BLOCK DECOMPOSITION OF THE CATEGORY OF ℓ-MODULAR SMOOTH REPRESENTATIONS OF $\mathrm{GL}_n(\mathrm{F})$ AND ITS INNER FORMS

by

Vincent Sécherre & Shaun Stevens

Abstract. — Soit F un corps commutatif localement compact non archimédien de caractéristique résiduelle p, soit D une F-algèbre à division centrale de dimension finie et soit R un corps algébriquement clos de caractéristique différente de p. A toute représentation lisse irréductible du groupe $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$, $m \geqslant 1$ à coefficients dans R correspond une classe d'inertie de paires supercuspidales de G. Ceci définit une partition de l'ensemble des classes d'isomorphisme de représentations irréductibles de G. Notons $\mathscr{R}(\mathrm{G})$ la catégorie des représentations lisses de G à coefficients dans R et, pour toute classe d'inertie Ω de paires supercuspidales de G, notons $\mathscr{R}(\Omega)$ la sous-catégorie formée des représentations lisses dont tous les sous-quotients irréductibles appartiennent au sous-ensemble déterminé par cette classe d'inertie. Nous prouvons que $\mathscr{R}(\mathrm{G})$ est le produit des $\mathscr{R}(\Omega)$, où Ω décrit les classes d'inertie de paires supercuspidales de G, et que chaque facteur $\mathscr{R}(\Omega)$ est indécomposable.

Résumé. — Let F be a nonarchimedean locally compact field of residue characteristic p, let D be a finite dimensional central division F-algebra and let R be an algebraically closed field of characteristic different from p. To any irreducible smooth representation of $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$, $m \geqslant 1$ with coefficients in R, we can attach a uniquely determined inertial class of supercuspidal pairs of G. This provides us with a partition of the set of all isomorphism classes of irreducible representations of G. We write $\mathscr{R}(\mathrm{G})$ for the category of all smooth representations of G with coefficients in R. To any inertial class Ω of supercuspidal pairs of G, we can attach the subcategory $\mathscr{R}(\Omega)$ made of smooth representations all of whose irreducible subquotients are in the subset determined by this inertial class. We prove that the category $\mathscr{R}(\mathrm{G})$ decomposes into the product of the $\mathscr{R}(\Omega)$'s, where Ω ranges over all possible inertial class of supercuspidal pairs of G, and that each summand $\mathscr{R}(\Omega)$ is indecomposable.

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Introduction

When considering a category of representations of some group or algebra, a natural step is to attempt to decompose the category into *blocks*; that is, into subcategories which are indecomposable summands. Thus any representation can be decomposed uniquely as a direct sum of pieces, one in each block; any morphism comes as a product of morphisms, one in each block;

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and this decomposition of the category is the finest decomposition for which these properties are satisfied. Then a full understanding of the category is equivalent to a full understanding of all of its blocks.

In the case of representations of a finite group G, over an algebraically closed field R, there is always a block decomposition. In the simplest case, when the characteristic of R is prime to the order of G, this is particularly straightforward: all representations are semisimple so each block consists of representations isomorphic to a direct sum of copies of a fixed irreducible representation. In the general case, there is a well-developed theory, beginning with the work of Brauer and Nesbitt, and understanding the block structure is a major endeavour.

Now suppose G is the group of rational points of a connected reductive algebraic group over a nonarchimedean locally compact field F, of residue characteristic p. When R has characteristic zero, a block decomposition of the category $\mathscr{R}_{\mathrm{R}}(\mathrm{G})$ of smooth R-representations of G was given by Bernstein [**1**], in terms of the classification of representations of G by their cuspidal support. Any irreducible representations π of G is a quotient of some (normalized) parabolically induced representation $\boldsymbol{i}_{\mathrm{M}}^{\mathrm{G}}\varrho$, with ϱ a cuspidal irreducible representation of a Levi subgroup M of G; the pair (M,ϱ) is determined up to G-conjugacy by π and is called its *cuspidal support*. Then each such pair (M,ϱ) determines a block, whose objects are those representations of G all of whose subquotients have cuspidal support $(\mathrm{M},\varrho\chi)$, for some unramified character χ of M.

One important tool in proving this block decomposition is the equivalence of the following two properties of an irreducible R-representation π of G:

- π is not a quotient of any properly parabolically induced representation; equivalently, all proper Jacquet modules of π are zero (π is *cuspidal*);
- π is not a *sub*quotient of any properly parabolically induced representation $\boldsymbol{i}_{\mathrm{M}}^{\mathrm{G}}\varrho$ with ϱ an irreducible representation (π is *supercuspidal*).

When R is an algebraically closed field of positive characteristic different from p (the *modular* case), these properties are no longer equivalent and the methods used in the characteristic zero case cannot be applied. Instead, one can attempt to define the *supercuspidal support* of a smooth irreducible R-representation π of G: it is a pair (M,ϱ) consisting of an irreducible supercuspidal representation ϱ of a Levi subgroup M of G such that π is a *subquotient* of $\boldsymbol{i}_{\mathrm{M}}^{\mathrm{G}}\varrho$. However, for a general group G, it is not known whether the supercuspidal support of a representation is well-defined up to conjugacy; indeed, the analogous question for finite reductive groups of Lie type is also open.

In any case, one can define the notion of an *inertial supercuspidal class* $\Omega=[\mathrm{M},\varrho]_{\mathrm{G}}$: it is the set of pairs (M',ϱ'), consisting of a Levi subgroup M' of G and a supercuspidal representation ϱ' of M', which are G-conjugate to $(\mathrm{M},\varrho\chi)$, for some unramified character χ of M. Given such a class Ω, we denote by $\mathscr{R}_{\mathrm{R}}(\Omega)$ the full subcategory of $\mathscr{R}_{\mathrm{R}}(\mathrm{G})$ whose objects are those representations all of whose subquotients are isomorphic to a subquotient of $\boldsymbol{i}_{\mathrm{M}'}^{\mathrm{G}}\varrho'$, for some $(\mathrm{M}',\varrho')\in\Omega$.

The main purpose of this paper is then to prove the following result:

Theorem . — *Let* G *be an inner form of* $\mathrm{GL}_n(\mathrm{F})$ *and let* R *be an algebraically closed field of characteristic different from* p*. Then there is a block decomposition*

$$\mathscr{R}_{\mathrm{R}}(\mathrm{G})=\prod_{\Omega}\mathscr{R}_{\mathrm{R}}(\Omega),$$

where the product is taken over all inertial supercuspidal classes.

This theorem generalizes the Bernstein decomposition in the case that R has characteristic zero, and also a similar statement, for general R, stated by Vignéras [**24**] in the split case $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$; however, the authors were unable to follow all the steps in [**24**] so our proof is independent, even if some of the ideas come from there.

Our proof builds on work of Mínguez and the first author [**15, 16**], in which they give a classification of the irreducible R-representations of G, in terms of supercuspidal representations, and of the supercuspidal representations in terms of the theory of types. In particular, they prove that supercuspidal support is well-defined up to conjugacy, so that the irreducible objects in $\mathscr{R}_\mathrm{R}(\Omega)$ are precisely those with supercuspidal support in Ω.

One question we do not address here is the structure of the blocks $\mathscr{R}_\mathrm{R}(\Omega)$. Given the explicit results on supertypes here, it is not hard to construct a progenerator Π for $\mathscr{R}_\mathrm{R}(\Omega)$ as a compactly-induced representation; for $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ this was done (independently) by Guiraud [**11**] (for level zero blocks) and Helm [**12**]. Then $\mathscr{R}_\mathrm{R}(\Omega)$ is equivalent to the category of $\mathrm{End}_\mathrm{G}(\Pi)$-modules. In the case that R has characteristic zero, the algebra $\mathrm{End}_\mathrm{G}(\Pi)$ was described as a tensor product of affine Hecke algebras of type A in [**22**] (or [**7**] in the split case); indeed, we use this description in our proof here. For R an algebraic closure $\overline{\mathbf{F}}_\ell$ of a finite field of characteristic $\ell \neq p$, and a block $\mathscr{R}_\mathrm{R}(\Omega)$ with $\Omega = [\mathrm{GL}_n(\mathrm{F}), \varrho]_{\mathrm{GL}_n(\mathrm{F})}$, Dat [**9**] has described this algebra; it is an algebra of Laurent polynomials in one variable over the R-group algebra of a cyclic ℓ-group. It would be interesting to obtain a description in the general case.

We now describe the proof of the theorem, which relies substantially on the theory of semisimple types developed in [**22**] (see [**7**] for the split case). Given an inner form G of $\mathrm{GL}_n(\mathrm{F})$, in [**22**] the authors constructed a family of pairs $(\mathbf{J}, \boldsymbol{\lambda})$, consisting of a compact open subgroup $\mathbf{J}$ of G and an irreducible complex representation $\boldsymbol{\lambda}$ of $\mathbf{J}$. This family of pairs $(\mathbf{J}, \boldsymbol{\lambda})$, called semisimple types, satisfies the following condition: for every inertial cuspidal class Ω, there is a semisimple type $(\mathbf{J}, \boldsymbol{\lambda})$ such that the irreducible complex representations of G with cuspidal support in Ω are exactly those whose restriction to $\mathbf{J}$ contains $\boldsymbol{\lambda}$.

In [**15**], Mínguez and the first author extended this construction to the modular case: they constructed a family of pairs $(\mathbf{J}, \boldsymbol{\lambda})$, consisting of a compact open subgroup $\mathbf{J}$ of G and an irreducible complex representation $\boldsymbol{\lambda}$ of $\mathbf{J}$, called semisimple supertypes. However, they did not give the relation between these semisimple supertypes and inertial supercuspidal classes of G. In this paper, we prove:

– for each inertial supercuspidal class Ω, there is a semisimple supertype $(\mathbf{J}, \boldsymbol{\lambda})$ such that the irreducible R-representations of G with supercuspidal support in Ω are precisely those which appear as subquotients of the compactly induced representation $\mathrm{ind}_\mathbf{J}^\mathrm{G}(\boldsymbol{\lambda})$;

– two semisimple supertypes $(\mathbf{J}, \boldsymbol{\lambda})$ and $(\mathbf{J}', \boldsymbol{\lambda}')$ correspond to the same inertial supercuspidal class if and only if the compactly induced representations $\mathrm{ind}_\mathbf{J}^\mathrm{G}(\boldsymbol{\lambda})$ and $\mathrm{ind}_{\mathbf{J}'}^\mathrm{G}(\boldsymbol{\lambda}')$ are isomorphic, in which case we say the supertypes are equivalent.

Thus we get a bijective correspondence between the inertial supercuspidal classes for G and the equivalence classes of semisimple supertypes.

To each semisimple supertype, we attach a crucial tool, already used in [**16**] to obtain the classification of the irreducible R-representations of G. This is a functor which associates, to each smooth R-representation of G, a representation of the finite reductive quotient of $\mathbf{J}$. More precisely, given a semisimple supertype $(\mathbf{J}, \boldsymbol{\lambda})$, there is a normal compact open subgroup $\mathbf{J}^1$ of $\mathbf{J}$ such that:

– the quotient $\mathbf{J}/\mathbf{J}^1$ is isomorphic to a group of the form $\mathrm{GL}_{n_1}(\mathfrak{k}_1) \times \cdots \times \mathrm{GL}_{n_r}(\mathfrak{k}_r)$, where $\mathfrak{k}_i$ is a finite extension of the residue field of F and n_i is a positive integer, for $i \in \{1, \ldots, r\}$;

– the representation $\boldsymbol{\lambda}$ decomposes (non-canonically) as $\boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$, where $\boldsymbol{\kappa}$ is a particular irreducible representation of $\mathbf{J}$ and $\boldsymbol{\sigma}$ is the inflation to $\mathbf{J}$ of a supercuspidal irreducible representation of $\mathrm{GL}_{n_1}(\mathfrak{k}_1) \times \cdots \times \mathrm{GL}_{n_r}(\mathfrak{k}_r)$;

– in the particular case where the semisimple supertype is homogeneous (see §6.2), there is a normal compact open subgroup $\mathbf{H}^1$ of $\mathbf{J}^1$ such that the restriction of $\boldsymbol{\kappa}$ to $\mathbf{H}^1$ is a direct sum of copies of a certain character $\boldsymbol{\theta}$, called a simple character.

Given a choice of decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$, we define a functor

$$\mathsf{K} = \mathsf{K}_{\boldsymbol{\kappa}} : \pi \mapsto \mathrm{Hom}_{\mathbf{J}^1}(\boldsymbol{\kappa}, \pi)$$

from $\mathscr{R}(\mathrm{G})$ to $\mathscr{R}(\mathbf{J}/\mathbf{J}^1)$, with $\mathbf{J}$ acting on $\mathrm{Hom}_{\mathbf{J}^1}(\boldsymbol{\kappa}, \pi)$ via

$$x \cdot f = \pi(x) \circ f \circ \boldsymbol{\kappa}(x)^{-1},$$

for all $x \in \mathbf{J}$ and $f \in \mathrm{Hom}_{\mathbf{J}^1}(\boldsymbol{\kappa}, \pi)$. Since $\mathbf{J}^1$ is a pro-p group, this functor is exact.

An important property of this functor K is its behaviour with respect to parabolic induction (see Theorem 6.2): for a parabolic subgroup of G compatible with the data involved in the construction of $(\mathbf{J}, \boldsymbol{\lambda})$, this functor commutes with parabolic induction. This result is related to a remarkable property of simple characters (see Lemma 4.2) which, to our knowledge, was not previously known even in the split case.

This allows a somewhat surprising back-and-forth argument between the complex case, where the compatibility of K with parabolic induction was already known (see [**17**]), and the modular case; this is because, in the case of a homogeneous supertype, the condition on the simple character $\boldsymbol{\theta}$ holds for R-representations if and only if it holds for complex representations, since $\mathbf{H}^1$ is a pro-p group (see the proof of Proposition 5.6). This is the objective of sections 2 to 8, and requires the notion of endo-class developed in [**21**] (see [**4**] in the split case).

Now we need to define the subcategories of $\mathscr{R}_{\mathrm{R}}(\mathrm{G})$ which will be the blocks we seek, which we do in section 9. To each semisimple supertype $(\mathbf{J}, \boldsymbol{\lambda})$ we associate a full subcategory $\mathscr{R}_{\mathrm{R}}(\mathbf{J}, \boldsymbol{\lambda})$, whose objects are those smooth representations V which are generated by the maximal subspace of $\mathsf{K}(\mathrm{V})$ all of whose irreducible subquotients have supercuspidal support in a fixed set determined by $\boldsymbol{\sigma}$ (see Definition 1.14). This subcategory is independent of the choice of decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$. Note that the existence of a maximal subspace of $\mathsf{K}(\mathrm{V})$ with the required property depends on a decomposition of the category of representations of the finite reductive group

$$\mathbf{J}/\mathbf{J}^1 \simeq \mathrm{GL}_{n_1}(\mathfrak{k}_1) \times \cdots \times \mathrm{GL}_{n_r}(\mathfrak{k}_r)$$

in terms of supercuspidal support (the unicity of which is one of the principal results of [**14**]). Moreover, it follows from this decomposition that $\mathscr{R}_{\mathrm{R}}(\mathrm{G})$ decomposes as a product of the subcategories $\mathscr{R}_{\mathrm{R}}(\mathbf{J}, \boldsymbol{\lambda})$, where $(\mathbf{J}, \boldsymbol{\lambda})$ runs through the equivalence classes of semisimple supertypes.

It remains only to prove that the $\mathscr{R}_{\mathrm{R}}(\mathbf{J}, \boldsymbol{\lambda})$ are indecomposable and coincide with the $\mathscr{R}_{\mathrm{R}}(\Omega)$, which is the purpose of section 10. To prove the indecomposability of the $\mathscr{R}_{\mathrm{R}}(\mathbf{J}, \boldsymbol{\lambda})$ we use the endomorphism algebra of the compactly induced representation $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$, whose structure was determined in [**22**] (and [**15**] for the modular case). The centre of this algebra is an integral domain, which implies that $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$ is indecomposable. Since its irreducible subquotients coincide with the irreducible objects of $\mathscr{R}_{\mathrm{R}}(\mathbf{J}, \boldsymbol{\lambda})$, it follows that this subcategory is indecomposable.

We end the paper, in section 11, by proving a remarkable property of supercuspidality: if an irreducible representation of G does not appear as a subquotient any properly parabolically induced representation $\boldsymbol{i}_{\mathrm{M}}^{\mathrm{G}}\varrho$, with ϱ irreducible, then it also does not appear as a subquotient of *any* properly parabolically induced representation.

Notation

Throughout the paper, we fix a prime number p and an algebraically closed field R of characteristic different from p.

All representations are supposed to be smooth representations on R-vector spaces. If G is a topological group, we write $\mathscr{R}(\mathrm{G})$ for the abelian category of all representations of G and Irr(G) for the set of all isomorphism classes of irreducible representations of G. A *character* of G is a homomorphism from G to $\mathrm{R}^\times$ with open kernel.

For G the group of points of a connected reductive group over either a finite field of characteristic p or a nonarchimedean locally compact field of residual characteristic p, and $\mathrm{P} = \mathrm{MN}$ a parabolic subgroup of G together with a Levi decomposition, we will write $\boldsymbol{i}_{\mathrm{P}}^{\mathrm{G}}$ for the *normalized* parabolic induction functor from $\mathscr{R}(\mathrm{M})$ to $\mathscr{R}(\mathrm{G})$, and $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}$ for the *unnormalized* parabolic induction functor from $\mathscr{R}(\mathrm{M})$ to $\mathscr{R}(\mathrm{G})$; these coincide in the finite field case.

§1. Extensions and blocks

We begin with some general results which apply to connected reductive groups over both finite and nonarchimedean locally compact fields. In the finite case, we give some further results towards a block decomposition, in particular for the group GL_n; these will be needed in the nonarchimedean case later.

1.1.

Let G be the group of points of a connected reductive group over either a finite field of characteristic p or a nonarchimedean locally compact field of residual characteristic p.

Definition 1.1. — An irreducible representation π of G is *supercuspidal* if it does not appear as a subquotient of any representation of the form $\boldsymbol{i}_{\mathrm{P}}^{\mathrm{G}}(\tau)$, where P is a proper parabolic subgroup of G with Levi component M and τ is an *irreducible* representation of M.

A *supercuspidal pair* of G is a pair (M, ϱ) made of a Levi subgroup $\mathrm{M} \subseteq \mathrm{G}$ and an irreducible supercuspidal representation ϱ of M.

For π an irreducible representation of G, the *supercuspidal support* of π is the set:

$$\mathrm{scusp}(\pi)$$

of supercuspidal pairs (M, ϱ) of G such that π occurs as a subquotient of $\boldsymbol{i}_{\mathrm{P}}^{\mathrm{G}}(\varrho)$, for some parabolic subgroup P with Levi component M.

Remark 1.2. — In the finite field case, the word *irreducible* may be omitted from the definition of supercuspidal (see Proposition 1.10); we will see that, for G an inner form of GL_n over a nonarchimedean locally compact field, the same is true (see Proposition 11.1).

Similarly, an irreducible representation π of G is *cuspidal* if it does not appear as a quotient of any representation of the form $\boldsymbol{i}_{\mathrm{P}}^{\mathrm{G}}(\tau)$, and we have the notion of *cuspidal pair* and *cuspidal support* $\mathrm{cusp}(\pi)$. It is known that the cuspidal support $\mathrm{cusp}(\pi)$ consists of a single G-conjugacy class of cuspidal pairs ([**16**, Théorème 2.1]) but there is no such general result for supercuspidal support; indeed, it is not even known that the possible supercuspidal supports form a partition of the set of supercuspidal pairs.

In this section, we make the following hypotheses:

(H1) for π, π' irreducible representations of G, if $\mathrm{scusp}(\pi) \cap \mathrm{scusp}(\pi') \neq \varnothing$ then $\mathrm{scusp}(\pi) = \mathrm{scusp}(\pi')$.

(H2) for supercuspidal pairs (M, ϱ), (M, ϱ') of G, if the space $\mathrm{Ext}^i_{\mathrm{M}}(\varrho', \varrho)$ is nonzero for some $i \geqslant 0$, then $\varrho' \simeq \varrho$;

Proposition 1.3. — *Assume hypotheses* (H1) *and* (H2) *are satisfied. Let* π *and* π' *be irreducible representations of* G *with unequal supercuspidal supports. Then* $\mathrm{Ext}^i_{\mathrm{G}}(\pi', \pi) = 0$ *for all* $i \geqslant 0$.

The idea of computing all the Ext^i rather than Ext^1 only (which allows us to reduce to the case where π, π' are supercuspidal) comes from Emerton–Helm [**10**, Theorem 3.2.13].

Proof. — Let π and π' be irreducible representations of G with unequal supercuspidal supports.

Lemma 1.4. — *Assume that* π' *is cuspidal and* π *is not. Then we have* $\mathrm{Ext}^i_{\mathrm{G}}(\pi', \pi) = 0$ *for all* $i \geqslant 0$.

Proof. — The proof is by induction on i, the case where $i = 0$ being immediate. Let us embed π in $\boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau)$ with τ an irreducible cuspidal representation of a proper Levi subgroup M and P a parabolic subgroup of Levi component M and unipotent radical N. We have an exact sequence

$$\mathrm{Ext}^{i-1}_{\mathrm{G}}(\pi', \xi) \to \mathrm{Ext}^i_{\mathrm{G}}(\pi', \pi) \to \mathrm{Ext}^i_{\mathrm{G}}(\pi', \boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau)),$$

where ξ is the quotient of $\boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau)$ by π. Since π, π' have unequal supercuspidal supports, we have, by the inductive hypothesis, $\mathrm{Ext}^{i-1}_{\mathrm{G}}(\pi', \lambda) = 0$ for all the irreducible subquotients λ of ξ, thus we have $\mathrm{Ext}^{i-1}_{\mathrm{G}}(\pi', \xi) = 0$. By [**23**, I.A.2], we have an isomorphism:

$$\mathrm{Ext}^i_{\mathrm{G}}(\pi', \boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau)) \simeq \mathrm{Ext}^i_{\mathrm{M}}(\pi'_{\mathrm{N}}, \tau) = 0$$

(where π'_{N} is the Jacquet module of π' with respect to $\mathrm{P} = \mathrm{MN}$). This gives us $\mathrm{Ext}^i_{\mathrm{G}}(\pi', \pi) = 0$ as expected. □

In the case where π is cuspidal and π' is not, we reduce to Lemma 1.4 by taking contragredients. Indeed, we have:

$$\mathrm{Ext}^i_{\mathrm{G}}(\pi', \pi) \simeq \mathrm{Ext}^i_{\mathrm{G}}(\pi^{\vee}, \pi'^{\vee})$$

and this is 0 by the previous case. We now treat the case where π and π' are both cuspidal.

Lemma 1.5. — *Assume that* π *is not supercuspidal. Then* $\mathrm{Ext}^i_{\mathrm{G}}(\pi', \pi) = 0$ *for all* $i \geqslant 0$.

Proof. — The proof is by induction on i, the case where $i = 0$ being immediate. By assumption, π occurs as a subquotient of $\boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau)$, with τ an irreducible supercuspidal representation of a proper Levi subgroup M and P a parabolic subgroup of Levi component M and unipotent radical N.

Let V be the minimal subrepresentations of $\mathrm{X} = \boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau)$ such that π is a (sub)quotient of V, and let $\mathrm{W} \subseteq \mathrm{V}$ be a subrepresentation such that $\mathrm{V}/\mathrm{W} \simeq \pi$; thus π is not a subquotient of W. Denote by $k = k(\pi)$ the number of irreducible cuspidal subquotients of W. Now we proceed by induction on k, noting that any irreducible subquotient π'' of W must have $k(\pi'') \leqslant k(\pi) - 1$.

We have an exact sequence:

$$\mathrm{Ext}^{i-1}_{\mathrm{G}}(\pi', \mathrm{X}/\mathrm{V}) \to \mathrm{Ext}^i_{\mathrm{G}}(\pi', \mathrm{V}) \to \mathrm{Ext}^i_{\mathrm{G}}(\pi', \boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau)) = 0.$$

We claim that $\mathrm{Ext}^{i-1}_{\mathrm{G}}(\pi',\lambda)=0$ for all the irreducible subquotients λ of X. Indeed, this follows from Lemma 1.4 if λ is not cuspidal and from the inductive hypothesis (on i) if λ is a cuspidal irreducible subquotient of X. This gives us $\mathrm{Ext}^{i-1}_{\mathrm{G}}(\pi',\mathrm{X}/\mathrm{V})=0$, and it follows from the above exact sequence that $\mathrm{Ext}^{i}_{\mathrm{G}}(\pi',\mathrm{V})=0$. Now we have an exact sequence:

$$0=\mathrm{Ext}^{i}_{\mathrm{G}}(\pi',\mathrm{V})\to\mathrm{Ext}^{i}_{\mathrm{G}}(\pi',\pi)\to\mathrm{Ext}^{i+1}_{\mathrm{G}}(\pi',\mathrm{W}).$$

If $k=0$ then all the irreducible subquotients of W are non-cuspidal and Lemma 1.4 implies that we have $\mathrm{Ext}^{i+1}_{\mathrm{G}}(\pi',\mathrm{W})=0$; thus $\mathrm{Ext}^{i}_{\mathrm{G}}(\pi',\pi)=0$, which completes the base step of the induction on k. For the general case, since every irreducible subquotient π'' of W is either non-cuspidal or has $k(\pi'')<k$, we again have $\mathrm{Ext}^{i+1}_{\mathrm{G}}(\pi',\mathrm{W})=0$, by Lemma 1.4 and the inductive hypothesis (on k). □

We have the same result when π' is not supercuspidal, by taking contragredients as above.

Corollary 1.6. — *Suppose that π, π' are cuspidal. Then $\mathrm{Ext}^{i}_{\mathrm{G}}(\pi',\pi)=0$ for all $i\geqslant 0$.*

Proof. — If either π or π' is not supercuspidal then the result follows from Lemma 1.5. If both are supercuspidal then this is the hypothesis (H2). □

We now treat the general case. The proof is by induction on i, the case $i=0$ being trivial. We have an exact sequence:

$$0=\mathrm{Ext}^{i-1}_{\mathrm{G}}(\pi',\boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau))\to\mathrm{Ext}^{i}_{\mathrm{G}}(\pi',\pi)\to\mathrm{Ext}^{i}_{\mathrm{G}}(\pi',\boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau))\simeq\mathrm{Ext}^{1}_{\mathrm{M}}(\pi'_{\mathrm{N}},\tau)$$

where π embeds in $\boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}(\tau)$ with τ an irreducible cuspidal representation of M. From the cuspidal case, we have $\mathrm{Ext}^{i}_{\mathrm{M}}(\sigma,\tau)=0$ for all irreducible representations σ of M that are nonisomorphic to τ. If we prove that τ does not appear as a subquotient of π'_{N}, then we will get $\mathrm{Ext}^{i}_{\mathrm{M}}(\pi'_{\mathrm{N}},\tau)=0$ and the result will follow.

Assume that τ appears as a subquotient of π'_{N}. Let λ' be an irreducible supercuspidal representation of a Levi subgroup M′ such that π' occurs as a subquotient of $\boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}'}(\lambda')$, for some parabolic subgroup P′ with Levi component M′. By the Geometric Lemma (see for example [**16**, (1.3)]), there is a permutation matrix w such that τ occurs in:

$$\boldsymbol{i}^{\mathrm{M}}_{\mathrm{M}\cap\mathrm{P}'^{w}}(\lambda'^{w}).$$

By replacing λ' by λ'^{w}, we may assume that $w=1$, so that τ occurs in $\boldsymbol{i}^{\mathrm{M}}_{\mathrm{M}\cap\mathrm{P}'}(\lambda')$. By applying $\boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}}$, we deduce that π occurs in $\boldsymbol{i}^{\mathrm{G}}_{\mathrm{P}'}(\lambda')$. This contradicts the fact that π,π' have unequal supercuspidal supports. □

Proposition 1.7. — *Assume hypotheses* (H1) *and* (H2) *are satisfied. Let* V *be a representation of* G *of finite length. There is a decomposition:*

$$\mathrm{V}=\mathrm{V}_1\oplus\cdots\oplus\mathrm{V}_r$$

of V *as a direct sum of subrepresentations where, for each $i\in\{1,\dots,r\}$, all irreducible subquotients of* V_i *have the same supercuspidal support.*

Proof. — Write n for the length of V and r for the number of distinct sets $\mathrm{scusp}(\pi)$, for π an irreducible subquotient of V. We may and will assume that $r>1$. The proof is by induction on n.

Since $r\leqslant n$, the minimal case with $r>1$ is $r=n=2$. Assume we are in this case. Then the result follows from Proposition 1.3 with $i=1$.

Assume now that $n > 2$. Let ω_0 be the supercuspidal support of an irreducible subrepresentation of V and V_0 be the maximal subrepresentation of V all of whose irreducible subquotients have supercuspidal support ω_0. By the inductive hypothesis, V/V_0 decomposes as a direct sum:

$$\mathrm{W}_1 \oplus \cdots \oplus \mathrm{W}_s$$

of nonzero subrepresentations, with $s \leqslant r$ and where, for each $i \in \{1, \ldots, s\}$, there is a supercuspidal support ω_i such that all irreducible subquotients of W_i have supercuspidal support ω_i. If there is $i \geqslant 1$ such that $\omega_i = \omega_0$, then the preimage of W_i in V would contradict the maximality of V_0. Thus we have $\omega_0 \notin \{\omega_1, \ldots, \omega_s\}$ and it follows that $r = s + 1$.

Lemma 1.8. — *For each $i \in \{1, \ldots, s\}$, there is an injective homomorphism $f_i : \mathrm{W}_i \to \mathrm{V}$.*

Proof. — For $i \in \{1, \ldots, s\}$, write Y_i for the preimage of W_i in V. If $\mathrm{Y}_i \neq \mathrm{V}$, then it follows from the inductive hypothesis that Y_i decomposes into the direct sum of V_0 and a subrepresentation isomorphic to W_i.

Assume now that $\mathrm{Y}_i = \mathrm{V}$, thus $r = 2$ and $i = 1$. By passing to the contragredient if necessary (and thus exchanging the roles of V_0 and V_1) we may assume that V_0 is reducible. Let π denote an irreducible subrepresentation of V_0. By the inductive hypothesis, V/π has a direct summand isomorphic to W_1. Its preimage in V is denoted X_1 and we can apply the inductive hypothesis to it. Thus W_1 embeds in V. □

We thus have injective homomorphisms $f_1, \ldots, f_s$, and write f_0 for the canonical inclusion of V_0 in V. We write $\mathrm{V}_i = f_i(\mathrm{W}_i)$ for all $i \in \{0, \ldots, s\}$ and claim that we have:

$$\mathrm{V} = \mathrm{V}_0 \oplus \cdots \oplus \mathrm{V}_s.$$

Indeed, we have a homomorphism:

$$f : \mathrm{V}_0 \oplus \big(\bigoplus_{i=1}^{s} \mathrm{W}_i\big) = \mathrm{X} \to \mathrm{V}.$$

Since X and V have the same length, it is enough to prove that f is injective.

Lemma 1.9. — *We have:*

$$\mathrm{Ker}(f) = (\mathrm{Ker}(f) \cap \mathrm{V}_0) \oplus \left(\bigoplus_{i=1}^{s} (\mathrm{Ker}(f) \cap \mathrm{W}_i)\right).$$

Proof. — Since f is nonzero, we have $\mathrm{Ker}(f) \subsetneq \mathrm{V}$, thus we can apply the inductive hypothesis to $\mathrm{Ker}(f)$. The decomposition that we obtain is the right hand side of the expected equality. □

Since $f_1, \ldots, f_s$ are injective, we get $\mathrm{Ker}(f) \cap \mathrm{W}_i = 0$ for all $i \in \{1, \ldots, s\}$. Thus f is injective and the result is proved. □

1.2.

Now we specialize to the case that G is a connected reductive group over a finite field. We begin with a general result which is independent of the hypotheses (H1) and (H2).

Proposition 1.10. — *Let* P *be a proper parabolic subgroup of* G *and σ be a representation of a Levi component* M *of* P. *Then $\boldsymbol{i}_{\mathrm{P}}^{\mathrm{G}}(\sigma)$ has no supercuspidal irreducible subquotient.*

Proof. — When σ is irreducible, the result follows from the definition of a supercuspidal representation. Assume $\mathrm{E} = \boldsymbol{i}_{\mathrm{P}}^{\mathrm{G}}(\sigma)$ contains a supercuspidal irreducible subquotient π, and let us fix a projective envelope Π of π in $\mathscr{R}(\mathrm{G})$. By [**13**, Proposition 2.3], all its irreducible subquotients are cuspidal (indeed, this is a characterization of supercuspidal representations). Let V be a subrepresentation of E having a quotient isomorphic to π. As Π is projective, we get a nonzero homomorphism from Π to V, whence it follows that some irreducible subquotient π' of Π occurs as a subrepresentation of V, thus of E. By Frobenius reciprocity, we get that the space π'_{N} of N-coinvariants, where N is the unipotent radical of P, is nonzero, contradicting the cuspidality of π'. ☐

Let R[G] be the group algebra of G over R. It decomposes as a direct sum:

$$\mathrm{R}[\mathrm{G}] = \mathrm{B}_1 \oplus \cdots \oplus \mathrm{B}_t$$

of indecomposable two-sided ideals, called blocks of R[G]. This corresponds to a decomposition:

$$1 = e_1 + \cdots + e_t$$

of the identity of R[G] as a sum of indecomposable central idempotents. This implies a decomposition:

$$\mathscr{R}(\mathrm{G}) = \mathscr{R}(\mathrm{B}_1) \oplus \cdots \oplus \mathscr{R}(\mathrm{B}_t)$$

of the category $\mathscr{R}(\mathrm{G})$ of R-representations of G (that is, of left R[G]-modules) into the direct sum of the subcategories $\mathscr{R}(\mathrm{B}_i)$, $i \in \{1, \dots, t\}$, where $\mathscr{R}(\mathrm{B}_i)$ is made of all representations V of G such that $e_i\mathrm{V} = \mathrm{V}$.

Lemma 1.11. — *Assume that hypotheses* (H1) *and* (H2) *are satisfied. Let* $\mathrm{V} \in \mathscr{R}(\mathrm{B}_i)$ *for some* $i \in \{1, \dots, t\}$. *Then all the irreducible subquotients of* V *have the same supercuspidal support.*

Proof. — If we apply Proposition 1.7 to the regular representation R[G], which has finite length, we get that all the irreducible subquotients of B_i have the same supercuspidal support. Since all the irreducible subquotients of V are isomorphic to subquotients of B_i, we get the result. ☐

We deduce the following decomposition theorem.

Theorem 1.12. — *Assume hypotheses* (H1) *and* (H2) *are satisfied. Let* V *be a representation of* G. *For any supercuspidal support* ω *of* G, *let* $\mathrm{V}(\omega)$ *denote the maximal subrepresentation of* V *all of whose irreducible subquotients have supercuspidal support* ω. *Then we have:*

$$\mathrm{V} = \bigoplus_{\omega} \mathrm{V}(\omega).$$

1.3.

Finally, we specialize to the case where G is the finite group $\mathrm{GL}_n(q)$, with $n \geqslant 1$ an integer and q a power of p. In this case, it is known ([**14**]) that the supercuspidal support consists of a single G-conjugacy class of supercuspidal pairs, so (H1) is satisfied. We prove that (H2) is also satisfied.

Lemma 1.13. — *Let* π, π' *be irreducible supercuspidal representations of* G *such that the space* $\mathrm{Ext}^i_{\mathrm{G}}(\pi', \pi)$ *is nonzero for some* $i \geqslant 0$. *Then* $\pi' \simeq \pi$.

Proof. — The proof is by induction on i, the case $i = 0$ being trivial. Let us fix a projective envelope Π of π in $\mathscr{R}(G)$. By [**23**, III.2.9], it has finite length, and all its irreducible subquotients are isomorphic to π. Consider the exact sequence:

$$0 \to \Pi_1 \to \Pi \to \pi \to 0$$

where Π_1 is the kernel of $\Pi \to \pi$. Then we have an exact sequence:

$$\mathrm{Ext}_G^{i-1}(\pi', \pi) \to \mathrm{Ext}_G^i(\pi', \Pi_1) \to \mathrm{Ext}_G^i(\pi', \Pi).$$

By the inductive hypothesis, we have $\mathrm{Ext}_G^{i-1}(\pi', \pi) = 0$. Since Π is projective in $\mathscr{R}(G)$, we have $\mathrm{Ext}_G^i(\pi', \Pi) = 0$. It follows that we have $\mathrm{Ext}_G^i(\pi', \Pi_1) = 0$. Since all irreducible subquotients of Π_1 are isomorphic to π, we can consider an exact sequence:

$$0 \to \Pi_2 \to \Pi_1 \to \pi \to 0$$

where Π_2 is the kernel of $\Pi_1 \to \pi$. By induction, we define a finite decreasing sequence:

$$\Pi = \Pi_0 \supsetneq \Pi_1 \supsetneq \Pi_2 \supsetneq \cdots \supsetneq \Pi_r \supsetneq \Pi_{r+1} = 0$$

of subrepresentations of Π such that $\Pi_j/\Pi_{j+1} \simeq \pi$ and $\mathrm{Ext}_G^i(\pi', \Pi_j) = 0$ for all $j \geqslant 0$. For $j = r$, we get the expected result. ☐

In particular, since every Levi subgroup of G is isomorphic to a product of smaller general linear groups, the hypothesis (H2) is satisfied and the conclusion of Theorem 1.12 holds for G.

As a corollary, we will need a weaker result in Section 9, in which we allow for the action of a Galois group. Fix Γ be a group of automorphisms of the finite field $\mathbf{F}_q$.

Definition 1.14. — Let (M, ϱ) be a supercuspidal pair of G, with

$$M \simeq GL_{n_1}(q) \times \cdots \times GL_{n_r}(q), \qquad \varrho \simeq \varrho_1 \otimes \cdots \otimes \varrho_r.$$

The *equivalence class* of (M, ϱ) is the set, denoted $[M, \varrho]$, of all supercuspidal pairs (M', ϱ') of G for which there are elements $\gamma_i \in \Gamma$, for each $i = 1, \ldots, r$, such that (M', ϱ') is G-conjugate to $(M, \bigotimes_{i=1}^r \varrho_i^{\gamma_i})$.

Corollary 1.15. — *Let* V *be a representation of* G *and, for any equivalence class of supercuspidal pairs* $[\omega]$*, write* $V[\omega]$ *for the maximal subrepresentation of* V *all of whose irreducible subquotients have supercuspidal support contained in* $[\omega]$*. Then* V *decomposes into the direct sum of the* $V[\omega]$*, where* $[\omega]$ *ranges over the set of equivalence classes of supercuspidal pairs of* G*.*

Further notation

Throughout the rest of the paper, we fix a nonarchimedean locally compact field F of residue characteristic p. All F-algebras are supposed to be finite-dimensional with a unit. By an F-*division algebra* we mean a central F-algebra which is a division algebra.

For K a finite extension of F, or more generally a division algebra over a finite extension of F, we denote by $\mathcal{O}_K$ its ring of integers, by $\mathfrak{p}_K$ the maximal ideal of $\mathcal{O}_K$ and by $\mathfrak{k}_K$ its residue field.

For A a simple central algebra over a finite extension K of F, we denote by $N_{A/K}$ and $\mathrm{tr}_{A/K}$ respectively the reduced norm and trace of A over K.

For u a real number, we denote by $\lfloor u \rfloor$ the greatest integer which is smaller than or equal to u, that is its integer part.

A *composition* of an integer $m \geqslant 1$ is a finite family of positive integers whose sum is m.

Given H a closed subgroup of a topological group G and σ a representation of H, write $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\sigma)$ for the representation of G compactly induced from σ.

We fix once and for all an additive character $\psi_{\mathrm{F}} : \mathrm{F} \to \mathrm{R}^{\times}$ that we assume to be trivial on $\mathfrak{p}_{\mathrm{F}}$ but not on $\mathcal{O}_{\mathrm{F}}$.

§2. Preliminaries

We fix an F-division algebra D, with reduced degree d. For all $m \geqslant 1$, we write $\mathrm{A}_m = \mathrm{M}_m(\mathrm{D})$ and $\mathrm{G}_m = \mathrm{GL}_m(\mathrm{D})$.

Let $m \geqslant 1$ be a positive integer and write $\mathrm{A} = \mathrm{A}_m$ and $\mathrm{G} = \mathrm{G}_m$. We will recall briefly the objects associated to the explicit construction of representations of G; we refer to [**18, 19, 20, 21, 22**] for more details on the notions of simple stratum, character and type.

Recall that, for P = MN a parabolic subgroup of G together with a Levi decomposition, we write $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}$ for the unnormalized parabolic induction functor from $\mathscr{R}(\mathrm{M})$ to $\mathscr{R}(\mathrm{G})$.

2.1.

Recall (see [**16**, Théorème 8.16]) that, for π an irreducible representation of G, the supercuspidal support $\mathrm{scusp}(\pi)$ consists of a single G-conjugacy class of supercuspidal pairs of G.

Definition 2.1. — The *inertial class* of a supercuspidal pair (M, ϱ) of G is the set, denoted by $[\mathrm{M}, \varrho]_{\mathrm{G}}$, of all supercuspidal pairs (M', ϱ') that are G-conjugate to $(\mathrm{M}, \varrho\chi)$ for some unramified character χ of M.

2.2.

Let Λ be an $\mathcal{O}_{\mathrm{D}}$-lattice sequence of D^m. It defines an hereditary $\mathcal{O}_{\mathrm{F}}$-order $\mathfrak{A}(\Lambda)$ of A and an $\mathcal{O}_{\mathrm{F}}$-lattice sequence:

$$\mathfrak{a}_k(\Lambda) = \{a \in \mathrm{A} \mid a\Lambda(i) \subseteq \Lambda(i+k), \text{ for all } i \in \mathbf{Z}\}$$

of A. For $i \geqslant 1$, we write $\mathrm{U}_i(\Lambda) = 1 + \mathfrak{a}_i(\Lambda)$. This defines a filtration $(\mathrm{U}_i(\Lambda))_{i\geqslant 1}$ of the compact open subgroup $\mathrm{U}(\Lambda) = \mathfrak{A}(\Lambda)^{\times}$ of G.

Let $[\Lambda, n, 0, \beta]$ be a simple stratum in A (see for instance [**21**, §1.6]). The element $\beta \in \mathrm{A}$ generates a field extension $\mathrm{F}[\beta]$ of F, denoted E, and we write B for its centralizer in A. Attached to this stratum, there are two compact open subgroups:

$$\mathrm{J} = \mathrm{J}(\beta, \Lambda), \quad \mathrm{H} = \mathrm{H}(\beta, \Lambda)$$

of G. For all $i \geqslant 1$, we set:

$$\mathrm{J}^i = \mathrm{J}^i(\beta, \Lambda) = \mathrm{J} \cap \mathrm{U}_i(\Lambda), \quad \mathrm{H}^i = \mathrm{H}^i(\beta, \Lambda) = \mathrm{H} \cap \mathrm{U}_i(\Lambda).$$

Together with the choice of ψ_{F}, the simple stratum defines a finite set $\mathcal{C}(\Lambda, 0, \beta)$ of characters of H^1, called simple characters. We do not recall here the definition of these characters, only the following basic property. Write $\psi_{\mathrm{A}} = \psi_{\mathrm{F}} \circ \mathrm{tr}_{\mathrm{A}/\mathrm{F}}$ and, for $b \in \mathrm{A}$, set:

$$\psi_b : x \mapsto \psi_{\mathrm{A}}(b(x-1))$$

for all $x \in \mathrm{A}$. If $b \in \mathfrak{a}_{-k}(\Lambda)$ for some $k \geqslant 1$, then ψ_b defines a character on $\mathrm{U}_{\lfloor k/2 \rfloor+1}(\Lambda)$. Then any simple character $\theta \in \mathcal{C}(\Lambda, 0, \beta)$ satisfies $\theta|_{\mathrm{U}_{\lfloor n/2 \rfloor+1}(\Lambda)} = \psi_{\beta}$.

Given θ a simple character attached to $[\Lambda, n, 0, \beta]$, there is, up to isomorphism, a unique irreducible representation η of J^1 whose restriction to H^1 contains θ. Moreover, the representation

η extends to an irreducible representation of the group J that is intertwined by the whole of $B^\times$. Such extensions of η to J are called β-extensions.

As B is a central simple E-algebra, there are a positive integer $m' \geqslant 1$, an E-division algebra D' and an isomorphism of E-algebras Φ from B to $M_{m'}(D')$. Moreover, we can choose Φ so that $\Phi(\mathfrak{A}(\Lambda) \cap B)$ is a standard order, that is, it is contained in $M_{m'}(\mathcal{O}_{D'})$ and its reduction mod $\mathfrak{p}_{D'}$ is upper block triangular. Since $J = (U(\Lambda) \cap B^\times)J^1$, we thus have group isomorphisms:

$$J/J^1 \simeq (U(\Lambda) \cap B^\times)/(U_1(\Lambda) \cap B^\times) \simeq GL_{m'_1}(\mathfrak{k}_{D'}) \times \cdots \times GL_{m'_r}(\mathfrak{k}_{D'})$$

for suitable positive integers $m'_1, \dots, m'_r$. It allows us to identify these groups and we denote by $\mathcal{G}$ the latter group.

A *simple type* attached to $[\Lambda, n, 0, \beta]$ is an irreducible representation λ of J of the form $\kappa \otimes \sigma$, where κ is a β-extension and σ is an irreducible representation of J trivial on J^1 which identifies with a cuspidal representation of $\mathcal{G}$ of the form $\tau \otimes \cdots \otimes \tau$ where τ is a cuspidal representation of $GL_{m'/r}(\mathfrak{k}_{D'})$ (this implies $m'_1 = \cdots = m'_r = m'/r$). When the representation τ is supercuspidal, λ is called a *simple supertype*.

We introduce the following useful definition.

Definition 2.2. — A simple character (or a β-extension, or a simple type) is said to be *maximal* if $U(\Lambda) \cap B^\times$ is a maximal compact open subgroup in $B^\times$.

§3. An abstract $\mathbf{K}$-functor

A main tool for us will be a family of functors from $\mathscr{R}(G)$ to the category of representations of some finite reductive group. Such functors were first introduced in the split case for complex representations in [**17**], where they were used just for simple types; in [**15**] these were generalized to apply to any G in the modular case. Since we will need several variants of these functors, it is convenient to give a general setup which applies to all situations.

Let $P = MN$ be a parabolic subgroup of G, together with a Levi decomposition. Given $g \in G$, K a compact open subgroup of G and π a representation of M, write:

$$\mathrm{Ind}_P^{PgK}(\pi) = \{f \in \mathrm{Ind}_P^G(\pi) \mid f \text{ is supported in } PgK\}.$$

This defines a functor from $\mathscr{R}(M)$ to $\mathscr{R}(K)$ denoted Ind_P^{PgK}.

We have the following easy but useful lemma.

Lemma 3.1. — *Let* K *be a compact open subgroup of* G*. For all representation* π *of* M *and all* $g \in G$*, there is an isomorphism:*

$$\mathrm{Ind}_P^{PgK}(\pi) \simeq \mathrm{Ind}_{K \cap P^g}^K(\pi^g)$$

of representations of K*, where* P^g, π^g *denote the conjugates of* P, π *by* g.

Proof. — The isomorphism is given by $f \mapsto f_g$, where $f_g(k) = f(gk)$ for all $k \in K$. ☐

Now we are given a compact open subgroup $\mathbf{J}$ of G, together with a normal pro-p subgroup $\mathbf{J}^1$, and an irreducible representation $\boldsymbol{\kappa}$ of $\mathbf{J}$. We define a functor:

$$\mathbf{K}_{\boldsymbol{\kappa}} : \pi \mapsto \mathrm{Hom}_{\mathbf{J}^1}(\boldsymbol{\kappa}, \pi)$$

from $\mathscr{R}(G)$ to $\mathscr{R}(\mathbf{J}/\mathbf{J}^1)$, by making $\mathbf{J}$ act on $\mathbf{K}_{\boldsymbol{\kappa}}(\pi)$ by the formula:

$$x \cdot f = \pi(x) \circ f \circ \boldsymbol{\kappa}(x)^{-1}$$

for all $x \in \mathbf{J}$ and $f \in \mathbf{K}_{\boldsymbol{\kappa}}(\pi)$. Note that $\mathbf{J}^1$ acts trivially. Since $\mathbf{J}^1$ is a pro-p-group, this functor is exact, and it sends admissible representations of G to finite dimensional representations of $\mathbf{J}/\mathbf{J}^1$.

Proposition 3.2. — *Let $g \in \mathrm{G}$. The following are equivalent:*

(i) *the functor $\mathbf{K}_{\boldsymbol{\kappa}} \circ \mathrm{Ind}_{\mathrm{P}}^{\mathrm{P}g\mathbf{J}}$ is nonzero on $\mathscr{R}(\mathrm{M})$;*

(ii) *the functor $\mathbf{K}_{\boldsymbol{\kappa}} \circ \mathrm{Ind}_{\mathrm{P}}^{\mathrm{P}g\mathbf{J}}$ is nonzero on $\mathrm{Irr}(\mathrm{M})$;*

(iii) $\mathrm{Hom}_{\mathbf{J}^1 \cap \mathrm{N}^g}(\boldsymbol{\kappa}, 1) \neq 0$ *(or, equivalently, κ has a non-zero $\mathbf{J}^1 \cap \mathrm{N}^g$-fixed vector).*

Proof. — Given $\pi \in \mathscr{R}(\mathrm{M})$, by Lemma 3.1 we have an isomorphism:

$$\mathrm{Ind}_{\mathrm{P}}^{\mathrm{P}g\mathbf{J}}(\pi) \simeq \mathrm{Ind}_{\mathbf{J}\cap\mathrm{P}^g}^{\mathbf{J}}(\pi^g)$$

of representations of $\mathbf{J}$. Applying Mackey's formula and Frobenius reciprocity, and writing $\boldsymbol{\eta}$ for the restriction of $\boldsymbol{\kappa}$ to $\mathbf{J}^1$, we get:

$$\mathbf{K}_{\boldsymbol{\kappa}}(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{P}g\mathbf{J}}(\pi)) \simeq \bigoplus_{x \in (\mathbf{J}\cap\mathrm{P}^g)\backslash\mathbf{J}/\mathbf{J}^1} \mathrm{Hom}_{\mathbf{J}^1\cap\mathrm{P}^{gx}}(\boldsymbol{\eta}, \pi^{gx}).$$

As $\boldsymbol{\eta}$ is normalized by $\mathbf{J}$, this implies that:

$$\mathbf{K}_{\boldsymbol{\kappa}}(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{P}g\mathbf{J}}(\pi)) \neq 0 \quad \Leftrightarrow \quad \mathrm{Hom}_{\mathbf{J}^1\cap\mathrm{P}^g}(\boldsymbol{\eta}, \pi^g) \neq 0.$$

As π is trivial on N, we have:

$$\mathrm{Hom}_{\mathbf{J}^1\cap\mathrm{P}^g}(\boldsymbol{\eta}, \pi^g) \subseteq \mathrm{Hom}_{\mathbf{J}^1\cap\mathrm{N}^g}(\boldsymbol{\eta}, 1)$$

Therefore, if $\mathbf{K}_{\boldsymbol{\kappa}} \circ \mathrm{Ind}_{\mathrm{P}}^{\mathrm{P}g\mathbf{J}}$ is nonzero on $\mathscr{R}(\mathrm{M})$, then $\mathrm{Hom}_{\mathbf{J}^1\cap\mathrm{N}^g}(\boldsymbol{\eta}, 1) \neq 0$. Thus (i) implies (iii), and it is clear that (ii) implies (i).

Now we assume that $\mathrm{Hom}_{\mathbf{J}^1\cap\mathrm{N}^g}(\boldsymbol{\eta}, 1) \neq 0$ and write $\mathrm{P}' = \mathrm{P}^g$, $\mathrm{N}' = \mathrm{N}^g$, $\mathrm{M}' = \mathrm{M}^g$. Define the compactly induced representation

$$\mathrm{V} = \mathrm{ind}_{\mathbf{J}^1\cap\mathrm{P}'}^{\mathrm{P}'}(\boldsymbol{\eta}).$$

For any $\pi \in \mathscr{R}(\mathrm{M})$, as π^g is trivial on N', we have

$$\mathrm{Hom}_{\mathbf{J}^1\cap\mathrm{P}'}(\boldsymbol{\eta}, \pi^g) \simeq \mathrm{Hom}_{\mathrm{P}'}(\mathrm{V}, \pi^g) \simeq \mathrm{Hom}_{\mathrm{M}'}(\mathrm{V}_{\mathrm{N}'}, \pi^g),$$

where $\mathrm{V}_{\mathrm{N}'}$ denotes the space of N'-coinvariants of V. But

$$\mathrm{V}_{\mathrm{N}'} \simeq \bigoplus_{l \in (\mathbf{J}^1\cap\mathrm{M}')\backslash\mathrm{M}'} \left(\mathrm{ind}_{\mathrm{N}'\cap(\mathbf{J}^1)^l}^{\mathrm{N}'}(\boldsymbol{\eta}^l)\right)_{\mathrm{N}'} \simeq \bigoplus_{l \in (\mathbf{J}^1\cap\mathrm{M}')\backslash\mathrm{M}'} (\boldsymbol{\eta}^l)_{\mathrm{N}'\cap(\mathbf{J}^1)^l},$$

by Shapiro's lemma, and the term corresponding to $l = 1$ is nonzero. Thus $\mathrm{V}_{\mathrm{N}'}$ is nonzero and, moreover, it is of finite type since V is of finite type and Jacquet functors preserve finite type. Thus $(\mathrm{V}_{\mathrm{N}'})^{g^{-1}}$ has an irreducible quotient $\pi \in \mathrm{Irr}(\mathrm{M})$ and $\mathbf{K}_{\boldsymbol{\kappa}} \circ \mathrm{Ind}_{\mathrm{P}}^{\mathrm{P}g\mathbf{J}}(\pi)$ is nonzero. Hence (iii) implies (ii). □

In some situations, we know more about the representation $\boldsymbol{\kappa}$ and can conveniently rephrase the final condition of Proposition 3.2.

Corollary 3.3. — *Write $\boldsymbol{\eta}$ for the restriction of $\boldsymbol{\kappa}$ to $\mathbf{J}^1$, and suppose that we have a normal pro-p subgroup $\mathbf{H}^1$ of $\mathbf{J}^1$ and a character $\boldsymbol{\theta}$ of $\mathbf{H}^1$ such that the restriction of $\boldsymbol{\eta}$ to $\mathbf{H}^1$ is $\boldsymbol{\theta}$-isotypic and that $\boldsymbol{\eta}$ is the unique irreducible representation of $\mathbf{J}^1$ which contains $\boldsymbol{\theta}$. Then the conditions of Proposition 3.2 are also equivalent to:*

(iv) *the character $\boldsymbol{\theta}$ is trivial on $\mathbf{H}^1 \cap \mathrm{N}^g$.*

Proof. — (iii) is equivalent to (iv) since $\mathrm{ind}_{\mathbf{H}^1}^{\mathbf{J}^1}(\boldsymbol{\theta})$ is a finite sum of copies of $\boldsymbol{\eta}$ and the restriction of $\boldsymbol{\eta}$ to $\mathbf{H}^1$ is $\boldsymbol{\theta}$-isotypic. □

The usefulness of conditions (iii) and (iv) is that they do not depend on characteristic of the ground field R; that is, if $\boldsymbol{\kappa}$ is a $\overline{\mathbf{Z}}_\ell$-representation then $\mathrm{Hom}_{\mathbf{J}^1\cap \mathrm{N}^g}(\boldsymbol{\kappa}, 1) \neq 0$ if and only if the same is true for the reduction modulo ℓ of $\boldsymbol{\kappa}$ (see [**15**, Lemme 5.7]).

§4. A lemma on simple characters

Let θ be a simple character with respect to a simple stratum $[\Lambda, n, 0, \beta]$ in A. Let $\mathrm{P} = \mathrm{MN}$ be a parabolic subgroup of G together with a Levi decomposition. The purpose of this section is to show that, under certain conditions, the criterion of Corollary 3.3 is satisfied.

Given a subset X of A, write X^* for the set of $a \in \mathrm{A}$ such that $\psi_{\mathrm{A}}(ax) = 1$ for all $x \in \mathrm{X}$.

Definition 4.1. — The pair (M, P) is *subordinate to* the simple stratum $[\Lambda, n, 0, \beta]$ if the idempotents in A that correspond to M are in B and if there is an isomorphism $\Phi : \mathrm{B} \to \mathrm{M}_{m'}(\mathrm{D}')$ of E-algebras such that $\Phi(\mathfrak{A}(\Lambda) \cap \mathrm{B})$ is a standard order and $\Phi(\mathrm{P} \cap \mathrm{B}^\times)$ is a standard parabolic subgroup corresponding to a composition of m' finer than or equal to that of $\Phi(\mathfrak{A}(\Lambda) \cap \mathrm{B})$.

Assume this is the case. For $k \geqslant 1$ and $i \in \mathbf{Z}$, write $\mathrm{H}^k = \mathrm{H}^k(\beta, \Lambda)$ and $\mathfrak{a}_i = \mathfrak{a}_i(\Lambda)$, and:

$$\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{A}(\Lambda) \mid \beta x - x\beta \in \mathfrak{a}_k\}.$$

Write q for the greatest integer $i \leqslant n$ such that $\mathfrak{n}_{1-i}(\beta, \Lambda) \subseteq \mathfrak{A}(\Lambda) \cap \mathrm{B} + \mathfrak{a}_1$ and $s = \lfloor (q+1)/2 \rfloor$. For $k \geqslant 1$, set:

$$\Omega^k = \Omega^k(\beta, \Lambda) = 1 + \mathfrak{a}_k \cap \mathfrak{n}_{k-q}(\beta, \Lambda) + \mathfrak{j}^s(\beta, \Lambda),$$

where $\mathfrak{j}^s = \mathfrak{j}^s(\beta, \Lambda)$ is defined by $\mathrm{J}^s = 1 + \mathfrak{j}^s(\beta, \Lambda)$. Write N^- for the unipotent radical opposite to N with respect to M.

Lemma 4.2. — *Let $g \in \mathrm{U}_1(\Lambda) \cap \mathrm{N}^-$ and $0 \leqslant m < q$. Assume that θ is trivial on the intersection $(\mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times)\mathrm{H}^{m+1} \cap \mathrm{N}^g$. Then $g \in (\mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times)\Omega^{q-m}$.*

Proof. — First note that it is enough to prove the result when $m \geqslant \lfloor q/2 \rfloor$. Indeed, if $m < \lfloor q/2 \rfloor$, then the result for $\lfloor q/2 \rfloor$ implies that:

$$g \in (\mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times)\Omega^s = \mathrm{J}^1(\beta, \Lambda) = (\mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times)\Omega^{q-m}.$$

The proof is by induction on both q and m with $\lfloor q/2 \rfloor \leqslant m < q$. Write $\mathfrak{n}$, $\mathfrak{p}$ for the Lie algebras of N, P in A, and also $\mathfrak{n}^-$ for that of N^-.

Assume first that $q = n$. Then g normalizes $\mathrm{H}^{m+1} = \mathrm{U}_{m+1}(\Lambda)$. Since we have $\mathrm{U}_{m+1}(\Lambda) \cap \mathrm{N}^g = (\mathrm{U}_{m+1}(\Lambda) \cap \mathrm{N})^g$, and since θ is trivial on $\mathrm{U}_{m+1}(\Lambda) \cap \mathrm{N}$, the condition on θ implies that

$$\theta([g^{-1}, 1 + y]) = 1,$$

for all $y \in \mathfrak{a}_{m+1} \cap \mathfrak{n}$. Recall that, for $b, x \in \mathrm{A}$, we have $\psi_b(x) = \psi_{\mathrm{A}}(b(x-1))$.

Lemma 4.3. — *We have $\psi_{g\beta g^{-1}-\beta}(1 + y) = 1$ for all $y \in \mathfrak{a}_{m+1} \cap \mathfrak{n}$.*

Proof. — Since $\lfloor q/2\rfloor \leqslant m$, the restriction of θ to H^{m+1} is given by ψ_β. Now:

$$\begin{aligned}\psi_\beta(g^{-1}(1+y)g) &= \psi_{\mathrm{A}}(\beta g^{-1}yg)\\ &= \psi_{\mathrm{A}}(g\beta g^{-1}y)\\ &= \psi_{g\beta g^{-1}}(1+y)\end{aligned}$$

for all $y \in \mathfrak{a}_{m+1} \cap \mathfrak{n}$, which gives us the desired result. □

If we write $g = 1+u$, with $u \in \mathfrak{a}_1 \cap \mathfrak{n}^-$, this gives us:

$$g\beta g^{-1} - \beta = -a_\beta(u)g^{-1} \in (\mathfrak{a}_{m+1} \cap \mathfrak{n})^* = \mathfrak{a}_{-m} + \mathfrak{n}^*,$$

where a_β is the map $x \mapsto \beta x - x\beta$ from A to A. Note that, since $\mathfrak{n}$ is an F-vector space, we have for all $a \in \mathrm{A}$:

$$\mathrm{tr}_{\mathrm{A/F}}(a\mathfrak{n}) \subseteq \mathrm{Ker}(\psi_{\mathrm{F}}) \quad\Leftrightarrow\quad \mathrm{tr}_{\mathrm{A/F}}(a\mathfrak{n}) = \{0\}.$$

It follows that $\mathfrak{n}^* = \mathfrak{p}$. Together with the fact that $a_\beta(u)g^{-1} \in \mathfrak{n}^-$ and $g \in \mathrm{U}_1(\Lambda)$, we get:

$$a_\beta(u) \in \mathfrak{a}_{-m}.$$

This gives us:

$$u \in \mathfrak{n}_{-m}(\beta,\Lambda) \cap \mathfrak{a}_1 = (\mathfrak{A}(\Lambda) \cap \mathrm{B} + \mathfrak{a}_{n-m}) \cap \mathfrak{a}_1,$$

where the last equality follows from [**21**, Proposition 2.29]. But:

$$\Omega^{n-m} = 1 + \mathfrak{a}_{n-m} \cap \mathfrak{n}_{-m}(\beta,\Lambda) + \mathfrak{j}^s(\beta,\Lambda) = 1 + \mathfrak{a}_{n-m} + \mathfrak{a}_s = 1 + \mathfrak{a}_{n-m}.$$

We thus get the expected result.

We now assume that $q < n$, and we fix a simple stratum $[\Lambda, n, q, \gamma]$ that is equivalent to the pure stratum $[\Lambda, n, q, \beta]$. First assume that $m = q-1$ and write:

$$\theta|_{\mathrm{H}^q \cap \mathrm{N}^g} = \psi_c\theta_\gamma = 1,$$

where $c = \beta - \gamma \in \mathfrak{a}_{-q}$ and $\theta_\gamma \in \mathcal{C}(\Lambda, q-1, \gamma)$. Now write $g = 1+u$.

Lemma 4.4. — *The character ψ_c is trivial on $\mathrm{H}^q \cap \mathrm{N}^g$.*

Proof. — Let $x = g^{-1}yg \in \mathfrak{h}^q \cap \mathfrak{n}^g$, where $\mathfrak{h}^k$ is defined for $k \geqslant 1$ by $\mathrm{H}^k = 1 + \mathfrak{h}^k$. Then:

$$\begin{aligned}\psi_c(1+x) &= \psi_{\mathrm{F}}(\mathrm{tr}_{\mathrm{A/F}}(gcg^{-1}y))\\ &= \psi_{\mathrm{F}}(\mathrm{tr}_{\mathrm{A/F}}(cy))\psi_{\mathrm{F}}(\mathrm{tr}_{\mathrm{A/F}}(-a_c(u)g^{-1}y))\\ &= \psi_{\mathrm{F}}(\mathrm{tr}_{\mathrm{A/F}}(-a_c(u)xg^{-1}))\end{aligned}$$

since $cy \in \mathfrak{n}$ has trace 0. Now $c \in \mathfrak{a}_{-q}$ and $u \in \mathfrak{a}_1$ and $xg^{-1} \in \mathfrak{a}_q$. Since ψ_{F} is trivial on $\mathfrak{p}_{\mathrm{F}}$, we get the expected result. □

Thus θ_γ is trivial on $\mathrm{H}^q \cap \mathrm{N}^g$. Note that $\mathrm{H}^q = \mathrm{H}^q(\gamma,\Lambda)$. By the inductive hypothesis, we get:

$$g \in (\mathrm{U}_1(\Lambda) \cap \mathrm{B}_\gamma^\times)\Omega^{q'-(q-1)}(\gamma,\Lambda) = (\mathrm{U}_1(\Lambda) \cap \mathrm{B}_\gamma^\times)\big(1 + \mathfrak{a}_{q'-(q-1)} \cap \mathfrak{n}_{1-q}(\gamma,\Lambda) + \mathfrak{j}^s(\gamma,\Lambda)\big)$$

where $q' = -k_0(\gamma,\Lambda)$ and B_γ is the centralizer of $\mathrm{F}[\gamma]$ in A.

The following lemma generalizes [**5**, (8.1.12)].

Lemma 4.5. — *Let $[\Lambda, n, m, \beta]$ be a simple stratum in* A *and $\theta \in \mathcal{C}(\Lambda, m, \beta)$ be a simple character. Let $z \in \mathfrak{a}_{q-m} \cap \mathfrak{n}_{-m}(\beta,\Lambda)$ and ϑ be a character of H^m whose restriction to H^{m+1} is θ. Then $1+z$ normalizes H^m and $\vartheta^{1+z} = \vartheta \cdot \psi_{a_\beta(z)}$.*

Proof. — We follow the proof of [**5**, (8.1.12)], replacing the results from [**5**] used there by their analogues in [**18, 21**]. □

If we apply Lemma 4.5 to the stratum $[\Lambda, n, q-1, \gamma]$, the simple character θ_γ, the element $g^{-1} = 1 + u'$ and the character θ, then g normalizes $\mathrm{H}^{q-1}(\gamma, \Lambda) = \mathrm{H}^{q-1}$ and $\mathrm{H}^{q}(\gamma, \Lambda) = \mathrm{H}^{q}$, and we have:

$$\theta^{1+u'} = \theta \cdot \psi_{a_\gamma(u')}$$

on H^q. Since $c \in \mathfrak{a}_{-q}$ and $u' \in \mathfrak{a}_1$, we have $\psi_{a_\gamma(u')} = \psi_{a_\beta(u')}$ on H^q. We thus get:

$$\theta([g^{-1}, 1+y]) = \psi_{a_\beta(u')}(1+y) = \psi_{\mathrm{A}}(a_\beta(u')y) = 1$$

for all $y \in \mathfrak{h}^q \cap \mathfrak{n}$. We need the following lemma.

Lemma 4.6. — *We have* $(\mathfrak{h}^q)^* = a_\beta(\mathfrak{j}^s) + \mathfrak{a}_{1-q}$.

Proof. — It is straightforward to check that we have the containment $\supseteq$, so suppose $x \in (\mathfrak{h}^q)^*$. We denote by s a tame corestriction on A relative to E/F (see for example [**21**, Définition 2.25]). By [**21**, Proposition 2.27], $\mathsf{s}(x) \in \mathfrak{a}_{1-q} \cap \mathrm{B}$ so, by [**21**, Proposition 2.29], there exists $y \in \mathfrak{a}_{1-q}$ such that $\mathsf{s}(x) = \mathsf{s}(y)$. Thus $x - y \in (\mathfrak{h}^q)^* \cap \ker(\mathsf{s})$ and, again by [**21**, Proposition 2.27], there is $z \in \mathfrak{a}_1 \cap \mathfrak{n}_{1-q}(\beta, \Lambda) + \mathfrak{j}^s$ such that $x - y = a_\beta(z)$. Since $a_\beta(\mathfrak{a}_1 \cap \mathfrak{n}_{1-q}(\beta, \Lambda)) \subseteq \mathfrak{a}_{1-q}$, the result follows. □

Therefore we have:

$$a_\beta(u') \in (\mathfrak{h}^q)^* + \mathfrak{p} = a_\beta(\mathfrak{j}^s) + \mathfrak{a}_{1-q} + \mathfrak{p}.$$

As it is also in $\mathfrak{n}^-$, we get:

$$a_\beta(u') \in a_\beta(\mathfrak{j}^s) + \mathfrak{a}_{1-q}.$$

This implies $u' \in \mathfrak{a}_1 \cap \mathfrak{n}_{1-q}(\beta, \Lambda) + \mathfrak{j}^s$, thus $g \in \Omega^1$.

Assume now that the result is true for some $m \leqslant q-1$, and that θ is trivial on $\mathrm{H}^m \cap \mathrm{N}^g$. Then it is trivial on $\mathrm{H}^{m+1} \cap \mathrm{N}^g$. From the inductive hypothesis, we thus get $g \in (\mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times)\Omega^{q-m}$. By Lemma 4.5, this implies that g normalizes H^m and that:

$$\theta^{1+u'} = \theta \cdot \psi_{a_\beta(u')}$$

on H^m, with $g^{-1} = 1 + u'$. This implies:

$$\theta([g^{-1}, 1+y]) = 1$$

for all $y \in \mathfrak{h}^m \cap \mathfrak{n}$. Therefore:

$$a_\beta(u') \in ((\mathfrak{h}^m)^* + \mathfrak{p}) \cap \mathfrak{n}^- = (a_\beta(\mathfrak{j}^s) + \mathfrak{a}_{1-m} + \mathfrak{p}) \cap \mathfrak{n}^- \subseteq a_\beta(\mathfrak{j}^s) + \mathfrak{a}_{1-m}.$$

Thus there is $j \in \mathfrak{j}^s$ such that:

$$u' + j \in \mathfrak{n}_{1-m}(\beta, \Lambda) \cap \mathfrak{a}_1.$$

From [**21**, Proposition 2.29] we have:

$$\mathfrak{n}_{1-m}(\beta, \Lambda) = \mathfrak{A}(\Lambda) \cap \mathrm{B} + \mathfrak{a}_{q-m+1} \cap \mathfrak{n}_{1-m}(\beta, \Lambda).$$

This implies the expected result, that is $g \in (\mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times)\Omega^{q-m+1}$. □

Continuing with the same notation, we will also need the following variant of Lemma 4.2. We put $\mathrm{H}^1_\mathrm{P} = \mathrm{H}^1(\mathrm{J}^1 \cap \mathrm{N})$, which is a normal subgroup of J^1, and define the character θ_P of H^1_P by

$$\theta_\mathrm{P}(hj) = \theta(h),$$

for $h \in \mathrm{H}^1$ and $j \in \mathrm{J}^1 \cap \mathrm{N}$. By [**21**, Proposition 5.4], if we write $\mathrm{J}^1_\mathrm{P} = \mathrm{H}^1(\mathrm{J}^1 \cap \mathrm{P})$, the intertwining of the character θ_P is $\mathrm{J}^1_\mathrm{P}\mathrm{B}^\times\mathrm{J}^1_\mathrm{P}$.

Corollary 4.7. — *Let $g \in \mathrm{U}_1(\Lambda) \cap \mathrm{N}^-$ and assume that θ_P is trivial on the intersection $\mathrm{H}^1_\mathrm{P} \cap \mathrm{N}^g$. Then $g \in \mathrm{J}^1_\mathrm{P}$.*

Proof. — Suppose that $g \in \mathrm{U}_1(\Lambda) \cap \mathrm{N}^-$ and θ_P is trivial on $\mathrm{H}^1_\mathrm{P} \cap \mathrm{N}^g$. In particular, intersecting with H^1, we see that θ is trivial on $\mathrm{H}^1 \cap \mathrm{N}^g$ so, by Lemma 4.2, we find $g \in \mathrm{J}^1 \cap \mathrm{N}^-$. Since g then normalizes θ, we see that it also normalizes θ_P, so lies in $\mathrm{J}^1_\mathrm{P}\mathrm{B}^\times\mathrm{J}^1_\mathrm{P} \cap \mathrm{J}^1 = \mathrm{J}^1_\mathrm{P}$. □

§5. Parabolic induction and the functor K in the simple case

The main result of this section is Theorem 5.6, which says that, in the simple case, the functor **K** commutes with parabolic induction; in the next section we will extend this result to the semisimple case. This fact has been claimed in [**15**] for representations of finite length (see [**15**], Proposition 5.9) but it appears that the proof of *ibid.*, Lemme 5.10 requires more details.

We give a different proof here, based on our Lemma 4.2, which works for all smooth representations and not only for representations of finite length.

5.1.

Let $[\Lambda_{\max}, n, 0, \beta]$ be a simple stratum in $\mathrm{M}_m(\mathrm{D})$ and assume that $\mathrm{U}(\Lambda_{\max}) \cap \mathrm{B}^\times$ is a maximal compact open subgroup in $\mathrm{B}^\times$. Let $\theta_{\max}$ be a simple character in $\mathcal{C}(\Lambda_{\max}, 0, \beta)$ and $\kappa_{\max}$ be a β-extension of $\theta_{\max}$. We write $\mathrm{J}_{\max} = \mathrm{J}(\beta, \Lambda_{\max})$ and $\mathrm{J}^1_{\max} = \mathrm{J}^1(\beta, \Lambda_{\max})$. Let **K** be the functor:

$$\pi \mapsto \mathrm{Hom}_{\mathrm{J}^1_{\max}}(\kappa_{\max}, \pi)$$

from $\mathscr{R}(\mathrm{G})$ to $\mathscr{R}(\mathrm{J}_{\max}/\mathrm{J}^1_{\max})$ and set $\mathscr{G} = \mathrm{J}_{\max}/\mathrm{J}^1_{\max}$; this is the functor denoted $\mathbf{K}_{\kappa_{\max}}$ in §3.

Let M be a standard Levi subgroup of G, associated with a composition $\alpha = (m_1, \dots, m_r)$ of m. We assume that it is *β-admissible*, that is, the F-algebra $\mathrm{F}[\beta]$, denoted E, can be embedded in A_{m_i} for all i. Equivalently, $m_i d$ is a multiple of the degree of E over F for all i. Let P be the corresponding standard parabolic subgroups of G, and write N for its unipotent radical.

We fix an isomorphism of E-algebras Φ between B and $\mathrm{M}_{m'}(\mathrm{D}')$ that identifies $\mathfrak{A}(\Lambda_{\max}) \cap \mathrm{B}$ with the maximal standard order made of matrices with integer entries. We choose an E-pure lattice sequence Λ such that:

$$\mathrm{U}(\Lambda) \cap \mathrm{B}^\times = (\mathrm{U}_1(\Lambda_{\max}) \cap \mathrm{B}^\times)(\mathrm{P} \cap \mathrm{U}(\Lambda_{\max}) \cap \mathrm{B}^\times). \tag{5.1}$$

The image $\Phi(\mathrm{U}(\Lambda) \cap \mathrm{B}^\times)$ is the standard parahoric subgroup of $\mathrm{GL}_{m'}(\mathrm{D}')$ whose reduction mod $\mathfrak{p}_{\mathrm{D}'}$ is made of upper block triangular matrices of sizes $(m'_1, \dots, m'_r)$, with:

$$m'_i d' = \frac{m_i d}{[\mathrm{E} : \mathrm{F}]}, \quad i \in \{1, \dots, r\},$$

where d' is the reduced degree of D' over E. Moreover, Λ can be chosen such that it satisfies the conditions of the following lemma.

Lemma 5.1. — *There is an* E*-pure lattice sequence* Λ *on* D^m *satisfying* (5.1) *and such that:*

$$\begin{aligned}\mathrm{U}(\Lambda) &\subseteq \mathrm{U}(\Lambda_{\max});\\ \mathrm{U}_1(\Lambda)\cap\mathrm{N}^- &= \mathrm{U}_1(\Lambda_{\max})\cap\mathrm{N}^-.\end{aligned}$$

Proof. — We fix a simple left $\mathrm{E}\otimes_{\mathrm{F}}\mathrm{D}$-module V_0, and form the simple left B-module

$$\mathrm{V}_{\mathrm{B}} = \mathrm{Hom}_{\mathrm{E}\otimes_{\mathrm{F}}\mathrm{D}}(\mathrm{V}_0, \mathrm{D}^m).$$

The E-algebra opposite to $\mathrm{End}_{\mathrm{B}}(\mathrm{V}_{\mathrm{B}})$ is isomorphic to D'. Write $\mathrm{A}_0 = \mathrm{End}_{\mathrm{D}}(\mathrm{V}_0)$ and $\mathfrak{A}_0$ for the unique hereditary order in A_0 normalized by $\mathrm{E}^\times$, and $\mathfrak{P}_0$ for its Jacobson radical. If we identify A with $\mathrm{M}_{m'}(\mathrm{A}_0)$, then $\mathfrak{A}(\Lambda_{\max})$ identifies with $\mathrm{M}_{m'}(\mathfrak{A}_0)$. Then choose Λ such that:

$$\mathfrak{A}(\Lambda) = \begin{pmatrix}\mathfrak{A}_0 & \cdots & \mathfrak{A}_0\\ \vdots & \ddots & \vdots\\ \mathfrak{P}_0 & \cdots & \mathfrak{A}_0\end{pmatrix} \subseteq \begin{pmatrix}\mathfrak{A}_0 & \cdots & \mathfrak{A}_0\\ \vdots & \ddots & \vdots\\ \mathfrak{A}_0 & \cdots & \mathfrak{A}_0\end{pmatrix} = \mathfrak{A}(\Lambda_{\max})$$

(see [**20**]). We have:

$$\mathfrak{a}_1(\Lambda) = \begin{pmatrix}\mathfrak{P}_0 & \cdots & \mathfrak{A}_0\\ \vdots & \ddots & \vdots\\ \mathfrak{P}_0 & \cdots & \mathfrak{P}_0\end{pmatrix} \supseteq \begin{pmatrix}\mathfrak{P}_0 & \cdots & \mathfrak{P}_0\\ \vdots & \ddots & \vdots\\ \mathfrak{P}_0 & \cdots & \mathfrak{P}_0\end{pmatrix} = \mathfrak{a}_1(\Lambda_{\max}).$$

Therefore both $\mathfrak{a}_1(\Lambda)\cap\mathfrak{n}^-$ and $\mathfrak{a}_1(\Lambda_{\max})\cap\mathfrak{n}^-$ are made of blocks with values in $\mathfrak{P}_0$. $\square$

Write θ for the transfer of $\theta_{\max}$ to $\mathcal{C}(\Lambda, 0, \beta)$ in the sense of [**21**], and κ for the unique β-extension of θ such that:

$$\mathrm{Ind}_{\mathrm{J}}^{(\mathrm{U}(\Lambda)\cap\mathrm{B}^\times)\mathrm{U}_1(\Lambda)}(\kappa) \simeq \mathrm{Ind}_{(\mathrm{U}(\Lambda)\cap\mathrm{B}^\times)\mathrm{J}^1_{\max}}^{(\mathrm{U}(\Lambda)\cap\mathrm{B}^\times)\mathrm{U}_1(\Lambda)}(\kappa_{\max}) \tag{5.2}$$

where $\mathrm{J} = \mathrm{J}(\beta, \Lambda)$. We also write $\mathrm{J}_{\mathrm{P}} = \mathrm{H}^1(\mathrm{J}\cap\mathrm{P})$ and κ_{P} for the unique irreducible representation of J_{P} that is trivial on $\mathrm{J}_{\mathrm{P}}\cap\mathrm{N}$ and $\mathrm{J}_{\mathrm{P}}\cap\mathrm{N}^-$ and such that, if we restrict κ_{P} to $\mathrm{J}\cap\mathrm{M}$, we get:

$$\mathrm{J}\cap\mathrm{M} = \mathrm{J}_1\times\cdots\times\mathrm{J}_r, \quad \kappa_{\mathrm{P}}|_{\mathrm{J}\cap\mathrm{M}} = \kappa_1\otimes\cdots\otimes\kappa_r,$$

where $\mathrm{J}_i = \mathrm{J}(\beta, \Lambda_i)$ and κ_i is a β-extension with respect to some simple stratum $[\Lambda_i, n_i, 0, \beta]$ in A_{m_i}. We have an isomorphism of representations of J:

$$\mathrm{Ind}_{\mathrm{J}_{\mathrm{P}}}^{\mathrm{J}}(\kappa_{\mathrm{P}}) \simeq \kappa. \tag{5.3}$$

We write $\mathrm{J}_{\max,\alpha} = \mathrm{J}\cap\mathrm{M}$, $\mathrm{J}^1_{\max,\alpha} = \mathrm{J}^1\cap\mathrm{M}$ and $\kappa_{\max,\alpha} = \kappa_{\mathrm{P}}|_{\mathrm{J}\cap\mathrm{M}}$. We have a functor:

$$\mathbf{K}_{\mathrm{M}} : \pi \mapsto \mathrm{Hom}_{\mathrm{J}^1_{\max,\alpha}}(\kappa_{\max,\alpha}, \pi)$$

from $\mathscr{R}(\mathrm{M})$ to $\mathscr{R}(\mathrm{J}_{\max,\alpha}/\mathrm{J}^1_{\max,\alpha})$.

The groups $\mathrm{J}\cap\mathrm{M}/\mathrm{J}^1\cap\mathrm{M}$, $(\mathrm{U}(\Lambda)\cap\mathrm{B}^\times)\mathrm{J}^1_{\max}/(\mathrm{U}_1(\Lambda)\cap\mathrm{B}^\times)\mathrm{J}^1_{\max}$ and $\mathrm{J}_{\max,\alpha}/\mathrm{J}^1_{\max,\alpha}$ will all be identified, and all of them will be denoted $\mathscr{M}$. For simplicity, we will write:

$$\begin{aligned}\mathrm{U} &= (\mathrm{U}(\Lambda)\cap\mathrm{B}^\times)\mathrm{U}_1(\Lambda),\\ \mathrm{U}^1 &= \mathrm{U}_1(\Lambda)\cap\mathrm{U} = \mathrm{U}_1(\Lambda),\\ \mathrm{S} &= (\mathrm{U}(\Lambda)\cap\mathrm{B}^\times)\mathrm{J}^1_{\max},\\ \mathrm{S}^1 &= \mathrm{U}_1(\Lambda)\cap\mathrm{S} = (\mathrm{U}_1(\Lambda)\cap\mathrm{B}^\times)\mathrm{J}^1_{\max}.\end{aligned}$$

5.2.

We write $\mathbf{K}_\mathrm{S}$ for the functor:

$$\pi \mapsto \mathrm{Hom}_{\mathrm{S}^1}(\kappa_{\max}|_\mathrm{S}, \pi)$$

from $\mathscr{R}(\mathrm{S})$ to $\mathscr{R}(\mathscr{M})$. Note that this fits in the framework of §3, with:

$$\mathbf{J} = \mathrm{S}, \quad \mathbf{J}^1 = \mathrm{S}^1, \quad \mathbf{H}^1 = (\mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times)\mathrm{H}^1_{\max} \quad \boldsymbol{\kappa} = \kappa_{\max}|_\mathrm{S},$$

since, by the construction of β-extensions in [**19**]:

(i) the restriction of $\kappa_{\max}$ to S^1 is the unique (irreducible) representation $\tilde{\eta}$ which extends $\eta_{\max}$ and such that $\mathrm{Ind}_{\mathrm{S}^1}^{\mathrm{U}^1}(\tilde{\eta})$ is equivalent to $\mathrm{Ind}_{\mathrm{J}^1}^{\mathrm{U}^1}(\eta)$;

(ii) the restriction of $\tilde{\eta}$ to $(\mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times)\mathrm{H}^1_{\max}$ is a multiple of the character $\tilde{\theta}$ given by:

$$\tilde{\theta}(uh) = \theta(u)\theta_{\max}(h),$$

for $u \in \mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times$ and $h \in \mathrm{H}^1_{\max}$. (Note that this is well-defined, by [**21**, Théorème 2.13].)

Proposition 5.2. — *For any smooth representation π of* M, *we have*

$$\mathbf{K}_\mathrm{M}(\pi) \simeq \mathbf{K}_\mathrm{S}\left(\mathrm{Ind}_\mathrm{P}^{\mathrm{PS}}(\pi)\right)$$

as representations of $\mathscr{M}$.

Proof. — Let π be a smooth representation of M. Then, by inflation, we have

$$\mathbf{K}_\mathrm{M}(\pi) = \mathrm{Hom}_{\mathrm{J}^1_{\max,\alpha}}(\kappa_{\max,\alpha}, \pi) \simeq \mathrm{Hom}_{\mathrm{J}^1\cap\mathrm{P}}(\kappa_\mathrm{P}, \pi).$$

By Frobenius reciprocity and the Mackey formula, this is isomorphic to

$$\mathrm{Hom}_{\mathrm{J}^1_\mathrm{P}}(\kappa_\mathrm{P}, \mathrm{Ind}_{\mathrm{J}\cap\mathrm{P}}^{\mathrm{J}_\mathrm{P}}(\pi)).$$

Again we are in the situation of §3, with $\mathbf{J} = \mathrm{J}_\mathrm{P}$, $\mathbf{J}^1 = \mathrm{J}^1_\mathrm{P}$, $\boldsymbol{\kappa} = \kappa_\mathrm{P}$, and $\boldsymbol{\theta} = \theta_\mathrm{P}$, the character of Corollary 4.7. Thus, using the notation of §3 and Lemma 3.1, we get

$$\mathbf{K}_\mathrm{M}(\pi) \simeq \mathbf{K}_{\kappa_\mathrm{P}} \circ \mathrm{Ind}_\mathrm{P}^{\mathrm{PJ}_\mathrm{P}}(\pi). \tag{5.4}$$

We decompose PU as a disjoint union of double cosets $\mathrm{P}u\mathrm{J}_\mathrm{P}$, where the double coset representatives u may, and will, be chosen in $\mathrm{U} \cap \mathrm{N}^- = \mathrm{U}_1(\Lambda) \cap \mathrm{N}^-$; then $\mathrm{Ind}_\mathrm{P}^{\mathrm{PU}}(\pi) = \bigoplus_u \mathrm{Ind}_\mathrm{P}^{\mathrm{P}u\mathrm{J}_\mathrm{P}}(\pi)$.

By Corollary 3.3, we have that $\mathbf{K}_{\kappa_\mathrm{P}} \circ \mathrm{Ind}_\mathrm{P}^{\mathrm{P}u\mathrm{J}_\mathrm{P}}$ is non-zero if and only if θ_P is trivial on $\mathrm{H}^1_\mathrm{P} \cap \mathrm{N}^u$, which, by Corollary 4.7, implies $u \in \mathrm{J}^1_\mathrm{P}$. Thus (5.4) implies

$$\mathbf{K}_\mathrm{M}(\pi) \simeq \mathbf{K}_{\kappa_\mathrm{P}} \circ \mathrm{Ind}_\mathrm{P}^{\mathrm{PU}}\pi \simeq \mathrm{Hom}_{\mathrm{J}^1_\mathrm{P}}(\kappa_\mathrm{P}, \mathrm{Ind}_{\mathrm{P}\cap\mathrm{U}}^{\mathrm{U}}(\pi)).$$

Write ρ for the irreducible induced representation $\mathrm{Ind}_{\mathrm{J}_\mathrm{P}}^{\mathrm{U}}(\kappa_\mathrm{P})$ which, by (5.2) and (5.3), is isomorphic to $\mathrm{Ind}_\mathrm{S}^\mathrm{U}(\kappa_{\max}|_\mathrm{S})$. Then, again by Frobenius and Mackey, we get

$$\mathbf{K}_\mathrm{M}(\pi) \simeq \mathrm{Hom}_{\mathrm{U}^1}(\rho, \mathrm{Ind}_{\mathrm{P}\cap\mathrm{U}}^{\mathrm{U}}(\pi)) \simeq \mathrm{Hom}_{\mathrm{S}^1}(\kappa_{\max}|_\mathrm{S}, \mathrm{Ind}_{\mathrm{P}\cap\mathrm{U}}^{\mathrm{U}}(\pi)) \simeq \mathbf{K}_\mathrm{S} \circ \mathrm{Ind}_\mathrm{P}^{\mathrm{PU}}(\pi),$$

applying Lemma 3.1 again.

As before, we decompose PU as a disjoint union of double cosets $\mathrm{P}u\mathrm{S}$, where the double coset representatives u lie in $\mathrm{U}\cap\mathrm{N}^-$ which, by Lemma 5.1, is $\mathrm{U}_1(\Lambda)\cap\mathrm{N}^-$; then $\mathrm{Ind}_\mathrm{P}^{\mathrm{PU}}\pi = \bigoplus_u \mathrm{Ind}_\mathrm{P}^{\mathrm{P}u\mathrm{S}}\pi$. Now Corollary 3.3 shows that the functor $\mathbf{K}_\mathrm{S} \circ \mathrm{Ind}_\mathrm{P}^{\mathrm{P}u\mathrm{S}}$ is nonzero on $\mathscr{R}(\mathrm{M})$ if and only if $\tilde{\theta}$ is trivial on $(\mathrm{U}_1(\Lambda) \cap \mathrm{B}^\times)\mathrm{H}^1_{\max} \cap \mathrm{N}^u$; in particular, restricting to $\mathrm{H}^1_{\max}$ and applying Lemma 4.2, we see that $u \in \mathrm{PS}$ so

$$\mathbf{K}_\mathrm{M}(\pi) \simeq \mathbf{K}_\mathrm{S} \circ \mathrm{Ind}_\mathrm{P}^{\mathrm{PU}}(\pi) = \mathbf{K}_\mathrm{S} \circ \mathrm{Ind}_\mathrm{P}^{\mathrm{PS}}(\pi).$$

This ends the proof of Proposition 5.2. □

The following lemma relates the functor K_S back to our functor K. We put $\mathscr{P} = S/J^1_{\max}$, which is a parabolic subgroup of $\mathscr{G} = J_{\max}/J^1_{\max}$ with Levi component $\mathscr{M}$. We regard representations of $\mathscr{M}$ as representations of $\mathscr{P}$ by inflation.

Lemma 5.3. — *For any smooth representation π of* M*, we have*

$$\mathsf{K}_S\left(\mathrm{Ind}_P^{PS}(\pi)\right) \simeq \mathsf{K}\left(\mathrm{Ind}_P^{PS}(\pi)\right)$$

as representations of $\mathscr{P}$.

Proof. — We clearly have an inclusion of spaces $\mathrm{Hom}_{S^1}(\kappa_{\max}, \mathrm{Ind}_P^{PS}\pi) \subseteq \mathrm{Hom}_{J^1_{\max}}(\kappa_{\max}, \mathrm{Ind}_P^{PS}\pi)$ and, if we check that we have equality here, it is then straightforward that the actions of $\mathscr{P}$ are the same. Write $\mathcal{V}$ for the space of $\kappa_{\max}$.

The action of $U_1(\Lambda) \cap B^\times$ on $\mathcal{V}$ is a multiple of $\tilde{\theta}|_{U_1(\Lambda)\cap B^\times}$, which factors through the reduced norm. Thus, for $u \in U_1(\Lambda) \cap B^\times \cap N$, we have $\kappa_{\max}(u) = \mathrm{id}_{\mathcal{V}}$. Now let:

$$f \in \mathrm{Hom}_{J^1_{\max}}(\kappa_{\max}, \mathrm{Ind}_P^{PS}\pi)$$

and $v \in \mathcal{V}$, and put $\varphi = f(v)$. For $j \in J^1_{\max}$ and $u \in U_1(\Lambda) \cap B^\times \cap N$, we have $\eta_{\max}(u^{-1}ju) = \eta_{\max}(j)$ and $\pi(u)$ acts trivially on the space of π so

$$(u \cdot \varphi)(j) = \varphi(ju) = \varphi(u^{-1}ju) = f(\eta_{\max}(u^{-1}ju)v)(1) = f(\eta_{\max}(j)v)(1) = \varphi(j).$$

Since $PS = PJ^1_{\max}$, this implies that $u \cdot \varphi = \varphi$. Thus

$$f(\kappa_{\max}(u)v) = f(v) = u \cdot f(v)$$

and $f \in \mathrm{Hom}_{S^1}(\kappa_{\max}, \mathrm{Ind}_P^{PS}\pi)$ since $S^1 = (U_1(\Lambda) \cap B^\times \cap N)J^1_{\max}$. □

5.3.

Then next step is to relate parabolic induction in the finite reductive group $\mathscr{G}$ to induction inside $J_{\max}$.

Lemma 5.4. — *For any smooth representation τ of* S*, we have:*

$$\mathsf{K}\left(\mathrm{Ind}_S^{J_{\max}}(\tau)\right) \simeq \mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}(\mathsf{K}(\tau))$$

as representations of $\mathscr{G}$.

Note that $\mathsf{K}(\tau) = \mathrm{Hom}_{J^1_{\max}}(\kappa_{\max}, \tau)$ is viewed here as a representation of $\mathscr{P}$ by restriction.

Proof. — As above, write $\mathcal{V}$ for the space of $\kappa_{\max}$. Given $f \in \mathsf{K}(\mathrm{Ind}_S^{J_{\max}}(\tau))$, we define a function $\bar{f}$ by:

$$\bar{f}(\dot{x}) : v \mapsto f(x^{-1} \cdot v)(x)$$

for all $x \in J_{\max}$ and $v \in \mathcal{V}$, where $\dot{x}$ is the class of x in $\mathscr{G}$.

We first need to check that $\bar{f}$ is well defined. Let $z \in J^1_{\max}$. For $v \in \mathcal{V}$ and $x \in J_{\max}$, we have:

$$\begin{aligned} f(z^{-1}x^{-1} \cdot v)(xz) &= [z^{-1} \cdot f(x^{-1} \cdot v)](xz) \\ &= f(x^{-1} \cdot v)(xz \cdot z^{-1}) \\ &= f(x^{-1} \cdot v)(x). \end{aligned}$$

We now check that $\bar{f}$ takes its values in $\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}(\mathrm{Hom}_{\mathrm{J}^1_{\max}}(\kappa_{\max},\tau))$. Given $v\in\mathcal{V}$, $x\in\mathrm{J}_{\max}$ and $j\in\mathrm{J}^1_{\max}$, we first have:

$$\begin{aligned}\bar{f}(\dot{x})(j\cdot v) &= f(x^{-1}j\cdot v)(x)\\ &= f(x^{-1}j\cdot v)(j\cdot j^{-1}x)\\ &= \tau(j)[f(x^{-1}j\cdot v)(j^{-1}x)]\end{aligned}$$

which is equal to $\tau(j)[\bar{f}(\dot{x})(v)]$ since $j^{-1}x$ and x have the same image in $\mathscr{G}$. Now given $s\in\mathrm{S}$, $x\in\mathrm{J}_{\max}$ and $v\in\mathcal{V}$, we have:

$$\begin{aligned}\bar{f}(\dot{s}\dot{x})(v) &= f(x^{-1}s^{-1}\cdot v)(sx)\\ &= \tau(s)[f(x^{-1}s^{-1}\cdot v)(x)].\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}[\dot{s}\cdot\bar{f}(\dot{x})](v) &= [\tau(s)\circ\bar{f}(\dot{x})\circ\kappa_{\max}(s)^{-1}](v)\\ &= \tau(s)[\bar{f}(\dot{x})(s^{-1}\cdot v)]\end{aligned}$$

and this coincides with $\bar{f}(\dot{s}\dot{x})(v)$.

We now check that $f\mapsto\bar{f}$ is a $\mathscr{G}$-homomorphism. Given $x,y\in\mathrm{J}_{\max}$ and $v\in\mathcal{V}$, we have:

$$\begin{aligned}\overline{\dot{y}\cdot f}(\dot{x})(v) &= [\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}(\tau)(y)\circ f\circ\kappa_{\max}(y)^{-1}](x^{-1}\cdot v)(x)\\ &= f(y^{-1}x^{-1}\cdot v)(xy)\end{aligned}$$

which is equal to $\bar{f}(\dot{x}\dot{y})(v)$ and gives us $\overline{\dot{y}\cdot f}(\dot{x})=\bar{f}(\dot{x}\dot{y})$, thus the expected relation $\overline{\dot{y}\cdot f}=\dot{y}\cdot\bar{f}$.

The map $f\mapsto\bar{f}$ is clearly injective. Now let ϕ be some function in $\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}(\mathrm{Hom}_{\mathrm{S}^1}(\kappa_{\max}|_{\mathrm{S}},\tau))$. We define a function f from $\mathcal{V}$ to $\mathrm{Ind}_{\mathrm{S}}^{\mathrm{J}_{\max}}(\tau)$ by:

$$f(v)(x)=\phi(\dot{x})(x\cdot v).$$

Checking that $f\in\mathbf{K}(\mathrm{Ind}_{\mathrm{S}}^{\mathrm{J}_{\max}}(\tau))$ and that $\bar{f}=\phi$ is similar to the calculations above, and this completes the proof of the lemma. □

Putting this together with the results of the previous subsection, we get:

Corollary 5.5. — *For any smooth representation π of* M, *we have*

$$\mathbf{K}\left(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{PJ}_{\max}}(\pi)\right)\simeq\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}\left(\mathbf{K}_{\mathrm{M}}(\pi)\right)$$

as representations of $\mathscr{G}$.

Proof. — Putting together Proposition 5.2 with Lemmas 5.3, 5.4, we get

$$\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}\left(\mathbf{K}_{\mathrm{M}}(\pi)\right)\simeq\mathbf{K}\left(\mathrm{Ind}_{\mathrm{S}}^{\mathrm{J}_{\max}}\left(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{PS}}(\pi)\right)\right),$$

while $\mathrm{Ind}_{\mathrm{S}}^{\mathrm{J}_{\max}}\left(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{PS}}(\pi)\right)\simeq\mathrm{Ind}_{\mathrm{P}}^{\mathrm{PJ}_{\max}}(\pi)$, from Lemma 3.1 and the fact that $\mathrm{P}\cap\mathrm{S}=\mathrm{P}\cap\mathrm{J}_{\max}$. □

Proposition 5.6. — *For any smooth representation π of* M, *we have an isomorphism*

$$\mathbf{K}\left(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi)\right)\simeq\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}\left(\mathbf{K}_{\mathrm{M}}(\pi)\right)\tag{5.5}$$

as representations of $\mathscr{G}$.

Proof. — Assume first that R is the field of complex numbers. In that case, we may assume that π belongs to a single Bernstein block of M. If π does not contain the simple character $\theta_{\max}$, then both sides of (5.5) are zero. Otherwise, the method used by Schneider and Zink in [**17**], based on equivalences of categories given by the theory of types for complex representations, applies *mutatis mutandis*, replacing the reference to [**6**, (11.4)] by [**8**, Theorem 1.5]. Therefore, for any irreducible complex representation π of M, the canonical inclusion:

$$\mathbf{K}(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{PJ}_{\max}}(\pi)) \subseteq \mathbf{K}(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi))$$

is an equality by Corollary 5.5, since the right hand side is finite-dimensional. Thus the functor $\mathbf{K} \circ \mathrm{Ind}_{\mathrm{P}}^{\mathrm{P}g\mathrm{J}_{\max}}$ is zero on Irr(M), for any $g \notin \mathrm{PJ}_{\max}$. By Corollary 3.3, this implies that, for $g \in \mathrm{G}$,

$$\theta_{\max} \text{ is trivial on } \mathrm{H}^1_{\max} \cap \mathrm{N}^g \quad \Leftrightarrow \quad g \in \mathrm{PJ}_{\max} \tag{5.6}$$

for any complex maximal simple character $\theta_{\max}$. As $\mathrm{H}^1_{\max}$ is a pro-p-group, (5.6) holds also for any *modular* maximal simple character. Thus, by Corollary 3.3 again, the equality

$$\mathbf{K}\left(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi)\right) = \mathbf{K}\left(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{PJ}_{\max}}(\pi)\right)$$

holds for all smooth R-representations π of M. The result follows from Corollary 5.5. □

Remark 5.7. — We have proved that the functors $\mathbf{K}\circ\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}$ and $\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}\circ\mathbf{K}_{\mathrm{M}}$ from $\mathscr{R}(\mathrm{M})$ to $\mathscr{R}(\mathrm{G})$ behave in the same way on objects. It seems likely that similar proofs would show that they behave in the same way on morphisms so that the two functors are in fact isomorphic.

§6. Semisimple supertypes

In this section, we first recall briefly the basic properties of, and data attached to, semisimple supertypes, for which we refer to [**22, 15**] for more details, and we explain the functor $\mathbf{K}$ in this situation. The main result is Theorem 6.2, which extends to the semisimple case the main result of the previous section: the functor $\mathbf{K}$ commutes with parabolic induction.

6.1.

Let $\alpha = (m_1, \dots, m_r)$ be a composition of m. For all $i \in \{1, \dots, r\}$, let $(\mathrm{J}_i, \lambda_i)$ be a maximal simple type attached to a simple stratum $[\Lambda_i, n_i, 0, \beta_i]$ in A_{m_i}. We write M for the standard Levi subgroup $\mathrm{G}_{m_1} \times \cdots \times \mathrm{G}_{m_r}$ in G and:

$$\mathrm{J}_\alpha = \mathrm{J}_1 \times \cdots \times \mathrm{J}_r, \quad \lambda_\alpha = \lambda_1 \otimes \cdots \otimes \lambda_r.$$

A pair of the form $(\mathrm{J}_\alpha, \lambda_\alpha)$ is called a maximal simple type of M. Associated to it, there is a pair $(\mathbf{J}, \boldsymbol{\lambda})$ called a semisimple type of G (see [**22, 15**]). For any parabolic subgroup P of G with Levi component M, the pair $(\mathbf{J}, \boldsymbol{\lambda})$ satisfies the following properties:

(i) the kernel of $\boldsymbol{\lambda}$ contains $\mathbf{J} \cap \mathrm{N}$ and $\mathbf{J} \cap \mathrm{N}^-$, where N and N^- denote the unipotent radicals of P and P^-, the parabolic subgroup opposite to P with respect to M;

(ii) one has $\mathbf{J} \cap \mathrm{M} = \mathrm{J}_\alpha$ and $\boldsymbol{\lambda}|_{\mathbf{J}\cap\mathrm{M}} = \lambda_\alpha$;

(these two conditions say that $(\mathbf{J}, \boldsymbol{\lambda})$ is *decomposed* above the pair $(\mathrm{J}_\alpha, \lambda_\alpha)$ with respect to (M, P) in the sense of [**6**, Definition 6.1]), plus another technical condition saying that the pair $(\mathbf{J}, \boldsymbol{\lambda})$ is a cover of $(\mathrm{J}_\alpha, \lambda_\alpha)$ in the sense of [**6**, Definition 8.1]. Note that there is considerable flexibility in the construction of semisimple types; in particular, there is a (not entirely arbitrary) choice of lattice sequence Λ on D^m such that:

$$\mathrm{U}(\Lambda) \cap \mathrm{M} = \mathrm{U}(\Lambda_1) \times \cdots \times \mathrm{U}(\Lambda_r)$$

(see [**22**, §7.1] and [**15**, §2.8-9] for the precise condition). In particular, we may and will assume that the lattice sequences $\Lambda_1, \ldots, \Lambda_r$ and Λ all have the same period.

Given π_i a representation of G_{m_i} for all $i \in \{1, \ldots, r\}$, we write $\pi_1 \times \cdots \times \pi_r$ for the representation $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi_1 \otimes \cdots \otimes \pi_r)$, where P is the parabolic subgroup of G with Levi component M made of upper triangular matrices.

An important relationship between $(\mathbf{J}, \boldsymbol{\lambda})$ and $(\mathrm{J}_1, \lambda_1), \ldots, (\mathrm{J}_r, \lambda_r)$ is that there is an isomorphism of representations of G:

$$\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}) \simeq \mathrm{ind}_{\mathrm{J}_1}^{\mathrm{G}_{m_1}}(\lambda_1) \times \cdots \times \mathrm{ind}_{\mathrm{J}_1}^{\mathrm{G}_{m_r}}(\lambda_r)$$

(see [**2**]). Note, in particular, that this is independent of any choices made in the construction of $(\mathbf{J}, \boldsymbol{\lambda})$.

Definition 6.1. — (i) When $(\mathrm{J}_1, \lambda_1), \ldots, (\mathrm{J}_r, \lambda_r)$ are maximal simple supertypes, $(\mathbf{J}, \boldsymbol{\lambda})$ is called a *semisimple supertype* of G.

(ii) The *equivalence class* of a semisimple type $(\mathbf{J}, \boldsymbol{\lambda})$ is the set $[\mathbf{J}, \boldsymbol{\lambda}]$ of all semisimple supertypes $(\mathbf{J}', \boldsymbol{\lambda}')$ of G such that $\mathrm{ind}_{\mathbf{J}'}^{\mathrm{G}}(\boldsymbol{\lambda}')$ is isomorphic to $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$.

Together with $\mathbf{J}$, we also have a normal open subgroup $\mathbf{J}^1$ and an irreducible representation $\boldsymbol{\eta}$ of $\mathbf{J}^1$ (see [**15**, §2.10]). When restricting $\boldsymbol{\lambda}$ to $\mathbf{J}^1$, we get a direct sum of copies of $\boldsymbol{\eta}$. There is a decomposition of the form:

$$\boldsymbol{\lambda} \simeq \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}, \tag{6.1}$$

where $\boldsymbol{\kappa}$ is an irreducible representation of $\mathbf{J}$ extending $\boldsymbol{\eta}$ and $\boldsymbol{\sigma}$ is an irreducible representation of $\mathbf{J}$ trivial on $\mathbf{J}^1$. The representation $\boldsymbol{\kappa}$ has the property that its intertwining is the same as that of $\boldsymbol{\eta}$, but is not uniquely determined by this condition; thus there is a choice of $\boldsymbol{\kappa}$ to be made in the decomposition (6.1).

For each $i \in \{1, \ldots, r\}$, we have a maximal simple character θ_i attached to the simple stratum $[\Lambda_i, n_i, 0, \beta_i]$, an isomorphism of $\mathrm{F}[\beta_i]$-algebras $\mathrm{B}_i \simeq \mathrm{M}_{m'_i}(\mathrm{D}'_i)$ for a suitable $\mathrm{F}[\beta_i]$-division algebra D'_i, and isomorphisms of groups:

$$\mathbf{J}/\mathbf{J}^1 \simeq \mathrm{J}_1/\mathrm{J}_1^1 \times \cdots \times \mathrm{J}_r/\mathrm{J}_r^1 \simeq \mathrm{GL}_{m'_1}(\mathfrak{k}_{\mathrm{D}'_1}) \times \cdots \times \mathrm{GL}_{m'_r}(\mathfrak{k}_{\mathrm{D}'_r});$$

we denote by $\mathscr{M}$ this latter group. The representation $\boldsymbol{\kappa}$ is trivial on $\mathbf{J} \cap \mathrm{N}$ and $\mathbf{J} \cap \mathrm{N}^-$, and its restriction to $\mathbf{J} \cap \mathrm{M} = \mathrm{J}_\alpha$ is of the form $\kappa_\alpha = \kappa_1 \otimes \cdots \otimes \kappa_r$, where κ_i is a maximal β_i-extension of θ_i.

For each i, there is a decomposition $\lambda_i = \kappa_i \otimes \sigma_i$, where σ_i is an irreducible representation of J_i trivial on J_i^1 that identifies with a cuspidal representation of $\mathrm{GL}_{m'_i}(\mathfrak{k}_{\mathrm{D}'_i})$, and $\boldsymbol{\sigma}$ identifies with the irreducible cuspidal representation $\sigma_1 \otimes \cdots \otimes \sigma_r$ of $\mathscr{M}$.

6.2.

We will need to recall some more detail of the structure of semisimple supertypes $(\mathbf{J}, \boldsymbol{\lambda})$, which we begin in this section.

We write $\boldsymbol{\Theta}_i$ for the endo-class of θ_i (see [**3**] for the definition of endo-class) and assume first that the endo-classes $\boldsymbol{\Theta}_i$ all coincide, the so-called *homogeneous case*. In this case, we may and will assume that the elements $\beta_1, \dots, \beta_r$ are all equal to (the image of) a single element β and that the characters θ_i are related by the transfer maps (in other words, they are realizations of the same ps-character – see [**3**]). We put $\mathrm{E} = \mathrm{F}[\beta]$ and denote by B the centralizer of E in A, so that $\mathrm{B} \simeq \mathrm{M}_{m'}(\mathrm{D}')$, where D' is a suitable E-division algebra. Similarly, we write $\mathrm{B}_i \simeq \mathrm{M}_{m'_i}(\mathrm{D}')$ for the centralizer of E in A_{m_i}.

We choose a simple stratum $[\Lambda_{\max}, n_{\max}, 0, \beta]$ in A and an isomorphism of E-algebras Φ from B to $\mathrm{M}_{m'}(\mathrm{D}')$ with the following properties:

(i) $\mathrm{U}(\Lambda_{\max}) \cap \mathrm{B}^\times$ is a maximal compact subgroup of $\mathrm{B}^\times$ that contains $\mathrm{U}(\Lambda) \cap \mathrm{B}^\times$;

(ii) $\Phi(\mathrm{U}(\Lambda_{\max}) \cap \mathrm{B}^\times)$ and $\Phi(\mathrm{U}(\Lambda) \cap \mathrm{B}^\times)$ are both standard parahoric subgroups of $\mathrm{GL}_{m'}(\mathrm{D}')$;

By passing to an equivalent type if necessary, we will assume that $\mathrm{U}(\Lambda) \subseteq \mathrm{U}(\Lambda_{\max})$ as in Lemma 5.1.

We are now in the situation of §5.1, with θ the transfer of θ_i to $\mathcal{C}(\Lambda, 0, \beta)$ (which is independent of i), and we take the notation from there. We have $\mathbf{J} = \mathrm{J}_{\mathrm{P}}$ and $\boldsymbol{\kappa} = \kappa_{\mathrm{P}}$ for some choice of β-extension $\kappa_{\max}$ of $\theta_{\max}$; it is thus this choice of $\kappa_{\max}$ which imposes the choice of $\boldsymbol{\kappa}$ in §6.1. The group $\mathscr{M}$ is a Levi subgroup of:

$$\mathscr{G} = \mathrm{GL}_{m'}(\mathfrak{k}_{\mathrm{D}'}) \simeq \mathrm{J}_{\max}/\mathrm{J}^1_{\max}$$

so we get a supercuspidal pair $(\mathscr{M}, \boldsymbol{\sigma})$ of $\mathscr{G}$, where $\boldsymbol{\sigma} = \sigma_1 \otimes \cdots \otimes \sigma_r$ is as above. Taking Γ to be the group $\mathrm{Gal}(\mathfrak{k}_{\mathrm{D}'}/\mathfrak{k}_{\mathrm{E}})$, we also get an equivalence class $[\mathscr{M}, \boldsymbol{\sigma}]$ of supercuspidal pairs, in the sense of Definition 1.14.

The group $\mathscr{G}$ and the conjugacy class of $\mathscr{M} \subseteq \mathscr{G}$ are uniquely determined by the semisimple type $(\mathbf{J}, \boldsymbol{\lambda})$, independently of the decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$. The representation $\boldsymbol{\kappa}$ is not uniquely determined but, once it is fixed (or, equivalently, the representation $\kappa_{\max}$ is fixed), it determines the equivalence class $[\mathscr{M}, \boldsymbol{\sigma}]$, as well as the functor:

$$\mathsf{K} = \mathsf{K}_{\kappa_{\max}} : \mathscr{R}(\mathrm{G}) \to \mathscr{R}(\mathscr{G}).$$

Moreover, every equivalence class $[\mathscr{M}', \boldsymbol{\sigma}']$ arises from some homogeneous semisimple supertype: $\mathscr{M}'$ determines a composition α' of m' and hence a Levi subgroup M' of G with standard parabolic subgroup P'; then we may make the constructions of §5.1 to get a pair $(\mathbf{J}', \boldsymbol{\lambda}')$, with $\mathbf{J}' = \mathrm{J}_{\mathrm{P}'}$ and $\boldsymbol{\lambda}' = \kappa_{\mathrm{P}'} \otimes \boldsymbol{\sigma}'$, which is a homogeneous semisimple supertype with the required property.

6.3.

Now we consider the general case, when the endo-classes $\boldsymbol{\Theta}_i$ may differ. Let $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\mathbf{J}, \boldsymbol{\lambda})$ be the formal sum:

$$\sum_{i=1}^{r} \frac{m_i d}{[\mathrm{F}[\beta_i] : \mathrm{F}]} \cdot \boldsymbol{\Theta}_i$$

in the semigroup of finitely supported maps $\{\text{endo-classes over F}\} \to \mathbf{N}$ (with $\mathbf{N}$ the semigroup of nonnegative integers). The fibers of the map $i \mapsto \boldsymbol{\Theta}_i$ define a partition:

$$\{1, \dots, r\} = \mathrm{I}_1 \cup \cdots \cup \mathrm{I}_l$$

for some $s \geqslant 1$. Renumbering, we may assume that the I_j (for $j \in \{1, \dots, l\}$) are of the form:

$$\mathrm{I}_j = \{i \in \{1, \dots, r\} \mid a_{j-1} < i \leqslant a_j\}$$

for some integers $0 = a_0 < a_1 < \cdots < a_l = r$. For all $j \in \{1, \dots, l\}$, we write:

$$n_j = \sum_{i \in \mathrm{I}_j} m_i, \quad \mathrm{M}_j = \prod_{i \in \mathrm{I}_j} \mathrm{G}_{m_i},$$

and P_j the standard parabolic subgroup of G_{n_j} with Levi subgroup M_j. Let L be the standard Levi subgroup $\mathrm{G}_{n_1} \times \cdots \times \mathrm{G}_{n_l}$ in G; thus we have $\mathrm{P} \cap \mathrm{L} = \mathrm{P}_1 \times \cdots \times \mathrm{P}_l$. From the construction of semisimple types, and by passing to an equivalent semisimple type as before if necessary, we have:

$$\mathbf{J} \cap \mathrm{L} = \mathbf{J}_1 \times \cdots \times \mathbf{J}_l, \quad \boldsymbol{\lambda}_{\mathbf{J}\cap\mathrm{L}} = \boldsymbol{\lambda}_1 \otimes \cdots \otimes \boldsymbol{\lambda}_l,$$

where each $(\mathbf{J}_j, \boldsymbol{\lambda}_j)$ is a homogeneous semisimple supertype, as described in the previous section. In particular, for each $j \in \{1, \ldots, l\}$, we choose a pair $(\mathrm{J}_{\max,j}, \kappa_{\max,j})$ and have the group $\mathscr{G}_j$ and the supercuspidal equivalence class $[\mathscr{L}_j, \boldsymbol{\sigma}_j]$. The choice of the representations $\kappa_{\max,j}$ imposes the choice of $\boldsymbol{\kappa}$ in §6.1 (and vice versa).

Now write $\mu = (n_1, \ldots, n_l)$ and:

$$\mathrm{J}_{\max,\mu} = \mathrm{J}_{\max,1} \times \cdots \times \mathrm{J}_{\max,l}, \quad \kappa_{\max,\mu} = \kappa_{\max,1} \otimes \cdots \otimes \kappa_{\max,l},$$

so that:

$$\mathrm{J}_{\max,\mu}/\mathrm{J}^1_{\max,\mu} \simeq \mathscr{G}_1 \times \cdots \times \mathscr{G}_l;$$

we denote the latter group by $\mathscr{G}$. We also get an isomorphism of groups $\mathscr{M} \simeq \mathscr{M}_1 \times \cdots \times \mathscr{M}_l$ which identifies $\boldsymbol{\sigma}$ with $\boldsymbol{\sigma}_1 \otimes \cdots \otimes \boldsymbol{\sigma}_l$. Then $(\mathscr{M}, \boldsymbol{\sigma})$ is a supercuspidal pair of $\mathscr{G}$ and we define the equivalence class $[\mathscr{M}, \boldsymbol{\sigma}]$ to be the product of the equivalence classes $[\mathscr{M}_j, \boldsymbol{\sigma}_j]$ (see Definition 1.14).

The formal sum $\boldsymbol{\Theta}$, the group $\mathscr{G}$ and the conjugacy class of $\mathscr{M} \subseteq \mathscr{G}$ are uniquely determined by $(\mathbf{J}, \boldsymbol{\lambda})$ (independently of the decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$). In fact, the group $\mathscr{G}$ depends only on $\boldsymbol{\Theta}$, since $\mathscr{G}_j \simeq \mathrm{GL}_{n'_j}(\mathfrak{k}_{\mathrm{D}'_j})$, where:

$$n'_j \cdot [\mathfrak{k}_{\mathrm{D}'_j} : \mathfrak{k}_{\mathrm{E}_j}] = \frac{n_j d}{[\mathrm{E}_j : \mathrm{F}]} = \sum_{i \in \mathrm{I}_j} \frac{m_i d}{[\mathrm{F}[\beta_i] : \mathrm{F}]},$$

which is the coefficient of $\boldsymbol{\Theta}_i$ in $\boldsymbol{\Theta}$, for $i \in \mathrm{I}_j$.

As in the previous case, the representation $\boldsymbol{\kappa}$ is not uniquely determined by $\boldsymbol{\lambda}$, but once it is fixed (or, equivalently, once $\kappa_{\max,\mu}$ is fixed), it determines the equivalence class $[\mathscr{M}, \boldsymbol{\sigma}]$. Further, there is a decomposed pair $(\mathbf{J}_{\max}, \boldsymbol{\kappa}_{\max})$ above $(\mathrm{J}_{\max,\mu}, \kappa_{\max,\mu})$ (see [**15**]) and we let $\mathbf{J}^1_{\max}$ denote the pro-p radical of $\mathbf{J}_{\max}$; we are now in the situation of §3, with $\mathbf{J} = \mathbf{J}_{\max}$ and $\boldsymbol{\kappa} = \boldsymbol{\kappa}_{\max}$ so we have the functor:

$$\mathsf{K} = \mathsf{K}_{\boldsymbol{\kappa}_{\max}} : \mathscr{R}(\mathrm{G}) \to \mathscr{R}(\mathscr{G}),$$

which is also determined by the choice of $\boldsymbol{\kappa}$. As in the homogeneous case, every equivalence class $[\mathscr{M}', \boldsymbol{\sigma}']$ arises from some semisimple supertype $(\mathbf{J}', \boldsymbol{\lambda}')$, by taking a cover.

We will see below that K induces a bijection between the set of equivalence classes $[\mathbf{J}, \boldsymbol{\lambda}]$ of semisimple supertypes for G such that $\boldsymbol{\Theta}(\mathbf{J}, \boldsymbol{\lambda}) = \boldsymbol{\Theta}$ and the set of equivalence classes $[\mathscr{M}, \boldsymbol{\sigma}]$ of supercuspidal pairs in $\mathscr{G}$ (see Proposition 10.7); it might be possible to prove this directly but in fact we deduce it as a consequence of our block decomposition of $\mathscr{R}(\mathrm{G})$.

6.4.

We continue with a semisimple supertype $(\mathbf{J}, \boldsymbol{\lambda})$ and all the notation of the previous section, making a choice of decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$. In particular we have Levi subgroups $\mathrm{M} \subseteq \mathrm{L} \subseteq \mathrm{G}$; a decomposed pair $(\mathbf{J}_{\max}, \boldsymbol{\kappa}_{\max})$ in G of $(\mathrm{J}_{\max,\mu}, \kappa_{\max,\mu})$ in L; a pair $(\mathrm{J}_\alpha, \kappa_\alpha)$ in M; and a Levi subgroup $\mathscr{M}$ of $\mathscr{G}$. This gives us functions:

$$\mathbf{K} = \mathbf{K}_{\boldsymbol{\kappa}_{\max}} : \mathscr{R}(\mathrm{G}) \to \mathscr{R}(\mathscr{G}),$$
$$\mathbf{K}_{\mathrm{L}} = \mathbf{K}_{\kappa_{\max,\mu}} : \mathscr{R}(\mathrm{L}) \to \mathscr{R}(\mathscr{G}),$$
$$\mathbf{K}_{\mathrm{M}} = \mathbf{K}_{\kappa_{\alpha}} : \mathscr{R}(\mathrm{M}) \to \mathscr{R}(\mathscr{M}),$$

using the notation of §3. Denote by $\mathrm{Q} = \mathrm{LU}$ the standard parabolic subgroup of G with Levi component L, and by $\mathscr{P}$ the standard parabolic subgroup of $\mathscr{G}$ with Levi component $\mathscr{M}$.

Theorem 6.2. — *For any smooth representation π of* M, *one has:*

$$\mathbf{K}(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi)) \simeq \mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}(\mathbf{K}_{\mathrm{M}}(\pi)).$$

Proof. — First note that it is enough to prove the result when M = L. Indeed, assuming that the theorem is true for M = L, we set $\pi_0 = \mathrm{Ind}_{\mathrm{P}\cap\mathrm{L}}^{\mathrm{L}}(\pi)$ and get:

$$\mathbf{K}(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi)) \simeq \mathbf{K}(\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{G}}(\pi_0)) \simeq \mathbf{K}_{\mathrm{L}}(\mathrm{Ind}_{\mathrm{P}\cap\mathrm{L}}^{\mathrm{L}}(\pi))$$

and the latter representation of $\mathscr{G}$ is isomorphic to $\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}(\mathbf{K}_{\mathrm{M}}(\pi))$ thanks to Proposition 5.6.

Assume now that M = L. Given $\pi \in \mathscr{R}(\mathrm{L})$, by Lemma 3.1, we have an isomorphism:

$$\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{Q}\mathbf{J}_{\max}}(\pi) \simeq \mathrm{Ind}_{\mathbf{J}_{\max}\cap\mathrm{Q}}^{\mathbf{J}_{\max}}(\pi)$$

of representations of $\mathbf{J}_{\max}$. Since $\mathbf{J}_{\max} = \mathbf{J}_{\max}^1(\mathbf{J}_{\max} \cap \mathrm{Q})$, we get:

$$\mathbf{K}(\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{Q}\mathbf{J}_{\max}}(\pi)) \simeq \mathrm{Hom}_{\mathbf{J}_{\max}^1\cap\mathrm{Q}}(\boldsymbol{\kappa}|_{\mathbf{J}_{\max}\cap\mathrm{Q}}, \pi) \simeq \mathrm{Hom}_{\mathrm{J}_{\max,\mu}^1}(\kappa_{\max,\mu}, \pi) \tag{6.2}$$

which is $\mathbf{K}_{\mathrm{L}}(\pi)$. Therefore it is enough to prove that:

$$\mathbf{K}(\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{G}}(\pi)) = \mathbf{K}(\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{Q}\mathbf{J}_{\max}}(\pi)) \tag{6.3}$$

for all smooth representations π of L.

First assume R is the field of complex numbers and π is irreducible. Define a representation V of $\mathscr{G}$ by the following exact sequence:

$$0 \to \mathbf{K}(\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{Q}\mathbf{J}_{\max}}(\pi)) \xrightarrow{\iota} \mathbf{K}(\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{G}}(\pi)) \to \mathrm{V} \to 0 \tag{6.4}$$

of representations of $\mathscr{G}$, where ι is the inclusion map, and assume that V is nonzero. Then it has an irreducible subquotient, with some supercuspidal support $(\mathscr{M}', \boldsymbol{\sigma}')$. Let $\mathscr{P}'$ be the standard parabolic subgroup of $\mathscr{G}$ with Levi component $\mathscr{M}'$ and write $\mathscr{N}'$ for its unipotent radical. There is a standard parabolic subgroup $\mathrm{P}' = \mathrm{M}'\mathrm{N}'$ of G contained in Q, having the following property: the intersection $\mathrm{P}' \cap \mathrm{L} = \mathrm{M}'(\mathrm{N}' \cap \mathrm{L})$ is a parabolic subgroup of L such that:

$$(\mathrm{U}(\Lambda_{\max}) \cap \mathrm{B}^{\times} \cap \mathrm{N}' \cap \mathrm{L})(\mathrm{U}^1(\Lambda_{\max}) \cap \mathrm{B}^{\times})/(\mathrm{U}^1(\Lambda_{\max}) \cap \mathrm{B}^{\times}) = \mathscr{N}'.$$

Let $[\Lambda', n', 0, \beta]$ be a simple stratum such that:

(i) the image of $\mathrm{U}^1(\Lambda') \cap \mathrm{B}^{\times} \cap \mathrm{L}$ in $\mathscr{G}$ is $\mathscr{N}'$;

(ii) $\mathrm{U}(\Lambda') \cap \mathrm{L} \subseteq \mathrm{U}(\Lambda_{\max}) \cap \mathrm{L}$ and $\mathrm{U}(\Lambda') \cap \mathrm{N}' \cap \mathrm{L} = \mathrm{U}(\Lambda_{\max}) \cap \mathrm{N}' \cap \mathrm{L}$ as in Lemma 5.1.

(Note that this makes sense because it is happening in L, where we just have a direct sum of simple strata so we can do it separately in each block of L and then take the sum.)

By using (5.2) and (5.3) in L, there is an irreducible representation $\kappa_{\mathrm{P}'\cap\mathrm{L}}$ of a group $\mathrm{J}_{\mathrm{P}'\cap\mathrm{L}}$ which is compatible with $\kappa_{\max,\mu}$, that is, we have an isomorphism:

$$\mathrm{Ind}_{\mathrm{J}_{\mathrm{P}'\cap\mathrm{L}}}^{(\mathrm{U}(\Lambda')\cap\mathrm{B}^{\times}\cap\mathrm{L})(\mathrm{U}^1(\Lambda')\cap\mathrm{L})}(\kappa_{\mathrm{P}'\cap\mathrm{L}}) \simeq \mathrm{Ind}_{(\mathrm{U}(\Lambda')\cap\mathrm{B}^{\times}\cap\mathrm{L})\mathrm{J}_{\max,\mu}^1}^{(\mathrm{U}(\Lambda')\cap\mathrm{B}^{\times}\cap\mathrm{L})(\mathrm{U}^1(\Lambda')\cap\mathrm{L})}(\kappa_{\max,\mu}),$$

and these induced representations are irreducible. In particular, by the Mackey formula, there is an element $g \in (\mathrm{U}(\Lambda') \cap \mathrm{B}^\times \cap \mathrm{L})(\mathrm{U}^1(\Lambda') \cap \mathrm{L})$ that intertwines $\kappa_{\mathrm{P}'\cap\mathrm{L}}$ with $\kappa_{\max,\mu}$. Moreover, the representation $\kappa_{\mathrm{P}'\cap\mathrm{L}}$ is decomposed above the restriction of $\kappa_{\max,\mu}$ to $\mathrm{J}_{\mathrm{P}'\cap\mathrm{L}} \cap \mathrm{L}$, denoted κ_{L}, which is a maximal β-extension of J_{L} in L.

By [**15**, Proposition 2.33], we get a representation $\boldsymbol{\kappa}'$ of a compact open subgroup $\mathbf{J}'$ which is decomposed above $\kappa_{\mathrm{P}'\cap\mathrm{L}}$ in G, so also above $(\mathrm{J}_{\mathrm{L}}, \kappa_{\mathrm{L}})$.

Lemma 6.3 **(cf.** [**6**, Proposition 6.3]**)**. — *For $i = 1, 2$, let K_i be a subgroup of* G *with an Iwahori decomposition with respect to* (L, Q)*, and let ρ_i be an irreducible representation of K_i which is trivial on* U *and* U^-*. Then, for $g \in \mathrm{L}$, we have:*

$$\mathrm{Hom}_{\mathrm{K}_1\cap(\mathrm{K}_2)^g}(\rho_1, (\rho_2)^g) = \mathrm{Hom}_{(\mathrm{K}_1\cap\mathrm{L})\cap(\mathrm{K}_2\cap\mathrm{L})^g}(\rho_1, (\rho_2)^g).$$

Proof. — One inclusion is obvious and the other follows from the fact that $\mathrm{K}_1 \cap (\mathrm{K}_2)^g$ has an Iwahori decomposition with respect to (L, Q). □

Applying this lemma with $\boldsymbol{\kappa}'$ and the restriction of $\boldsymbol{\kappa}_{\max}$ to $(\mathrm{U}(\Lambda') \cap \mathrm{B}^\times)\mathbf{J}^1_{\max,\mu}$, we see that g intertwines these two representations. Thus, by Mackey, there is a non-zero morphism:

$$\mathrm{Ind}_{\mathbf{J}'}^{(\mathrm{U}(\Lambda')\cap\mathrm{B}^\times\cap\mathrm{L})(\mathrm{U}^1(\Lambda')\cap\mathrm{L})}(\boldsymbol{\kappa}') \to \mathrm{Ind}_{(\mathrm{U}(\Lambda')\cap\mathrm{B}^\times)\mathbf{J}^1_{\max}}^{(\mathrm{U}(\Lambda')\cap\mathrm{B}^\times\cap\mathrm{L})(\mathrm{U}^1(\Lambda')\cap\mathrm{L})}(\boldsymbol{\kappa}_{\max}).$$

Moreover, the intertwining formula given by [**15**, Proposition 2.31] (together with an analogue of [**15**, Lemme 2.2]) implies that both of these representations are irreducible. Thus they are isomorphic, and we have a compatibility property analogous to (5.2).

We now go back to (6.4). By taking the $\mathscr{N}'$-fixed vectors and then the $\boldsymbol{\sigma}'$-isotypic component, and thanks to (6.2), we get an exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{J}^1_{\max,\mu}}(\kappa_{\max,\mu}, \pi)^{\mathscr{N}',\boldsymbol{\sigma}'} \to \mathrm{Hom}_{\mathbf{J}^1_{\max}}(\boldsymbol{\kappa}_{\max}, \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi))^{\mathscr{N}',\boldsymbol{\sigma}'} \to \mathrm{V}^{\mathscr{N}',\boldsymbol{\sigma}'} \to 0$$

of complex vector spaces, which are finite-dimensional since π is admissible. Now

$$\begin{aligned}\mathrm{Hom}_{\mathbf{J}^1_{\max}}(\boldsymbol{\kappa}_{\max}, \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi))^{\mathscr{N}',\boldsymbol{\sigma}'} &\simeq \mathrm{Hom}_{(\mathrm{U}^1(\Lambda')\cap\mathrm{B}^\times)\mathbf{J}^1_{\max}}(\boldsymbol{\kappa}_{\max}, \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi))^{\boldsymbol{\sigma}'}\\ &\simeq \mathrm{Hom}_{(\mathrm{U}(\Lambda')\cap\mathrm{B}^\times)\mathbf{J}^1_{\max}}(\boldsymbol{\kappa}_{\max}\otimes\boldsymbol{\sigma}', \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi))\\ &\simeq \mathrm{Hom}_{\mathbf{J}'}(\boldsymbol{\kappa}'\otimes\boldsymbol{\sigma}', \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi)),\end{aligned}$$

where $\boldsymbol{\kappa}'$ is compatible with $\boldsymbol{\kappa}_{\max}$ as above. Similarly, we have

$$\mathrm{Hom}_{\mathrm{J}^1_{\max,\mu}}(\kappa_{\max,\mu}, \pi)^{\mathscr{N}',\boldsymbol{\sigma}'} \simeq \mathrm{Hom}_{\mathrm{J}_{\mathrm{P}'\cap\mathrm{L}}}(\kappa_{\mathrm{P}'\cap\mathrm{L}}\otimes\boldsymbol{\sigma}', \pi)$$

Now, by [**22**], the semisimple type $\boldsymbol{\lambda}' = \boldsymbol{\kappa}'\otimes\boldsymbol{\sigma}'$ is a cover of $\kappa_{\mathrm{P}'\cap\mathrm{L}}\otimes\boldsymbol{\sigma}'$, which is itself a cover of $\kappa_{\mathrm{L}}\otimes\boldsymbol{\sigma}'$. Thus the algebra $\mathcal{H} = \mathrm{End}_{\mathrm{G}}(\mathrm{ind}_{\mathbf{J}'}^{\mathrm{G}}(\boldsymbol{\kappa}'\otimes\boldsymbol{\sigma}')$ is a free module of rank 1 over:

$$\mathcal{H}_{\mathrm{L}} = \mathrm{End}_{\mathrm{L}}(\mathrm{ind}_{\mathrm{J}_{\mathrm{P}'\cap\mathrm{L}}}^{\mathrm{L}}(\kappa_{\mathrm{P}'\cap\mathrm{L}}\otimes\boldsymbol{\sigma}'))$$

(see [**15**, Corollaire 2.32]) and there is an isomorphism of $\mathcal{H}$-modules

$$\mathrm{Hom}_{\mathbf{J}'}(\boldsymbol{\kappa}'\otimes\boldsymbol{\sigma}', \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\pi)) \simeq \mathrm{Hom}_{\mathcal{H}_{\mathrm{L}}}(\mathcal{H}, \mathrm{Hom}_{\mathrm{J}_{\mathrm{P}'\cap\mathrm{L}}}(\kappa_{\mathrm{P}'\cap\mathrm{L}}\otimes\boldsymbol{\sigma}', \pi)).$$

Since these are finite-dimensional, we deduce that $\mathrm{V}^{\mathscr{N}',\boldsymbol{\sigma}'} = 0$, a contradiction.

We deduce from Proposition 3.2 that, for $g \in \mathrm{G}$, we have:

$$\mathrm{Hom}_{\mathbf{J}^1_{\max}\cap\mathrm{U}^g}(\boldsymbol{\kappa}_{\max}, 1) \neq 0 \quad\Leftrightarrow\quad g \in \mathrm{P}\mathbf{J}_{\max}. \tag{6.5}$$

As $\mathbf{J}^1_{\max}$ is a pro-p-group, (6.5) also holds when R has positive characteristic. Thus, by Proposition 3.2 again, the equality (6.3) holds for all smooth R-representations π of L. □

§7. A semisimple computation

As in Section 6, the notation of which we use, $(\mathbf{J},\boldsymbol{\lambda})$ is a semisimple supertype of G. We fix a decomposition $\boldsymbol{\lambda}=\boldsymbol{\kappa}\otimes\boldsymbol{\sigma}$ and write $\mathsf{K}=\mathsf{K}_{\boldsymbol{\kappa}_{\max}}$ and $[\mathscr{M},\boldsymbol{\sigma}]$ for the functor and the equivalence class of supercuspidal pairs associated with it.

Proposition 7.1. — *Every irreducible subquotient of* $\mathsf{K}(\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}))$ *has its supercuspidal support in* $[\mathscr{M},\boldsymbol{\sigma}]$.

Proof. — Assume first that $(\mathbf{J},\boldsymbol{\lambda})$ is a maximal simple type. Then $r=l=1$ and we have:

$$\mathsf{K}(\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}))\simeq\bigoplus_{\mathbf{J}\backslash\mathrm{G}/\mathbf{J}}\mathsf{K}(\mathrm{ind}_{\mathbf{J}\cap\mathbf{J}^g}^{\mathbf{J}}(\boldsymbol{\lambda}^g)).$$

By reciprocity, one see that the $g\in\mathrm{G}$ that contribute to this sum intertwine $\boldsymbol{\eta}$. Therefore one may assume that they are in $\mathrm{B}^\times$. Since $\mathbf{J}\cap\mathrm{B}^\times$ is a maximal compact open subgroup in $\mathrm{B}^\times$, by the Cartan decomposition one may assume that the g that contribute are diagonal matrices in $\mathrm{B}^\times$. As $\boldsymbol{\sigma}$ is cuspidal, only those g which normalize $\mathbf{J}\cap\mathrm{B}^\times$ contribute to this sum. Fix $\varpi\in\mathrm{B}^\times$ such that the $\mathrm{B}^\times$-normalizer of $\mathbf{J}\cap\mathrm{B}^\times$ is generated by $\mathbf{J}\cap\mathrm{B}^\times$ and ϖ. We get:

$$\mathsf{K}(\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}))\simeq\bigoplus_{n\in\mathbf{Z}}\mathsf{K}(\boldsymbol{\lambda}^{\varpi^n})=\bigoplus_{\mathbf{Z}}(\boldsymbol{\sigma}\oplus\boldsymbol{\sigma}^{\phi}\oplus\cdots\oplus\boldsymbol{\sigma}^{\phi^{b-1}})=\bigoplus_{\mathbf{Z}}\bigoplus_{j=0}^{b-1}\boldsymbol{\sigma}^{\phi^j},$$

where ϕ is a generator of $\mathrm{Gal}(\mathfrak{k}_{\mathrm{D}'}/\mathfrak{k}_{\mathrm{E}})$ and b is the cardinality of the $\mathrm{Gal}(\mathfrak{k}_{\mathrm{D}'}/\mathfrak{k}_{\mathrm{E}})$-orbit of $\boldsymbol{\sigma}$ (see [**15**, Lemme 5.3])

We treat the general case. Recall that we have the standard parabolic subgroup P of G, with standard Levi component M. We have an isomorphism:

$$\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})\simeq\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathrm{ind}_{\mathbf{J}\cap\mathrm{M}}^{\mathrm{M}}(\lambda_\alpha)).$$

As K commutes with parabolic induction (see Theorem 6.2), we get:

$$\begin{aligned}\mathsf{K}(\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}))&\simeq\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}\big(\mathsf{K}_{\mathrm{M}}(\mathrm{ind}_{\mathbf{J}\cap\mathrm{M}}^{\mathrm{M}}(\lambda_\alpha))\big)\\&\simeq\mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}\big(\mathsf{K}_1(\mathrm{ind}_{\mathrm{J}_1}^{\mathrm{G}_{m_1}}(\lambda_1))\otimes\cdots\otimes\mathsf{K}_r(\mathrm{ind}_{\mathrm{J}_r}^{\mathrm{G}_{m_r}}(\lambda_r))\big)\end{aligned}$$

where we have $\mathsf{K}_i=\mathsf{K}_{\kappa_i}$. For each $i\in\{1,\dots,r\}$ we have:

$$\mathsf{K}_i(\mathrm{ind}_{\mathrm{J}_i}^{\mathrm{G}_i}(\lambda_i))\simeq\bigoplus_{\mathbf{Z}}\bigoplus_{j=0}^{b_i-1}\sigma_i^{\phi_i^j},$$

where ϕ_i is a generator of $\Gamma_i=\mathrm{Gal}(\mathfrak{k}_{\mathrm{D}'_i}/\mathfrak{k}_{\mathrm{F}[\beta_i]})$ and b_i is the cardinality of the orbit of σ_i under Γ_i. Thus:

$$\mathsf{K}(\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}))\simeq\bigoplus_{\mathbf{Z}^r}\bigoplus_{\boldsymbol{j}}(\sigma_1^{\phi_1^{j_1}}\times\cdots\times\sigma_r^{\phi_r^{j_r}})$$

where $\boldsymbol{j}$ ranges over the r-tuples $(j_1,\dots,j_r)$ with $j_i\in\{0,\dots,b_i-1\}$ for all $i\in\{1,\dots,r\}$, and where $\times$ stands for parabolic induction. The result follows by unicity of supercuspidal support in $\mathscr{G}$. □

§8. Supercuspidal inertial classes and supertypes

Given $(\mathbf{J},\boldsymbol{\lambda})$ a semisimple supertype of G, write $\mathrm{Irr}(\mathbf{J},\boldsymbol{\lambda})$ for the set of all classes of irreducible subquotients of $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$.

Given Ω an inertial class of supercuspidal pairs of G, write $\mathrm{Irr}(\Omega)$ for the set of all classes of irreducible representations of G having their supercuspidal support in Ω.

Proposition 8.1. — *Let* (M,ϱ) *be a supercuspidal pair of* G *and* $(\mathbf{J},\boldsymbol{\lambda})$ *be a semisimple supertype of* G *associated with a maximal simple type* $(\mathrm{J}_\alpha,\lambda_\alpha)$ *of* M *contained in* ϱ. *Write* Ω *for the inertial class of* (M,ϱ). *Then we have* $\mathrm{Irr}(\Omega)=\mathrm{Irr}(\mathbf{J},\boldsymbol{\lambda})$.

Proof. — We begin by proving the containment $\mathrm{Irr}(\Omega)\subseteq\mathrm{Irr}(\mathbf{J},\boldsymbol{\lambda})$. Assume M is standard and write $\varrho=\rho_1\otimes\cdots\otimes\rho_r$, where ρ_i is a supercuspidal irreducible representation of G_{m_i} for $m_i\geqslant 1$. For $i\in\{1,\dots,r\}$, fix an unramified character χ_i of G_{m_i}. Then $\rho_i\chi_i$ is a quotient of the compact induction of λ_i to G_{m_i}. It follows that $\rho_1\chi_1\times\cdots\times\rho_r\chi_r$ is a quotient of:

$$\mathrm{ind}_{\mathrm{J}_1}^{\mathrm{G}_{m_1}}(\lambda_1)\times\cdots\times\mathrm{ind}_{\mathrm{J}_1}^{\mathrm{G}_{m_r}}(\lambda_r)\simeq\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}). \tag{8.1}$$

Thus any irreducible subquotient of $\rho_1\chi_1\times\cdots\times\rho_r\chi_r$ appears in $\mathrm{Irr}(\mathbf{J},\boldsymbol{\lambda})$.

For the opposite containment, we need the following lemma.

Lemma 8.2. — *Let* Ω *and* $(\mathbf{J},\boldsymbol{\lambda})$ *be as in Proposition 8.1, and assume that* $\mathrm{Irr}(\mathbf{J},\boldsymbol{\lambda})$ *contains a cuspidal representation* π. *Then we have* $\pi\in\mathrm{Irr}(\Omega)$.

Proof. — Let (J_0,λ_0) be a maximal simple type of G contained in π. It is attached to a simple stratum $[\Lambda_0,n_0,0,\beta_0]$ and we write θ_0 for the simple character occurring in the restriction of λ_0 to $\mathrm{H}_0^1=\mathrm{H}^1(\beta_0,\Lambda_0)$. This character occurs as a subquotient (hence a subrepresentation since H_0^1 is a pro-p group) of the restriction of $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$ to H_0^1. Recall that we have an isomorphism (8.1) and that the compact induction of λ_i to G_{m_i} is isomorphic to

$$\rho_i\otimes\mathrm{R}[\mathrm{X},\mathrm{X}^{-1}],$$

with G_{m_i} acting on $\mathrm{R}[\mathrm{X},\mathrm{X}^{-1}]$ by $g\cdot\mathrm{X}^k=\mathrm{X}^{k+v(g)}$, for all $k\in\mathbf{Z}$, where $v(g)$ is the valuation of the reduced norm of $g\in\mathrm{G}_{m_i}$. Therefore, when restricting (8.1) to H_0^1, we deduce that θ_0 occurs as a subrepresentation of

$$\bigoplus_{\mathbf{Z}^r}(\rho_1\times\cdots\times\rho_r).$$

Thus θ_0 occurs as a subrepresentation of $\rho_1\times\cdots\times\rho_r$, and it follows from [**15**, Proposition 5.6] that the sum $\boldsymbol{\Theta}=\boldsymbol{\Theta}(\mathbf{J},\boldsymbol{\lambda})$ is equal to

$$\boldsymbol{\Theta}(\mathrm{J}_0,\lambda_0)=\frac{md}{[\mathrm{F}[\beta_0]:\mathrm{F}]}\cdot\boldsymbol{\Theta}_0,$$

where $\boldsymbol{\Theta}_0$ is the endo-class of π. We thus are in the homogeneous situation of Section 6.2 so that a decomposition $\boldsymbol{\lambda}=\boldsymbol{\kappa}\otimes\boldsymbol{\sigma}$ is determined by a pair $(\mathrm{J}_{\max},\kappa_{\max})$. Then the simple character $\theta_{\max}$ contained in $\kappa_{\max}$ is the transfer of the simple character θ_0 in λ_0.

We fix a decomposition $\lambda_0=\kappa_0\otimes\sigma_0$ and write $\mathsf{K}_0=\mathsf{K}_{\kappa_0}$. By [**3**], the characters θ_0 and $\theta_{\max}$ are in fact conjugate and, replacing the pair $(\mathbf{J},\boldsymbol{\lambda})$ by a suitable G-conjugate, we may assume that the pairs $(\mathrm{J}_{\max},\kappa_{\max})$ and (J_0,κ_0) coincide. Thus the functor $\mathsf{K}=\mathsf{K}_{\kappa_{\max}}$ of section 6.2 coincides with K_0. This also induces a decomposition $\lambda_i=\kappa_i\otimes\sigma_i$ for all $i\in\{1,\dots,r\}$.

We now apply this functor to the subquotient π of $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$. By [**15**, Lemme 5.3], the representation $\mathsf{K}(\pi)$ is a sum of cuspidal irreducible representations of $\mathscr{G}=\mathrm{GL}_{m'}(\mathfrak{k}_{\mathrm{D}'})$. By Proposition

7.1, these cuspidal representations have their supercuspidal support in $[\mathscr{M},\boldsymbol{\sigma}]$. By the classification of cuspidal irreducible representations of $\mathscr{G}$ in terms of supercuspidal representations (see for instance [**16**, Proposition 3.7]), there is a supercuspidal representation σ of $\mathrm{GL}_{m'/r}(\mathfrak{k}_{\mathrm{D}'})$ such that

$$\sigma_i = \sigma^{\gamma_i}, \quad \gamma_i \in \mathrm{Gal}(\mathfrak{k}_{\mathrm{D}'}/\mathfrak{k}_{\mathrm{F}[\beta_0]}), \quad i \in \{1,\dots,r\},$$

and an integer $u \geqslant 0$ such that we have $r = e(\sigma)\ell^u$, where $e(\sigma)$ is a positive integer attached to σ (see [**16**, Remarque 3.6]). Since $\kappa_i \otimes \sigma$ can be obtained from λ_i by conjugacy in G_{m_i}, we may assume without changing $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$ that we have:

$$\boldsymbol{\Theta}_i = \cdots = \boldsymbol{\Theta}_r = \boldsymbol{\Theta}_0, \quad \sigma_1 = \cdots = \sigma_r = \sigma.$$

By [**15**, Corollaire 5.5], it follows that $\rho_1,\dots,\rho_r$ are inertially equivalent to a given supercuspidal representation ρ. It also follows from [**16**, §6] that π is inertially equivalent to $\mathrm{St}(\rho,r)$, the unique cuspidal irreducible subquotient of the product $\rho \times \rho\nu_\rho \times \cdots \times \rho\nu_\rho^{r-1}$ (where ν_ρ is the unramified character associated with ρ in [**15**, §4.5]). It follows that the supercuspidal pair (M,ϱ) is inertially equivalent to $(\mathrm{M},\rho\otimes\cdots\otimes\rho)$ and that π appears in $\mathrm{Irr}(\Omega)$. □

We return to the proof of Proposition 8.1. Let π be an irreducible subquotient of $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$, and let (L,τ) be its cuspidal support. Write:

$$\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}) \simeq \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\mathrm{ind}_{\mathrm{J}_\alpha}^{\mathrm{L}}(\lambda_\alpha)) = \mathrm{ind}_{\mathrm{J}_1}^{\mathrm{G}_{m_1}}(\lambda_1) \times \cdots \times \mathrm{ind}_{\mathrm{J}_r}^{\mathrm{G}_{m_r}}(\lambda_r).$$

For $i \in \{1,\dots,r\}$, note that $\Pi_i = \mathrm{ind}_{\mathrm{J}_i}^{\mathrm{G}_{m_i}}(\lambda_i)$ is made of supercuspidal irreducible subquotients all of whose are unramified twists of a given supercuspidal irreducible representation ρ_i of G_{m_i}. Let $\mathrm{Q} = \mathrm{LU}$ be a parabolic subgroup of G with Levi component L. We compute the Jacquet module $(\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}))_{\mathrm{U}}$. Since it contains π_{U}, it contains an irreducible cuspidal subquotient which is G-conjugate to τ. By the geometric lemma, there are a permutation w of $\{1,\dots,r\}$ and integers $0 = a_0 < a_1 < \cdots < a_t = r$ such that, if we write $\tau = \tau_1 \otimes \cdots \otimes \tau_t$ with τ_j cuspidal, then τ_j appears, for each $j \in \{1,\dots,t\}$, as a subquotient of:

$$\Sigma_j = \Pi_{w(a_{j-1}+1)} \times \cdots \times \Pi_{w(a_j)}.$$

It follows from Lemma 8.2 that τ_j has its supercuspidal support in Ω_j, the inertial class of the supercuspidal pair:

$$(\mathrm{G}_{w(a_{j-1}+1)} \times \cdots \times \mathrm{G}_{w(a_j)}, \rho_{w(a_{j-1}+1)} \otimes \cdots \otimes \rho_{w(a_j)}).$$

It follows that π has its supercuspidal support in Ω, as required. □

Proposition 8.3. — *Let* $(\mathbf{J},\boldsymbol{\lambda})$ *and* $(\mathbf{J}',\boldsymbol{\lambda}')$ *be semisimple supertypes of* G. *The representations* $\mathrm{ind}_{\mathbf{J}'}^{\mathrm{G}}(\boldsymbol{\lambda}')$, $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$ *have an irreducible subquotient in common if and only if* $[\mathbf{J},\boldsymbol{\lambda}] = [\mathbf{J}',\boldsymbol{\lambda}']$.

Proof. — Since the $\mathrm{Irr}(\Omega)$ form a partition of the set of all isomorphism classes of irreducible representations of G, it follows from Proposition 8.1 that $\mathrm{ind}_{\mathbf{J}'}^{\mathrm{G}}(\boldsymbol{\lambda}')$, $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$ have an irreducible subquotient in common if and only if $\mathrm{Irr}(\mathbf{J},\boldsymbol{\lambda}) = \mathrm{Irr}(\mathbf{J}',\boldsymbol{\lambda}')$.

Suppose that $\mathrm{Irr}(\mathbf{J},\boldsymbol{\lambda}) = \mathrm{Irr}(\mathbf{J}',\boldsymbol{\lambda}') = \mathrm{Irr}(\Omega)$, with $\Omega = [\mathrm{M},\varrho]_{\mathrm{G}}$. If $\mathrm{M} = \mathrm{G}$ then, by following the proof of Lemma 8.2, we find that $(\mathbf{J},\boldsymbol{\lambda})$ and $(\mathbf{J}',\boldsymbol{\lambda}')$ are both equivalent to maximal simple supertypes; by unicity (up to conjugacy) of maximal simple supertypes in a supercuspidal representation (see [**15**, Théorème 3.11] and [**16**, Proposition 6.10]), we deduce that $[\mathbf{J},\boldsymbol{\lambda}] = [\mathbf{J}',\boldsymbol{\lambda}']$. In the general case, we have

$$\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}) \simeq \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{G}}(\mathrm{ind}_{\mathrm{J}_\alpha}^{\mathrm{M}}(\lambda_\alpha)) \simeq \mathrm{Ind}_{\mathrm{Q}}^{\mathrm{G}}(\mathrm{ind}_{\mathrm{J}'_\alpha}^{\mathrm{M}}(\lambda'_\alpha)) \simeq \mathrm{ind}_{\mathbf{J}'}^{\mathrm{G}}(\boldsymbol{\lambda}'),$$

where the middle isomorphism follows from the previous case. □

It also follows that there is a bijection:

$$\Omega \leftrightarrow [\mathbf{J}, \boldsymbol{\lambda}] \tag{8.2}$$

between inertial classes of supercuspidal pairs of G and equivalence classes of semisimple supertypes of G, characterized by the equality $\mathrm{Irr}(\Omega) = \mathrm{Irr}(\mathbf{J}, \boldsymbol{\lambda})$.

§9. Splitting of the category

Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a semisimple supertype of G, together with a decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$. Associated with it, there are a formal sum $\boldsymbol{\Theta}$ of endo-classes, a functor $\mathbf{K} = \mathbf{K}_{\boldsymbol{\kappa}_{\max}}$ and the group $\mathscr{G} = \mathbf{J}_{\max}/\mathbf{J}^1_{\max}$.

9.1.

We now fix $\boldsymbol{\Theta}$ and $\mathbf{K}$, and make $[\mathscr{M}, \boldsymbol{\sigma}]$ vary among the equivalence classes of supercuspidal pairs of $\mathscr{G}$. By Corollary 1.15, we have, for all $\mathrm{V} \in \mathscr{R}(\mathrm{G})$, a decomposition:

$$\mathbf{K}(\mathrm{V}) = \bigoplus_{[\mathscr{M}, \boldsymbol{\sigma}]} \mathrm{V}(\boldsymbol{\Theta}, \boldsymbol{\sigma}), \tag{9.1}$$

where $\mathrm{V}(\boldsymbol{\Theta}, \boldsymbol{\sigma})$ is the maximal subspace of $\mathbf{K}(\mathrm{V})$ all of whose composition factors have supercuspidal support in $[\mathscr{M}, \boldsymbol{\sigma}]$.

Definition 9.1. — Given $\mathrm{V} \in \mathscr{R}(\mathrm{G})$ a smooth representation, we write:
(i) $\mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$ for the G-subspace of V generated by $\mathrm{V}(\boldsymbol{\Theta}, \boldsymbol{\sigma})$;
(ii) $\mathrm{V}[\boldsymbol{\Theta}]$ for the G-subspace of V generated by $\mathbf{K}(\mathrm{V})$.

Thus $\mathrm{V}[\boldsymbol{\Theta}]$ is the sum of all the $\mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$, as $[\mathscr{M}, \boldsymbol{\sigma}]$ ranges over the set of equivalence classes of supercuspidal pairs of $\mathscr{G}$. We claim that $\mathrm{V}[\boldsymbol{\Theta}]$ is in fact the direct sum of the $\mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$.

Lemma 9.2. — *Given $[\mathscr{M}, \boldsymbol{\sigma}]$, $[\mathscr{M}', \boldsymbol{\sigma}']$ equivalence classes of supercuspidal pairs of $\mathscr{G}$, we have:*

$$\mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}](\boldsymbol{\Theta}, \boldsymbol{\sigma}') = \begin{cases} \mathrm{V}(\boldsymbol{\Theta}, \boldsymbol{\sigma}) & \textit{if } [\mathscr{M}', \boldsymbol{\sigma}'] = [\mathscr{M}, \boldsymbol{\sigma}]; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. — We have the containment $\mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}](\boldsymbol{\Theta}, \boldsymbol{\sigma}) \subseteq \mathrm{V}(\boldsymbol{\Theta}, \boldsymbol{\sigma})$. Since $\mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$ contains $\mathrm{V}(\boldsymbol{\Theta}, \boldsymbol{\sigma})$, this containment is an equality. Write $\mathbf{T}$ for the functor $\xi \mapsto \mathbf{K}(\mathrm{ind}_{\mathbf{J}_{\max}}^{\mathrm{G}}(\boldsymbol{\kappa}_{\max} \otimes \xi))$. We have a surjective map:

$$\mathrm{ind}_{\mathbf{J}_{\max}}^{\mathrm{G}}(\boldsymbol{\kappa}_{\max} \otimes \mathrm{V}(\boldsymbol{\Theta}, \boldsymbol{\sigma})) \to \mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$$

thus a surjective map:

$$\mathbf{T}(\mathrm{V}(\boldsymbol{\Theta}, \boldsymbol{\sigma})) \to \mathbf{K}(\mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}]).$$

To prove the remaining part of the lemma, it is enough to prove that any irreducible subquotient of the left hand side has supercuspidal support in $[\mathscr{M}, \boldsymbol{\sigma}]$. As $\mathbf{T}$ is exact, it is enough to prove that, for all irreducible representation ξ with supercuspidal support in $[\mathscr{M}, \boldsymbol{\sigma}]$, any irreducible subquotients of $\mathbf{T}(\xi)$ has supercuspidal support in $[\mathscr{M}, \boldsymbol{\sigma}]$. By the same exactness argument, it is enough to prove the following lemma.

Lemma 9.3. — *Let $(\mathscr{M}', \boldsymbol{\sigma}') \in [\mathscr{M}, \boldsymbol{\sigma}]$ and $\mathrm{X} = \mathrm{Ind}_{\mathscr{M}'}^{\mathscr{G}}(\boldsymbol{\sigma}')$. Then all irreducible subquotients of $\mathbf{T}(\mathrm{X})$ have supercuspidal support in $[\mathscr{M}, \boldsymbol{\sigma}]$.*

Proof. — We may and will assume that $\mathscr{M}' = \mathscr{M}$. We see $\boldsymbol{\sigma}'$ as a representation of $\mathbf{J}$ trivial on $\mathbf{J}^1$ and write $\boldsymbol{\lambda}'$ for the semisimple supertype $\boldsymbol{\kappa} \otimes \boldsymbol{\sigma}'$. Then we have:

$$\mathrm{ind}_{\mathbf{J}_{\max}}^{\mathrm{G}}(\boldsymbol{\kappa}_{\max} \otimes \mathrm{X}) \simeq \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\kappa} \otimes \boldsymbol{\sigma}') = \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}').$$

Then the lemma follows from Proposition 7.1. □

This ends the proof of Lemma 9.2. □

As a corollary, we have the following result.

Corollary 9.4. — *For all smooth representations* V *of* G*, we have:*

$$\mathrm{V}[\boldsymbol{\Theta}] = \bigoplus_{[\mathscr{M},\boldsymbol{\sigma}]} \mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}].$$

Remark 9.5. — Note that, given $\mathrm{V} \in \mathscr{R}(\mathrm{G})$, the subrepresentation $\mathrm{V}[\boldsymbol{\Theta}]$ does not depend on the choice of the functor K; a different choice of $\boldsymbol{\kappa}$ simply permutes the equivalence classes of supercuspidal pairs $[\mathscr{M}, \boldsymbol{\sigma}]$ so permutes the terms $\mathrm{V}[\boldsymbol{\Theta}, \boldsymbol{\sigma}]$ in $\mathrm{V}[\boldsymbol{\Theta}]$.

9.2.

We now make $\boldsymbol{\Theta}$ vary among all possible formal sums of endo-classes arising from a semisimple supertype of G.

Theorem 9.6. — *For all smooth representation* V *of* G*, there is an isomorphism:*

$$\mathrm{V} \simeq \bigoplus_{\boldsymbol{\Theta}} \mathrm{V}[\boldsymbol{\Theta}]$$

of representations of G.

Proof. — Let V be a smooth representation of G. We have a morphism:

$$f : \bigoplus_{\boldsymbol{\Theta}} \mathrm{V}[\boldsymbol{\Theta}] = \mathrm{Y} \to \mathrm{V}.$$

Write W for its kernel.

Lemma 9.7. — *We have:*

$$\mathrm{W} = \bigoplus_{\boldsymbol{\Theta}} (\mathrm{W} \cap \mathrm{V}[\boldsymbol{\Theta}]).$$

Proof. — Let Z denote the quotient of W by the right hand side, and assume that it is nonzero. Let π be an irreducible subquotient of Z. For all sums of endo-classes $\boldsymbol{\Theta}$, the representation π is an irreducible subquotient of $\mathrm{W}/(\mathrm{W} \cap \mathrm{V}[\boldsymbol{\Theta}])$, thus of:

$$\mathrm{V}/\mathrm{V}[\boldsymbol{\Theta}] = \bigoplus_{\boldsymbol{\Theta}' \neq \boldsymbol{\Theta}} \mathrm{V}[\boldsymbol{\Theta}'],$$

which implies that $\pi[\boldsymbol{\Theta}] = 0$. Since π contains some semisimple supertype $(\mathbf{J}, \boldsymbol{\lambda})$ by [**22, 15**], for any decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$ with associated functor K and formal sum $\boldsymbol{\Theta}$, we have $\mathsf{K}(\pi) \neq 0$ so that $\pi[\boldsymbol{\Theta}] \neq 0$, a contradiction. □

Since f is injective on each $\mathrm{V}[\boldsymbol{\Theta}]$, we have $\mathrm{W}\cap\mathrm{V}[\boldsymbol{\Theta}]=0$ for all $\boldsymbol{\Theta}$ and it follows that we have $\mathrm{W}=0$. Assume that f is not surjective, and let π be an irreducible subquotient in its cokernel. Write Ω for the inertial class of its supercuspidal support. Its corresponds to some semisimple supertype $(\mathbf{J},\boldsymbol{\lambda})$. Write $\boldsymbol{\Theta}=\boldsymbol{\Theta}(\mathbf{J},\boldsymbol{\lambda})$ and fix a decomposition $\boldsymbol{\lambda}=\boldsymbol{\kappa}\otimes\boldsymbol{\sigma}$. By applying K, we get that $\mathsf{K}(\pi)$ is a subquotient of:

$$\mathsf{K}(\mathrm{V})/\mathsf{K}(\mathrm{Y})=\mathsf{K}(\mathrm{V})/\mathsf{K}(\mathrm{V}[\boldsymbol{\Theta}])=\mathsf{K}(\mathrm{V})/\bigoplus_{[\mathscr{M},\boldsymbol{\sigma}]}\mathsf{K}(\mathrm{V}[\boldsymbol{\Theta},\boldsymbol{\sigma}])=\mathsf{K}(\mathrm{V})/\bigoplus_{[\mathscr{M},\boldsymbol{\sigma}]}\mathrm{V}(\boldsymbol{\Theta},\boldsymbol{\sigma})$$

by Corollary 9.4 and Lemma 9.2. But the right hand side is zero by (9.1): contradiction. ☐

§10. Blocks of the category

Recall that an abelian category $\mathscr{A}$ is the *direct sum* of two full subcategories $\mathscr{A}_1,\mathscr{A}_2$ if every object V of $\mathscr{A}$ decomposes uniquely as $\mathrm{V}=\mathrm{V}_1\oplus\mathrm{V}_2$, with V_i an object of $\mathscr{A}_i$ for $i=1,2$, and $\mathrm{Hom}_{\mathscr{A}}(\mathrm{V}_1,\mathrm{V}_2)=0$. In this case, we say that $\mathscr{A}_1,\mathscr{A}_2$ are *direct summands* of $\mathscr{A}$. We say that $\mathscr{A}$ is *indecomposable* if it cannot be expressed as the direct sum of two proper subcategories.

Definition 10.1. — A *block* in $\mathscr{R}(\mathrm{G})$ is a direct summand of $\mathscr{R}(\mathrm{G})$ which is indecomposable.

10.1.

Given Ω an inertial class of a supercuspidal pair of G, we write $\mathscr{R}(\Omega)$ for the full subcategory of representations all of whose irreducible subquotients have their supercuspidal support in Ω.

Given $(\mathbf{J},\boldsymbol{\lambda})$ a semisimple supertype of G, we fix a decomposition $\boldsymbol{\lambda}=\boldsymbol{\kappa}\otimes\boldsymbol{\sigma}$ and associate to it the sum $\boldsymbol{\Theta}$, the functor $\mathsf{K}=\mathsf{K}_{\boldsymbol{\kappa}_{\max}}$ and the equivalence class $[\mathscr{M},\boldsymbol{\sigma}]$. We write $\mathscr{R}(\mathbf{J},\boldsymbol{\lambda})$ for the full subcategory of representations $\mathrm{V}\in\mathscr{R}(\Omega)$ such that $\mathrm{V}=\mathrm{V}[\boldsymbol{\Theta},\boldsymbol{\sigma}]$. This does not depend on the choice of the decomposition of $\boldsymbol{\lambda}$.

Assume that $\Omega=[\mathrm{L},\varrho]_{\mathrm{G}}$ and $[\mathbf{J},\boldsymbol{\lambda}]$ correspond to each other (see Section 8).

Proposition 10.2. — *One has* $\mathscr{R}(\Omega)=\mathscr{R}(\mathbf{J},\boldsymbol{\lambda})$.

Proof. — Given $\mathrm{V}\in\mathscr{R}(\Omega)$, we apply Theorem 9.6 and thus get a decomposition:

$$\mathrm{V}=\bigoplus_{\boldsymbol{\Theta}'}\mathrm{V}[\boldsymbol{\Theta}']. \tag{10.1}$$

Assume $\mathrm{V}[\boldsymbol{\Theta}']$ is nonzero for some sum $\boldsymbol{\Theta}'$, and let W be an irreducible subquotient of it. Note that W has supercuspidal support in Ω. We first prove that $\boldsymbol{\Theta}'=\boldsymbol{\Theta}$. For this, it is enough to prove the following lemma.

Lemma 10.3. — *We have* $\mathsf{K}(\mathrm{W})\neq0$.

Proof. — If Ω is homogeneous, that is, if Ω is the inertial class of a tensor product of copies of a given supercuspidal representation, the result is given by [**15**, Proposition 5.8]. In general, we use [**16**, Théorème 8.19] together with Theorem 6.2 to reduce to the homogeneous case. ☐

We thus have $\boldsymbol{\Theta}'=\boldsymbol{\Theta}$, and $\mathsf{K}(\mathrm{W})$ is a subquotient of:

$$\mathsf{K}(\mathrm{V}[\boldsymbol{\Theta}])=\bigoplus_{[\mathscr{M}',\boldsymbol{\sigma}']}\mathrm{V}(\boldsymbol{\Theta},\boldsymbol{\sigma}').$$

But there is also an unramified character χ of L such that $\mathbf{K}(\mathrm{W})$ is a subquotient of:

$$\mathbf{K}(\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{G}}(\varrho\chi)) \simeq \mathrm{Ind}_{\mathscr{M}}^{\mathscr{G}}(\mathbf{K}_{\mathrm{L}}(\varrho\chi)),$$

which is a finite direct sum of representations of the form $\mathrm{Ind}_{\mathscr{M}'}^{\mathscr{G}}(\boldsymbol{\sigma}')$ for $(\mathscr{M}',\boldsymbol{\sigma}') \in [\mathscr{M},\boldsymbol{\sigma}]$. Thus all irreducible subquotients of $\mathbf{K}(\mathrm{W})$ have supercuspidal support in $[\mathscr{M}',\boldsymbol{\sigma}']$, and the decomposition (10.1) reduces to $\mathrm{V} = \mathrm{V}[\boldsymbol{\Theta},\boldsymbol{\sigma}]$. Conversely, let $\mathrm{V} \in \mathscr{R}(\mathbf{J},\boldsymbol{\lambda})$ and let W be an irreducible subquotient of V. All irreducible subquotients of $\mathbf{K}(\mathrm{W})$ have supercuspidal support in $[\mathscr{M},\boldsymbol{\sigma}]$. Write φ for the canonical surjective map:

$$\mathrm{ind}_{\mathrm{J}_{\max}}^{\mathrm{G}}(\kappa_{\max} \otimes \mathbf{K}(\mathrm{W})) \to \mathrm{W}.$$

Choose a composition series $0 = \mathrm{Z}_0 \subsetneq \mathrm{Z}_1 \subsetneq \cdots \subsetneq \mathrm{Z}_n = \mathbf{K}(\mathrm{W})$ and write $\mathrm{W}_i = \mathrm{ind}_{\mathrm{J}_{\max}}^{\mathrm{G}}(\kappa_{\max}\otimes\mathrm{Z}_i)$. There is a minimal i such that φ is nonzero on W_{i+1}. Thus W is isomorphic to a quotient of:

$$\mathrm{W}_{i+1}/\mathrm{W}_i \simeq \mathrm{ind}_{\mathrm{J}_{\max}}^{\mathrm{G}}(\kappa_{\max} \otimes (\mathrm{Z}_{i+1}/\mathrm{Z}_i))$$

and $\mathrm{Z}_{i+1}/\mathrm{Z}_i$ has supercuspidal support in $[\mathscr{M},\boldsymbol{\sigma}]$. Thus W is a subquotient of $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$. Now the result follows from Proposition 8.1. □

10.2.

Theorem 9.6 and Corollary 9.4 can now be restated as follows.

Theorem 10.4. — *The category $\mathscr{R}(\mathrm{G})$ decomposes into the product of the subcategories $\mathscr{R}(\Omega)$, where Ω ranges over all possible inertial classes of supercuspidal pairs of* G.

The following result says that the decomposition given by Theorem 10.4 is the best possible.

Proposition 10.5. — *Each subcategory $\mathscr{R}(\Omega)$ is indecomposable.*

Proof. — Assume this is not the case. There are two subcategories $\mathscr{A}$ and $\mathscr{A}'$ such that:

$$\mathscr{R}(\Omega) = \mathscr{A} \oplus \mathscr{A}'.$$

Let $[\mathbf{J},\boldsymbol{\lambda}]$ be the equivalence class of semisimple supertypes which corresponds to Ω and consider $\mathrm{V} = \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$. By Proposition 10.2, we have $\mathrm{V} \in \mathscr{R}(\Omega)$, and there is a decomposition $\mathrm{V} = \mathrm{W}\oplus\mathrm{W}'$ with $\mathrm{W} \in \mathscr{A}$ and $\mathrm{W}' \in \mathscr{A}'$, and with no nonzero intertwining between W and W′. We get:

$$\mathrm{End}_{\mathrm{G}}(\mathrm{V}) = \mathrm{End}_{\mathrm{G}}(\mathrm{W}) \oplus \mathrm{End}_{\mathrm{G}}(\mathrm{W}').$$

This implies that $\mathrm{End}_{\mathrm{G}}(\mathrm{V})$ possesses a nontrivial central idempotent. By [**22, 15**], this algebra is isomorphic to a finite tensor product of affine Hecke algebras $\mathcal{H}(n_i, q^{f_i})$, with $1 \leqslant i \leqslant r$. Thus its centre is isomorphic to the finite tensor product of the centres of the algebras $\mathcal{H}(n_i, q^{f_i})$, with $1 \leqslant i \leqslant r$. The centre of $\mathcal{H}(n, q^f)$ is isomorphic to $\mathrm{R}[\mathrm{X}_1^{\pm1},\dots,\mathrm{X}_n^{\pm1}]^{\mathfrak{S}_n}$, where $\mathfrak{S}_n$ is the nth symmetric group acting on $\mathrm{X}_1,\dots,\mathrm{X}_n$. This is an integral domain. Thus the centre of $\mathrm{End}_{\mathrm{G}}(\mathrm{V})$ does not contain any nontrivial idempotent. Therefore W′, say, is zero. Now let X be a simple object in $\mathscr{A}'$. There is a G-subspace Y of V such that X is a quotient of Y. As $\mathrm{V} \in \mathscr{A}$, we get $\mathrm{Y} \in \mathscr{A}$. But $\mathrm{Hom}(\mathrm{Y},\mathrm{X})$ is nonzero: contradiction. □

Remark 10.6. — We remark that the representation $\mathrm{V} = \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$ used in the proof of Proposition 10.5 is *not*, in general, a progenerator for the subcategory $\mathscr{R}(\Omega)$: in general this representation is not projective, nor is every irreducible subquotient isomorphic to a quotient. However, given the explicit results on supertypes here, it is not hard to construct a progenerator as a compactly-induced representation; for $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ this was done (independently) by Guiraud [**11**] (for level zero blocks) and Helm [**12**].

10.3.

Let π be a supercuspidal irreducible representation of G. The endo-class of a simple character in π is well-defined (see [**3**, §9.2]) and we denote it $\boldsymbol{\Theta}_\pi$. Moreover, if $(\mathbf{J}, \boldsymbol{\lambda})$ is a maximal simple supertype of G occurring in π and attached to a simple stratum $[\Lambda, n, 0, \beta]$, then we have:

$$\boldsymbol{\Theta}(\mathbf{J}, \boldsymbol{\lambda}) = \frac{md}{[\mathrm{F}[\beta] : \mathrm{F}]} \cdot \boldsymbol{\Theta}_\pi.$$

It does not depend on the choice of the simple type $(\mathbf{J}, \boldsymbol{\lambda})$ nor of the simple stratum $[\Lambda, n, 0, \beta]$, and we denote it $\boldsymbol{\Theta}(\pi)$. In fact, it depends only on the inertial class $[\mathrm{G}, \pi]_{\mathrm{G}}$.

Now let Ω be the inertial class of a supercuspidal pair (M, ϱ) of G. We may (and will) assume that $\mathrm{M} = \mathrm{G}_{m_1} \times \cdots \times \mathrm{G}_{m_r}$ and $\varrho = \rho_1 \otimes \cdots \otimes \rho_r$ with $m_1 + \cdots + m_r = m$ and ρ_i an irreducible supercuspidal representation of G_{m_i}, for each $i \in \{1, \ldots, r\}$. Then the formal sum:

$$\boldsymbol{\Theta}(\Omega) = \sum_{i=1}^{r} \boldsymbol{\Theta}(\rho_i)$$

is well-defined. Moreover, if $(\mathbf{J}, \boldsymbol{\lambda})$ is a semisimple supertype of G such that $[\mathbf{J}, \boldsymbol{\lambda}]$ corresponds to Ω in the sense of (8.2), then we have $\boldsymbol{\Theta}(\mathbf{J}, \boldsymbol{\lambda}) = \boldsymbol{\Theta}(\Omega)$.

Proposition 10.7. — *Let $(\mathbf{J}_0, \boldsymbol{\lambda}_0)$ be a semisimple supertype, put $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\mathbf{J}_0, \boldsymbol{\lambda}_0)$ and write $\mathscr{G}$ for the finite reductive group associated with it. Then the following finite sets have the same cardinality:*

(i) *the set of supercuspidal inertial classes Ω of* G *with $\boldsymbol{\Theta}(\Omega) = \boldsymbol{\Theta}$;*
(ii) *the set of equivalence classes $[\mathbf{J}, \boldsymbol{\lambda}]$ of semisimple supertypes of* G *with $\boldsymbol{\Theta}(\mathbf{J}, \boldsymbol{\lambda}) = \boldsymbol{\Theta}$;*
(iii) *the set of equivalence classes $[\mathscr{M}, \boldsymbol{\sigma}]$ of supercuspidal pairs in $\mathscr{G}$.*

Moreover any choice of functor K *associated with $(\mathbf{J}_0, \boldsymbol{\lambda}_0)$ induces a bijection between the sets in* (ii) *and* (iii).

Proof. — We have already seen the bijection between the first two sets. We make a choice of a functor K associated with $(\mathbf{J}_0, \boldsymbol{\lambda}_0)$. We have already seen that K induces a surjective map from the set in (ii) to that in (iii). Thus it is enough to check that the sets in (i) and (iii) have the same cardinality. Moreover, it is enough to treat the case where $\boldsymbol{\Theta}$ is homogeneous, thus

$$\boldsymbol{\Theta} = \frac{md}{[\mathrm{E} : \mathrm{F}]} \cdot \boldsymbol{\Theta}_1 = m'd' \cdot \boldsymbol{\Theta}_1$$

as in §6.2.

By the unicity (up to conjugacy) of maximal simple supertypes in a supercuspidal representation (see [**22**, Theorem 7.2] and also [**15**, Corollaire 5.5]), the number of inertial classes $[\mathrm{G}, \pi]_{\mathrm{G}}$ of supercuspidal representations with a given endo-class $\boldsymbol{\Theta}_1$ is precisely the number of $\mathrm{Gal}(\mathfrak{k}_{\mathrm{D}'}/\mathfrak{k}_{\mathrm{E}})$-conjugacy classes of supercuspidal representations of $\mathrm{GL}_{m'}(\mathfrak{k}_{\mathrm{D}'})$, where the notation is as in §5.1.

We think of an inertial class of supercuspidal pairs of G as a finitely supported map:

$$\phi : \bigcup_{k \geqslant 1} \{\text{inertial classes } [\mathrm{G}_k, \pi]_{\mathrm{G}_k} \text{ of supercuspidal irreducible representations of } \mathrm{G}_k\} \to \mathbf{N}$$

such that

$$\sum_{k\geqslant 1} k \sum_{[\mathrm{G}_k,\pi]} \phi([\mathrm{G}_k,\pi]_{\mathrm{G}_k}) = m.$$

We deduce that the number of inertial classes of supercuspidal pairs Ω with a given homogeneous $\boldsymbol{\Theta}$ is precisely the number of finitely supported maps:

$$\psi : \bigcup_{f\geqslant 1} \{\mathrm{Gal}(\mathfrak{k}_{\mathrm{D}'}/\mathfrak{k}_{\mathrm{E}})\text{-conjugacy classes } [\sigma] \text{ of supercuspidal representations of } \mathrm{GL}_f(\mathfrak{k}_{\mathrm{D}'})\} \to \mathbf{N}$$

such that

$$\sum_{f\geqslant 1} f \sum_{[\sigma]} \psi([\sigma]) = m',$$

where we are again using the notation of §5.1. But this is also the number of equivalence classes of supercuspidal pairs in $\mathscr{G} = \mathrm{GL}_{m'}(\mathfrak{k}_{\mathrm{D}'})$. □

§11. A remarkable property of supercuspidal representations

We end this article by the following result. When G is split, that is when $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$, $n \geqslant 1$, it is proven by Dat [**9**, Corollaire B.1.3] in a different manner.

Proposition 11.1. — *Let* P *be a proper parabolic subgroup of* G *and* σ *be a representation of a Levi component* M *of* P. *Then* $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)$ *has no supercuspidal irreducible subquotient.*

Proof. — When σ is irreducible, the result follows from the definition of a supercuspidal representation (Definition 1.1). Assume $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)$ contains a supercuspidal irreducible subquotient π. There is a simple stratum $[\Lambda_{\max}, n_{\max}, 0, \beta]$ in $\mathrm{A} = \mathrm{M}_m(\mathrm{D})$ such that the restriction of π to the pro-p-subgroup $\mathrm{H}^1_{\max} = \mathrm{H}^1(\beta, \Lambda_{\max})$ contains a simple character $\theta_{\max} \in \mathcal{C}(\Lambda_{\max}, 0, \beta)$.

Lemma 11.2. — *There is an irreducible subquotient* τ *of* σ *such that* $\theta_{\max}$ *occurs in the restriction of* $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\tau)$ *to* $\mathrm{H}^1_{\max}$.

Proof. — Since any representation of $\mathrm{H}^1_{\max}$ is semisimple, $\theta_{\max}$ is a direct summand of the restriction of $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)$ to $\mathrm{H}^1_{\max}$. We fix an embedding ι of $\theta_{\max}$ in $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)$ and write W for the (one-dimensional) image of $\theta_{\max}$ by ι. Write V for the representation of finite type $\mathrm{ind}_{\mathrm{H}^1_{\max}}^{\mathrm{G}}(\theta_{\max})$. If we write N for the unipotent radical of P, Frobenius reciprocity gives us a nonzero homomorphism:

$$\iota_* : \mathrm{V}_{\mathrm{N}} \to \sigma.$$

Write σ_1 for the image of this homomorphism. It has the following properties:

(i) if σ' is a proper subrepresentation of σ_1 then $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma') \cap \mathrm{W} = 0$;

(ii) it is of finite type, since V is of finite type and Jacquet functors preserve finite type.

This implies that σ_1 has a maximal proper subrepresentation σ_2 and that the image of V in the representation $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma_1/\sigma_2)$ is non-zero. In particular $\theta_{\max}$ occurs in $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma_1/\sigma_2)$ and σ_1/σ_2 is an irreducible subquotient of σ. □

We may assume that M is a standard Levi subgroup, attached to a composition $(m_1, \dots, m_r)$ of m. Thus τ can be written on the form $\tau_1 \otimes \cdots \otimes \tau_r$, with τ_i an irreducible representation of G_{m_i}, for each $i \in \{1, \dots, r\}$. Let $(\mathbf{J}_i, \boldsymbol{\lambda}_i)$ be a semisimple supertype of G_{m_i} occurring in τ_i. Then $\theta_{\max}$ occurs in:

$$\mathrm{ind}_{\mathbf{J}_1}^{\mathrm{G}_{m_1}}(\boldsymbol{\lambda}_1) \times \cdots \times \mathrm{ind}_{\mathbf{J}_r}^{\mathrm{G}_{m_r}}(\boldsymbol{\lambda}_r) \simeq \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$$

where $(\mathbf{J}, \boldsymbol{\lambda})$ is a suitable semisimple supertype of G. We fix a decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$ and thus get a functor $\mathbf{K}$. As in the first part of the proof of Lemma 8.2, it follows that $\mathbf{K}(\pi)$ is nonzero. By [**15**, Lemme 5.3], it is a finite direct sum of supercuspidal irreducible representations of $\mathscr{G} = \mathbf{J}/\mathbf{J}^1$. By Theorem 6.2, it is a subquotient of:

$$\mathbf{K}(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\sigma)) \simeq \mathrm{Ind}_{\mathscr{P}}^{\mathscr{G}}(\mathbf{K}_{\mathrm{M}}(\sigma)).$$

Thus Proposition 1.10 gives us a contradiction. □

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Vincent Sécherre, Université de Versailles Saint-Quentin-en-Yvelines, Laboratoire de Mathématiques de Versailles, 45 avenue des Etats-Unis, 78035 Versailles cedex, France
E-mail : vincent.secherre@math.uvsq.fr

Shaun Stevens, School of Mathematics, University of East Anglia, Norwich NR4 7TJ, United Kingdom
E-mail : Shaun.Stevens@uea.ac.uk