A note on mixed boundary value problems involving triple trigonometrical series

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Abstract

This study was motivated by the two-dimensional hydrodynamic slamming problem of a steep wave hitting a vertical wall. The fundamental problem considers dual impact on the wall at the lower and upper region resembling the impact of a wave at the time of its breaking. The solution method results into a mixed-boundary value problem that involves a triplet of trigonometrical series which, to the author's best knowledge, has not been investigated in the past. The formulation of the mixed-boundary value problem is generic and could be used in different fields as well.

Key words: mixed-boundary value problems; triple trigonometrical series; triple integral equations

1. Introduction

Trigonometrical series that hold in different regions are often encountered in mixed-boundary value problems in potential theory. The vast majority of relevant studies considers dual trigonometrical series. On the contrary, triple trigonometrical series have not been investigated to the extent that dual trigonometrical series have been studied.

Looking back in the literature one could find that the first solution to dual series of this kind was given by Shepherd [1]. Analytical solutions have been given by Tranter [2-4]. The solutions provided in [2, 3] were admittedly complicated and accordingly were simplified by the same author [4]. Srivastav [5] showed that certain dual trigonometrical relations can be reduced to a Fredholm integral equation of the second kind and under specific conditions can admit closed forms.

Studies approaching the analytic solution of triple trigonometrical series are rarely found in the literature. For example the classical book of Sneddon [6] has no reference and surprisingly the only book written after

Sneddon's book on mixed-boundary value problems, that of Duffy [7], has only one specific example on sine series. Examples on triple trigonometrical series are the studies of Tranter [8] and Kerr et al. [9]. In reference [8] Tranter showed that the solution of triple trigonometrical series can be sought by solving an equivalent system of three integral equations. Although Tranter [8] considered both sine and cosine series involving harmonics $\sin[(n-1/2)\theta]$ and $\cos[(n-1/2)\theta]$, he didn't considered the case when the argument $(n-1/2)$ that multiplies the expansion coefficients is reversed.

The present study is specifically dedicated to the analytic solution of mixed-boundary value problems involving triple trigonometrical series. The task of the present study is twofold. Firstly, to complement Tranter's [8] work on the transformation of mixed-boundary value problems involving triple trigonometrical series to associated problems involving triple integral equations and secondly, to provide an analytical solution for the concerned problem.

2. Triple trigonometrical series - Transformation to triple integral equations

The mixed-boundary value problem under investigation is composed by the following triplet of trigonometrical series:

$$\sum_{n=1}^{\infty} A_n \cos[(n-1/2)x] = 0 \qquad (0 < x < a) \tag{1}$$

$$\sum_{n=1}^{\infty} (n-1/2)^{-1} A_n \cos[(n-1/2)x] = G(x) \qquad (a < x < b) \tag{2}$$

$$\sum_{n=1}^{\infty} A_n \cos[(n-1/2)x] = 0 \qquad (b < x < \pi) \tag{3}$$

Equations (1)-(3) can be considered as a special (homogeneous) case of the more general (non-homogeneous) triple series given for example in [10-11]. The sums of the left hand sides of (1) and (3) are expected to be singular at the boundaries $x = a$ and $x = b$. Our first goal is to show that this system can be transformed into a group of three integral equations. The sought system is widely referred in the literature as *triple integral equations of Titchmarsh type* [12]. Tranter's [8] method cannot be employed to this particular system as a straightforward application of it would require the analytical form of the integral $\int \sin[t \sin(x/2)] \mathrm{d}x$ which apparently is not known.

For the solution of the system (1)-(3) we start by letting $u = \sin(x/2)$ and accordingly $x = 2\arcsin(u)$ and we rely on the validity of the following expressions (see [13], p. 717, equation 6.671.2; p. 728, equation 6.693.2):

$$\int_0^\infty J_s(t)\cos(ut)\mathrm{d}t = \begin{cases} \dfrac{\cos[s\arcsin(u)]}{\sqrt{1-u^2}} & u<1 \\ \\ \infty & u=1 \end{cases} \tag{4}$$

for Re(s)>-2 and

$$\int_0^\infty t^{-1} J_s(t)\cos(ut)\mathrm{d}t = \frac{1}{s}\cos[s\arcsin(u)] \quad u \le 1\,. \tag{5}$$

In (4)-(5) J_s denotes the Bessel function of the first kind with order s.

Letting s = 2n-1 and introducing (4) into (1) and (3) and (5) into (2) we arrive at

$$\sum_{n=1}^\infty A_n \int_0^\infty J_{2n-1}(t)\cos(ut)\mathrm{d}t = 0 \qquad [0 < u < \sin(a/2)] \tag{6}$$

$$\sum_{n=1}^\infty A_n \int_0^\infty t^{-1} J_{2n-1}(t)\cos(ut)\mathrm{d}t = \frac{1}{2} G[2\arcsin(u)] \qquad [\sin(a/2) < u < \sin(b/2)] \tag{7}$$

$$\sum_{n=1}^\infty A_n \int_0^\infty J_{2n-1}(t)\cos(ut)\mathrm{d}t = 0 \qquad [\sin(b/2) < u < \infty] \tag{8}$$

where the last interval in (8) extends to infinity due to (4). Interchanging the summations and the integrals and letting

$$A(t) = \sum_{n=1}^\infty A_n J_{2n-1}(t) \tag{9}$$

the system of (6)-(8) will finally read

$$\int_0^\infty A(t)\cos(ut)\mathrm{d}t = 0 \qquad [0 < u < \sin(a/2)] \tag{10}$$

$$\int_0^\infty t^{-1} A(t)\cos(ut)\mathrm{d}t = \frac{1}{2} G[2\arcsin(u)] \qquad [\sin(a/2) < u < \sin(b/2)] \tag{11}$$

$$\int_0^\infty A(t)\cos(ut)\mathrm{d}t = 0 \qquad [\sin(b/2) < u < \infty] \tag{12}$$

The concerned mixed-boundary value problem has not yet become a problem of *Titchmarsh type*. That is achieved by using (see [14], p. 438, equation 10.1.12; p. 437, equation 10.1.1; p. 358, equation 9.1.2)

$$\cos(ut) = \sqrt{\frac{\pi ut}{2}} J_\nu(ut) \tag{13}$$

where ν = -1/2. Thus, the initial mixed-boundary value problem of (1)-(3) is transformed into

$$\int_0^\infty t^{1/2} A(t) J_\nu(ut)\mathrm{d}t = 0 \qquad \left(0 < u < a^*\right) \tag{14}$$

$$\int_0^\infty t^{-1/2} A(t) J_\nu(ut)\mathrm{d}t = \frac{1}{\sqrt{2\pi u}} G[2\arcsin(u)] \qquad \left(a^* < u < b^*\right) \tag{15}$$

$$\int_0^\infty t^{1/2} A(t) J_\nu(ut)\mathrm{d}t = 0 \qquad \left(b^* < u < \infty\right) \tag{16}$$

where we set $a^* = \sin(a/2)$, $b^* = \sin(b/2)$. The system of (14)-(16) is the final form of the integral equation representation of the original mixed-boundary value problem of the triple trigonometrical series (1)-(3).

3. Reduction to dual series

In order to solve the system (14)-(16) we exploit the relation [15]

$$\int_0^\infty t^{-q} J_{\nu+2n+1+q}\left(b^* t\right) J_\nu(ut)\mathrm{d}t = 0 \tag{17}$$

which is valid in the interval $b^* < u < \infty$ provided that n is a positive integer. Note that in equation (17), $q = \pm 1/2$. Here we assume that $q = -1/2$ whilst letting

$$A(t) = \sum_{n=1}^{\infty} C_n J_{\nu+2n+1+q}\left(b^* t\right) \tag{18}$$

our goal becomes to calculate the unknown expansion coefficients C_n. Introducing (18) into (16) and rearranging the summation and the integration, it is easily seen that (16) is indeed satisfied due to (17). Hence, having satisfied one of the three integral equations, we introduce (18) into (14) and (15) to yield

$$\sum_{n=1}^{\infty} C_n \int_0^\infty t^{1/2} J_{2n}\left(b^* t\right) J_\nu(ut)\mathrm{d}t = 0 \qquad \left(0 < u < a^*\right) \tag{19}$$

$$\sum_{n=1}^{\infty} C_n \int_0^\infty t^{-1/2} J_{2n}\left(b^* t\right) J_\nu(ut)\mathrm{d}t = \sqrt{\frac{1}{2\pi u}} G[2\arcsin(u)] \qquad \left(a^* < u < b^*\right) \tag{20}$$

Thus we have succeeded to reduce the original system (1)-(3) into a mixed-boundary value problem that involves dual integral equations. The system (19)-(20) is further processed assuming $a^*/b^* = d$, $u/b^* = y$ and $b^* t = \tau$. Accordingly, the system is transformed into

$$\sum_{n=1}^{\infty} C_n \int_0^\infty \tau^{1/2} J_{2n}(\tau) J_\nu(y\tau)\mathrm{d}\tau = 0 \qquad \left(0 < y < d\right) \tag{21}$$

$$\sum_{n=1}^{\infty} C_n \int_0^{\infty} \tau^{-1/2} J_{2n}(\tau) J_\nu(y\tau)\mathrm{d}\tau = \sqrt{\frac{1}{2\pi y}} G\left[2\arcsin(yb^*)\right] \qquad (d < y < 1) \qquad (22)$$

The integral involved in (21)-(22) is the so called Sonine-Schafheitlin integral (Gradshteyn & Ryzhik [13], p. 683, equations 6.574.1 & 6.574.3; Watson [16], p. 401, equation 2) which admits analytic solutions in the interval $0 < y < 1$. In our case $\nu = -1/2$ and accordingly all conditions of the Sonine-Schafheitlin integral are satisfied for $n \geq 1$. Hence, we have

$$\int_0^{\infty} \tau^{1/2} J_{2n}(u) J_\nu(y\tau)\mathrm{d}\tau = \sqrt{\frac{2}{\pi y}}\, {}_2F_1\left(n+\frac{1}{2}, -n+\frac{1}{2}; \frac{1}{2}; y^2\right) \qquad (23)$$

$$\int_0^{\infty} \tau^{-1/2} J_{2n}(\tau) J_\nu(y\tau)\mathrm{d}\tau = \frac{n^{-1}}{\sqrt{2\pi y}}\, {}_2F_1\left(n, -n+\frac{1}{2}; \frac{1}{2}; y^2\right) \qquad (24)$$

where ${}_2F_1$ is the hypergeometric function. The hypergeometric functions in equations (23) and (24) can be significantly simplified by setting $y = \sin\theta$. From Abramowitz and Stegun [14] (p. 556, equations 15.1.17 and 15.1.18) we have

$${}_2F_1\left(n+\frac{1}{2}, -n+\frac{1}{2}; \frac{1}{2}; \sin^2\theta\right) = \frac{\cos(2n\theta)}{\cos\theta} \qquad (25)$$

$${}_2F_1\left(n, -n; \frac{1}{2}; \sin^2\theta\right) = \cos(2n\theta) \qquad (26)$$

Further, we define new parameters $\theta = \vartheta/2$; $n^{-1}C_n = c_n$; $d^* = 2\arcsin(d)$ so that the dual series (21) and (22) eventually become

$$\sum_{n=1}^{\infty} n c_n \cos(n\vartheta) = 0 \qquad (0 < \vartheta < d^*) \qquad (27)$$

$$\sum_{n=1}^{\infty} c_n \cos(n\vartheta) = G^*(\vartheta) = G\left\{2\arcsin\left[b^* \sin(\vartheta/2)\right]\right\} \qquad (d^* < \vartheta < \pi) \qquad (28)$$

4. The solution of the mixed-boundary value problem

Dual trigonometrical series of the type displayed in (27)-(28) have been considered by several authors in the past. For a review the reader is referred to the classical book of Sneddon [6]. However, the solution suggested by Sneddon [6] is complicated in the sense that the method requires the computation of the derivative of a scalar function. Hence, here we will proceed differently using the method proposed by Tranter [4] who unexpectedly considered all possible dual trigonometrical series except the form of (27)-(28).

First we take the derivative of (28) with respect to ϑ to obtain

$$\sum_{n=1}^{\infty} n c_n \sin(n\vartheta) = -\frac{\mathrm{d}G^*(\vartheta)}{\mathrm{d}\vartheta} \qquad (d^* < \vartheta < \pi) \tag{29}$$

Accordingly, (27) and (29) are multiplied by $2\cos(\vartheta/2)$ to exploit the properties of the trigonometrical functions. In particular (27) yields

$$\sum_{n=1}^{\infty} n c_n \cos[(n+1/2)\vartheta] + \sum_{n=1}^{\infty} n c_n \cos[(n-1/2)\vartheta] = 0 \tag{30}$$

In the first term we let $n = r$ and since for $r = 0$ the corresponding term is zero we start the summation from $r = 0$. In the second term we let $n = r + 1$ and we finally obtain

$$\sum_{r=0}^{\infty} [(r+1)c_{r+1} + r c_r] \cos[(r+1/2)\vartheta] = 0 \qquad (0 < \vartheta < d^*) \tag{31}$$

Using a similar procedure in (29) we obtain

$$\sum_{r=0}^{\infty} [(r+1)c_{r+1} + r c_r] \sin[(r+1/2)\vartheta] = -2\frac{\mathrm{d}G^*(\vartheta)}{\mathrm{d}\vartheta}\cos(\vartheta/2) \qquad (d^* < \vartheta < \pi) \tag{32}$$

We are now able to employ Mehler's integrals [17], p.52

$$P_r(\cos t) = \frac{\sqrt{2}}{\pi}\int_0^t \frac{\cos[(r+1/2)\vartheta]}{\sqrt{\cos\vartheta - \cos t}}\mathrm{d}\vartheta = \frac{\sqrt{2}}{\pi}\int_t^\pi \frac{\sin[(r+1/2)\vartheta]}{\sqrt{\cos t - \cos\vartheta}}\mathrm{d}\vartheta \tag{33}$$

where P_r stands for the Legendre polynomial. That means that we can multiply equation (31) by $\frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{\cos\vartheta - \cos t}}$ and integrate with respect to ϑ over the interval [0, t] and equation (32) by $\frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{\cos t - \cos\vartheta}}$ and integrate over the interval [t, π]. Doing so one gets

$$\sum_{r=0}^{\infty}[(r+1)c_{r+1} + rc_r]P_r(\cos t) = \begin{cases} 0, & 0 < t < d^* \\ 2F^*(t), & d^* < t < \pi \end{cases} \tag{34}$$

where

$$F^*(t) = -\frac{\sqrt{2}}{\pi}\int_t^\pi \frac{\mathrm{d}G^*(\vartheta)}{\mathrm{d}\vartheta}\frac{\cos(\vartheta/2)}{\sqrt{\cos t - \cos\vartheta}}\mathrm{d}\vartheta \tag{35}$$

Finally, making use of the orthogonality relation of the Legendre polynomials (e.g. [14], p. 338, equation 8.14.13) the following is obtained

$$(r+1)c_{r+1} + rc_r = (2r+1)\int_{d^*}^{\pi} F^*(t)P_r(\cos t)\sin t\mathrm{d}t \tag{36}$$

In fact, (36) represents a recurrence relation that allows calculating the current order r+1 based on the previous order r up to the required truncation. Clearly, the zeroth order expansion coefficient can be assumed equal to zero. The derivation of (36) completes the solution of the concerned problem. Having calculated the coefficients c_r by (36), the original expansion coefficients A_n in (1)-(3) can be obtained by following the reverse process.

Indicating results which are obtained for a specific case study following the outlined procedure are shown in Fig. 1. The plot literarily represents the velocity potential of the associated hydrodynamic slamming problem due to a steep wave hitting a vertical wall. It is evident that the inner boundaries of the mixed-boundary value problem are $a/\pi = 0.6$ and $b/\pi = 0.8$. The results, and in this particular case the potential, is smooth as expected. Between $a/\pi < x/\pi < b/\pi$, the solution is equal to the, assumed, constant function $G(x)$ [see (2)]. It is remarked that Fig. 1 is used as a validation for the convergence of the employed methodology. It shows that convergence is achieved using a relatively large number of modes for the infinite series under investigation, here equal to 100. However, the procedure is relatively fast, as only 10 modes approximate quite satisfactorily the final variation. Fig. 2 shows the sum $\sum_{n=1}^{\infty} A_n \cos[(n-1/2)x]$ [left hand side of (1) and (3)] in the entire interval $x/\pi \in [0, 1]$ for the details assumed in Fig. 1. It is immediately evident that on the intermediate points on $x = a$ and $x = b$ the solution is singular.

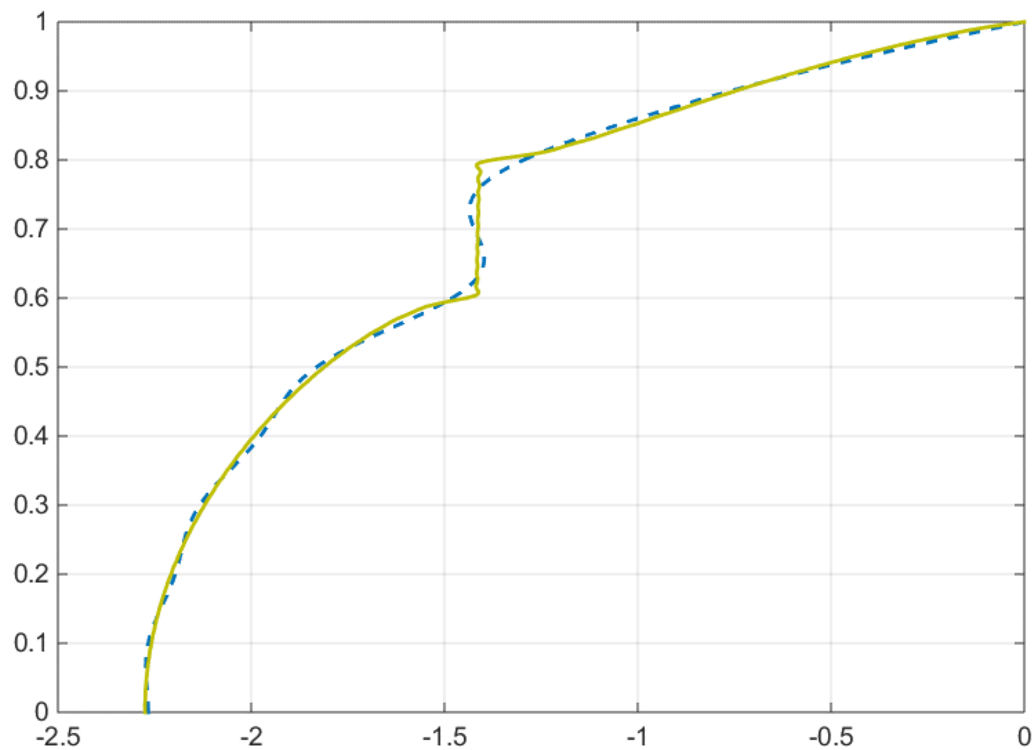

Fig. 1. Convergence of the solution of the triple trigonometrical series. The vertical axis is x/π [see (1)-(3)]. The horizontal axis shows the sum $\sum_{n=1}^{\infty} (n-1/2)^{-1} A_n \cos[(n-1/2)x]$. The dashed line corresponds to a truncation of 10 modes and the solid line to a truncation of 100 modes.

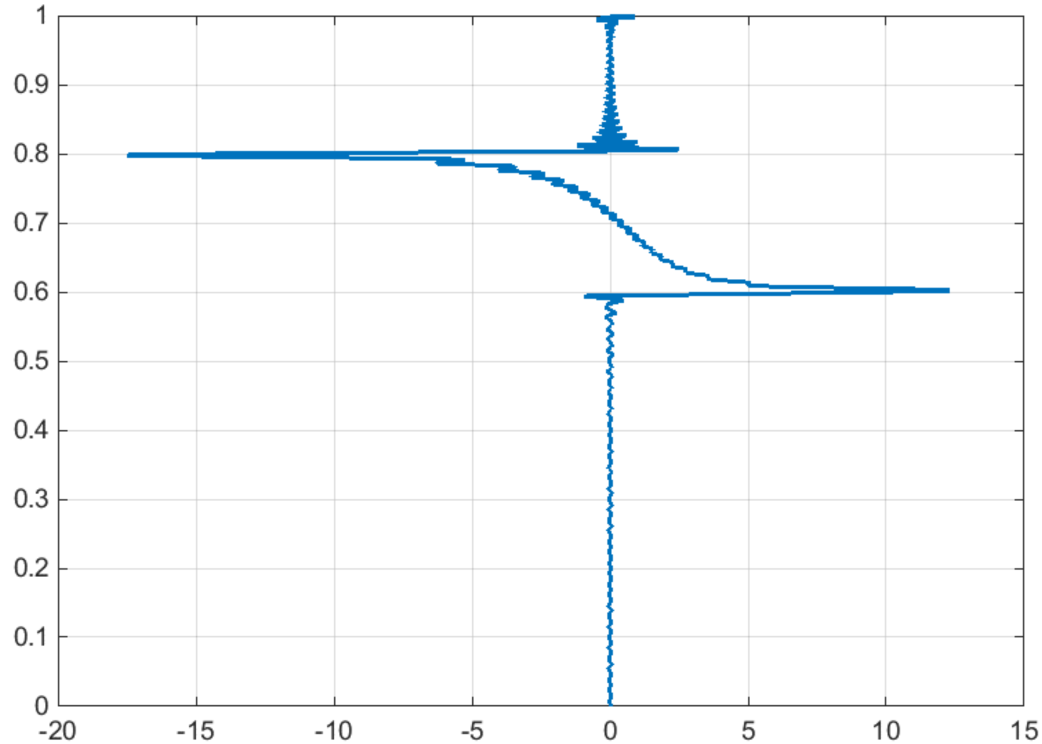

Fig. 2. The sum $\sum_{n=1}^{\infty} A_n \cos\left[(n-1/2)x\right]$ (horizontal axis) against x/π (vertical axis) [see (1)-(3)].

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