

ENDOMORPHISMS OF CELL 2-REPRESENTATIONS

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ABSTRACT. We determine the endomorphism categories of cell 2-representations of fiat 2-categories associated with strongly regular two-sided cells and classify, up to biequivalence, \mathcal{J} -simple fiat 2-categories which have only one two-sided cell \mathcal{J} apart from the identities.

1. INTRODUCTION AND DESCRIPTION OF THE RESULTS

Classically, Schur's Lemma asserts that the endomorphism algebra of a simple module (say for a finite dimensional algebra A over some algebraically closed field \mathbb{k}) is isomorphic to \mathbb{k} . It might happen that the algebra A is obtained by decategorifying some 2-category and that the simple module in question is the decategorification of some 2-representation of A . It is then natural to ask whether the assertion of Schur's Lemma is the 1-shadow of some 2-analogue. Put differently, this is a question about the endomorphism category of a 2-representation of some 2-category.

In [MM1] we defined a class of 2-categories, which we call fiat 2-categories, forming a natural 2-analogue of finite dimensional cellular algebras. Examples of fiat 2-categories appear (sometimes in disguise) in e.g. [BG, CR, FKS, KhLa, La, Ro2]. Fiat 2-categories have certain 2-representations called *cell 2-representations*, which were also defined in [MM1]. These 2-representations satisfy some natural generalizations of the concept of simplicity for representations of finite dimensional algebras. The main objective of the present paper is to study the endomorphism categories of these cell 2-representations with the ultimate goal to establish a 2-analogue of Schur's Lemma.

We start the paper by extending the 2-setup from [MM1] to accommodate non-strict 2-natural transformations between 2-representations of fiat 2-categories. This is done in Section 2, which also contains all necessary preliminaries. The advantage of our new setup is the fact that 2-natural transformations become closed under isomorphism of functors and under taking inverses of equivalences (see Subsection 2.4).

Cell 2-representations of fiat 2-categories have particularly nice properties for so-called strongly regular cells, see Subsection 2.7. In particular, the main result of [MM4] asserts that in this cases cell 2-representations exhaust all simple transitive 2-representations. This is the main case of our study in this paper. Our main result is that the endomorphism category of such a cell 2-representation is equivalent to $\mathbb{k}\text{-mod}$, see Theorem 5.3.1 in Section 5.

Along the way, we prove two further interesting results. Firstly, we establish 2-fullness for cell 2-representations with respect to the class of 1-morphisms in the two-sided cell, see Corollary 4.4.2 in Subsection 4.4. Secondly, we completely describe fiat 2-categories which have only one two-sided cell \mathcal{J} apart from the identities, in the case when our 2-category is \mathcal{J} -simple in the sense of [MM2], see

Theorem 4.6.1 in Subsection 4.6. This can be viewed as a 2-analogue of Artin-Wedderburn Theorem.

We present various examples in Section 6, including the fiat 2-category of Soergel bimodules acting on the principal block of the BGG category \mathcal{O} and the fiat 2-category associated with the \mathfrak{sl}_2 -categorification of Chuang and Rouquier. Finally, in Section 7, we introduce and investigate a natural setup for the study of graded fiat 2-categories.

Remark. The original version of the paper appeared on arxiv in July 2012. The present version is a substantial revision of the original one which takes into account that since the publication of the original version several results and assumptions became obsolete due to further developments presented in [MM3, MM4, MM5].

Acknowledgment. A substantial part of the paper was written during a visit of the second author to Uppsala University, whose hospitality is gratefully acknowledged. The visit was supported by the Swedish Research Council and the Department of Mathematics. The first author is partially supported by the Swedish Research Council and the Royal Swedish Academy of Sciences. The second author is partially supported by ERC grant PERG07-GA-2010-268109 and EPSRC grant EP/K011782/1.

2. PRELIMINARIES

We denote by \mathbb{N} and \mathbb{N}_0 the sets of positive and non-negative integers, respectively.

2.1. Various 2-categories. In this paper by a 2-category we mean a strict locally small 2-category (see [Le] for a concise introduction to 2-categories and bi-categories). Let \mathcal{C} be a 2-category. We will use i, j, \dots to denote objects in \mathcal{C} ; 1-morphisms in \mathcal{C} will be denoted by F, G, \dots ; 2-morphisms in \mathcal{C} will be denoted by α, β, \dots . For $i \in \mathcal{C}$ we will denote by $\mathbb{1}_i$ the corresponding identity 1-morphisms. For a 1-morphism F we will denote by id_F the corresponding identity 2-morphisms.

Denote by **Cat** the 2-category of all small categories. Let \mathbb{k} be an algebraically closed field. Denote by $\mathfrak{A}_{\mathbb{k}}$ the 2-category whose objects are small \mathbb{k} -linear fully additive categories; 1-morphisms are additive \mathbb{k} -linear functors and 2-morphisms are natural transformations. Denote by $\mathfrak{A}_{\mathbb{k}}^f$ the full 2-subcategory of $\mathfrak{A}_{\mathbb{k}}$ whose objects are fully additive categories \mathcal{A} such that \mathcal{A} has only finitely many isomorphism classes of indecomposable objects and all morphisms spaces in \mathcal{A} are finite dimensional. We also denote by $\mathfrak{R}_{\mathbb{k}}$ the full subcategory of $\mathfrak{A}_{\mathbb{k}}$ containing all objects which are equivalent to $A\text{-mod}$ for some finite dimensional associative \mathbb{k} -algebra A .

2.2. Finitary and fiat 2-categories. A 2-category \mathcal{C} is called *finitary (over \mathbb{k})*, see [MM1], if the following conditions are satisfied:

- \mathcal{C} has finitely many objects;
- for any $i, j \in \mathcal{C}$ we have $\mathcal{C}(i, j) \in \mathfrak{A}_{\mathbb{k}}^f$ and horizontal composition is both additive and \mathbb{k} -linear;

- for any $i \in \mathcal{C}$ the 1-morphism $\mathbb{1}_i$ is indecomposable.

We will call \mathcal{C} *weakly fiat* provided that it has a weak object preserving anti-autoequivalence $*$ and for any 1-morphism $F \in \mathcal{C}(i, j)$ there exist 2-morphisms $\alpha : F \circ F^* \rightarrow \mathbb{1}_j$ and $\beta : \mathbb{1}_i \rightarrow F^* \circ F$ such that $\alpha_F \circ_1 F(\beta) = \text{id}_F$ and $F^*(\alpha) \circ_1 \beta_{F^*} = \text{id}_{F^*}$. If $*$ is involutive, then \mathcal{C} is called *fiat*, see [MM1].

2.3. 2-representations. From now on \mathcal{C} will denote a finitary 2-category. By a 2-representation of \mathcal{C} we mean a strict 2-functor from \mathcal{C} to either \mathfrak{A}_k (additive 2-representation), \mathfrak{A}_k^f (finitary 2-representation), or \mathfrak{R}_k (abelian 2-representation). In this paper we define the 2-categories of 2-representations of \mathcal{C} extending the setup (from the one in [MM1, MM2]) by considering non-strict 2-natural transformations between two 2-representations \mathbf{M} and \mathbf{N} . Such a 2-natural transformation Ψ consists of the following data: a map, which assigns to every $i \in \mathcal{C}$ a functor $\Psi_i : \mathbf{M}(i) \rightarrow \mathbf{N}(i)$, and for any 1-morphism $F \in \mathcal{C}(i, j)$ a natural isomorphism $\eta_F = \eta_F^\Psi : \Psi_j \circ \mathbf{M}(F) \rightarrow \mathbf{N}(F) \circ \Psi_i$, where naturality means that for any $G \in \mathcal{C}(i, j)$ and any $\alpha : F \rightarrow G$ we have

$$\eta_G \circ_1 (\text{id}_{\Psi_j} \circ_0 \mathbf{M}(\alpha)) = (\mathbf{N}(\alpha) \circ_0 \text{id}_{\Psi_i}) \circ_1 \eta_F.$$

In other words, the left diagram on the following picture commutes up to η_F while the right diagram commutes (compare with [Kh, Subsection 2.2]):

$$\begin{array}{ccc} \mathbf{M}(i) & \xrightarrow{\mathbf{M}(F)} & \mathbf{M}(j) \\ \Psi_i \downarrow & \nearrow \eta_F & \downarrow \Psi_j \\ \mathbf{N}(i) & \xrightarrow{\mathbf{N}(F)} & \mathbf{N}(j) \end{array} \quad \begin{array}{ccc} \Psi_j \circ \mathbf{M}(F) & \xrightarrow{\eta_F} & \mathbf{N}(F) \circ \Psi_i \\ \text{id}_{\Psi_j} \circ_0 \mathbf{M}(\alpha) \downarrow & & \downarrow \mathbf{N}(\alpha) \circ_0 \text{id}_{\Psi_i} \\ \Psi_j \circ \mathbf{M}(G) & \xrightarrow{\eta_G} & \mathbf{N}(G) \circ \Psi_i \end{array}$$

Moreover, the isomorphisms η should satisfy

$$(1) \quad \eta_{F \circ_0 G} = (\text{id}_{\mathbf{N}(F)} \circ_0 \eta_G) \circ_1 (\eta_F \circ_0 \text{id}_{\mathbf{M}(G)})$$

for all composable 1-morphisms F and G .

Given two 2-natural transformations Ψ and Φ as above, a modification $\theta : \Psi \rightarrow \Phi$ is a map which assigns to each $i \in \mathcal{C}$ a natural transformation $\theta_i : \Psi_i \rightarrow \Phi_i$ such that for any $F, G \in \mathcal{C}(i, j)$ and any $\alpha : F \rightarrow G$ we have

$$(2) \quad \eta_G^\Phi \circ_1 (\theta_j \circ_0 \mathbf{M}(\alpha)) = (\mathbf{N}(\alpha) \circ_0 \theta_i) \circ_1 \eta_F^\Psi.$$

Proposition 2.3.1. *Together with non-strict 2-natural transformations and modifications as defined above, 2-representations of \mathcal{C} form a 2-category.*

Our notation for these 2-categories is $\mathcal{C}\text{-amod}$ in the case of additive representations and $\mathcal{C}\text{-afmod}$ in the case of finitary representations. To define the 2-category $\mathcal{C}\text{-mod}$ for abelian representations we additionally assume that all Ψ_i are right exact (this assumption is missing in [MM1]).

Proof. To check that these are 2-categories, we have to verify that (strict) composition of non-strict 2-natural transformations is a non-strict 2-natural transformation and that both horizontal and vertical compositions of modifications are modifications. The first fact follows by defining

$$\eta_F^{\Psi' \circ \Psi} := (\eta_F^{\Psi'} \circ_0 \text{id}_{\Psi_i}) \circ_1 (\text{id}_{\Psi'_j} \circ_0 \eta_F^\Psi)$$

and then checking (1) (which is a straightforward computation). Since the diagrams

$$\begin{array}{ccccc}
\Psi'_j \circ \Psi_j \circ \mathbf{M}(F) & \xrightarrow{\text{id}_{\Psi'_j} \circ \theta_j \circ \text{id}_{\mathbf{M}(F)}}} & \Psi'_j \circ \Phi_j \circ \mathbf{M}(F) & \xrightarrow{\theta'_j \circ \text{id}_{\Phi_j} \circ \text{id}_{\mathbf{M}(F)}}} & \Phi'_j \circ \Phi_j \circ \mathbf{M}(F) \\
\text{id}_{\Psi'_j} \circ \eta_F^\Psi \downarrow & & \text{id}_{\Psi'_j} \circ \eta_F^\Phi \downarrow & & \text{id}_{\Phi'_j} \circ \eta_F^\Phi \downarrow \\
\Psi'_j \circ \mathbf{N}(F) \circ \Psi_i & \xrightarrow{\text{id}_{\Psi'_j} \circ \text{id}_{\mathbf{N}(F)} \circ \theta_i} & \Psi'_j \circ \mathbf{N}(F) \circ \Phi_i & \xrightarrow{\theta'_j \circ \text{id}_{\mathbf{N}(F)} \circ \text{id}_{\Phi_i}} & \Phi'_j \circ \mathbf{N}(F) \circ \Phi_i \\
\eta_F^{\Psi'} \circ \text{id}_{\Psi_i} \downarrow & & \eta_F^{\Psi'} \circ \text{id}_{\Phi_i} \downarrow & & \eta_F^{\Phi'} \circ \text{id}_{\Phi_i} \downarrow \\
\mathbf{K}(F) \circ \Psi'_i \circ \Psi_i & \xrightarrow{\text{id}_{\mathbf{K}(F)} \circ \text{id}_{\Psi'_i} \circ \theta_i} & \mathbf{K}(F) \circ \Psi'_i \circ \Phi_i & \xrightarrow{\text{id}_{\mathbf{K}(F)} \circ \theta'_i \circ \text{id}_{\Phi_i}} & \mathbf{K}(F) \circ \Phi'_i \circ \Phi_i
\end{array}$$

$$\begin{array}{ccccc}
\Psi_j \circ \mathbf{M}(F) & \xrightarrow{\theta_j \circ \mathbf{M}(\alpha)} & \Phi_j \circ \mathbf{M}(G) & \xrightarrow{\tau_j \circ \text{id}_{\mathbf{M}(G)}} & \Sigma_j \circ \mathbf{M}(G) \\
\eta_F^\Psi \downarrow & & \eta_G^\Phi \downarrow & & \eta_G^\Sigma \downarrow \\
\mathbf{N}(F) \circ \Psi_i & \xrightarrow{\mathbf{N}(\alpha) \circ \theta_i} & \mathbf{N}(G) \circ \Phi_i & \xrightarrow{\text{id}_{\mathbf{N}(G)} \circ \tau_i} & \mathbf{N}(G) \circ \Sigma_i
\end{array}$$

commute, the latter two facts also follow. \square

2.4. Properties of 2-natural transformations. Let \mathbf{M} and \mathbf{N} be two 2-representations of \mathcal{C} and $\Psi : \mathbf{M} \rightarrow \mathbf{N}$ a 2-natural transformation. Given, for every $i \in \mathcal{C}$, a functor Φ_i and an isomorphism $\xi_i : \Phi_i \rightarrow \Psi_i$, define, for every 1-morphism $F \in \mathcal{C}(i, j)$

$$\eta_F^\Phi := (\text{id}_{\mathbf{N}(F)} \circ \xi_i^{-1}) \circ \eta_F^\Psi \circ (\xi_j \circ \text{id}_{\mathbf{M}(F)}).$$

Then it is straightforward to check that this extends Φ to a 2-natural transformation.

Proposition 2.4.1. *Let \mathbf{M} and \mathbf{N} be two 2-representations of \mathcal{C} and $\Psi : \mathbf{M} \rightarrow \mathbf{N}$ a 2-natural transformation. Assume that for every $i \in \mathcal{C}$ the functor Ψ_i is an equivalence. Then there exists an inverse 2-natural transformation.*

Proof. For any $i \in \mathcal{C}$ choose an inverse equivalence Φ_i of Ψ_i . Let

$$\xi_i : \text{Id}_{\mathbf{M}(i)} \rightarrow \Phi_i \circ \Psi_i \quad \text{and} \quad \zeta_i : \Psi_i \circ \Phi_i \rightarrow \text{Id}_{\mathbf{N}(i)}$$

be some isomorphisms. Define

$$\eta_F^\Phi := ((\text{id}_{\Phi_j \circ \mathbf{N}(F)} \circ \zeta_i) \circ (\text{id}_{\Phi_j} \circ \eta_F^\Psi \circ \text{id}_{\Phi_i}) \circ (\xi_j \circ \text{id}_{\mathbf{M}(F) \circ \Phi_i}))^{-1}.$$

It is obvious that this produces a natural transformation, but we have to check that

$$(3) \quad \eta_{F \circ G}^\Phi = (\text{id}_{\mathbf{N}(F)} \circ \eta_G^\Phi) \circ (\eta_F^\Phi \circ \text{id}_{\mathbf{M}(G)}).$$

This follows from commutativity of the diagram

$$\begin{array}{ccccc}
 & & \mathbf{M}(\mathbf{F})\mathbf{M}(\mathbf{G})\Phi_i & & \\
 & \swarrow & & \searrow & \\
 & \Phi_k\Psi_k\mathbf{M}(\mathbf{F})\mathbf{M}(\mathbf{G})\Phi_i & & \mathbf{M}(\mathbf{F})\Phi_j\Psi_j\mathbf{M}(\mathbf{G})\Phi_i & \\
 & \swarrow & & \searrow & \\
 \Phi_k\mathbf{N}(\mathbf{F})\Psi_j\mathbf{M}(\mathbf{G})\Phi_i & & \Phi_k\Psi_k\mathbf{M}(\mathbf{F})\Phi_j\Psi_j\mathbf{M}(\mathbf{G})\Phi_i & & \mathbf{M}(\mathbf{F})\Phi_j\mathbf{N}(\mathbf{G})\Psi_i\Phi_i \\
 & \swarrow & \searrow & \swarrow & \searrow \\
 & \Phi_k\mathbf{N}(\mathbf{F})\Psi_j\Phi_j\Psi_j\mathbf{M}(\mathbf{G})\Phi_i & & \Phi_k\Psi_k\mathbf{M}(\mathbf{F})\Phi_j\mathbf{N}(\mathbf{G})\Psi_i\Phi_i & \mathbf{M}(\mathbf{F})\Phi_j\mathbf{N}(\mathbf{G}) \\
 & \swarrow & \searrow & \swarrow & \searrow \\
 \Phi_k\mathbf{N}(\mathbf{F})\Psi_j\mathbf{M}(\mathbf{G})\Phi_i & & \Phi_k\mathbf{N}(\mathbf{F})\Psi_j\Phi_j\mathbf{N}(\mathbf{G})\Psi_i\Phi_i & & \Phi_k\Psi_k\mathbf{M}(\mathbf{F})\Phi_j\mathbf{N}(\mathbf{G}) \\
 & \swarrow & \searrow & \swarrow & \searrow \\
 & \Phi_k\mathbf{N}(\mathbf{F})\mathbf{N}(\mathbf{G})\Psi_i\Phi_i & & \Phi_k\mathbf{N}(\mathbf{F})\Psi_j\Phi_j\mathbf{N}(\mathbf{G}) & \\
 & \swarrow & & \searrow & \\
 & & \Phi_k\mathbf{N}(\mathbf{F})\mathbf{N}(\mathbf{G}) & &
 \end{array}$$

where the maps are the obvious ones (each of the maps has exactly one component of the form ξ, ζ or η^Ψ and identities elsewhere). Commutativity of all squares is immediate. Then reading along the right border gives (the inverse of) the right hand side of (3). Computing (the inverse of) the left hand side of (3) directly, using the definition of η^Φ and property (1) of $\eta_{\mathbf{F} \circ \mathbf{G}}^\Psi$, gives the left border of the diagram, after noting that the third and fourth morphism in this path compose to the identity on $\Phi_k\mathbf{N}(\mathbf{F})\Psi_j\mathbf{M}(\mathbf{G})\Phi_i$ by adjunction. Therefore (3) holds and this extends Φ to a 2-natural transformation. \square

In this scenario we will say that the 2-representations \mathbf{M} and \mathbf{N} are *equivalent*.

2.5. Abelianization and identities. Denote by $\bar{\cdot} : \mathcal{C}\text{-afmod} \rightarrow \mathcal{C}\text{-mod}$ the abelianization 2-functor defined as in [MM2, Subsection 4.2]: for $\mathbf{M} \in \mathcal{C}\text{-afmod}$ and $\mathbf{i} \in \mathcal{C}$, the category $\overline{\mathbf{M}}(\mathbf{i})$ consists of all diagrams of the form $X \xrightarrow{\alpha} Y$, where $X, Y \in \mathbf{M}(\mathbf{i})$ and α is a morphism in $\mathbf{M}(\mathbf{i})$. Morphisms in $\overline{\mathbf{M}}(\mathbf{i})$ are commutative squares modulo factorization of the right downwards arrow using a homotopy. The 2-action of \mathcal{C} on $\overline{\mathbf{M}}(\mathbf{i})$ is defined component-wise.

For any 2-representation \mathbf{M} of \mathcal{C} and any non-negative integer k , we denote by \spadesuit_k the 2-natural transformation from \mathbf{M} to \mathbf{M} given by assigning to each $\mathbf{i} \in \mathcal{C}$ the functor

$$\underbrace{\text{Id}_{\mathbf{M}(\mathbf{i})} \oplus \text{Id}_{\mathbf{M}(\mathbf{i})} \oplus \cdots \oplus \text{Id}_{\mathbf{M}(\mathbf{i})}}_{k \text{ summands}}$$

and defining $\eta_{\mathbf{F}}^{\spadesuit_k}$ as $\text{id}_{\mathbf{F}} \oplus \cdots \oplus \text{id}_{\mathbf{F}}$ (again with k summands).

2.6. Principal 2-representations and additive subrepresentations. For $\mathbf{i} \in \mathcal{C}$ we denote by $\mathbf{P}_{\mathbf{i}}$ the principal 2-representation $\mathcal{C}(\mathbf{i}, -) \in \mathcal{C}\text{-afmod}$. For any $\mathbf{M} \in \mathcal{C}\text{-afmod}$ we have the usual Yoneda Lemma (see [Le, Subsection 2.1] and compare to [MM2, Lemma 9]):

Lemma 2.6.1.

$$(4) \quad \text{Hom}_{\mathcal{C}\text{-afmod}}(\mathbf{P}_{\mathbf{i}}, \mathbf{M}) \cong \mathbf{M}(\mathbf{i}).$$

Proof. Let $\Psi : \mathbf{P}_{\mathbf{i}} \rightarrow \mathbf{M}$ be a 2-natural transformation and set $X := \Psi_{\mathbf{i}}(\mathbb{1}_{\mathbf{i}})$. Denote by $\Phi : \mathbf{P}_{\mathbf{i}} \rightarrow \mathbf{M}$ the unique strict 2-natural transformation sending $\mathbb{1}_{\mathbf{i}}$ to X (see [MM2, Lemma 9]). Then, for any 1-morphism $F \in \mathcal{C}(\mathbf{i}, \mathbf{j})$, we have the natural isomorphism

$$(\theta_j)_F := (\eta_F^\Psi)_{\mathbf{i}} : \Psi_j(F) \rightarrow \mathbf{M}(F) \Psi_{\mathbf{i}}(\mathbb{1}_{\mathbf{i}}) = \mathbf{M}(F) X = \Phi_j(F).$$

This gives us an (invertible) modification θ from Ψ to Φ and the claim follows. \square

Given $\mathbf{M} \in \mathcal{C}\text{-mod}$ and $X \in \mathbf{M}(\mathbf{i})$ for some $\mathbf{i} \in \mathcal{C}$, define $\mathbf{M}_X \in \mathcal{C}\text{-afmod}$ by restricting \mathbf{M} to the full subcategories $\text{add}(F X)$, where F runs through the set of all 1-morphisms in $\mathcal{C}(\mathbf{i}, \mathbf{j})$, $\mathbf{j} \in \mathcal{C}$.

2.7. The multisemigroup of \mathcal{C} and cells. The set $\mathcal{S}[\mathcal{C}]$ of isomorphism classes of indecomposable 1-morphisms in \mathcal{C} has the natural structure of a multisemigroup induced by horizontal composition, see [MM2, Subsection 3.1] (see also [KM] for more details on multisemigroups). Let \leq_L , \leq_R and \leq_J denote the natural left, right and two-sided orders on $\mathcal{S}[\mathcal{C}]$, respectively. For example, $F \leq_L G$ means that for some 1-morphism H the composition $H \circ F$ contains a direct summand isomorphic to G . Equivalence classes with respect to \leq_L are called *left cells*. Right and two-sided cells are defined analogously. Cells correspond exactly to Green's equivalence classes for the multisemigroup $\mathcal{S}[\mathcal{C}]$.

A two-sided cell \mathcal{J} is called *regular* if different left (right) cells in \mathcal{J} are not comparable with respect to the left (right) order. A two-sided cell \mathcal{J} is called *strongly regular* if it is regular and, moreover, the intersection of any left and any right cell inside \mathcal{J} consists of exactly one element.

Given a left cell \mathcal{L} , there exists an $\mathbf{i}_{\mathcal{L}} \in \mathcal{C}$ such that every 1-morphism $F \in \mathcal{L}$ belongs to $\mathcal{C}(\mathbf{i}_{\mathcal{L}}, \mathbf{j})$ for some $\mathbf{j} \in \mathcal{C}$. Similarly, given a right cell \mathcal{R} , there exists a $\mathbf{j}_{\mathcal{R}} \in \mathcal{C}$ such that every 1-morphism $F \in \mathcal{R}$ belongs to $\mathcal{C}(\mathbf{i}, \mathbf{j}_{\mathcal{R}})$ for some $\mathbf{i} \in \mathcal{C}$.

2.8. Cell 2-representations. Let \mathcal{L} be a left cell and $\mathbf{i} = \mathbf{i}_{\mathcal{L}}$. Consider $\overline{\mathbf{P}}_{\mathbf{i}}$. For an indecomposable 1-morphism F in some $\mathcal{C}(\mathbf{i}, \mathbf{j})$ denote by L_F the unique simple top of the indecomposable projective module $0 \rightarrow F$ in $\overline{\mathbf{P}}_{\mathbf{i}}(\mathbf{j})$. By [MM1, Proposition 17], there exists a unique $G_{\mathcal{L}} \in \mathcal{L}$ (called the *Duflo involution* in \mathcal{L}) such that the indecomposable projective module $0 \rightarrow \mathbb{1}_{\mathbf{i}}$ has a unique quotient N such that the simple socle of N is isomorphic to $L_{G_{\mathcal{L}}}$ and $F N / L_{G_{\mathcal{L}}} = 0$ for any $F \in \mathcal{L}$. Set $Q := G_{\mathcal{L}} L_{G_{\mathcal{L}}}$. Then the additive 2-representation $\mathbf{C}_{\mathcal{L}} := (\overline{\mathbf{P}}_{\mathbf{i}})_Q$ is called the *additive cell 2-representation* of \mathcal{C} associated to \mathcal{L} . The abelianization $\overline{\mathbf{C}}_{\mathcal{L}}$ of $\mathbf{C}_{\mathcal{L}}$ is called the *abelian cell 2-representation* of \mathcal{C} associated to \mathcal{L} . For $F \in \mathcal{L}$ we set $P_F := F L_{G_{\mathcal{L}}}$, which we also identify with the indecomposable projective object $0 \rightarrow F L_{G_{\mathcal{L}}}$ in $\overline{\mathbf{C}}_{\mathcal{L}}$.

3. A SPECIAL CASE OF 2-SCHUR'S LEMMA

In this section we prove a special case of Theorem 5.1.1 under one additional assumption of surjectivity of the action of the center. It turns out that this assumption of surjectivity allows us to use a short and elegant argument.

3.1. The claim. The following is a special case of Theorem 5.1.1:

Theorem 3.1.1. *Let \mathcal{C} be a fiat 2-category, \mathcal{J} a strongly regular two-sided cell of \mathcal{C} and \mathcal{L} a left cell in \mathcal{J} . Set $\mathbf{i} = \mathbf{i}_{\mathcal{L}}$ and $\mathbf{G} = \mathbf{G}_{\mathcal{L}}$. Assume that the natural map*

$$(5) \quad \begin{array}{ccc} \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathbf{i}}) & \longrightarrow & \mathrm{End}_{\mathbf{C}_{\mathcal{L}}}(P_{\mathbf{G}}) \\ \varphi & \mapsto & \mathbf{C}_{\mathcal{L}}(\varphi)_{P_{\mathbf{G}}} \end{array}$$

is surjective. Then any endomorphism of $\mathbf{C}_{\mathcal{L}}$ is isomorphic to \spadesuit_k for some k (in the category $\mathrm{End}_{\mathcal{C}\text{-afmod}}(\mathbf{C}_{\mathcal{L}})$). Similarly, any endomorphism of $\overline{\mathbf{C}}_{\mathcal{L}}$ is isomorphic to \spadesuit_k for some k (in the category $\mathrm{End}_{\mathcal{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$).

3.2. Annihilators of various objects in $\overline{\mathbf{C}}_{\mathcal{L}}$. For any 2-representation \mathbf{M} of \mathcal{C} and $X \in \mathbf{M}(\mathbf{j})$ for some \mathbf{j} , let $\mathrm{Ann}_{\mathcal{C}}(X)$ denote the left 2-ideal of \mathcal{C} consisting of all 2-morphisms α which annihilate X . The key observation to prove Theorem 3.1.1 is the following:

Lemma 3.2.1. *Under the assumption of Theorem 3.1.1, if $X \in \overline{\mathbf{C}}_{\mathcal{L}}(\mathbf{i})$ is such that $\mathrm{Ann}_{\mathcal{C}}(X) \supset \mathrm{Ann}_{\mathcal{C}}(L_{\mathbf{G}})$, then $X \in \mathrm{add}(L_{\mathbf{G}})$.*

Proof. Let $\mathbf{F} \in \mathcal{L}$ be different from \mathbf{G} . Then $\mathbf{F}^* L_{\mathbf{F}} \neq 0$ by [MM1, Lemma 15]. At the same time, from the fact that \mathcal{J} is strongly simple it follows that $\mathbf{F}^* \notin \mathcal{L}$. Therefore $\mathbf{F}^* L_{\mathbf{G}} = 0$ by [MM1, Lemma 15]. Hence $\mathrm{id}_{\mathbf{F}^*} \in \mathrm{Ann}_{\mathcal{C}}(L_{\mathbf{G}})$ and at the same time $\mathrm{id}_{\mathbf{F}^*} \notin \mathrm{Ann}_{\mathcal{C}}(L_{\mathbf{F}})$.

Since \mathbf{F}^* is exact, the previous paragraph implies that for any X satisfying $\mathrm{Ann}_{\mathcal{C}}(X) \supset \mathrm{Ann}_{\mathcal{C}}(L_{\mathbf{G}})$, every simple subquotient of X is isomorphic to $L_{\mathbf{G}}$. Assume now that X is indecomposable such that there is a short exact sequence

$$0 \rightarrow L_{\mathbf{G}} \rightarrow X \rightarrow L_{\mathbf{G}} \rightarrow 0.$$

Then there is a short exact sequence $K \hookrightarrow P_{\mathbf{G}} \rightarrow X$ and an endomorphism of $P_{\mathbf{G}}$ which induces a non-trivial nilpotent endomorphism of X . From (5), it follows that the natural map

$$\begin{array}{ccc} \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathbf{i}}) & \longrightarrow & \mathrm{End}_{\mathbf{C}_{\mathcal{L}}}(X) \\ \varphi & \mapsto & \mathbf{C}_{\mathcal{L}}(\varphi)_X \end{array}$$

is surjective. Let $\alpha \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathbf{i}})$ be a 2-morphism which produces a non-trivial nilpotent endomorphism of X . Then $\alpha \notin \mathrm{Ann}_{\mathcal{C}}(X)$ while $\alpha^2 \in \mathrm{Ann}_{\mathcal{C}}(X)$. At the same time, $\mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathbf{i}})$ is a local finite dimensional \mathbb{k} -algebra (see Subsection 2.2), and hence α is either nilpotent or invertible. But α cannot be invertible as α^2 annihilates X . Therefore, α is nilpotent. This implies that $\alpha \in \mathrm{Ann}_{\mathcal{C}}(L_{\mathbf{G}})$ as any nonzero endomorphism of $L_{\mathbf{G}}$ is invertible by Schur's lemma.

Finally, if Y is an indecomposable module, every simple subquotient of which is isomorphic to $L_{\mathbf{G}}$, then Y has a subquotient X as in the previous paragraph. Therefore $\mathrm{Ann}_{\mathcal{C}}(L_{\mathbf{G}}) \not\subset \mathrm{Ann}_{\mathcal{C}}(Y)$. The claim of the lemma follows. \square

3.3. Proof of Theorem 3.1.1. Let $\Psi \in \text{End}_{\mathcal{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$. By Lemma 3.2.1, we have $\Psi_i(L_G) \cong L_G^{\oplus k}$ for some non-negative integer k . Now for any $F \in \mathcal{L}$ we have an isomorphism

$$\Psi_j(P_F) = \Psi_j(F L_G) \cong F L_G^{\oplus k} \cong P_F^{\oplus k},$$

natural in F . As Ψ_j is right exact, every indecomposable projective is of the form P_F , and 2-morphisms in \mathcal{C} surject onto homomorphisms between indecomposable projectives (see [MM1, Subsection 4.5]), we have that Ψ_j is isomorphic to $\text{Id}_{\overline{\mathbf{C}}_{\mathcal{L}}^{(j)}}^{\oplus k}$. Clearly, k does not depend on j . Now we repeat the argument from the proof of Lemma 2.6.1. We have the natural isomorphisms

$$(\theta_j)_{F L_G} := (\eta_F^{\Psi})_{L_G} : \Psi_j \circ \overline{\mathbf{C}}_{\mathcal{L}}(F) L_G \rightarrow \overline{\mathbf{C}}_{\mathcal{L}}(F) \circ (\spadesuit_k)_i L_G = \overline{\mathbf{C}}_{\mathcal{L}}(F) L_G^{\oplus k},$$

which give us an invertible modification θ from Ψ to \spadesuit_k . This proves the abelian part of Theorem 3.1.1.

To prove the additive part we just note that any $\Psi \in \text{End}_{\mathcal{C}\text{-mod}}(\mathbf{C}_{\mathcal{L}})$ abelianizes to $\overline{\Psi} \in \text{End}_{\mathcal{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$. Now the additive claim of Theorem 3.1.1 follows from the abelian claim by restricting to projective modules. \square

4. DESCRIPTION OF \mathcal{J} -SIMPLE FIAT 2-CATEGORIES

4.1. Definition of 2-full 2-representations. Let \mathcal{C} be a finitary category and \mathbf{M} a 2-representation of \mathcal{C} . We will say that \mathbf{M} is *2-full* provided that for any 1-morphisms $F, G \in \mathcal{C}$ the representation map

$$(6) \quad \text{Hom}_{\mathcal{C}}(F, G) \rightarrow \text{Hom}_{\mathfrak{X}}(\mathbf{M}(F), \mathbf{M}(G)),$$

where $\mathfrak{X} \in \{\mathfrak{A}_k, \mathfrak{A}_k^f, \mathfrak{R}_k\}$ is the target 2-category of \mathbf{M} , is surjective. In other words, 2-morphisms in \mathcal{C} surject onto the space of natural transformations between functors.

If \mathcal{J} is a 2-sided cell of \mathcal{C} , we will say that \mathbf{M} is *\mathcal{J} -2-full* provided that for any 1-morphisms $F, G \in \mathcal{J}$ the representation map (6) is surjective.

4.2. The 2-category associated with \mathcal{J} . Let now \mathcal{C} be a fiat 2-category and \mathcal{J} a two-sided cell in \mathcal{C} . Let \mathcal{L} be a left cell of \mathcal{J} , $G := G_{\mathcal{L}}$ and $i := i_{\mathcal{L}}$. Let \mathcal{I} be the unique maximal 2-ideal of \mathcal{C} which does not contain id_F for any $F \in \mathcal{J}$ (see [MM2, Theorem 15]). Then the quotient 2-category \mathcal{C}/\mathcal{I} is \mathcal{J} -simple (see [MM2, Subsection 6.2]). Denote by $\mathcal{C}^{(\mathcal{J})}$ the 2-full 2-subcategory of \mathcal{C}/\mathcal{I} generated by $\mathbb{1}_{i_{\mathcal{L}}}$ and all $F \in \mathcal{J}$ (and closed with respect to isomorphism of 1-morphisms). We will call $\mathcal{C}^{(\mathcal{J})}$ the *\mathcal{J} -simple 2-category associated to \mathcal{J}* .

The cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of \mathcal{C} factors over \mathcal{C}/\mathcal{I} by [MM2, Theorem 19] and hence restricts to a 2-representation of $\mathcal{C}^{(\mathcal{J})}$. Assume now that \mathcal{J} is strongly regular. Then, by [MM1, Proposition 32], \mathcal{J} remains a strongly regular two-sided cell in $\mathcal{C}^{(\mathcal{J})}$. Moreover, using [MM2, Subsection 6.5], the restriction of $\mathbf{C}_{\mathcal{L}}$ to $\mathcal{C}^{(\mathcal{J})}$ is equivalent to the corresponding cell 2-representation of $\mathcal{C}^{(\mathcal{J})}$.

For the remainder of this section we fix a strongly regular cell \mathcal{J} and assume that $\mathcal{C} = \mathcal{C}^{(\mathcal{J})}$.

4.3. Detecting 2-fullness. We consider the cell 2-representation $\mathbf{M} := \overline{\mathbf{C}}_{\mathcal{L}}$. We start our analysis with the following observation:

Proposition 4.3.1. *For $F \in \mathcal{J}$ and $j \in \mathcal{C}$ consider the representation map*

$$(7) \quad \text{Hom}_{\mathcal{C}}(F, \mathbb{1}_j) \rightarrow \text{Hom}_{\mathfrak{A}_{\mathbb{k}}}(\mathbf{M}(F), \mathbf{M}(\mathbb{1}_j)).$$

If this map is surjective for $F = G$ and $j = i$, then it is surjective for any F and j .

Note that both sides of (7) are empty unless $F \in \mathcal{C}(j, j)$. As usual, to simplify notation we will use the module notation and write $F X$ instead of $\mathbf{M}(F)(X)$.

Proof. Let $H, K \in \mathcal{L}$ and assume that $H, K \in \mathcal{C}(i, j)$. By strong regularity of \mathcal{J} we have $HK^* = aF$ for some $F \in \mathcal{J}$ and $a \in \mathbb{N}$, moreover, if we vary H and K , we can obtain any $F \in \mathcal{J}$ in this way. To see that $HK^* \neq 0$, one evaluates HK^* on L_K obtaining $K^*L_K = P_G$ (by [MM1, Corollary 38(a)]), and $HP_G \neq 0$ since $HL_G = P_H \neq 0$.

Similarly, we have $K^*H = bG$ for some $b \in \mathbb{N}$ since K^*H is in the same left cell as H (which is \mathcal{L}) and the same right cell as K^* (which is \mathcal{L}^*), and $\mathcal{L} \cap \mathcal{L}^* = \{G\}$ since \mathcal{J} is strongly regular. Using the involution $*$ we have

$$\text{Hom}_{\mathcal{C}}(H, K) \cong \text{Hom}_{\mathcal{C}}(K^*, H^*).$$

By adjunction, we have

$$(8) \quad \text{Hom}_{\mathcal{C}}(H, K) \cong b\text{Hom}_{\mathcal{C}}(G, \mathbb{1}_i), \quad \text{Hom}_{\mathcal{C}}(K^*, H^*) \cong a\text{Hom}_{\mathcal{C}}(F, \mathbb{1}_j).$$

Evaluating $\text{Hom}_{\mathcal{C}}(H, K)$ at L_G (which is surjective by [MM1, Subsection 4.5]) and using adjunction, we get

$$\text{Hom}_{\mathbf{M}(j)}(H L_G, K L_G) \cong b\text{Hom}_{\mathbf{M}(i)}(G L_G, L_G).$$

As $G L_G \cong P_G$, the space $\text{Hom}_{\mathbf{M}(i)}(G L_G, L_G)$ is one-dimensional, and thus

$$(9) \quad b = \dim \text{Hom}_{\mathbf{M}(j)}(H L_G, K L_G)$$

On the other hand, evaluating $\text{Hom}_{\mathcal{C}}(K^*, H^*)$ at a multiplicity free direct sum L of all simple modules in $\mathbf{M}(j)$ and using adjunction, we have

$$(10) \quad \text{Hom}_{\mathbf{M}(i)}(K^* L, H^* L) \cong a\text{Hom}_{\mathbf{M}(j)}(F L, L).$$

By [MM1, Lemma 12], $K^* L_Q \neq 0$ for a direct summand L_Q of L , labeled by $Q \in \mathcal{L}$, implies that K is in the same right cell as Q . Strong regularity implies $Q = K$ and by [MM1, Corollary 38(a)], we have $K^* L \cong P_G$. Similarly $H^* L \cong P_G$ and the left hand side of (10) is isomorphic to $\text{End}_{\mathbf{M}(i)}(P_G)$.

As F is a direct summand of HK^* , again L_K is the only simple module which is not annihilated by F . By [MM5, Theorem 31], the module $F L_K$ is an indecomposable projective in $\mathbf{M}(j)$, namely P_H . This means that $\dim \text{Hom}_{\mathbf{M}(j)}(F L, L) = 1$ and hence

$$(11) \quad a = \dim \text{End}_{\mathbf{M}(i)}(P_G).$$

To proceed we need the following claim:

Lemma 4.3.2. *Let A be a finite dimensional \mathbb{k} -algebra and $e, f \in A$ primitive idempotents. Assume that F is an exact endofunctor of $A\text{-mod}$ such that $F L_f \cong Ae$ and $F L_g = 0$ for any simple $L_g \not\cong L_f$. Then F is isomorphic to the functor F' given by tensoring with the bimodule $Ae \otimes_{\mathbb{k}} fA$ and, moreover,*

$$\text{Hom}_{\mathfrak{A}_{\mathbb{k}}}(F, \text{Id}_{A\text{-mod}}) \cong \text{Hom}_A(Ae, Af).$$

Proof. Let L be a multiplicity free sum of all simple A -modules. As $F L_f$ has simple top L_e , it follows that F is a quotient of F' , which gives us a surjective natural transformation $\alpha : F' \rightarrow F$. Further, $F L \cong F' L$, meaning that α is an isomorphism when evaluated on simple modules. Using induction on the length of a module and the 3-Lemma we obtain that α is an isomorphism, which proves the first claim. The second claim follows by adjunction. \square

From Lemma 4.3.2 and surjectivity of (7) for G , we get

$$\dim \operatorname{Hom}_{\mathcal{C}}(G, \mathbb{1}_i) = \dim \operatorname{End}_{\mathbf{M}(i)}(P_G).$$

Using (8), (9) and Lemma 4.3.2, we have

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{C}}(H, K) &= \dim \operatorname{Hom}_{\mathbf{M}(j)}(H L_G, K L_G) \cdot \dim \operatorname{End}_{\mathbf{M}(i)}(P_G) \\ &= \dim \operatorname{Hom}_{\mathbf{M}(j)}(P_H, P_K) \cdot \dim \operatorname{End}_{\mathbf{M}(i)}(P_G). \end{aligned}$$

On the other hand, using (8) and (11) we have

$$\dim \operatorname{Hom}_{\mathcal{C}}(K^*, H^*) = \dim \operatorname{Hom}_{\mathcal{C}}(F, \mathbb{1}_j) \cdot \dim \operatorname{End}_{\mathbf{M}(i)}(P_G).$$

As \mathcal{C} is \mathcal{J} -simple, $\dim \operatorname{Hom}_{\mathcal{C}}(F, \mathbb{1}_j) \leq \dim \operatorname{Hom}_{\mathfrak{A}_k}(\mathbf{M}(F), \mathbf{M}(\mathbb{1}_j))$ and the latter by Lemma 4.3.2 is equal to $\dim \operatorname{Hom}_{\mathbf{M}(j)}(P_H, P_K)$. Dividing through by $\dim \operatorname{End}_{\mathbf{M}(i)}(P_G)$ yields

$$\begin{aligned} \dim \operatorname{Hom}_{\mathbf{M}(j)}(P_H, P_K) &= \dim \operatorname{Hom}_{\mathcal{C}}(F, \mathbb{1}_j) \\ &\leq \dim \operatorname{Hom}_{\mathfrak{A}_k}(\mathbf{M}(F), \mathbf{M}(\mathbb{1}_j)) \\ &= \dim \operatorname{Hom}_{\mathbf{M}(j)}(P_H, P_K) \end{aligned}$$

and hence

$$\dim \operatorname{Hom}_{\mathcal{C}}(F, \mathbb{1}_j) = \dim \operatorname{Hom}_{\mathfrak{A}_k}(\mathbf{M}(F), \mathbf{M}(\mathbb{1}_j)).$$

Injectivity of the representation map, which follows from \mathcal{J} -simplicity of \mathcal{C} , now implies surjectivity and hence the statement of the proposition. \square

Proposition 4.3.3. *Let $H, K \in \mathcal{C}(j, k) \cap \mathcal{J}$. If the representation map (7) is surjective for $F = G$ and $i = j$, then the representation map*

$$\operatorname{Hom}_{\mathcal{C}}(H, K) \rightarrow \operatorname{Hom}_{\mathfrak{A}_k}(\mathbf{M}(H), \mathbf{M}(K))$$

is surjective.

Proof. As \mathcal{J} is strongly regular, we have $K^*H = Q^{\oplus m}$ for some $m \in \mathbb{N}_0$, where Q is in the intersection of the left cell of H and the right cell of K^* . We have the commutative diagram

$$\begin{array}{ccccc} \operatorname{Hom}_{\mathcal{C}}(H, K) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathcal{C}}(K^*H, \mathbb{1}_j) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathcal{C}}(Q, \mathbb{1}_j)^{\oplus m} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathfrak{A}_k}(\mathbf{M}(H), \mathbf{M}(K)) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathfrak{A}_k}(\mathbf{M}(K^*H), \operatorname{Id}_{\mathbf{M}(j)}) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathfrak{A}_k}(\mathbf{M}(Q), \operatorname{Id}_{\mathbf{M}(j)})^{\oplus m} \end{array}$$

where the vertical arrows are the representation maps, the left horizontal arrows are isomorphisms given by adjunction, and the right horizontal arrows are isomorphisms given by additivity. Then the rightmost vertical arrow is an isomorphism by Proposition 4.3.1 and \mathcal{J} -simplicity of \mathcal{C} . This implies that all vertical arrows are isomorphisms and the claim follows. \square

4.4. Cell 2-representations are \mathcal{J} -2-full.

Theorem 4.4.1. *The cell 2-representation $\mathbf{M} := \overline{\mathbf{C}}_{\mathcal{L}}$ is \mathcal{J} -2-full.*

Proof. Thanks to Proposition 4.3.3, we have only to show that the representation map (7) is surjective for $F = G$ and $\mathbf{i} = \mathbf{j}$. In order to show this it suffices, by Lemma 4.3.2 and \mathcal{J} -simplicity of \mathcal{C} , to show that

$$\dim \mathrm{Hom}_{\mathcal{C}}(G, \mathbb{1}_{\mathbf{i}}) = \dim \mathrm{End}_{\mathbf{M}(\mathbf{i})}(P_G).$$

By Lemma 4.3.2 and \mathcal{J} -simplicity of \mathcal{C} , we have

$$\dim \mathrm{Hom}_{\mathcal{C}}(G, \mathbb{1}_{\mathbf{i}}) \leq \dim \mathrm{End}_{\mathbf{M}(\mathbf{i})}(P_G).$$

Recall from [MM1, Proposition 17] that there is a unique submodule K of the indecomposable projective module $0 \rightarrow \mathbb{1}_{\mathbf{i}}$ in $\overline{\mathbf{P}}_{\mathbf{i}}(\mathbf{i})$ which has simple top L_G and such that the quotient of the projective by K is annihilated by G . We denote by β some 2-morphism from G to $\mathbb{1}_{\mathbf{i}}$ which gives rise to a surjection from $0 \rightarrow G$ to K in $\overline{\mathbf{P}}_{\mathbf{i}}(\mathbf{i})$. Then the $\mathrm{End}_{\mathcal{C}}(G)$ -module $\mathrm{Hom}_{\mathcal{C}}(G, \mathbb{1}_{\mathbf{i}})$ has simple top and β is a representative for this simple top.

Let A be a basic finite dimensional associative \mathbb{k} -algebra such that $\mathbf{M}(\mathbf{i}) \cong A\text{-mod}$. Let $1 = \sum_{i=1}^n e_i$ be a decomposition of $1 \in A$ into a sum of pairwise orthogonal primitive idempotents. We assume that $e = e_1$ is a primitive idempotent corresponding to L_G . From Lemma 4.3.2, we have that the functor $\mathbf{M}(G)$ is isomorphic to tensoring with $Ae \otimes_{\mathbb{k}} eA$. Clearly, $\mathbf{M}(\mathbb{1}_{\mathbf{i}})$ is isomorphic to tensoring with A .

Since \mathcal{J} is strongly regular, Duflo involutions in $\mathcal{J} \cap \mathcal{C}(\mathbf{i}, \mathbf{i})$ are in bijection with $\{e_1, e_2, \dots, e_n\}$. Let G_i be the Duflo involution corresponding to e_i . Similarly to the existence of β , there is a β_i for each i , which we can put into the 2-morphism

$$\gamma := (\beta_1, \beta_2, \dots, \beta_n) : \bigoplus_i G_i \rightarrow \mathbb{1}_{\mathbf{i}}.$$

The cokernel $\mathrm{Coker}(\gamma)$, as an object of $\overline{\mathbf{P}}_{\mathbf{i}}$, is annihilated by all 1-morphisms in \mathcal{J} . This implies that $\mathbf{M}(\mathrm{Coker}(\gamma))$ annihilates L_F for every $F \in \mathcal{L}$ and hence $\mathbf{M}(\mathrm{Coker}(\gamma)) = 0$ by right exactness of $\mathbf{M}(\mathrm{Coker}(\gamma))$. From this we derive that $\mathbf{M}(\gamma)$ is surjective and hence we can choose β and the above identifications of functors with bimodules such that $\mathbf{M}(\beta)$ is the multiplication map $Ae \otimes_{\mathbb{k}} eA \rightarrow A$.

In order to show that $\dim \mathrm{Hom}_{\mathcal{C}}(G, \mathbb{1}_{\mathbf{i}}) \geq \dim \mathrm{End}_{\mathbf{M}(\mathbf{i})}(P_G)$, we show that no $\varphi \in \mathrm{End}_{\mathcal{C}}(G)$ that induces a nonzero endomorphism of P_G when evaluated at L_G , is sent to zero under composition with β .

In order to see this, let $\varphi \in \mathrm{End}_{\mathcal{C}}(G)$ be such that $\mathbf{M}(\varphi) \in eAe \otimes eAe$ is not killed under the map $eAe \otimes eAe \rightarrow eAe \otimes eAe / \mathrm{Rad}(eAe) \cong eAe$. In other words, writing $\mathbf{M}(\varphi) = \sum_j (\psi_j \otimes (c_j e + r_j))$ for some $c_j \in \mathbb{k}, r_j \in \mathrm{Rad}(eAe)$, and where ψ_j runs over a basis of eAe , chosen in accordance with radical powers, we have that $\psi := \sum_j c_j \psi_j$ is nonzero in eAe . Then $\mathbf{M}(\beta \circ \varphi) = \psi + (\sum_j c_j \psi_j r_j) \in eAe$. As $\psi \in \mathrm{Rad}^k(eAe)$ implies $\psi_j \in \mathrm{Rad}^k(eAe)$ for all ψ_j such that $c_j \neq 0$, the summand $\sum_j c_j \psi_j r_j$ is in $\mathrm{Rad}^{k+1}(eAe)$ and hence $\mathbf{M}(\beta \circ \varphi) \in \mathrm{Hom}_{\mathfrak{N}_{\mathbb{k}}}(\mathbf{M}(G), \mathbf{M}(\mathbb{1}_{\mathbf{i}}))$ is nonzero. Therefore $\beta \circ \varphi \in \mathrm{Hom}_{\mathcal{C}}(G, \mathbb{1}_{\mathbf{i}})$ is nonzero for any $\varphi \in \mathrm{End}_{\mathcal{C}}(G)$ that is not killed by evaluation at L_G . By surjectivity of the map from $\mathrm{End}_{\mathcal{C}}(G)$ onto $\mathrm{End}_{\mathbf{M}(\mathbf{i})}(P_G)$ given by evaluation at L_G (see [MM1, Subsection 4.5]), this implies

$$\dim \mathrm{Hom}_{\mathcal{C}}(G, \mathbb{1}_{\mathbf{i}}) \geq \dim \mathrm{End}_{\mathbf{M}(\mathbf{i})}(P_G)$$

and completes the proof of the proposition. \square

Corollary 4.4.2. *Assume that \mathcal{C} is any fiat 2-category and \mathcal{J} is a strongly regular 2-sided cell of \mathcal{C} . Then for any left cell \mathcal{L} in \mathcal{J} the cell 2-representation $\overline{\mathcal{C}}_{\mathcal{L}}$ is \mathcal{J} -2-full.*

Proof. This follows directly from Theorem 4.4.1 and [MM1, Corollary 33]. \square

4.5. Construction of \mathcal{J} -simple 2-categories $\mathcal{C}^{(\mathcal{J})}$. Let $n \in \mathbb{N}$ and $A := (A_1, A_2, \dots, A_n)$ be a collection of pairwise non-isomorphic, basic, connected, weakly symmetric finite dimensional associative \mathbb{k} -algebras. For $i \in \{1, 2, \dots, n\}$ choose some small category \mathcal{C}_i equivalent to $A_i\text{-mod}$, and let Z_i denote the center of A_i . Set $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n)$. Denote by $\mathcal{C}_{\mathcal{C}}$ the 2-full fiat 2-subcategory of $\mathfrak{R}_{\mathbb{k}}$ with objects \mathcal{C}_i , which is closed under isomorphisms of 1-morphisms and generated by functors that are isomorphic to tensoring with projective A_i - A_j bimodules.

We identify Z_i with $\text{End}_{\mathcal{C}_i}(\mathbb{1}_{\mathcal{C}_i})$ and denote by Z'_i the subalgebra of Z_i generated by $\text{id}_{\mathbb{1}_{\mathcal{C}_i}}$ and all elements which factor through 1-morphisms given by tensoring with projective A_i - A_i bimodules.

Remark 4.5.1. In general, $Z'_i \neq Z_i$. For example, let $n = 1$ and $A = A_1 = \mathbb{k}[x]/(x^3)$. Then $Z = Z_1 = A$ while Z'_1 is the linear span of 1 and x^2 in Z . Indeed, we have only one projective bimodule $A \otimes_{\mathbb{k}} A$, which has Loewy length 5 and unique Loewy filtration. As A has Loewy length 3, any nonzero composition $A \rightarrow A \otimes_{\mathbb{k}} A \rightarrow A$ must map the top of A to the socle of A . It is easy to check that the composition of the unique (up to scalar) injection $A \hookrightarrow A \otimes_{\mathbb{k}} A$ and the unique (up to scalar) surjection $A \otimes_{\mathbb{k}} A \rightarrow A$ is nonzero.

Choose subalgebras X_i in Z_i containing Z'_i and let $X = (X_1, X_2, \dots, X_n)$. Consider the additive 2-subcategory $\mathcal{C}_{\mathcal{C}, X}$ of $\mathcal{C}_{\mathcal{C}}$ defined as follows: $\mathcal{C}_{\mathcal{C}, X}$ has the same objects and the same 1-morphisms as $\mathcal{C}_{\mathcal{C}}$; all 2-morphism spaces between indecomposable 1-morphisms in $\mathcal{C}_{\mathcal{C}, X}$ are the same as for $\mathcal{C}_{\mathcal{C}}$ except for $\text{End}_{\mathcal{C}_{\mathcal{C}, X}}(\mathbb{1}_{\mathcal{C}_i}) := X_i$.

Lemma 4.5.2. *The 2-category $\mathcal{C}_{\mathcal{C}, X}$ is well-defined and fiat.*

Proof. To prove that $\mathcal{C}_{\mathcal{C}, X}$ is well-defined we have to check that it is closed under both horizontal and vertical composition of 2-morphisms. That it is closed under vertical composition follows directly from the fact that X_i is a subalgebra. To check that it is closed under horizontal composition, we first observe that if $\mathbb{1}_{\mathcal{C}_i}$ appears (up to isomorphism) as a direct summand of $F \circ G$ for some indecomposable 1-morphisms F and G , then both F and G are isomorphic to $\mathbb{1}_{\mathcal{C}_i}$. For $x, y \in X_i$, we have

$$\begin{array}{ccccccc} A & \xrightarrow{\sim} & A \otimes_A A & \xrightarrow{x \otimes y} & A \otimes_A A & \xrightarrow{\sim} & A \\ 1 & \mapsto & 1 \otimes 1 & \mapsto & x \otimes y & \mapsto & xy \end{array}$$

from which the claim follows, again using that X_i is a subalgebra.

To prove that $\mathcal{C}_{\mathcal{C}, X}$ is fiat we have to check that it contains all adjunction morphisms. The adjunction morphism from $\mathbb{1}_{\mathcal{C}_i}$ to $\mathbb{1}_{\mathcal{C}_i}$ is $\text{id}_{\mathbb{1}_{\mathcal{C}_i}}$ and thus contained in $\mathcal{C}_{\mathcal{C}, X}$. All other adjunction morphisms are between $\mathbb{1}_{\mathcal{C}_i}$ and direct sums of indecomposable 1-morphisms none of which is isomorphic to $\mathbb{1}_{\mathcal{C}_i}$ and therefore contained in $\mathcal{C}_{\mathcal{C}, X}$ by definition. \square

4.6. Description of \mathcal{J} -simple 2-categories $\mathcal{C}^{(\mathcal{J})}$. Now we are ready to prove the main result of this section, which gives a description, up to biequivalence, of fiat 2-categories that are “simple” in some sense.

Theorem 4.6.1. *Let $\mathcal{C} = \mathcal{C}(\mathcal{J})$ be a skeletal fiat \mathcal{J} -simple 2-category and assume that \mathcal{J} is strongly regular. Then \mathcal{C} is biequivalent to $\mathcal{C}_{\mathcal{C}, X}$ for appropriate \mathcal{C} and X .*

Proof. Let \mathcal{L} be a left cell in \mathcal{J} and $\mathbf{M} := \overline{\mathbf{C}}_{\mathcal{L}}$ be the corresponding cell 2-representation. Set $\mathcal{C}_i := \mathbf{M}(\mathbf{i})$ and let A_i be a basic algebra such that $A_i\text{-mod}$ is equivalent to $\mathbf{M}(\mathbf{i})$. Let Z_i be the center of A_i which we identify with $\text{End}_{\mathfrak{R}_k}(\mathbb{1}_{\mathbf{M}(\mathbf{i})})$. Set $X_i := \mathbf{M}(\text{End}_{\mathcal{C}}(\mathbb{1}_i)) \subset Z_i$. Then the representation map \mathbf{M} is a 2-functor from \mathcal{C} to $\mathcal{C}_{\mathcal{C}, X}$, which is a biequivalence by Theorem 4.4.1, \mathcal{J} -simplicity of \mathcal{C} and construction of X . \square

5. 2-SCHUR'S LEMMA

5.1. The first layer of 2-Schur's lemma. Here we prove the following generalization of Theorem 3.1.1.

Theorem 5.1.1. *Let \mathcal{C} be a fiat 2-category and \mathcal{J} a strongly regular two-sided cell of \mathcal{C} . Let \mathcal{L} be a left cell of \mathcal{J} . Then any endomorphism of $\mathbf{C}_{\mathcal{L}}$ is isomorphic to \spadesuit_k for some k (in the category $\text{End}_{\mathcal{C}\text{-afmod}}(\mathbf{C}_{\mathcal{L}})$). Similarly, any endomorphism of $\overline{\mathbf{C}}_{\mathcal{L}}$ is isomorphic to \spadesuit_k for some k (in the category $\text{End}_{\mathcal{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$).*

Proof. We follow the proof of Theorem 3.1.1 described in Section 3. What we need is an analogue of Lemma 3.2.1 in the new situation. More precise, we have to prove that given a non-split short exact sequence

$$0 \rightarrow L_G \rightarrow X \rightarrow L_G \rightarrow 0$$

in $\overline{\mathbf{C}}_{\mathcal{L}}(\mathbf{i})$, the obvious inclusion $\text{Ann}_{\mathcal{C}}(X) \subset \text{Ann}_{\mathcal{C}}(L_G)$ is strict.

As in Subsection 4.4, $\overline{\mathbf{C}}_{\mathcal{L}}(\mathbf{i})$ is equivalent to $A\text{-mod}$ for some finite dimensional associative \mathbb{k} -algebra A and the functor $\overline{\mathbf{C}}_{\mathcal{L}}(\mathbf{G})$ can be identified with tensoring with $Ae \otimes_{\mathbb{k}} eA$ for some primitive idempotent $e \in A$. By Theorem 4.4.1, this identification is fully faithful on 2-morphisms. Clearly,

$$\text{Ann}_{\mathcal{C}}(L_G) \cap \text{End}_{A \otimes_{\mathbb{k}} A^{\text{op}}}(Ae \otimes_{\mathbb{k}} eA) = eAe \otimes_{\mathbb{k}} \text{Rad}(eAe).$$

At the same time, as X is a non-split self-extension of L_G , we have

$$\text{Ann}_{\mathcal{C}}(X) \cap \text{End}_{A \otimes_{\mathbb{k}} A^{\text{op}}}(Ae \otimes_{\mathbb{k}} eA) = eAe \otimes_{\mathbb{k}} U,$$

where U is a proper subalgebra of $\text{Rad}(eAe)$ (since $eA \otimes_A X = eX = X$ as a vector space). The rest of the proof follows precisely the proof of Theorem 3.1.1. \square

5.2. Endomorphisms of the identity functor. So far we have only determined the *objects* in the endomorphism category of a cell 2-representation (Theorems 3.1.1 and 5.1.1) up to isomorphism. Now we would like to describe morphisms in this category.

Proposition 5.2.1. *Let \mathcal{C} be a fiat 2-category, \mathcal{J} a strongly regular two-sided cell of \mathcal{C} and \mathcal{L} a left cell in \mathcal{J} . For any $k \in \mathbb{N}$, consider $\spadesuit_k \in \text{End}_{\mathcal{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})$ (or $\spadesuit_k \in \text{End}_{\mathcal{C}\text{-mod}}(\mathbf{C}_{\mathcal{L}})$). Then there are isomorphisms*

$$\text{End}_{\text{End}_{\mathcal{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}})}(\spadesuit_k) \cong \text{Mat}_{k \times k}(\mathbb{k}) \quad \text{and} \quad \text{End}_{\text{End}_{\mathcal{C}\text{-mod}}(\mathbf{C}_{\mathcal{L}})}(\spadesuit_k) \cong \text{Mat}_{k \times k}(\mathbb{k}).$$

Proof. We prove the statement for $\overline{\mathbf{C}}_{\mathcal{L}}$, the other case being analogous. For $\mathbf{i} \in \mathcal{C}$, let $A_{\mathbf{i}}$ be a finite dimensional associative \mathbb{k} -algebra such that $\overline{\mathbf{C}}_{\mathcal{L}}(\mathbf{i})$ is equivalent to $A_{\mathbf{i}}\text{-mod}$. Let $\theta : \spadesuit_k \rightarrow \spadesuit_k$ be a modification. As endomorphisms of $\text{Id}_{\overline{\mathbf{C}}_{\mathcal{L}}(\mathbf{i})}$ can be identified with the center $Z_{\mathbf{i}}$ of $A_{\mathbf{i}}$, we can view $\theta_{\mathbf{i}}$ as an element of $\text{Mat}_{k \times k}(Z_{\mathbf{i}})$.

First consider the case $k = 1$. Clearly, scalars belong to the endomorphism ring of \spadesuit_1 . We would like to show that the radical of Z_i does not. Let e be a primitive idempotent of A_i . From [MM1, Corollary 38(b)] it follows that there is $F \in \mathcal{J}$ such that $\overline{\mathcal{C}}_{\mathcal{L}}(F)$ can be described by tensoring with a direct sum of bimodules of the form $A_i e \otimes_{\mathbb{k}} e A_i$. The action of \spadesuit_1 on $\overline{\mathcal{C}}_{\mathcal{L}}(i)$ is described as tensoring with A_i , and the isomorphism $\eta_{\mathbb{F}^1}$ is a direct sum of morphisms

$$A_i \otimes_{A_i} A_i e \otimes_{\mathbb{k}} e A_i \cong A_i e \otimes_{\mathbb{k}} e A_i \otimes_{A_i} A_i$$

sending $1 \otimes e \otimes e$ to $e \otimes e \otimes 1$.

Let $0 \neq z \in e \text{Rad}(Z_i)e$. Then applying z after η sends $1 \otimes e \otimes e$ to $e \otimes e \otimes z$, which is identified with $e \otimes z$ in $A_i e \otimes_{\mathbb{k}} e A_i$. Applying z before η sends $1 \otimes e \otimes e$ to $z \otimes e \otimes 1$, which is identified with $z \otimes e$ in $A_i e \otimes_{\mathbb{k}} e A_i$. We have $e \otimes z \neq z \otimes e$ as $z \in e \text{Rad}(Z_i)e$.

Now consider arbitrary k . From the above it follows that we can view θ_i as an element of $\text{Mat}_{k \times k}(\mathbb{k})$ (here $\mathbb{k} \cong Z_i / \text{Rad}(Z_i)$). That every element $M \in \text{Mat}_{k \times k}(\mathbb{k})$ indeed defines an element of $\text{End}_{\text{End}_{\mathcal{C}\text{-mod}}(\overline{\mathcal{C}}_{\mathcal{L}})}(\spadesuit_k)$ can be seen from the commutative diagram

$$\begin{array}{ccc} A^{\oplus k} \otimes_A Ae \otimes_{\mathbb{k}} eA & \xrightarrow{\eta_k} & Ae \otimes_{\mathbb{k}} eA \otimes_A A^{\oplus k} \\ \downarrow M \otimes \text{id} & & \downarrow \text{id} \otimes M \\ A^{\oplus k} \otimes_A Ae \otimes_{\mathbb{k}} eA & \xrightarrow{\eta_k} & Ae \otimes_{\mathbb{k}} eA \otimes_A A^{\oplus k} \end{array}$$

where $A := A_i$ and η_k is the diagonal $k \times k$ -matrix with η on the diagonal. This completes the proof. \square

5.3. The second layer of 2-Schur's lemma. Our main result is the following statement.

Theorem 5.3.1. *Let \mathcal{C} be a fiat 2-category, \mathcal{J} a strongly regular two-sided cell of \mathcal{C} and \mathcal{L} a left cell in \mathcal{J} . Then both categories $\text{End}_{\mathcal{C}\text{-mod}}(\overline{\mathcal{C}}_{\mathcal{L}})$ and $\text{End}_{\mathcal{C}\text{-amod}}(\mathcal{C}_{\mathcal{L}})$ are equivalent to $\mathbb{k}\text{-mod}$.*

Proof. This follows directly from Theorems 3.1.1 and 5.1.1 and Proposition 5.2.1. \square

6. EXAMPLES

6.1. Category \mathcal{O} in type A . Consider the simple complex Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ with the standard triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and a small category \mathcal{O}_0 equivalent to the principal block of the BGG-category \mathcal{O} for \mathfrak{g} (see [Hu]). Let \mathcal{S} be the 2-category of projective functors associated to \mathcal{O}_0 as in [MM1, Subsection 7.1]. Indecomposable 1-morphisms in \mathcal{S} are in natural bijection with elements of the symmetric group S_n (the Weyl group of \mathfrak{g}) and left, right and two-sided cells are Kazhdan-Lusztig right, left and two-sided cells, respectively. As shown in [MM1, Subsection 7.1], all two-sided cells are strongly regular. Hence Theorem 5.3.1 completely describes the endomorphism category of all cell 2-representations for \mathcal{S} (the latter were first constructed in [MS2]). As cell 2-representations corresponding to the same two-sided cell are equivalent (see [MS2, MM1]), it follows that this equivalence is unique (as a functor) up to isomorphism of functors. In [MS2], equivalence of cell 2-representations corresponding to the same two-sided cell was

obtained using Arkhipov's twisting functors and the fact that they naturally commute with projective functors, see [AS]. Our present result shows that the shadows of Arkhipov's twisting functors act, on a cell 2-representation, simply as a direct sum of the identity.

We also would like to note that in this example we can also apply Theorem 3.1.1. A very special feature of S_n is that every two-sided Kazhdan-Lusztig cell of S_n contains the longest element $w := w_0^P$ in some parabolic subgroup P in S_n . Then w is the Duflo involution in its Kazhdan-Lusztig right cell and hence the corresponding projective in the cell 2-representation is isomorphic to $\theta_w L_w$. From [MS1, Theorem 6.3] it follows that the center of \mathcal{O}_0 surjects onto the endomorphism algebra of $\theta_w L_w$ and hence we can apply Theorem 3.1.1.

6.2. Category \mathcal{O} in type B_2 . Consider the previous example for \mathfrak{g} of type B_2 . Let W be the Weyl group of type B_2 with elements $\{e, s, t, st, ts, sts, tst, stst\}$ (here $s^2 = t^2 = e$ and $stst = tsts$). We have the 2-category \mathcal{S} with 1-morphisms θ_w , $w \in W$. Cells are again given by Kazhdan-Lusztig combinatorics, the two-sided cells are $\mathcal{J}_e = \{e\}$, $\mathcal{J}_{s,t} = \{s, t, st, ts, sts, tst\}$ and $\mathcal{J}_{stst} = \{stst\}$. The middle cell splits into two left cells $\mathcal{L}_1 = \{s, st, sts\}$ and $\mathcal{L}_2 = \{t, ts, tst\}$ (recall that our left cells are Kazhdan-Lusztig's right cells and vice versa) as shown in the following picture:

		\mathcal{L}_1		\mathcal{L}_2	
\mathcal{L}_1^*		$\{s, sts\}$		$\{ts\}$	
\mathcal{L}_2^*		$\{st\}$		$\{t, tst\}$	

Since strong regularity fails, we cannot apply Theorem 5.3.1 and, indeed, it turns out that the cell 2-representation $\overline{\mathcal{C}}_{\mathcal{L}_1}$ has more endomorphisms than just the identity, as we now show.

For $w \in \mathcal{L}_i$, $i = 1, 2$, set $L_w := L_{\theta_w}$. Let T_s and T_t be Arkhipov's twisting functors corresponding to s and t . Starting from $\overline{\mathcal{C}}_{\mathcal{L}_1}$ we apply T_s , project onto $\overline{\mathcal{C}}_{\mathcal{L}_2}$, apply T_t and project onto $\overline{\mathcal{C}}_{\mathcal{L}_1}$. This maps L_s to $L_s \oplus L_{sts}$. As twisting functors naturally commute with projective functors, it follows that $\text{Ann}_{\mathcal{S}}(L_s) = \text{Ann}_{\mathcal{S}}(L_{sts})$ and hence mapping L_s to L_{sts} extends to an endomorphism of $\overline{\mathcal{C}}_{\mathcal{L}_1}$ which is clearly not isomorphic to the identity functor.

6.3. \mathfrak{sl}_2 -categorification. Consider the 2-category \mathcal{B}_n associated with the \mathfrak{sl}_2 -categorification of Chuang and Rouquier (see [CR]) as described in detail in [MM2, Subsection 7.1]. This is a fiat 2-category with strongly regular cells. Hence Theorem 5.3.1 completely describes endomorphisms for each cell 2-representation of \mathfrak{sl}_2 (compare [CR, Proposition 5.26]). However, we would like to point out that in the case of \mathcal{B}_n describing the endomorphism category for cell 2-representations is much easier (than e.g. for the example in Subsection 6.1). Indeed, as explained in [MM2, Subsection 7.1], each two-sided cell of \mathcal{B}_n has a left cell with Duflo involution G such that, in the corresponding cell 2-representation, the simple module L_G is projective (the corresponding Duflo involution has the form $\mathbb{1}_i$). Due to this, any endomorphism of the cell 2-representation maps L_G to a direct sum of copies of L_G and is uniquely determined by the image of L_G up to isomorphism.

6.4. A non-symmetric local algebra. In this subsection we describe an example for which the additional assumption of Theorem 3.1.1 fails, while the conditions in Theorem 5.3.1 are satisfied. Let $A := \mathbb{k}\langle x, y \rangle / (x^2, y^2, xy + yx)$ and \mathcal{C} be a small category equivalent to $A\text{-mod}$. The center Z of A is the linear span of 1 and xy .

Consider the fiat 2-category $\mathcal{C}_{\mathcal{C}, Z}$. This category has two two-sided cells, one consisting of the identity and the other one, say \mathcal{J} , consisting of the 1-morphism G given by tensoring with $A \otimes_{\mathbb{k}} A$. Then G is the Duflo involution in \mathcal{J} and the corresponding cell 2-representation is equivalent to the defining 2-representation. Therefore, the projective module P_G is isomorphic to ${}_A A$. Since A is not commutative, Z does not surject on the endomorphism algebra of P_G . Hence the additional assumption of Theorem 3.1.1 is not satisfied. On the other hand, the conditions in Theorem 5.3.1 are satisfied as explained in [MM1, Subsection 7.3].

7. GRADED FIAT 2-CATEGORIES

In the original version of the paper, the main results of this paper were stated under an additional numerical assumption which was shown to be redundant in [MM5]. The original version of this section contained an argument that the numerical assumption is satisfied for graded fiat 2-categories. Although the result itself is no longer interesting, the setup of graded fiat 2-categories is of interest (as most of the natural examples of fiat 2-categories are graded) and this is what is presented in this section, leading up to an analogue of Lusztig's \mathbf{a} -function for graded fiat 2-categories.

In this section, by *graded* we always mean \mathbb{Z} -graded.

7.1. 2-categories with free \mathbb{Z} -action. Let \mathcal{A} be 2-category. Assume that, for each $\mathbf{i}, \mathbf{j} \in \mathcal{A}$, we are given an automorphism $(\cdot)_1$ of $\mathcal{A}(\mathbf{i}, \mathbf{j})$. For $k \in \mathbb{Z}$, set $(\cdot)_k := (\cdot)_1^k$ and, for $F \in \mathcal{A}(\mathbf{i}, \mathbf{j})$, set $F_k := (F)_k$. We will say that this datum defines a *free* action of \mathbb{Z} on \mathcal{A} provided that, for any $F \in \mathcal{A}(\mathbf{i}, \mathbf{j})$, the equality $F_k = F_m$ implies $k = m$ and, moreover, for any composable 1-morphisms F and G , we have

$$(12) \quad F_k \circ G_m = (F \circ G)_{k+m}.$$

Example 7.1.1. Let A be a graded, connected, weakly symmetric finite dimensional associative \mathbb{k} -algebra and \mathcal{C} a small category equivalent to the category $A\text{-gmod}$ of finite dimensional graded A -modules. The algebra $A \otimes_{\mathbb{k}} A^{\text{op}}$ inherits the structure of a graded algebra from A . Let $\langle 1 \rangle$ denote the functor which shifts the grading such that $(M(1))_i = M_{i+1}$, $i \in \mathbb{Z}$. Consider the 2-category $\mathcal{C}_{\mathcal{C}}$ defined as follows: It has one object (which we identify with \mathcal{C}), its 1-morphisms are closed under isomorphism of functors and are generated by $\langle \pm 1 \rangle$ and functors induced by tensoring with projective A - A -bimodules (the latter are naturally graded), its 2-morphisms are natural transformations of functors (which correspond to homogeneous bimodule morphisms of degree zero). The group \mathbb{Z} acts on $\mathcal{C}_{\mathcal{C}}$ by shifting the grading and this is free in the above sense.

7.2. Graded fiat 2-categories. Assume that \mathcal{A} is a 2-category equipped with a free action of \mathbb{Z} . Assume further that \mathcal{A} satisfies the following conditions:

- \mathcal{A} has finitely many objects;
- for any $\mathbf{i}, \mathbf{j} \in \mathcal{A}$, we have $\mathcal{A}(\mathbf{i}, \mathbf{j}) \in \mathfrak{A}_{\mathbb{k}}$ and horizontal composition is both additive and \mathbb{k} -linear;
- the set of \mathbb{Z} -orbits on isomorphism classes of indecomposable objects in $\mathcal{A}(\mathbf{i}, \mathbf{j})$ is finite;
- all spaces of 2-morphisms are finite dimensional;

- for each 1-morphism F , there are only finitely many indecomposable 1-morphisms G (up to isomorphism) such that $\text{Hom}_{\mathcal{A}}(F, G) \neq 0$;
- for each 1-morphism F , there are only finitely many indecomposable 1-morphisms G (up to isomorphism) such that $\text{Hom}_{\mathcal{A}}(G, F) \neq 0$;
- for any $i \in \mathcal{C}$ the 1-morphism $\mathbb{1}_i$ is indecomposable;
- \mathcal{A} has a weak object preserving involution and adjunction morphisms.

We will call such \mathcal{A} *pro-fiat*.

Define the quotient 2-category $\mathcal{C} = \mathcal{A}/\mathbb{Z}$ to have the same objects as \mathcal{A} , and as morphism categories the categorical quotients $\mathcal{C}(i, j) := \mathcal{A}(i, j)/\mathbb{Z}$. Recall that objects of $\mathcal{A}(i, j)/\mathbb{Z}$ are orbits of \mathbb{Z} acting on objects of $\mathcal{A}(i, j)$ (for $F \in \mathcal{A}(i, j)$, we will denote the corresponding orbit by F_{\bullet}) and, for $F, G \in \mathcal{A}(i, j)$, the space $\text{Hom}_{\mathcal{C}}(F_{\bullet}, G_{\bullet})$ is the quotient of $\bigoplus_{k, l \in \mathbb{Z}} \text{Hom}_{\mathcal{A}(i, j)}(F_k, G_l)$ modulo the subspace generated by the expressions $\alpha - \alpha_l$ for $l \in \mathbb{Z}$. Horizontal composition in \mathcal{C} is induced by the one in \mathcal{A} in the natural way (which is well-defined due to (12)). We denote by $\Omega : \mathcal{A} \rightarrow \mathcal{C}$ the projection 2-functor.

Thanks to our assumptions on \mathcal{A} , the 2-category \mathcal{C} is a fiat 2-category. We will say that \mathcal{C} is a *graded fiat* 2-category. If we fix a representative F_s in each F_{\bullet} , then, by construction, the category $\mathcal{C}(i, j)$ becomes graded (in the sense that for any 1-morphisms F_{\bullet}, G_{\bullet} we have

$$\text{Hom}_{\mathcal{C}}(F_{\bullet}, G_{\bullet}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}^i(F_{\bullet}, G_{\bullet}),$$

where G_t is our fixed representative for G_{\bullet} and $\text{Hom}_{\mathcal{C}}^i(F_{\bullet}, G_{\bullet}) = \text{Hom}_{\mathcal{A}}(F_s, G_{t+i})$, vertical composition being additive on degrees). We will say that this grading is *positive* provided that the following condition is satisfied: for any indecomposable 1-morphisms $F_{\bullet}, G_{\bullet} \in \mathcal{C}$, the inequality $\text{Hom}_{\mathcal{C}}^i(F_{\bullet}, G_{\bullet}) \neq 0$ implies $i > 0$ unless $F_{\bullet} = G_{\bullet}$. In the latter case we require $\text{End}_{\mathcal{C}}^0(F_{\bullet}) = \mathbb{k} \text{id}_{F_{\bullet}}$.

Example 7.2.1. Let $D = \mathbb{k}[x]/(x^2)$ with x in degree 2 and consider $\mathcal{C}_{\mathcal{C}}$ as in Example 7.1.1 for some \mathcal{C} equivalent to $D\text{-gmod}$. Choosing the representatives $\text{Id}_{D\text{-gmod}}$ and $(D \otimes_{\mathbb{k}} D \otimes_D -)(1)$ makes $\mathcal{C}_{\mathcal{C}}/\mathbb{Z}$ into a positively graded 2-category.

7.3. From 2-representations of \mathcal{A} to 2-representations of \mathcal{C} . Let \mathcal{A} be a pro-fiat 2-category and $\mathcal{C} := \mathcal{A}/\mathbb{Z}$. Let \mathbf{M} be a 2-representation of \mathcal{A} and $i \in \mathcal{A}$. Then the group \mathbb{Z} acts (strictly) on $\mathbf{M}(i)$ via isomorphisms $\mathbb{1}_{i, k}$, $k \in \mathbb{Z}$. We call \mathbf{M} *pro-graded* if this action is free (i.e. the stabilizer of every object is trivial) for every i .

Let \mathbf{M} be a pro-graded 2-representation of \mathcal{A} . We define a 2-representation $\underline{\mathbf{M}}$ of \mathcal{C} as follows: For $i \in \mathcal{C}$, we set $\underline{\mathbf{M}}(i) := \mathbf{M}(i)/\mathbb{Z}$, that is objects of $\underline{\mathbf{M}}(i)$ are orbits of \mathbb{Z} acting on objects of $\mathbf{M}(i)$ (for $Q \in \underline{\mathbf{M}}(i)$, we will denote the corresponding orbit by (Q)). For $F \in \mathcal{A}(i, j)$ and $Q \in \underline{\mathbf{M}}(i)$, we define $\underline{\mathbf{M}}(F_{\bullet})(Q) := (\mathbf{M}(F)Q)$ while, for $f : Q \rightarrow P$, mapping the class $\hat{f} : (Q) \rightarrow (P)$ to the class

$$\widehat{\mathbf{M}(F)}f : (\mathbf{M}(F)Q) \rightarrow (\mathbf{M}(F)P)$$

defines the action of $\underline{\mathbf{M}}(F_{\bullet})$ on morphisms (this is well-defined because of the strictness of our \mathbb{Z} -action). Functoriality of $\underline{\mathbf{M}}(F_{\bullet})$ follows directly from the definition. Each $\alpha : F \rightarrow G$ induces a morphism from F_{\bullet} to G_{\bullet} and we define

$$\underline{\mathbf{M}}(\alpha)_{(Q)} : \underline{\mathbf{M}}(F_{\bullet})(Q) \rightarrow \underline{\mathbf{M}}(G_{\bullet})(Q)$$

as the class of $\mathbf{M}(\alpha)_Q : \mathbf{M}(F)Q \rightarrow \mathbf{M}(G)Q$. This extends to all 2-morphisms by additivity. It follows directly from the definitions that $\underline{\mathbf{M}}$ becomes a 2-representation of \mathcal{C} .

7.4. Functoriality of $\underline{\cdot}$. Unfortunately, $\underline{\cdot}$ is not a 2-functor between the 2-categories of 2-representations of \mathcal{A} and $\mathcal{C} = \mathcal{A}/\mathbb{Z}$. However, it turns out to be a 2-functor on a suitably defined subcategory of 2-representations of \mathcal{A} . Define the 2-category $\mathcal{A}\text{-pgamod}$ as follows: objects are pro-graded additive 2-representations of \mathcal{A} ; 1-morphisms are 2-natural transformations satisfying the condition that $\eta_{\mathbf{i},n}$ is the identity map for all \mathbf{i} and n (that is, our 2-natural transformations commute *strictly* with all shifts of the identity); 2-morphisms are modifications. This clearly forms a 2-subcategory in the category of additive 2-representations of \mathcal{A} .

Proposition 7.4.1. *The operation $\underline{\cdot}$ defines a 2-functor from $\mathcal{A}\text{-pgamod}$ to $\mathcal{C}\text{-amod}$.*

Proof. Let $\mathbf{M}, \mathbf{N} \in \mathcal{A}\text{-pgamod}$ and $\Psi \in \text{Hom}_{\mathcal{A}\text{-pgamod}}(\mathbf{M}, \mathbf{N})$. Define $\underline{\Psi} : \underline{\mathbf{M}} \rightarrow \underline{\mathbf{N}}$ by $\underline{\Psi}_{\mathbf{i}}(Q) := (\Psi_{\mathbf{i}}Q)$. This is well defined as $\Psi_{\mathbf{i}}$ commutes strictly with the action of $\mathbb{1}_{\mathbf{i},n}$ and each element in (Q) is obtained by applying some $\mathbb{1}_{\mathbf{i},n}$ to Q . We have to check commutativity of the diagram

$$\begin{array}{ccc} \underline{\Psi}_j \circ \underline{\mathbf{M}}(F_{\bullet}) & \xrightarrow{\eta_{F_{\bullet}}} & \underline{\mathbf{N}}(F_{\bullet}) \circ \underline{\Psi}_{\mathbf{i}} \\ \text{id}_{\underline{\Psi}_j} \circ \text{id}_{\underline{\mathbf{M}}(\alpha)} \downarrow & & \downarrow \underline{\mathbf{N}}(\alpha) \circ \text{id}_{\underline{\Psi}_{\mathbf{i}}} \\ \underline{\Psi}_j \circ \underline{\mathbf{M}}(G_{\bullet}) & \xrightarrow{\eta_{G_{\bullet}}} & \underline{\mathbf{N}}(G_{\bullet}) \circ \underline{\Psi}_{\mathbf{i}} \end{array}$$

for any $\alpha : F \rightarrow G$ in \mathcal{A} (here $\eta_{F_{\bullet}}$ is the class of η_F and similarly for $\eta_{G_{\bullet}}$). To check commutativity of this diagram, we have to evaluate it at any object and it is straightforward to check commutativity there using strict commutativity of Ψ with shifts of the identity. Condition (1) for $\eta_{F_{\bullet}}$ is automatic. This verifies the first level of 2-functoriality.

For a modification $\theta : \Psi \rightarrow \Phi$ in $\mathcal{A}\text{-pgamod}$, we define $\underline{\theta}$ by $\underline{\theta}_{\mathbf{i},(Q)} := \widehat{\theta_{\mathbf{i},Q}}$. We have to check (2), that is commutativity of the diagram

$$\begin{array}{ccc} \underline{\Psi}_j \circ \underline{\mathbf{M}}(F_{\bullet}) & \xrightarrow{\eta_{F_{\bullet}}^{\Psi}} & \underline{\mathbf{N}}(F_{\bullet}) \circ \underline{\Psi}_{\mathbf{i}} \\ \underline{\theta}_j \circ \text{id}_{\underline{\mathbf{M}}(\alpha)} \downarrow & & \downarrow \underline{\mathbf{N}}(\alpha) \circ \text{id}_{\underline{\Psi}_{\mathbf{i}}} \\ \underline{\Phi}_j \circ \underline{\mathbf{M}}(G_{\bullet}) & \xrightarrow{\eta_{G_{\bullet}}^{\Phi}} & \underline{\mathbf{N}}(G_{\bullet}) \circ \underline{\Phi}_{\mathbf{i}} \end{array}$$

which again follows by evaluating it at any object and using strict commutativity of Ψ and Φ with shifts of the identity. \square

7.5. Principal and cell 2-representations of \mathcal{A} . For $\mathbf{i} \in \mathcal{A}$, consider the principal 2-representation $\mathbf{P}_{\mathbf{i}}^{\mathcal{A}}$ of \mathcal{A} .

Proposition 7.5.1. *The 2-representations $\underline{\mathbf{P}}_{\mathbf{i}}^{\mathcal{A}}$ and $\mathbf{P}_{\mathbf{i}}$ of \mathcal{C} are equivalent.*

Proof. First we note that $\mathbf{P}_{\mathbf{i}}^{\mathcal{A}}$ is pro-graded by definition. For $\mathbf{j} \in \mathcal{C}$, the orbits of \mathbb{Z} on $\underline{\mathbf{P}}_{\mathbf{i}}^{\mathcal{A}}(\mathbf{j})$ coincide with the fibers of Ω on $\mathcal{C}(\mathbf{i}, \mathbf{j})$. The equivalence is then defined by mapping the fiber to its image under Ω . \square

Directly from the definitions, we have that $\overline{(\mathbf{M})} = \overline{(\mathbf{M})}$ for any 2-representation \mathbf{M} of \mathcal{A} . Consider the 2-representation $\overline{\mathbf{P}_i^{\mathcal{A}}}$. By definition, each $\overline{\mathbf{P}_i^{\mathcal{A}}}(j)$ is a length category with enough projective objects. For any j , there is a bijection between isomorphism classes of simple objects in $\overline{\mathbf{P}_i^{\mathcal{A}}}(j)$ and \mathbb{Z} -orbits on isomorphism classes of simple objects in $\overline{\mathbf{P}_i^{\mathcal{A}}}(j)$.

The 2-functor Ω induces a bijection between left, right and two-sided cells of \mathcal{A} and \mathcal{C} . Let \mathcal{L} be a left cell in \mathcal{C} and G a 1-morphism in \mathcal{A} such that G_{\bullet} is the Duflo involution in \mathcal{L} . Setting $Q := GL_G$ as in Subsection 2.8, we consider the 2-representation $\mathbf{C}_{\mathcal{L}}^{\mathcal{A}} := (\overline{\mathbf{P}_i^{\mathcal{A}}}(j))_Q$. We leave it to the reader to check that this is the cell 2-representation of \mathcal{A} associated with $\Omega^{-1}(\mathcal{L})$.

Proposition 7.5.2. *The 2-representations $\underline{\mathbf{C}_{\mathcal{L}}^{\mathcal{A}}}$ and $\mathbf{C}_{\mathcal{L}}$ of \mathcal{C} are equivalent.*

Proof. The fact that $\mathbf{C}_{\mathcal{L}}^{\mathcal{A}}$ is pro-graded follows from the definition of $\mathbf{C}_{\mathcal{L}}^{\mathcal{A}}$ and the fact that $\mathbf{P}_i^{\mathcal{A}}$ is pro-graded. Similarly to Proposition 7.5.1, the equivalence is induced by Ω . \square

7.6. Graded adjunctions. Let \mathcal{A} be a pro-fiat 2-category and $\mathcal{C} := \mathcal{A}/\mathbb{Z}$. Let \mathcal{L} be a strongly regular left cell of \mathcal{C} and $\mathbf{i} := \mathbf{i}_{\mathcal{L}}$. We assume that we have chosen some representatives in \mathbb{Z} -orbits such that the induced grading on \mathcal{C} is positive. We also assume that $\mathbb{1}_{\mathbf{i},\bullet}$ is represented by the identity 1-morphism $\mathbb{1}_{\mathbf{i},0}$ in $\mathcal{A}(\mathbf{i}, \mathbf{i})$. Let G_{\bullet} be the Duflo involution for \mathcal{L} and let G be its chosen representative in $\mathcal{A}(\mathbf{i}, \mathbf{i})$.

We have $\text{Hom}_{\mathcal{C}}(G_{\bullet}, \mathbb{1}_{\mathbf{i},\bullet}) \neq 0$ by [MM1, Proposition 17] and hence it makes sense to define \mathbf{a} as the smallest integer such that

$$\text{Hom}_{\mathcal{C}}^{\mathbf{a}}(G_{\bullet}, \mathbb{1}_{\mathbf{i},\bullet}) = \text{Hom}_{\mathcal{A}}(G_{-\mathbf{a}}, \mathbb{1}_{\mathbf{i},0}) \neq 0.$$

This should be thought of as an analogue of Lusztig's \mathbf{a} -function.

Consider the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of \mathcal{C} . By Proposition 7.5.2, we have a positive grading on $\mathbf{C}_{\mathcal{L}}(\mathbf{i})$. Denote by \mathbf{l} the maximal $i \in \mathbb{Z}$ such that $\text{End}^i(P_{G_{\bullet}}) \neq 0$.

Lemma 7.6.1. *We have $G^* \cong G_{1-2\mathbf{a}}$.*

Proof. As $G^* \cong G_{\bullet}$, we have $G^* \cong G_x$ for some $x \in \mathbb{Z}$. As in [MM1, Subsection 4.7], we denote by Δ the unique quotient of $0 \rightarrow \mathbb{1}_{\mathbf{i},0}$ which has simple socle $L_{G_{-\mathbf{a}}}$. We compute:

$$\begin{aligned} 0 &\neq \text{Hom}(G \mathbb{1}_{\mathbf{i},0}, L_G) \\ &\subset \text{Hom}(G \Delta, L_G) \\ &= \text{Hom}(G L_{G_{-\mathbf{a}}}, L_G) \\ &= \text{Hom}(L_{G_{-\mathbf{a}}}, G_x L_G) \\ &= \text{Hom}(L_{G_{-\mathbf{a}}}, G_{x+\mathbf{a}} L_{G_{-\mathbf{a}}}). \end{aligned}$$

Here the third line follows from the fact that G annihilates all subquotients of Δ apart from $L_{G_{-\mathbf{a}}}$ (see [MM1, Proposition 17]), and the fourth line uses adjunction. The module $G_{x+\mathbf{a}} L_{G_{-\mathbf{a}}}$ has simple socle $L_{G_{x+\mathbf{a}-1}}$. Therefore, the inequality $\text{Hom}(L_{G_{-\mathbf{a}}}, G_{x+\mathbf{a}} L_{G_{-\mathbf{a}}}) \neq 0$ means that $-\mathbf{a} = x + \mathbf{a} - \mathbf{l}$, that is $x = \mathbf{l} - 2\mathbf{a}$. \square

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