, 191–225. MR 760676

[DR09] Stephen DeBacker and Mark Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169** (2009), no. 3, 795–901. MR 2480618

[Fin15] Jessica Fintzen, *On the Moy–Prasad filtration*.

[Gar97] Paul Garrett, *Buildings and classical groups*, Chapman & Hall, London, 1997. MR 1449872

[GK82] S. S. Gelbart and A. W. Knapp, *L-indistinguishability and R groups for the special linear group*, Adv. in Math. **43** (1982), no. 2, 101–121. MR 644669 (83j:22009)

[GR02] David Goldberg and Alan Roche, *Types in* SL_n, Proc. London Math. Soc. (3) **85** (2002), no. 1, 119–138. MR 1901371 (2003d:22018)

[GR10] Benedict H. Gross and Mark Reeder, *Arithmetic invariants of discrete Langlands parameters*, Duke Math. J. **154** (2010), no. 3, 431–508. MR 2730575

[GT11] Wee Teck Gan and Shuichiro Takeda, *The local Langlands conjecture for* GSp(4), Ann. of Math. (2) **173** (2011), no. 3, 1841–1882. MR 2800725

[Hen00] Guy Henniart, *Une preuve simple des conjectures de Langlands pour* GL(n) *sur un corps* p*-adique*, Invent. Math. **139** (2000), no. 2, 439–455. MR 1738446

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802

[Kim07] Ju-Lee Kim, *Supercuspidal representations: an exhaustion theorem*, J. Amer. Math. Soc. **20** (2007), no. 2, 273–320 (electronic). MR 2276772 (2008c:22014)

[Lat16a] Peter Latham, *On the unicity of types in special linear groups*, arXiv preprint (2016).

[Lat16b] ———, *The unicity of types for depth zero supercuspidal representations*, arXiv preprint (2016).

[Lat16c] ———, *Unicity of types for supercuspidal representations of* p*-adic* $\mathbf{SL}_2$, J. Number Theory **162** (2016), 376–390.

[LL79] J.-P. Labesse and R. P. Langlands, L*-indistinguishability for* SL(2), Canad. J. Math. **31** (1979), no. 4, 726–785. MR 540902 (81b:22017)

[Mac80] I. G. Macdonald, *Zeta functions attached to finite general linear groups*, Math. Ann. **249** (1980), no. 1, 1–15. MR 575444

[Mor93] Lawrence Morris, *Tamely ramified intertwining algebras*, Invent. Math. **114** (1993), no. 1, 1–54. MR 1235019

[Mor99] ———, *Level zero* $\mathbf{G}$*-types*, Compositio Math. **118** (1999), no. 2, 135–157. MR 1713308 (2000g:22029)

[MP94] Allen Moy and Gopal Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), no. 1-3, 393–408. MR 1253198 (95f:22023)

[MP96] ______, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121. MR 1371680 (97c:22021)

[MS14] Michitaka Miyauchi and Shaun Stevens, *Semisimple types for p-adic classical groups*, Math. Ann. **358** (2014), no. 1-2, 257–288. MR 3157998

[Nad14] Santosh Nadimpalli, *Typical representations of $GL_3(F)$*, arXiv:1405.7286 (2014).

[Nad15] ______, *Typical representations of $GL_N(F)$*, PhD thesis, Université Paris-Sud (2015).

[Pas05] Vytautas Paskunas, *Unicity of types for supercuspidal representations of* GL_N, Proc. London Math. Soc. (3) **91** (2005), no. 3, 623–654. MR 2180458 (2007b:22018)

[Ren10] David Renard, *Représentations des groupes réductifs p-adiques*, Cours Spécialisés [Specialized Courses], vol. 17, Société Mathématique de France, Paris, 2010. MR 2567785 (2011d:22019)

[RY14] Mark Reeder and Jiu-Kang Yu, *Epipelagic representations and invariant theory*, J. Amer. Math. Soc. **27** (2014), no. 2, 437–477. MR 3164986

[Séc05] Vincent Sécherre, *Représentations lisses de* $\mathrm{GL}_m(D)$*. III. Types simples*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 6, 951–977. MR 2216835 (2007a:22010)

[SS97] Peter Schneider and Ulrich Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 85, 97–191. MR 1471867

[SS08] V. Sécherre and S. Stevens, *Représentations lisses de* $\mathrm{GL}_m(D)$*. IV. Représentations supercuspidales*, J. Inst. Math. Jussieu **7** (2008), no. 3, 527–574. MR 2427423 (2009d:22023)

[SS12] Vincent Sécherre and Shaun Stevens, *Smooth representations of $GL_m(D)$ VI: semisimple types*, Int. Math. Res. Not. IMRN (2012), no. 13, 2994–3039. MR 2946230

[Ste08] Shaun Stevens, *The supercuspidal representations of p-adic classical groups*, Invent. Math. **172** (2008), no. 2, 289–352. MR 2390287 (2010e:22008)

[Yu01] Jiu-Kang Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622 (electronic). MR 1824988 (2002f:22033)