Supertasks and Numeral Systems

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Abstract. Physical supertasks are completed, infinite sequences of events or interactions that occur within a finite amount of time. Examples thereof have been constructed to show that infinite physical systems may violate conservation laws. It is shown in this paper that this conclusion may be critically sensitive to a selection of numeral system. Weaker numeral systems generate physical reports whose inaccuracy simulates the violation of a conservation law. Stronger numeral systems can confirm this effect by allowing a direct computation of the quantities conserved. The supertasks presented in [2], [4] are used to illustrate this phenomenon from the point of view of the new numeral system introduced in [6].

SUPERTASKS AND PHYSICAL SUPERTASKS

A supertask is an actually infinite sequence of tasks carried out in a finite amount of time. A famous example is Thomson's lamp (see [13]). In this supertask, a switch, assumed initially on, is turned off after $1/2$ minutes, then on after $1/2 + 1/4$ minutes, then off again after $1/2 + 1/4 + 1/8$ minutes, and so on. For any number n expressible in decimal form (i.e., in base ten), n operations are carried out within one minute. It cannot however be determined what the state of the switch after one minute will be. The reason is that the numeral reports about this process that can be given in decimal form only convey information about its finite, albeit arbitrarily long, initial segments. Such numeral reports sustain the type of sequential argument familiar from analysis, which rests on relations between finite quantities. Because a supertask is a completed, sequential infinity of tasks, sequential arguments based on numeral reports in decimal form do not offer enough information to describe its completion. In particular, they cannot supply any specification of the supertask's length. In the absence of such specification, the final state of the switch cannot be determined. Note the difference between concluding that the final state of the switch is inherently indeterminate (a feature of the object under study) and concluding that the numeral reports describing Thomson's lamp do not offer enough information to reach a determinate conclusion (an effect of the reports' low accuracy, not a feature of the object under study).

Given the definition of a supertask as a completed process, the first conclusion can be ruled out in favour of the second. As a consequence, a numeral system that can specify infinite lengths in a computationally amenable manner[1] may enable the production of numeral reports about Thomson's lamp that suffice to determine its outcome.

The new numeral system introduced by Yaroslav Sergeyev (see e.g. [6], [7], [8]) achieves this goal. In fact, on account of the ease with which it handles computations involving infinitely large and small numbers, it affords a general framework to produce more accurate reports about any supertask. This possibility is of special interest when the stages in a supertask are physical events. Physical supertasks have been constructed by physicists and philosophers to show that actually infinite physical systems (e.g. within the framework classical, relativistic or quantum mechanics) may violate conservation laws or exhibit indeterministic behaviour (see e.g. [1], [2], [3], [4], [5]). Such conclusions can be shown to depend on identifying the object of study (an infinite physical system) with the means adopted to describe it (e.g. a numeral system in base ten). The application of Sergeyev's stronger numeral system makes the distinction between the two apparent and leads to a simple verification that no anomalous physical behaviour has occurred. A

[1] Such is not the case of Cantorian cardinals, for instance, on account of the essential triviality of transfinite cardinal arithmetic, in presence of the Axiom of Choice. Infinite ordinals violate arithmetical laws that hold in the finite.

detailed discussion of observations of infinite processes using Sergeyev's numerals can be found in [10, 12] where Turing machines are studied

A STRONGER NUMERAL SYSTEM

Sergeyev's numeral system is based on the infinite unit ①, to be regarded as a measure of the length of $\mathbb{N}$ (0 not included). It is naturally suited to the study of supertasks, which are usually assumed to be sequential processes of precisely the length of $\mathbb{N}$. In Sergeyev's system, the stages in a supertask can be counted along the following sequence:

$$\{1, 2, 3, \ldots, ① - 3, ① - 2, ① - 1, ①\}.$$

Because Sergeyev's approach is developed within the axiomatic characterisation of the real field (see [6]), ① obeys all of the familiar laws of arithmetic. For present purposes, the fact that ① specifies the length of a supertask and that ordinary arithmetic can be carried out with it suffices. It is however important to remark that Sergeyev's numeral system allows a sharp generalisation of the ordinary notion of supertask, because it is capable of specifying the lengths of sequential processes that numerals in decimal form or even Cantorian numerals could not have measured (a detailed discussion on relations of Cantor's and Sergeyev's numerals can be found, e.g. in [9]). In particular, on the basis of Sergeyev's assumptions, every arithmetical progression of the form $\mathbb{N}_{k,n} = k + n, k + 2n, k + 3n, \ldots\}$, with $1 \leq k \leq n$, singles out a subset of $\mathbb{N}$ of size $①/n$. Thus, the set of even numbers $\mathbb{N}_{2,2}$ has size $①/2$ and this measure denotes a number in $\mathbb{N}$, since it determines the size of a collection of objects. In a similar manner $①/3, ①/4, ①/5, \ldots$ denote numbers in $\mathbb{N}$. It is therefore possible to describe supertasks of any one among these lengths, which will be shorter than a standard supertask of length ①. Longer supertasks can also be specified (e.g. including $① + 1$ stages).

A resolution of Thomson's lamp is now easy to compute (it was originally given in [6]). If the switch is turned on in the first stage of the supertask, it will be off at every even stage: since ① is even ($①/2 \in \mathbb{N}$ and $2(①/2) = ①$), it will be eventually off. The completion of Thomson's lamp, which the reports baed on the potential infinity of numeral labels $1, 2, 3, \ldots$ left indeterminate, becomes visible when Sergeyev's stronger numeral system is introduced. A similar analysis applies to physical supertasks, with the difference that the expressive limitations of traditional numeral systems do not simulate, in such cases, an indeterminate outcome but instead the failure of a conservation law.

ELASTIC COLLISIONS

Simple mechanical supertasks involving an infinite sequence of elastic collision (either between point particles or between rigid bodies) have been studied by Laraudogoitia in [2] and, recently, in [4]. The following subsections show in what way numeral reports in decimal form give rise to the impression that kinetic energy is not conserved in these systems. A transition to more accurate numeral reports, formulated in Sergeyev's system, allows a direct verification that it is.

Point Particles

The one-dimensional, kinematic supertask constructed in [2] may be located within a length of 1 metre, marked by the numeral labels 0 and 1. For every n expressible in decimal form, the point particle p_n, of mass m, is at $1/2^n$ metres from 0 and at rest. A force is applied to p_1, which sets it into motion at a uniform, finite velocity, say 1m/s. This action triggers a chain of elastic collisions. When p_1 collides with p_2 at $1/4$ metres away from 0, it comes to rest there and pushes p_2 towards p_3 at the uniform velocity 1m/s, and so on.

If one restricts attention to reports about this supertask that only use numerals in decimal form, it is possible to bound the duration of the process. Every particle for which numeral reports can be produced comes to rest before 1/2s. It is, however, not possible to specify the length of the process, i.e., the number of successive collisions or of particles involved. In the absence of this specification, if the numeral reports available are taken to yield a complete description of the given physical system, it is possible to infer that the whole system is at rest within 1/2s. Then, its kinetic energy 2^{-1}J has vanished. A conservation law seems to fail. This conclusion depends on the fact that no information other than that supplied by the available numeral reports could be obtained. Sergeyev's system, on the other hand, generates more reports about the successive collisions involved in the supertask. First, there are ① of them. Furthermore, there is a particle $p_{①-2}$ that collides with $p_{①-1}$ after $1 - 1/2^{①-1}$ units of time. Finally, it is possible to produce a report about the final collision, between $p_{①-1}$ and $p_{①}$, which takes place at $1/2^{①}$ units of distance from position 0 and after which

$p_{①}$ starts moving at the uniform velocity 1m/s towards 0. These reports make apparent that the kinetic energy of the system is not lost.

It may be objected that one could reproduce Laraudogoitia's setup even in presence of Sergeyev's numeral system by requiring that the given physical system should contain exactly as many particles as there are numbers expressible in decimal form. The size of such a system cannot be specified in base ①: reports about its completion are not available within the stronger numeral system. It does not follow from this that kinetic energy is not conserved. Physical laws are not violated because of expressive limitations in the language of physical reports. The restriction that there should be as many particles as there are numbers expressible in decimal form tacitly presupposes that Sergeyev's numeral system should yield a complete description of $\mathbb{N}$, so that the subsets of $\mathbb{N}$ for which it does not provide a measure are to be thought of as absolutely devoid of length. This amounts to identifying the means of representation with the object they represent, a move that contradicts one of the three postulates in Sergeyev's methodology (see e.g. [8], p.e1690), to the effect that only stronger or weaker, but always finite, resources can be used to study infinite entities. There is no hope to attain, by finite means, a complete description of the infinite, but there is hope to refine it.

Rigid bodies

In [4], Laraudogoitia considers again a physical system of the size of $\mathbb{N}$, in which a sequence of collisions takes place between two points marked by $0, 1$ and, say, one metre apart. The features of the system are so arranged that every object in the middle of the supertask undergoes exactly two collisions and, after the second collision, it recoils at the same, constant velocity. It is also ensured that the total mass is finite (details are worked out below). Furthermore, the colliding objects are now taken to be concentric, hollow shells $p_1, p_2, p_3, \ldots$. If their common centre is at point 1, then the sum of the differences between their radii is supposed to be bounded by 1. Laraudogoitia's system of shells is at first considered at rest in a reference frame ρ. The innermost sphere p_1 is then pushed towards 0 at 1m/s. If m_i is the mass of p_i, set $\mu_i = m_{i+1}/m_1$ $(i \leq 1)$. Since both momentum and kinetic energy are conserved in any finite sequence of elastic collisions, Laraudogoitia shows that the velocities of p_i after the first and the second collision, respectively v_i'', v_i' are determined by the following equations:

$$v_i'' = \left(\frac{2}{\mu_{i-1}+1}\right)\cdots\left(\frac{2}{\mu_2+1}\right)\left(\frac{2}{\mu_1+1}\right)v, \tag{1}$$

$$v_i' = \left(\frac{1-\mu_i}{1+\mu_i}\right)\left(\frac{2}{\mu_{i-1}+1}\right)\cdots\left(\frac{2}{\mu_2+1}\right)\left(\frac{2}{\mu_2+1}\right)v. \tag{2}$$

In order to guarantee that, after their last collision, shells in the middle of the supertask recoil at the same, constant velocity, the relation $v_i' = v_{i+1}'$ must be satisfied, which is equivalent to:

$$\mu_{i+1} = \frac{1+\mu_i}{3-\mu_i}. \tag{3}$$

Setting $m_1 = 1kg$, where m_1 is the mass of p_1, and adopting the solution $\mu_i = i/(i+2)$ to (3) (suggested by Laraudogoitia in his paper), it is easy to see that:

$$m_i = \frac{1}{1+2+\ldots+(i-1)+i}kg. \tag{4}$$

The value for the total mass of the system, when determined as the limit of a sequence of partial sums, is $2kg$. This numeral report is inexact up to an infinitesimal error. It is refined in Sergeyev's numeral system, as follows, by fixing the number of shells equal to grossone (for a detailed discussion on summation of a fixed infinite number of addends see [8, 11]):

$$\begin{aligned}
M = \sum_{i=1}^{①} m_i &= 2\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{(①-1)\cdot ①}+\frac{1}{①\cdot(①+1)}\right)kg \\
&= 2\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{①-1}-\frac{1}{①}+\frac{1}{①}-\frac{1}{①+1}\right)kg \\
&= \left(2-\frac{2}{①+1}\right)kg.
\end{aligned}$$

For the given values of m_i, it is easy to verify that:

$$\frac{1-\mu_i}{1+\mu_i} = \frac{1}{i+1} \quad ; \quad \frac{2}{\mu_i + 1} = \frac{i+2}{i+1} \tag{5}$$

Using (5) in (2), it is found that every shell in the middle of the sequential process recoils at 2^{-1}m/s after its last collision. This is true of p_1 as well, the only object not in the middle of the supertask for which reports can be compiled in the numeral system presupposed by Laraudogoitia. On account of this expressive limitation of the numeral system, it is plausible to consider a reference frame ρ^* that moves, relative to ρ, at 2^{-1}m/s in the direction from 0 to 1. After the collisions describable by reports in decimal form have taken place, the whole system of shells is at rest in ρ^*: a net loss of kinetic energy seems to have occurred. One does not need to reach this conclusion by abstracting from the dynamical character of the system because 'the gravitational attraction between two hollow spheres in our model is null' ([4], p.9). The conclusion that kinetic energy is not conserved follows, again, from the fact that numeral reports in decimal form only describe arbitrarily long but finite segments of the given sequential process, whilst they are insufficient to describe its completion. The only available marker of completion, if values are given for the shells' radii, is an estimate of the supertask's duration, which does not provide any insight into the mechanical state of the physical system. The required insight can be supplied in Sergeyev's numeral system. Within the frame of reference ρ, equation (1) allows one to compute the velocity of $p_{①}$ after the last collision, which is $[(①+1)/2]$ m/s. Moreover, since the sum of all natural numbers, which can be computed in Sergeyev's system, is $[①(①+1)]/2$, equation (4) yields $m_{①} = 2/[①(①+1)]$. These values are all that is needed to compute the kinetic energy in ρ of the system of hollow shells described by Laraudogoitia, both before any collision has taken place and after every collision has taken place. The kinetic energy of the whole system before any collision has taken place is 2^{-1}J. The kinetic energy of the system after every collision has taken place can now be computed by recalling that every shell p_i with $2 \le i \le ① - 1$ recoils at the constant velocity 2^{-1}m/s after its last collision (note that this report is not available in decimal form). The kinetic energy of the system when all ① collisions have occurred is then:

$$\frac{1}{2}\left[(M - m_{①})\frac{1}{4}\right]J + \frac{1}{2}\left[m_{①}\frac{(①+1)^2}{4}\right]J \quad = \quad \frac{1}{4}\left(\frac{①-1}{①}\right)J + \frac{1}{4}\left(\frac{①+1}{①}\right)J = 2^{-1}J$$

and kinetic energy is conserved in ρ. Thus, it is conserved in ρ^*.

CONCLUSIONS

Laraudogoitia's argument against the conservation of kinetic energy in ρ^* allows him to infer, using time-symmetry, that a net gain in kinetic energy may also occur. He then uses this remark to envisage an infinite chain of supertasks, in which gains and losses alternate infinitely many times and increase without bounds (in decimal form). This scenario leads him to the conclusion that, after a finite amount of time, the whole system of shells must vanish out of existence, if only the world-lines of its points are continuous (details are in [4], pp.16-18). Such a conclusion is undercut by the previous discussion. In Sergeyev's stronger numeral system enough reports can be produced to rule out this scenario: after the first collision supertask is completed, $p_{①}$ is moving infinitely fast towards 0 (in ρ or ρ^*) and it takes an external force to reverse the direction of its motion.

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