CHARACTERIZING BLOCK GRAPHS IN TERMS OF THEIR VERTEX-INDUCED PARTITIONS

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Abstract. Block graphs are a generalization of trees that arise in areas such as metric graph theory, molecular graphs, and phylogenetics. Given a finite connected simple graph $G = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$, we will show that the (necessarily unique) smallest block graph with vertex set V whose edge set contains E is uniquely determined by the V-indexed family $\mathbf{P}_G = \big(\pi_v\big)_{v\in V}$ of the partitions π_v of the set V into the set of connected components of the graph $(V, \{e \in E : v \notin e\})$. Moreover, we show that an arbitrary V-indexed family $\mathbf{P} = (\mathbf{p}_v)_{v\in V}$ of partitions $\mathbf{p}_v$ of the set V is of the form $\mathbf{P} = \mathbf{P}_G$ for some connected simple graph $G = (V, E)$ with vertex set V as above if and only if, for any two distinct elements $u, v \in V$, the union of the set in $\mathbf{p}_v$ that contains u and the set in $\mathbf{p}_u$ that contains v coincides with the set V, and $\{v\} \in \mathbf{p}_v$ holds for all $v \in V$. As well as being of inherent interest to the theory of block graphs, these facts are also useful in the analysis of compatible decompositions of finite metric spaces.

1. Introduction

A *block graph* is a graph in which every maximal 2-connected subgraph or *block* is a clique [1, 8]. Block graphs are a natural generalization of trees, and they arise in areas such as metric graph theory [1], molecular graphs [2] and phylogenetics [7]. They have been characterized in various ways, for example, as certain intersection graphs [8], in terms of distance conditions [2, 9] and also by forbidden graph configurations [1]. Here we shall present an alternative approach to describing the set of block graphs.

More specifically, given a finite set V we call any partition of V a *V-partition*, and we define a V-indexed family of V-partitions $\mathbf{P}_V = (\mathbf{p}_v)_{v\in V}$ to be a *compatible family of V-partitions* if, for any two distinct elements $u, v \in V$, the union of the set in $\mathbf{p}_v$ that contains u and the set in $\mathbf{p}_u$ that contains v coincides with the set V, and $\{v\} \in \mathbf{p}_v$ holds for all $v \in V$. In addition, we let $\mathbf{P}(V)$ denote the set of all compatible families of V-partitions. Note that compatibility of partitions is a concept that naturally arises when analyzing phylogenetic trees (cf. e.g. [10]). In particular, if a V-indexed family of V-partitions is compatible, then every pair of partitions in this family is *strongly compatible* in the sense defined in [7].

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In this note, we consider connected simple graphs $G = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$. For any vertex $v \in V$ of such a graph, let π_v denote the family of connected components of the graph $(V, \{e \in E : v \notin e\})$. We show that the map that takes the graph G to the V-indexed family $\mathbf{P}_G := \big(\pi_v\big)_{v\in V}$ of partitions of the set V induces a bijection from the set of connected block graphs with vertex set V onto the set $\mathbf{P}(V)$. We prove this in Theorem 1 below. Defining two graphs G_1 and G_2 with vertex set V to be *block-equivalent* if the "smallest" block graphs that contain G_1 and G_2 coincide, this bijection implies in particular the following observation: The set of block-equivalence classes of connected simple graphs G with vertex set V is in bijective correspondence with the set $\mathbf{P}(V)$.

As well as contributing to the tasks of phylogenetic combinatorics outlined in [5], this result is part of a broader investigation into so-called *compatible decompositions* and *block realizations* of finite metric spaces [3, 4] which was first mentioned in [6, Section 4]. In particular, it is key to proving that there is a unique "finest" compatible decomposition of any finite metric space (cf. [3, p.1619] for a more precise statement of this result).

The rest of this note is organized as follows. After presenting some preliminaries in the next section and establishing three supporting lemmas in Section 3, we prove our main result in Section 4.

2. Preliminaries

From now on, we will consider connected simple graphs G with a fixed finite vertex set V. Following [4], we will use the following notations and definitions.

Given any set Y, we denote
– by $Y - y$ the complement $Y - \{y\}$ of a one-element subset $\{y\}$ of Y,
– and by $\mathbf{p}[y]$, for any Y-partition $\mathbf{p}$ and any element $y \in Y$, that subset $Z \in \mathbf{p}$ of Y which contains y.

Further, given a simple graph G with vertex set V and edge set $E \subseteq \binom{V}{2}$, we denote
– by $\pi_0(G)$ the V-partition formed by the connected components of G,
– by $G[v] := \pi_0(G)[v]$, for any vertex $v \in V$ of G, the connected component of G containing v,
– by $G^{(v)}$ the largest subgraph of G with vertex set V for which v is an isolated vertex, that is, the graph with vertex set V and edge set $\{e \in E : v \notin e\}$,
– by $[G]$ the smallest block graph with vertex set V that contains G as a subgraph, i.e., the graph $(V, [E])$ with vertex set V whose edge set $[E]$ is the union of E and all 2-subsets $\{u, v\}$ of V that are contained in a circuit of G (i.e., a connected subgraph of G all of whose vertices have degree 2) (see e.g. [8]), or, equivalently, the graph whose edge set $[E]$ consists of all 2-subsets $\{u, v\}$ of V for which there is no vertex $w \in V - \{u, v\}$ with $G^{(w)}[u] \neq G^{(w)}[v]$ (see e.g. [11, Theorem 4.2.3]),
– and by $\mathbf{P}_G$ the V-indexed family

$$\mathbf{P}_G := \big(\pi_0(G^{(v)})\big)_{v\in V}$$

of partitions of V.

In the remainder of this section we establish some basic properties of compatible families of V-partitions that will be used later. To state these properties, we will say that an element $w \in V$ *separates* two elements $u, v \in V$ (relative to $\mathbf{P}$) or, for short, that "$u|w|v$" holds if $w \neq u, v$ and $\mathbf{p}_w[u] \neq \mathbf{p}_w[v]$ (and, therefore, also $u \neq v$).

Lemma 1. *Let $\mathbf{P} = (\mathbf{p}_v)_{v \in V} \in \mathbf{P}(V)$. Then, for any three distinct elements $u, v, w \in V$, the following nine assertions all are equivalent:*

(i)	$u\|w\|v$,		
(ii)	$\mathbf{p}_w[u] \subsetneq \mathbf{p}_v[w]$,	**(vi)**	$\mathbf{p}_w[v] \subsetneq \mathbf{p}_u[w]$,
(iii)	$\mathbf{p}_w[u] \subsetneq \mathbf{p}_v[u]$,	**(vii)**	$\mathbf{p}_w[v] \subsetneq \mathbf{p}_u[v]$,
(iv)	$\mathbf{p}_w[u] \subseteq \mathbf{p}_v[u]$,	**(vii)**	$\mathbf{p}_w[v] \subseteq \mathbf{p}_u[v]$,
(v)	$v \notin \mathbf{p}_w[u]$,	**(ix)**	$u \notin \mathbf{p}_w[v]$,

and they all imply that also

(x)	$w \in \mathbf{p}_v[u] \cap \mathbf{p}_u[v]$,		
(xi)	$\mathbf{p}_v[w] = \mathbf{p}_v[u]$,	**(xii)**	$\mathbf{p}_u[w] = \mathbf{p}_u[v]$

must hold.

Proof: It is clear that, in view of $V = \mathbf{p}_w[v] \cup \mathbf{p}_v[w]$ and $w \notin \mathbf{p}_w[u]$, we have

$$\mathbf{p}_w[u] \neq \mathbf{p}_w[v] \Rightarrow \mathbf{p}_w[u] \cap \mathbf{p}_w[v] = \emptyset \Rightarrow \mathbf{p}_w[u] \subseteq V - (\mathbf{p}_w[v] \cup \{w\})$$
$$\Rightarrow \mathbf{p}_w[u] \subsetneq \mathbf{p}_v[w] \Rightarrow \mathbf{p}_w[u] \subseteq \mathbf{p}_v[w] \Rightarrow v \notin \mathbf{p}_w[u] \Rightarrow \mathbf{p}_w[u] \neq \mathbf{p}_w[v].$$

So, all these assertions must be equivalent to each other, and they imply also that $u \in \mathbf{p}_w[u] \subseteq \mathbf{p}_v[w]$ and, hence, $\mathbf{p}_v[w] = \mathbf{p}_v[u]$ and, therefore, also $w \in \mathbf{p}_v[w] = \mathbf{p}_v[u]$ must hold. In other words, the implications listed above yield that

$$\textbf{(i)} \iff \textbf{(ii)} \iff \textbf{(iii)} \iff \textbf{(iv)} \iff \textbf{(v)} \Longrightarrow \textbf{(xi)} \Longrightarrow w \in \mathbf{p}_v[u]$$

holds. And, switching u and v, we also get

$$\textbf{(i)} \iff \textbf{(vi)} \iff \textbf{(vii)} \iff \textbf{(viii)} \iff \textbf{(ix)} \Longrightarrow \textbf{(xii)} \Longrightarrow w \in \mathbf{p}_u[v]$$

and, therefore, also "**(i)** ⇒ **(x)**", as claimed. ■

Note that the last three assertions **(x) - (xii)** in Lemma 1 are not equivalent to the former nine assertions **(i) - (ix)**. To see this, consider, for example, the following compatible family $\mathbf{P} = (\mathbf{p}_v)_{v \in V}$ of V-partitions for the 4-element set $V = \{u, v, w, z\}$:

$$\mathbf{p}_u = \{\{u\}, \{v, w, z\}\}, \qquad \mathbf{p}_w = \{\{w\}, \{u, v, z\}\},$$
$$\mathbf{p}_v = \{\{v\}, \{u, w, z\}\}, \qquad \mathbf{p}_z = \{\{u\}, \{v\}, \{w\}, \{z\}\}.$$

3. Some useful lemmas

To describe the correspondence between compatible families of V-partitions and connected block graphs with vertex set V, we associate to each family $\mathbf{P} = (\mathbf{p}_v)_{v \in V}$

in $\mathbf{P}(V)$, the graph $B_{\mathbf{P}} := (V, E_{\mathbf{P}})$ with vertex set V and edge set

$$E_{\mathbf{P}} := \Big\{\{u,v\} \in \binom{V}{2} : \forall_{w \in V - \{u,v\}} \mathbf{p}_w[u] = \mathbf{p}_w[v]\Big\}.$$

In this section, we present three supporting lemmas that establish some properties of the graph $B_{\mathbf{P}}$ that will be used to prove our main result (Theorem 1 below), starting with a lemma concerning the existence of certain triangles in $B_{\mathbf{P}}$.

Lemma 2. *Let $\mathbf{P} = (\mathbf{p}_v)_{v \in V} \in \mathbf{P}(V)$. Then, for any three distinct elements $u, v, w \in V$ with $\{u,w\}, \{w,v\} \in E_{\mathbf{P}}$, we have $\{u,v\} \in E_{\mathbf{P}}$ if and only if $\mathbf{p}_w[u] = \mathbf{p}_w[v]$ holds.*

Proof: Indeed, $\{u,w\}, \{w,v\} \in E_{\mathbf{P}}$ implies that $\mathbf{p}_{w'}[u] = \mathbf{p}_{w'}[w] = \mathbf{p}_{w'}[v]$ holds for all $w' \in V - \{u,v,w\}$ and that, therefore, $\{u,v\} \in E_{\mathbf{P}}$ or, equivalently, "$\forall_{w' \in V - \{u,v\}} \mathbf{p}_{w'}[u] = \mathbf{p}_{w'}[v]$" holds if and only if we have $\mathbf{p}_{w'}[u] = \mathbf{p}_{w'}[v]$ also for the only element $w' \in V - \{u,v\}$ not in $V - \{u,v,w\}$, i.e., for $w' := w$. ■

The next lemma characterizes those 2-subsets $\{u,v\} \subseteq V$ that are contained in $E_{\mathbf{P}}$.

Lemma 3. *Let $\mathbf{P} = (\mathbf{p}_v)_{v \in V} \in \mathbf{P}(V)$. Then, for two distinct elements $u, v \in V$, we have $\{u,v\} \in E_{\mathbf{P}}$ if and only if $\mathbf{p}_v[u]$ is a minimal set in the collection*

$$\mathbf{P}_{\mathbf{p}_u[v] \cap \mathbf{p}_v[u]}[u] := \big\{\mathbf{p}_w[u] : w \in (\mathbf{p}_u[v] \cap \mathbf{p}_v[u]) \cup \{v\}\big\}$$

of subsets of V or, equivalently, in the collection

$$\mathbf{P}_{\mathbf{p}_u[v]}[u] := \{\mathbf{p}_w[u] : w \in \mathbf{p}_u[v]\}$$

or, still equivalently, in

$$\mathbf{P}[u] := \{\mathbf{p}_w[u] : w \in V - u\}.$$

Proof: First note that

$$\begin{aligned} \{u,v\} \notin E_{\mathbf{P}} \quad &\iff \quad \exists_{w \in V - \{u,v\}} \ \mathbf{p}_w[u] \neq \mathbf{p}_w[v] \qquad \text{(by definition)} \\ &\iff \quad \exists_{w \in \mathbf{p}_v[u] \cap \mathbf{p}_u[v]} \ \mathbf{p}_w[u] \subsetneq \mathbf{p}_v[u] \quad \text{(by Lemma 1)} \\ &\iff \quad \mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v] \cap \mathbf{p}_v[u]}[u]\big) \end{aligned}$$

holds for any two distinct elements $u, v \in V$ and we clearly have

$$\mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v] \cap \mathbf{p}_v[u]}[u]\big) \Longrightarrow \mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v]}[u]\big) \Longrightarrow \mathbf{p}_v[u] \notin \min\big(\mathbf{P}[u]\big).$$

Hence, it remains to establish the following implication:

$$\mathbf{p}_v[u] \notin \min\big(\mathbf{P}[u]\big) \Longrightarrow \mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v] \cap \mathbf{p}_v[u]}[u]\big)$$

To this end, note that $w \in V - u$ and $\mathbf{p}_w[u] \subsetneq \mathbf{p}_v[u]$ implies $w \neq u, v$ as well as $u|w|v$ and, therefore, also $w \in \mathbf{p}_u[v] \cap \mathbf{p}_v[u]$ in view of Lemma 1. Thus, we must have

$$\mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v] \cap \mathbf{p}_v[u]}[u]\big) \iff \mathbf{p}_v[u] \notin \min\big(\mathbf{P}_{\mathbf{p}_u[v]}[u]\big),$$

as claimed. So,

$$\begin{aligned}\{u,v\}\in E_{\mathbf{P}} &\iff \mathbf{p}_v[u]\in\min\big(\mathbf{P}_{\mathbf{p}_u[v]\cap\mathbf{p}_u[v]}[u]\big)\\ &\iff \mathbf{p}_v[u]\in\min\big(\mathbf{P}_{\mathbf{p}_u[v]}[u]\big)\\ &\iff \mathbf{p}_v[u]\in\min\big(\mathbf{P}[u]\big)\end{aligned}$$

must hold, as claimed. ■

The final supporting lemma is concerned with the existence of certain paths in the graph $B_{\mathbf{P}}$.

Lemma 4. *Let $\mathbf{P}=(\mathbf{p}_v)_{v\in V}\in\mathbf{P}(V)$. Then, for any two distinct elements $u,v\in V$, and any sequence $\mathfrak{p}:=(u_0:=u,u_1,\dots,u_n:=v)$ of elements of V such that*

$$\mathbf{p}_{u_1}[u]\subsetneq\mathbf{p}_{u_2}[u]\subsetneq\cdots\subsetneq\mathbf{p}_{u_n}[u]=\mathbf{p}_v[u]$$

is a maximal chain of subsets of $\mathbf{p}_v[u]$ in

$$\mathbf{P}^{\subseteq\mathbf{p}_v[u]}[u]:=\{\mathbf{p}_w[u]:w\in V-u,\mathbf{p}_w[u]\subseteq\mathbf{p}_v[u]\}$$

ending with $\mathbf{p}_v[u]=\mathbf{p}_{u_n}[u]$, the sequence $\mathfrak{p}$ forms a path from u to v in the graph $B_{\mathbf{P}}=(V,E_{\mathbf{P}})$, i.e., the 2-subsets $\{u_0,u_1\},\{u_1,u_2\},\dots,\{u_{n-1},u_n\}$ of V are all contained in $E_{\mathbf{P}}$. Moreover, we have $u_i|u_j|u_k$ for all $i,j,k\in\{0,1,\dots,n\}$ with $i<j<k$ and, therefore, also $u_1,\dots,u_{n-1}\in\mathbf{p}_u[v]\cap\mathbf{p}_v[u]$. In particular, we must have $u|u_j|v$ for all $j\in\{1,\dots,n-1\}$ and $\mathbf{p}_{u_j}[u]=\mathbf{p}_{u_j}[u_i]$ and $\mathbf{p}_{u_i}[v]=\mathbf{p}_{u_i}[u_j]$ for all $i,j=1,\dots,n$ with $i<j$.

Proof: Our assumption that $\mathbf{p}_{u_j}[u]\subsetneq\mathbf{p}_{u_k}[u]$ holds for all $j,k\in\{1,2,\dots,n\}$ with $j<k$ implies, in view of Lemma 1, that also $u|u_j|u_k$ and, therefore, also $\mathbf{p}_{u_k}[u_j]=\mathbf{p}_{u_k}[u]$ must hold for all $j,k=1,2,\dots,n$ with $j<k$. In consequence, we must also have $\mathbf{p}_{u_j}[u_i]=\mathbf{p}_{u_j}[u]\subsetneq\mathbf{p}_{u_k}[u]=\mathbf{p}_{u_k}[u_i]$ and, therefore (again by Lemma 1), also $u_i|u_j|u_k$ as well as $\mathbf{p}_{u_k}[u_i]=\mathbf{p}_{u_k}[u_j]$ for all $i,j,k\in\{0,1,\dots,n\}$ with $i<j<k$. In particular, we must have $u|u_j|v$ for all $j\in\{1,\dots,n-1\}$ and, hence (again by Lemma 1), $u_1,\dots,u_{n-1}\in\mathbf{p}_u[v]\cap\mathbf{p}_v[u]$ and $\mathbf{p}_{u_j}[u]=\mathbf{p}_{u_j}[u_i]$ and $\mathbf{p}_{u_i}[v]=\mathbf{p}_{u_i}[u_j]$ for all $i,j=1,\dots,n$ with $i<j$, as claimed.

Next, to establish that $\{u_0,u_1\},\{u_1,u_2\},\dots,\{u_{n-1},u_n\}\in E_{\mathbf{P}}$ holds, note first that $\mathbf{p}_{u_1}[u]$ is, by assumption, a minimal set in the set system $\mathbf{P}^{\subseteq\mathbf{p}_v[u]}[u]$. We claim that this implies that $\mathbf{p}_{u_1}[u]$ is also a minimal set in $\mathbf{P}[u]$. To see this, note that $w\in V-u$ and $\mathbf{p}_w[u]\subseteq\mathbf{p}_{u_1}[u]$ implies $\mathbf{p}_w[u]\subseteq\mathbf{p}_v[u]$ or, equivalently, $\mathbf{p}_w[u]\in\mathbf{P}^{\subseteq\mathbf{p}_v[u]}[u]$ and therefore, in view of the minimality of $\mathbf{p}_{u_1}[u]$ in $\mathbf{P}^{\subseteq\mathbf{p}_v[u]}[u]$, also $\mathbf{p}_w[u]=\mathbf{p}_{u_1}[u]$, as claimed. So, by Lemma 3, $\{u_0,u_1\}\in E_{\mathbf{P}}$ must hold.

Similarly, our choice of the elements $u_0,u_1,\dots,u_n$ implies also that

$$\mathbf{p}_{u_i}[u]\in\min\{\mathbf{p}_w[u]:w\in V-u\text{ and }\mathbf{p}_{u_{i-1}}[u]\subsetneq\mathbf{p}_w[u]\subseteq\mathbf{p}_v[u]\}\tag{1}$$

must hold for all $i=2,3,\dots,n$. Therefore, recalling that we have already established that $\mathbf{p}_{u_i}[u]=\mathbf{p}_{u_i}[u_{i-1}]$, we claim that

$$\mathbf{p}_{u_i}[u]=\mathbf{p}_{u_i}[u_{i-1}]\in\min\big(\mathbf{P}_{\mathbf{p}_{u_i}[u_{i-1}]\cap\mathbf{p}_{u_{i-1}}[u_i]}[u_{i-1}]\big).\tag{2}$$

To see this, assume for a contradiction that there is some $w\in\mathbf{p}_{u_i}[u_{i-1}]\cap\mathbf{p}_{u_{i-1}}[u_i]$ with $\mathbf{p}_w[u_{i-1}]\subsetneq\mathbf{p}_{u_i}[u]=\mathbf{p}_{u_i}[u_{i-1}]$. This would imply $u_i\notin\mathbf{p}_w[u_{i-1}]$ and $w\notin$

$\mathbf{p}_{u_{i-1}}[u]$ (in view of $w \in \mathbf{p}_{u_{i-1}}[u_i] = \mathbf{p}_{u_{i-1}}[v] \neq \mathbf{p}_{u_{i-1}}[u]$) and, therefore (by Lemma 1), $u_{i-1}|w|u_i$ as well as $u|u_{i-1}|w$ which (using again Lemma 1) would imply

$$\mathbf{p}_{u_{i-1}}[u] \subsetneq \mathbf{p}_w[u] = \mathbf{p}_w[u_{i-1}] \subsetneq \mathbf{p}_{u_i}[u] \subseteq \mathbf{p}_v[u]$$

in contradiction to (1). So, (2) or, equivalently (again by Lemma 3), $\{u_{i-1}, u_i\} \in E_{\mathbf{P}}$ must hold also for all $i \in \{2, \ldots, n\}$. ■

4. Statement and proof of main result

We now state and prove our main result:

Theorem 1. *Associating to each connected simple graph $G = (V, E)$ with vertex set V the V-indexed family $\mathbf{P}_G$ induces a one-to-one map from the set $\mathbf{B}(V)$ of connected block graphs with vertex set V (or, equivalently, from the set of block-equivalence classes of connected simple graphs G with that vertex set) onto the set $\mathbf{P}(V)$. The inverse of this map is given by associating, to each family $\mathbf{P} = (\mathbf{p}_v)_{v\in V} \in \mathbf{P}(V)$, the graph $B_{\mathbf{P}}$. In particular, for any family $\mathbf{P} = (\mathbf{p}_v)_{v\in V} \in \mathbf{P}(V)$, we have $\pi_0(B_{\mathbf{P}}^{(v)}) = \mathbf{p}_v$ for every element $v \in V$.*

Proof: It is easy to see that, given any connected simple graph $G = (V, E)$ with vertex set V, the V-indexed family $\mathbf{P}_G = \big(\pi_0(G^{(v)})\big)_{v\in V}$ is a compatible family of V-partitions: Indeed, we have obviously $\pi_0(G^{(v)})[v] = \{v\}$ for every $v \in V$, and we have $\pi_0(G^{(v)})[u] \cup \pi_0(G^{(u)})[v] = V$ for any two distinct elements v, u in V as, given any vertex $w \in V$, there must exist a path $\mathfrak{p} = (u_0 := u, u_1, \ldots, u_k := w)$ connecting u and w in G implying that $w \in \pi_0(G^{(v)})[u]$ holds in case $v \notin \{u_1, u_2 \ldots, u_k\}$ and $w \in \pi_0(G^{(u)})[v]$ in case $v \in \{u_1, u_2, \ldots, u_k\}$.

We also have $[E] = E_{\mathbf{P}_G}$ for every connected graph $G = (V, E)$ as, by definition of $[E]$, a 2-subset $\{u, v\} \subseteq V$ is an edge in $[E]$ if and only if $G^{(w)}[u] = G^{(w)}[v]$ holds for all $w \in V - \{u, v\}$. This shows that the map from $\mathbf{B}(V)$ into the set $\mathbf{P}(V)$ given by associating to each connected simple graph $G = (V, E)$ with vertex set V the V-indexed family $\mathbf{P}_G$ is a well-defined injective map, and that $B_{\mathbf{P}_G} = (V, E_{\mathbf{P}_G}) = (V, [E]) = [G]$ holds for every connected graph $G = (V, E)$.

To establish the theorem, it therefore remains to show that, conversely, $\mathbf{P}_{B_{\mathbf{P}}} = \mathbf{P}$ holds for every compatible family $\mathbf{P}$ of V-partitions. So, assume that $\mathbf{P}$ is a fixed compatible family $\mathbf{P} = (\mathbf{p}_v)_{v\in V}$ of V-partitions. It suffices to show that $\mathbf{p}_v[u] = B_{\mathbf{P}}^{(v)}[u]$ holds for any two distinct elements $u, v \in V$. Clearly, we have $\{u, v\} \in E_{\mathbf{P}}$ for two distinct elements $u, v \in V$ if and only if there is no $w \in V - \{u, v\}$ that separates u and v. Thus, we must have $B_{\mathbf{P}}^{(v)}[u] \subseteq \mathbf{p}_v[u]$ for any two distinct elements $u, v \in V$ since, otherwise, there would exist $u', u'' \in B_{\mathbf{P}}^{(v)}[u]$ with $\{u', u''\} \in E_{\mathbf{P}}$, but $\mathbf{p}_v[u'] \neq \mathbf{p}_v[u'']$, in contradiction to the definition of $E_{\mathbf{P}}$.

Now, to finish the proof, it remains to show that we have $\mathbf{p}_v[u] \subseteq B_{\mathbf{P}}^{(v)}[u]$ for any two distinct elements $u, v \in V$. To this end, consider any element $u' \in \mathbf{p}_v[u]$, $u' \neq u$. Lemma 4 implies that there exist two paths $\mathfrak{p} := (u_0 := u, u_1, \ldots, u_n := v)$

and $\mathfrak{p}' := (u'_0 := u', u'_1, \ldots, u'_{n'} := v)$ connecting u and u' with v in $B_{\mathbf{P}}$. Moreover, Lemma 2 implies that also either $u_{n-1} = u'_{n'-1}$ or $\{u_{n-1}, u'_{n'-1}\} \in E_{\mathbf{P}}$ holds. As a consequence, there exists also a path in $B_{\mathbf{P}}^{(v)}$ from u to u' and, therefore, $u' \in B_{\mathbf{P}}^{(v)}[u]$, as required. ■

To conclude the paper, we note that it could be interesting to try and find a concise characterization of those ternary relations "..|..|.."$\subseteq V^3$ that correspond to compatible families of V-partitions and, hence, to block graphs with vertex set V.

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