AMENABILITY AND GEOMETRY OF SEMIGROUPS

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ABSTRACT. We study the connection between amenability, Følner conditions and the geometry of finitely generated semigroups. Using results of Klawe, we show that within an extremely broad class of semigroups (encompassing all groups, left cancellative semigroups, finite semigroups, compact topological semigroups, inverse semigroups, regular semigroups, commutative semigroups and semigroups with a left, right or two-sided zero element), left amenability coincides with the strong Følner condition. Within the same class, we show that a finitely generated semigroup of subexponential growth is left amenable if and only if it is left reversible. We show that the (weak) Følner condition is a left quasi-isometry invariant of finitely generated semigroups, and hence that left amenability is a left quasi-isometry invariant of left cancellative semigroups. We also give a new characterisation of the strong Følner condition, in terms of the existence of *weak* Følner sets satisfying a local injectivity condition on the relevant translation action of the semigroup.

1. INTRODUCTION

What are now called amenable groups were introduced in 1929 by von Neumann [25], motivated by the desire for a group-theoretic understanding of paradoxical decompositions such as the Banach-Tarski paradox. The term "amenable" was coined by Day [9], who also broadened consideration to encompass semigroups. In the decades that followed, amenability — in groups, semigroups and also Banach algebras — has developed into a major topic of study, forming a remarkably deep vein of connections between different areas of mathematics including algebra, analysis, geometry, combinatorics and dynamics; see [20] for a comprehensive introduction and for example [3] for more recent developments. Important recent research on amenable semigroups, specifically, includes for example [2, 4, 10, 26], while the extensive memoir [8] studies the relationship between amenability in semigroups and in related Banach algebras.

In the context of amenability, groups and semigroups have many similarities, with many of the key results being shared. A notable exception, however, appears where finitely generated objects are concerned. Within the study of amenable groups, there is a distinct strand of research focussing on amenability of *finitely generated* groups, and hence linking the subject to combinatorial and geometry group theory. Notable results include

the quasi-isometry invariance of amenability (see [11, Theorem 10.23]) and (building on important work of Kesten [18]) the beautiful characterisation of amenability, due independently to Cohen [7] and Grigorchuk [16], in terms of *cogrowth*. Despite widespread interest in finitely generated semigroups, both within algebra and because of applications in theoretical computer science, amenability of finitely generated semigroups has yet to receive a comparable level of attention. This is, we suggest, because results about amenability for finitely generated groups are largely built on elementary characterisations in terms of *Følner sets* which tend to manifest themselves in natural ways in Cayley graphs, and hence lend themselves to study using the tools and techniques of geometric group theory. This approach poses two problems in the semigroup setting.

The first is that the theory of finitely generated semigroups has in recent decades been more combinatorial and less geometric than that of finitely generated groups: the ideal structure of a semigroup is notoriously difficult to capture in a geometric way, leading researchers to prefer other tools such as rewriting systems and automata. However, recent advances such as the authors' use of *asymmetric geometry* [12, 14, 13, 15] seem to be bringing a genuinely geometric aspect to the subject.

The second problem is that amenability in semigroups is not characterised by the Følner set property, or indeed by any known elementary combinatorial condition. To be precise, the Følner condition admits two formulations which are trivially equivalent in a group (or indeed, a left cancellative semigroup), but not in a general semigroup. The weaker of these (generally known as "the Følner condition" or *FC*) was studied by Day [9] who showed that it was a necessary but not a sufficient condition for left amenability. The stronger form ("the strong Følner condition" or *SFC*) was considered by Argabright and Wilde [1]; they showed that SFC is sufficient for left amenability, and that necessity would follow from a conjecture of Sorenson [22, 23], asserting that every right cancellative, left amenable semigroup was also left cancellative. Subsequently, Klawe [19] showed that necessity was actually equivalent to Sorenson's conjecture, before disproving both, by producing a (non-finitely-generated) example of a left amenable semigroup which was right but not left cancellative; this was subsequently refined to a finitely generated example by Takahashi [24]. Despite considerable further work in this area (see for example [27]), an elementary combinatorial characterisation of (left or two-sided) amenability for semigroups remains elusive.

While SFC does not provide an exact characterisation of left amenability for semigroups in full generality, it can do so within large and important classes of semigroups. We have already noted that in the presence of left cancellativity SFC is trivially equivalent to FC (which in general is weaker that left amenability), and so both FC and SFC characterise left amenability within the class of left cancellative semigroups. In fact, we shall see below (Theorem 2.6 and Corollary 2.7) that SFC is equivalent to left amenability within an extremely large class of (not necessarily finitely generated) semigroups including not only left cancellative and commutative semigroups, but

also semigroups in which every ideal contains an idempotent: the latter condition encompasses for example all groups, finite semigroups, compact left or right topological semigroups, inverse semigroups, regular semigroups and semigroups with a left, right or two-sided zero element, and the result therefore applies to the overwhelming majority of widely studied semigroups.

We also present (in Section 3) a new characterisation of SFC, in terms of the existence of *weak* Følner sets satisfying a local injectivity condition on the relevant translation action of the semigroup. As well as having potential applications as a technical lemma for studying SFC, this result is conceptually interesting, because it gives a new insight into the difference between FC and SFC, and what exactly it is about left cancellativity which is important for amenability.

The equivalence of FC and/or SFC with left amenability in such a wide range of semigroups provides a strong motivation for studying these conditions, and how they relate to the geometry of finitely generated semigroups: results so obtained will bear directly upon amenability for a very large range of semigroups, and are also likely to give clues as to how amenability itself could be directly studied for finitely generated semigroups in even greater generality. The chief aim of the present paper is to begin this study.

One of our main results (Theorem 4.4) is that, for semigroups in the same broad class described above, a finitely generated semigroup of subexponential growth has SFC (and hence is left amenable) if and only if it is left reversible (the latter being a trivially necessary condition for amenability in all semigroups). We also show (Theorem 5.5) that FC is a left quasi-isometry invariant of finitely generated semigroups, and hence that left amenability is a left quasi-isometry invariant of finitely generated left cancellative semigroups. Neither amenability nor SFC is a left quasi-isometry invariant of finitely generated semigroups more generally, but it remains open whether these properties can be seen in the right quasi-isometry class, or in the left and right quasi-isometry classes together.

2. Analytic, Algebraic and Geometric Conditions

In this section we briefly recall the definitions of left amenability, the Følner condition and the strong Følner condition for semigroups. For a more detailed introduction we direct the reader to the monograph of Paterson [20]. We then show that, mainly as a consequence of work of Klawe [19], the strong Følner condition exactly characterises left amenability within an extremely broad class of semigroups. Semigroups are assumed to be discrete unless indicated otherwise.

2.1. **Amenability and Følner Conditions.** A semigroup S is called *left amenable* if there is a mean on $l_\infty(S)$ which is invariant under the natural left action of S on the dual space $l_\infty(S)'$ [20, Section 0.18]. Equivalently [20, Problem 0.32], S is left amenable if it admits a finitely additive probability measure μ, defined on all the subsets of S, which is *left invariant*, in the sense that $\mu(a^{-1}X) = \mu(X)$ for all $X \subseteq S$ and $a \in S$. Here, $a^{-1}X$ denotes the set $\{s \in S \mid as \in X\}$. (Note that left invariance is strictly weaker than requiring $\mu(aX) = \mu(X)$ for all $X \subseteq S$ and $a \in S$; semigroups admitting a finitely additive probability measure satisfying this much stronger property

are called *left measurable* — see [23] and [19, Section 5] for more on this property.)

A semigroup S satisfies *the Følner condition* (*FC*) if for every finite subset H of S and every $\epsilon > 0$, there is a finite non-empty subset F of S with $|sF \setminus F| \leq \epsilon |F|$ for all $s \in H$. A semigroup S satisfies *the strong Følner condition* (*SFC*) if for every finite subset H of S and every $\epsilon > 0$, there is a finite non-empty subset F of S with $|F \setminus sF| \leq \epsilon |F|$ for all $s \in H$.

In a left cancellative semigroup FC and SFC are trivially equivalent, but without left cancellativity SFC is strictly stronger, because left translation by an element s can map many elements in a set F onto a few elements, allowing $sF \setminus F$ to be small but $F \setminus sF$ large. For semigroups in general, it is known that SFC implies left amenability [1], which in turn implies FC [9], but neither of these implications is reversible [9, 19]. (For left cancellative semigroups, of course, it follows that FC and SFC both exactly characterise amenability.)

2.2. **Left Thick Subsets and Left Reversibility.** Recall that a subset E of a semigroup S is called *left thick* if for every finite subset F of S there is an element t of S such that $Ft \subseteq E$. Left thickness is of interest in the theory of amenability, because it provides an abstract algebraic characterisation of those subsets capable of having "full measure": more precisely, a subset of a left amenable semigroup S is left thick if and only if it has measure 1 in some left invariant, finitely additive probability measure on S [20, Proposition 2.1].

In any semigroup, it is immediate from the definition that every left ideal (and hence every two-sided ideal) is left thick. Right ideals in a general semigroup need not be left thick, but in left amenable semigroups it transpires that they are. Recall that a semigroup is called *left reversible* if every pair of right ideals (or equivalently, every pair of principal right ideals) intersects. Left reversibility is easily seen to be a necessary precondition for left amenability [20, Proposition 1.23]. Examples of left reversible semigroups include inverse semigroups, commutative semigroups, and cancellative semigroups which embed in groups of left quotients. The following elementary result connects left reversibility with left thickness:

Proposition 2.1. *For any semigroup S, the following conditions are equivalent:*

(i) *S is left reversible;*
(ii) *every principal right ideal of S is left thick;*
(iii) *every right ideal of S is left thick;*

Proof. Suppose first that (i) holds, and let $aS^1 = \{a\} \cup aS$ be a principal right ideal of S. Let $F = \{f_1, \ldots, f_n\}$ be a finite subset of S. Since S is left reversible, the right ideal $f_1 S$ intersects the right ideal aS, so we may choose $t_1 \in S$ with $f_1 t_1 \in aS$. Similarly, $f_2 t_1 S$ intersects aS, so we may choose $t_2 \in S$ so that $f_2 t_1 t_2 \in aS$. Continuing in this way, we define a sequence of elements $t_1, \cdots, t_n \in S$ so that $f_i t_1 \ldots t_i \in aS$ for each i. But since aS is a right ideal, setting $t = t_1 \ldots t_n$, it follows that $f_i t \in aS$ for all i, i.e. $Ft \subseteq aS$. This shows that aS, and hence aS^1, is left thick, and hence that (ii) holds.

It is immediate from the definition that a set containing a left thick set is left thick, so the fact that (ii) implies (iii) follows from the fact that every right ideal contains a principal right ideal.

Finally, suppose (iii) holds, and let aS^1 and bS^1 be principal right ideals. Since aS^1 is left thick and $\{b\}$ is a finite set, there is a $t \in S$ so that $\{b\}t \subseteq aS^1$. But now $bt \in aS^1 \cap bS^1$, as required to show that (i) holds. $\quad\square$

2.3. **Near Left Cancellativity.** In this section we introduce a new definition which will prove useful for understanding the relationship between amenability, Følner conditions and cancellativity conditions. We say that a semigroup S is *near left cancellative* if for every element s of S there is a left thick subset E on which the left translation map by s restricts to an injective map, that is, such that $sx \neq sy$ whenever $x, y \in E$ with $x \neq y$.

(Near left cancellativity is stronger than that property informally termed *almost left cancellativity* in [19, p.104], which is that for every s the set of elements t such that there is no other element t' with $st = st'$ is left thick. The term *almost left cancellative* is also used in a completely different sense in [20, Section 7.22].)

It is immediate that a left cancellative semigroup is near left cancellative, since the whole semigroup will always be a left thick subset of itself. Perhaps a more surprising observation is that any semigroup with a right zero element is near left cancellative: indeed, the singleton set containing a right zero is easily seen to be a left thick subset on which the left translation action of every element of S cannot help but be injective. This may seem initially troubling to the reader versed in the algebraic theory of semigroups, where the existence of a zero element is often regarded as the antithesis of cancellativity; from this perspective to declare *everything* with a zero to be close to left cancellative may seem bizarre. However, when viewing semigroups in a *dynamic* context the logic is clearer: intuitively, while a semigroup with zero may have arbitrary algebraic complexity "above" the zero, random walks on the semigroup eventually end at zero with probability 1, meaning the asymptotic dynamical behaviour is the same as that of the (left cancellative) trivial monoid.

The following proposition shows that the class of near left cancellative semigroups is very large, including for example all left reversible regular semigroups (and hence all inverse semigroups) and all left reversible finite semigroups.

Proposition 2.2. *Let S be a left reversible semigroup in which every ideal contains an idempotent (for example, a left reversible regular and/or left reversible finite semigroup). Then S is near left cancellative.*

Proof. Let $s \in S$, and consider the ideal SsS. Since every ideal contains an idempotent, we may choose an idempotent $e \in SsS$, say $e = xsy$. Now $xsyxsy = xsy$ implies $e = xsy\mathcal{R}xsyxs$. Thus, $xsyxs$ is a regular element and so is also $\mathcal{L}$-related to an idempotent, say $xsyxs\mathcal{L}f = f^2$. In particular, f is $\mathcal{L}$-below s, so we may write $f = ts$ for some t.

Now consider the right ideal fS. By Proposition 2.1 this is left thick. Moreover, for any element $fz \in fS$ we have $t(s(fz)) = (ts)(fz) = f^2z = fz$ which tells us that left translation by s is injective on fS as required. $\quad\square$

Examples of semigroups which are *not* near left cancellative include non-trivial left zero semigroups, or more generally, semigroups which are right cancellative but not left cancellative. (This claim will be justified in the remarks following Proposition 2.3 below.)

2.4. **The Klawe Condition and Right Cancellative Quotients.** We say that a semigroup S satisfies the *Klawe condition* if whenever s, x and y in S are such that $sx = sy$, there exists $t \in S$ so that $xt = yt$. This condition was implicitly introduced and used, although not given a name, by Klawe [19]. Every left cancellative semigroup satisfies this condition vacuously; in fact, it transpires that near left cancellativity suffices:

Proposition 2.3. *Every near left cancellative semigroup satisfies the Klawe condition.*

Proof. Let S be near left cancellative. Suppose $sx = sy$. Let E be a left thick subset of S on which s acts injectively by left translation. Since E is left thick, we may choose a $t \in S$ such that $\{x, y\}t \subseteq E$. Now $s(xt) = (sx)t = (sy)t = s(yt)$ where $xt, yt \in E$. But left translation by s is injective on E, so we must have $xt = yt$. $\qquad\square$

Note that a right cancellative semigroup satisfying the Klawe condition must clearly be left cancellative, so Proposition 2.3 justifies the claim in the previous section that a semigroup which is right cancellative but not left cancellative cannot be near left cancellative.

On any semigroup S, we may define a binary relation by $x \cong y$ if and only if there exists an s with $xs = ys$. In general this relation is not transitive, but in the case S is left reversible it is actually a congruence, and the quotient $S/\cong$ is a right cancellative semigroup [20, Proposition 1.24]. (Note that left reversibility is not a *necessary* condition for the relation to be a congruence or the quotient to be right cancellative, as witnessed for example by a free semigroup of rank 2.)

We shall need the following elementary fact, one implication of which was shown by Klawe [19].

Proposition 2.4. *Let S be a left reversible semigroup. Then S satisfies the Klawe condition if and only if $S/\cong$ is left cancellative.*

Proof. The direct implication is the contrapositive of [19, Lemma 2.1].

For the converse, suppose $S/\cong$ is left cancellative, and that $sx = sy$. Then writing $[a]$ for the $\cong$-equivalence class of a, we have $[s][x] = [s][y]$, so by left cancellativity of the quotient we must have $[x] = [y]$, that is, $x \cong y$. But by definition this means that there is a t with $xt = yt$, as required. $\quad\square$

Notice that the left reversibility hypothesis is required only to ensure that $\cong$ is a congruence so that $S/\cong$ is actually well-defined. The Klawe condition does not suffice to ensure left reversibility (consider for example a free semigroup of rank 2, or any cancellative semigroup which does not embed in a group) or indeed even for $\cong$ to be transitive. For example, the semigroup $\langle a, b, c, x, y \mid ax = bx, by = cy \rangle$ is left cancellative, and hence trivially satisfies the Klawe condition, but we have $a \cong b$ and $b \cong c$ but $a \not\cong c$.

Klawe [19, Theorem 2.2] showed that a left amenable semigroup satisfies SFC if and only if the right cancellative quotient $S/\cong$ is also left cancellative. Combining this with Proposition 2.4 yields a three-way equivalence:

Theorem 2.5. *Let S be a left amenable semigroup. Then the following are equivalent:*

   (i) *S satisfies the strong Følner condition;*
  (ii) *the right cancellative quotient $S/\cong$ is left cancellative;*
 (iii) *S satisfies the Klawe condition.*

Combining with a result of Argabright and Wilde [1] we obtain the fact that for the (very large) class of semigroups satisfying the Klawe condition, the strong Følner condition gives an exact characterisation of amenability.

Theorem 2.6. *Let S be a semigroup satisfying the Klawe condition. Then S is left amenable if and only if S satisfies SFC.*

Proof. One implication is immediate from Theorem 2.5; the other is [1, Theorem 1]. $\square$

Corollary 2.7. *Let S be a semigroup in which every ideal contains an idempotent (for example a regular semigroup, inverse semigroup, finite semigroup, compact left or right topological semigroup, or semigroup with a left, right or two-sided zero). Then S is left amenable if and only if S satisfies SFC. Moreover, if S is left amenable then for every $s \in S$ there is a left invariant finitely additive probability measure on S such that the left translation map by s is injective when restricted to some set of full measure.*

Proof. If S is left amenable then in particular it is left reversible, so we deduce by Proposition 2.2 that S is near left cancellative, by Proposition 2.3 that S satisfies the Klawe condition and so by Theorem 2.6 that S satisfies SFC. The converse is again [1, Theorem 1].

Moreover, if $s \in S$ then since S is near left cancellative, there is a left thick subset E of S such that the left translation map of s on E is injective. But since E is left thick, by [20, Proposition 1.21], there exists a left invariant finitely additive probability measure on S such that E has full measure. $\square$

Of course, it is not the case that every left thick subset of a left amenable semigroup can be made to have full measure simultaneously, with respect to the same measure; consider for example a non-trivial finite right zero semigroup, which is left amenable but has disjoint left thick subsets. However, with reference to the latter part of Corollary 2.7, one may ask whether it is necessary for the measure to be chosen differently for different translation maps, or if a single measure suffices:

Question 2.8. *If a semigroup is left amenable and near left cancellative, is there necessarily a left invariant finitely additive probability measure such that every element acts injectively by left translation on some set of full measure?*

3. A NEW CHARACTERISATION OF SFC

In this section we present a new characterisation of SFC, in terms of the existence of *weak* Følner sets satisfying a local injectivity condition on the

relevant translation action of the semigroup. As well as having potential applications as a technical lemma for studying SFC, this result is conceptually interesting, as it gives a new insight into what exactly it is about left cancellativity which is important for amenability. The proof is a straightforward direct argument.

Theorem 3.1. *Let S be a semigroup. Then S satisfies the strong Følner condition if and only if for every finite set $H \subseteq S$ and $\epsilon > 0$ there is a finite non-empty set $F \subseteq S$ such that for each $s \in H$ we have*

$$|sF \setminus F| \leq \epsilon|F|$$

and for all $x, y \in F$ and $s \in H$, if $sx = sy$ then $x = y$.

Proof. Suppose the given condition is satisfied, and given H and ϵ choose F as in the condition. Then for each $s \in H$, the injectivity of the action of s on F implies that $|sF| = |F|$, whence

$$|F \setminus sF| \;=\; |sF \setminus F| \leq \epsilon|F|$$

as required to show that SFC holds.

Conversely, suppose S satisfies SFC and let H and ϵ be given. Let $\mu > 0$ be small. By SFC we can choose a finite non-empty set A such that $|A \setminus sA| < \mu|A|$ for all $s \in H$. Define

$$B \;=\; \{a \in A \mid \neg\,(\exists s \in H, b \in A, b \neq a, sb = sa)\}$$

to be the set of all elements of A which form singleton fibres under the action of left translation by each element of H. Clearly by definition, elements of H act injectively by left translation on B. We claim that

$$|A \setminus B| \;\leq\; 2|H|\mu|A|. \tag{1}$$

Indeed, clearly we have

$$A \setminus B = \bigcup_{s \in H} \{a \in A \mid \exists b \in A, b \neq a, sa = sb\}.$$

We write C_s for the component of the union on the right-hand side corresponding to $s \in H$. Now if (1) does not hold then there must be some $s \in H$ such that $|C_s| > 2\mu|A|$. By the definition of C_s, it is clear that $|sC_s| \leq \frac{1}{2}|C_s|$ so we have

$$|sA| \leq |sC_s| + |s(A \setminus C_s)|$$
$$\leq \frac{1}{2}|C_s| + |A \setminus C_s|$$
$$= |A| - \frac{1}{2}|C_s|$$
$$< |A| - \mu|A|$$
$$= (1 - \mu)|A|.$$

But this means that $|A \setminus sA| > \mu|A|$, contradicting the choice of A. This completes the proof of (1).

It follows also from (1) that

$$|B| \;\geq\; |A| - 2|H|\mu|A| \;=\; (1 - 2|H|\mu)|A|. \tag{2}$$

Now for any $s \in H$ we have

$$
\begin{aligned}
|sB \setminus B| &= |B \setminus sB| \\
&\leq |A \setminus sB| && \text{(since } B \subseteq A) \\
&\leq |A \setminus sA| + |sA \setminus sB| \\
&\leq |sA \setminus A| + |A \setminus B| \\
&\leq \mu|A| + 2|H|\mu|A| && \text{(by the definition of } A \text{ and (1))} \\
&= (1 + 2|H|)\mu|A| \\
&\leq \frac{(1 + 2|H|)\mu}{1 - 2|H|\mu}|B| && \text{(by (2))}.
\end{aligned}
$$

By choosing $\mu > 0$ sufficiently small we may make

$$
\frac{(1 + 2|H|)\mu}{1 - 2|H|\mu} < \epsilon
$$

(since the left hand-side as a function of μ takes the value 0 at $\mu = 0$ and is clearly continuous away from $\mu = 1/(2|H|)$, so that B is a non-empty set satisfying

$$
|sB \setminus B| \leq \epsilon|B|
$$

for all $s \in H$, completing the proof.   □

4. Growth and Amenability

Our aim in this section is to show that for finitely generated semigroups satisfying the Klawe condition, sub-exponential growth is a sufficient condition for SFC, and hence for left amenability.

Recall that if M is a semigroup generated by a finite subset X, the *growth function* of M with respect to X is the function which maps a natural number n to the number of distinct elements of M which can be written as a product of n or fewer generators from X. Although the growth function of M depends on the choice of finite generating set, its asymptotic behaviour does not, and is an invariant of the monoid. We say that M has *polynomial growth* if its growth function is bounded above by a polynomial, or *subexponential growth* if its growth function is eventually bounded above by every increasing exponential function. Growth of finitely generated semigroups is a major topic in both abstract semigroup theory and application areas — see for example [5, 17, 21] for work in this area.

We shall need the following lemma about semigroups satisfying the Klawe condition.

Lemma 4.1. *Let a and b be elements of a semigroup S satisfying the Klawe condition. Then either aS and bS intersect, or a and b freely generate a free subsemigroup of rank 2.*

Proof. Consider the natural map $\psi : \{A, B\}^+ \to S$ from the free semigroup on symbols A and B to S, taking A to a and B to b. If a and b do not freely generate a free subsemigroup then this map cannot be injective, so we may choose distinct words $u, v \in \{A, B\}^+$ such that $\psi(u) = \psi(v)$.

We may assume without loss of generality that neither u nor v is a prefix of the other. Indeed, if u is a prefix of v, then choose $c \in \{A, B\}$ such that

c is not the letter of v immediately following the prefix u. Then clearly we have $\psi(uc) = \psi(vc)$ where neither uc nor vc is a prefix of the other, so we may simply replace u by uc and v by vc. A symmetric argument applies if v is a prefix of u.

Now let w be the longest common prefix of u and v, and write $u = wu'$ and $v = wv'$. The assumption from the previous paragraph ensures that u' and v' are non-empty. If w is non-empty then $\psi(w)\psi(u') = \psi(w)\psi(v')$ in S, so by the Klawe condition we may choose $s \in S$ such that $\psi(u')s = \psi(v')s$. If w is empty then $\psi(u') = \psi(v')$ so we may choose $s \in S$ arbitrarily to obtain the same property. But by construction, u' and v' begin with different letters from $\{A, B\}$, so the element $\psi(u')s = \psi(v')s$ lies in both aS and bS. $\square$

An immediate corollary is a useful fact about semigroups satisfying the Klawe condition:

Corollary 4.2. *Let S be a semigroup satisfying the Klawe condition. Then either S is left reversible or S contains a free subsemigroup of rank 2.*

Note that the two possibilities in Corollary 4.2 are not mutually exclusive: for example, adjoining a zero element to a free semigroup of rank 2 yields a semigroup which satisfies both conditions (and also the Klawe condition).

Our next result says that all finitely generated semigroups of subexponential growth satisfy the weak condition FC. In itself this is not especially interesting — indeed, it might be thought of as simply indicating just how weak a condition FC is — but combined with other results it will allow us to establish sufficient conditions for SFC.

Theorem 4.3. *Let S be a finitely generated semigroup of subexponential growth. Then S satisfies the Følner condition FC.*

Proof. Given a finite subset H of S, we may choose a finite generating set X for S containing H. Let B_i denote the ball of radius i in S with respect to the generating set X, that is, the set of all elements which can be written as a product of i or fewer generators from X. We claim that

$$\inf_{i \in \mathbb{N}} \frac{|B_{i+1}|}{|B_i|} = 1.$$

Indeed, because $B_i \subseteq B_{i+1}$ for each i the given infimum is certainly at least 1, so if the claim were false it would be bounded below by some $\lambda > 1$. But then we have $|B_{i+1}|/|B_i| > \lambda$ for all i and certainly $|B_1| > 0$, so $|B_i| \geq |B_1|\lambda^{i-1}$ for all i, contradicting the subexponential growth of S.

Hence, given $\epsilon > 0$, we may choose i with $|B_{i+1}|/|B_i| < 1 + \epsilon$. For any $s \in H$ we have $s \in X$ so $sB_i \subseteq B_{i+1}$ and now

$$|sB_i \setminus B_i| \ \leq \ |B_{i+1} \setminus B_i| \ \leq \ \epsilon|B_i|$$

as required to show that S satisfies FC. $\square$

Since the Følner condition suffices for left amenability within the class of left cancellative semigroups, it follows immediately from Theorem 4.3 that left cancellative semigroups of subexponential growth are left amenable. But in fact, by combining with previously known results, we are in a position to prove something much stronger:

Theorem 4.4. *Let S be a finitely generated semigroup of subexponential growth and satisfying the Klawe condition. Then S is left amenable and satisfies the strong Følner condition.*

Proof. Since S has subexponential growth it cannot have free subsemigroups of rank greater than 1, so Corollary 4.2 tells us that S is left reversible. Thus, the quotient semigroup $S' = S/\cong$ is well-defined and right cancellative by [20, Proposition 1.24], and left cancellative by Proposition 2.4.

It is immediate that the quotient S' is finitely generated with growth function bounded above by that of S (hence in particular subexponential). Hence, by Theorem 4.3, S' satisfies FC. But since S' is left cancellative, this means S' it satisfies SFC, which by [1, Theorems 1 and 5] means that S is left amenable and satisfies SFC. $\square$

We also recover from Theorem 4.4 a new elementary proof of the well-known fact that commutative semigroups are amenable (or equivalently, satisfy SFC).

Corollary 4.5 ([1, Theorem 4])**.** *All commutative semigroups are amenable.*

Proof. Commutative semigroups trivially satisfy the Klawe condition, and finitely generated ones clearly have growth functions bounded above by a polynomial of degree the number of generators. Hence, given a commutative semigroup S, Theorem 4.4 tells us that all of its finitely generated subsemigroups are left amenable, which by [20, Problem 0.30] suffices for S to be left amenable.

Finally, right amenability is easily observed to be trivially equivalent to left amenability in the commutative case. $\square$

We have shown that subexponential growth suffices for left amenability in a large class of finitely generated semigroups. We know it does not suffice for semigroups in absolute generality, because not all finite semigroups are left amenable [20, Corollary 1.19]. We do not know if lack of left reversibility is the only possible obstruction here, that is, whether a left reversible, finitely generated semigroup of subexponential growth is necessarily left amenable. We conjecture that it is not, even if subexponential growth is replaced by the stronger condition of polynomial growth:

Conjecture 4.6. *There is a left reversible, finitely generated semigroup of polynomial growth which is not left amenable.*

Just as for the Sorenson conjecture, the question (for both polynomial and subexponential cases) reduces to the case of semigroups which are right cancellative but not left cancellative. Indeed, if S is an example as postulated by the conjecture (or a corresponding example with subexponential in place of polynomial growth) then $S' = S/\cong$ is well defined and right cancellative by [20, Proposition 1.24]. Moreover, S' is finitely generated, of polynomial (or subexponential) growth, left reversible (since this property is clearly inherited by quotients) and also not left amenable [20, Proposition 1.25]. Finally, S' is not left cancellative, since if it were then by Proposition 2.4 S would satisfy the Klawe condition, but then by Theorem 4.4 S would be left amenable, giving a contradiction.

5. Quasi-Isometry Invariance of Følner Conditions

In this section we explore the extent to which amenability and the Følner conditions are "geometric" properties of finitely generated semigroups, in the sense of being invariant under left and/or right quasi-isometry.

5.1. **Digraphs, Quasi-Isometry and Semigroups.** If Γ is a digraph we write $V\Gamma$ for its vertex set and $E\Gamma$ for its edge set, which we view as a subset of $V\Gamma \times V\Gamma$. We write $P_f(X)$ for the set of finite non-empty subsets of a set X. We write $\overline{\mathbb{R}}$ for the set of non-negative real numbers with ∞ adjoined, and extend multiplication, addition and the usual order on $\mathbb{R}$ to $\overline{\mathbb{R}}$ in the obvious way, leaving 0∞ undefined.

We briefly recall the definition of quasi-isometry for digraphs and semigroups — see [13] for a detailed introduction. Given a digraph Γ, we define a *semimetric* $d_\Gamma : V\Gamma \times V\Gamma \to \overline{\mathbb{R}}$ by setting $d_\Gamma(x,y)$ to be the length of the shortest directed path from x to y in Γ, or ∞ if there is no such path. We say that two digraphs Γ and Δ are *quasi-isometric* if there is a map $\phi : V\Gamma \to V\Delta$ and a real number $\lambda > 0$ such that for all vertices x and y of Γ we have

$$\frac{1}{\lambda}d_\Delta(\phi(x),\phi(y)) - \lambda \;\leq\; d_\Gamma(x,y) \;\leq\; \lambda d_\Delta(\phi(x),\phi(y)) + \lambda,$$

and for every vertex y of Δ there is a vertex x of Γ with $d_\Delta(\phi(x),y) \leq \lambda$ and $d_\Delta(y,\phi(x)) \leq \lambda$. The map ϕ is called a λ-*quasi-isometry*. Quasi-isometry gives rise to an equivalence relation on the class of all digraphs [13, Proposition 1].

If S is a semigroup with a finite generating set X, the *right* [respectively, *left*] *Cayley graph of S with respect to X*, denoted $\Gamma_r(S,X)$ [respectively, $\Gamma_l(S,X)$], is the digraph with vertex set S and an edge from s to t if and only if $sx = t$ [respectively, $xs = t$] for some $x \in X$. Although the right (or left) Cayley graph depends on the choice of generators, its quasi-isometry class does not [13, Proposition 4]; thus we may say that two finitely generated semigroups are *right* [*left*] *quasi-isometric* if their right [left] Cayley graphs are quasi-isometric, without concerning ourselves about choice of generators.

Unlike in a group, the left and right Cayley graphs of a semigroup need not be isomorphic, or indeed even quasi-isometric. The left and right quasi-isometry classes of a semigroup thus form two distinct natural invariants which in some sense capture its "large-scale geometry", and it is natural to ask what properties of a finitely generated semigroup can be seen in either or both of these quasi-isometry classes. Our first observation is that the key property of left reversibility can be seen in one of the quasi-isometry classes (although not the other):

Proposition 5.1. *Left reversibility is a right (but not a left) quasi-isometry invariant of semigroups.*

Proof. From the definition it follows immediately that a semigroup is left reversible if and only if every pair of principal right ideals contain a common principal right ideal. Thus, left reversibility can be seen in the structure of the containment order on the set of principal right ideals. By [13, Proposition 5], this order is a right quasi-isometry invariant of semigroups.

On the other hand, the 2-element left-zero semigroup (with both elements as generators) is not left reversible, but its left Cayley graph is a complete digraph on 2 vertices, so isomorphic (and in particular quasi-isometric) to that of $\mathbb{Z}_2$, which is left reversible, with both elements as generators. $\square$

5.2. **Isoperimetric Numbers of Digraphs.** Let Γ be a digraph. For any subset A of $V\Gamma$ we define the *out-boundary* ∂A of A to be the set

$$\partial A = \{x \in V\Gamma : \exists (y, x) \in E\Gamma \text{ with } y \in A \text{ and } x \notin A\}.$$

Now we define the (*outward*) *isoperimetric number* $\iota(\Gamma)$ of Γ by:

$$\iota(\Gamma) = \inf_{A \in P_f(V\Gamma)} \left(\frac{|\partial A|}{|A|} \right).$$

We shall be interested in digraphs with isoperimetric number zero, that is, graphs Γ for which

$$(\forall \epsilon > 0)\,(\exists A \in \mathcal{P}_f(V\Gamma))\ :\ \frac{|\partial A|}{|A|} < \epsilon. \tag{3}$$

We note that the notions of outboundary and isoperimetric number of a digraph defined here arise in the literature in the definitions of Cheeger constants and Cheeger inequalities as part of the spectral theory of directed graphs; see [6].

The following result shows the relationship between FC and isoperimetric number.

Proposition 5.2. *Let S be a semigroup generated by a finite set X. Then S satisfies FC if and only if the left Cayley graph $\Gamma_l(S, X)$ has isoperimetric number zero.*

Proof. Suppose first that S satisfies FC. We must show that Γ satisfies (3). Let $\epsilon > 0$ be given and choose $\delta > 0$ with $\delta < \epsilon/|X|$. Since S satisfies FC and X is a finite set, we may choose a finite non-empty subset F of S such that

$$(\forall x \in X)(|xF \setminus F| < \delta|F|).$$

We claim that by taking $A = F$ condition (3) will be satisfied. Indeed, by the definition of Γ we have

$$\partial A = \bigcup_{x \in X} (xA \setminus A) = \bigcup_{x \in X} (xF \setminus F),$$

and so

$$\begin{aligned}
|\partial A| &= \left| \bigcup_{x \in X} (xF \setminus F) \right| \leq \sum_{x \in X} |xF \setminus F| \\
&< |X|\delta|F| < |X|(\epsilon/|X|)|F| = \epsilon|F| = \epsilon|A|,
\end{aligned}$$

which shows that (3) holds.

Conversely, suppose that Γ has isoperimetric number zero or, equivalently, suppose that (3) holds. We must show that S satisfies FC. To this end, let $H \in \mathcal{P}_f(S)$ and $\epsilon > 0$ be given. Let $r \in \mathbb{N}$ be such that every element $h \in H$ can be written as a product of elements from X of length at most r; such an r exists because H is finite and X is a generating set for S. Write $X^{\leq r}$ for the set of elements of S which can be written as a product of elements of

X of length at most r. Set $C = |X^{\leq r}|$ and $\delta = \epsilon/C$. By (3) we may choose a finite non-empty set $F \in \mathcal{P}_f(S)$ such that $|\partial F|/|F| < \delta$.

Let $h \in H$ be arbitrary and consider the set $hF \setminus F$. Write $h = x_1 x_2 \ldots x_t$ where $x_i \in X$ and $t \leq r$. This is possible by the definition of r. Consider an arbitrary element of $hF \setminus F$, say hf where $f \in F$. Then there is a path of length t in the left Cayley graph from f to $hf = x_1 x_2 \ldots x_t f$ with the edges labelled $x_1, x_2, \cdots, x_t$. Since this path leads from a vertex in F to a vertex not in F, it must have at least one vertex in ∂F (say $x_i x_{i+1} \ldots x_t f$, with $1 \leq i \leq t$). It follows that hf can be written as the product of an element of $X^{\leq r}$ and an element of ∂F (namely of $x_1 x_2 \ldots x_{i-1}$ and $x_i x_{i+1} \ldots x_t f$). Thus, there are at most $C|\partial F|$ choices of $hf \in hF \setminus F$, so we have

$$|hF \setminus F| \ \leq \ C|\partial F| \ < \ C\left(\delta|F|\right) \ < \ C\left(\frac{\epsilon|F|}{C}\right) \ = \ \epsilon|F|.$$

This holds for all $h \in H$ and therefore S satisfies FC. $\qquad\square$

Next we shall show that the property of having isoperimetric number zero is a quasi-isometry invariant of directed graphs with bounded out-degree. We first need a lemma.

Lemma 5.3. *Let $\phi : V\Gamma \to V\Delta$ be a quasi-isometry between digraphs. If Γ has bounded out-degree then ϕ has bounded fibre sizes.*

Proof. Let k be an upper bound on the out-degree of Γ, and suppose ϕ is a λ-quasi-isometry. Fix an element $x \in V\Gamma$ and consider the set of elements $y \in V\Gamma$ with $\phi(y) = \phi(x)$. For each such y we have

$$d_\Gamma(x,y) \ \leq \ \lambda d_\Delta(\phi(x),\phi(y)) + \lambda \ = \ \lambda d_\Delta(\phi(x),\phi(x)) + \lambda \ = \ \lambda.$$

Thus, every such y is reachable by a directed path of length at most λ starting from x. But there are clearly no more than

$$\sum_{i=0}^{\lambda'} k^i$$

such paths, where λ' is the integer part of λ, so this number, which is independent of x, forms a bound on the fibre size. $\qquad\square$

Theorem 5.4. *Let Γ and Δ be directed graphs both with bounded out-degree. If Γ and Δ are quasi-isometric then $\iota(\Gamma) = 0$ if and only if $\iota(\Delta) = 0$.*

Proof. Let $\phi : \Gamma \to \Delta$ be a λ-quasi-isometry. Let C be the bound on the fibre size of ϕ given by Lemma 5.3. Since Δ has bounded out-degree, we may choose an upper bound (call it E) on the number of directed paths in Δ originating at any one vertex and having length at most λ. Similarly, let D be an upper bound on the number of directed paths in Γ originating at any one vertex and having length at most $2\lambda^2 + 2\lambda$.

Suppose that $\iota(\Gamma) = 0$. We claim that $\iota(\Delta) = 0$, so, we must show that in the digraph Δ we have

$$(\forall \epsilon > 0)\ (\exists Q \in \mathcal{P}_f(V\Delta)) \ : \ \frac{|\partial Q|}{|Q|} < \epsilon.$$
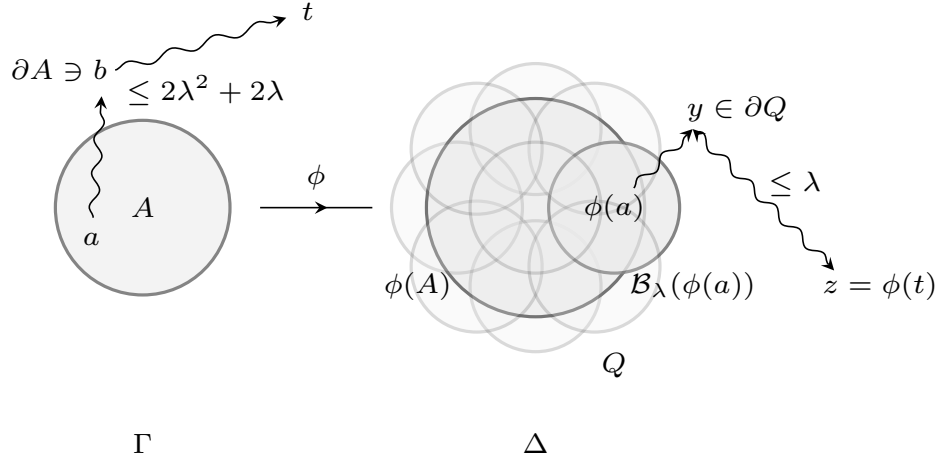
FIGURE 1. Diagram showing the inequality $|\partial Q| \leq DE|\partial A|$ in the proof of Theorem 5.4.

Let $\epsilon > 0$ be given. Since $\iota(\Gamma) = 0$ we may choose a finite subset A of $V\Gamma$ such that

$$\frac{|\partial A|}{|A|} < \frac{\epsilon}{CDE}. \tag{4}$$

Set

$$Q = \bigcup_{a \in A} \mathcal{B}_\lambda(\phi(a)) \subseteq V\Delta,$$

where

$$\mathcal{B}_\lambda(x) = \{a \in V\Delta \mid d(a, x) \leq \lambda \text{ and } d(x, a) \leq \lambda\}.$$

We claim that $\frac{|\partial Q|}{|Q|} < \epsilon$ and the rest of the proof will be devoted to establishing this fact.

Firstly, since $\phi(A) \subseteq Q$, by Lemma 5.3 we have

$$|Q| \geq |\phi(A)| \geq \frac{|A|}{C}. \tag{5}$$

Now we want to estimate the size of the set ∂Q, seeking an upper bound in terms of $|\partial A|$.

In Δ let $y \in \partial Q$ be an arbitrary element of the out-boundary of the set Q. Since ϕ is a λ-quasi-isometry there is a vertex $z \in \phi(V\Gamma)$ such that

$$d_\Delta(z, y) \leq \lambda \text{ and } d_\Delta(y, z) \leq \lambda. \tag{6}$$

Let $t \in V\Gamma$ be such that $\phi(t) = z$. Notice that $t \notin A$; indeed, $y \in \mathcal{B}_\lambda(z) = \mathcal{B}_\lambda(\phi(t))$ so if t were in A then, by the definition of Q, we would have $y \in Q$ contradicting $y \in \partial Q$.

Now by the definitions of Q and ∂Q, there exists $a \in A$ such that

$$d_\Delta(\phi(a), y) \leq \lambda + 1.$$

Together with (6) and the directed triangle inequality this gives

$$d_\Delta(\phi(a), \phi(t)) = d_\Delta(\phi(a), z) \leq 2\lambda + 1.$$

Applying the quasi-isometry inequality to this, we obtain

$$d_\Gamma(a,t) \ \leq \ \lambda \, d_\Delta(\phi(a), \phi(t)) + \lambda \ \leq \ \lambda(2\lambda + 1) + \lambda \ = \ 2\lambda^2 + 2\lambda.$$

So in the digraph Γ we have $a \in A$, $t \notin A$, and there is a directed path from a to t in Γ of length at most $2\lambda^2 + 2\lambda$. Any such path must include at least one vertex from the out-boundary ∂A. It follows that there is a directed path of length at most $2\lambda^2 + 2\lambda$ from some vertex $b \in \partial A$ to the vertex t. The number of possibilities for b is $|\partial A|$, and once $b \in \partial A$ is chosen, there are at most D possible paths of the given length. We conclude that the number of possibilities for the vertex t is bounded above by $D|\partial A|$. Since $z = \phi(t)$ it follows that the number of distinct vertices z that can occur in the above argument is also bounded above by $D|\partial A|$. But every $y \in \partial Q$ is reachable from some such z by a path of length at most λ, and there are at most E such paths from each z, so we obtain

$$|\partial Q| \ \leq \ DE|\partial A|. \tag{7}$$

Finally, combining equations (4), (5) and (7) we obtain

$$\frac{|\partial Q|}{|Q|} \ \leq \ \left(\frac{C}{|A|}\right)(DE|\partial A|) \ < \ \epsilon$$

as required to complete the proof. $\square$

An immediate consequence of the above results is the following.

Theorem 5.5. *The Følner condition FC is a left Cayley graph quasi-isometry invariant of finitely generated semigroups.*

Proof. This follows from Proposition 5.2 and Theorem 5.4. $\square$

For left cancellative semigroups, where the conditions of left amenability, FC and SFC coincide, this yields a direct generalisation of a well-known result for groups:

Corollary 5.6. *Left amenability is a left quasi-isometry invariant of finitely generated left cancellative semigroups.*

In contrast, the fact that left reversibility is not visible in the left Cayley graph (Proposition 5.1) means that neither SFC nor left amenability are left quasi-isometry invariants for finitely generated semigroups in general. Indeed, we have already seen that the 2-element left zero semigroup, which is not left reversible (and hence not left amenable), is left quasi-isometric to the amenable (and hence SFC) group $\mathbb{Z}_2$. However, since left reversibility is a right quasi-isometry invariant (Proposition 5.1), it is natural to ask whether SFC and/or left amenability are invariants of the right quasi-isometry class, or of the two quasi-isometry classes considered together.

Question 5.7. *Can SFC and/or left amenability of a finitely generated semigroup be determined from (i) the right quasi-isometry class or (ii) the left and right quasi-isometry classes together?*

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