FINITELY ADDITIVE MEASURES ON TOPOLOGICAL SPACES AND BOOLEAN ALGEBRAS

by

ZANYAR A. AMEEN

A thesis submitted to the University of East Anglia in partial

fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

School of Mathematics

University of East Anglia

Norwich, UK

June 2015

Dedication

I dedicate this work to my family. Specially my mother for her support, encouragement, and constant love that have sustained me during the whole of my life, and my wonderful wife, Dilveen Ahmed, for being here (in the UK) with me throughout the entirety of my study. She also has been a great source of motivation and inspiration.

This thesis is dedicated to my lovely little son, Sidar, who came into existence during the period of my study.

I also dedicate this thesis and express my special thanks to all my best friends whose words of encouragement pushed me to reach this degree.

Declaration

We hereby confirm that all the work in this thesis without citing is our own contribution excluding the chapter one.

Abstract

The thesis studies some problems in measure theory. In particular, a possible generalization corresponding to Maharam Theorem for finitely additive measures (charges).

In the first Chapter, we give some definitions and results on different areas of Mathematics that will be used during this work.

In Chapter two, we recall the definitions of nonatomic, continuous and Darboux charges, and show their relations tWe hereby confirm that all work without reference are our work excluding the chapter one. We hereby confirm that all work without reference are our work excluding the chapter one. o each other. The relation between charges on Boolean algebras and the induced measures on their Stone spaces is mentioned in this chapter. We also show that for any charge algebra, there exists a compact zero-dimensional space such that its charge algebra is isomorphic to the given charge algebra.

In Chapter three, we give the definition of Jordan measure and some of its outcomes. We define another measure on an algebra of subsets of some set called Jordanian measure, and investigate it. Then we define the Jordan algebras and Jordanian algebras, and study some of their properties.

Chapter four is mostly devoted to the investigation of uniformly regular measures and charges (on both Boolean algebras and topological spaces). We show how the properties of a charge on a Boolean algebra can be transferred to the induced measure on its Stone space. We give a different proof to a result by Mercourakis in [36, Remark 1.10]. In 2013, Borodulin-Nadzieja and Džamonja [10, Theorem 4.1] proved the countable version of Maharam Theorem for charges using uniform regularity. We show that this result can be proved under weaker assumption and further extended.

The final Chapter is concerned with the higher versions of uniform regularity which are called uniform κ-regularity. We study these types of measures and obtain several results and characterizations. The major contribution to this work is that we show we cannot hope for a higher analogue of Maharam Theorem for charges using uniform κ-regularity. In particular, Theorem 4.1 in [10] and Remark 1.10 in [36] cannot be extended for all cardinals. We prove that a higher version of Theorem 4.1 in [10] (resp. Remark 1.10 in [36]) can be proved only for charges on free algebras on κ many generators (resp. measures on a product of compact metric spaces). We also generalize Proposition 2.10 in [26].

Acknowledgements

Firstly, I would like to express my sincere gratitude to primary supervisor Professor Mirna Džamonja for her guidance and for sharing with me her deep understanding of Mathematics and knowledge. It is often said that the relationship between student and his/her supervisor is a crucial reason for the success of a thesis. Surely, it applies to my case. Mirna has managed to make a balance between providing me continuous support, encouragement and advice, and giving me the freedom to explore new ideas in my own way. I am honoured to have been a student of her, and I respect her friendship.

I would like to express my gratitude to my secondary supervisor Professor Alexander Korobkin for his support and advice. He was always ready when I needed him.

I wish to express my thanks to all staff members in the School of Mathematics at the University of East Anglia.

I would also like to thank my family for the support they provided through my entire life. Particularly my mother and my wife, without their love and encouragement, I would not have finished this thesis.

Finally, I would like to thank all my colleagues and everyone who helped me for completing this work.

List of Symbols

Symbol	Description
ω	the set of natural numbers.
$\lvert A\rvert$	the cardinality of a set A.
$\mathfrak{c}$	the cardinality of the continuum.
$\mathrm{Int}(A)$	the set of all interior points of A.
$\mathrm{Cl}(A)$	the closure of A.
$\mathcal{N}_d$	the set of all nowhere dense sets.
$\mathcal{M}_g$	the set of all meager sets.
$\bar{\mathfrak{A}}$	the completion of $\mathfrak{A}$.
$\hat{\mu}$	the completion of a measure μ.
$\mathfrak{A} \cong \mathfrak{B}$	$\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.
$\mathfrak{A} \sqsubseteq \mathfrak{B}$	$\mathfrak{A}$ is embeddable into $\mathfrak{B}$.
$a \triangle a$	the symmetric difference between a, b, i.e., $(a - b) \cup (b - a)$.
$[a]$	the class of an element a.
$\hat{a}$	deotes the clopen set in the Stone space of a Boolean algebra that corresponds an element a in the Boolean algebra.
$\mathrm{Clop}(X)$	the set of all clopen sets in X.
$\mathrm{RO}(X)$	the set of all regular open sets in X.
$\mathrm{Clop}(X)$	the set of all clopen sets in X.
$\omega(X)$	the weight of a space X.
$\pi(X)$	the pseudo weight of a space X.
$\mathcal{N}$	the set of all null sets.

$\mathscr{N}$	the set of all negligible sets.
$\Sigma \cup \mathscr{N}$	$:= \{E \cup N : E \in \Sigma, N \in \mathscr{N}\}$.
$\sigma(\mathcal{A})$	the σ-algebra generated by an algebra $\mathcal{A}$.
$\sigma(\Sigma \cup \mathscr{N})$	the σ-algebra generated by $\{E \cup N : E \in \Sigma, N \in \mathscr{N}\}$.
$\sigma(\Sigma \triangle \mathscr{N})$	the σ-algebra generated by $\{E \triangle N : E \in \Sigma, N \in \mathscr{N}\}$.
$\mathcal{B}(X)$	the Borel algebra of a space X.
$\mathcal{B}_0(X)$	the Baire algebra of a space X.
$[s]$	the measurable rectangle generated by a finite sequence s.
$\mathrm{supp}(\mu)$	the support of μ.
$\mathrm{dom}(\mu)$	the domain of μ.
$\mathrm{Rng}_{\mathcal{A}}(\mu)$	the range of μ on an algebra $\mathcal{A}$.
$\mathcal{E}(\mathbb{I})$	an elementary algebra.
$\mathcal{A}_o(\mathbb{I})$	an open interval algebra.
$\mathfrak{I}(\mathbb{I})$	the interval algebra.
$\mathrm{Free}(\kappa)$	the free algebra on κ generators.
$\mathfrak{A}_c$	the Cantor algebra.
$\mathfrak{C}$	the Cohen algebra.
$\mathcal{J}$	the Jordan measure.
$\mathcal{J}_*$	the inner Jordan measure.
$\mathcal{J}^*$	the our Jordan measure.
∂	the topological boundary.
χ_A	the indicator function of A.
ℓ	the length function.
$\mathrm{Ma}(X, \mu)$	the measure algebra of μ on X.
$\mathscr{C}(X, \mu)$	the charge algebra of μ on X.
$\mathcal{J}(X, \mu)$	the Boolean algebra of Jordan μ-measurable sets in X.
$\mathcal{L}(\mathbb{I}, \lambda)$	the Boolean algebra of Lebesgue measurable sets in $\mathbb{I}$.
$J(X, \mu)$	the Boolean algebra of Jordanian μ-measurable sets in X.
$\mathcal{J}_\mu(X)$	the Jordan algebra of μ on X.
$\mathcal{J}_0(X)$	the algebra of open representatives in $\mathcal{J}_\mu(X)$.

$\mathcal{J}_c(X)$	the algebra of closed representatives in $\mathcal{J}_\mu(X)$.
$J_\mu(X)$	the Jordanian algebra of μ on X.
$\mathcal{L}_\lambda(\mathbb{I})$	the Lebesgue measure algebra of $\mathbb{I}$.
$\mathcal{B}_\lambda(\mathbb{I})$	the Borel algebra modulo null subsets of $\mathbb{I}$.
$\sigma(\mathcal{J}_\lambda(\mathbb{I}))$	the σ-algebra generated by elements of $\mathcal{J}_\lambda(\mathbb{I})$.
$\mathcal{A}_E$	the algebra generated by $\{A \cap E : E \subseteq X, A \in \mathcal{A}\}$, where $\mathcal{A}$ is algebra of sets in a space X.
$\mathrm{ur}(\mu)$	the smallest cardinality of a uniformly μ-dense set.
$\mathrm{mt}(\mu)$	the smallest cardinality of a μ-dense set.
$\omega_P(X)$	pure weight of a space X.

Contents

Chapter 1

Introduction, Background and Notation

This chapter is devoted to an introduction and basic results in the relevant mathematical areas used in this thesis.

1.1 Introduction

Measure theory is the study of measures, it generalizes the notions of (arc) length, (surface) area and volume. The earliest and most important examples are Peano-Jordan measure (also known as Jordan measure) and Lebesgue measure. The former was found by the French mathematician Camille Jordan and the Italian mathematician Giuseppe Peano at the end of the nineteenth century and named after them. The latter was developed by Henri Lebesgue in about 1900 as an extension of Jordan measure. Measure theory became the foundation for the integration theory used in advanced mathematics and for modern probability theory. It is well known that there are countably additive and finitely additive measures. Finitely additive measures have not received as much attention as the countably additive ones. We do not see lots of text books of measure theory on finitely additive measures. A reason could be that countably additive measures are easier to manage than finitely additive ones. According to S. Bochner (as he remarked to D. Maharam [34]), finitely additive measures are more interesting, and possibly more important, than countably additive ones. Finitely additive measures arise quite naturally in many areas of analysis and are a wide class of measures. Our focus will be on finitely additive (probability) measures. The only book on finitely additive measures we are aware of is “Theory of Charges” by Rao and Rao [43].

It is known that Jordan measure is defined on some algebra of subsets of the unit interval $[0, 1]$. The unit interval topologically is a compact metrizable space. The question arises: Can a similar measure be defined on an arbitrary topological space? Before answering this question, if we look at the Jordan measure, it can be constructed in two ways: firstly, it is the (Caratheodory) extension of the length function on the algebra of elementary subsets of $[0, 1]$. Secondly, it is the restricted Lebesgue measure to the algebra of Jordan measurable subsets of $[0, 1]$. A Jordan measurable set is defined as any Lebesgue measurable set that has boundary of measure zero. Coming back to our question, we can define the Jordan measure on arbitrary topological spaces, we only dispose of the second method, because an arbitrary charge on an algebra of subsets of a topological space might not have the properties that the length function has on the elementary algebra. We define an analogue to Jordan measure constructed in the first way on any algebra of subsets of a topological space and call it Jordanian measure. We investigate some of its properties and behavior, for more details about Jordan and Jordanian measures, see Chapter three.

Nonatomicity, continuity and Darboux are properties of measures. These three properties turn out to be equivalent on countably additive measures. Interestingly, they are not the same for finitely additive measures. Their relation is the following:

$$\text{Darboux} \Longrightarrow \text{continuity} \Longrightarrow \text{nonatomicity [43].}$$

More details on these properties are given and possible cases when these properties agree are also studied in Chapter two.

The separable measure is an interesting type of measure. This class of measures has been studied by many authors, for a detailed historical review, see pages 446-459 in [7]. A good characterization of separable measures is obtained by Halmos and von

Neumann in [28]. In fact, separable is the countable case of Maharam type of measures.

Another type of measure which is stronger than separability has been defined and called uniformly regular. Uniformly regular measures were first introduced by Berezanskii [5] in the context of pseudo-metric spaces. Then, Babiker [2] generalized them to the context of topological spaces. He made a series of articles on this kind of measures, with coauthors. In 1996, Mercourakis [36] claimed that uniform regularity is much superior to separability. He gave a nice characterization of uniformly regular measures on a compact Hausdorff space, see Theorem 4.2.2. Then he proved that every strictly positive nonatomic uniformly regular measure on a compact Hausdorff space has a Jordan algebra isomorphic to the usual Jordan algebra, see Theorem 4.3.6. Using our results we provide a different proof to this theorem.

Recently, Borodulin-Nadzieja and Džamonja [10] defined uniform κ-regularity with respect to finitely additive measures on Boolean algebras. When $\kappa = \omega$, they obtained a countable version of the most celebrated result of measure theory called Maharam Theorem for charges. Namely: Given a Boolean algebra $\mathfrak{A}$ and a uniformly regular, continuous, strictly positive charge μ on it. Then the charge algebra $(\mathfrak{A}, \mu)$ is (metrically) isomorphic to a subalgebra of the Jordan algebra of the Lebesgue measure on the unit interval. Consequently, a Boolean algebra supports a continuous uniformly regular charge if and only if it is isomorphic to a subalgebra of the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ containing a dense Cantor subalgebra, see Theorem 4.1 in [10]. We show that this result can be proved under weaker assumptions, i.e.: Suppose that $\mathfrak{A}$ is a Boolean algebra and μ is a uniformly regular, nonatomic, strictly positive charge on $\mathfrak{A}$. Then $(\mathfrak{A}, \mu)$ is (metrically) isomorphic to a subalgebra of the Jordan algebra. Consequently, a Boolean algebra supports a nonatomic uniformly regular charge if and only if it is isomorphic to a subalgebra of the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ containing a dense Cantor subalgebra, see Theorem 4.4.3. Also one of our aims in this thesis is to extend this result to have the full Jordan algebra. We find out if the charge μ is, in addition, μ-complete, then the charge algebra $(\mathfrak{A}, \mu)$ is (metrically) isomorphic to the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$, see Corollary 4.4.15. Equivalently, given an algebra $\mathfrak{A}$ of subsets of a topological space X

and a strictly positive uniformly regular nonatomic charge μ on it, then the Jordanian algebra $J_\mu(X)$ is (metrically) isomorphic to the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$, see Theorem 4.4.11.

Moving on to a more general case, the main purpose of this work is to extend Theorem 4.4.2 given by Borodulin-Nadzieja and Džamonja, Theorem 4.3.6 by Mercourakis [36] or Theorem 4.4.11 to a higher version to have the corresponding Maharam Theorem for charges for all cardinals. We show that none of these results can be directly extended to a higher version. We observe that for any cardinal κ such that $\kappa^\omega = \kappa$, there exist two strictly positive nonatomic uniformly κ-regular measures (even of Maharam type κ) on a compact Hausdorff space that have non-isomorphic Jordan algebras, see Theorem 5.4.2. We show that Theorem 5.4.2 is true in the special context for measures on (an arbitrary) product of compact metric spaces. That is, given a cardinal κ and any strictly positive homogeneous Radon measure μ of Maharam type κ on the generalized Cantor space $X = \{0,1\}^\kappa$, then the Jordan algebra of μ is isomorphic to the Jordan algebra of λ, the standard product measure on X. In fact, this was posed as an open question in both [36] and [26]. Also, we prove a higher version of Theorem 4.1 in [10] is only true for charges on free algebras. That is, any two strictly positive s-homogeneous (defined later) charges of Maharam type κ on the free algebra Free(κ) on κ generators have isomorphic Jordanian algebras.

In [26, Proposition 2.10], Grekas and Mercourakis proved that for a family $\{X_\alpha : \alpha < \kappa\}$ of compact metric spaces, each space with at least two points, and on each X_α there is a strictly positive nonatomic Radon measure μ_α, the Jordan algebra of $\mu = \bigotimes_{\alpha<\kappa} X_\alpha$ on $X = \prod_{\alpha<\kappa} X_\alpha$ is isomorphic to the Jordan algebra of the usual (Lebesgue) product measure ν on the generalized Cantor space $\{0,1\}^\kappa$. We generalize this result and show that we can replace each compact metric space X_α by a compact Hausdorff space assuming that each μ_α is uniformly regular. That is, given a family $\{X_\alpha : \alpha < \kappa\}$ of compact Hausdorff spaces, each space with at least two points and on each X_α there is a strictly positive nonatomic uniformly regular Radon measure μ_α, then the Jordan algebra of $\mu = \bigotimes_{\alpha<\kappa} X_\alpha$ on $X = \prod_{\alpha<\kappa} X_\alpha$ is isomorphic to the Jordan algebra of the usual

(Lebesgue) product measure λ on the generalized Cantor space $\{0,1\}^\kappa$, see Theorem 5.3.4.

1.2 Preliminaries, Background and Notation

Throughout the present work, "space" always means "Hausdorff topological space" on which no other property is assumed unless explicitly stated. We shall denote by $\omega(= 0, 1, 2, 3, ...)$ the natural numbers, by $\mathfrak{c} = 2^\omega$ the cardinality of the continuum and by κ any infinite cardinal. We say that a Boolean algebra $\mathfrak{A}$ carries a charge (or a measure) μ if μ is defined on the whole of $\mathfrak{A}$. Furthermore, if μ is strictly positive on $\mathfrak{A}$, then we say that $\mathfrak{A}$ supports μ.

We start with some sections on background of different areas of Mathematics that will be used in this thesis.

1.2.1 Some General Topology

Recall that a subset A of a topological space X is called **regular open** if A =Int(Cl(A)). Its complement A =Cl(Int(A)) is called a **regular closed set** set. A is said to be **dense** [49] if its closure is the whole space X, i.e. (Cl(A)$= X$). A is called a **nowhere dense** set [49] if the interior of its closure is empty, i.e. (Int(Cl(A)$= \emptyset$). A is said to be **meager** [49] if it can be expressed as a countable union of nowhere dense subsets of X. It is known that the class of all nowhere dense (resp. meager) sets in X, denoted by $\mathcal{N}_d$ (resp. $\mathcal{M}_g$), is an ideal (resp. σ-ideal) of $\mathcal{P}(X)$, see the properties given below. A is called G_δ if it is a countable intersection of open sets. The complement of a G_δ set is an F_σ set, that is, a countable union of closed sets. A is said to be a **zero set** in X if there exists a continuous real valued function f such $A = f^{-1}(0)$. A complement of a zero set is a **cozero** set.

Now we give some properties of nowhere dense sets:

- Every subset of a nowhere dense set is nowhere dense.
- A finite union of nowhere dense sets is nowhere dense.
- The complement of a closed nowhere dense set is an open dense set, and thus the complement of a nowhere dense set is a set with dense interior.

- The boundary of every open or closed set is a closed nowhere dense set.
- Every closed nowhere dense set is the boundary of an open set.
- Every nowhere dense set is contained in a closed nowhere dense set.
- A set is nowhere dense if and only if its complement contains an open dense set.

Definition 1.2.1.1. A topological space (X, τ) is **compact** if every open cover of X has a finite subcover.

Definition 1.2.1.2. A topological space (X, τ) is **Hausdorff** if for every two distinct points in X there exist two disjoint open sets containing them.

Lemma 1.2.1.3. Let X be a topological space and $A \subseteq X$.

(1) If X is a metrizable and if A is closed, then A is a G_δ set, [17].

(2) If X is a compact (Hausdorff) space, then A is a zero set if and only if it is a closed G_δ set, [17].

(3) Every cozero set is the union of a non-decreasing sequence of zero sets, ([21], 4A2C-(b)).

Definition 1.2.1.4 (Countable Chain Condition, ccc). A topological space X is said to satisfy **ccc** if every family of pairwise disjoint open subsets of X is at most countable.

Definition 1.2.1.5. Let X be a topological space. A **net** is a function from a directed set A (= every two elements have an upper bound) to X.

Definition 1.2.1.6. Let (X, τ) be a topological space. X said to be **normal** if for every two disjoint closed subsets of X, there exist two disjoint open sets containing them.

Lemma 1.2.1.7. [17] A topological space X is normal if and only if for every open set U containing a closed set E, there is an open set V such that

$$E \subseteq V \subseteq \mathrm{Cl}(V) \subseteq U.$$

Definition 1.2.1.8. A topological space (X, τ) is said to be **connected** if it cannot be expressed as a union of two disjoint nonempty open sets. A subset A of X is connected if it is connected as a subspace.

Definition 1.2.1.9. A topological space (X, τ) is called **totally disconnected** if the only connected subsets of X are singletons.

Definition 1.2.1.10. A topological space (X, τ) is said to be **extremally disconnected** if the closure of every open subset of X is open.

Definition 1.2.1.11. A topological space is called **separable** if it has a countable dense subset.

Theorem 1.2.1.12. [31, Theorem 4.10] Let I be any set of cardinality not exceeding $\mathfrak{c}$, i.e. $|I| \leq \mathfrak{c}$. Then for any separable space X, the power X^I is separable.

Definition 1.2.1.13. A space X is **completely regular** (or **Tychonoff**) if and only if for every closed sets E and every point $x \in X \setminus E$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(E) = 1$.

Lemma 1.2.1.14 (Urysohn's Lemma). [49] A space X is normal if and only if for every two disjoint closed sets E and F in X, there is a continuous function $f : X \to [0, 1]$ such that $f(E) = 0$ and $f(F) = 1$.

Lemma 1.2.1.15. [17]

(1) Every compact Hausdorff space is normal.

(2) Every compact metric space is second countable, i.e. has a countable base.

Definition 1.2.1.16. Let X be a topological space. A family $\mathscr{B}$ of nonempty open sets in X is called a **π-base** of X if for each open set U in X there exists $B \in \mathscr{B}$ such that $B \subseteq$.

Definition 1.2.1.17. Let X be a topological space, the **pseudo weight** $\pi(X)$ of X is defined to be the smallest cardinality of a π-basis of X.

Definition 1.2.1.18. Let X be a topological space, the **weight** $\omega(X)$ of X is defined to be the smallest cardinality of a basis of X.

Theorem 1.2.1.19 (Tychonoff Embedding Theorem). A space X is Tychonoff if and only if it is homeomorphic to a (closed) subspace of a cube, (see Theorem 3.2.6 [17]). In particular, every completely regular space of weight κ is embeddable into the cube $[0,1]^\kappa$.

Definition 1.2.1.20. [46] A continuous surjective function $f : X \longrightarrow Y$ is called **irreducible** if X is the only subset for which $f(X) = Y$.

Proposition 1.2.1.21. [46, Proposition 7.1.12] Let X, Y be two compact spaces and let $f : X \longrightarrow Y$ be continuous and irreducible. Then both X, Y have the same density character (= the smallest cardinality of a dense subset of a topological space).

Theorem 1.2.1.22. [25, Theorem 3.2] Every compact Hausdorff space X is the continuous image of an extremally disconnected compact Hausdorff space. Among the pairs (Y, f) consisting of an extremally disconnected compact space Y and a continuous function f from Y onto X, there is one for which

$$f(Y_0) \neq X \text{ for any proper closed subset } Y_0 \text{ of } Y. \qquad (*)$$

Remark 1.2.1.23. From the proof the above theorem, we notice that the pair (Y, f) that satisfies the condition $(*)$ consists of: the Stone space Y of a complete Boolean algebra of subsets of X and the continuous surjection f that maps each utrafilter in Y to its limit in X.

Lemma 1.2.1.24. [46, Exercise 25.2.3 (D)] Let X, Y be two compact spaces and let $f : X \longrightarrow Y$ be an irreducible continuous surjection. The following conditions hold:

(1) $\mathrm{Int}(f^{-1}(B)) = \emptyset$ for every closed $B \subseteq Y$ with $\mathrm{Int}(B) = \emptyset$.

(2) $f^{-1}(N)$ is nowhere dense in X for every nowhere dense subset $N \subseteq Y$.

(3) $f(M)$ is nowhere dense in Y for every nowhere dense subset $M \subseteq X$.

(4) If X satisfies the condition "every meager set is nowhere dense", then Y also satisfies it.

1.2.2 Boolean algebras

Definition 1.2.2.1. An **(abstract) Boolean algebra** $(\mathfrak{A}, \vee, \wedge, \neg, 0, 1)$ is a nonempty set $\mathfrak{A}$ together with two special elements $0, 1$, two binary operations $\vee$, $\wedge$ and one unary operation $\neg$ satisfying the following conditions:

(1) $\neg 1 = 0; \qquad \neg 0 = 1;$

(2) $\neg(\neg a) = a;$

(3) $a \wedge 0 = 0; \qquad a \vee 1 = 1;$

(4) $a \wedge 1 = a; \qquad a \vee 0 = a;$

(5) $a \wedge \neg a = 0; \qquad a \vee \neg a = 1;$

(6) $a \wedge a = a; \qquad a \vee a = a;$

(7) $\neg(a \wedge b) = \neg a \vee \neg b; \qquad \neg(a \vee b) = \neg a \wedge \neg b;$

(8) $a \wedge b = b \wedge a; \qquad a \vee b = b \vee a;$

(9) $a \wedge (b \wedge c) = (a \wedge b) \wedge c; \qquad a \vee (b \vee c) = (a \vee b) \vee c;$

(10) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c); \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);$

for all $a, b, c \in \mathfrak{A}$.

A subset $\mathfrak{B}$ of a Boolean algebra $\mathfrak{A}$ is called **subalgebra** if it is closed under the operations on $\mathfrak{A}$ and contains both 0 and 1.

Definition 1.2.2.2. A **(concrete) Boolean algebra** is an ordered pair $(X, \mathcal{A})$, where X is a set and $\mathcal{A}$ is a family of subsets of X that satisfies the following:

(1) $\emptyset, X \in \mathcal{A}$;

(2) for any $A \in \mathcal{A}$, A^c belongs to $\mathcal{A}$; and

(3) if $A_1, A_2 \in \mathcal{A}$, then $A_1 \cap A_2 \in \mathcal{A}$.

In the literature $\mathcal{A}$ is known as an **algebra of sets** or **field of sets**. Obviously, every algebra of sets (concrete Boolean algebra) is a (abstract) Boolean algebra. Stone's Representation Theorem gives a converse, which states "every Boolean algebra is isomorphic to an algebra of sets". We may interchangeably use $\vee$ and $\cup$, as well as, $\wedge$ and $\cap$ and $\leq$ and $\subseteq$, but we prefer to fix the complement symbol to $\bullet^c$ (instead of $\neg$) for both notions.

Definition 1.2.2.3. Let $\mathfrak{A}$ be a Boolean algebra. A subset $\mathcal{I} \subseteq \mathfrak{A}$ is called an **ideal** if:

(1) $0 \in \mathcal{I}$,

(2) $b \in \mathcal{I}$, $a \in \mathfrak{A}$ and $a \leq b$, then $a \in \mathcal{I}$, and

(3) $a, b \in \mathcal{I}$, then $a \vee b \in \mathcal{I}$.

A proper ideal $\mathcal{M}$ of $\mathfrak{A}$ (that is $\mathcal{M} \neq \mathfrak{A}$) is called **maximal** if there is no any other proper ideal $\mathcal{I}$ such that $\mathcal{M} \subsetneq \mathcal{I}$. A dual notion to ideal is the notion of a filter:

Definition 1.2.2.4. Let $\mathfrak{A}$ be a Boolean algebra. A subset $\mathcal{F} \subseteq \mathfrak{A}$ is called a **filter** if:

(1) $1 \in \mathcal{F}$,

(2) $a \in \mathcal{F}$, $b \in \mathfrak{A}$ and $a \leq b$, then $b \in \mathcal{F}$, and

(3) $a, b \in \mathcal{F}$, then $a \wedge b \in \mathcal{F}$.

A filter $\mathcal{U}$ of $\mathfrak{A}$ is an **ultrafilter** (also called maximal) if $\mathcal{F}$ is any other filter such that $\mathcal{U} \subseteq \mathcal{F}$ implies that $\mathcal{U} = \mathcal{F}$, or equivalently, if $a \in \mathfrak{A}$, then either $a \in \mathcal{U}$ or $a^c \in \mathcal{U}$.

Definition 1.2.2.5. A filter $\mathcal{U}$ in a topological space X **converges** to a point $x \in X$ if every neighborhood U of x belongs to $\mathcal{U}$.

Definition 1.2.2.6. A filter $\mathcal{F}$ in a topological space X **accumulates** to a point $x \in X$ if $U \cap F \neq \emptyset$ for every open set U containing x and every $F \in \mathcal{F}$.

Lemma 1.2.2.7. If $\mathcal{U}$ is an ultrafilter in a topological space X, then $\mathcal{U}$ converges to a point $x \in X$ if and only if it accumulates to x.

Lemma 1.2.2.8. Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous at $x \in X$ if and only if whenever $\mathcal{U}$ an ultrafilter on X converges to x, then the ultrafilter $f(\mathcal{U})$ on Y converges to $f(x)$.

Definition 1.2.2.9. Given Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$, and a mapping $f : \mathfrak{A} \to \mathfrak{B}$. We say that f is a **Boolean homomorphism** if it satisfies the following equalities for all $a, b \in \mathfrak{A}$:

$$f(a \wedge b) = f(a) \wedge f(b),$$
$$f(a \vee b) = f(a) \vee f(b),$$
$$f(a^c) = f(a)^c.$$

A **monomorphism**, also called an **embedding**, is an injective (one-to-one) homomorphism: if $f(a) = f(b)$, then $a = b$. We write $\mathfrak{B} \sqsubseteq \mathfrak{A}$ if $\mathfrak{B}$ is embeddable into $\mathfrak{A}$. An **epimorphism** is a surjective (onto) homomorphism, that is $f(\mathfrak{A}) = \mathfrak{B}$. A bijective homomorphism, that is to say, it is both one-to-one and onto, is called an **isomorphism**. If there is an isomorphism from a Boolean algebra $\mathfrak{A}$ onto $\mathfrak{B}$, then the two algebras are said to be isomorphic, and denoted by $\mathfrak{A} \cong \mathfrak{B}$. An isomorphism from a Boolean algebra onto itself is called an **automorphism**. Every homomorphism f is **monotonic** ($a \leq b$ implies that $f(a) \leq f(b)$ for all $a, b \in \mathfrak{A}$) and the image $f(\mathfrak{A})$ of $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.

Lemma 1.2.2.10. If $f : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism of Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$, then the so called kernel of f, $f^{-1}(0) = \{a \in \mathfrak{A} : f(a) = 0\}$ is an ideal of $\mathfrak{A}$, and the dual of the kernel, $f^{-1}(1) = \{a \in \mathfrak{A} : f(a) = 1\}$, is a filter of $\mathfrak{A}$.

Fact 1.2.2.11. Given an ideal $\mathcal{I}$ of a Boolean algebra $\mathfrak{A}$, we define a binary relation $\sim$ on $\mathfrak{A}$ by the following formula:

$$a \sim b \text{ if and only if } a \bigtriangleup b \in \mathcal{I}.$$

By properties of the symmetric difference $\bigtriangleup$, it follows that $\sim$ is an equivalence relation, that splits $\mathfrak{A}$ into equivalence classes which are denoted by $[\bullet]$. Lemma 5.22, [37] states that $\mathfrak{A}/\sim$ is a quotient (Boolean) algebra and the canonical map $\mathfrak{A} \mapsto \mathfrak{A}/\sim$ has kernel $\mathcal{I}$. This means that the following operations on the quotient $\mathfrak{A}/\sim$ are well-defined:

$$[a] \vee [b] = [a \vee b]$$
$$[a] \wedge [b] = [a \wedge b]$$
$$[a]^c = [a^c].$$

Also, it is worth saying that $[0] = \mathcal{I}$ and $[1] = \mathcal{I}^c := \{a^c : a \in \mathcal{I}\}$. Therefore, the structure $(\mathfrak{A}/\sim, \wedge, \vee, \bullet^c, [0], [1])$ is a Boolean algebra, which is called **quotient Boolean algebra** (with respect to the given ideal $\mathcal{I}$) and denoted by $\mathfrak{A}/\mathcal{I}$. The **canonical surjection**, say, $\phi : \mathfrak{A} \to \mathfrak{A}/\mathcal{I}$ defined by:

$$\phi(a) = [a] \text{ for every } a \in \mathfrak{A}$$

is a homomorphism of Boolean algebras and $ker(f) = \{a \in \mathfrak{A} : \phi(a) = [0]\} = \mathcal{I}$.

Lemma 1.2.2.12 (First Homomorphism Theorem). Let $f : \mathfrak{A} \to \mathfrak{B}$ be an epimorphism of Boolean algebras with kernel $\mathcal{I}$. Then there is a unique isomorphism $g : \mathfrak{A}/\mathcal{I} \to \mathfrak{B}$ such that $g \circ \phi = f$, where ϕ is the canonical map (see [37]).

Definition 1.2.2.13. [42, Definition 3.2.1] A family $\mathcal{S}$ of subsets of a set X is called a **semi-algebra** if it contains both empty set and X, is closed under finite intersections and the complement of any element can be expressed as a finite union of a mutually disjoint set of elements of $\mathcal{S}$.

Definition 1.2.2.14. A **σ-algebra** or **σ-field** on a set X is a family of subsets of X which contains the empty set and is closed under the complement and countable unions of its members.

Definition 1.2.2.15. Let $\mathcal{C}$ be a collection of subsets of a set X. Then there exists a (unique) smallest algebra (resp., σ-algebra) which contains every set in $\mathcal{C}$ (maybe $\mathcal{C}$ itself is an algebra (resp., σ-algebra)). This algebra (resp., σ-algebra) is called the **algebra** (resp., **σ-algebra**) **generated** by $\mathcal{C}$.

Definition 1.2.2.16. Let (X, τ) be a topological space. The **Borel algebra** (or **Borel σ-algebra**) $\mathcal{B}$ is the σ-algebra generated by all open subsets of X, or equivalently, all closed subsets of X. Members of $\mathcal{B}$ are called **Borel sets**. They are named after Êmile Borel (1871-1956).

Definition 1.2.2.17. Let (X, τ) be a locally compact Hausdorff topological space. The **Baire algebra** (or **Baire σ-algebra**) $\mathcal{B}_0$ is the σ-algebra generated by all compact G_δ subsets of X, or equivalently, all open F_σ subsets of X. Members of $\mathcal{B}_0$ are called **Baire sets** in René-Louis Baire's (1874-1932) honor.

Lemma 1.2.2.18. [27, Theorem D] A closed subset A of a compact space X is Baire if and only if it is G_δ.

Definition 1.2.2.19. A distinguished set of elements of a Boolean algebra is called **generators** if each element of the Boolean algebra can be written as a finite combination of generators, using the Boolean operations, and the generators are as independent as possible, in the sense that there are no relationships among them.

Definition 1.2.2.20. A generating set E of a Boolean algebra $\mathfrak{A}$ is called **free** if the elements of E satisfy no non-trivial equation of Boolean algebras. A Boolean algebra is called free if it has a free set of generators.

We usually denote by $\mathrm{Free}(\kappa)$ (or shortly $\mathrm{F}(\kappa)$) the free algebra on κ generators. It can be realized topologically as the collection of all clopen subsets of the product space $\{0,1\}^\kappa$, see Theorem 14.3 in [47].

Definition 1.2.2.21. The **Cantor algebra** $\mathfrak{A}_c$ is defined to be the countably infinite atomless (Boolean) algebra.

Remark 1.2.2.22. Since all countable atomless algebras that have more than one element are isomorphic (Theorem 10, [24]), the Cantor algebra is unique (up isomorphism).

Definition 1.2.2.23. A Boolean algebra $\mathfrak{A}$ is said to be **complete** if every subset of $\mathfrak{A}$ has a union (or supremum).

Definition 1.2.2.24. A subset $\mathfrak{B}$ of a Boolean algebra $\mathfrak{A}$ is called **dense** if for every $0 \neq a \in \mathfrak{A}$, there is $0 \neq b \in \mathfrak{B}$ such that $b \leq a$.

Definition 1.2.2.25. The **completion** of $\mathfrak{B}$ is the unique complete Boolean algebra $\mathfrak{A}$ (up to isomorphism) such that $\mathfrak{B}$ is dense in $\mathfrak{A}$, or equivalently, is the complete Boolean algebra $\mathfrak{A}$ containing $\mathfrak{B}$ such that every nonzero element of $\mathfrak{A}$ is the supremum of some subset of $\mathfrak{B}$.

Definition 1.2.2.26. The completion of the Cantor algebra $\mathfrak{A}_c$ is called the **Cohen algebra** and denoted by $\mathfrak{C}$.

Definition 1.2.2.27 (Countable Chain Condition, ccc). A Boolean algebra $\mathfrak{A}$ satisfies **ccc** if every disjoint set of non-zero elements of $\mathfrak{A}$ is countable.

Definition 1.2.2.28. For a Boolean algebra $\mathfrak{A}$, the **pseudo weight** $\pi(\mathfrak{A})$ is defined to be the smallest cardinality of a set B of positive elements in $\mathfrak{A}$ such that for all $a \in \mathfrak{A} \setminus \{0\}$, there is $b \in B$ with $b \leq a$.

M. Stone [50] proved an unexpected connection between Boolean algebras and certain topological spaces, now called Stone spaces in his honor. It states that every Boolean algebra $\mathfrak{A}$ has an associated topological space, denoted by Stone($\mathfrak{A}$), called its **Stone space**. Stone($\mathfrak{A}$) is a compact totally disconnected Hausdorff space, or compact zero-dimensional (zero-dimensional is a space that has a base consisting of clopen sets). Stone's representation theorem in fact shows that if we start with an abstract Boolean algebra $\mathfrak{A}$ then through construction of its Stone space X we get a concrete Boolean algebra of clopen sets of X which is isomorphic to $\mathfrak{A}$ (see Theorem given below). A Stone space is also called a **Boolean space**.

Theorem 1.2.2.29 (Stone's Representation Theorem). [50] Every Boolean algebra is isomorphic to an algebra of sets. More precisely, every Boolean algebra $(\mathfrak{A}, \vee, \wedge, \neg, \mathbf{0}, \mathbf{1})$ is isomorphic to the algebra of clopen subsets of a compact Hausdorff totally disconnected space $Z = \text{Stone}(\mathfrak{A})$.

The space Z is identified as the set of ultrafilters on $\mathfrak{A}$ or equivalently the homomorphisms from $\mathfrak{A}$ to the two-element Boolean algebra. Its topology is generated by the collection

$$\{\widehat{a} : a \in \mathfrak{A}\}, \text{ where } \widehat{a} := \{z \in Z : a \in z\}.$$

Every element in the above family is both open and closed because every ultrafilter contains either a or $\neg a$ and both of them belong to $\mathfrak{A}$. So both $\widehat{a}$ and $\widehat{(\neg a)} = Z \setminus \widehat{a}$ are open and consequently, they are closed sets (as each one is complement of the other).

The assignment $a \longrightarrow \widehat{a}$, called **Stone isomorphism**, forms a Boolean isomorphism from $\mathfrak{A}$ into $\mathcal{P}(Z)$ whose image, $\widehat{\mathfrak{A}} = \mathrm{Clop}(Z)$, can be identified with $\mathfrak{A}$. If $a \in \mathfrak{A}$, then clearly $a \longrightarrow \widehat{a}$ is an isomorphism embedding (by Stone the isomorphism). On the other hand, if C is any clopen set, since C is open, it can be written as a union of clopen sets in the basis $\{\widehat{a} : a \in \mathfrak{A}\}$. By compactness, C can be written as $C = \widehat{a}_1 \cup \widehat{a}_2 \cdots \cup \widehat{a}_n$ for some $a_1, a_2, \ldots, a_n \in \mathfrak{A}$. But $a_1 \vee a_2 \cdots \vee a_n \in \mathfrak{A}$ and so $a_1 \vee a_2 \cdots \vee a_n = a'$ for some a'. Hence, $\widehat{a'}$ belong to the basis $\mathrm{Clop}(Z)$. Also, this assignment preserves the following basic operations:

(i) $\widehat{\mathbf{0}} = \emptyset$ and $\widehat{\mathbf{1}} = Z$;

(ii) $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$;

(iii) $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$;

(iv) $a \leq b$ if and only if $\widehat{a} \subseteq \widehat{b}$; and

(v) $\widehat{(\neg a)} = Z \setminus \widehat{a}$.

Proofs of the above statements can be found in ([47], page 23 and [20], page 16).

Proposition 1.2.2.30. [20, Proposition 313C] Let Z be the Stone space of a Boolean algebra $\mathfrak{A}$.

(1) If $A \subseteq \mathfrak{A}$ and $a_0 \in \mathfrak{A}$, then $a_0 = \sup A$ in $\mathfrak{A}$ if and only if $\widehat{a}_0 = \mathrm{Cl}\big(\bigcup_{a \in A} \widehat{a}\big)$.

(2) If $A \subseteq \mathfrak{A}$ is nonempty and $a_0 \in \mathfrak{A}$, then $a_0 = \inf A$ in $\mathfrak{A}$ if and only if $\widehat{a}_0 = \mathrm{Int}\big(\bigcap_{a \in A} \widehat{a}\big)$.

(3) If $A \subseteq \mathfrak{A}$ is nonempty, then $\inf A = 0$ in $\mathfrak{A}$ if and only if $\bigcap_{a \in A} \widehat{a}$ is nowhere dense in Z.

Theorem 1.2.2.31. [20, Theorem 314S] Let $\mathfrak{A}$ be a Boolean algebra and Z be its Stone space. The following are equivalent:

(1) $\mathfrak{A}$ is complete.

(2) Z is extremally disconnected.

(3) $\mathrm{Clop}(Z) = \mathrm{RO}(Z)$, the family of all regular open sets in Z.

Let us recall some relations between Boolean homomorphisms on Boolean algebras and continuous functions on their Stone spaces.

Lemma 1.2.2.32. Let $\mathfrak{A}, \mathfrak{B}$ be two algebras, Z, Y their Stone spaces and $\mathrm{Clop}(Z), \mathrm{Clop}(Y)$ the clopen algebras of Z, Y, respectively. Then there is a one-to-one correspondence between Boolean homomorphisms $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ and continuous functions $f : Y \longrightarrow Z$, given by the following formula

$$\varphi(a) = b \Longleftrightarrow f^{-1}(\widehat{a}) = \widehat{b},$$

where $\widehat{a} \in \mathrm{Clop}(Z)$ corresponds to $a \in \mathfrak{A}$ and $\widehat{b} \in \mathrm{Clop}(Y)$ corresponds to $b \in \mathfrak{B}$. Such φ and f possess the following properties:

(a) f is onto if and only if φ is one-to-one.

(b) f is one-to-one if and only if φ is onto.

Proof. See Theorem 312P and Corollary 312R in [20]. ■

As a consequence of the above lemma, one can obtain the following:

Lemma 1.2.2.33. For Boolean algebras $\mathfrak{A}, \mathfrak{B}$ and their Stone spaces Z, Y, respectively and φ and f as in Lemma 1.2.2.32, then $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is an isomorphism if and only if $f : Y \longrightarrow Z$ is a homeomorphism.

Theorem 1.2.2.34. The Stone space of a Boolean algebra $\mathfrak{A}$ is metrizable if and only if $\mathfrak{A}$ is countable.

Proof. Apply the Stone construction. A compact Hausdorff space is metrizable if and only if it is second-countable (see [17], Theorem 4.2.8), the proof follows. For more details, (see [47], page 25). ■

Definition 1.2.2.35. Let $\{\mathfrak{A}_i : i \in I\}$ be a family of Boolean algebras. Let Z_i be the Stone space of $\mathfrak{A}_i$ for every i. Then the **free product** $\bigotimes_{i\in I} \mathfrak{A}_i$ of $\{\mathfrak{A}_i\}_{i\in I}$ is the algebra $\mathfrak{A}$ of clopen subsets of the product topology $Z = \prod_{i\in I} Z_i$.

1.2.3 Some Measure Theory

Definition 1.2.3.1. Let $\mathcal{A}$ be a σ-algebra of subsets of a space X. A **countably additive measure** μ on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty]$ that satisfies the following conditions:

(1) $\mu(\emptyset) = 0$,

(2) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ for every countable collection $\{A_i\}$ of pairwise disjoint sets in $\mathcal{A}$.

If $\mathcal{A}$ is an algebra of subsets of X (not necessarily σ-algebra) and we restrict the second condition to finite collections only, that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$, then μ is said to be a **finitely additive measure** on $\mathcal{A}$. The pair $(X, \mathcal{A})$ is called a **measurable space** (resp. **chargeable space**) and the triple $(X, \mathcal{A}, \mu)$ is called a **measure space** (resp. **charge space**).

By a **measure** we mean a countably additive one, and by a **charge** we mean a finitely additive measure. It is obvious that every measure is a charge but not vice versa as shown in the following example:

Example 1.2.3.2. Consider a set $X = \{x_n : n < \omega\}$ and an algebra $\mathcal{A} = \{A \subseteq X : A$ is finite or A^c is finite$\}$. Let μ on $\mathcal{A}$ be defined by

$$\mu(A) = \begin{cases} 0, & \text{if } A \text{ is finite,} \\ 1, & \text{if } A^c \text{ is finite.} \end{cases}$$

Clearly μ is a charge on $\mathcal{A}$ but not a measure because $\mu(\{x_n\}) = 0$ for all n, and $0 = \sum\limits_{n=1}^{\infty} \mu(\{x_n\}) \neq \mu(\bigcup\limits_{n=1}^{\infty} \{x_n\}) = \mu(X) = 1$.

Definition 1.2.3.3. Let $\mathcal{A}$ be a σ-algebra (resp. algebra) of subsets of a space X. A measure (resp. charge) μ on $\mathcal{A}$ is said to be

(1) **finite** if $\mu(A) < \infty$ for every $A \in \mathcal{A}$.

(2) **σ-finite** if there is a countable family $(A_n)_{n \in \mathbb{N}}$ of subsets of $\mathcal{A}$ with $\mu(A_n) < \infty$ for every n such that $X = \bigcup A_n$.

(3) **probability** if $\mu(X) = 1$.

(4) **strictly positive** if $\mu(A) = 0$ only for $A = \emptyset$.

Definition 1.2.3.4. Let μ be a measure (resp. charge) on an algebra of sets $\mathcal{A}$. A set $A \in \mathcal{A}$ is called **null** if it is of zero measure (resp. charge) (i.e. $\mu(A) = 0$). Any subset of a null set is also null and countable (resp. finite) union of null sets is null set with respect to the measure (resp. the charge) μ, so the class of all null sets, denoted by $\mathcal{N}$, forms a σ-ideal (resp. ideal) of $\mathcal{A}$.

Definition 1.2.3.5. Let μ be a measure (resp. charge) on an algebra $\mathcal{A}$ of subsets of X. A set $B \subseteq X$ is called **negligible** if there is $A \in \mathcal{A}$ such that $B \subseteq A$ and $\mu(A) = 0$. Also, the class of all negligible sets in X forms a σ-ideal (resp. ideal) in $P(X)$ with respect to the measure (resp. the charge) μ, and is denoted by $\mathscr{N}$.

Fact 1.2.3.6. (i) Every null set is negligible.

(ii) A negligible set is null if and only if it is measurable or chargeable.

Definition 1.2.3.7. Let (X, Σ, μ) be a measure space (resp. charge space). μ is said to be **complete** if every subset of every null set is measurable (resp. chargeable), i.e., every negligible set is measurable (resp. chargeable).

Definition 1.2.3.8. Let X be a set. A set function μ^* defined on the family of all subsets of X is called an **outer measure** whenever μ^* satisfies the following:

(i) $\mu^*(\emptyset) = 0$;

(ii) if $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$;

(iii) $\mu^*(\bigcup\limits_{i=1}^{\infty} A_i) \leq \sum\limits_{i=1}^{\infty} \mu^*(A_i)$ for $A_1, A_2, \dots \in \mathcal{P}(X)$.

μ^* is called an **outer charge** if it satisfies (i), (ii) and $\mu^*(\bigcup\limits_{i=1}^{n} A_i) \leq \sum\limits_{i=1}^{n} \mu^*(A_i)$ for $A_1, A_2, \dots, A_n \in \mathcal{P}(X)$.

One can define an outer measure (resp. outer charge) from another function as follows:

Remark 1.2.3.9. Suppose that we have a family $\mathcal{A}$ of subsets of a set X such that $\emptyset, X \in \mathcal{A}$. Let $m : \mathcal{A} \longrightarrow [0, \infty]$ be such that $m(\emptyset) = 0$. For $E \subseteq X$, define

$$\mu^*(E) = \inf\Big\{\sum_{i=1}^{\infty} m(A_i) : E \subseteq \bigcup_{i=1}^{\infty} A_i \text{ and } A_i \in \mathcal{A} \text{ for all } i\Big\}. \qquad (\heartsuit)$$

$$(\text{resp. } \mu^*(E) = \inf\Big\{\sum_{i=1}^{n} m(A_i) : E \subseteq \bigcup_{i=1}^{n} A_i \text{ and } A_i \in \mathcal{A} \text{ for } 1 \leq i \leq n\Big\}). \qquad (\spadesuit)$$

Then μ^* is an outer measure (resp. outer charge) on $P(X)$.

Definition 1.2.3.10. Let $\mathcal{A}$ be an algebra (not necessarily a σ-algebra) of subsets of a space X and let $\mu : \mathcal{A} \longrightarrow [0, \infty]$ be a function. Then μ is called a **premeasure** if

(1) $\mu(\emptyset) = 0$; and

(2) $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, for every finite or countable family $A_1, A_2, \ldots$ of pairwise disjoint sets in $\mathcal{A}$ whose union belongs to $\mathcal{A}$.

Definition 1.2.3.11. Let μ^* be an outer measure (resp. outer charge) on a set X. A subset $A \subseteq X$ is **μ^*-measurable** (resp. **μ^*-chargeable**) if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \qquad (\bigstar)$$

for all $E \subseteq X$.

Theorem 1.2.3.12 (Caratheodory's Theorem for Measures). Let μ^* be an outer measure on X. If $\mathcal{M}$ is the family of all μ^*-measurable subsets of X, then $\mathcal{M}$ is a σ-algebra, and if μ is the restriction of μ^* to $\mathcal{M}$, then μ is a measure. Furthermore, $\mathcal{M}$ contains all μ-null sets, *i.e.* μ is complete.

Proof. See Theorem 1.11.4 in [8]. ∎

Notice that the above theorem is valid for charges in the following context (to be constructed from outer charges):

Theorem 1.2.3.13 (Caratheodory's Theorem for Charges). Let μ^* be an outer charge on a set X. If $\mathcal{A}$ is the family of all μ^*-chargeable subsets of X, then $\mathcal{A}$ is an algebra, and if μ is the restriction of μ^* to $\mathcal{A}$, then μ is a charge. Furthermore, $\mathcal{A}$ contains all μ-null sets, *i.e.* μ is complete.

Proof. See Corollary 8.3 in [29], page 233. ∎

Lemma 1.2.3.14 (Caratheodory's Extension Theorem). Let $\mathcal{A}$ be an algebra of sets in X and $\mu : \mathcal{A} \longrightarrow [0,\infty]$ a premeasure on $\mathcal{A}$. Define

$$\mu^*(E) = \inf\Big\{\sum_{i=1}^{\infty}\mu(A_i) : E \subseteq \bigcup_{i=1}^{\infty} A_i \text{ and } A_i \in \mathcal{A} \text{ for all } i\Big\}, \qquad (\dagger)$$

for all $E \subseteq X$. Then

(i) μ^* is an outer measure on X;

(ii) $\mu^*(A) = \mu(A)$ for every $A \in \mathcal{A}$. In particular, every $A \in \mathcal{A}$ is μ^*-measurable;

(iii) μ has a unique extension to $\mathcal{B} = \sigma(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$.

Proof. The proof of (i) follows from Remark 1.2.3.9.
(ii) Let $E \in \mathcal{A}$. Take a sequence $A_1, A_2, \dots$ in $\mathcal{A}$ such that $A_1 = E$ and $A_n = \emptyset$ for all $n \geq 2$ and then applying ($\dagger$) yields $\mu^*(E) \leq \mu(E)$. For the other direction, suppose that $E \subset \bigcup\limits_{i=1}^{\infty} A_i$ with $A_i \in \mathcal{A}$. Take $A_n^* = E \cap (A_n \setminus \bigcup\limits_{i=1}^{n-1} A_i)$. So A_n^* are pairwise disjoint elements in $\mathcal{A}$ (as it is an algebra) and $E = \bigcup\limits_{n=1}^{\infty} A_n^*$. Now, we have

$$\mu(E) = \sum_{i=1}^{\infty}\mu(A_i^*) \leq \sum_{i=1}^{\infty}\mu(A_i).$$

Taking the infimum over all such sequences $A_1, A_2, \dots$, we obtain $\mu(E) \leq \mu^*(E)$, hence

$$\mu^*(A) = \mu(A) \text{ for every } A \in \mathcal{A}.$$

We now show that all $A \in \mathcal{A}$ are μ^*-measurable. Let $A \in \mathcal{A}$ and $E \subseteq X$. By subadditivity of μ^* and using the fact that $E = (E \cap A) \cup (E \cap A^c)$, we get

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

For the other direction, let $\epsilon > 0$, there is a sequence $A_1, A_2, A_3 \dots \in \mathcal{A}$ such that $E \subset \bigcup\limits_{i=1}^{\infty} A_i$ and $\sum\limits_{n=1}^{\infty}\mu(A_i) \leq \mu^*(E) + \epsilon$. Therefore,

$$\begin{aligned}\mu^*(E) + \epsilon &\geq \sum_{n=1}^{\infty}\mu(A_i) = \sum_{n=1}^{\infty}\mu(A_i \cap A) + \sum_{n=1}^{\infty}\mu(A_i \cap A^c)\\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c).\end{aligned}$$

As ϵ was taken arbitrarily, we have $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Thus, A is μ^*-measurable.

(iii) Assume that μ has two extensions on the σ-algebra $\mathcal{B}$ generated by $\mathcal{A}$, (by Theorem 1.2.3.12). One of them is μ^* (as in Theorem 1.2.3.12) and let the other be ν. We now have to prove that $\mu^*(B) = \nu(B)$ for every $B \in \mathcal{B}$. Given $B \in \mathcal{B}$, as $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{A}$ and the family of all μ^*-measurable sets is a σ-algebra, by Theorem 1.2.3.12, B must be μ^*-measurable and

$$\mu^*(B) = \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) : B \subseteq \bigcup_{i=1}^{\infty} A_i \text{ and } A_i \in \mathcal{A} \text{ for all } i \right\}.$$

But μ and ν agree on $\mathcal{A}$, so $\sum\limits_{i=1}^{\infty} \mu(A_i) = \sum\limits_{i=1}^{\infty} \nu(A_i)$ and then for $B \subseteq \bigcup\limits_{i=1}^{\infty} A_i$ where $A_i \in \mathcal{A}$ for all i, we have

$$\nu(B) \leq \sum_{i=1}^{\infty} \nu(A_i) = \sum_{i=1}^{\infty} \mu(A_i),$$

which implies that

$$\nu(B) \leq \mu^*(B). \qquad (\bigstar)$$

To obtain the reverse of $(\bigstar)$, we have to use some other arguments because we do not know whether ν is constructed from an outer measure or not. Let $\epsilon > 0$ and let $A_1, A_2, A_3 \cdots \in \mathcal{A}$ be a sequence such that $B \subset \bigcup\limits_{i=1}^{\infty} A_i$ and $\sum\limits_{i=1}^{\infty} \mu(A_i) \leq \mu^*(B) + \epsilon$. Let $D = \bigcup\limits_{i=1}^{\infty} A_i$ and let $C_n = \bigcup\limits_{i=1}^{n} A_i$. Now,

$$\mu^*(B) + \epsilon \geq \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \mu^*(A_i) \geq \mu^*(\bigcup_{i=1}^{\infty} A_i) = \mu^*(D),$$

and so $\mu^*(D \setminus B) \leq \epsilon$. On the other hand,

$$\mu^*(D) = \lim_{n\to\infty} \mu^*(C_n) = \lim_{n\to\infty} \nu(C_n) = \nu(D).$$

Therefore, since $B \subseteq D$,

$$\begin{aligned} \mu^*(B) &\leq \mu^*(D) = \nu(D) = \nu(B) + \nu(D \setminus B) \\ &\leq \nu(B) + \mu^*(D \setminus B) \leq \nu(B) + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, then $\mu^*(B) \leq \nu(B)$. Together with $(\bigstar)$, this yields $\mu^*(B) = \nu(B)$ for an arbitrary $B \in \mathcal{B}$. Hence, μ has a unique extension. ■

Now, we observe the following:

Remark 1.2.3.15. Note that the premeasure μ in Lemma 1.2.3.14 cannot be replaced by a charge. More precisely, a charge μ_0 (which is not a measure) defined on an algebra $\mathcal{A}$ cannot always be extended to a measure on the σ-algebra generated by $\mathcal{A}$, as shown in the next example. On the other hand, any charge on a Boolean algebra can be extended to a measure on its Stone space, which will be explained in detail in the upcoming sections.

Example 1.2.3.16. Consider the algebra $\mathcal{A}$ and the charge μ given in Example 1.2.3.2. That is, $\mathcal{A} = \{A \subseteq \mathbb{N} : A \text{ is finite or } A^c \text{ is finite}\}$ and μ is defined to be

$$\mu(A) = \begin{cases} 0, & \text{if } A \text{ is finite,} \\ 1, & \text{if } A^c \text{ is finite.} \end{cases}$$

It is shown in Example 1.2.3.2 that μ is a charge but not a measure. It is easy to show that the σ-algebra Σ generated by $\mathcal{A}$ coincides with the power set of $\mathbb{N}$ (which is the σ-algebra generated by $\{\{n\} : n \in \mathbb{N}\}$). Assume that $\hat{\mu}$ is an extension of μ to Σ. Clearly, Σ contains singletons $\{n\}$ and $\hat{\mu}(\{n\}) = \mu(\{n\}) = 0$ because $\{n\}$ is finite. So, now we have $0 = \sum\limits_{n\in\mathbb{N}} \hat{\mu}(\{n\}) \neq \hat{\mu}(\bigcup\limits_{n\in\mathbb{N}} \{n\}) = \hat{\mu}(\mathbb{N}) = 1$. Therefore, $\hat{\mu}$ is not a measure. This shows that μ cannot be extended to a measure $\hat{\mu}$.

Theorem 1.2.3.17. [42, Theorem 3.3.3] Let μ' be a charge on a semi-algebra $\mathcal{S}$. There is a unique charge μ on the algebra $\mathcal{A}$ generated by $\mathcal{S}$ such that $\mu(E) = \mu'(E)$ for all $E \in \mathcal{S}$.

Lemma 1.2.3.18. Let (X, Σ, μ) be a measure space, and let $\mathscr{N}$ be the family of all negligible sets, i.e., $\mathscr{N} = \{N \subseteq X : N \subseteq M, M \in \Sigma, \mu(M) = 0\}$. Set $\hat{\Sigma} = \{A \cup N : A \in \Sigma, N \in \mathscr{N}\}$ and define a function $\hat{\mu} : \hat{\Sigma} \longrightarrow [0, \infty]$ by

$$\hat{\mu}(A \cup N) = \mu(A).$$

Then

(1) $\hat{\Sigma}$ is a σ-algebra.

(2) $\Sigma \subset \hat{\Sigma}$.

(3) $\hat{\Sigma}$ is exactly the σ-algebra generated by $\Sigma \cup \mathscr{N}$, i.e., $\hat{\Sigma} = \sigma(\Sigma \cup \mathscr{N})$

(4) $\hat{\mu}$ is a measure on $\hat{\Sigma}$ such that $\hat{\mu}(A) = \mu(A)$ for every $A \in \Sigma$.

(5) $(X, \hat{\Sigma}, \hat{\mu})$ is a complete measure space.

Proof. (1) Observe that $\emptyset \in \mathscr{N}$, so $X = X \cup \emptyset \in \hat{\Sigma}$. Let $B \in \hat{\Sigma}$. Then $B = A \cup N$ for some $A, M \in \Sigma$ with $\mu(M) = 0$ and $N \subseteq M$. We shall show that $B^c \in \hat{\Sigma}$. Now, $B^c = A^c \cap N^c$. Since $N = M \setminus (M \setminus N) = M \cap (M \setminus N)^c$, we have $N^c = M^c \cup (M \setminus N)$. So $B^c = A^c \cap \big(M^c \cup (M \setminus N)\big) = (A^c \cap M^c) \cup \big(A^c \cap (M \setminus N)\big)$. Since $A, M \in \Sigma$, then $A^c \cap M^c \in \Sigma$ (as Σ is a σ-algebra). On the other hand, $A^c \cap (M \setminus N) \subseteq M \setminus N \subseteq M$ which implies that $A^c \cap (M \setminus N) \in \mathscr{N}$. Hence, $B^c \in \hat{\Sigma}$. It remains to show that $\hat{\Sigma}$ is closed under countable unions. Let $\langle B_n : n < \omega \rangle$ be a sequence of elements of $\hat{\Sigma}$. Then for every n there are $A_n, M_n \in \Sigma$ with $\mu(M_n) = 0$ and $N_n \in \mathscr{N}$ with $N_n \subseteq M_n$ such that $B_n = A_n \cup N_n$. Then $\bigcup\limits_{n<\omega} B_n = \bigcup\limits_{n<\omega} (A_n \cup N_n) = \Big(\bigcup\limits_{n<\omega} A_n\Big) \cup \Big(\bigcup\limits_{n<\omega} N_n\Big)$ and $\bigcup\limits_{n<\omega} N_n \subseteq \bigcup\limits_{n<\omega} M_n$. Since null sets are closed under countable unions, we have $\mu(\bigcup\limits_{n<\omega} M_n) = 0$. Therefore, $\bigcup\limits_{n<\omega} N_n \in \mathscr{N}$. But $\bigcup\limits_{n<\omega} A_n \in \Sigma$. Hence $\bigcup\limits_{n<\omega} B_n \in \hat{\Sigma}$. Thus $\hat{\Sigma}$ is a σ-algebra.

(2) Let $A \in \Sigma$. Since $\emptyset \in \mathscr{N}$ and $A = A \cup \emptyset$, then $A \in \hat{\Sigma}$. Thus $\Sigma \subset \hat{\Sigma}$.

(3) Let $N \in \mathscr{N}$. Since $\emptyset \in \hat{\Sigma}$ and $N = \emptyset \cup N$, we have $N \in \hat{\Sigma}$, and $\Sigma \subset \hat{\Sigma}$ by (2), thus $\Sigma \cup \mathscr{N} \subset \hat{\Sigma}$. Then $\sigma(\Sigma \cup \mathscr{N}) \subseteq \sigma(\hat{\Sigma}) = \hat{\Sigma}$. If we show that $\hat{\Sigma}$ is the smallest σ-algebra containing $\Sigma \cup \mathscr{N}$, then we are done. Take any σ-algebra Λ of sets in X containing $\Sigma \cup \mathscr{N}$, then it should contain all sets of the form $A \cup N$ where $A \in \Sigma$ and $N \in \mathscr{N}$. But this means that $\hat{\Sigma} \subset \Lambda$. Thus, $\hat{\Sigma} = \sigma(\Sigma \cup \mathscr{N})$, the smallest σ-algebra containing $\Sigma \cup \mathscr{N}$.

(4) We first need to show that $\hat{\mu}$ is well-defined on $\hat{\Sigma}$. Let $B = A_1 \cup N_1$ and $B = A_2 \cup N_2$ where $A_1, A_2 \in \Sigma$ and $N_1, N_2 \in \mathscr{N}$. Since N_1 and N_2 are negligible sets, there are $M_1, M_2 \in \Sigma$ with $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$ such that $\mu(M_1) = 0$ and $\mu(M_2) = 0$. Then $A_1 \subseteq A_1 \cup N_1 = A_2 \cup N_2 \subseteq A_2 \cup M_2$ and so $\mu(A_1) \leq \mu(A_2) + \mu(M_2) = \mu(A_2) + 0 = \mu(A_2)$. Hence $\mu(A_1) \leq \mu(A_2)$. Similarly, we get $\mu(A_2) \leq \mu(A_1)$. Therefore $\mu(A_1) = \mu(A_2)$. This shows that $\hat{\mu}$ is well-defined.

Next, we show that $\hat{\mu}$ is a measure on $\hat{\Sigma}$. Clearly $\hat{\mu}(\emptyset) = 0$ because $\emptyset \cup \emptyset = \emptyset$ and $\mu(\emptyset) = 0$. Let $\langle E_n : n < \omega \rangle$ be a sequence of pairwise disjoint elements of $\hat{\Sigma}$. Then $E_n = A_n \cup N_n$ where $A_n \in \Sigma$ and $N_n \in \mathscr{N}$ for $n < \omega$. Now

$$\begin{aligned}
\hat{\mu}(\bigcup_{n<\omega} E_n) &= \hat{\mu}\Big(\bigcup_{n<\omega}(A_n \cup N_n)\Big) \\
&= \hat{\mu}\Big[\Big(\bigcup_{n<\omega} A_n\Big) \cup \Big(\bigcup_{n<\omega} N_n\Big)\Big] \\
&= \mu\Big(\bigcup_{n<\omega} A_n\Big) \\
&= \sum_{n<\omega} \mu(A_n) \\
&= \sum_{n<\omega} \hat{\mu}(A_n \cup N_n) \\
&= \sum_{n<\omega} \hat{\mu}(E_n).
\end{aligned}$$

Thus, $\hat{\mu}$ is measure.

Let $A \in \Sigma$. We have $A \cup \emptyset = A$ and so $\hat{\mu}(A) = \hat{\mu}(A \cup \emptyset) = \hat{\mu}(A) + \hat{\mu}(\emptyset) = \mu(A)$.

(5) Finally, we now show that $\hat{\mu}$ is complete. Let $D \in \hat{\Sigma}$ with $\hat{\mu}(D) = 0$. We need to prove that any subset $D_0 \subseteq D$ is $\hat{\mu}$-measurable and of measure zero, i.e., $\hat{\mu}(D_0) = 0$. Now, $D = A \cup N$ where $A \in \Sigma$ and $N \in \mathscr{N}$ with $N \subseteq M$ for some $M \in \Sigma$ such that $\mu(M) = 0$. Then $D = A \cup N \subseteq A \cup M$ and by assumption, $0 = \hat{\mu}(D) = \mu(A)$. Therefore A is μ-null set and so $A \cup M$ is also μ-null set (union of two null sets). This implies that $D_0 \subseteq D \subseteq A \cup M$ is negligible set. So $\hat{\mu}(\emptyset \cup D_0) = \mu(\emptyset) = 0$. This shows that any subset $D_0 \subseteq D$ is $\hat{\mu}$-measurable and $\hat{\mu}(D_0) = 0$. Thus $\hat{\mu}$ is complete and $(X, \hat{\Sigma}, \hat{\mu})$ is the complete measure space. ■

Definition 1.2.3.19. The measure space $(X, \hat{\Sigma}, \hat{\mu})$ constructed above is called the **completion** of the measure space (X, Σ, μ).

Lemma 1.2.3.20. [19, Proposition 212C] Let (X, Σ, μ) be a measure space and let $(X, \hat{\Sigma}, \hat{\mu})$ be its completion. A set A belongs to $\hat{\Sigma}$ if and only if there are $A_1, A_2 \in \Sigma$ with $A_1 \subseteq A \subseteq A_2$ such that $\mu(A_2 \setminus A_1) = 0$.

Proposition 1.2.3.21. [19, Proposition 212E] Let (X, Σ, μ) be a measure space and let $(X, \hat{\Sigma}, \hat{\mu})$ be its completion. Then we have the following:

(1) The outer measures $\hat{\mu}^*$ and μ^* defined from $\hat{\mu}$ and μ coincide.

(2) $\hat{\mu}$ and μ have the same negligible sets.

(3) $\hat{\mu}$ is the only measure on $\hat{\Sigma}$ which agrees with μ on Σ.

(4) Every $E \in \hat{\Sigma}$ can be expressed as $A \triangle N$ where $A \in \Sigma$ and N is μ-negligible.

From Lemma 1.2.3.18 and Proposition 1.2.3.21 we point out the following:

Remark 1.2.3.22. (1) $\hat{\mu}$ is the smallest extension of μ for which every $\hat{\mu}$-negligible set is $\hat{\mu}$-null.

(2) $\hat{\Sigma}$ is the smallest σ-algebra containing $\Sigma \triangle \mathscr{N} := \{E \triangle N : E \in \Sigma, N \in \mathscr{N}\}$. Thus $\hat{\Sigma} = \sigma(\Sigma \cup \mathscr{N}) = \sigma(\Sigma \triangle \mathscr{N})$.

Proposition 1.2.3.23. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a space, and $f : X \longrightarrow Y$ a function. Set $\mathcal{B} = \{B : B \subseteq Y, f^{-1}(B) \in \mathcal{A}\}$ and $f(\mu)(B) = \mu(f^{-1}(B)) = \nu(B)$. Then $(Y, \mathcal{B}, \nu)$ is a measure space.

Proof. See Proposition 112E in [18]. ■

The measure ν is called the **image measure** under f.

Given a measure μ on a topological space X, the **support of** μ, denoted by supp(μ), is defined to be the set of all points $x \in X$ for which every open set U_x containing x has positive measure.

The support of μ has the following properties:

- supp(μ) is closed because it is the complement of the largest open set of measure zero (largest means union of all open sets of measure zero).
- Every nonempty open set in supp(μ) has positive measure.

Definition 1.2.3.24 (Strictly Positive Measure on Topological Spaces). A **strictly positive measure** on a topological space X is a measure in which every nonempty open set has a positive measure.

A measure μ is a strictly positive on X if X is equal to its support.

Definition 1.2.3.25. Let μ_0 be a measure on a topological space X. μ_0 is said to be a **Baire measure** if it is defined on the Baire algebra $\mathcal{B}_0(X)$ of subsets of X.

Definition 1.2.3.26. Let μ be a measure on a topological space X. μ is called a **Borel measure** if it is defined on the Borel algebra $\mathcal{B}(X)$ of subsets of X.

Definition 1.2.3.27. Let μ be a Borel measure on a topological space X and let $\mathcal{B}(X)$ be the Borel algebra in X, the smallest σ-algebra containing all open subsets of X. It is said that μ is **inner regular** for a set $B \in \mathcal{B}(X)$, with respect to the family $\mathcal{F}$ of closed sets in X, if

$$\mu(B) = \sup\{\mu(F) : F \subseteq B, F \in \mathcal{F}\}.$$

It is said that μ is **outer regular** for a set $B \in \mathcal{B}(X)$, with respect to the family $\mathcal{G}$ of open sets in X, if

$$\mu(B) = \inf\{\mu(G) : B \subseteq G, G \in \mathcal{G}\}.$$

So μ is respectively inner regular and outer regular if it is inner regular and outer regular for all $B \in \mathcal{B}(X)$. Therefore, μ is said to be **regular** if it is both an inner regular and an outer regular measure.

Definition 1.2.3.28. A measure μ on some topological space X is said to be **locally finite** if every point of X has a neighborhood of finite measure.

Definition 1.2.3.29. Let μ be a measure on a topological space X. μ is called a **Radon measure** if it is locally finite and inner regular with respect to the family of compact subsets of X.

A finite regular Borel measure on a compact Hausdorff space is always Radon.

Definition 1.2.3.30. [7] A (Borel) measure μ on a topological space X is τ**-additive** if for every increasing net of open sets $\langle U_\alpha : \alpha \in \Delta\rangle$ on X, we have

$$\sup_{\alpha\in\Delta}(\mu(U_\alpha)) = \mu(\bigcup_{\alpha\in\Delta} U_\alpha).$$

Lemma 1.2.3.31. [7]

(1) Every Radon measure is τ-additive.

(2) Every τ-additive measure on a regular space is regular.

(3) Every τ-additive measure on a compact space is Radon.

Theorem 1.2.3.32. [15, Theorem 7.3.1] Let X be a compact Hausdorff space and μ any finite Baire measure on X. Then μ has an extension to a Borel measure on X, and a unique regular Borel extension.

Theorem 1.2.3.33. [7, Theorem 7.3.11] Let $\mathcal{A}$ be a subalgebra of the Borel algebra $\mathcal{B}(X)$ of a Hausdorff space, and let μ be a non-negative additive function on $\mathcal{A}$ such that for every $A \in \mathcal{A}$ and every $\epsilon > 0$, there exists a compact $K \subset A$ such that $\mu(A \setminus K) < \epsilon$.Then μ extends to a Radon measure on X.

Definition 1.2.3.34. Let (X, Σ, μ) be a probability measure space. A family $\langle A_i \rangle_{i<I}$ of events of Σ (for some index set I) is said to be **stochastically independent** with respect to μ if for every non-empty finite subset $\{i_1, i_2, ..., i_n\}$ of distinct elements of I,

$$\mu(A_1 \cap A_2 \cap ... \cap A_n) = \mu(A_1) \cdot \mu(A_2) \cdot ... \cdot \mu(A_n).$$

Proposition 1.2.3.35. Let X be a compact Hausdorff topological space, Y a subspace of X, and ν a (finite) Radon measure on Y. Then there is a Radon measure μ on X such that $\mu(E) = \nu(E \cap Y)$ whenever μ measures E.

In [21, Proposition 415J], the proposition is stated for quasi-Radon measures (which we do not use here) agrees with Radon measures if X is compact (Hausdorff), see Propositions 416A and 416G in [21].

Definition 1.2.3.36. [19] Let (X, Σ, μ) and (Y, T, ν) be two measure spaces. A function $\phi : X \longrightarrow Y$ is called **inverse-measure-preserving** if $\phi^{-1}(E) \in \Sigma$ and $\nu(\phi^{-1}(E)) = \mu(E)$ for every $E \in T$.

Definition 1.2.3.37. Let (X, Σ, μ) and (Y, T, ν) be two measure spaces. A function $\phi : X \longrightarrow Y$ is called **measure-preserving** if $\nu(\phi(A)) = \mu(A)$ for every $A \in \Sigma$.

Definition 1.2.3.38. Two measure spaces (X, Σ, μ) and (Y, T, ν) are said to be **isomorphic** if there is a bijective function $f : X \longrightarrow Y$ such that f and f^{-1} are measurable, and $\nu(f^{-1}(E)) = \mu(E)$ for every $E \in T$.

Here we mention some definitions and notes regarding the product measure on product spaces. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. Subsets of $X \times Y$ of the form $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are called **measurable cylinder sets** or **measurable rectangles**. It is known that the family of all measurable rectangles $\mathcal{R} = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$ in $X \times Y$ forms a semi-algebra, (see §7.2 [42]). The family of finite (disjoint) unions of measurable rectangles $\mathcal{R}$ is an algebra. The σ-algebra generated by the family of all measurable rectangles $\mathcal{R}$ is usually denoted by $\mathcal{A} \otimes \mathcal{B}$. Define a function $\rho_0 : \mathcal{R} \longrightarrow [0, 1]$ by

$$\rho_0(A \times B) = \mu(A) \cdot \nu(B).$$

By Theorem 7.2.1 in [42], ρ_0 is well-defined and is a premeasure. By Carathéodory's Extension Theorem (Lemma 1.2.3.14), ρ_0 can be extended to a (complete) measure ρ on the σ-algebra $\mathcal{A} \otimes \mathcal{B}$ generated by $\mathcal{R}$. The measure ρ is called the **product measure** of μ and ν and is denoted by $\mu \otimes \nu$, and $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ is called the **product measure space**.

Let us extend the above case to infinity. Given an index set I, let $\{(X_i, \Sigma_i, \mu_i) : i \in I\}$ be a family of probability measure spaces. Let $X = \prod_{i \in I} X_i$ be the Cartesian product of $X_{i's}$. The measurable rectangles in X are defined as follows:

$$R = R_1 \times R_2 \times \cdots \times R_n \times \prod_{\substack{i \in I \\ j \neq i}} X_i,$$

where $R_j \in \Sigma_j$ for $j = 1, 2, ..., n$.

The premeasure ρ_0 can be defined on the set all measurable rectangles $\mathcal{R}$ of X by

$$\rho_0(R) = \bigotimes_{j=1}^{n} \mu_j(R_j).$$

The extended measure ρ on the σ-algebra $\bigotimes_{i \in I} \Sigma_i$ generated by $\mathcal{R}$ (from ρ_0 by Carathéodory method) is the product measure on X and denoted by $\bigotimes_{i \in I} \mu_i$.

The most interesting example of a product measure for us in this thesis is the (probability) measure on the product space $\{0,1\}^\kappa$, where κ is any cardinal. We try to give a detailed construction of this measure. We start with the finite case, then infinite countable and then uncountable. Let us first consider the finite case. Let $X_n = \{0,1\}^n$ for some $n \in \omega$. This is a finite product of discrete spaces $X_i = \{0,1\}$ with $\tau_i = \{\emptyset, \{0\}, \{1\}, X_i\}$ for $i = 1, 2, \cdots, n$. If we think of 0 as the Tail and 1 as the Head of a fair coin, then the possible outcomes give us a measure μ_i as follows:

$$\mu_i(\emptyset) = 0,\ \mu_i(X_i) = 1,\ \mu_i(\{0\}) = 1/2 \text{ and } \mu_i(\{1\}) = 1/2.$$

Now,

$$X_n = \{s = (s_1, s_2, ..., s_n) : s_i = 0 \text{ or } 1\}.$$

Then all (finite) sequences $s_n \in X_n$ have measure

$$\mu_n(s) = \mu_1(s_1) \times \mu_2(s_2) \times \cdots \times \mu_n(s_n) = \underbrace{1/2 \times 1/2 \times \cdots \times 1/2}_{n-times} = 1/2^n.$$

It is not difficult to show that μ_n is a charge on the family $\mathcal{A}_n$ of all subsets of X_n. Thus, for all subsets A of X_n the measure is given by

$$\mu_n(A) = \sum_{s \in A} \mu_n(s) = |A|/2^n.$$

Let us extend the above case to (countable) infinity. $X = \{0,1\}^\omega$ is the infinite product space of discrete spaces, or equivalently, the space of countable infinite sequences of 0's and 1's. So

$$X = \{x = (x_1, x_2, x_3, ...) : x_i = 0 \text{ or } 1\}.$$

Let $s = s_1, s_2, ..., s_n \in \{0,1\}^n$ for some $n \in \omega$. Define

$$[s] = \{x \in X : x_1 = s_1,\ x_2 = s_2, ...,\ x_n = s_n\}.$$

We call $[s]$ the **measurable rectangle generated** by s. Let $\mathcal{T}$ be the collection of all measurable rectangles of X together with X and $\emptyset$. So $\mathcal{T}$ is a semi-algebra (but not algebra) because for any two measurable rectangles $[s]$ and $[t]$ their intersection is either one of them, if either t is an extension of s or vice versa, or empty. The complement of a measurable rectangle is the union of all other measurable rectangles of the same length. Define a set function $\mu : \mathcal{T} \longrightarrow [0,1]$ by

$$\mu([s]) = 1/2^n, \quad \text{for } [s] \in \mathcal{T}.$$

(This corresponds to $\mu_n(s) = 1/2^n$). We try to show that μ is a charge on $\mathcal{T}$. The most important property to satisfy is finite additivity. Since μ is a probability, then the additivity is already satisfied. The reason is, given two (disjoint) measurable rectangles $[s]$ and $[t]$ of length n and m respectively. We consider two cases: (1) suppose that $s \neq t$ with $n = m$. Then $\mu([s]) = \mu([t]) = 1/2^n$. For $[s] \cup [t] = [s \cup t]$ we have freedom to choose either s or t, the probability will be doubled, i.e., $\mu([s] \cup [t]) = 2/2^n = 1/2^{(n-1)} = \mu([s]) + \mu([t])$. (2) if $m > n$, let us think of $m = n + k$ (for some $k \in \omega$) to understand the situation better, then for $[s] \cup [t] = [s \cup t]$, we have one choice of t from 2^m, and 2^k choices of t from 2^m because we know that $s = \underbrace{s_1, s_2, \ldots, s_n}_{\text{until } n \text{ is fixed}}, \underbrace{s_{n+1}, s_{n+2}, \ldots, s_{n+k}}_{\text{free to choose}}$, so we have freedom of choosing k-terms, either 0 or 1, to reach the length of t. In conclusion, we have $2^k + 1$ choices from 2^m, i.e., $\mu([s] \cup [t]) = (1 + 2^k)/2^m$. Therefore

$$\mu([s] \cup [t]) = \frac{(1+2^k)}{2^m} = \frac{2^k}{2^m} + \frac{1}{2^m} = \frac{2^k}{2^{(n+k)}} + \frac{1}{2^m} = \frac{1}{2^n} + \frac{1}{2^m} = \mu([s]) + \mu([t]).$$

This shows that μ is a charge on the semi-algebra $\mathcal{T}$.

Let $\mathfrak{A}$ be the algebra generated by $\mathcal{T}$. By Theorem 1.2.3.17, there is a unique charge μ (we will call it μ again). Notice that μ is defined as follows:

$$\mu(A) = \sum_{i=1}^{n} \mu(C_i),$$

because every A can be represented as a finite disjoint union of members C_i of $\mathcal{T}$.

Our next task is to show that μ is a premeasure on $\mathfrak{A}$. Since X is compact and all measurable rectangles are both open and closed (clopen), then all measurable rectangles are also compact. Let $\{A_n : n \in \omega\}$ be a countable family of mutually disjoint members of $\mathfrak{A}$ whose union is $A \in \mathfrak{A}$. That is, $A = \bigcup_{n=1}^{\infty} A_n$. There exists a finite $N \in \omega$ such that $A = \bigcup_{n=1}^{N} A_n$, as A is compact and each A_i is open. This implies

A_i must be empty for $i > N$ because $A_{i's}$ are mutually disjoint. Thus, μ is automatically a measure because no measurable rectangle can be a countable union of disjoint measurable rectangles. By Lemma 1.2.3.14, μ can be extended to a unique (complete) measure λ on the σ-algebra Σ generated by $\mathfrak{A}$, which is the product measure on 2^ω.

Notice that the Lebesgue measure λ on the unit interval $\mathbb{I} = [0, 1]$ is almost the same as the measure μ constructed above on $X = \{0, 1\}^\omega$. The reason is as follows: every point $x \in \mathbb{I}$ can be mapped to its binary expansion. That is,

$$x = .s_1 s_2 s_3 s_4 \cdots = \sum_{n=1}^{\infty} s_n/2^n \text{ where } s_n \in \{0, 1\}.$$

For justifying the above comment, we recall the following result proved by Fremlin.

Proposition 1.2.3.39. [19, Proposition 254K] Let λ be the Lebesgue measure on $\mathbb{I} = [0, 1]$, and let μ be the standard product measure on $X = \{0, 1\}^\omega$. We have the following:

(1) For $x \in X$, let $\phi : X \longrightarrow \mathbb{I}$ such that $\phi(x) = \sum_{n=1}^{\infty} x(n)/2^n$. Then

   (a) $\phi^{-1}(B) \in \Sigma$ and $\mu(\phi^{-1}(B)) = \lambda(B)$ for every $B \in T$; and

   (b) $\phi(A) \in T$ and $\lambda(\phi(A)) = \mu(A)$ for every $A \in \Sigma$, where $\Sigma = \mathrm{dom}(\mu)$ and $T = \mathrm{dom}(\lambda)$.

(2) There is a bijection $\psi : X \longrightarrow \mathbb{I}$ such that ϕ and ψ are equal for all except countably many points, and any such bijection is an isomorphism between (X, Σ, μ) and $(\mathbb{I}, T, \lambda)$.

In the light of [13, Exercise 7(c), § 18, Chapter (XIII), page 211], ψ^{-1} can be chosen as homeomorphism preserving measure $\psi^{-1} : \mathbb{I} \setminus N \longleftrightarrow X \setminus M$ for some μ-negligible subset M of X and some λ-negligible N subset of $\mathbb{I}$.

Before moving to uncountably infinite, we shall remark the following:

Remark 1.2.3.40. (1) By Remark 1.2.2.22, the Cantor algebra $\mathfrak{A}_c$ is unique (up to isomorphism). Using the fact that the free algebra Free(ω) on countably many generators is countable and also atomless (by Corollary 2 in [24]), so Free(ω) is isomorphic to the clopen algebra of the product space $\{0,1\}^\omega$ (by Theorem 14.3 in [47]). The clopen algebra $\mathrm{Clop}(\{0,1\}^\omega)$ is the algebra generated by the semi-algebra $\mathcal{T} = \{[s] : s \in 2^{<\omega}\}$, discussed in the above construction, which is exactly the generator for the Cantor algebra. Hence, $\mathrm{Free}(\omega)$ and $\mathrm{Clop}(\{0,1\}^\omega)$ can be identified as the Cantor algebra.

(2) In the setting given in (1), the Cohen algebra can be seen as the Boolean algebra of all Borel subsets of $\{0,1\}^\omega$ modulo meager subsets of it, or equivalently, the Boolean algebra of regular open subsets of $\{0,1\}^\omega$, (see Theorem 29 in [24]).

(3) The Lebesgue measure λ on $\mathfrak{A}_c$ for us is the measure that is defined on $\mathcal{T}$ as $\lambda([s]) = 1/2^n$ if $[s] \in \mathcal{T}$ and $s \in 2^n$.

Now, let us consider the uncountable case. First we give a brief explanation and then we mention a nice result proved by Fremlin that covers everything we deal in this work. The product (Lebesgue) measure λ_κ or simply λ on $X = \{0,1\}^\kappa$ is the measure that is extended by Caratheodory's Method from the premeasure λ_0 defined on the semi-algebra of measurable rectangles $[s]$ as follows:

$$\lambda_0([s]) = 1/2^{|\mathrm{dom}(s)|},$$

for all $[s] = \{f : f \in 2^\kappa, s \subseteq f\}$ where $s : \kappa \longrightarrow 2$ is a finite partial function.

The above extension theorem is due to Kakutani [32, Theorem 3].

Definition 1.2.3.41. Let $\langle X_i \rangle_{i\in I}$ be a family of sets for some index set I, and $X = \prod_{i\in I} X_i$. If Σ_i^* is a σ-subalgebra of subsets of X_i, we denote $\bigotimes_{i\in I} \Sigma_i^*$ for the σ-algebra of subsets of X generated by $\{\{x : x \in X, x(i) \in E\} : i \in I, E \in \Sigma_i^*\}$

Theorem 1.2.3.42. [19, Theorem 254F] Given an index set I, let $\{(X_i, \Sigma_i, \mu_i) : i \in I\}$ be a family of probability measure spaces and let ρ be the product measure on $X = \prod_{i\in I} X_i$. Denote by Λ the domain of ρ.

(1) $\rho(X) = 1$.

(2) ρ is complete.

(3) If $E_i \in \Sigma_i$ for every $i \in I$, and $\{i : E_i \neq X_i\}$ is countable, then $\prod_{i\in I} E_i \in \Lambda$ and $\rho(\prod_{i\in I} E_i) = \bigotimes_{i\in I} \mu_i(E_i)$. In particular, $\rho(R) = \rho_0(R)$ for all measurable rectangles R, and for every $j \in I$, the projection $\pi_j : X \longrightarrow X_j$ is inverse measure preserving.

(4) $\bigotimes_{i\in I} \Sigma_i^* \subseteq \Lambda$.

(5) For every $\epsilon > 0$ and every $A \in \Lambda$, there is a finite family $R_1, R_2, \ldots, R_n$ of measurable rectangles such that $\rho(A \triangle \bigcup_{i=1}^{n} R_i) < \epsilon$.

(6) For every $A \in \Lambda$, there exist $A_1, A_2 \in \bigotimes_{i\in I} \Sigma_i^*$ such that $A_1 \subseteq A \subseteq A_2$ and $\rho(A_2 \setminus A_1) = 0$.

Definition 1.2.3.43. [14] A (Radon) measure μ on some topological space X is said to be **normal** if $\mu(N) = 0$ for every nowhere dense subset N of X.

Definition 1.2.3.44. [12] Let μ be a complete Radon measure on a compact Hausdorff space X and μ_0 its restriction to the Baire algebra of X, and let $\hat{\mu}$ be the completion of μ_0. μ is said to be **completion regular** if every Borel subset of X is $\hat{\mu}$-measurable.

Therefore, if $(X, \mathcal{B}_0, \mu_0)$, $(X, \mathcal{B}, \mu)$ and $(X, \hat{\mathcal{B}}, \hat{\mu})$ denote the Baire measure space, complete Radon measure space and the completion measure space of $(X, \mathcal{B}_0, \mu_0)$ respectively, then μ is completion regular if $(X, \mathcal{B}, \mu) = (X, \hat{\mathcal{B}}, \hat{\mu})$.

Lemma 1.2.3.45. [19, Lemma 254G] Let (X, Σ, μ) be the product measure space of a family $\{(X_i, \Sigma_i, \mu_i) : i \in I\}$ of probability spaces for some index I (possibly uncountable), and (Y, T, ν) be a complete probability space. A function $\phi : Y \longrightarrow X$ is inverse measure preserving if and only if $\phi^{-1}(C) \in T$ and $\nu(\phi^{-1}(C)) = \mu(C)$ for every measurable rectangle $C \subseteq X$.

Remark 1.2.3.46. As a comment on the above lemma, if $\phi : Y \longrightarrow X$ is a measure preserving bijection, then both ϕ and ϕ^{-1} are inverse-measure preserving.

Definition 1.2.3.47. A **measure algebra** is a Boolean algebra with a strictly positive (countably additive) measure. A (probability) measure space produces a measure algebra, which is the Boolean algebra of measurable sets modulo null sets.

Definition 1.2.3.48. The **random algebra** $\mathfrak{R}$ is the quotient algebra $\mathcal{B}(2^\omega)/\{A \subseteq 2^\omega : \lambda(A) = 0\}$, where $\mathcal{B}(2^\omega)$ is the Borel σ-algebra.

Lemma 1.2.3.49. Let κ be a cardinal and λ the product measure on $X = \{0,1\}^\kappa$. If the measure algebra $\mathfrak{A}$ of λ is embedded as a subalgebra into the measure algebra $\mathfrak{B}$ of a complete probability space (Y, T, μ), then there is an inverse-measure preserving function from Y to X.

Proof. By Lemma 1.2.3.45. See proof in [20], Example 343C(a). ∎

Remark 1.2.3.50. By Lemma 1.2.3.49, given the product measure λ and a complete probability measure μ on $X = \{0,1\}^\kappa$, if the measure algebras of λ and μ are isomorphic, then there is a measure preserving bijection between (X, Σ, λ) and (X, T, μ), where $\Sigma = \mathrm{dom}(\lambda)$ and $T = \mathrm{dom}(\mu)$.

Definition 1.2.3.51. A **charge algebra** $(\mathfrak{A}, \mu)$ is a Boolean algebra $\mathfrak{A}$ with a strictly positive charge μ. As an example, one can construct a charge algebra from a charge space $(X, \mathcal{A}, \mu)$ after quotienting out all subsets of charge zero (see Proposition 2.3.2). In this case we denote it either by $\mathscr{C}(X, \bar{\mu})$ or $(\mathfrak{A}, \bar{\mu})$, where $\mathfrak{A} = \mathcal{A}/\mathcal{N}$.

Definition 1.2.3.52. [10] Let μ be a charge on a Boolean algebra $\mathfrak{A}$. A family $\mathfrak{B} \subseteq \mathfrak{A}$ is **μ-dense** in $\mathfrak{A}$, if for every $a \in \mathfrak{A}$ and $\epsilon > 0$ there is $b \in \mathfrak{B}$ such that $\mu(a \triangle b) < \epsilon$.

Definition 1.2.3.53. [10] Let μ be a charge on a Boolean algebra $\mathfrak{A}$. A family $\mathfrak{B} \subseteq \mathfrak{A}$ is **uniformly μ-dense** in $\mathfrak{A}$, if for every $a \in \mathfrak{A}$ and $\epsilon > 0$ there is $b \in \mathfrak{B}$ such that $b \leq a$ and $\mu(a \setminus b) < \epsilon$.

Definition 1.2.3.54 (Separable Measure). A measure μ on a compact space X is **separable** if there is a countable μ-dense family $\mathfrak{B}$ contained in the measure algebra of μ.

Definition 1.2.3.55 (Separable Charge). A charge μ on a Boolean algebra $\mathfrak{A}$ is **separable** if there is a countable μ-dense family $\mathfrak{B}$ contained in $\mathfrak{A}$.

Definition 1.2.3.56. [10] A charge μ on a Boolean algebra $\mathfrak{A}$ is **uniformly regular** if there is a countable uniformly μ-dense family $\mathfrak{B}$ contained in $\mathfrak{A}$.

Definition 1.2.3.57 (Isomorphism & Embedding Between Measure Algebras). Let $(\mathfrak{A}, \mu)$ and $(\mathfrak{B}, \nu)$ be two charge algebras (or measure algebras). A map $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is called **isomorphism** if it is a bijection homomorphism such that $\nu(\varphi(A)) = \mu(A)$ for all $A \in \mathfrak{A}$. If φ is a one-to-one measure preserving homomorphism, then it is called **embedding**. We say $(\mathfrak{A}, \mu)$ is (metrically) isomorphic to $(\mathfrak{B}, \nu)$ (resp. $(\mathfrak{A}, \mu)$ embeddable into $(\mathfrak{B}, \nu)$) if there is an isomorphism (resp. embedding) from $(\mathfrak{A}, \mu)$ into $(\mathfrak{B}, \nu)$ and denoted by $\mathfrak{A} \cong \mathfrak{B}$ (resp. $\mathfrak{A} \sqsubseteq \mathfrak{B}$).

Maharam Theorem is a nice classification in measure theory. It is the result on decomposability of measure spaces. Also, it plays an important role in classical Banach space theory.

Theorem 1.2.3.58 (Maharam Theorem). Let (X, Σ, μ) be a finite measure space. Then X is a countable union of measurable subspaces on which the measure algebras induced by μ are either finite or isomorphic to the measure algebra of the product measure λ on $\{0, 1\}^\kappa$ for some infinite cardinal κ.

Chapter 2

Some Properties of Charges

In this chapter we discuss some properties of charges and give a characterization of charge algebras. In Section 2.1 we collect some known results on nonatomic, continuous and Darboux charges, and explain them further. We give different proofs to some results, for instance, Lemma 2.1.11 and Theorem 2.1.22. We also prove Lemma 2.1.7, Lemma 2.1.15 and Theorem 2.1.18, and establish Examples 2.1.19, 2.1.21 and 2.1.23. In Section 2.2 we explain the relation between a charge on a Boolean Algebra $\mathfrak{B}$ and its induced measure on the Stone space of $\mathfrak{B}$. In Section 2.3 we prove Theorem 2.3.1 which shows that for any charge algebra, there is a compact zero-dimensional space such that its charge algebra is isomorphic to it.

2.1 Nonatomicity, Continuity & Darboux Property of Charges

This section is devoted to discussion of nonatomic, continuous and Darboux charges, and their relationships.

Definition 2.1.1. An **atom** is a minimal non-zero element in a Boolean algebra. That is, an element a is said to be an **atom** in a Boolean algebra $\mathfrak{A}$ if for any $b \in \mathfrak{A}$ with $b < a$, either $b = a$ or $b = 0$.

Definition 2.1.2. A Boolean algebra $\mathfrak{A}$ is called **atomless** if it has no atoms. i.e, for any $0 \neq a \in \mathfrak{A}$, there is $b \in \mathfrak{A}$ such that $0 < b < a$.

Definition 2.1.3. Let $\mathfrak{A}$ be a Boolean algebra that carries a charge (or a measure) μ. An element a with $\mu(a) > 0$ is said to be a **μ-atom** in $\mathfrak{A}$ if for any $b \in \mathfrak{A}$ with $b < a$, either $\mu(b) = \mu(a)$ or $\mu(b) = 0$.

Definition 2.1.4. A charge (or a measure) μ on a Boolean algebra $\mathfrak{A}$ is called **nonatomic** if it has no μ-atoms. i.e, for any $a \in \mathfrak{A}$ with $\mu(a) > 0$, there is $b \in \mathfrak{A}$ such that $b < a$ and $0 < \mu(b) < \mu(a)$.

Lemma 2.1.5. Let μ be a charge (or a measure) on a Boolean algebra $\mathfrak{A}$. Every atom a with $\mu(a) > 0$ is a μ-atom.

Proof. Directly from the definitions. ■

In general, a μ-atom need not be an atom, as shown in the following example:

Example 2.1.6. Let $\mathcal{A} = \{A : A \subseteq \mathbb{R}, A \text{ is countable or } A^c \text{ is countable}\}$. Let μ on $\mathcal{A}$ be defined by

$$\mu(A) = \begin{cases} 0, & \text{if } A \text{ is countable,} \\ 1, & \text{if } A^c \text{ is countable.} \end{cases}$$

If $A = \mathbb{R} \setminus \mathbb{N}$, then A is a μ-atom but not an atom because for example with $B = \mathbb{R} \setminus \mathbb{Q}$ we have $\emptyset \subseteq B \subseteq A$. Actually all singletons are atoms.

Lemma 2.1.7. Let μ be a charge on a Boolean algebra $\mathfrak{A}$ and $\mathcal{I}$ the ideal of all null sets in $\mathfrak{A}$. Then μ is nonatomic on $\mathfrak{A}$ if and only if $\mathfrak{A}/\mathcal{I}$ is atomless.

Proof. Suppose that $\mathfrak{A}/\mathcal{I}$ is atomless. Given an element $a \in \mathfrak{A}$ with $\mu(a) > 0$, this means that $a \notin \mathcal{I}$ and hence $0 \neq a \in \mathfrak{A}/\mathcal{I}$. By the assumption, there is $b \in \mathfrak{A}/\mathcal{I}$ such that $0 < b < a$. Thus, by monotonicity of μ, we have $0 < \mu(b) < \mu(a)$. Therefore, μ is nonatomic.
Conversely, suppose that μ is nonatomic on $\mathfrak{A}$. Let $0 \neq a \in \mathfrak{A}/\mathcal{I}$. Then $a \notin \mathcal{I}$ implies that $\mu(a) > 0$. By the assumption, there is $0 \neq b \in \mathfrak{A}$ such that $b < a$ and $0 < \mu(b) < \mu(a)$. This implies that $b \in \mathfrak{A}/\mathcal{I}$. Therefore, $\mathfrak{A}/\mathcal{I}$ is atomless. ■

From the above lemma, we remark the following:

Remark 2.1.8. Let μ be a strictly positive charge on a Boolean algebra $\mathfrak{A}$. Then μ is nonatomic on $\mathfrak{A}$ if and only if $\mathfrak{A}$ is atomless.

Here we recall two stronger notions of nonatomicity with respect to a measure or a charge on a Boolean algebra, and then show their relations.

Definition 2.1.9. A charge (or a measure) μ on a Boolean algebra $\mathfrak{A}$ is called **continuous** [48] if for any $\epsilon > 0$, there is a finite partition $\mathscr{P} = \{a_1, a_2, \ldots, a_n\}$ of the unity $\mathbf{1}$ such that $\mu(a_i) < \epsilon$ for $i = 1, 2, \ldots, n$. This type of charge is also called strongly continuous [43] or atomless [10].

Definition 2.1.10. A charge (or a measure) μ on a Boolean algebra $\mathfrak{A}$ is said to have the **Darboux property** [39] if for any $a \in \mathfrak{A}$ and $0 < t < \mu(a)$, there is $b \in \mathfrak{A}$ such that $b < a$ and $\mu(b) = t$. This property also has two other names in the literature, which are full-valued [34] and strongly nonatomic charge [43]. For the purpose of simplicity, we call a charge μ **Darboux** if it has the Darboux property.

Lemma 2.1.11. [43] For a (finite) charge μ on a Boolean algebra $\mathfrak{A}$, each of the following conditions implies the succeeding condition:

(1) μ is Darboux.

(2) μ is continuous.

(3) μ is nonatomic.

Proof. (1) $\implies$ (2) Given $\epsilon > 0$, take an n large enough such that $1/n < \epsilon$. Now, if $\mu(\mathbf{1}) \leq 1/n$, then we are done. Otherwise, by (1), choose $a_1 \in \mathfrak{A}$ such that $\mu(a_1) = 1/n$, then choose $a_2 < \mathbf{1} \setminus a_1$ with $\mu(a_2) = 1/n$ and so on. After finitely many steps this should stop because $\mu(\mathbf{1}) < \infty$, and thus we find our partition $\mathscr{P} = \{a_1, a_2, \ldots, a_n\}$ of $\mathbf{1}$ with $\mu(a_i) < \epsilon$ for each $1 \leq i \leq n$. This completes the proof.
(2) $\implies$ (3) Let $a \in \mathfrak{A}$ with $\mu(a) > 0$. To show μ is nonatomic, we have to find $b < a$ such that $0 < \mu(b) < \mu(a)$. Assume, without loss of generality, that $\mu(a) > \epsilon$ for a fixed $\epsilon > 0$. By (2) the unity $\mathbf{1}$ can be partitioned as $\mathbf{1} = a_1 \vee a_2 \vee \cdots \vee a_n$ such that

$0 < \mu(a_i) < \epsilon$. Then, $b = \mathbf{1} \wedge b = (b \wedge a_1) \vee (b \wedge a_2) \vee \cdots \vee (b \wedge a_n)$ for $i = 1, 2, \ldots, n$, and hence

$$0 < \mu(b) = \sum_{i=1}^{n} \mu(b \wedge a_i).$$

Therefore, there must be an i_0 with $1 \le i_0 \le n$ for which $\mu(b \wedge a_{i_0}) > 0$. But $b \wedge a_{i_0} \le a_{i_0}$, so $0 < \mu(b \wedge a_{i_0}) \le \mu(a_{i_0}) < \epsilon < \mu(a)$. Setting $b = b \wedge a_{i_0}$ we are done.

■

Note that all the above conditions are pairwise equivalent when μ is a measure and $\mathfrak{A}$ is a σ-algebra, (see Theorem 5.1.6, [43]).

The non implications of (3) $\implies$ (2) and (2) $\implies$ (1) are shown in the following examples:

Example 2.1.12. [43] Let $\mathbb{I}$ be the unit interval $[0, 1]$. Consider the Lebesgue measure λ_0 restricted to the algebra $\mathcal{A}$ on $\mathbb{I}$ generated by $(a, b] \subseteq [1/4, 3/4)$. λ_0 is nonatomic (because Lebesgue measure is nonatomic) but not continuous on $\mathcal{A}$, as we shall now show. For $\epsilon = 1/2$, suppose there is a partition $\{A_1, A_2, ..., A_n\}$ of $\mathbb{I}$ that satisfies the required properties, that is

$$\mathbb{I} = A_1 \vee A_2 \vee \cdots \vee A_n \text{ such that } \lambda_0(A_i) < 1/2.$$

There must be an i with $1 \le i \le n$ such that $A_i \bigcap \big([0, 1/4) \cup [3/4, 1]\big) \ne \emptyset$, otherwise, our partition will not cover the unity. Setting $B_i = A_i \bigcap \big([0, 1/4) \cup [3/4, 1]\big)$, we have $B_i \ne \emptyset$ and $B_i \subseteq [0, 1/4) \cup [3/4, 1]$. Thus $B_i = [0, 1/4) \cup [3/4, 1]$ because if $\emptyset \ne A \in \mathcal{A}$ and $A \subseteq [0, 1/4) \cup [3/4, 1]$, then $A^c \supseteq [1/4, 3/4)$ and $A^c \in \mathcal{A}$. But $\mathcal{A}$ is the algebra generated by subintervals of $[1/4, 3/4)$, so $A^c = [1/4, 3/4)$ implies that $A = [0, 1/4) \cup [3/4, 1]$. Therefore $A_i \supseteq [0, 1/4) \cup [3/4, 1]$, so $\lambda_0(A_i) \ge \lambda_0\big(0, 1/4) \cup [3/4, 1]\big) = 1/2$. Contradiction!

Example 2.1.13. [43] Let $\mathbb{I} = [0, 1)$ and let $\mathcal{B}$ be the algebra generated by the collection $\{[a, b) : 0 \le a \le b \le 1, a, b \in \mathbb{Q} \cap \mathbb{I}\}$ of subsets of $\mathbb{I}$. Consider the Lebesgue measure λ_1 restricted to $\mathcal{B}$. λ_1 is continuous but not Darboux as we now explain. Choose $n < \omega$ such that $1/2^n < \epsilon < 1/n$. Since $1/2^n$ is a rational number, there are always some sets of λ_1-charge $1/2^n$. Divide $\mathbb{I}$ into n disjoint sets of λ_1-charge $1/2^n$, surely these n sets will be a partition of $\mathbb{I}$ and n cannot be infinite. Otherwise $\epsilon = 0$ which contradicts the assumption that $\epsilon > 0$. Thus, λ_1 is continuous. But λ_1 is not Darboux, because $\mathcal{B}$ is the collection of subsets of X of the form $A = \bigcup_{i=1}^{n}[a_i, b_i)$, so $\lambda_1(A) = \lambda_1(\bigcup_{i=1}^{n}[a_i, b_i)) = \sum_{i=1}^{n}\lambda_1(b_i - a_i)$, so for a given $A \in \mathcal{B}$ and $0 < \alpha < \lambda_1(A)$, there is no set B such that $B \subseteq A$ and $\lambda_1(B) = \alpha$ where α is irrational.

Remark 2.1.14. In this remark, we present the results when replacing algebra by a σ-algebra or a countable algebra in Lemma 2.1.11:

(i) Given a σ-algebra $\mathcal{A}$ with a charge μ. Then μ is continuous if and only if it is Darboux, (the proof is given in [43] and [45]).

(ii) A nonatomic charge μ on a σ-algebra need not be continuous. Consider the example given in ([43], page 143). Let $\mathbb{I} = [0, 1]$, $\mathcal{B}$ the Borel σ-algebra of sets in $\mathbb{I}$ and λ the Lebesgue measure on $\mathbb{I}$. Suppose that ν is a $0-1$ valued charge on $\mathbb{I}$ such that $\nu(A) = 0$ if $\lambda(A) = 0$, otherwise $\nu(A) = 1$. Then $\mu = \lambda + 2\nu$ defines a charge on $\mathcal{B}$. We now show that μ is nonatomic but not continuous. Take $A \in \mathcal{B}$ with $\mu(A) > 0$ which implies that $\lambda(A) > 0$, (because if $\lambda(A) = 0$, then $\nu(A) = 0$ and so $\mu(A) = 0$). But λ is nonatomic, so there exists $B \subseteq A$ such that $0 < \lambda(B) < \lambda(A)$. Since $\nu(B) \leq \nu(A)$, then $0 < \mu(B) = \lambda(B) + 2\nu(B) < \lambda(A) + 2\nu(A) = \mu(A)$. Thus, μ is nonatomic.

Since ν takes values only either 0 or 1, given any $A \in \mathcal{B}$, either $\mu(A) = \lambda(A) \leq 1$ or $\mu(A) = \lambda(A) + 2 \geq 2$ and obviously, $\mu(\emptyset) = 0$ and $\mu(\mathbb{I}) = 3$. So μ does not take any value in the interval $[1, 2]$, for if there is $A \in \mathcal{B}$ such that $\mu(A) = 2$, then $\lambda(A) = 0$ and $\nu(A) = 1$, but this is a contradiction, if $\lambda(A) = 0$, then $\nu(A) = 0$ by definition of μ. Thus μ is not Darboux and by the result in (i) it is not continuous, (*cf.* Theorem 1, [39]).

(iii) A nonatomic charge μ need not be continuous even on a countable Boolean algebra $\mathcal{A}$. Let $\mathbb{I} = [0, 1]$ and λ the Lebesgue measure. Consider the restricted Lebesgue measure λ_0 to the algebra $\mathcal{A}$ generated by intervals with rational endpoints in S, where S is the family of open intervals $I_n, n \in \omega$, that are the complements of the removed intervals in the Cantor set, such that $I_i \cap I_j = \emptyset$ for every $i \neq j$ and $\lambda_0(S = \bigcup_{n=1}^{\infty} I_n) = 1$. Notice that $0 \leq \lambda_0(A) < 1$ for each $A \in \mathcal{A}$ (no A can take $\lambda_0(A) = 1$ because A is a finite union of intervals in S and cannot be equal to S itself). Let $\mathcal{B}$ be the algebra generated by the family of the complements of all $A \in \mathcal{A}$ with respect to $\mathbb{I}$. That is, each $B \in \mathcal{B}$ is $B = \mathbb{I} \setminus A$. Define a charge μ on $\mathcal{B}$ to be $\mu(B) = 2 - \lambda_0(A)$ for each B and so it takes the values $1 < \mu(B) \leq 2$. Now, let $\mathcal{C}$ be the algebra generated by $\mathcal{A} \cup \mathcal{B}$ and let us define $\nu = \mu + \lambda_0$ on $\mathcal{C}$. Clearly $\mathcal{C}$ is countable and ν is composed from the Lebesgue measure, so it is nonatomic. On the other hand, $\mathbb{I}$ cannot be partitioned into a finite disjoint union of sets with charge < 1 because the members in $\mathcal{A}$ can only posses charges in the interval $(0, 1)$ and $\mathcal{A}$ does not cover the whole $\mathbb{I}$. Thus, ν is not continuous.

(iv) Obviously, no charge on a countable Boolean algebra can be Darboux, because otherwise it will take uncountably many values.

Lemma 2.1.15. Let μ be a probability charge on the Cantor algebra $\mathfrak{A}_c$. Then μ is continuous if and only if it is nonatomic.

Proof. By Lemma 2.1.11, continuity implies nonatomicity.
Conversely, assume that μ is nonatomic. Given $\epsilon > 0$ such that $1/2^m < \epsilon$ for some $m < \omega$. Consider the family $\mathcal{S} = \{[s] : s \text{ is a sequence of 0s and 1s of length } m\}$. Equivalently, each s can be considered as a partial function from ω to $2 = \{0, 1\}$ whose domain $|\text{dom}(s)| = m$. Each $[s] \in \mathcal{S}$ has the charge $\mu([s]) = 1/2^m$ (see § 1.2.3). But every $[s] \in \mathcal{S}$ is open in the space 2^ω. So, by compactness, $2^\omega = [s_1] \cup [s_2] \cup ... \cup [s_n]$ for some $n \in \omega$. Let

$$\begin{aligned}
a_1 &= [s_1],\\
a_2 &= [s_2] \setminus [s_1],\\
&\vdots\\
a_n &= [s_n] \setminus \big([s_1] \cup [s_2] \cup ... \cup [s_{n-1}]\big).
\end{aligned}$$

So, $2^\omega = [s_1] \cup [s_2] \cup ... \cup [s_n] = a_1 \vee a_2 \vee \cdots \vee a_n$. Thus, $\{a_1, a_2, ..., a_n\}$ is partition of 2^ω such that $\mu(a_i) \leq 1/2^m < \epsilon$. This shows that μ is continuous.

■

Next, we present some results regarding a charge μ on an algebra, say $\mathcal{B}$, and its extension μ (we still denote the extension by μ) over another algebra (not necessarily a σ-algebra) $\mathcal{A}$ containing $\mathcal{B}$.

Theorem 2.1.16. [40] Let $\mathcal{A}, \mathcal{B}$ be two algebras and $\mathcal{B} \subseteq \mathcal{A}$. A charge μ on $\mathcal{A}$ is continuous if and only if it is continuous on $\mathcal{B}$.

Theorem 2.1.17. [40] Let $\mathcal{A}, \mathcal{B}$ be two algebras and $\mathcal{B} \subseteq \mathcal{A}$. A charge μ on $\mathcal{A}$ is nonatomic if and only if it is nonatomic on $\mathcal{B}$.

Theorem 2.1.18. Let $\mathcal{A}, \mathcal{B}$ be two algebras and $\mathcal{B} \subseteq \mathcal{A}$. A (probability) charge μ on $\mathcal{A}$ is Darboux if it is Darboux on $\mathcal{B}$.

Proof. By assumption $\mathcal{B} \subseteq \mathcal{A}$. Then the range $\text{Rng}_{\mathcal{B}}(\mu)$ of μ on $\mathcal{B}$ is also a subset of the range $\text{Rng}_{\mathcal{A}}(\mu)$ of μ on $\mathcal{A}$. That is, $\text{Rng}_{\mathcal{B}}(\mu) \subseteq \text{Rng}_{\mathcal{A}}(\mu)$. Since μ is Darboux on $\mathcal{B}$, then $\text{Rng}_{\mathcal{B}}(\mu) = [0, 1]$. But $\text{Rng}_{\mathcal{B}}(\mu) = [0, 1] \subseteq \text{Rng}_{\mathcal{A}}(\mu)$. Hence $\text{Rng}_{\mathcal{A}}(\mu) = [0, 1]$ and so μ is Darboux on $\mathcal{A}$. ■

The converse of the above theorem need not be true in general, as shown in the following example:

Example 2.1.19. Given the unit interval $\mathbb{I}$, and let $\mathcal{A}$ be the algebra generated by the family $\{[a, b) : 0 \leq a < b \leq 1, a, b \in \mathbb{I}\}$ of subsets of $\mathbb{I}$ and $\mathcal{B}$ be the algebra generated by the family $\{[a, b) : 0 \leq a < b \leq 1, a, b \in \mathbb{Q} \cap \mathbb{I}\}$ of subsets of $\mathbb{I}$. Consider the Lebesgue measure λ_0 restricted to $\mathcal{A}$. Let $\lambda_1 = \lambda_0|_{\mathcal{B}}$. By Remark 2.1.14 (iv), λ_1 is not Darboux. It remains to show that λ_0 is Darboux. This is obvious because for any set $A \in \mathcal{A}$ with $\lambda_1(A) > 0$ and any $0 < t < \lambda_1(A)$, there is always $[0, t)$ that satisfies the required property.

Here are some results concerning a measure on algebras and its extension to σ-algebras generated by the algebras.

Theorem 2.1.20. If a measure μ on an algebra of sets $\mathcal{A}$ is Darboux, then its extension to $\hat{\mu}$ on the σ-algebra $\mathcal{B}$ generated by $\mathcal{A}$ is Darboux.

Proof. Given $B \in \mathcal{B}$ with $\hat{\mu}(B) > 0$, we need to show that for every $0 < t < \hat{\mu}(B)$, there exists B^* such that $B^* \subseteq B$ and $\hat{\mu}(B^*) = t$. Since every member of $\mathcal{B}$ can be expressed as a countable union of members of $\mathcal{A}$, so $B = \bigcup_{i=1}^{\infty} A_i$ where $A_1, A_2, \ldots A_n, \ldots$ and thus $0 < t < \hat{\mu}(\bigcup_{i=1}^{\infty} A_i)$. Suppose that n is large enough such that $t \le \hat{\mu}(\bigcup_{i=1}^{n} A_i)$. If $\hat{\mu}(\bigcup_{i=1}^{n} A_i) = t$, we are done. If not, since $\mu(\bigcup_{i=1}^{n} A_i) = \hat{\mu}(\bigcup_{i=1}^{n} A_i)$ and μ is Darboux, for every $0 < t < \mu(\bigcup_{i=1}^{n} A_i) = \hat{\mu}(\bigcup_{i=1}^{n} A_i)$, there exists $A \in \mathcal{A}$ and consequently $A \in \mathcal{B}$ such that $\hat{\mu}(A) = t$. This completes the proof. ■

The converse of the above theorem is not true in general.

Example 2.1.21. In Example 2.1.13 it is shown that the restricted Lebesgue measure λ_0 to the algebra $\mathcal{A}$ generated by intervals $[a, b) \cap (\mathbb{Q} \cap [0, 1])$ is not Darboux. Moreover the σ-algebra $\mathcal{B}$ generated by $\mathcal{A}$ is the Borel algebra and the extension of λ_0 is the full Lebesgue measure λ. It is known that λ is nonatomic and thus Darboux.

Theorem 2.1.22. [6] Let $\mathcal{A}$ be an algebra of subsets of a set X and $\mathcal{B} = \sigma(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$. If a measure μ on $\mathcal{B}$ is nonatomic, then its restriction to (a premeasure) μ_0 on $\mathcal{A}$ is also nonatomic.
Note that this result is mentioned in [6], but there is no explicit proof. We give a detailed proof.

Proof. Given an algebra $\mathcal{A}$ and $\mathcal{B} = \sigma(\mathcal{A})$. Suppose for contradiction that μ_0 is atomic. Let A' be an atom. By Definition of atom, $\mu_0(A') > 0$ and for each $B \subseteq A'$ ($B \in \mathcal{A}$), either $\mu_0(B) = \mu_0(A')$ or $\mu_0(B) = 0$. But μ is an extension of μ_0, so

$$\mu(A) = \mu_0(A) \text{ for all } A \in \mathcal{A}.$$

Since μ is nonatomic, by assumption, there is $C \in \mathcal{B}$ such that

$$C \subseteq A' \text{ and } 0 < \mu(C) < \mu(A').$$

By Lemma 1.2.3.14, for $\epsilon = (\mu(A') - \mu(C))/2$, there is a sequence $\langle A_n : n \in \omega \rangle$ in $\mathcal{A}$ such that

$$C \subset \bigcup_{n=1}^{\infty} A_n \text{ and } \sum_{n=1}^{\infty} \mu_0(A_n) = \sum_{n=1}^{\infty} \mu(A_n) < \mu(C) + \epsilon.$$

Since $\bigcup\limits_{n=1}^{\infty} A_n \cap A' \in \mathcal{A}$ and μ_0 is a premeasure, then

$$\mu_0(\bigcup_{n=1}^{\infty} A_n \cap A') = \mu(\bigcup_{n=1}^{\infty} A_n \cap A') \leq \mu(\bigcup_{n=1}^{\infty} A_n) < \mu(C) + \epsilon < \mu(A') = \mu_0(A').$$

On the other hand, $\mu_0(\bigcup\limits_{n=1}^{\infty} A_n \cap A') \geq \mu(C) > 0$.

Therefore, now we have $0 < \mu_0(\bigcup\limits_{n=1}^{\infty} A_n \cap A') < \mu_0(A')$, which contradicts the assumption that A' is μ_0-atom. Hence, μ_0 is nonatomic. ■

The converse of the above theorem need not be true in general, as shown in the following example:

Example 2.1.23. Let $X = \mathbb{Q} \cap \mathbb{I}$. The power set of X, $\mathcal{P}(X) = \{A : A \subseteq X\}$, forms a σ-algebra of subsets of X. As X is countable, we can enumerate it like $X = \{x_1, x_2, x_3, ...\}$. We define a generalization of the weighted measure given by Maharam (in [34], page 1) on the natural numbers. That is, for each n we choose a non-negative real number r_n such that $\sum_n r_n \leq 1$, and define $\mu(A) = \sum\{r_n : x_n \in A\}$ for each $A \subseteq X$. With this μ every singleton is an atom. Let $\mathcal{A}$ be the algebra generated by $\{[a, b) \cap \mathbb{Q} : a, b \in \mathbb{I}\}$. If μ_0 is the restriction of μ to the algebra $\mathcal{A}$, then μ_0 is nonatomic because $\mathcal{A}$ does not contain singletons which are the only minimal positive elements, and for each $A = \bigcup\limits_{i=1}^{n} [a_i, b_i)$, one can find a subset $B = \bigcup\limits_{i=1}^{n} [c_i, d_i)$ such that $0 \leq a_i < c_i < d_i < b_i \leq 1$ for $i = 1, 2, ..., n$ and thus $B \subseteq A$ and $0 < \mu_0(B) < \mu_0(A)$. Finally, we need to show that $\mathcal{P}(X)$ is exactly the σ-algebra generated by $\mathcal{A}$. But $\mathcal{P}(X)$ is certainly generated by (a, b) and $[c, d]$, so it is enough to see that every (a, b) and $[c, d]$ belong to $\sigma(\mathcal{A})$. Now, observe that

$$(a, b) = \bigcup_{n<\omega} \left(a, b - \frac{1}{n}\right] \quad \text{and} \quad [c, d] = \bigcap_{n<\omega} \left[c, d + \frac{1}{n}\right).$$

Thus, $(a, b), [c, d] \in \sigma(\mathcal{A})$. Surely, $\sigma(\mathcal{A})$ contains singletons which are atoms. This completes the proof.

Another counterexample can be seen in Remark 3.2 [6].

Theorem 2.1.24. [44] A measure μ on a σ-algebra $\mathcal{B}$ is continuous if and only if it is continuous on the algebra $\mathcal{A}$ which generates $\mathcal{B}$.

Theorem 2.1.25. [40] A charge μ on a σ-algebra $\mathcal{A}$ is Darboux if and only if its restriction μ_0 to a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is Darboux.

2.2 Relation Between Charges on Boolean Algebras and Measures on Their Stone Spaces

In earlier sections of this thesis, we said that any charge μ on a Boolean algebra $\mathfrak{A}$ can be extended to a Radon measure $\hat{\mu}$ on the Stone space Z of $\mathfrak{A}$, in a natural way. We shall explain this in the following remark:

Remark 2.2.1. Let μ be a charge on a Boolean algebra $\mathfrak{A}$ and Z the Stone space of $\mathfrak{A}$. If $\mathcal{C} = \text{Clop}(Z)$ is the algebra of clopen subsets of Z, then the charge μ on $\mathfrak{A}$ can be transferred to a charge $\hat{\mu}_1$ on $\mathcal{C}$ by the Stone isomorphism $\varphi : \mathfrak{A} \longrightarrow \mathcal{C}$, that is, $\hat{\mu}_1(\varphi(a)) = \hat{\mu}_1(\hat{a}) = \mu(a)$ for all $a \in \mathfrak{A}$. Since no infinite (countable) union of clopen sets is clopen in Z, then the charge $\hat{\mu}_1$ is also a premeasure on $\mathcal{C}$ and it can be extended to a measure $\hat{\mu}_0$ on the Baire σ-algebra $\mathcal{B}_0(Z)$ generated by $\mathcal{C}$ by Lemma 1.2.3.14. This measure is called the Baire measure. Again, since Z is a compact Hausdorff space, by Theorem 1.2.3.32, $\hat{\mu}_0$ can also be extended uniquely to a measure $\hat{\mu}$ on $\mathcal{B}(Z)$, the Borel σ-algebra generated by the collection of open sets in Z. Thus, $\hat{\mu}$ is a Borel measure on $\mathcal{B}(Z)$. By Theorem 1.2.3.33, this extension $\hat{\mu}$ is a Radon measure and both $\hat{\mu}_0$ and $\hat{\mu}$ agree on $\mathcal{B}_0(Z)$.

We see that it is important to answer the following inquiries: (i) How is it possible that μ is a charge and $\hat{\mu}_1$ is a premeasure, and yet both are isomorphic? This seems contradiction. Besides, it is not true in general that one can extend a charge (not premeasure) to a measure, (see Remark 1.2.3.15).
(ii) What about a charge on a complete Boolean algebra?

We now explain the above issues. Regarding (i), yes $\hat{\mu}_1$ is a charge but also a premeasure because every $C \in \mathcal{C}$ is clopen and so is compact. Therefore, for any open cover, we can find a finite subcover that covers C (choose cover of clopen sets). So no countably infinite union of clopen sets is clopen in Z (as stated above). Hence, $\hat{\mu}_1$ is automatically a premeasure.

Regarding (ii), we shall warn that the Stone isomorphism does not always preserve countably infinite unions. Let us look at how a charge or measure on a Boolean algebra transfers to a measure on its Stone space.

Example 2.2.2. Consider the set of natural numbers $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$ is a σ-complete Boolean algebra. Define a set function μ on $\mathcal{P}(\mathbb{N})$ as follows:

$$\mu(A) = \begin{cases} 0, & \text{if } A \text{ is finite,} \\ \infty, & \text{if } A \text{ is infinite,} \end{cases}$$

where $A \in \mathcal{P}(\mathbb{N})$. We show that μ is a charge but not measure. Simply take $A, B \in \mathcal{P}(\mathbb{N})$. If both A and B are finite, then $A \cup B$ is finite and so $\mu(A \cup B) = 0 = \mu(A) + \mu(B)$. When both A and B are infinite or one of them is infinite, the conclusion can be obtained similarly. On the other hand, we have $\mathbb{N} = \bigcup\limits_{n\in\mathbb{N}} \{n\}$. But $\mu(\mathbb{N}) = \mu(\bigcup\limits_{n\in\mathbb{N}} \{n\}) = \infty \neq 0 = \sum\limits_{n\in\mathbb{N}} \mu(\{n\})$. Thus, μ is not a measure.

Let Z be the Stone space of $\mathcal{P}(\mathbb{N})$, or equivalently, the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ and ν the induced measure on $\beta\mathbb{N}$. Let $\{a_n : \text{ the numbers } < n\}$ be a sequence in $\mathcal{P}(\mathbb{N})$, since $\mathcal{P}(\mathbb{N})$ is σ-complete, so there is $a \in \mathcal{P}(\mathbb{N})$ such that $a = \bigvee\limits_{n\in\mathbb{N}} a_n$. That a is $\mathbb{N}$ itself. Consider $\langle \hat{a}_n : n < \omega \rangle$, so we have $\beta\mathbb{N} = \widehat{\mathbb{N}} \neq \bigcup\limits_n \widehat{a}_n$ because $\bigcup\limits_n \widehat{a}_n$ can be seen as the set of all principal ultrafilters on a_n, which is not the whole $\beta\mathbb{N}$, and the measure $\nu(\widehat{a}_n) = \mu(a_n) = 0$ for all n. Therefore, their sum is also zero, while $\nu(\beta\mathbb{N}) = \infty$. On the other hand, $\bigcup\limits_n \widehat{a}_n \subseteq \beta\mathbb{N}$ (or rather, $Y = \bigcup\limits_n \widehat{a}_n$ is a subspace of $\beta\mathbb{N}$) and so $\nu(\beta\mathbb{N}) \geq \nu(\bigcup\limits_n \widehat{a}_n) = \sum\limits_n \nu(\widehat{a}_n) = \sum\limits_n \mu(a_n)$. If μ happens to be a measure on $\mathcal{P}(\mathbb{N})$, then we would have the following equalities:

$$\nu(\beta\mathbb{N}) = \nu(\bigcup_n \widehat{a}_n) = \mu(\bigcup_n a_n) = \sum_n \mu(a_n) = \sum_n \nu(\widehat{a_n}) = \nu(Y).$$

Lemma 2.2.3. [47, 20] Let Z be the Stone space of an algebra $\mathcal{A}$, and $\widehat{A} \subseteq Z$ the clopen set corresponding to $A \in \mathcal{A}$. We have the following:

(1) There is a one-to-one correspondence between ideals $\mathcal{I}$ of $\mathcal{A}$ and open sets $G \subseteq Z$, given by the formula

$$G = \bigcup_{A\in\mathcal{I}} \widehat{A} \text{ and } \mathcal{I} = \{A : \widehat{A} \subseteq G\}.$$

(2) There is a one-to-one correspondence between filters $\mathcal{F}$ of $\mathcal{A}$ and closed sets $E \subseteq Z$, given by the formula

$$E = \bigcap_{A \in \mathcal{F}} \widehat{A} \text{ and } \mathcal{F} = \{A : E \subseteq \widehat{A}\}.$$

Fact 2.2.4. Let $\mathcal{A}$ be an algebra of subsets of a space X, and let E be any subset of X. The collection of all sets of the form $E \cap A$, where $A \in \mathcal{A}$, forms an algebra and is denoted by $\mathcal{A}_E$. Suppose that $\mathcal{I}$ is the principal ideal generated by E^c, i.e. $\mathcal{I} = \{A \in \mathcal{A} : A \cap E = \emptyset\}$. Then,

$$\mathcal{A}_E \text{ is isomorphic to } \mathcal{A}/\mathcal{I} : \text{via } E \cap A \mapsto [A].$$

Remark 2.2.5. [47] Sikorski remarked that one can determine the Stone space of $\mathcal{A}/\mathcal{I}$ by constructing the Stone space of $\mathcal{A}$. This shows that Stone spaces of quotient algebras $\mathcal{A}/\mathcal{I}$ are (up to homeomorphism) closed subspaces of the Stone space of $\mathcal{A}$, and conversely.

Lemma 2.2.6. Let μ, ν be two charges on Boolean algebras $\mathfrak{A}, \mathfrak{B}$ and Z, Y their Stone spaces, respectively. Then, $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is an isomorphism such that $\mu(\varphi^{-1}(b)) = \nu(b)$ if and only if $f : Y \longrightarrow Z$ is homeomorphism with $f(\hat{\nu}) = \hat{\mu}$, where $\hat{\mu}, \hat{\nu}$ are extensions of μ, ν, respectively.

Proof. This lemma has a routine proof by applying Lemma 1.2.2.33. ∎

Proposition 2.2.7. Let μ be a charge (premeasure) on a Boolean algebra $\mathfrak{A}$ and $\mathcal{N} = \{b : b \in \mathfrak{A}, \mu(b) = 0\}$. If $Z = \text{Stone}(\mathfrak{A})$ and $Y = \text{Stone}(\mathfrak{A}/\mathcal{N})$, then Y can be identified with the support of $\hat{\mu}$, where $\hat{\mu}$ in the induced Radon measure on Z.

Proof. From the definition of the support of a measure, it is enough to show that Y is homeomorphic to a closed subset of Z in which every nonempty open set is of positive measure. Let

$$G = \bigcup_{b \in \mathcal{N}} \hat{b}.$$

By Lemma 2.2.3, G is open in Z. Let $W = Z \setminus G$. By Remark 2.2.5, there is an isomorphism φ from $\widehat{\mathfrak{A}/\mathcal{N}} = \widehat{[\mathfrak{A}]}$ (the clopen algebra of Y) to $\widehat{\mathfrak{A}}_W$ (the clopen algebra of W), (W is zero-dimensional as it is a subspace of the zero-dimensional space Z). By Lemma 1.2.2.33, there is a homeomorphism f between Y and W such that $\varphi = \widehat{f^{-1}}$ (this means that $\varphi(a) = f^{-1}(\widehat{a}) \quad \forall a \in \widehat{[\mathfrak{A}]}$). We claim that f is also measure preserving. By Lemma 2.2.6, it is enough to show that φ is measure preserving. Let ν be the induced measure on $\mathfrak{A}/\mathcal{N}$. Then,

$$\nu([a]) = \mu(a) \text{ for all } a \in \mathfrak{A}. \tag{i}$$

By Remark 2.2.1, ν can be extended to a Radon measure $\hat{\nu}$ on Y and so

$$\hat{\nu}(\widehat{[a]}) = \nu([a]) \text{ for all } [a] \in \mathfrak{A}/\mathcal{N}. \tag{ii}$$

Since $\hat{\mu}$ is the induced measure on Z (from μ), by Stone isomorphism,

$$\hat{\mu}(\hat{a}) = \mu(a) \text{ for all } a \in \mathfrak{A}. \tag{iii}$$

On the other hand, if $\hat{\mu}_W$ is the restriction of $\hat{\mu}$ to W, then $\hat{\mu}(A) = \hat{\mu}_W(A)$ for all $A \subseteq W$. Therefore,

$$\hat{\mu}(\hat{a}) = \hat{\mu}(\widehat{a} \cap W) = \hat{\mu}_W(\hat{a}) \text{ for all } \hat{a} \in \widehat{\mathfrak{A}}_W. \tag{iv}$$

From (i) to (iv), we conclude that φ preserves measure, i.e. $\hat{\nu}(\widehat{[a]}) = \hat{\mu}(\hat{a} \cap W) = \hat{\mu}_W(\hat{a}_W)$. Thus, f is preserving measure. Since $\hat{\nu}$ is strictly positive on Y (as it is the induced measure from a strictly positive charge ν on $\mathfrak{A}/\mathcal{N}$), then the image measure $\hat{\mu}_W$ on W under a homeomorphism measure preserving has to be strictly positive. Hence, $W = \text{supp}(\hat{\mu})$. This completes the proof. ■

2.3 Representation of 0-Dimensional Spaces with Charge by Charge Algebras

In this section we prove a possible generalization of the Stone's representation theorem for charges.

Theorem 2.3.1. Let μ a charge on a Boolean algebra $\mathfrak{A}'$. Then there is a compact zero-dimensional space Z and a charge ν on Z such that the charge algebra of μ is isomorphic to the charge algebra of ν.

Before starting to prove our theorem, we have to demonstrate some results which help us to complete the proof.

Proposition 2.3.2. Let μ a charge on a Boolean algebra $\mathfrak{A}'$. Denote by $\mathfrak{A}$ the quotient Boolean algebra $\mathfrak{A}'/\mathcal{N}$, where $\mathcal{N}$ is the ideal of null sets in $\mathfrak{A}'$. Then

$$\bar{\mu} : \mathfrak{A} \to [0, \infty] \text{ defined by}$$
$$\bar{\mu}([A]) = \mu(A) \text{ for all } A \in \mathfrak{A}'$$

is a function and $(\mathfrak{A}, \bar{\mu})$ is a charge algebra.

Proof. Let us first show that $\bar{\mu}$ is well defined. Let $A, B \in \mathfrak{A}'$ such that $[A] = [B] \in \mathfrak{A}$, i.e both A and B belong to the same class. This means that $A \triangle B \in \mathcal{N}$. So $\mu(A \triangle B)$ $= 0 \implies \mu(A \setminus B \cup B \setminus A) = \mu(A \setminus B) + (B \setminus A) = 0$. This implies that $\mu(A) =$ $\mu(A) + \mu(B \setminus A) = \mu(B) + \mu(A \setminus B) = \mu(B)$.

To prove that $(\mathfrak{A}, \bar{\mu})$ is a charge algebra, we have to show that $\bar{\mu}$ is a strictly positive charge on the algebra $\mathcal{A}$ of chargeable sets in X modulo null sets, that is:

(a) If $a = 0 \in \mathfrak{A}$, $\bar{\mu}(a) = \bar{\mu}(0) = \bar{\mu}([\emptyset]) = \mu(\emptyset) = 0$.

(b) If $a > 0 \in \mathfrak{A}$, there is $A \in \mathcal{A}$ such that $a = [A]$ and $A \notin \mathcal{N}$. So $\bar{\mu}(a) = \mu(A) > 0$.

(c) If $a, b \in \mathfrak{A}$ with $a \wedge b = 0$, we can choose two members $A, B \in \mathcal{A}$ such that $a = [A]$ and $b = [B]$. Note that even if $a \wedge b = 0$, the sets A, B may not be disjoint, but we can make them disjoint as follows. Without loss of generality, let us take $a = [A]$ and $b = [B \setminus A]$. Since $A \cup B = A \cup (B \setminus A)$, we have $\bar{\mu}(a \vee b) = \bar{\mu}([A \cup B]) =$ $\mu(A \cup B) = \mu(A) + \mu(B \setminus A) = \bar{\mu}(a) + \bar{\mu}(b)$. Thus, $(\mathfrak{A}, \bar{\mu})$ is a charge algebra.

■

Proposition 2.3.3. Let Z be the Stone space of a Boolean algebra $\mathfrak{A}$. Denote by $\text{Clop}(Z)$ the algebra of all clopen subsets of Z and $\mathcal{N}_d$ the ideal of nowhere dense subsets of Z. Then $\mathcal{B} := \{U \triangle N : U \in \text{Clop}(Z) \text{ and } N \in \mathcal{N}_d\}$ is an algebra of subsets of Z. Moreover, $\mathcal{N}_d \subseteq \mathcal{B}$ is an ideal of $\mathcal{B}$ and $\mathfrak{A}$ is isomorphic to $\mathcal{B}/\mathcal{N}_d$.

Proof. Let us first show that $\mathcal{B}$ is an algebra. Clearly $\emptyset \triangle \emptyset = \emptyset \in \mathcal{B}$.

If $A \in \mathcal{B}$, so $A = U \triangle N$ where $U \in \text{Clop}(Z)$ and $N \in \mathcal{N}_d$, we have

$$\begin{aligned}
A^c = (U \triangle N)^c &= [(U \cup N) \setminus (U \cap N)]^c \\
&= [(U \cup N) \cap (U \cap N)^c]^c \\
&= (U \cup N)^c \cup (U \cap N) \\
&= (U^c \cap N^c) \cup (U \cap N) \\
&= \big[U^c \cup (U \cap N)\big] \cap \big[N^c \cup (U \cap N)\big] \\
&= \big[(U^c \cup U) \cap (U^c \cup N)\big] \cap \big[(N^c \cup U) \cap (N^c \cup N)\big] \\
&= (U^c \cup N) \cap (U \cup N^c) \\
&= (U^c \cup N) \cap (U^c \cap N)^c \\
&= (U^c \cup N) \setminus (U^c \cap N) \\
&= U^c \triangle N
\end{aligned}$$

Therefore, $U^c \triangle N \in \mathcal{B}$, because $\text{Clop}(Z)$ is an algebra ($U^c \in \text{Clop}(Z)$).

Let $A_1, A_2 \in \mathcal{B}$, they can be represented as $A_1 = U_1 \triangle N_1$ and $A_2 = U_2 \triangle N_2$ where $U_1, U_2 \in \text{Clop}(Z)$ and $N_1, N_2 \in \mathcal{N}_d$. So $U_1 \triangle A_1 = N_1$ and $U_2 \triangle A_2 = N_2$. By properties of the symmetric difference operator we have

$$(U_1 \cup U_2) \triangle (A_1 \cup A_2) \subseteq (U_1 \triangle A_1) \cup (U_2 \triangle A_2) \in \mathcal{N}_d.$$

Putting $U = U_1 \cup U_2$ and $U \triangle (A_1 \cup A_2) = N$ for some $N \in \mathcal{N}_d$, we have

$$A_1 \cup A_2 = U \triangle N \in \mathcal{B}.$$

This shows that $\mathcal{B}$ is an algebra.

Clearly $\mathcal{N}_d \subseteq \mathcal{B}$, since $\emptyset \in \text{Clop}(Z)$. Applying the Stone theorem, there is a surjective Boolean homomorphism $f : \mathcal{B} \to \mathfrak{A}$ with kernel $\mathcal{N}_d$, and by Fact 1.2.2.12 the canonical map $g : \mathcal{B} \to \mathcal{B}/\mathcal{N}_d$ is a homomorphism.

Consequently, by *first homomorphism theorem* 1.2.2.12, $h : \mathcal{B}/\mathcal{N}_d \to \mathfrak{A}$ is a Boolean isomorphism (unique) such that $h \circ g = f$. Thus, the claim follows. ■

Proof of Theorem 2.3.1. Consider $\mathfrak{A}$ defined as in Proposition 2.3.2, i.e. $\mathfrak{A} = \mathcal{A}/\mathcal{N}$. Let $\mathcal{B}$ be defined as in Proposition 2.3.3. By that proposition, $\mathfrak{A}$ and $\mathcal{B}/\mathcal{N}_d$ are isomorphic, as Boolean algebras, so there is an isomorphism $\Psi : \mathcal{B}/\mathcal{N}_d \to \mathfrak{A}$, and put $\Phi(A) = \Psi([A])$. Then $\Phi : \mathcal{B} \to \mathfrak{A}$ is a corresponding surjective Boolean homomorphism with kernel $\mathcal{N}_d$. Set $\nu = \bar{\mu} \circ \Phi : \mathcal{B} \to [0, \infty]$. So we have

$$\nu(A) = \bar{\mu}(\Phi(A)) \text{ for every } A \in \mathcal{B}.$$

If $A = \emptyset$, so $\nu(\emptyset) = \bar{\mu}(\Phi(\emptyset)) = \bar{\mu}(0) = 0$. Let A and B be two disjoint members of $\mathcal{B}$. Since Φ is a Boolean homomorphism, then

$$\Phi(A \cup B) = \Phi(A) \cup \Phi(B) \text{ and } \Phi(A) \cap \Phi(B) = \emptyset, \text{ so}$$
$$\nu(A \cup B) = \bar{\mu}(\Phi(A \cup B)) = \bar{\mu}(\Phi(A)) + \bar{\mu}(\Phi(B)) = \nu(A) + \nu(B).$$

Since $\mathcal{B}$ is an algebra of subsets of Z (proved in Proposition 2.3.3) and ν satisfies the charge conditions (from the above steps), so $(Z, \mathcal{B}, \nu)$ is a charge space.

One can see that the family of sets with zero ν-charge is exactly the family of nowhere dense sets (in Z), since $\nu(A) = 0 \iff \bar{\mu}(\Phi(A)) = 0 \iff \Phi(A) = 0 \iff A \in \mathcal{N}_d$ for any A. So by the same construction as in Proposition 2.3.2, the charge algebra of Z is the algebra $\mathcal{B}/\mathcal{N}_d$ with $\bar{\nu}$ such that

$$\bar{\nu}([A]) = \nu(A) = \bar{\mu}(\Phi(A)) = \bar{\mu}(\Psi([A])) \text{ for every } A \in \mathcal{B}.$$

Therefore, $(\mathfrak{A}, \bar{\mu}) \cong (\mathcal{B}/\mathcal{N}_d, \bar{\nu})$ because Ψ is a Boolean isomorphism between them. ■

Chapter 3

Jordan Measure, Jordanian Measure, Jordan Algebra & Jordanian Algebra

This chapter deals with the study of the Jordan measure, Jordanian measure, Jordan algebra and Jordanian algebra. In Section 3.1 we give an introduction to the Jordan measure on the unit interval and arbitrary topological spaces, and investigate its construction. We show that the family of open Jordan μ-measurable subsets of a compact Hausdorff space X with respect to a Radon measure μ is a basis for its topology. In Section 3.2 we define a charge, similar to the Jordan measure, on arbitrary algebras of sets, and call it Jordanian measure. In Section 3.3 we define Jordan algebra and Jordanian algebra, and study some of their properties. We also study the nature of the Stone space of the Jordan algebra in this section.

3.1 The Jordan Measure

3.1.1 Jordan Measure on [0,1]

At the beginning of this section, we devote a small part for recalling some types of known algebras of subsets of the unit interval $\mathbb{I}$ that will assist us when we introduce this type of charges.

Consider the unit interval $\mathbb{I} = [0, 1]$, we have the following types of subintervals of it:

- Empty interval (set) $(a, a) = [a, a) = (a, a] = \emptyset$, or $[b, a] = \emptyset$ when $a < b$.
- Degenerate interval (singleton set) $[a, a] = \{a\}$.
- Open interval $(a, b) = \{x : a < x < b\}$.
- Closed interval $[a, b] = \{x : a \leq x \leq b\}$.
- Right-open (semiopen) interval $[a, b) = \{x : a \leq x < b\}$.
- Right-closed (semiopen) interval $(a, b] = \{x : a < x \leq b\}$.

An **elementary set** (in $\mathbb{I}$) is defined to be a finite union of (any type of) the above intervals.

Fact 3.1.1.1. Here are some facts about these sets:

(1) Every interval is an elementary set.

(2) The intersection of finitely many elementary sets is an elementary set.

(3) The union of finitely many elementary sets is an elementary set.

(4) The set difference of two elementary set is an elementary set.

(5) The symmetric difference of two elementary sets is an elementary set.

Thus, the family of all elementary sets in $\mathbb{I}$ forms an algebra. This algebra is called **elementary algebra** and is denoted by $\mathcal{E}(\mathbb{I})$, or simply $\mathcal{E}$.

Furthermore, we have two more algebras,

- **Open interval algebra** $\mathcal{A}_o(\mathbb{I})$, or simply $\mathcal{A}_o$, is the algebra generated by all open intervals in $\mathbb{I}$, or equivalently, the algebra generated by all closed intervals in $\mathbb{I}$.
- **Interval algebra** $\mathfrak{I}(\mathbb{I})$, or simply $\mathfrak{I}$, is the algebra generated by all half-open (or half-closed) intervals in $\mathbb{I}$.

Remark 3.1.1.2. In this remark, we present some basic properties of these algebras and their relations.

- The open interval algebra is atomic. It has infinitely many atoms, for instance, singletons.
- The interval algebra is atomless.
- Open interval algebras and interval algebras are not isomorphic, because the former have infinitely many atoms and the latter are atomless.
- Elementary algebras also have atoms.
- Interval algebra is a subalgebra of elementary algebra. Since each element A of the interval algebra $\mathfrak{I}(\mathbb{I})$ is of the form $A = [a_0, b_0) \cup [a_1, b_1) \cup ... \cup [a_n, b_n)$ where $0 \leq a_0 < b_0 < a_1 < b_1 < ... < a_n < b_n$, then A is an elementary set and so A belongs to $\mathcal{E}(\mathbb{I})$.
- All these algebras happen to be countable if we restrict the generators to intervals with rational end points. Otherwise, they are uncountable in general.

The **Jordan measure** $\mathcal{J}$ on the unit interval $\mathbb{I}$ is defined on the algebra of elementary subsets of $\mathbb{I}$. Given any elementary set E, E can be represented as:

$$E = I_1 \cup I_2 \cup \cdots \cup I_n,$$

where I_i is one of these subintervals mentioned above and $\bigcap_{i=1}^{n} I_i = \emptyset$. No matter which type of interval we choose, we would have the same length. The Jordan measure is defined to agree with the length function on intervals in $\mathbb{I}$, so

$$\mathcal{J}(E) = \ell(E) = \sum_{i=1}^{n} \ell(I_i).$$

For an arbitrary subset $A \subseteq \mathbb{I}$, the **Jordan measure** is defined as follows:

The **inner Jordan measure** of A is

$$\mathcal{J}_*(A) = \sup\{\ell(E) : E \subseteq A\},$$

and the **outer Jordan measure** of A is

$$\mathcal{J}^*(A) = \inf\{\ell(E) : A \subseteq E\}.$$

Here E is an elementary set and ℓ is the length function.
For any A, if $\mathcal{J}^*(A) = \mathcal{J}_*(A)$, then A is said to be **Jordan measurable** with the Jordan measure $\mathcal{J}(A) = \mathcal{J}^*(A) = \mathcal{J}_*(A)$.

Note that we can replace the elementary algebra by any algebras that stated above and obtain the same Jordan measure. This is the classical way of constructing the Jordan measure.

On the other hand, because of the nice properties that Jordan measurable sets have, the Jordan measure can be constructed in another way. This construction can be done using the Lebesgue measure λ on $\mathbb{I}$ as it is the unique extension of the Jordan measure itself to the σ-algebra of Borel sets. If A is a Jordan measurable set, then both its (topological) interior and closure are also Jordan measurable and have the same measure as A, (see [23] and [51], Exercise 1.1.18), and since the Lebesgue measure and Jordan measure agree on A, then

$$\lambda(\mathrm{Cl}(A)) = \lambda(\mathrm{Int}(A)) \Longrightarrow \lambda(\mathrm{Cl}(A) \setminus \mathrm{Int}(A)) = 0.$$

But $\mathrm{Cl}(A) \setminus \mathrm{Int}(A) = \partial(A)$, the topological boundary of A. Thus,

$$A \text{ is Jordan measurable if } \lambda(\partial(A)) = 0.$$

This is one of the characterizations for Jordan measurable sets and the following is another characterization which will be relied on during this work.

Lemma 3.1.1.3. [51] For a subset A of $\mathbb{I}$, the following are equivalent:

(1) A is Jordan measurable.

(2) χ_A (indicator function of A) is Riemann integrable on $\mathbb{I}$.

(3) For any $\epsilon > 0$, there exist two elementary sets E, F with $E \subseteq A \subseteq F$ such that

$$\mathcal{J}(F \setminus E) < \epsilon.$$

(4) For any $\epsilon > 0$, there exists an elementary set E such that

$$\mathcal{J}^*(A \triangle E) < \epsilon.$$

(5) $\mathcal{J}^*(E) = \mathcal{J}^*(A \cap E) + \mathcal{J}^*(A^c \cap E)$ for all $E \subseteq \mathbb{I}$.

(6) $\mathcal{J}_*(E) = \mathcal{J}_*(A \cap E) + \mathcal{J}_*(A^c \cap E)$ for all $E \subseteq \mathbb{I}$.

Since the Lebesgue measure λ is the unique extension of the Jordan measure $\mathcal{J}$, every Jordan measurable subset of $\mathbb{I}$ is Lebesgue measurable, but not conversely. For example, the set of rational numbers in $\mathbb{I}$, i.e. $A = \mathbb{Q} \cap \mathbb{I}$, is not Jordan measurable.

Let us recall some properties of the topological boundary of a set to better understand the nature of Jordan measurable sets.

Lemma 3.1.1.4. For any subset A of a space X, we have

(1) $\partial(Int(A)) \subseteq \partial(A)$.

(2) $\partial(Cl(A)) \subseteq \partial(A)$.

(3) $\partial(A^c) = \partial(A)$.

(4) $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

(5) $\partial(A \cap B) \subseteq \partial A \cup \partial B$.

Since $\emptyset$ and $\mathbb{I}$ are Jordan measurable sets, so applying (3), (4) and (5) in Lemma 3.1.1.4, we conclude that the class of Jordan measurable sets is closed under finite unions, intersections and complements. Thus the class of all Jordan measurable sets in $\mathbb{I}$ forms a Boolean algebra and is denoted by $\mathcal{J}(\mathbb{I}, \lambda)$. The countable union of Jordan measurable sets need not be Jordan measurable, (see example given below or consider the example of the rationals). In contrast, the class of Lebesgue measurable sets is closed under countable unions, intersections and complements and so this class forms a (Boolean) σ-algebra which is denoted by $\mathcal{L}(\mathbb{I}, \lambda)$.

Not all subsets of $\mathbb{I}$ are Jordan measurable. The most noticeable ones are open or compact sets. Let $\{q_1, q_2, q_3, ...\}$ be the set of rational numbers in $\mathbb{I}$. If $U = \bigcup_{i=1}^{\infty}(q_i - \epsilon/2^i, q_i + \epsilon/2^i)$, then U is open because it is the countable union of open intervals but not Jordan measurable, for the proof see Remark 1.2.8 in [51]. The fat Cantor set (Smith–Volterra–Cantor set, [1], pages 140-141) is compact but not Jordan measurable because its Lebesgue measure is 1/2. Most non-Jordan measurable sets have a problem with inner Jordan measure which is strictly less than or equal to inner Lebesgue measure as shown in the following inequality.
For a subset A of $\mathbb{I}$, we have

$$\mathcal{J}_*(A) \leq \lambda_*(A) \leq \lambda^*(A) \leq \mathcal{J}^*(A).$$

The Boolean algebra $\mathcal{J}(\mathbb{I}, \lambda)$ of Jordan measurable sets is a subalgebra of the Boolean algebra of Lebesgue measurable sets $\mathcal{L}(\mathbb{I}, \lambda)$.

Remark 3.1.1.5. The Boolean algebra of all Lebesgue measurable sets and the Boolean algebra of all Jordan measurable sets in $\mathbb{I}$ have cardinality $2^{\mathfrak{c}}$, (see Exercise 1.9.13 [11]).

3.1.2 Jordan Measure on Arbitrary Topological Spaces

In the previous section, we gave the construction of Jordan measure on the unit interval $\mathbb{I}$. We have seen that there were two ways of constructing it. One of them was from a charge on an algebra, which was the length function on the elementary algebra. The second was from a (complete) measure on a σ-algebra, which was the Lebesgue measure on the Borel σ-algebra on the compact metrizable topological space $\mathbb{I}$. So we try to give analogue constructions in a more general case.

We observe that the first method may not work with an arbitrary charge on an algebra of subsets of any topological space, i.e., we cannot have an analogue of Jordan measure from an arbitrary charge on an algebra, because we cannot always have an analogue of the length function ℓ on an elementary algebra $\mathcal{E}(\mathbb{I})$ that satisfies $\ell(E) = \ell(\mathrm{Cl}(E)) = \ell(\mathrm{Int}(E))$ for every elementary set $E \in \mathcal{E}(\mathbb{I})$. Therefore, we try to give such a construction under the different name "Jordanian measure" (see the next section) and then show its relation to the (usual) Jordan measure.

On the other hand, we can define the Jordan measure on an arbitrary topological space with respect to a measure, as follows: Assume that X is any topological space and μ is a measure defined on the Borel σ-algebra $\mathcal{B}(X, \mu)$ of subsets of X. A bounded subset A of X is **μ-Jordan measurable** if the boundary of A ($\partial(A) = \mathrm{Cl}(A) \setminus \mathrm{Int}(A)$) has μ-measure zero. Jordan measure is also closely related to the Riemann integral. If we have a topological space X and a Radon (probability) measure μ on it, then a bounded function $f : X \to \mathbb{R}$ is **μ-Riemann integrable** if the set of points of discontinuity of f has μ-measure zero. Therefore, using μ-Riemann integrable functions one can obtain Jordan μ-measurable sets, as follows: a subset A of X is μ-Jordan measurable if its characteristic function χ_A is μ-Riemann integrable. By $\mathcal{J}(X, \mu)$ denote the class of all Jordan measurable subsets of a given topological measure space X. For the same reason as in usual Jordan measure, $\mathcal{J}(X, \mu)$ forms an algebra but not σ-algebra.

The Boolean algebra $\mathcal{J}(X, \mu)$ may have properties different from $\mathcal{J}(\mathbb{I}, \lambda)$, as shown in the following:

Remark 3.1.2.1. [35] The σ-algebra generated by $\mathcal{J}(X, \mu)$ contains all Baire sets in X but not all Borel sets. For example, consider the space X of all ordinals $\leq \omega_1$ with the usual order topology. For every Borel set A, define a measure μ by

$$\mu(A) = \begin{cases} 1, & \text{if } \omega_1 \in A \\ 0, & \text{otherwise.} \end{cases}$$

Then, by a remark on page 171 in [35], $\mathcal{J}(X, \mu)$ and the σ-algebra generated by $\mathcal{J}(X, \mu)$ is identical to the Baire σ-algebra. Then we need find a set in X that is Borel but not Baire. Consider the singleton $\{\omega_1\}$. Clearly it is Borel because it is closed. We now show that it is not a Baire set. By Lemma 1.2.2.18, it is enough only to show that $\{\omega_1\}$ is not G_δ. Let $\{[a_n, \omega_1] : n < \omega\}$ be a countable collection of (basic) open sets in X containing ω_1. Let $a > a_n$ for all $n < \omega$. This implies that $[a, \omega_1] \subseteq \bigcap_{n<\omega} [a_n, \omega_1]$ and so $\bigcap_{n<\omega} [a_n, \omega_1] \neq \{\omega_1\}$. Hence $\{\omega_1\}$ is not G_δ. This completes the proof.

Definition 3.1.2.2. [26] Let μ be a Radon (probability) measure on a compact Hausdorff space X. A subset $\mathcal{B} \subseteq \mathcal{J}(X, \mu)$ is said to be a **set of generators** for $\mathcal{J}(X, \mu)$ if for every $A \in \mathcal{J}(X, \mu)$ and every $\epsilon > 0$, there exist $U, V \in \mathcal{B}$ such that $U \subseteq A \subseteq V$ and $\mu(V \setminus U) < \epsilon$.

Lemma 3.1.2.3. Let X be a compact Hausdorff space and μ a nonatomic Radon probability measure on X. Then

(1) for every $\epsilon > 0$ and every open set U, there is an open Jordan measurable subset A of X such that

$$A \subseteq \mathrm{Cl}(A) \subseteq U \text{ and } \mu(U \setminus A) < \epsilon$$

(2) for every two open Jordan measurable sets U and W such that $\mathrm{Cl}(U) \subseteq W$, there is an open Jordan measurable set V such that

$$\mathrm{Cl}(U) \subseteq V \subseteq \mathrm{Cl}(V) \subseteq W \text{ and } \tfrac{1}{3}\mu(W - U) < \mu(V - U) < \tfrac{2}{3}\mu(W - U).$$

Proof. For (1) Let U be an open set and $\epsilon > 0$. Since μ is regular, there is a closed set E such that $E \subseteq U$ and $\mu(U \setminus E) < \epsilon$. Now, set $F = U^c$, so we have two disjoint closed set E and F. By Lemma 1.2.1.15, X is normal and by Urysohn Lemma 1.2.1.14, there is a continuous function $f : X \to [0, 1]$ such that

$$f(E) = 1, \qquad \text{i.e. } f(x) = 1 \text{ for all } x \in E;$$
$$f(F) = 0, \qquad \text{i.e. } f(x) = 0 \text{ for all } x \in F.$$

For $r : 0 \leq r \leq 1$, we let $U(r) = \{x \in X : f(x) = r\}$. Since $X = \bigcup_{0 \leq r \leq 1} U(r)$, there are at most countably many U_r with positive measure, $\mu(U(r)) > 0$. So there is $r \in [0, 1]$ such that $\mu(U(r)) = 0$. Let $A = \{x \in X : f(x) > r\}$, the closure of A is $\mathrm{Cl}(A) = \{x \in X : f(x) \geq r\}$. So $\partial(A) = \mathrm{Cl}(A) \setminus \mathrm{Int}(A) = \{x \in X : f(x) = r\} \subseteq U(r)$. This implies that $\mu(\partial(A)) \subseteq \mu(U(r)) = 0$ and so $\mu(\partial(A)) = 0$. This shows that A is Jordan measurable. Now, we have $E \subseteq U$ and $\mu(U \setminus E) < \epsilon$, but $E \subseteq A$ and $\mathrm{Cl}(A) \subseteq U$. Therefore, $E \subseteq A \subseteq \mathrm{Cl}(A) \subseteq U$ and $\mu(U \setminus A) < \epsilon$.

For (2) The proof is similar to that of (1), but here choose W so that $\mathrm{Cl}(U) \subseteq W$ and ϵ appropriately. ■

Remark 3.1.2.4. From the above result, we conclude that for any nonempty open set U containing a point x, there is an open Jordan measurable set B of X such that

$$x \in B \subseteq U.$$

So the set of all open Jordan measurable sets forms a base for the topology on X, (*cf.* [4] Proposition 7).

3.2 Jordanian Measure on Topological Spaces

Let μ_0 be a charge on some algebra $\mathcal{B}$ of subsets of a topological space X. For $A \subseteq X$, we define the inner and the outer charge by

$$\mu^*(A) = \inf\{\mu_0(E) : E \in \mathcal{B}, A \subseteq E\},$$
$$\mu_*(A) = \sup\{\mu_0(E) : E \in \mathcal{B}, E \subseteq A\}.$$

Then A is said to be **Jordanian μ-measurable** (or simply Jordanian measurable) if $\mu^*(A) = \mu_*(A)$. The value of **Jordanian measure** is $\mu(A) = \mu^*(A) = \mu_*(A)$.

The Jordanian measure has a similar characterization to that of the Jordan measure on $\mathbb{I}$.

Lemma 3.2.1. For a subset $A \subseteq X$, the following statements are equivalent:

(1) A is Jordanian μ-measurable

(2) For any $\epsilon > 0$, there exist B_1, B_2 in $\mathcal{B}$ such that $B_1 \subseteq A \subseteq B_2$ and $\mu(B_2 \setminus B_1) < \epsilon$.

(3) $\mu^*(Y) = \mu^*(A \cap Y) + \mu^*(A^c \cap Y)$ for all $Y \subseteq X$.

(4) $\mu_*(Y) = \mu_*(A \cap Y) + \mu_*(A^c \cap Y)$ for all $Y \subseteq X$.

Proof. Follows from Lemma 3.1.1.3. ∎

Similar properties to the above can be found on ([52], page 15) under the name of **completion of content** on families of sets called $(\emptyset, \cup f, \cap c)$-**paving** (:= is a family of subsets of a set X that closed under finite unions and countable intersections together with $\emptyset$).

Note that the set of all Jordanian μ-measurable sets in X is a Boolean algebra, let us denote it by $J(X, \mu)$ or $J(\mathcal{B}, \mu)$, and the induced μ is a (complete) charge (the proof is in [29], page 232). We prefer to call μ the **Jordan extension** of μ_0 on $\mathcal{B}$, and $\mathcal{B}$ is called the **set of generators** for $J(X, \mu)$.

The Boolean algebra $J(X, \mu)$ is the largest algebra in the sense that any Jordan extension of μ with respect to $J(X, \mu)$ itself will give us the same algebra $J(X, \mu)$.

3.3 Jordan Algebra & Jordanian Algebra

This section is dedicated to study the Jordan algebra and Jordanian algebra. The reason why we study the Jordan algebra is because it is the most natural algebra of sets on which there is an extension of Riemann integration and that it was invented in order to introduce the Jordan measure, which is a precedent to the Lebesgue measure. The Jordan measure is exactly charge but not measure, so in the context of charges, the Jordan algebra is the most natural object of investigation. Then we choose to study the Jordanian algebra in order to know how they are closely related.

We start defining the Jordan algebra on arbitrary topological spaces. Let X be a topological space and μ a measure thereon. The **Jordan algebra** $\mathcal{J}_\mu(X)$ is a new algebra constructed from the Boolean algebra $\mathcal{J}(X, \mu)$ by quotienting out by the ideal of all null sets $\mathcal{N}$. This is done by the equivalence relation $\sim$ on $\mathcal{J}(X, \mu)$ as follows:

$$A \sim B \text{ if and only if } A \triangle B \in \mathcal{N} \text{ for } A, B \in \mathcal{J}(X, \mu).$$

Therefore, $\mathcal{J}_\mu(X)$ is the collection of all equivalence classes on $\mathcal{J}(X, \mu)$ determined by $\sim$. The equivalence class of A is an element $[A]$ in the Jordan algebra which is given by:

$$[A] = \{B \in \mathcal{J}(X, \mu), B \sim A\}.$$

Similarly, given a charge μ on any algebra $\mathcal{A}$ of subsets of a topological space X, the so called **Jordanian algebra** $J_\mu(X)$ (or $J_\mu(\mathcal{A})$) is the quotient algebra of $J(X, \mu)$ (or $J(\mathcal{A}, \mu)$) by the ideal of null sets $\mathcal{N}$. Equivalently, the set of all equivalence classes on $J(X, \mu)$ determined by the relation $\sim$, where $\sim$ is defined as follows:

$$A \sim B \text{ if and only if } A \triangle B \in \mathcal{N} \text{ for } A, B \in J(X,\mu).$$

The equivalence class of A is an element $[A]$ in the Jordanian algebra which is given by:

$$[A] = \{B \in J(X,\mu), B \sim A\}.$$

Observe that by saying the Jordanian algebra of a charge μ on an algebra $\mathcal{A}$ of subsets of X we mean the Boolean algebra of all measurable subsets of X with respect to the Jordan extension of μ modulo (the extended) measure zero.

Note that for the sake of reducing the level of confusion between operations on both $\mathcal{J}(X,\mu)$ (resp. $J(X,\mu)$) and $\mathcal{J}_\mu(X)$ (resp. $J_\mu(X)$), we prefer to use $\vee, \wedge$ and $\bullet^c$ on $\mathcal{J}_\mu(X)$ (resp. $J_\mu(X)$). These operations $\vee, \wedge$ and $\bullet^c$ on $\mathcal{J}_\mu(X)$ (resp. $J_\mu(X)$) are defined as follows:

$$[A] \vee [B] = [A \cup B]$$
$$[A] \wedge [B] = [A \cap B]$$
$$[A]^c = [A^c],$$

where $A^c = X \setminus A$. The unit and zero element of this algebra are $[0] = \mathcal{N}$ and $[1] = [X]$, i.e. every Jordan (resp. Jordanian) measurable sets of measure 1. From the definition of the above operations, we can define an ordering on $\mathcal{J}_\mu(X)$ (resp. $J_\mu(X)$) in such a way that for any $[A], [B] \in \mathcal{J}_\mu(X)$ (resp. $J_\mu(X)$),

$$[A] \leq [B] \Longleftrightarrow A \setminus B \in \mathcal{N}.$$

This ordering follows from the fact that $[A]$ is zero element in $\mathcal{J}_\mu(X)$ (resp. $J_\mu(X)$) if and only if $A \in \mathcal{N}$. So $[A] \leq [B]$ holds if and only if $[A] \setminus [B] = [A \setminus B]$ is zero element in $\mathcal{J}_\mu(X)$ (resp. $J_\mu(X)$) which is equivalent to $A \setminus B \in \mathcal{N}$.

For $X = \mathbb{I}$ and $\mu = \lambda$, the Lebesgue measure on $\mathbb{I}$, we call $\mathcal{J}_\lambda(\mathbb{I})$ the **usual Jordan algebra**.

Lemma 3.3.1. For every $A, B \in \mathcal{J}(X,\mu)$ (resp. $J(X,\mu)$), $A \sim B$ if and only if $[A] = [B]$ and then $\mu(A) = \mu(B)$.

Proof. Directly from the definition of $\sim$. ∎

Lemma 3.3.2. The elements of the Jordan algebra have the following properties:

(1) If $B \in [A]$, then $[B] = [A]$.

(2) Every two classes either have empty intersection or they are the same. So if $[A] \cap [B] \neq \emptyset$, there is $C \in [A] \cap [B]$, which implies that $[C] = [A] = [B]$ and $[C] = [A \cap B]$.

(3) For every $[A] \in \mathcal{J}_\mu(X)$, $\mathrm{Int}(A) \in [A]$.

(4) For every $[A] \in \mathcal{J}_\mu(X)$, $\mathrm{Cl}(A) \in [A]$.

Proof. (1), (2) Straightforward.
(3) Suppose that $[A] \in \mathcal{J}_\mu(X)$. From properties of the Jordan measurable sets we can say that $\mathrm{Int}(A)$ is Jordan measurable for any $A \in \mathcal{J}(X, \mu)$ and $\mu(\mathrm{Int}(A)) = \mu(A)$. This implies that $\mathrm{Int}(A) \sim A$ and thus $\mathrm{Int}(A) \in [A]$.
(4) Similar to (3). ■

Definition 3.3.3. Elements $A \in [A]$ are called **representatives** of $[A]$.

The Lemma 3.3.1 shows that all representatives of a given element of the Jordan algebra have the same measure.

Lemma 3.3.4. For any two elements $[A], [B]$ in the Jordan algebra (resp. Jordanian algebra), the following are equivalent:

(1) $[A] \leq [B]$.

(2) For any representatives A of $[A]$ and B of $[B]$, there is $N \in \mathcal{N}$ such that $A \subseteq (B \cup N)$.

(3) For any representative A of $[A]$ there is a representative B of $[B]$ such that $A \subseteq B$.

Proof. (1) $\Rightarrow$ (2) Assume that $[A] \leq [B]$, then $[A \cap B] = [A] \wedge [B] = [A]$, so by the above definition $A \cap B \sim A$. Thus, $((A \cap B) \setminus A) \cup (A \setminus (A \cap B)) = N$ for some $N \in \mathcal{N}$. This implies that $A \setminus (A \cap B) = N$ and hence $A \subseteq (N \cup B)$.
(2) $\Rightarrow$ (3) Since $B \cup N \sim B$ for some $N \in \mathcal{N}$ and $A \subseteq (B \cup N)$ by assumption, therefore, $A \subseteq B$.
(3) $\Rightarrow$ (1) Suppose that $A \subseteq B$, we have $[A] = [A \cap B] = [A] \wedge [B]$, and thus $[A] \leq [B]$. ■

Definition 3.3.5. An element $[A] \in \mathcal{J}_\mu(X)$ is called **open** (reps. **closed**) if one representative A of $[A]$ is an open (reps. closed) Jordan measurable subset of X, i.e A is open (reps. closed) in $\mathcal{J}(X, \mu)$. We denote the family of all open (reps. closed) elements in $\mathcal{J}_\mu(X)$ by $\mathcal{J}_0(X)$ (reps. $\mathcal{J}_c(X)$)

Proposition 3.3.6. Given a measure μ on some topological space X, we have $\mathcal{J}_\mu(X) = \mathcal{J}_0(X) = \mathcal{J}_c(X)$.

Proof. The direction $\mathcal{J}_0(X) \subseteq \mathcal{J}_\mu(X)$ is straightforward.
Let $[A] \in \mathcal{J}_\mu(X)$. By Lemma 3.3.2, $\text{Int}(A) \in [A]$. This implies that $[A]$ has an open representative and so $[A] \in \mathcal{J}_0(X)$. Thus $\mathcal{J}_\mu(X) \subseteq \mathcal{J}_0(X)$. This completes the proof. In same way we can get $\mathcal{J}_\mu(X) = \mathcal{J}_c(X)$.

■

Lemma 3.3.7. (1) The Lebesgue measure algebra $\mathcal{L}_\lambda(\mathbb{I})$ of the unit interval $\mathbb{I}$ has cardinality $\mathfrak{c}$.
(2) The Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ of the unit interval $\mathbb{I}$ also has cardinality $\mathfrak{c}$.

Proof. (1) Since Lebesgue measure λ is regular, for every $n < \omega$ and for every measurable set A, there is an open set G_n and a closed set F_n such that

$$F_n \subseteq A \subseteq G_n \text{ and } \lambda(G_n \setminus F_n) < 1/n.$$

This implies that $\cup_{n<\omega} F_n = F_\sigma \simeq A \simeq G_\delta = \cap_{n<\omega} G_n$. So each class $[A]$ has a Borel representative, but there are only $\mathfrak{c}$ Borel sets in $\mathbb{I}$. Thus, $\mathcal{L}_\lambda(\mathbb{I})$ has cardinality at most $\mathfrak{c}$.

On the other hand, for $x \neq y$, the sets $A = (0, x)$ and $B = (0, y)$ are nonequivalent representatives for two different classes. So the sets $(0, x)$ are all nonequivalent for different values of $x \in \mathbb{I}$, and the cardinality of $\mathbb{I}$ is $\mathfrak{c}$. So the cardinality of $\mathcal{L}_\lambda(\mathbb{I})$ will be at least $\mathfrak{c}$ and consequently, the cardinality of $\mathcal{L}_\lambda(\mathbb{I})$ is exactly $\mathfrak{c}$.

(2) For any $\epsilon > 0$ and any Jordan measurable set A in $\mathbb{I}$, there are two elementary sets E and F such that

$$E \subseteq A \subseteq F \text{ and } \lambda(F \setminus E) < \epsilon.$$

But all elementary sets are Borel. So each class $[A]$ must contain a Borel set (as in case (1)), but we have only $\mathfrak{c}$ Borel sets in $\mathbb{I}$. We obtain that the cardinality of $\mathcal{J}_\lambda(\mathbb{I})$ is at most $\mathfrak{c}$.

On the other hand, for $x \neq y$, the open intervals $A = (0, x)$ and $B = (0, y)$ are Jordan measurable (as their boundaries are finite) and are nonequivalent representatives for two different classes. So the sets $(0, x)$ are all nonequivalent for different values of $x \in \mathbb{I}$, and the cardinality of $\mathbb{I}$ is $\mathfrak{c}$. So the cardinality of $\mathcal{J}_\lambda(\mathbb{I})$ is at least $\mathfrak{c}$. Hence, the cardinality of $\mathcal{J}_\lambda(\mathbb{I})$ is exactly $\mathfrak{c}$. ■

Lemma 3.3.8. For the Lebesgue measure λ on $\mathbb{I}$, we have the following:

(1) $\mathcal{L}_\lambda(\mathbb{I}) \cong \mathscr{B}_\lambda(\mathbb{I})$, the Borel algebra in $\mathbb{I}$ modulo null sets.

(2) $\sigma(\mathcal{J}_\lambda(\mathbb{I})) = \mathcal{L}_\lambda(\mathbb{I})$, where $\sigma(\mathcal{J}_\lambda(\mathbb{I})) := \{ \bigvee_{n<\omega} [a_n] = [\bigcup_{n<\omega} a_n] : a_n \in \mathcal{J}(\mathbb{I}, \lambda)\}$.

Proof. (1) Since every Borel set is Lebesgue measurable, obviously, $\mathscr{B}_\lambda(\mathbb{I}) \subseteq \mathcal{L}_\lambda(\mathbb{I})$. On the other hand, from the part (1) in Lemma 3.3.7, every class in $\mathcal{L}_\lambda(\mathbb{I})$ has a Borel representative, and so $\mathcal{L}_\lambda(\mathbb{I}) \subseteq \mathscr{B}_\lambda(\mathbb{I})$. Hence, $\mathcal{L}_\lambda(\mathbb{I}) \cong \mathscr{B}_\lambda(\mathbb{I})$.
(2) By Remark 3.1.2.1, the σ-algebra generated by $\mathcal{J}_\lambda(\mathbb{I})$ contains all Baire sets. But in $\mathbb{I}$ Baire and Borel sets coincide. So $\sigma(\mathcal{J}_\lambda(\mathbb{I}))$ contains all Borel classes and consequently it is exactly $\mathcal{L}_\lambda(\mathbb{I})$ by (1). ■

Note that, in an arbitrary topological measure space X, the σ-algebra generated by the Jordan algebra of X is a subalgebra of the measure algebra of X.

Lemma 3.3.9. (1) The Lebesgue measure algebra $\mathcal{L}_\lambda(\mathbb{I})$ is complete.
(2) The Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ is not complete.

Proof. (1) Follows from the fact that "Every measure algebra is complete", ([24], page 295).
(2) Let $q_1, q_2, q_3, ...$ be the rational numbers in $\mathbb{I}$. For $1/2 > \epsilon > 0$, consider the collection $\mathcal{G} = \{(q_i - \epsilon/2^i, q_i + \epsilon/2^i) : i = 1, 2, ...\}$ of open intervals. Obviously, every open interval in $\mathcal{G}$ is Jordan measurable and acts as representative of a class. So $\mathcal{G}$ is a subset of the

Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ but the union of $\mathcal{G}$ is not Jordan measurable (see § 3.1.1, page 63), and so does not belong to $\mathcal{J}_\lambda(\mathbb{I})$. ■

Lemma 3.3.10. Let μ be a normal Radon measure on a compact Hausdorff space X. Then every measurable subset of X is Jordan measurable. That is, the measure algebra and the Jordan algebra of μ coincide.

Proof. Let A be any measurable set in X. Clearly $A \setminus \mathrm{Int}(A)$ is measurable and does not contain any open set. By regularity of μ,

$$\mu(A \setminus \mathrm{Int}(A)) = \sup\{\mu(K) : \text{ compact } K \subseteq A \setminus \mathrm{Int}(A)\}.$$

But every compact set $K \subseteq A \setminus \mathrm{Int}(A)$ is nowhere dense and $\mu(K) = 0$ by hypothesis. So $\mu(A \setminus \mathrm{Int}(A)) = 0$. Hence

$$\mu(A) = \mu(\mathrm{Int}(A)). \tag{1}$$

By the same argument, we will obtain

$$\mu(A) = \mu(\mathrm{Cl}(A)). \tag{2}$$

Equations (1) & (2) imply that $\mu(\partial(A)) = 0$. Thus, A is Jordan measurable.

Now, we obtain that $\mathrm{Ma}(X, \mu) \subseteq \mathcal{J}_\mu(X)$. The other way round is always true, i.e., every Jordan measurable set is measurable, $\mathcal{J}_\mu(X) \subseteq \mathrm{Ma}(X, \mu)$. Hence, $\mathcal{J}_\mu(X) = \mathrm{Ma}(X, \mu)$. ■

From the above result and the fact that every measure algebra is complete (in the sense of Boolean algebras) we conclude the following:

Remark 3.3.11. If μ is a normal Radon measure on a compact Hausdorff space X, then $\mathcal{J}_\mu(X)$ is a complete algebra.

Lemma 3.3.12. Let μ be a (strictly positive) Radon measure on some compact Hausdorff space X. If μ is not normal, then the Jordan algebra $\mathcal{J}_\mu(X)$ in X is not complete.

Proof. By assumption, there is at least one closed nowhere dense set N in X such that $\mu(N) > 0$. There exists an open set U in X whose boundary is N, by properties of nowhere dense sets (page 16). So U is not Jordan measurable. On the other hand, by Remark 3.1.2.4, the collection $\mathcal{B}$ of open Jordan μ-measurable subsets of X forms a basis for the topology. So $U = \bigcup_{\alpha} B_\alpha$ for some B_α in $\mathcal{B}$. This shows that $\mathcal{J}_\mu(X)$ is not complete because U is a union of members of $\mathcal{J}_\mu(X)$ but U itself does not belong to it. ■

Now, we discuss the relationships between Jordan algebra and some other types of algebras that were mentioned in the previous section.

Remark 3.3.13. We observe the following relations:

(1) The open interval algebra $\mathcal{A}_o(\mathbb{I})$ is not isomorphic the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$, and even not a subalgebra of the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$. $\mathcal{A}_o(\mathbb{I})$ contains infinitely many atoms but $\mathcal{J}_\lambda(\mathbb{I})$ is atomless.

(2) The elementary algebra $\mathcal{E}(\mathbb{I})$ is not a subalgebra of the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ and not isomorphic to it because $\mathcal{E}(\mathbb{I})$ contains infinitely many atoms while $\mathcal{J}_\lambda(\mathbb{I})$ is atomless.

(3) The interval algebra $\mathfrak{I}(\mathbb{I})$ is a subalgebra of the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ but not isomorphic to it, (see Question 2 in [30]).

(4) All of those three algebras are subalgebras of the Boolean algebra of Jordan measurable sets in $\mathbb{I}$, but none of them is isomorphic to it because the latter algebra has a very large cardinality ($2^{\mathfrak{c}}$) (see Remark 3.1.1.5).

Lemma 3.3.14. The restricted of the Lebesgue measure λ_0 to the interval algebra $\mathfrak{I}$ is a Darboux charge.

Proof. Consider the function $f(a) = \lambda_0(A \cap [0, a))$. Since every continuous function taking values 0 and 1 is Darboux (by the intermediate value theorem), it is enough to show that the function f is continuous. Given $\epsilon > 0$ and take $0 < \delta \leq \epsilon$. Suppose that $|b-a| < \delta$. Now, $|f(b)-f(a)| = |\lambda_0(A\cap[0, b))-\lambda_0(A\cap[0, a))| = |\lambda_0(A\cap([0, b)-[0, a)))| \leq |\lambda_0([0, b) - [0, a))| = |b - a| < \delta < \epsilon$. Therefore, $|f(b) - f(a)| < \epsilon$ and so f is continuous. ■

Definition 3.1.2.2 can be redefined for strictly positive measures, as follows:

Definition 3.3.15. [26] Let X be a compact Hausdorff space and μ a strictly positive Radon (probability) measure. A subalgebra $\mathcal{B}$ of $\mathcal{J}_\mu(X)$ is said to be a **set of generators** for $\mathcal{J}_\mu(X)$ if for every $A \in \mathcal{J}_\mu(X)$ and every $\epsilon > 0$ there exist $U, V \in \mathcal{B}$ such that $U \subseteq A \subseteq V$ and $\mu(V \setminus U) < \epsilon$.

We remark that a set of generators for $\mathcal{J}_\mu(X)$ may exist in the following way:

Remark 3.3.16. Given μ and X as defined above. A subalgebra $\mathcal{B}$ of $\mathcal{J}(X, \mu)$ is a set of generators for $\mathcal{J}_\mu(X)$ if for every $A \in \mathcal{J}_\mu(X)$ and every $\epsilon > 0$ there exist $U, V \in \mathcal{B}$ with $[U], [V] \in \mathcal{B}$ such that $U \subseteqq A \subseteqq V$ and $\mu(V \setminus U) < \epsilon$, where $A \subseteqq B$ means $\mu([A] \setminus [B]) = 0$ for Jordan measurable sets A, B.

If $\mathcal{B}$ is a set of generators for $\mathcal{J}(X, \mu)$, then $\mathcal{B}^* = \{[B] : B \in \mathcal{B}\}$ is a set of generators for $\mathcal{J}_\mu(X)$.

Notice that the above definition is also valid for Jordanian algebras.

Definition 3.3.17. [26] Let X, Y be compact Hausdorff spaces and μ, ν Radon probability measures on X, Y respectively. A measure preserving function $f : X \to Y$ is said to be **Jordan measurable** if for every ν-Jordan measurable subset B of Y, there is a μ-Jordan measurable subset A of X such that

$$\mu(A \triangle f^{-1}(B)) = 0.$$

An invertible measure preserving function $f : X \to Y$ is said to be a **Jordan isomorphism** if both f and f^{-1} are Jordan measurable. In that case, it is said that the measure spaces (X, μ) and (Y, ν) are **Jordan isomorphic**. The isomorphism is **mod(0)** if f becomes one-to-one and onto after removing sets of measure zero from X and Y.

Remark 3.3.18. We remark that the notion of Jordan isomorphism is stronger than Jordan isomorphism mod(0), and Jordan isomorphism mod(0) between two spaces is equivalent to saying that the Jordan algebras of such spaces are isomorphic.

Example 3.3.19. Let μ, ν be two measures on topological spaces X, Y respectively. If $f : X \longrightarrow Y$ is continuous surjection such that $f(\mu) = \nu$, then f is Jordan measurable. More precisely, if the given f is a homeomorphism, then f would be a Jordan isomorphism, see Example 2.7 in [26].

Proposition 3.3.20. [26, Proposition 2.3] Let μ, ν be two strictly positive measures on compact Hausdorff spaces X, Y respectively, and $\mathfrak{A}, \mathfrak{B}$ Boolean subalgebras of $\mathcal{J}_\mu(X), \mathcal{J}_\nu(Y)$, which are sets of generators for $\mathcal{J}_\mu(X), \mathcal{J}_\nu(Y)$. If $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, then $\mathcal{J}_\mu(X)$ and $\mathcal{J}_\nu(Y)$ are isomorphic, and hence their measure algebras are also isomorphic.

The converse of the above proposition is not true as shown in the following example:

Example 3.3.21. Consider the restriction of the Lebesgue measure to the interval algebra $\mathfrak{I}$ and the algebra $\mathcal{A}$ generated by closed intervals in $\mathbb{I}$ whose endpoints are rationals. By Remark 3.3.13, $\mathfrak{I}$ and $\mathcal{A}$ are not isomorphic but they both generate the usual Jordan algebra.

Using Fact 2.2.4 and Remark 2.2.5, we conclude the following:

Remark 3.3.22. Let Z be the Stone space of $\mathcal{J}(X, \mu)$. By Stone's theorem, $\mathcal{J}(X, \mu)$ is isomorphic to the algebra $\mathcal{C}$ of clopen subsets of Z by the isomorphism $\mathcal{J}(X, \mu) \ni A \mapsto \widehat{A} \in \mathcal{C}$. Let G be the union of all $\widehat{A}$ such that $A \in \mathcal{N}$. By Lemma 2.2.3, G is an open set in Z. Then set $E = G^c$, so E being closed implies that it is a compact zero-dimensional space (as a subspace), and $\mathcal{C}_E$ is the algebra of clopen subsets of E. Notice that $A \in \mathcal{N}$ if and only if $\widehat{A} \subseteq G$ or equivalently $\widehat{A} \cap E = \emptyset$. By Fact 2.2.4, there is an isomorphism from $\mathcal{J}(X, \mu)/\mathcal{N}$ onto $\mathcal{C}_E$, i.e.

$$\widehat{[A]} \mapsto \widehat{A} \cap E.$$

Thus, the space E is the Stone space of $\mathcal{J}(X, \mu)/\mathcal{N} = \mathcal{J}_\mu(X)$.

Lemma 3.3.23. Let Z be the Stone space of the Jordan algebra $\mathcal{J}_\mu(X) = \mathcal{J}(X, \mu)/\mathcal{N}$. We claim that Z can be identified with the set of all ultrafilters on $\mathcal{J}(X, \mu)$ that contain no sets of μ-measure zero (or simply measure zero), or equivalently, the set of all ultrafilters on $\mathcal{J}(X, \mu)$ that contain all sets of measure one.

Proof. Since Z is the set of all ultrafilters on $\mathcal{J}_\mu(X)$, it suffices to show that every ultrafilter on $\mathcal{J}_\mu(X)$ corresponds to an ultrafilter on $\mathcal{J}(X,\mu)$ which contains all sets of μ-measure one and conversely. Assume that $\mathcal{U}$ is an ultrafilter on $\mathcal{J}_\mu(X)$. Set $\mathcal{V} = \{A : [A] \in \mathcal{U}\}$ (i.e., the family of all Jordan measurable sets that quotient out by sets of measure zero). Then clearly $\mathcal{V}$ is an ultrafilter on $\mathcal{J}(X,\mu)$, and it contains every A of measure one (because if A has measure one, then $[A]$ is the unit element on $\mathcal{J}_\mu(X)$ and therefore belongs to every ultrafilter on $\mathcal{J}_\mu(X)$).
Conversely, if $\mathcal{V}$ is an ultrafilter on $\mathcal{J}(X,\mu)$ which contains all sets of measure one, then, we claim that for every class $[A] \in \mathcal{J}_\mu(X)$, either all its elements or none of them belong to $\mathcal{V}$. Suppose that B is any other element of the same class, so $[A] = [B]$ and $A \sim B$, then $\mu(A \backslash B) = 0 = \mu(B \backslash A)$. For, if $\mu(A \backslash B) = 0$, so its complement $C = (A \backslash B)^c$ has measure one and therefore belongs to $\mathcal{V}$. Now, if $A \in \mathcal{V}$ then $A \cap C \in \mathcal{V}$ but $A \cap C \subseteq B$, so B is also in $\mathcal{V}$. In the same way (using $\mu(B \setminus A) = 0$), if $B \in \mathcal{V}$ then so is A. Set $\mathcal{U} = \{[A] : A \in \mathcal{V}\}$. Then this $\mathcal{U}$ is an ultrafilter on $\mathcal{J}_\mu(X)$. The claim follows. ∎

Now, we can work on the Stone space of $\mathcal{J}(X,\mu)$ by means of the Stone space of the Jordan algebra $\mathcal{J}_\mu(X)$.

Lemma 3.3.24. If X is a compact Hausdorff space and μ is a nonatomic Radon measure on X, then every ultrafilter on the Boolean algebra of Jordan measurable subsets $\mathcal{J}(X,\mu)$ of X converges in X.

Proof. Given an ultrafilter $\mathcal{U}$, and suppose it does not converge to any point of X. This means that for any $x \in X$, there exist an open (Jordan measurable) set U_x containing x such that $U_x \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, then this implies that $A = X \setminus U_x \in \mathcal{U}$. By compactness, there exists a finite number of open sets $U_{x_1}, U_{x_2}, .., U_{x_n}$ that cover X so that $\bigcap_{i=1}^n A_i = X \setminus \bigcup_{i=1}^n U_{x_i} = \emptyset$. Since $\mathcal{U}$ is a filter which is closed under finite intersections, we obtain that $\emptyset \in \mathcal{U}$, which is a contradiction.

∎

Note that on $\mathcal{J}(X,\mu)$ there are two types of utrafilters: principal ulftrafilters, $z \in Z$, which are generated by single elements $x \in X$ and are of the form $z = \{A \in \mathcal{J}(X,\mu) : x \in A\}$, and free (or nonprincipal) ultrafilters $z = \{A : \text{for some } A \in \mathcal{J}(X,\mu)\}$ for which $\bigcap z = \emptyset$.

The above lemma motivates us to introduce the following result.

Proposition 3.3.25. Let μ be a nonatomic Radon measure on compact Hausdorff space X and Z be the Stone space of the Boolean algebra $\mathcal{J}(X,\mu)$ of Jordan measurable sets in X. We claim that there is a function $f : Z \to X$ such that

(1) $f(\widehat{A}) = A$, for every $A \in \mathcal{J}(X,\mu)$, where $f(\widehat{A}) = \{f(z) : z \in \widehat{A}\}$ and $\widehat{A} = \{z \in Z : A \in z\}$ is the clopen subset of Z determined by A.

(2) $f(\hat{\mu}) = \mu$ (i.e. $\hat{\mu}(f^{-1}(A)) = \mu(A) \implies \hat{\mu}(\widehat{A}) = \mu(A)$),

(3) f is continuous, and

(4) f is surjective.

Proof. (1) Suppose that $\mathcal{A}_z = \{A : A \in \mathcal{J}(X,\mu), z \in \widehat{A}\}$ for every $z \in Z$. We claim that this family has the finite intersection property. Let $A_1, A_2, .., A_n \in \mathcal{A}_z$, so $z \in \bigcap_{i=1}^{n} \widehat{A_i}$ and hence $\widehat{A_1} \cap \widehat{A_2} \cap .. \cap \widehat{A_n} \neq \emptyset$. Since Z is compact and $\widehat{A_i}$ are closed, then $\bigcap_{i=1}^{n} \widehat{A_i} \neq \emptyset$. This implies that $\bigcap \mathcal{A}_z \neq \emptyset$.

Let $x \in \bigcap \mathcal{A}_z$ and $y \in X \setminus \{x\}$. Since X is Hausdorff, there exist two disjoint open sets U, V in X such that $x \in U$ and $y \in V$. Fix $A_0 \in \mathcal{A}_z$. Then $A_0 \setminus U \notin \mathcal{A}_z$ because $x \notin A_0 \setminus U$. So $z \notin \widehat{A_0 \setminus U}$ and $z \in \widehat{A_0 \cap U}$. But $\widehat{A_0 \cap U} \subset \widehat{A_0 \cap V^c} = \widehat{A_0 \setminus V}$ and $A_0 \setminus V \in \mathcal{A}_z$. Hence, $y \notin \bigcap \mathcal{A}_z$. Therefore, $\bigcap \mathcal{A}_z$ is a single point x.

Define a function $f : Z \longrightarrow X$ by the relation $f(z) \in \bigcap \mathcal{A}_z$, or equivalently, $\bigcap \mathcal{A}_z = \{f(z)\}$. From the preceding paragraph, we obtain that $f(\widehat{A}) = A$.

(2) Follows from (1).

(3) Given $z \in Z$. Let G be an open subset of X containing $f(z)$. Then $f(z) \in \bigcap \mathcal{A}_z \subseteq G$. Since X is a compact Radon measure space, by Lemma 3.1.2.3, there is an open Jordan μ-measurable subset of X such that H such that

$$\cap \mathcal{A}_z \subseteq H \subseteq G.$$

Now, if z' is any other point in $\widehat{H}$ with $H \in \mathcal{A}_{z'}$, then $f(z') \subseteq H \subseteq G$. Thus $\widehat{H}$ is an open neighborhood of z with $z \in \widehat{H} \subseteq f^{-1}(G)$ and so f is continuous.

(4) To show that f is surjective, since the collection of open Jordan measurable sets is a basis for the topology on X by Remark 3.1.2.4, for every $x \in X$ we can choose an open Jordan measurable A in $\mathcal{J}(X, \mu)$ such that there is $z \in \widehat{A} \subseteq Z$ and $f(z) = x$. The claim follows. ■

Lemma 3.3.26. Let (X, Σ, μ) be a measure space and let $(X, \hat{\Sigma}, \hat{\mu})$ be its completion.

(1) If μ is strictly positive, then every open set H in $\hat{\Sigma}$ is already in Σ.

(2) For every Jordan $\hat{\mu}$-measurable A subset of X, there is a Jordan μ-measurable B such that $B \subseteq A$ and $\hat{\mu}(A) = \mu(B)$.

Proof. (1) Let H be an open set in $\hat{\Sigma}$. By Lemma 1.2.3.20, there are $A, B \in \Sigma$ such that $A \subseteq H \subseteq B$ and $\mu(B \setminus A) = 0$. Therefore, $H = \mathrm{Int}(H) \subseteq \mathrm{Int}(B) \subseteq B$. Hence,

$$H \subseteq \mathrm{Int}(B) \quad \text{and} \quad \hat{\mu}(\mathrm{Int}(B) \setminus H) = 0 \tag{I}$$

On the other hand, by Proposition 1.2.3.21, one can find, $N \in \mathscr{N}$ such that $H = B \cup N$. Since μ is strictly positive and $\mu(N) = 0$, then $\mathrm{Int}(N) = \emptyset$. Now

$$\mathrm{Int}(B) = \mathrm{Int}(B) \cup \mathrm{Int}(N) \subseteq \mathrm{Int}(B \cup N) = \mathrm{Int}(H) = H.$$

This implies that

$$\mathrm{Int}(B) \subseteq H \quad \text{and} \quad \hat{\mu}(H \setminus \mathrm{Int}(B)) = 0 \tag{II}$$

From (I) and (II), we obtain that $H = \mathrm{Int}(B)$ which belongs to Σ.

(2) Let A be a Jordan $\hat{\mu}$-measurable set in X. Since $\mathrm{Int}(A)$ is open Jordan $\hat{\mu}$-measurable and $\hat{\mu}(A) = \hat{\mu}(\mathrm{Int}(A))$, by (1) we have that $\mathrm{Int}(A) \in \Sigma$ and $\hat{\mu}(\mathrm{Int}(A)) = \mu(\mathrm{Int}(A))$. Simply, let $B = \mathrm{Int}(A)$. Clearly B is Jordan μ-measurable. ■

Proposition 3.3.27. Let (X, Σ, μ) be a measure space and let $(X, \hat{\Sigma}, \hat{\mu})$ be its completion. Then the Jordan algebra of μ is the same as the Jordan algebra of $\hat{\mu}$. That is

$$\mathcal{J}_{\mu}(X) = \mathcal{J}_{\hat{\mu}}(X).$$

Proof. By Lemma 1.2.3.18, both μ and $\hat{\mu}$ agree on Σ (as $\hat{\mu}$ is an extension of μ). This implies that $\mathcal{J}_\mu(X) \subseteq \mathcal{J}_{\hat{\mu}}(X)$.

The other way round follows from Lemma 3.3.26. That is $\mathcal{J}_{\hat{\mu}}(X) \subseteq \mathcal{J}_\mu(X)$, and so $\mathcal{J}_\mu(X) = \mathcal{J}_{\hat{\mu}}(X)$. ■

Chapter 4

Uniform Regularity of Measures and Charges

In searching for a classification of charges along various cardinal versions of the Jordan algebra, in the way that we classify measures using the Lebesgue algebra, a surprise happens. The charges are provably not classifiable in a simple way (see [9]), but the uniformly regular ones are, and hence the interest in studying them. Interestingly, the uniformly regular measures give us the same outcome as the uniformly regular charges do.

This chapter is divided into five sections. Section 4.1 is dedicated to the definitions of uniform regularity of measures and charges, and their relations. We also show that uniform regularity is stronger than separability in this section. In Section 4.2 we study a various properties of uniformly regular measures and provide some results. In Section 4.3 we give a characterization of (nonatomic) uniformly regular Radon measures on compact Hausdorff spaces which classify this type of measures. In Section 4.4 we investigate uniform regular charges on Boolean algebras. In Section 4.5 we show that the Lebesgue measure on the unit interval is uniformly regular in the sense of topological spaces but not uniformly regular in the sense of Boolean algebras. In Section 4.6 we show that a countable product (or free product) of uniformly regular measures is uniformly regular.

4.1 Definitions and Relations

In this section we show the connection between uniformly regular measures and charges, in general, in the sense of topology and Boolean algebra.

Definition 4.1.1. Let $\mathcal{A}, \mathcal{B}$ be two families of μ-measurable sets. We say that $\mathcal{B}$ **approximates** $\mathcal{A}$ from below if for every $\epsilon > 0$ and every $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ with $B \subseteq A$ such that $\mu(A \setminus B) < \epsilon$.

Definition 4.1.2. Let $\mathcal{A}, \mathcal{B}$ be two families of μ-measurable sets. We say that $\mathcal{B}$ $\triangle$-**approximates** $\mathcal{A}$ if for every $\epsilon > 0$ and every $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $\mu(A \triangle B) < \epsilon$.

Definition 4.1.3. A measure μ on a topological space X is said to be **uniformly regular** if there is a countable family $\mathcal{G}$ of open subsets of X such that for every open set $U \subseteq X$ and every $\epsilon > 0$, there is $G \in \mathcal{G}$ with $G \subseteq U$ such that

$$\mu(U \setminus G) < \epsilon.$$

Notice that we can obtain uniform regularity of measures by approximating closed sets from above by a countable family of closed subsets of a space (using the complement).

Lemma 4.1.4. A measure μ on a compact X is uniformly regular if and only if there is a countable family $\mathcal{A}$ of open subsets of X such that $\mu(G \setminus H) = 0$ for every open set $G \subseteq X$ where $H = \bigcup\{A : A \in \mathcal{A}, A \subseteq G\}$.

Proof. Let $\mathcal{B}$ be a countable family of open subsets of X that makes μ uniformly regular measure. Given an open set U and $\epsilon > 0$, there is $B \in \mathcal{B}$ such that $B \subseteq U$ and $\mu(U \setminus B) < \epsilon$. So $\mu(\bigcup_{B \subseteq U} B(U)) = \mu(U)$ and we are done.

Conversely, given a countable family $\mathcal{A}$ as in (if case). Let $\mathcal{C} = \{\bigcup_{i=1}^{n} A_i : A_i \in \mathcal{A}\}$. Take any open set G, by assumption, there exists an increasing sequence $\{C_n : n < \omega\}$ in $\mathcal{C}$ such that $C_n \subseteq G$ for all n and $\mu(G) = \mu(\bigcup C_n)$, (this is possible because $\mathcal{C}$ is closed under finite unions). Let $C = \bigcup C_n$. Then $\mu(C) = \mu(\bigcup C_n) \leq \sup \mu(C_n)$, but $\sup \mu(C_n) \leq \mu(G)$. So

$$\mu(G) \leq \sup\{\mu(C_n), C_n \subseteq G\} \leq \mu(G).$$

This completes the proof.

■

Note that the (only if) case is the definition of uniformly regular given by Fremlin in [22].

We defined uniformly regular charge on Boolean algebras earlier, see Definition 1.2.3.56.

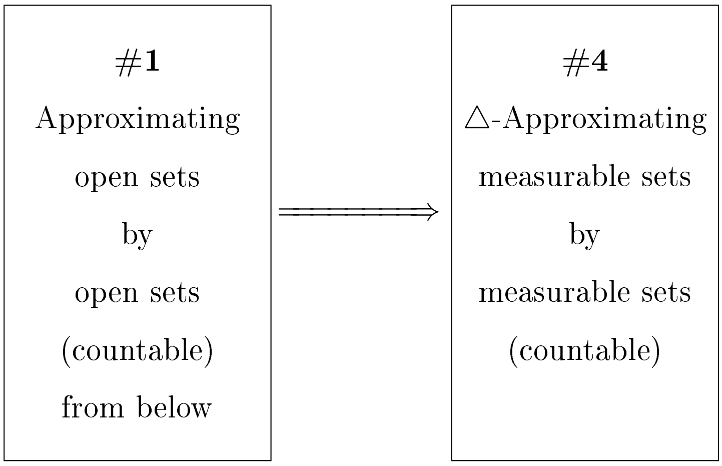
The following diagram demonstrates the relation between uniform regularity in the sense of Boolean algebra and topology.

#1 Approximating open sets by open sets (countable) from below	$\Longleftarrow$	**#2** Approximating measurable sets by open sets (countable) from below	$\Longrightarrow$	**#3** Approximating measurable sets by measurable sets (countable) from below

We have seen that it is important to give some details on the definitions stated in the diagram. Let us start from **#1**, this is our definition of uniformly regular measures (not charges) on topological spaces. The same definition cannot be given for charges because all open sets need not be chargeable. Even if we assumed such a condition, we would have the relation stated in the diagram. Regarding **#2**, we mention it for the sake of seeing the relation between **#1** and **#3**. This definition may not be reasonable as we cannot have such property in general. That is, given a measure μ on a space X and a countable family $\mathcal{G}$ of open sets in X, then for every $\epsilon > 0$ and every measurable set E of a measurable space (X, μ), there is an open set $G \in \mathcal{G}$ such that $G \subseteq E$ and $\mu(E \setminus G) < \epsilon$. **#3** is our definition of uniformly regular charges on Boolean algebras, but it is also true for measures. We also provide some connections between uniformly regular measures and charges, for instance, see Proposition 4.3.2 and Proposition 4.2.12.

But the situation is different when we compare "Approximating open sets by open sets (from a countable family)" with "$\triangle$-Approximating measurable sets by measurable sets (from a countable family)" as shown in the following:

Lemma 4.1.5. For any Radon measure μ on a topological space X, uniform regularity of μ implies separability. That is,



Proof. Given $\epsilon > 0$ and let $\mathcal{G}$ be a countable family of open subsets of X that makes μ uniformly regular. Clearly $\mathcal{G}$ is a countable family of measurable sets as every open is measurable. It remains to show that for every measurable set E in X, there is $G \in \mathcal{G}$ such that

$$\mu(E \triangle G) < \epsilon.$$

Let E be a measurable set in X. By regularity of μ, there is an open set U with $E \subseteq U$ such that

$$\mu(U \setminus E) < \epsilon/2.$$

By assumption, for such open set U, there is $G \in \mathcal{G}$ with $G \subseteq U$ such that

$$\mu(U \setminus G) < \epsilon/2.$$

Now, we have

$$\begin{aligned} \mu(E \triangle G) &= \mu(E \setminus G) + \mu(G \setminus E) \\ &\leq \mu(U \setminus G) + \mu(U \setminus E) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This shows that μ is separable. ■

4.2 Uniformly Regular Measures

In this section we study several properties of uniformly regular measures on topological spaces. We start with the following two known results concerning uniform regularity of measures and then give our results:

Theorem 4.2.1. [22, Lemma 533G(a)] For a Radon (probability) measure μ on a compact Hausdorff space X, the following are equivalent:

(1) μ is uniformly regular;

(2) there is a compact metric space M and a continuous (surjective) function $f : X \to M$ such that $\mu(f^{-1}(f(K))) = \mu(K)$ for every compact $K \subseteq X$;

(3) there is a countable family $\mathcal{A}$ of zero subsets of X such that for every $\epsilon > 0$ and every open set U of X, there is $A \in \mathcal{A}$ such that $A \subseteq U$ and $\mu(U \setminus A) < \epsilon$;

(4) there is a countable family $\mathcal{B}$ of cozero subsets of X such that for every $\epsilon > 0$ and every open set U of X, there is $B \in \mathcal{B}$ such that $B \subseteq U$ and $\mu(U \setminus B) < \epsilon$.

Notice that the above result is also mentioned in [2].

Uniformly regular measure is also called **strongly countably determined**, (see [41]).

Theorem 4.2.2. [36, Theorem 1.7] The following are equivalent for a Radon (probability) measure μ on a compact Hausdorff space X:

(1) μ is uniformly regular on X;

(2) there is a countable family $\mathcal{U}$ of open F_σ Jordan μ-measurable subsets of X such that for every $\epsilon > 0$ and every compact subset K of X, there is $U \in \mathcal{U}$ such that

$$K \subseteq U \text{ and } \mu(U \setminus K) < \epsilon;$$

(3) there is a compact metric space M and continuous function f from X onto M such that if we set $\lambda = f(\mu)$, then $f(A)$ is Jordan λ-measurable for every Jordan μ-measurable subset A of X and

$$\mu(K) = \lambda(f(K)) \text{ for every compact subset } K \text{ of } X;$$

(4) there is a countable family $\mathcal{F}$ of continuous functions on X such that for every $\epsilon > 0$ and every continuous (Riemann μ-integrable) function f on X, there are $g, h \in \mathcal{F}$ such that

$$g \leq f \leq h \text{ and } \int_x (h - g) d\mu < \epsilon$$

(5) there is a compact metric space M and strictly positive measure λ on M such that the Jordan algebra $\mathcal{J}_\nu(Y)$ is isomorphic to the Jordan algebra $\mathcal{J}_\lambda(M)$, where Y is support of μ and ν its restriction on Y.

Remark 4.2.3. In this remark recall some simple results on uniformly regular measures which can be seen also in [16] on page 2065.

(1) Every uniformly regular measure is separable and has a separable support.

(2) Every Radon measure on a compact metrizable space X is uniformly regular.

(3) If λ is the Lebesgue measure on $[0,1]$ then the corresponding measure $\hat{\lambda}$ on the Stone space of the measure algebra of λ is separable but not uniformly regular.

(4) No measure on $[0,1]^{\mathfrak{c}}$ can be uniformly regular. This is Example 5.5 in [2].

Proposition 4.2.4. Every Borel measure on a compact metric space is uniformly regular.

Proof. Let μ be a Borel measure on a compact metric space X. By Lemma 1.2.1.15 (2), X has a countable base $\{U_n : n \in \omega\}$ of open sets. Lemma 1.2.1.3 indicates that every open set is a union of countably many closed or compact sets. For every $n \in \omega$, consider a countable collection $\{K_n^m, m \in \omega\}$ of compact subsets of U_n such that $\mu(U_n \setminus K_n^m) < \frac{1}{m.2^n}$. Let U be an open set and $\epsilon > 0$. Then $U = \bigcup_{n\in N} U_n$ for some $N \subset \omega$. There is a finite $M \subseteq$ such that $\mu(U \setminus \bigcup_{n\in N\cap M} U_n) < \epsilon$. For $n \in M$, we take m such that $\frac{1}{m} < \epsilon$. Thus $K := \bigcup_{n=1}^{M} K_n^m$, which is compact G_δ and hence $\mu(U \setminus K) < \epsilon$. The claim follows. ∎

Example 4.2.5. Consider the Lebesgue measure λ on the algebra generated by open intervals in $\mathbb{I} = [0, 1]$. The Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ is a subalgebra of the random algebra (=Boolean algebra of all Lebesgue measurable modulo measure zero) $\mathfrak{R}$. Moving to the Stone space $Y = \text{Stone}(\mathfrak{R})$ of the random algebra $\mathfrak{R}$, we obtain a topological space on which the Lebesgue measure induces a separable measure but not uniformly regular (see Remark 4.2.3 (3)). While the induced measure $\hat{\lambda}$ on the Stone space $Z = \text{Stone}(\mathcal{J}_\lambda(\mathbb{I}))$ of the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ happens to be uniformly regular. By Proposition 3.3.25, $f : Z \to \mathbb{I}$ is a continuous surjective function such that $f(\hat{\lambda}) = \lambda$. From Lemma 2.2.3, we can deduce $\hat{\lambda}(f^{-1}(f(F) \setminus F) = 0$ for every compact set $F \subseteq Z$ (for more detail, see Lemma 2 in [33]), and thus $\hat{\lambda}$ is uniformly regular on Z because $\mathbb{I}$ is metrizable (see Theorem 4.2.1).

Proposition 4.2.6. Let μ be a uniformly regular measure on a topological space X and Y a subspace of X. The restricted measure $\mu_0 = \mu|_Y$ to Y is also uniformly regular.

Proof. Let $\mathcal{G}$ be a countable family of open subsets of X that makes μ uniformly regular. Set $\mathcal{H} = \{H : H = G \cap Y, G \in \mathcal{G}\}$, then $\mathcal{H}$ is a countable collection of open sets in Y. We now show that $\mathcal{H}$ is uniformly μ_0-dense in Y. Given an open set U in Y and $\epsilon > 0$, there is an open set V in X such that $U = V \cap Y$. By assumption, for that V there is $G \in \mathcal{G}$ with $G \subseteq V$ such that

$$\mu(V \setminus G) < \epsilon.$$

But

$$\mu_0(U \setminus H) = \mu(V \cap Y \setminus G \cap Y) \leq \mu(V \setminus G) < \epsilon.$$

We are done. ■

The other direction of the above proposition is not true in general. That is, if we have a uniformly regular measure μ on a subspace Y of a given space X, even if such Y is open or closed as a subspace, then μ cannot always be extended to a uniformly regular measure $\hat{\mu}$ on the whole X, as shown in the following example:

Example 4.2.7. Given the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ and let Y be its Stone space. By Example 4.2.5, the induced measure $\hat{\lambda}$ on Y is uniformly regular. Since Y has weight

$\mathfrak{c}$ as $|\mathcal{J}_\lambda(\mathbb{I})| = \mathfrak{c}$, by Lemma 3.3.7, Y is embeddable into $[0,1]^{\mathfrak{c}}$, by Theorem 1.2.1.19. Hence, $\hat{\lambda}$ can be extended to a measure $\bar{\lambda}$, say, on the whole $[0,1]^{\mathfrak{c}}$, by Proposition 1.2.3.35. But, by Remark 4.2.3 (4), no measure on $[0,1]^{\mathfrak{c}}$ can be uniformly regular. Thus, $\hat{\lambda}$ cannot be extended to a uniformly regular measure on $[0,1]^{\mathfrak{c}}$.

The image of a uniformly regular measure is preserved under an open continuous surjective function.

Proposition 4.2.8. Let X, Y be two topological spaces and $f : X \longrightarrow Y$ a continuous open surjection. If μ is a uniformly regular measure on X, then $f(\mu)$ is also a uniformly regular measure on Y. Furthermore, the converse is true whenever f is in addition a one-to-one function.

Proof. Let $\nu = \mu f^{-1}$ and let $\mathcal{G}$ be a countable family of open subsets of X that makes μ uniformly regular. If $\mathcal{H} = \{H : H = f(G), G \in \mathcal{G}\}$, then $\mathcal{H}$ is a countable family of open subsets of Y as f is an open function. We need to show that for every open set U in Y and every $\epsilon > 0$, there is $H \in \mathcal{H}$ with $H \subseteq U$ such that

$$\nu(U \setminus H) < \epsilon.$$

Given an open set U in Y, so $f^{-1}(U)$ is open in X. Since μ is uniformly regular, then there is $G \in \mathcal{G}$ with $G \subseteq f^{-1}(U)$ such that

$$\mu(f^{-1}(U) \setminus G) < \epsilon.$$

Now, we have $G \subseteq f^{-1}(U)$ which implies that $f(G) \subseteq U$. But $G \subseteq f^{-1}(f(G)) \subseteq f^{-1}(U)$ and so

$$\mu(f^{-1}(U) \setminus f^{-1}(f(G))) < \epsilon.$$

Equivalently,

$$\nu(U \setminus f(G)) = \nu(U \setminus H) < \epsilon.$$

On the other hand, set $\nu = \mu f^{-1}$. Suppose that $\mathcal{H}$ is a countable family of open subsets of Y that makes ν uniformly regular and f is one-to-one. We need to prove that μ is uniformly regular on X. Now, $f^{-1}(\mathcal{H}) = \{f^{-1}(H) : H \in \mathcal{H}\}$ is a countable set of open subsets of X. Let $\epsilon > 0$ and U be an open set in X. By assumption $f(U)$ is open in Y. Since ν is uniformly regular on Y, there is $H \in \mathcal{H}$ such that

$$H \subseteq f(U) \text{ and } \nu(f(U) \setminus H) < \epsilon.$$

Equivalently,

$$\mu f^{-1}(f(U) \setminus H) = \mu(f^{-1}(f(U)) \setminus f^{-1}(H)) < \epsilon.$$

But $f^{-1}(f(U)) = U$ as f is a one-to-one function and $f^{-1}(H) \in f^{-1}(\mathcal{H})$ is open in X. Hence,

$$\mu(U \setminus f^{-1}(H)) < \epsilon.$$

This completes the proof. ■

From the above proposition, we conclude the following:

Corollary 4.2.9. Let X, Y be two compact Hausdorff spaces and $f : X \longrightarrow Y$ a homeomorphism. Then μ is a uniformly regular measure on X if and only if $\nu = f(\mu)$ is a uniformly regular measure on Y.

Lemma 4.2.10. If μ is a Radon measure on a compact Hausdorff zero-dimensional space X, then we have the following:

(1) Every open subset of Z can be approximated by a clopen set from below. That is, for every $\epsilon > 0$ and every open set U in Z, there is a clopen set C with $C \subseteq U$ such that

$$\mu(U \setminus C) < \epsilon.$$

(2) Every μ-measurable (Borel) subset of Z can be $\triangle$-approximated by a clopen set. That is, for every $\epsilon > 0$ and every μ-measurable (Borel) set B in Z, there is a clopen set C such that

$$\mu(U \triangle C) < \epsilon.$$

Proof. (1) Let $\epsilon > 0$ and U be an open set in Z. By regularity of μ, there is a closed set F with $F \subseteq U$ such that

$$\mu(U \setminus F) < \epsilon.$$

But U can be expressed as union of clopen subsets C_i of X, $U = \bigcup_{i \in I} C_i$ for some index set I. Since C_i are basic open, for each $x \in F$, there exists C_i such that $x \in C_i \subseteq U$. Therefore $F \subseteq \bigcup_{x \in F} C_i(x) \subseteq U$. So $\bigcup_{x \in F} C_i(x)$ forms a cover for F. By compactness, there is a finite subcover $\{C_i(x) : i = 1, \ldots, n\}$ of $\{C_i(x) : x \in F\}$ such that $F \subseteq \bigcup_{i=1}^{n} C_i(x) \subseteq U$. Set $C = \bigcup_{i=1}^{n} C_i(x)$, then C is clopen. Hence $C \subseteq U$ and $\mu(U \setminus C) < \epsilon$.

(2) Given a measurable (Borel) subset B of Z and $\epsilon > 0$. By regularity of μ, there is an open set U with $B \subseteq U$ such that

$$\mu(U \setminus B) < \epsilon/2,$$

.

By part (1) for open set U, one can find a clopen set C with with $C \subseteq U$ such that

$$\mu(U \setminus C) < \epsilon/2.$$

Now, we have

$$\begin{aligned} \mu(B \triangle C) &= \mu(B \setminus C) + \mu(C \setminus B) \\ &\leq \mu(U \setminus C) + \mu(U \setminus B) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

This completes the proof. ■

Remark 4.2.11. Note that the above lemma is true for any compact Hausdorff space that carries a Radon measure when replacing clopen sets by basic open sets.

In the following result, we show how the properties of a charge on a Boolean algebra $\mathfrak{A}$ transfer to its induced measure on the Stone space of $\mathfrak{A}$:

Proposition 4.2.12. Let μ be a charge on a Boolean algebra $\mathfrak{A}$ and ν be the induced Radon measure on the Stone space Z of $\mathfrak{A}$. We have the following:

(1) If ν is nonatomic, then μ is nonatomic.

(2) μ is continuous if and only if ν is continuous.

(3) If μ is Darboux, then ν is Darboux.

(4) μ is uniformly regular if and only if ν is uniformly regular.

(5) μ is separable if and only if ν is separable.

(6) μ is strictly positive if and only if ν is strictly positive.

Proof. (1) and (3) follow directly from Theorem 2.1.22 and Theorem 2.1.20, respectively, and (6) is straightforward.

(2) Given $\epsilon > 0$. Assume that the charge μ is continuous. Then there is a finite partition $\mathscr{P} = \{a_1, a_2, \ldots, a_n\}$ of the unity $\mathbf{1}_{\mathfrak{A}}$ such that $\mu(a_i) < \epsilon$ for $i = 1, 2, \ldots, n$. By the Stone isomorphism, for each i, a_i corresponds to $\widehat{a}_i$ and $\mu(a_i) = \nu(\widehat{a}_i)$. So $\widehat{\mathscr{P}} = \{\widehat{a}_1, \widehat{a}_2, \ldots, \widehat{a}_n\}$ with $Z = \bigcup_{i=1}^{n} \widehat{a}_i$ and $\nu(\widehat{a}_i) < \epsilon$. Thus, $\widehat{\mathscr{P}}$ is the required partition. Hence, ν is continuous.

Conversely, let ν be a continuous measure and m a positive integer such that $1/m < \epsilon$. Then there is a finite partition $\mathscr{E} = \{E_1, E_2, \ldots, E_n\}$ of Z such that

$$0 < \nu(E_i) < \epsilon \cdot \left[\frac{m \cdot \nu(Z)}{1 + m \cdot \nu(Z)}\right],$$

for $i = 1, 2, \ldots, n$. Since Z is compact zero-dimensional, by Lemma 4.2.10 (2), for each i, there is a clopen set C_i such that

$$\nu(E_i \triangle C_i) < \frac{\nu(E_i)}{m \cdot \nu(Z)}.$$

But $C_i \subseteq E_i \cup (E_i \triangle C_i)$, so

$$\mu(C_i) = \nu(C_i) \leq \nu(E_i) \cdot \left[1 + \frac{1}{m \cdot \nu(Z)}\right] = \nu(E_i) \cdot \left[\frac{1 + m \cdot \nu(Z)}{m \cdot \nu(Z)}\right] < \epsilon.$$

Moreover, we have

$$\begin{aligned}
\mu\big(Z \setminus \bigcup_{i=1}^{n} C_i\big) &= \nu\big(Z \setminus \bigcup_{i=1}^{n} C_i\big) = \nu\big(\bigcup_{i=1}^{n} E_i \setminus \bigcup_{i=1}^{n} C_i\big) \\
&\leq \nu\Big[\bigcup_{i=1}^{n} (E_i \setminus C_i)\Big] = \sum_{i=1}^{n} \nu(E_i \setminus C_i) \\
&\leq \sum_{i=1}^{n} \nu(E_i \triangle C_i) < 1/m < \epsilon, \qquad \text{for } i = 1, 2, \ldots, n.
\end{aligned}$$

Setting, $\widehat{a}_1 = C_1$, $\widehat{a}_i = C_i \setminus (C_1 \cup C_2 \cup \cdots \cup C_{i-1})$ $i = 2, \ldots, n+1$, where $C_{n+1} = Z \setminus \bigcup_{i=1}^{n} C_i$. This is possible by the properties of Stone representation. So $\widehat{a}_i$ is clopen and $\nu(\widehat{a}_i) = \mu(a_i) < \epsilon$ for every $i = 1, 2, \ldots, n+1$ and $Z = \bigcup_{i=1}^{n+1} \widehat{a}_i$. Thus, by Stone isomorphism, $\widehat{a}_i \longleftrightarrow a_i$ and so the set $\{a_i : i = 1, 2, \ldots, n+1\}$ forms a partition of $\mathbf{1}_{\mathfrak{A}}$ with $\mu(a_i) < \epsilon$. Hence, μ is continuous.

(4) Let $\mathfrak{B}$ be a countable collection of elements of $\mathfrak{A}$ that makes μ uniformly regular on $\mathfrak{A}$. For $b \in \mathfrak{B}$, b corresponds to the clopen set $\hat{b}$ in Z. Let $\mathcal{C} = \{C : C = \hat{b}, b \in \mathfrak{B}\}$, so $\mathcal{C}$ is a countable collection of compact G_δ sets in Z because every closed set is compact and every open set is G_δ. We now show that $\mathcal{C}$ is uniformly ν-dense in Z. Let $\epsilon > 0$ and let U be an open set in Z. By Lemma 4.2.10, we can approximate U by a clopen set A in Z, that is

$$\nu(U \setminus A) < \epsilon/2.$$

But A corresponds to a for some $a \in \mathfrak{A}$ and μ is uniformly regular on $\mathfrak{A}$, so, there is $b \in \mathfrak{B}$ such that

$$b \leq a \text{ and } \mu(a \setminus b) < \epsilon/2.$$

This implies that

$$\hat{b} = C \subseteq A \text{ and } \nu(A \setminus C) < \epsilon/2,$$

for some C. Now, we have

$$\nu(U \setminus C) < \nu(U \setminus A) + \nu(A \setminus C) < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus, ν is uniformly regular.

Conversely, assume that ν is uniformly regular. Let $\{G_n, n < \omega\}$ be a countable family of open subsets of Z. Using Lemma 4.2.10, we let $\{C_m^n, m < \omega\}$ be a countable family of clopen sets in Z such that

$$\nu(G_n \setminus C_m^n) < 1/m.$$

We now have a countable family $\mathcal{C} = \{C_m^n : n < \omega, m < \omega\}$ of clopen subsets of Z which corresponds to a countable subset $\mathfrak{B} \subset \mathfrak{A}$. It remains to show that this family is uniformly μ-dense in $\mathfrak{A}$. Let $a \in \mathfrak{A}$. Then $\hat{a}$ is open in Z and $\mu(a) = \nu(\hat{a})$. We can choose $\epsilon > 0$ and n, m such that

$$\nu(\hat{a} \setminus C_m^n) < \epsilon.$$

But $\nu(\hat{a} \setminus C_m^n) = \mu(a \setminus b)$ for some $b \in \mathfrak{B}$.
Thus,

$$\mu(a \setminus b) < \epsilon.$$

This proves that μ is uniformly regular.

(5) Suppose that μ is separable on $\mathfrak{A}$. Then $\mathfrak{A}$ has a countable μ-dense $\mathfrak{B}$. We know that each $b \in \mathfrak{B}$ corresponds to a clopen set $\hat{b} = K$ in Z. Let $\mathcal{K} = \{K : K = \hat{b}, b \in \mathfrak{B}\}$, so $\mathcal{K}$ is a countable collection of (Borel) subsets of Z. Let B be a Borel set in Z, by Lemma 4.2.10, there is a clopen set C such that

$$\nu(B \triangle C) < \epsilon/2.$$

By Stone isomorphism, each clopen set C corresponds to $a \in \mathfrak{A}$. By the assumption, for every $\hat{a} = C$ there is $\hat{b} = K \in \mathfrak{B}$ such that

$$\mu(a \triangle b) = \nu(C \triangle K) < \epsilon/2.$$

Therefore,

$$\begin{aligned} \mu(B \triangle K) &\leq \mu(B \triangle C) + \mu(C \triangle K) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

This shows that ν is separable.

Conversely, assume that ν is separable. Let $\{B_n, n < \omega\}$ be a countable family of ν-measurable (Borel) subsets of Z. Using Lemma 4.2.10 and applying the same steps as in (4), one can prove that the charge μ is separable.

■

Remark 4.2.13. We shall further explain the above proposition in the following cases:

(1) The converse of (1) is true in a very special case as it can be seen in Lemma 4.3.5. We do not know if it is true in general.

(2) The converse of (3) is false in general. Consider the product measure μ on 2^ω. It is known that 2^ω is isomorphic to the Stone space of the Cantor algebra $\mathfrak{A}_c$ and μ is isomorphic to the Lebesgue measure λ on $\mathbb{I} = [0, 1]$. Clearly μ is Darboux, but the restricted measure $\mu|_{\mathfrak{A}_c}$ to $\mathfrak{A}_c$ is not because $\mathfrak{A}_c$ is isomorphic to the algebra $\mathcal{A}$ generated by $\{[a, b) : 0 \leq a < b < 1, a, b \in \mathbb{Q} \cap \mathbb{I}\}$ and Example 2.1.13 showed that $\lambda_{\mathcal{A}}$ is not Darboux.

(3) The converses of (1) and (3) will be true if we have one of the following hypotheses:

   (i) If μ is a charge on a Boolean algebra $\mathfrak{A}$ and ν its induced charge (not measure) on the Stone space Z of $\mathfrak{A}$, (see Theorem 2.3.1).

   (ii) If μ is a measure on a complete Boolean σ-algebra $\mathfrak{A}$ and ν its induced measure on the Stone space Z of $\mathfrak{A}$, (see Theorem 321J, [20]).

From Lemma 4.2.12 and Lemma 4.2.10, we conclude the following corollary:

Corollary 4.2.14. For a Radon measure μ on a compact zero-dimensional Hausdorff space X, we have

(1) μ is uniformly regular if and only if there is a countable family $\mathcal{K}$ of clopen sets in X such that for every $\epsilon > 0$ and every clopen set C in X, there is $K \in \mathcal{K}$ such that $K \subseteq C$ and $\mu(C \setminus K) < \epsilon$.

(2) μ is separable if and only if there is a countable family $\mathcal{K}$ of clopen sets in X such that for every $\epsilon > 0$ and every clopen set C in X, there is $K \in \mathcal{K}$ such that $\mu(C \triangle K) < \epsilon$.

4.3 Classification of Uniformly Regular Measures

In this section we give a characterization of (nonatomic) uniformly regular Radon measures on compact Hausdorff spaces which classify this type of measures. We begin by establishing some results that assist us to prove our characterization.

Lemma 4.3.1. [26, Proposition 2.4] Let X be a compact Hausdorff space and μ be a Radon probability measure on X. Then a base $\mathfrak{B}$ for the topology consisting of open Jordan μ-measurable subsets of X, that is closed under finite unions, is a set of generators for $\mathcal{J}(X, \mu)$.

Proposition 4.3.2. Let μ be a nonatomic uniformly regular Radon measure on a compact Hausdorff space X.

(1) There exists a countable family $\mathcal{U}$ of open (F_σ) Jordan measurable subsets of X which is uniformly μ-dense in X.

(2) The Boolean algebra $\mathcal{J}_0$ generated by $\mathcal{U}$ is a set of generators for $\mathcal{J}(X, \mu)$.

(3) The quotient algebra of $\mathcal{J}_0$ modulo null sets (in $\mathcal{J}_0$) is a set of generators for $\mathcal{J}_\nu(Y)$, where $Y = \operatorname{supp}(\mu)$ and ν the restriction of μ to Y.

(4) Both $\mathcal{J}(X, \mu)$ and $\mathcal{J}_\nu(Y)$ carry uniformly regular measure.

Proof. (1) Let $\mathcal{H} = \{H_n : n < \omega\}$ be a countable family of F_σ open subsets of X that makes μ uniformly regular. By assumption, for a given $\epsilon > 0$ and every compact set K in X, there is $H_n \in \mathcal{H}$ with $K \subseteq H_n$ such that

$$\mu(H_n \setminus K) < \epsilon.$$

Since X is normal and the collection of open Jordan μ-measurable sets forms a basis for X (by Remark 3.1.2.4), for each open set H_n containing a compact or closed set K, there is an open Jordan measurable set U_n such that

$$K \subseteq U_n \subseteq Cl(U_n) \subseteq H_n \text{ and } \mu(U_n \setminus K) < \epsilon. \tag{$\star$}$$

Obviously, the family $\mathcal{U}$ of all such open (F_σ) Jordan measurable sets U_n (as $\bigcup K = U_n$) that satisfies the above inequality $(\star)$ is uniformly μ-dense.

(2) Assume that $\mathcal{J}_0$ is the algebra generated by $\mathcal{U}$. Then $\mathcal{J}_0$ is a subalgebra of $\mathcal{J}(X,\mu)$. We now show that $\mathcal{J}_0$ is a set of generators for $\mathcal{J}(X,\mu)$. Let $\epsilon > 0$ and $A \in \mathcal{J}(X,\mu)$. By assumption (1), for each compact set, say, $\mathrm{Cl}(A)$, there is a $U \in \mathcal{J}_0$ with $A \subseteq \mathrm{Cl}(A) \subseteq U$ such that

$$\mu(U \setminus A) = \mu(U \setminus \mathrm{Cl}(A)) < \epsilon/2. \tag{$\dagger$}$$

On the other hand, in the same way as above and using the complement of (1) (because uniform regularity is self dual), for any open set, say $\mathrm{Int}(A)$, there is a Baire Jordan measurable $V \in \mathcal{J}_0$ with $V \subseteq \mathrm{Int}(A) \subseteq A$ such that

$$\mu(\mathrm{Int}(A) \setminus V) = \mu(A \setminus V) < \epsilon/2. \tag{$\ddagger$}$$

From (†) and (‡), we conclude that for an arbitrary $A \in \mathcal{J}(X,\mu)$, there exist Baire Jordan measurable sets $U, V \in \mathcal{J}_0$ with $V \subseteq A \subseteq U$ such that

$$\mu(U \setminus V) < \epsilon.$$

Hence, $\mathcal{J}_0$ is a set of generators for $\mathcal{J}(X,\mu)$.

(3) Let $\mathcal{J}'$ be the quotient algebra of $\mathcal{J}_0$ modulo the ideal of null sets in $\mathcal{J}_0$. Since all nonempty members of $\mathcal{J}'$ are classes of positive measure, $\mathcal{J}' \subseteq \mathcal{J}_\nu(Y)$. In the same way above and using Remark 3.3.16, we can prove that $\mathcal{J}'$ is a set of generators for $\mathcal{J}_\nu(Y)$. This is possible because the only members of positive measure in $\mathcal{J}(X,\mu)$ can lie between two members of $\mathcal{J}'$. This completes the proof.

(4) Since $|\mathcal{J}_0| = \aleph_0$ (because $\mathcal{U}$ is countable) and $|\mathcal{J}'| = \aleph_0$, the uniform regularity of charges on $\mathcal{J}(X,\mu)$ and $\mathcal{J}_\mu(X)$ follows from (2) and (3). ■

From Proposition 4.3.2, Lemma 2.1.7 and Theorem 2.1.22 we obtain the following:

Remark 4.3.3. For any strictly positive nonatomic uniformly regular measure on a compact Hausdorff space, there is a countable atomless algebra which is a set of generators for its Jordan algebra.

Also, notice that a measure μ on a topological space and its support have the same Jordan algebra if $\mathrm{supp}(\mu)$ has full measure, i.e. $\mu(\mathrm{supp}(\mu)) = 1$.

Lemma 4.3.4. [10, Theorem 4.9] Let μ be a strictly positive nonatomic charge on the Cantor algebra $\mathfrak{A}_c$. Then the Jordan extension algebra $J_\mu(\mathfrak{A}_c)$ (or equivalently, the Jordan algebra $\mathcal{J}_\mu(2^\omega)$) is isomorphic to the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$.

Lemma 4.3.5. Let μ be a strictly positive charge on a countable Boolean algebra $\mathfrak{A}$ and ν the induced measure on its Stone space Z. Then μ is nonatmoic if and only if ν is nonatomic.

Proof. Suppose that μ is nonatomic on $\mathfrak{A}$. By Remark 2.1.8, $\mathfrak{A}$ is atomless, so now $\mathfrak{A}$ is both countable and atomless. Hence it is isomorphic to the Cantor algebra. By Lemma 2.1.15, μ is continuous. Therefore its extension ν to Z is continuous, by Proposition 4.2.12 (2) and so ν is nonatomic, because the continuity and nonatomicity for measures are equivalent, (see Theorem 5.1.6, [43]).
Conversely, already proved in Proposition 4.2.12 (1). ■

The following result classify the class of (strictly positive nonatomic) uniformly regular Radon measures on compact spaces:

Theorem 4.3.6. Let μ be a strictly positive nonatomic uniformly regular Radon measure on a compact Hausdorff space X. Then the Jordan algebra of μ is isomorphic to the Jordan algebra of λ, the Lebesgue measure on $[0, 1]$. That is

$$\mathcal{J}_\mu(X) \cong \mathcal{J}_\lambda([0, 1]).$$

Proof. This result was proved by Mercourakis in [36, Remark 1.10] but we prove it in a different way. Let $\mathfrak{B} = \mathcal{J}'$, the quotient algebra constructed in Proposition 4.3.2 (3). By Remark 4.3.3, $\mathfrak{B}$ is countable and atomless. Obviously, the induced charge μ_0 on $\mathfrak{B}$ is strictly positive and then it is nonatomic by Remark 2.1.8. Let $Y = \text{Stone}(\mathfrak{B})$. By Theorem 1.2.2.34, Y is a compact metrizable space and by Lemma 4.3.5, the induced measure $\hat{\mu}$ on Y is nonatomic. It is known that $(Y, \hat{\mu})$ is isomorphic to the product measure space $(2^\omega, \lambda)$ and $\mathfrak{B} \cong \text{Clop}(Y) \cong \mathfrak{A}_c$ (see § 1.2). But $\mathfrak{B}$ is (isomorphic to) a set of generators for $\mathcal{J}_\mu(X)$ by Proposition 4.3.2. On the other hand, $\mathfrak{A}_c$ is a basis for Y and all its elements are clopen and thus open Jordan measurable sets. So, by Lemma 4.3.1, $\mathfrak{A}_c$ is a set of generators for $\mathcal{J}_\lambda(2^\omega)$. Therefore, by Proposition 3.3.20, $\mathcal{J}_\mu(X) \cong \mathcal{J}_\lambda(2^\omega)$ (as $\mathfrak{B} \cong \mathfrak{A}_c$) and by Lemma 4.3.4, $\mathcal{J}_\lambda(2^\omega) \cong \mathcal{J}_\lambda(\mathbb{I})$. Hence, $\mathcal{J}_\mu(X) \cong \mathcal{J}_\lambda(\mathbb{I})$. ■

From the above theorem we deduce the following corollary:

Corollary 4.3.7. Any two strictly positive nonatomic uniformly regular Radon measures on compact Hausdorff spaces have isomorphic Jordan algebras.

Proposition 4.3.8. Let μ be a strictly positive nonatomic measure on a compact metric space X. Then the Jordan algebra of μ is isomorphic to the Jordan algebra of λ, the Lebesgue measure on $[0,1]$. That is

$$\mathcal{J}_\mu(X) \cong \mathcal{J}_\lambda([0,1]).$$

Proof. Follows from Proposition 4.2.4 and Theorem 4.3.6. ∎

Notice that for a topological space carrying a strictly positive uniformly regular measure puts a limit on its weight, as shown in the following:

Proposition 4.3.9. Let μ be a strictly positive nonatomic Radon measure on a compact Hausdorff space X. If μ is uniformly regular, then the weight of X is at most $\mathfrak{c}$.

Proof. By Remark 3.1.2.4, the family of open Jordan μ-measurable subsets of X is a base $\mathscr{B}$ for the topology on X because X is compact Hausdorff and μ is Radon. Since μ is strictly positive, all nonempty members of $\mathscr{B}$ are of positive measure and so $\mathscr{B} \subseteq \mathcal{J}_\mu(X)$. By the uniform regularity of μ, $\mathcal{J}_\mu(X) \cong \mathcal{J}_\lambda([0,1])$ and the size of $\mathcal{J}_\lambda([0,1])$ is exactly $\mathfrak{c}$, (see Lemma 3.3.7). Therefore

$$|\mathscr{B}| \leq |\mathcal{J}_\mu(X)| = \mathfrak{c}$$

Thus, the weight of X cannot be greater than $\mathfrak{c}$. ∎

The following are some consequences of Theorem 4.3.6:

Proposition 4.3.10. Let μ be a nonatomic uniformly regular Radon measure on a compact Hausdorff space X, and let $\hat{\mu}$ be the induced measure on Y, the Stone space of the Boolean algebra generated by a countable family of open sets in X that makes μ uniformly regular. Then we have the following:

(1) The Boolean algebra of Jordan μ-measurable subsets of X is isomorphic to the Boolean algebra of Jordan $\hat{\mu}$-measurable subsets of Y.

(2) The Jordan algebra of μ is isomorphic to the Jordan algebra of the restricted measure to the support of $\hat{\mu}$, whenever μ is strictly positive.

Proof. (1) Let $\mathcal{K}$ be a countable family of open sets in X that makes μ uniformly regular. Suppose that $\mathcal{B}$ is the Boolean algebra generated by $\mathcal{K}$. $\mathcal{B}$ is countable and μ induces a charge μ_0 on $\mathcal{B}$. Let $Y = \mathrm{Stone}(\mathcal{B})$. Then μ_0 extends to a Radon measure $\hat{\mu}$ on Y. By Theorem 1.2.2.34, Y is metrizable, and by Proposition 4.2.4, such $\hat{\mu}$ is uniformly regular. In particular, the algebra $\mathrm{Clop}(Y)$ makes $\hat{\mu}$ uniformly regular, (see Lemma 4.2.10). Since $\mathrm{Clop}(Y)$ is closed under finite unions and each element is open Jordan $\hat{\mu}$-measurable, by Lemma 4.3.1, $\mathrm{Clop}(Y)$ is a set of generators for the Boolean algebra $\mathcal{J}(Y, \hat{\mu})$ of Jordan $\hat{\mu}$-measurable subsets of Y. On the other hand, by Proposition 4.3.2, $\mathcal{B}$ can be identified as a set of generators for the Boolean algebra $\mathcal{J}(X, \mu)$ of Jordan μ-measurable subsets of X. But $\mathcal{B} \cong \mathrm{Clop}(Y)$, by the Stone Representation Theorem. Hence, by Proposition 3.3.20,

$$\mathcal{J}(X, \mu) \cong \mathcal{J}(Y, \hat{\mu}).$$

(2) Suppose that μ is strictly positive nonatomic. Given the countable algebra $\mathcal{B}$ and μ_0 as constructed above. By Theorem 2.1.22, μ_0 is nonatomic. Let $\mathcal{B}'$ be the quotient algebra of $\mathcal{B}$ modulo null sets. Then, by Lemma 2.1.7, $\mathcal{B}'$ is atomless and countable (because every atomless with more than one point is infinite). Therefore, μ_0 induces a strictly positive charge ν on $\mathcal{B}'$ and so $\nu([b]) = \mu_0(b)$ for all $b \in \mathcal{B}$. Let $W = \mathrm{Stone}(\mathcal{B}')$. By Proposition 2.2.7, the induced Radon measure $\hat{\nu}$ on W is the same as the restriction of $\hat{\mu}$ to W and $W = \mathrm{supp}(\hat{\mu})$. By Theorem 1.2.2.34, W is metrizable, and by Proposition 4.2.4, $\hat{\mu}|_W$ is uniformly regular on W. Hence, by Corollary 4.3.7,

$$\mathcal{J}_\mu(X) \cong \mathcal{J}_{\hat{\mu}|_W}(W) = \mathcal{J}_{\hat{\nu}}(W).$$

This finishes the proof. ■

Proposition 4.3.11. Let μ be a strictly positive nonatomic Radon measure on a compact Hausdorff space X. Then μ is uniformly regular if and only if there is a compact metric space M and strictly positive measure ν on M such that the Jordan algebra $\mathcal{J}_\mu(X)$ is isomorphic to the Jordan algebra $\mathcal{J}_\nu(M)$.

Proof. Let μ be uniformly regular. Set $\nu = \hat{\mu}|_W$ and $M = W$. By Proposition 4.3.10, M is compact metric and ν is strictly positive on M for which

$$\mathcal{J}_\mu(X) \cong \mathcal{J}_\nu(Y).$$

Conversely, the same proof of (5) $\Longrightarrow$ (2) in Theorem 4.2.2 works well. ∎

Lemma 4.3.12. Assume that $\mathfrak{A}, \mathfrak{B}$ are Boolean algebras, and $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a bijection which preserves " $\leq$ " (i.e., $a \leq b \Longleftrightarrow \varphi(a) \leq \varphi(b)$). Then φ is a Boolean isomorphism.

Proof. To check that $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$: let $a, b \in \mathfrak{A}$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, then $\varphi(a \wedge b) \leq \varphi(a)$ and $\varphi(a \wedge b) \leq \varphi(b)$. Therefore, $\varphi(a \wedge b) \leq \varphi(a) \wedge \varphi(b)$.

For the other direction, let $c = a \wedge b$, so $c \leq a$ and $c \leq b$, and for all d, $d \leq a, b$ implies that $d \leq c$. For every $e \in \mathfrak{B}$: if $e \leq \varphi(a), e \leq \varphi(b)$ and $e = \varphi(d)$ for some d (as φ is bijective), then $d \leq a, b$ which implies that $d \leq a \wedge b$, so $\varphi(d) \leq \varphi(a \wedge b)$. Thus, $e \leq \varphi(a \wedge b)$. Hence, $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$. ∎

The next proposition shows that the Jordan algebra of any uniformly regular measure is unique up to isomorphism.

Proposition 4.3.13. Consider the Lebesgue measure λ on the unit interval $\mathbb{I} = [0, 1]$. Then $\mathcal{J}_{\hat{\lambda}}(Z) \cong \mathcal{J}_\lambda(\mathbb{I})$, where $\hat{\lambda}$ is the induced measure on $Z = \text{Stone}(\mathcal{J}_\lambda(\mathbb{I}))$.

Proof. We give two different proofs. (1) is obtained from our results. As follows: the Lebesgue measure on $\mathbb{I}$ is strictly positive uniformly regular. By Proposition 4.2.12, $\hat{\lambda}$ is uniformly regular on Z as the measure on $\mathcal{J}_\lambda(\mathbb{I})$ is always uniformly regular. By Corollary 4.3.7, $\mathcal{J}_{\hat{\lambda}}(Z) \cong \mathcal{J}_\lambda(\mathbb{I})$.

(2) by the Stone isomorphism we know that $\mathcal{J}_\lambda(\mathbb{I}) \cong \text{Clop}(Z)$, and $\text{Clop}(Z) \subseteq \mathcal{J}_{\hat{\lambda}}(Z)$ because $\hat{\lambda}(\partial(C)) = \hat{\lambda}(\emptyset) = 0$ for every $C \in \text{Clop}(Z)$, then $\mathcal{J}_\lambda(\mathbb{I}) \sqsubseteq \mathcal{J}_{\hat{\lambda}}(Z)$, i.e. $\mathcal{J}_\lambda(\mathbb{I})$ is embeddable into $\mathcal{J}_{\hat{\lambda}}(Z)$.

For the other direction, let $A_z \in \mathcal{J}_{\hat{\lambda}}(Z)$. Since $\mathcal{J}_\lambda(\mathbb{I})$ has a dense subalgebra, which is the Cantor algebra $\mathfrak{A}_c$, then λ is uniformly regular on $\mathcal{J}_\lambda(\mathbb{I})$. By Proposition 4.2.12, $\hat{\lambda}$ is uniformly regular on Z. In particular, $\hat{\mathfrak{A}}_c$ is uniformly $\hat{\lambda}$-dense in Z. By Proposition 4.3.2, this copy of $\hat{\mathfrak{A}}_c$ is a set of generators for $\mathcal{J}_{\hat{\lambda}}(Z)$. For every $\epsilon > 0$ and for the given A_z, there exist $\hat{a}_1, \hat{a}_2 \in \hat{\mathfrak{A}}_c$ such that $\hat{a}_1 \subseteq A_z \subseteq \hat{a}_2$ and $\hat{\lambda}(\hat{a}_2 \setminus \hat{a}_1) < \epsilon$. But $\hat{\lambda}(\hat{a}) = \lambda(a)$ for all $a \in \mathfrak{A}_c$. Find A, for all $n < \omega$ fix $a_n^1, a_n^2 \in \mathfrak{A}_c$ such that $\hat{a}_n^1 \subseteq A_z \subseteq \hat{a}_n^2$ and $\lambda(\hat{a}_n^2 \setminus \hat{a}_n^1) < 1/n$. Let $A = \bigvee_{n<\omega} a_n^1$, so $A \notin \mathfrak{A}_c$ however $A \in \mathcal{J}_\lambda(\mathbb{I})$ because $\lambda(\partial(A)) = 0$. Note that $\lambda(A) = \hat{\lambda}(A)$. Therefore, $\varphi : A_z \longmapsto A$ shows that $\mathcal{J}_{\hat{\lambda}}(Z) \sqsubseteq \mathcal{J}_\lambda(\mathbb{I})$. It remains to prove that the operations (between $\mathcal{J}_{\hat{\lambda}}(Z)$ and $\mathcal{J}_\lambda(\mathbb{I})$) are preserved. From the above statements, one can conclude that φ is a bijection because if $A_z \neq B_z$, then $\hat{\lambda}(A_z \triangle B_z) > 0$. So, if $\varphi(A_z) = \varphi(B_z) = A$ for some $A \in \mathcal{J}_\lambda(\mathbb{I})$, then $\varphi(A_z \triangle B_z) = 0$ which is contradiction. Thus, by Lemma 4.3.12, $\mathcal{J}_{\hat{\lambda}}(Z) \cong \mathcal{J}_\lambda(\mathbb{I})$. ■

From the above result we remark that for every Jordan $\hat{\lambda}$-measurable subset A of Z, there is a clopen set C in Z such that

$$\hat{\lambda}(A \triangle C) = 0.$$

This means that every class in $\mathcal{J}_{\hat{\lambda}}(Z)$ has a clopen representative.

Remark 4.3.14. A charge on Jordan algebras is always uniformly regular, so the Jordan algebra of the Stone space of a Jordan algebra is isomorphic to the usual Jordan algebra.

4.4 Uniformly Regular Charges and Their Classification

This section study several properties of uniformly regular charges on some algebra of sets and their classification.

Proposition 4.4.1. [10, Proposition 4.2] Let $\mathfrak{A}$ be a Boolean algebra that supports a uniformly regular continuous charge. Then $\mathfrak{A}$ is isomorphic to a subalgebra of the Cohen algebra.

Theorem 4.4.2. [10, Theorem 4.1] Suppose that $\mathfrak{A}$ is a Boolean algebra and μ is a strictly positive continuous uniformly regular charge on $\mathfrak{A}$. Then $(\mathfrak{A}, \mu)$ is (metrically) isomorphic to a subalgebra of the Jordan algebra (of the unit interval) with the Lebesgue measure. Consequently, a Boolean algebra supports a continuous uniformly regular charge if and only if it is isomorphic to a subalgebra of the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ containing a dense Cantor subalgebra.

We show that Theorem 4.4.2 can be proved under a weaker assumption, as follows:

Theorem 4.4.3. Suppose that $\mathfrak{A}$ is a Boolean algebra and μ is a strictly positive nonatomic uniformly regular charge on $\mathfrak{A}$. Then $(\mathfrak{A}, \mu)$ is (metrically) isomorphic to a subalgebra of the Jordan algebra (of the unit interval) with the Lebesgue measure. Consequently, a Boolean algebra supports a nonatomic uniformly regular charge if and only if it is isomorphic to a subalgebra of the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$ containing a dense Cantor subalgebra.

Proof. The same proof of Theorem 4.4.2 goes well. The point is that uniform regularity of μ on $\mathfrak{A}$ give us a countable subalgebra, say $\mathfrak{B}$. Then, the nonatomicity of μ tells us this subalgebra is atomless (by Lemma 2.1.7) and hence $\mathfrak{B}$ is isomorphic to the Cantor algebra. ■

From Theorem 4.4.2 and Theorem 4.4.3, we obtain the following corollary:

Corollary 4.4.4. A Boolean algebra supports a continuous uniformly regular charge if and only if it supports a nonatomic uniformly regular charge.

Theorem 4.4.5. [10, Proposition 4.10] Let μ be a strictly positive nonatomic separable charge on Boolean algebra $\mathfrak{A}$. Then $(\mathfrak{A}, \mu)$ is metrically isomorphic to a subalgebra of the random algebra.

Corollary 4.4.6. Let X be any topological space and μ a (nonatomic) uniformly regular charge on an algebra $\mathcal{A}$ of subsets of X. Then the charge algebra $\mathscr{C}(X, \mu)$ of X is isomorphic to a subalgebra of $\mathcal{J}_\lambda(\mathbb{I})$.

Proof. Follows from Proposition 2.3.2 and Theorem 4.4.2. ■

Theorem 4.4.7. [9, Theorem 4.6] Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ carries either a uniformly regular charge or a charge which is not separable.

Proposition 4.4.8. Let μ be a uniformly regular charge on algebra of subsets of a topological space X and Y a subspace of X. The restricted charge $\mu_0 = \mu|_Y$ to Y is also uniformly regular.

Proof. Follows from Proposition 4.2.6. ■

Proposition 4.4.9. Let $\mathcal{A}$ be an algebra of subsets of a space X and μ_0 a charge on $\mathcal{A}$. Assume that $\mathcal{B}$ is a uniformly μ_0-dense subalgebra of $\mathcal{A}$ and $\mu_1 = \mu_0|_{\mathcal{B}}$. For a subset $E \subseteq X$, define

$$\begin{aligned} \mu^*(E) &= \inf\{\mu_1(B) : E \subseteq B, B \in \mathcal{B}\} \text{ and} \\ \nu^*(E) &= \inf\{\mu_0(A) : E \subseteq A, A \in \mathcal{A}\}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{C} &= \{C : \nu^*(E) = \nu^*(E \cap C) + \nu^*(E \cap C^c), E \subseteq X\} \text{ and} \\ \mathcal{D} &= \{D : \mu^*(E) = \mu^*(E \cap D) + \mu^*(E \cap D^c), E \subseteq X\}. \end{aligned}$$

Then

(1) $\mathcal{C}$ and $\mathcal{D}$ coincide.

(2) The restriction of μ^* and ν^* to $\mathcal{D}$ and $\mathcal{C}$ are charges μ and ν, which are the Jordan extensions of μ_0 and μ_1, respectively.

(3) $\mu = \nu$.

(4) μ_0 (or μ_1) is uniformly regular if and only if μ is uniformly regular on X.

Proof. (1) Suppose that $\epsilon > 0$ and $D \in \mathcal{D}$. By Lemma 3.2.1 (4), there exist $B_1, B_2 \in \mathcal{B}$ with $B_1 \subseteq D \subseteq B_2$ such that $\mu_1(B_2 \setminus B_1) < \epsilon$. But $\mathcal{B}$ is subalgebra of $\mathcal{A}$ and μ_1 and μ_0 agree on $\mathcal{B}$, so $D \in \mathcal{C}$. On the other hand, let $C \in \mathcal{C}$, then there exist $A_1, A_2 \in \mathcal{A}$ with $A_1 \subseteq C \subseteq A_2$ such that $\mu_0(A_2 \setminus A_1) < \epsilon/3$. Since $\mathcal{B}$ is uniformly μ_0-dense in

$\mathcal{A}$, we can find $B_1, B_2 \in \mathcal{B}$ with $B_1 \subseteq A_1$ and $B_2 \supseteq A_2$ (using complement) such that $\mu_0(A_1 \setminus B_1) < \epsilon/3$ and $\mu_0(B_2 \setminus A_2) < \epsilon/3$. Therefore, $B_1 \subseteq A_1 \subseteq C \subseteq A_2 \subseteq B_2$ and so

$$\mu_0(B_2 \setminus B_1) = \mu_1(B_2 \setminus B_1) < \epsilon.$$

Thus, $C \in \mathcal{D}$.

(2) By Theorem 1.2.3.12, μ and ν are charges on X.

(3) Uniqueness follows from the completeness of μ and ν. Hence $\mu = \nu$, (see Theorem 1.2.3.12).

(4) Let $\mathcal{A}^*$ be a countable family of μ_0-chargeable sets in $\mathcal{A}$ that makes μ_0 uniformly regular. Let E be any μ-chargeable set in X and $\epsilon > 0$. By Lemma 3.2.1 (4), there exist $A_1, A_2 \in \mathcal{A}$ such that

$$A_1 \subseteq E \subseteq A_2 \text{ and } \mu(A_2 \setminus A_1) < \epsilon/2. \text{ Hence, } \mu(E \setminus A_1) < \epsilon/2$$

and by assumption, for the given set A_1, there is $A^* \in \mathcal{A}^*$ with $A^* \subseteq A_1$ such that

$$\mu_0(A_1 \setminus A^*) < \epsilon/2.$$

Now,

$$\begin{aligned} \mu(E \setminus A^*) &\leq \mu(E \setminus A_1) + \mu(A_1 \setminus A^*) \\ &\leq \mu(E \setminus A_1) + \mu_0(A_1 \setminus A^*) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Since $\mathcal{A}^*$ is also a (countable) family of μ-chargeable sets in X, the claim follows.

Conversely, consider the family $\mathcal{H} = \{H_1, H_2, \dots\}$ of μ-chargeable sets in X that makes μ uniformly regular. If all $H_i \in \mathcal{A}$, then we are done. Otherwise, for each H_i, one can find $A_i \in \mathcal{A}$ such that $A_i \subseteq H_i$ and $\mu(H_i \setminus A_i) < \epsilon$. Clearly, the family of all such A_i is countable and will approximate all μ_0-chargeable sets in $\mathcal{A}$ because all $A \in \mathcal{A}$ are also μ-chargeable and μ is uniformly regular. Thus, μ_0 is uniformly regular on $\mathcal{A}$. Consequently, μ_1 is also uniformly regular on $\mathcal{B}$ as $\mathcal{B}$ is uniformly μ_0-dense in $\mathcal{A}$. ■

Notice that, by Lemma 3.2.1, $\mathcal{C} = J(\mathcal{A}, \nu)$ and $\mathcal{D} = J(\mathcal{B}, \mu)$ in our notation.

We remark that the Jordan extension of ν with respect to $\mathcal{C}$ is exactly ν.

Proposition 4.4.10. For any strictly positive charge μ on an algebra $\mathcal{A}$ of subsets of a space X, the Jordanian algebra of μ (or the Jordan extension of μ with respect to $\mathcal{A}$) is isomorphic to the Jordan extension of $\hat{\mu}$ with respect to the clopen algebra $\text{Clop}(Y)$ of the Stone space Y of $\mathcal{A}$. That is, $J_\mu(\mathcal{A}) \cong J_{\hat{\mu}}(\text{Clop}(Y))$, where $\hat{\mu}$ is the premeasure on $\text{Clop}(Y)$ and the induced measure on Y. Furthermore, $J_{\hat{\mu}}(\text{Clop}(Y)) = \{\hat{\mu}(\partial(B)) = 0 : B \in \text{Borel}(Y)\}/\hat{\mu}\text{-nulls} = \mathcal{J}_{\hat{\mu}}(Y)$.

Proof. Given $\epsilon > 0$ and $A \in J_\mu(\mathcal{A})$, then there exist $A_1, A_2 \in \mathcal{A}$ with $A_1 \subseteq A \subseteq A_2$ such that $\mu(A_2 \setminus A_1) < \epsilon$. Hence, $\hat{A}_1 \subseteq \hat{A} \subseteq \hat{A}_2$. But $\mu(A^*) = \hat{\mu}(\hat{A}^*)$ for all $A^* \in \mathcal{A}$. So $\mu(A_2 \setminus A_1) = \hat{\mu}(\hat{A}_2 \setminus \hat{A}_1) < \epsilon$. Hence, by Lemma 3.2.1, $\hat{A} \in J_{\hat{\mu}}(\text{Clop}(Y))$. Reversing the above steps, we obtain the converse.

Let us prove the last part of the lemma, $\hat{A}$ is Jordanian $\hat{\mu}$-measurable if $\hat{\mu}(\hat{A}) = \hat{\mu}^*(\hat{A}) = \inf\{\mu(\hat{C}) : \hat{A} \subseteq \hat{C}\}$. Given $\epsilon > 0$, choose $\hat{C} \in \text{Clop}(Y)$ with $\hat{A} \subseteq \hat{C}$ such that $\mu(\hat{C}) \leq \hat{\mu}^*(\hat{A}) + \epsilon$. Now, $Cl(\hat{A}) \subseteq Cl(\hat{C}) = \hat{C}$, so

$$\hat{\mu}^*(\hat{A}) \leq \hat{\mu}^*(Cl(\hat{A})) \leq \mu(Cl(\hat{C})) = \mu(\hat{C}) \leq \hat{\mu}^*(\hat{A}) + \epsilon.$$

As ϵ was taken arbitrarily, let it go to zero. Thus

$$\hat{\mu}^*(\hat{A}) = \hat{\mu}^*(Cl(\hat{A})).$$

By the same steps using inner Jordanian measure, one can prove that $\hat{\mu}(\hat{A}) = \hat{\mu}(Int(\hat{A}))$ because the interior of any clopen set remains open. ■

We have seen from the above proposition that a subset A of Y is Jordanian μ-measurable if and only if it is Jordan $\hat{\mu}$-measurable.

Next, we find a similar result to Theorem 4.3.6 for uniformly regular charges in the sense of Boolean algebras.

Theorem 4.4.11. The Jordanian algebra of any strictly positive nonatomic uniformly regular probability charge on an algebra of subsets of a topological space is isomorphic to the usual Jordan algebra.

Proof. Given a nonatomic uniformly regular probability charge μ on an algebra $\mathcal{A}$ of subsets of a space X. Let $\mathcal{C}$ be a countable uniformly μ-dense subalgebra in $\mathcal{A}$ and μ_0 the induced nonatomic charge on $\mathcal{C}$. Let $Y = \text{Stone}(\mathcal{C})$. By Theorem 1.2.2.34, Y is compact metrizable and μ_0 induces a nonatomic uniformly regular measure $\hat{\mu}$ on Y, by Proposition 4.2.4 and Lemma 4.3.5. But $(Y, \hat{\mu})$ and $(2^\omega, \lambda)$ are isomorphic. By Theorem 4.3.4 and Proposition 4.4.10, we have $J_{\mu_0}(\mathcal{C}) = \mathcal{J}_{\hat{\mu}}(Y) \cong \mathcal{J}_\lambda(2^\omega) \cong \mathcal{J}_\lambda(\mathbb{I})$, but by Proposition 4.4.9, $J_\mu(\mathcal{A}) = J_{\mu_0}(\mathcal{C})$. Thus, $J_\mu(\mathcal{A}) \cong \mathcal{J}_\lambda(\mathbb{I})$. ■

From the above theorem, we deduce the following corollary:

Corollary 4.4.12. Any two strictly positive nonatomic uniformly regular charges on any algebras of subsets of some topological spaces have isomorphic Jordanian algebras.

Also, from Theorems 4.3.6 and 4.4.11, we conclude the following:

Corollary 4.4.13. Let X be a compact Hausdorff space and let $\mathcal{A}$ be an algebra of subsets of X. If μ is a strictly positive nonatomic uniformly regular probability charge on $\mathcal{A}$ and if ν is strictly positive nonatomic uniformly regular probability Radon measure on X, then

$$J_\mu(X) = \mathcal{J}_\nu(X).$$

Definition 4.4.14. A charge μ on a Boolean algebra $\mathfrak{A}$ is said to be a μ-**completion** if for every $\epsilon > 0$ and every $a \in \bar{\mathfrak{A}}$, there exist $a_1, a_2 \in \mathfrak{A}$ with $a_1 \leq a \leq a_2$ such that

$$\mu(a_2 \setminus a_1) < \epsilon.$$

Corollary 4.4.15. Suppose that μ is a μ-completion uniformly regular charge on a Boolean algebra $\mathfrak{A}$, then $(\mathfrak{A}, \mu)$ is isomorphic to the usual Jordan algebra.

Proof. Follows from Proposition 4.4.9 (4) and Theorem 4.4.11. ■

We remark that μ-completion can be constructed by Caratheodory Criteria from a charge on an algebra, (see Theorem 1.2.3.12).

4.5 Lebesgue Measure vs Uniform Regularity

Using the Lebesgue measure we present the difference between the two Definitions (1.2.3.56 & 4.1.3) of uniform regularity of measures (or charges) in the sense of topologies and Boolean algebras.

Lemma 4.5.1. The Lebesgue measure λ on the space $\mathbb{R}$ with the standard (usual) topology is not uniformly regular.

Proof. Let $\epsilon > 0$. Consider the open set $\mathbb{R}$, it clear that there is no compact set K in $\mathbb{R}$ such that $\lambda(\mathbb{R} \setminus K) < \epsilon$, because every compact set has a finite measure but the measure of $\mathbb{R}$ is ∞. ∎

Lemma 4.5.2. Consider the unit interval $\mathbb{I}$ as a topological subspace of $\mathbb{R}$. The Lebesgue measure λ on $\mathbb{I}$ is uniformly regular.

Proof. Let $\mathcal{K}$ be a family of finite unions of compact intervals in $\mathbb{I}$ whose endpoints are rationals. Then $\mathcal{K}$ is countable. Since $\mathbb{I}$ is a compact metrizable space, by Lemma 1.2.1.3, each compact set is compact G_δ. So we can use elements in $\mathcal{K}$ to approximate open sets. Given any open set U in $\mathbb{I}$, U is a countable union of some intervals. For a given $\epsilon > 0$, there is a finite number of these intervals whose union is V, say, such that $\lambda(U \setminus V) < \epsilon/2$. So for V, as now it is a finite union of intervals, we can find $K \in \mathcal{K}$ with $K \subseteq V$ such that $\lambda(V \setminus K) < \epsilon/2$.

Now, we have

$$K \subseteq U \text{ and } \lambda(U \setminus K) = \lambda(U \setminus V) + \lambda(V \setminus K) < \epsilon/2 + \epsilon/2 < \epsilon$$

Thus, λ is uniformly regular. ∎

Lemma 4.5.3. The Lebesgue measure λ on $\mathbb{I} = [0, 1]$ is not uniformly regular in the sense of Boolean algebras, *i.e,* the Lebesgue measure on the Boolean algebra $\mathcal{L}(\mathbb{I}, \lambda)$ of Lebesgue measurable sets in the unit interval $\mathbb{I}$ is not uniformly regular. The Lebesgue measure on the measure algebra $\mathcal{L}_\lambda(\mathbb{I})$ is not uniformly regular. In particular, $\mathcal{L}_\lambda(\mathbb{I})$ does not support any uniformly regular measure.

Proof. We argue by contradiction. Suppose that $\mathcal{C}_0$ is a countable family of Lebesgue measurable subsets of $\mathbb{I}$ that demonstrates the uniform regularity of λ. Let $A \subseteq \mathbb{I}$ be such that $\lambda(A) = t > 0$ and $t \notin \{\lambda(C) : C \in \mathcal{C}_0\}$. This is possible because λ takes all values in $\mathbb{I}$ and $\mathcal{C}_0$ is countable. Assume that $\mathcal{C} = \{C_i : C_i \in \mathcal{C}_0, \lambda(A \setminus C_i) > 0\}$. $\mathcal{C} \neq \emptyset$ because $\mathcal{C}_0$ approximates A from below and so if $C_i \subseteq A$, then $\lambda(C_i) \leq \lambda(A)$, hence, $\lambda(C_i) < \lambda(A) = t$ by the choice of t. Obviously, $\mathcal{C}$ is (at most) countable. So by nonatomicity of λ, for i such that $C_i \in \mathcal{C}$, choose $U_i \subseteq A \setminus \bigcup_{j=1}^{i} C_j$ with $0 < \mu(U_i) < t/2^{i+1}$. Let $U = \bigcup_{C_i \in \mathcal{C}} U_i$, so U is measurable with $0 < \lambda(U) < t/2$. However, no $C_i \in \mathcal{C}_0$ can approximate U. Namely, suppose there is $C_i \in \mathcal{C}$ which approximate U in such a way that $C_i \subseteq U_j \subseteq U$. Now, we have the following cases:

(i) if $i \leq j$, we have no i that satisfies $C_i \subseteq U_j$ because of choice of U_i;

(ii) if $i > j$, let us say, there is $C_i \subseteq U_j \subseteq U$. Consequently, C_i should be a subset of U, which is a contradiction!

On the other hand, if $C_i \in \mathcal{C}_0$ and $C_i \notin \mathcal{C}$, then by assumption, such C_i cannot approximate either. We are done.

For the second statement, suppose otherwise, so by Proposition 4.2.12, the induced measure $\hat{\lambda}$ on the Stone space of $\mathcal{L}_\lambda(\mathbb{I})$ is uniformly regular. In fact this contradicts to Lemma 4.2.3 (3). Therefore the measure on $\mathcal{L}_\lambda(\mathbb{I})$ is not uniformly regular. ■

Next, we shall check that what will be the case if we consider Lebesgue measure as a charge in the above results. We give a proof to Lemma 4.5.3 using Theorem 4.4.2, while Lemma 4.5.1 and Lemma 4.5.2 can be proved similarly when consider the Lebesgue measure as a charge.

Lemma 4.5.4. The Lebesgue measure on both Boolean algebra $\mathcal{L}(\mathbb{I}, \lambda)$ of Lebesgue measurable sets in $\mathbb{I}$ and Lebesgue measure algebra cannot be uniformly regular as a charge.

Proof. First, we start proving the second statement, while the first statement can be concluded consequently. Suppose not for contradiction. Let $\varphi : \mathcal{L}(\mathbb{I}, \lambda) \longrightarrow \mathcal{L}(\mathbb{I}, \lambda)/\mathcal{N} = \mathcal{L}_\lambda(\mathbb{I})$ be the canonical surjective homomorphism, where $\mathcal{N}$ is the set of all null sets. By Theorem 4.4.2, the Lebesgue measure algebra $\mathcal{L}_\lambda(\mathbb{I})$ is isomorphic to a subalgebra of the Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$. As both $\mathcal{L}_\lambda(\mathbb{I})$ and $\mathcal{J}_\lambda(\mathbb{I})$ have the cardinality $\mathfrak{c}$, this might be the case. But $\mathcal{L}_\lambda(\mathbb{I})$ is a complete Boolean algebra and $\mathcal{J}_\lambda(\mathbb{I})$ is not (see Lemma 3.3.9). Therefore, the charge λ on $\mathcal{L}_\lambda(\mathbb{I})$ is not uniformly regular. Consequently, it follows from φ that λ on $\mathcal{L}(\mathbb{I}, \lambda)$ cannot be uniformly regular. ■

Remark 4.5.5. From Lemma 4.5.4 and Theorem 4.4.5 we conclude that Theorem 4.4.2 is not valid for measures on complete algebras or even on σ-algebras.

Remark 4.5.6. Note that if one restricts the Lebesgue measure to any (infinite) algebra of subintervals of $\mathbb{I}$, the measure will be a charge on the given algebra and its charge algebra will be a subalgebra of the usual Jordan algebra. So the induced charge will be uniformly regular, as a consequence of Theorem 4.4.2.

In conclusion, we find out uniform regularity of a measure (or charge) depends on the domain of its measure (or charge).

4.6 Product of Uniformly Regular Measures

In this section we show that the product (resp. free product) of countably many uniformly regular measures (resp. charges) is uniformly regular.

Proposition 4.6.1. Let X, Y be (compact) topological spaces and μ, ν probability Radon measures on X, Y, respectively. If μ and ν are uniformly regular, then the product measure $\lambda = \mu \bigotimes \nu$ on $X \times Y$ is also uniformly regular Radon.

Proof. By Theorem 9.3 in [38] λ is also a Radon measure on $X \times Y$. We need to show that λ is uniformly regular. Let $\mathcal{A}$ and $\mathcal{B}$ be countable families of open subsets of X and Y that make μ and ν uniformly regular, respectively. Assume that $\mathcal{C}$ is a family of finite unions of sets $A \times B$ such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Clearly $\mathcal{C}$ is countable as it is the product of two countable families. It remains to show that $\mathcal{C}$ is uniformly λ-dense in $X \times Y$. By Remark 4.2.11, every open set can be approximated by a basic open set, so we can work on basic open sets in $X \times Y$. Let $\epsilon > 0$ and $G = U \times V$ be any basic open set. By hypothesis, for U and V, there exist $A_0 \in \mathcal{A}$ and $B_0 \in \mathcal{B}$ with $A_0 \subseteq U$ and $A_0 \subseteq U$ such that

$$\mu(U \setminus A_0) < \epsilon/2 \text{ and } \nu(V \setminus B_0) < \epsilon/2, \text{ respectively.}$$

Setting $C = A_0 \times B_0$, obviously $C \in \mathcal{C}$ such that $C \subseteq G$ and so now

$$\begin{aligned}
\lambda(G \setminus C) &= \lambda((U \times V) \setminus (A_0 \times B_0)) \\
&= \lambda\big((U \times (V \setminus B_0)) \cup ((U \setminus A_0) \times V)\big) \\
&= \lambda(U \times (V \setminus B_0)) + \lambda((U \setminus A_0) \times V)) \\
&\leq \lambda(X \times (V \setminus B_0)) + \lambda((U \setminus A_0) \times Y)) \\
&\leq \nu(V \setminus B_0) + \mu(U \setminus A_0) \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

This shows that λ is uniformly regular on $X \times Y$. ■

Now we consider the countable product of uniformly regular measures.

Proposition 4.6.2. Let μ_n be a probability uniformly regular Radon measure on a (compact) space X_n for $n \in \mathbb{N}$. Then the product measure $\mu = \bigotimes_{n=1}^{\infty} \mu_n$ is also uniformly regular Radon on $X = \prod_{n=1}^{\infty} X_n$.

Proof. We start by induction on n. For every n, let $\mathcal{B}_n$ be a countable family of open subsets of X_n that makes μ_n uniformly regular Radon.

Let

$$\mathcal{B} = \bigcup_{m=1}^{\infty} \left\{ B_m^n \times \prod_{\substack{k \in \mathbb{N} \\ m \neq k}} X_k : B_m^n \in \mathcal{B}_n, n = 1, 2, 3, ... \right\}.$$

Clearly, $\mathcal{B}$ is countable. Then applying the steps in the proof of Proposition 4.6.1 demonstrate that $\mathcal{B}$ is uniformly μ-dense in X. Hence μ is uniformly regular. ■

Proposition 4.6.3. Suppose that all measures on topological spaces X and Y are uniformly regular. Then all the measures on $X \times Y$ are also uniformly regular.

Proof. Let μ be a measure on $X \times Y$. We want to prove that μ is uniformly regular. Let ν be the measure on X induced from μ by projection map $\pi_1 : X \times Y \to X$, that is

$$\nu(A) = \mu(\pi_1^{-1}(A)) = \mu(A \times Y) \text{ for } A \subseteq X.$$

Since ν is uniformly regular, there is a countable family $\mathcal{A} = \{A_n : n \in \omega\}$ of open subsets of X, which is uniformly ν-dense in X. For each $n \in \omega$, let λ_n be the measure on Y induced from μ by projection map $\pi_2 \restriction_{(A_n \times Y)}: X \times Y \to Y$, that is

$$\lambda_n(B) = \mu(\pi_2^{-1} \restriction_{(A_n \times Y)} (B)) = \mu(A \times B) \text{ for } B_{A_n} \subseteq Y.$$

By hypothesis λ_n is also uniformly regular for every n, so there is a countable family $\mathcal{B} = \{B^n_m : m \in \omega\}$ of open subsets of Y which is uniformly λ_n-dense in Y. Set $\mathcal{C} = \{A_n \times B^n_m : A_n \in \mathcal{A}, B^n_m \in \mathcal{B}\}$. It suffices to show that $\mathcal{C}$ is uniformly μ-dense because obviously $\mathcal{C}$ is countable (as it is a product of two countable collections).

Given any $\epsilon > 0$ and any open set $A \times B \subseteq X \times Y$. Find an element $A_n \times B^n_m \in \mathcal{C}$ such that $A_n \times B^n_m \subseteq A \times B$ and $\mu((A \times B) \setminus (A_n \times B^n_m)) < \epsilon$. By hypothesis, for every open set $A \subseteq X$ there is $A_n \in \mathcal{A}$ with $A_n \subseteq A$ such that $\nu(A \setminus A_n) < \epsilon/2$ and for every open set $B \subseteq Y$ there is $B^n_m \in \mathcal{B}$ with $B^n_m \subseteq B$ such that $\lambda_n(B \setminus B^n_m) < \epsilon/2$. Since $A_n \subseteq A$ and $B^n_m \subseteq B$, then clearly $A_n \times B^n_m \subseteq A \times B$.

$$\begin{aligned} \text{Now,} \quad \mu((A \times B) \setminus (A_n \times B^n_m)) &= \mu\big((A \times (B \setminus B^n_m)) \cup ((A \setminus A_n) \times B)\big) \\ &= \mu(A \times (B \setminus B^n_m)) + \mu((A \setminus A_n) \times B)) \\ &\leq \mu(A \times (B \setminus B^n_m)) + \mu((A \setminus A_n) \times Y)) \\ &\leq \lambda_n(B \setminus B^n_m) + \nu(A \setminus A_n) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

Thus, μ is uniformly regular measure on $X \times Y$. ∎

Observe that we can further extend the above result to only countable product.

Proposition 4.6.4. Given a family $\{X_n : n \in \omega\}$ of topological spaces, suppose that all measure on X_n are uniformly regular. Then all measures on $X = \prod_{n\in\omega} X_n$ are also uniformly regular.

Proof. Let μ be a measure on X. For every n, let μ_n be the measure on Y_n induced from μ by projection map $\pi_n : X \to Y_n$, that is, $\mu_n = \mu\pi_n^{-1}$, where $Y_n = \prod_{i\leq n} X_i$. By assumption μ_n is uniformly regular for every n. By Proposition 4.6.3 and induction on n. For every n, there a countable family $\mathcal{A}_n$ of open subsets of Y_n, which is uniformly μ_n-dense in Y_n. Therefore $\mathcal{A} = \bigcup_{n\in\omega}\{\pi_n^{-1}(A) : A \in \mathcal{A}_n\}$ is a countable uniformly μ-dense in X. This witnesses that μ is uniformly regular. ■

The following example shows that neither of Propositions 4.6.2 and 4.6.4 can be extended further:

Example 4.6.5. Consider the Lebesgue measure λ on the unit interval $[0, 1]$. By Lemma 4.5.2, μ is uniformly regular. But if we multiply $[0, 1]$ $\mathfrak{c}$-times, then no measure on $[0, 1]^{\mathfrak{c}}$ will be uniformly regular, see Remark 4.2.3 (4).

Next we study the uniform regularity of a charge on the free product of a family of Boolean algebras.

The following remark shows that if we restrict ourselves to use algebras of sets instead of Boolean algebras, the construction of free product (of the family of algebra) will be much easier to describe.

Remark 4.6.6. Let $\{X_i : i \in I\}$ be a family of topological spaces, and $\mathcal{A}_i$ is an algebra of subsets of X_i for every i. Suppose that $X = \prod_{i\in I} X_i$ and let $\pi_i : X \to X_i$ be the projection mapping. Then the free product $\bigotimes_{i\in I} \mathcal{A}_i$ can be represented as an algebra $\mathcal{A}$ of subsets of X generated by the set $\{\pi_i^{-1}(A) : i \in I, A \in \mathcal{A}_i\}$. Moreover, if $\mathcal{I}_i$ is an ideal of $\mathcal{A}_i$ for every i, the $\bigotimes_{i\in I} \mathcal{A}_i/\mathcal{I}_i$ can be identified with the algebra $\mathcal{A}/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $\mathcal{A}$ generated by $\{\pi_i^{-1}(B) : i \in I, B \in \mathcal{I}_i\}$, for more details (see Proposition 315L [20]).

Proposition 4.6.7. Suppose that every charge on Boolean algebras $\{\mathfrak{A}_n : n \in \omega\}$ is uniformly regular. Then every charge on the free product $\bigotimes_{n\in\omega} \mathfrak{A}_n$ is also uniformly regular.

Proof. For every $n \in \omega$, suppose that μ_n is a uniformly regular charge on Boolean algebra $\mathfrak{A}_n$. By Proposition 4.2.12, the induced measure ν_n on the Stone space Z_n of $\mathfrak{A}_n$ is uniformly regular. Then by Proposition 4.6.2, the product measure ν on the product space $Z = \prod_{n\in\omega} Z_n$ is uniformly regular. Again by Proposition 4.2.12, the charge μ on Clop(Z) is uniformly regular. But Clop(Z) can be identified with the free product $\bigotimes_{n\in\omega} \mathcal{A}_n$ by Remark 4.6.6 and so the identified charge μ (again denoted by μ) on $\bigotimes_{i\in I} \mathcal{A}_i$ is uniformly regular. ■

Remark 4.6.8. In the categorical sense, the dual category of Stone spaces is equivalent to the category of Boolean algebras, so the product of Stone spaces corresponds to the coproduct of Boolean algebras. But we can only construct the coproduct (free product) of a finite family of Boolean algebras without using their Stone spaces as it is shown in the following:

Remark 4.6.9. If $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras, their free product $\mathfrak{A}\otimes\mathfrak{B}$ is given by: for $a \in \mathfrak{A}, b \in \mathfrak{B}$, the product of $a \otimes b = \pi_1(a) \cap \pi_2(b)$, where $\pi_1 : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{B}, \pi_2 : \mathfrak{B} \to \mathfrak{A}\otimes\mathfrak{B}$ are canonical maps. Observe that $(a_1\otimes b_1)\cap(a_2\otimes b_2) = (a_1\cap a_2)\otimes(b_1\cap b_2)$, and that the maps $a \mapsto a \otimes b_0, b \mapsto a_0 \otimes b$ are always Boolean homomorphisms. In the context of algebras of sets, say $\mathcal{A}, \mathcal{B}$, we can identify $A\otimes B$ with $A\times B$ for $A \in \mathcal{A}, B \in \mathcal{B}$, and $[A \otimes B]$ with $[A] \times [B]$. Note that above conclusion is only true for a finite family of Boolean algebras, for more details (see Notation 315M, [20]).

Proposition 4.6.10. Suppose that every charge on Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ are uniformly regular. Then every charge on the free product $\mathfrak{A} \otimes \mathfrak{B}$ is also uniformly regular.

Proof. Let μ be a charge on $\mathfrak{A} \otimes \mathfrak{B}$. We want to prove that μ is uniformly regular. By Remark 4.6.9, $\mathfrak{A} \otimes \mathfrak{B}$ can be identified with the usual (Cartesian) product, i.e. every member of $\mathfrak{A} \otimes \mathfrak{B}$ is a finite union of members of the form $a \times b$, where $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. We try to find the free product as a coproduct of Boolean algebras (see Remark 4.6.8). Let ν be the charge on $\mathfrak{A}$ induced from μ by canonical map $\pi_1 : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{B}$ which is defined by

$$\nu(a) = \mu(\pi_1(a)) = \mu(a \otimes \mathbf{1}).$$

Since ν is uniformly regular, there is a countable family $\mathcal{A} = \{a_n : n \in \omega\}$ of members of $\mathfrak{A}$, which is uniformly ν-dense. For each $n \in \omega$, let λ_n be the charge on $\mathfrak{B}$ induced from μ by canonical map $\pi_n \restriction_{(a_n \otimes \mathfrak{B})}: \mathfrak{B} \to \mathfrak{A} \otimes \mathfrak{B}$ which is defined by

$$\lambda_n(b) = \mu(\pi_n \restriction_{(a_n \otimes \mathfrak{B})} (b)) = \mu(a \otimes b).$$

By hypothesis λ_n is also uniformly regular for every n, so there is a countable family $\mathcal{B} = \{b^n_m : m \in \omega\}$ and that is uniformly λ_n-dense in $\mathfrak{B}$. Set $\mathcal{C} = \{a_n \otimes b^n_m : a_n \in \mathcal{A}, b^n_m \in \mathcal{B}\}$. It remains to show that $\mathcal{C}$ is countable uniformly μ-dense. Obviously, $\mathcal{C}$ is countable (as it is a product of two countable collections).

Given any $\epsilon > 0$ and any $a \otimes b \in \mathfrak{A} \otimes \mathfrak{B}$. We can find an element $a_n \otimes b^n_m \in \mathcal{C}$ such that $\mu((a \otimes b) \setminus (a_n \otimes b^n_m)) < \epsilon$. By hypothesis, for every $a \in \mathfrak{A}$, there is $a_n \in \mathcal{A}$ with $a_n \leq a$ such that $\nu(a \setminus a_n) < \epsilon/2$ and for every $b \in \mathfrak{B}$, there is $b^n_m \in \mathcal{B}$ with $b^n_m \leq b$ such that $\lambda_n(b \setminus b^n_m) < \epsilon/2$.

$$\begin{aligned}
\text{Now,}\quad \mu((a \otimes b) \setminus (a_n \otimes b^n_m)) &= \mu\big((a \otimes (b \setminus b^n_m)) \cup ((a \setminus a_n) \otimes b)\big) \\
&= \mu(a \otimes (b \setminus b^n_m)) + \mu((a \setminus a_n) \otimes b)) \\
&\leq \mu(a \otimes (b \setminus b^n_m)) + \mu((a \setminus a_n) \otimes \mathbf{1})) \\
&\leq \lambda_n(b \setminus b^n_m) + \nu(a \setminus a_n) \\
&\leq \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

Thus, μ is uniformly regular charge on $\mathfrak{A} \otimes \mathfrak{B}$.

■

Chapter 5

Higher Version of Jordan Algebra & Uniform Regularity

In Section 5.1 we give an introduction to the Jordan algebra in higher dimensional spaces $\{0,1\}^\kappa$, for all cardinals κ. In Section 5.2 we establish various results on uniformly κ-regular measures and charges. In Section 5.3 we generalize a result given by Grekas and Mercourakis. In the previous chapter, we proved Theorems 4.3.6 and 4.4.11, in Section 5.4, we show that none of these results is true for uniformly κ-regular when $\kappa > \omega$. We show that these results are true in the special context for measures on (an arbitrary) product of compact metric spaces or charges on free algebras, in Section 5.5.

5.1 Jordan Algebra in Higher Dimensional Spaces

In this brief section, we introduce higher analogues of the usual Jordan algebra. We follow the same notations as given in [10]. By λ_κ or simply λ we denote the usual product measure on 2^κ and by $\mathcal{T}_\kappa$ the family of (basic) clopen sets in 2^κ which can be identified with the free algebra on κ generators, then by Kakutani's theorem λ can be seen as an extension of the premeasure λ_κ on $\mathcal{T}_\kappa$ which is defined by:

$$\lambda_\kappa([s]) = 1/2^{|\mathrm{dom}(s)|},$$

for all $[s] = \{f : f \in 2^\kappa, s \subseteq f\}$ where $s : \kappa \longrightarrow 2$ is a finite partial function, for more details see page 43. The usual Jordan algebra $\mathcal{J}^\kappa_\lambda(2^\kappa)$ (simply $\mathcal{J}_\lambda(2^\kappa)$) is obtained as the algebra generated by those open sets in 2^κ whose boundaries have λ_κ-measure zero. The same algebra can be obtained if working with $[0,1]^\kappa$ in place of 2^κ, so both of these measures are denoted by λ_κ. Equivalently, $\mathcal{J}^\kappa$ is the same as the Jordan extension of λ_κ on Free(κ), (see Proposition 4.4.10). That is $\mathcal{J}^\kappa = J_{\lambda_\kappa}(\text{Free}(\kappa))$.

5.2 Uniformly κ-Regular Measures & Charges

In this section we investigate different kinds of properties of uniformly regular measures and charges that help us to achieve our aim.

A higher cardinal version of uniform regularity and Maharam type of measures and charges on Boolean algebras is defined with respect to cardinal invariants: density and uniform density. The **density** of a Boolean algebra $\mathfrak{A}$ with a strictly positive charge μ is the smallest cardinal κ such that there is a μ-dense $\mathfrak{B} \subseteq \mathfrak{A}$ of size κ. *i.e.*

$$\mathbf{mt}(\mu) = \min\{\kappa : \text{there is a } \mu\text{-dense family } \mathfrak{B} \subseteq \mathfrak{A} \text{ with } |\mathfrak{B}| = \kappa\}.$$

Notice that the density determines the Maharam type of measures, so we use the notation $\text{mt}(\mu)$ to refer to the Maharam type of μ.

The **uniform density** of a Boolean algebra $\mathfrak{A}$ with a strictly positive charge μ is the smallest cardinal κ such that there is a uniformly μ-dense $\mathfrak{B} \subseteq \mathfrak{A}$ of size κ. *i.e.*

$$\mathbf{ur}(\mu) = \min\{\kappa : \text{there is a uniformly } \mu\text{-dense family } \mathfrak{B} \subseteq \mathfrak{A} \text{ with } |\mathfrak{B}| = \kappa\}.$$

Notice that the uniform density determines the version of uniform regularity of measures, so we use the notation $\text{ur}(\mu)$ to refer to the version of uniform regularity of μ.

As remarked (in [10]), these cardinal invariants are always infinite if the algebra is atomless.

Now we give definitions of higher uniform regularity and Maharam type of charges on Boolean algebras.

Definition 5.2.1. [10] A charge μ on a Boolean algebra $\mathfrak{A}$ is **uniformly κ-regular** if $\mathrm{ur}(\mu) = \kappa$.

When $\kappa = \aleph_0$, then μ is called a **uniformly regular** charge which is already described in earlier sections.

Definition 5.2.2. A charge μ on a Boolean algebra $\mathfrak{A}$ is of **Maharam type** κ if $\mathrm{mt}(\mu) = \kappa$.

If $\mathrm{mt}(\mu) = \omega$, then the charge is said to be **separable**.

Now we define a higher version of uniform regularity and Maharam type of measures on topological spaces.

Definition 5.2.3. Let μ be a measure on a topological space X. Then μ is said to be **uniformly κ-regular** if κ is the minimal cardinal for which there is a family $\mathcal{A}$ of open subsets of X of size κ such that for every open set $U \subseteq X$ and every $\epsilon > 0$, there is $A \in \mathcal{A}$ with $A \subseteq U$ such that

$$\mu(U \setminus A) < \epsilon.$$

Lemma 5.2.4. A Radon measure μ on a compact Hausdorff space X is uniformly κ-regular if and only if κ is the minimal cardinal for which there is a family $\mathcal{A}$ of size κ of open subsets of X such that $\mu(G \setminus H) = 0$ for every open set $G \subseteq X$ where $H = \bigcup\{A : A \in \mathcal{A}, A \subseteq G\}$.

Proof. Follows from Lemmas 1.2.3.31 and 4.1.4. ■

Definition 5.2.5. Let μ be a measure on some topological spaces X. μ is said to have **Maharam type** κ if κ is the minimal cardinal for which there is a family $\mathcal{A}$ of size κ consisting of measurable subsets of X such that for every $\epsilon > 0$ and every measurable subset B of X, there is $A \in \mathcal{A}$ such that

$$\mu(B \triangle A) < \epsilon.$$

Definition 5.2.6. A measure μ on a topological space X is **homogeneous** if it has the same type on every measurable subset A of X that possesses a positive measure. *i.e.* $\mathrm{mt}(\mu) = \mathrm{mt}(\mu|_A)$.

Lemma 5.2.7. Let μ be a Radon measure on a topological space X. If μ is uniformly κ-regular, then it is of Maharam type $\leq \kappa$. That is,

Approximating open sets by open sets (of a family of size κ from below)	$\Longrightarrow$	$\triangle$-Approximating measurable sets by measurable sets (of a family of size $\leq \kappa$)

Proof. Follows from Lemma 4.1.5. ■

Remark 5.2.8. [10] In relation with the known cardinal invariants of Boolean algebras and charges, notice that $\mathrm{ur}(\mu) \geq \mathrm{mt}(\mu)$ and also $\mathrm{ur}(\mu) \geq \pi(\mathfrak{A})$. See Example 5.5.3 which shows that $\mathrm{ur}(\mu) > \mathrm{mt}(\mu)$ is possible.

Proposition 5.2.9. Let μ be a uniformly κ-regular measure on a topological space X and Y a subspace of X. Then $\mathrm{ur}(\mu_Y) \leq \kappa$, where μ_Y restricted measure to Y.

Proof. The same as Proposition 4.2.6. ■

Proposition 5.2.10. Let X, Y be two topological spaces and $f : X \longrightarrow Y$ a continuous open surjection. If μ is a uniformly κ-regular measure on X, then $f(\mu)$ is also uniformly κ-regular measure on Y. Furthermore, the converse is true whenever f is one-to-one.

Proof. The same proof as Proposition 4.2.8 hold by replacing countable families by families that are of size κ. ■

From the above proposition, we conclude the following result:

Corollary 5.2.11. Let X, Y be two compact Hausdorff spaces and $f : X \longrightarrow Y$ a homeomorphism. Then μ is a uniformly κ-regular measure on X if and only if $\nu = f(\mu)$ is a uniformly κ-regular measure on Y.

Theorem 5.2.12. For a Radon probability measure on a compact Hausdorff space X, the following statements are equivalent:

(1) μ is uniformly κ-regular;

(2) there is a family $\mathcal{G}$ of cozero subsets of X, of size κ, such that for every $\epsilon > 0$ and every open set U in X, there is $G \in \mathcal{G}$ with $G \subseteq U$ such that

$$\mu(U \setminus G) < \epsilon;$$

(3) there is a family $\mathcal{H}$ of zero subsets of X, of size κ, such that for every $\epsilon > 0$ and every open set U in X, there is $H \in \mathcal{H}$ with $H \subseteq U$ such that

$$\mu(U \setminus H) < \epsilon;$$

(4) there is a family $\mathcal{K}$ of compact G_δ subsets of X, of size κ, such that for every $\epsilon > 0$ and every open set U in X, there is $K \in \mathcal{K}$ with $K \subseteq U$ such that

$$\mu(U \setminus K) < \epsilon;$$

(5) there is a continuous surjection $f : X \to [0,1]^\kappa$ such that

$$\mu(f^{-1}(f(K))) = \mu(K)$$

for every compact $K \subseteq X$.

Proof. (1)$\Longrightarrow$(2). Let $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ be a family of open subsets of X that makes μ uniformly κ-regular. Since X is compact Hausdorff and so is completely regular, then the set of all cozero subsets of X is a basis for its topology. By Remark 4.2.11, any open set can be approximated by a basic open set from below. Given $\epsilon > 0$, we let $\mathcal{G} = \{G_\alpha : \alpha < \kappa\}$ be a family such that for any V_α, there is G_α with $G_\alpha \subseteq V_\alpha$ and

$$\mu(V_\alpha \setminus G_\alpha) < \epsilon/2. \qquad (\clubsuit)$$

We claim that $\mathcal{G}$ approximates all open sets from below. Let U be an open set. By assumption, there is $V_\alpha \in \mathcal{V}$ with $V_\alpha \subseteq U$ such that

$$\mu(U \setminus V_\alpha) < \epsilon/2.$$

By inequality ($\clubsuit$), we find G_α such that $G_\alpha \subseteq V_\alpha$ and $\mu(V_\alpha \setminus G_\alpha) < \epsilon/2$.

Now, we have $G_\alpha \subseteq V_\alpha \subseteq U$ and

$$\begin{aligned} \mu(U \setminus G_\alpha) &= \mu(U \setminus V_\alpha) + \mu(V_\alpha \setminus G_\alpha) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

This shows that $\mathcal{G}$ has the desired property.

(2)$\Longrightarrow$(3). Let $\mathcal{G} = \{G_\alpha : \alpha < \kappa\}$ be a family of cozero sets given in (2). By Lemma 1.2.1.3 (3), for every α, G_α can be written as the union of a non-decreasing sequence $\langle S(G_\alpha)_n : n < \omega\rangle$ of zero sets. Let $\mathcal{H} = \{S(G_\alpha)_n : G_\alpha \in \mathcal{G}, n < \omega\}$. Clearly, $\mathcal{H}$ is the family of zero sets and has size κ. Given an open set U, then we have

$$\begin{aligned} \mu(U) &= \sup\{\mu(G_\alpha) : G_\alpha \subseteq U, G_\alpha \in \mathcal{G}\} \\ &= \sup\{\mu(S[G_\alpha]_n) : G_\alpha \subseteq U, G_\alpha \in \mathcal{G}, n < \omega\} \\ &\leq \sup\{\mu(H) : H \subseteq U, H \in \mathcal{H}\} \\ &\leq \mu(U). \end{aligned}$$

This shows that $\mathcal{H}$ approximates open sets from below. We are done.

(3)$\Longleftrightarrow$(4) Follows from the fact that a subset A of X is zero set if and only if it is compact G_δ when X is compact Hausdorff, (see Lemma 1.2.1.3).

(4)$\Longrightarrow$(5). Let $\mathcal{H} = \{H_\alpha : \alpha < \kappa\}$ be a family of zero subsets of X as in (3), so for each $\alpha < \kappa$, there is a continuous function $f_\alpha : X \longrightarrow [0,1]$ such that $H_\alpha = f^{-1}(\{0\})$. Let $f(x) = (f_\alpha(x))_{\alpha<\kappa}$, then $f : X \to [0,1]^\kappa$ is a continuous function and $f^{-1}(f(H_\alpha)) = H_\alpha$ for each $H_\alpha \in \mathcal{H}$.

Let K be any compact subset of X and let $\mathcal{H}^*$ be a collection of $H_\alpha \in \mathcal{H}$ that does not intersect K. Then $\mu(\bigcup \mathcal{H}^*) = \mu(X \setminus K)$ and $(\bigcup \mathcal{H}^*) \cap (f^{-1}(f(K)) = \emptyset$. Thus,

$$\mu(f^{-1}(f(K)) = \mu(K).$$

As K was taken arbitrarily, the claim follows.

(5)$\Longrightarrow$(1). let $f : X \to [0,1]^\kappa$ be a continuous surjection. Since $[0,1]^\kappa$ has a base of size κ, let $\mathscr{B}$ be such a base. If $\mathscr{A} = \{f^{-1}(B) : B \in \mathscr{B}\}$, then $\mathscr{A}$ is a family of open subsets of X and $|\mathscr{A}| = \kappa$. Given an open subset G of X, we let $K = X \setminus G$, $\mathscr{B}^* = \{B : B \in \mathscr{B}, B \cap f(K) = \emptyset\}$ and $\mathscr{A}^* = \{f^{-1}(B) : B \in \mathscr{B}^*\}$. Then $[0,1]^\kappa \setminus f(K) = \bigcup \mathscr{B}^*$, and so $X \setminus f^{-1}(f(K)) = \bigcup \mathscr{A}^*$. Now, we have

$$\begin{aligned} \sup\{\mu(A) : A \in \mathscr{A}, A \subseteq G\} &\geq \sup\{\mu(A) : A \in \mathscr{A}^*, A \subseteq G\} \\ &= \mu(\bigcup \mathscr{A}^*) \qquad \text{(by } \tau\text{-additivity)} \\ &= \mu(X \setminus f^{-1}(f(K))) \\ &= \mu(X \setminus K) \\ &= \mu(G). \end{aligned}$$

This shows that $\mathscr{A}$ approximates all open sets from below and so μ is uniformly κ-regular on X. ■

Note that the above result was proved by Fremlin ([22], 533G(a)) for uniformly regular measures. We follow approximately the same steps proving the last three parts.

A base $\mathscr{B}$ of topological measure space $(X, \mathcal{S}, \tau, \mu)$ of size κ is said to be **pure** if it is closed under finite unions and contains no π-base of size $< \kappa$ which is uniformly μ-dense in $\mathscr{B}$. A topological measure space $(X, \mathcal{S}, \tau, \mu)$ is said to have a **pure weight** $\omega_P(X) = \kappa$ if X has a minimal pure base $\mathscr{B}$ of size κ. For any cardinal κ, the free algebra Free(κ) on κ generators is a pure base for its Stone space $\{0,1\}^\kappa$ with respect to a (strictly positive) Radon measure.

Clearly every pure base is a base, so $\omega_P(X) \geq \omega(X)$.

Proposition 5.2.13. Let μ be a Radon measure on a compact space X and κ any infinite cardinal. We have the following:

(1) If X has pure weight $\omega_P(X) = \kappa$, then μ is uniformly κ-regular.

(2) If μ is uniformly κ-regular, then X has pseudo weight $\pi(X) \leq \kappa$.

Proof. (1) Let $\mathscr{B} = \{B_\alpha : \alpha < \kappa\}$ be a pure basis of X. Given any open set U in X, by Remark 4.2.11, for every $\epsilon > 0$, there is $B \in \mathscr{B}$ with $B \subseteq U$ such that

$$\mu(U \setminus B) < \epsilon.$$

Since $\mathscr{B}$ is of size κ, it witnesses that μ is uniformly κ-regular.

(2) Suppose that μ is uniformly κ-regular on X. There is a family $\mathcal{G}$ of open subsets of X of size κ such that any non-empty open set U in X, there $G \in \mathcal{G}$ such that $G \subseteq U$ and their difference is of measure $< \epsilon$ for any given $\epsilon > 0$. Thus $\mathcal{G}$ is a π-base and so $\pi(X) \leq \kappa$.

In conclusion, we obtain that $\omega_P(X) \geq \omega(X) \geq \mathrm{ur}(\mu) \geq \pi(X)$. ■

From Proposition 5.2.13, we remark the following:

Remark 5.2.14. Every strictly positive Radon measure on $\{0,1\}^\kappa$ (or $[0,1]^\kappa$) is uniformly κ-regular.

Remark 5.2.15. Note that in Proposition 5.2.13 the pure weight cannot be replaced by the standard weight. We now give a counterexample. Consider the usual Jordan algebra $\mathcal{J}_\lambda(\mathbb{I})$, by Lemma 3.3.7, $\mathcal{J}_\lambda(\mathbb{I})$ has cardinality $\mathfrak{c}$. Let $Y = \mathrm{Stone}(\mathcal{J}_\lambda(\mathbb{I}))$ and $\hat{\lambda}$ be the extension of the Lebesgue measure λ on Y. Example 4.2.5 shows that $\hat{\lambda}$ is uniformly regular, *i.e.* $\mathrm{ur}(\hat{\lambda}) = \omega \neq \mathfrak{c}$. The reason is that $\mathcal{J}_\lambda(\mathbb{I})$ is not a pure base of Y because it has a countable uniformly λ-dense subbase, which is the Cantor algebra.

Proposition 5.2.16. Let X be a compact Hausdorff space and $P(X)$ the space of Radon probability measures on X. Then

$$\sup\{\mathrm{ur}(\mu) : \mu \in P(X)\} \leq \omega(X).$$

Proof. By Remark 4.2.11, the base of X is always uniformly μ-dense for every $\mu \in P(X)$, and $\mathrm{ur}(\mu)$ is the cardinality of the smallest uniformly μ-dense family or subalgebra of X. So it is supremum cannot be greater than the weight of X. Hence, $\sup\{\mathrm{ur}(\mu) : \mu \in P(X)\} \leq \omega(X)$. ■

Lemma 5.2.17. Let (X, Σ, μ) be a measure space and let $(X, \hat{\Sigma}, \hat{\mu})$ be its completion.

(1) If μ is strictly positive, then $\hat{\mu}$ is strictly positive.

(2) If μ is strictly positive uniformly κ-regular, then $\hat{\mu}$ is uniformly κ-regular.

(3) If μ has Maharam type κ, then $\hat{\mu}$ also has Maharam type κ.

(4) If μ is homogeneous, then $\hat{\mu}$ is homogeneous.

Proof. (1) Directly follows from Lemma 3.3.26 (1).

(2) Let $\mathcal{G}$ be a family of open μ-measurable subsets of X that makes μ uniformly κ-regular. Since $\hat{\mu}(A) = \mu(A)$ for all $A \in \Sigma$. $\mathcal{G}$ is a family of open $\hat{\mu}$-measurable subsets of X as well. We now show that $\mathcal{G}$ is uniformly $\hat{\mu}$-dense in X. Let $\epsilon > 0$ and U be an open subset of X that belongs to $\hat{\Sigma}$. Since μ is strictly positive, by Lemma 3.3.26, $U \in \Sigma$. By assumption, there is an open set $G \in \mathcal{G}$ with $G \subseteq U$ such that $\mu(U \setminus G) < \epsilon$. But $\hat{\mu}(U \setminus G) = \mu(U \setminus G)$ for all open $G, H \in \Sigma$. Therefore, $\hat{\mu}(U \setminus G) < \epsilon$. Hence, $\hat{\mu}$ is uniformly κ-regular.

(3) Let $\mathcal{E}$ be a family of size κ consisting of μ-measurable subsets of X that is μ-dense in Σ. By Lemma 1.2.3.18, $\hat{\mu}(A) = \mu(A)$ for every $A \in \Sigma$. Then $\mathcal{E}$ is also a family in $\hat{\Sigma}$. We now show that $\mathcal{E}$ is $\hat{\mu}$-dense in $\hat{\Sigma}$. Let $\epsilon > 0$ and $A \in \hat{\Sigma}$. By Lemma 1.2.3.20, there is $B \in \Sigma$ such that $B \subseteq A$ and $\hat{\mu}(A \setminus B) = 0$. By assumption, there is $E \in \mathcal{E}$ such that

$$\mu(B \triangle E) < \epsilon.$$

Now, we have

$$\begin{aligned}
\hat{\mu}(A \triangle E) &\leq \hat{\mu}\Big((A \triangle B) \cup (B \triangle E)\Big) \\
&\leq \hat{\mu}(\triangle B) + \hat{\mu}(B \triangle E) \\
&= \hat{\mu}(A \triangle B) + \mu(B \triangle E) \\
&< 0 + \epsilon = \epsilon.
\end{aligned}$$

This yields that $\mathcal{E}$ is $\hat{\mu}$-dense in $\hat{\Sigma}$ and shows that $\hat{\mu}$ of Maharam type $\leq \kappa$.

We shall show that $\hat{\mu}$ cannot be of Maharam type $< \kappa$. Suppose for contradiction that there is a $\hat{\mu}$-dense family $\mathcal{D}$ of size $< \kappa$ consisting of $\hat{\mu}$-measurable sets in X. By Lemma 1.2.3.20, for each $D \in \mathcal{D}$, there is $A_D \in \Sigma$ such that $A_D \subseteq D$ and $\hat{\mu}(D \setminus A_D) = \hat{\mu}(D \triangle A_D) = 0$. Let $\mathcal{A}$ be the family of all such A_D's. Clearly $\mathcal{A} \subset \Sigma$ and $|\mathcal{A}| < \kappa$. Let $\epsilon > 0$ and $B \in \Sigma$. Then $B \in \hat{\Sigma}$, by assumption, there is $A_D \in \mathcal{A}$ such that $\hat{\mu}(B \triangle A_D) < \epsilon$. But $\mu(B \triangle A_D) = \hat{\mu}(B \triangle A_D) < \epsilon$. This shows that μ is of Maharam type $< \kappa$ which is contradiction. We are done.

(4) It follows from Proposition 322D(a) in [20] that both $\hat{\mu}$ and μ have the same measure algebra. So if μ is homogeneous, then $\hat{\mu}$ has to be homogeneous. ■

Proposition 5.2.18. Let μ be a charge on a Boolean algebra $\mathfrak{A}$ and ν be the induced Radon measure on the Stone space Z of $\mathfrak{A}$. We have the following:

(1) μ is uniformly κ-regular if and only if ν is uniformly κ-regular.

(2) μ is of Maharam type κ if and only if ν is Maharam type κ.

Proof. Follows from (4) and (5) in Proposition 4.2.12 by replacing countable families by families of size κ. ■

Proposition 5.2.19. Let μ be a nonatomic uniformly κ-regular Radon measure on a compact Hausdorff space X.

(1) There exists a family $\mathcal{U}$ of size κ consisting of open Jordan measurable subsets of X which is uniformly μ-dense in X.

(2) The Boolean algebra $\mathcal{J}_0$ generated by $\mathcal{U}$ is a set of generators for $\mathcal{J}(X, \mu)$.

(3) The quotient algebra of $\mathcal{J}_0$ modulo null sets (in $\mathcal{J}_0$) is a set of generators for $\mathcal{J}_\nu(Y)$, where $Y = \text{supp}(\mu)$ and ν the restriction of μ to Y.

(4) Both $\mathcal{J}(X, \mu)$ and $\mathcal{J}_\nu(Y)$ carry uniformly κ-regular measure.

Proof. The proof can proceed as in Proposition 4.3.2 with minor changes (countable families to families of size κ). ■

Proposition 5.2.20. Let μ be a uniformly κ-regular Radon measure on a topological space X, and let $\hat{\mu}$ be the induced measure on Y, the Stone space of the Boolean algebra $\mathcal{B}$ generated by a family $\mathcal{G}$ of size κ consisting of open sets in X that makes μ uniformly κ-regular. Then we have the following:

(1) The Boolean algebra of Jordan μ-measurable subsets of X is isomorphic to the Boolean algebra of Jordan $\hat{\mu}$-measurable subsets of Y.

(2) The Jordan algebra of μ is isomorphic to the Jordan algebra of the support of $\hat{\mu}$, whenever μ is strictly positive nonatomic.

Proof. Follows from Proposition 4.3.10. The only thing to be worried about is that in proof of (2), the size of quotient algebra $\mathcal{B}'$ of $\mathcal{B}$ will remain κ because $\mathcal{G}$ containing all open sets of positive measure and is of size κ. So at least we will have κ many different classes, each contains an open set G from $\mathcal{G}$. ■

By higher (usual) Jordan algebra we mean the Jordan algebra of the product measure λ on 2^κ, $\kappa > \omega$.

Lemma 5.2.21. The Jordan algebra of the Stone space of the higher Jordan algebra is isomorphic to the higher Jordan algebra. In particular, the Jordan algebra of the Stone space of any Jordan algebra is isomorphic to the Jordan algebra itself.

Proof. Since the measure on higher (usual) Jordan algebra $\mathcal{J}_\lambda(2^\kappa)$ is always uniformly κ-regular, so there is minimal uniformly λ-dense subalgebra $\mathcal{J}'$ in $\mathcal{J}_\lambda(2^\kappa)$ that has size κ. By Proposition 5.2.19, $\mathcal{J}'$ is a set of generators for $\mathcal{J}_\lambda(2^\kappa)$ (because λ is strictly positive on $\mathcal{J}_\lambda(2^\kappa)$, see Remark 3.3.16). By Proposition 5.2.18 and Proposition 5.2.19, the image of $\mathcal{J}'$ under the Stone isomorphism is also a set of generators for $\mathcal{J}_{\hat{\lambda}}(Z)$, where $\hat{\lambda}$ is the induced measure on the Stone space $Z = \text{Stone}(\mathcal{J}_\lambda(2^\kappa))$. Consequently, by Proposition 3.3.20, $\mathcal{J}_\lambda(2^\kappa) \cong \mathcal{J}_{\hat{\lambda}}(Z)$. These steps can also be applied to other Jordan algebras. We are done. ■

Theorem 5.2.22. Let $\{X_\alpha : \alpha < \kappa\}$ be a family of compact Hausdorff spaces, each containing at least two points, and let μ_α be a Radon measure on X_α such that $\text{ur}(\mu_\alpha) \leq \kappa$. Then the product measure $\mu = \bigotimes_{\alpha<\kappa} \mu_\alpha$ is uniformly κ-regular on $X = \prod_{\alpha<\kappa} X_\alpha$.

Proof. For every α, we let $\mathcal{K}_\alpha = \{K_\delta^\alpha : \delta < \delta^*\}$ be a family of compact G_δ or open subsets of X_α that makes $\text{ur}(\mu_\alpha) \leq \kappa$ for some $\delta^* \leq \kappa$. Set

$$\mathcal{D} = \bigcup_{\alpha<\kappa} \left\{ K_\delta^\alpha \times \prod_{\substack{\gamma<\kappa \\ \gamma\neq\alpha}} X_\gamma : \delta < \delta^* \right\}.$$

Let $\mathcal{A}$ be the algebra generated by $\mathcal{D}$. Then $\mathcal{A}$ is of size κ because it is a union of κ many sets of size $\leq \kappa$. We claim that $\mathcal{A}$ is uniformly μ-dense in X. Let U be any open set in X, where $U = U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\substack{\alpha<\kappa \\ \alpha_i\neq\alpha}} X_\alpha$, and U_{α_i} is open in X_{α_i} for

$i = 1, 2, ..., n$. By assumption, for every $\epsilon > 0$ and every $U_{\alpha_i} \in X_{\alpha_i}$, there is $K_{\alpha_i} \in \mathcal{K}_{\alpha_i}$ such that $K_{\alpha_i} \subseteq U_{\alpha_i}$ and $\mu_{\alpha_i}(U_{\alpha_i} \setminus K_{\alpha_i}) < \epsilon/n$. Then we have

$$\begin{aligned} \mu(U \setminus K) &= \mu(U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha_i \neq \alpha} X_\alpha \setminus K_{\alpha_1} \times K_{\alpha_2} \times \cdots \times K_{\alpha_n} \times \prod_{\alpha_i \neq \alpha} X_\alpha) \\ &= \mu\Big((U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \setminus K_{\alpha_1} \times K_{\alpha_2} \times \cdots \times K_{\alpha_n}) \times \prod_{\alpha_i \neq \alpha} X_\alpha\Big) \\ &= \bigotimes_{i=1}^{n} \mu_{\alpha_i}(U_{\alpha_i} \setminus K_{\alpha_i}) \\ &< \epsilon. \end{aligned}$$

Where $K = K_{\alpha_1} \times K_{\alpha_2} \times \cdots \times K_{\alpha_n} \times \prod_{\substack{\alpha<\kappa \\ \alpha_i \neq \alpha}} X_\alpha$. Evidently $K \in \mathcal{A}$. This shows that the algebra $\mathcal{A}$ is uniformly μ-dense and so μ is uniformly κ-regular. ■

Theorem 5.2.23. Let $\{X_\alpha : \alpha < \kappa\}$ be a family of compact metric spaces, each space with at least two points, and let μ_α be any measure on X_α. Then the product measure $\mu = \bigotimes_{\alpha<\kappa} \mu_\alpha$ is uniformly κ-regular on $X = \prod_{\alpha<\kappa} X_\alpha$.

Proof. By Proposition 4.2.4, every μ_α is uniformly regular on X_α, and by Theorem 5.2.22, $\mu = \bigotimes_{\alpha<\kappa} \mu_\alpha$ is uniformly κ-regular on $X = \prod_{\alpha<\kappa} X_\alpha$. ■

Theorem 5.2.24. Let $\{\mathfrak{A}_\alpha : \alpha < \kappa\}$ be a family of Boolean algebras, each algebra with at least tour elements. If $\mathfrak{A}_\alpha$ supports a uniformly $_\leq\kappa$-regular charge μ_α, then their free product $\mathfrak{A} = \bigotimes_{\alpha<\kappa} \mathfrak{A}_\alpha$ supports a uniformly κ-regular charge $\mu = \bigotimes_{\alpha<\kappa} \mu_\alpha$.

Proof. Apply Proposition 4.6.7 and follow the same steps as in Theorem 5.2.22. ■

5.3 Generalization of a Result Proved by Grekas and Mercourakis

Grekas and Mercourakis deduced the following theorem from the Remark 1.10 in [26] (or Theorem 4.3.6).

Theorem 5.3.1. [26, Proposition 2.10] Let $\{X_\alpha : \alpha < \kappa\}$ be a family of compact metric spaces, each space with at least two points and let μ_α be a strictly positive, nonatomic, Radon measure on X_α. If $X = \prod_{\alpha<\kappa} X_\alpha$ and $\mu = \bigotimes_{\alpha<\kappa} X_\alpha$, then the Jordan algebra $\mathcal{J}_\mu(X)$ of μ is isomorphic to the Jordan algebra of the standard product measure ν on the generalized Cantor space $\mathcal{J}_\nu(\{0,1\}^\kappa)$.

For proving the above theorem, the authors have used a stochastically independent sequence $\langle U_n : n < \omega\rangle$ of Jordan μ-measurable sets with $\mu(U_n) = 1/2$ such that the Boolean algebras generated by the sequence and the isomorphic image of the given sequence are sets of generators for $\mathcal{J}_\mu(X)$ and $\mathcal{J}_\nu(\{0,1\}^\kappa)$, respectively. Using their technique and Proposition 4.3.2, we can prove a generalized version of the above theorem which can be found hereunder.

Before stating our theorem, we should give the following remark that will help us completing the proof.

Remark 5.3.2. Given a Radon measure μ on a compact Hausdorff space X, if μ is uniformly regular on X, then there is a homeomorphism $f : X \setminus M \longrightarrow \mathbb{I} \setminus N$ (where M is μ-measure zero and N is λ-measure zero) such that $f(\mu) = \lambda$, where λ is the Lebesgue measure on $\mathbb{I} = [0,1]$ (see Theorem 3.3 in [3]). By Example 3.3.19, if such a homeomorphism exists then f is a Jordan isomorphism in the sense of Definition 3.3.17. Thus, the set of generators of $\mathcal{J}_\mu(X)$ and $\mathcal{J}_\lambda(\mathbb{I})$ are isomorphic.

Let us generalize Lemma 4.3.1 to an arbitrary (possibly uncountable) product of topological spaces.

Lemma 5.3.3. Let $\{X_i : i \in I\}$ be a family of compact Hausdorff spaces for some index set I (possibly uncountable) and let μ_i be a probability Radon measure on X_i. If $X = \prod_{i\in I} X_i$ and $\mu = \bigotimes_{i\in I} \mu_i$, then the base $\mathscr{B}$ of open Jordan μ-measurable subsets of X is closed under finite unions and is a set of generators for $\mathcal{J}(X,\mu)$.

Proof. Consider the class $\mathscr{B}$ of all sets of the form $B \times \prod_{i\in I-J} X_i$ where J is a finite subset of I, and B is open Jordan μ_J-measurable set in $\prod_{i\in J} X_i$. For each i, by Remark 3.1.2.4, the class of open Jordan μ_i-measurable sets in X_i forms a base. By Exercise 3.7 in [31], the class of sets B in X such that $B = B_{i_1} \times B_{i_2} \times \cdots \times B_{i_m} \times \prod_{i\neq i_n} X_i$, where B_{i_n} is open Jordan μ_{i_n}-measurable set in X_{i_n} and $n = 1, 2, ..., m$, forms a base for the topology on X and is a subclass of $\mathscr{B}$. So $\mathscr{B}$ is base and is closed under finite unions.

We want to show $\mathscr{B}$ is the set of generators for $\mathcal{J}(X,\mu)$. Let $A \in \mathcal{J}(X,\mu)$ and $\epsilon > 0$. From the regularity of μ, for any open set in X, we choose $\mathrm{Int}(A)$, there is a compact set K such that $K \subseteq \mathrm{Int}(A)$

$$\mu(\mathrm{Int}(A) \setminus K) < \epsilon/2.$$

But $\mathscr{B}$ is a base and K is compact, so there is $B_1 \in \mathscr{B}$ such that

$$K \subseteq B_1 \subseteq \mathrm{Int}(A) \text{ and } \mu(A \setminus B_1) < \epsilon/2,$$

because $\mu(\mathrm{Int}(A)) = \mu(A)$.

Similarly, we can find $B_2 \in \mathscr{B}$ such that

$$A \subseteq \mathrm{Cl}(A) \subseteq B_2 \text{ and } \mu(B_2 \setminus A) < \epsilon/2,$$

Consequently, we find $B_1, B_2 \in \mathscr{B}$ such that

$$B_1 \subseteq A \subseteq B_2 \text{ and } \mu(B_2 \setminus B_1) < \epsilon.$$

Therefore, $\mathscr{B}$ is a set of generators for $\mathcal{J}(X,\mu)$.

■

Theorem 5.3.4. Let $\{X_\alpha : \alpha < \kappa\}$ be a family of compact Hausdorff spaces, each space with at least two points, and let μ_α be a strictly positive nonatomic uniformly regular probability Radon measure on X_α. If $X = \prod_{\alpha<\kappa} X_\alpha$ and $\mu = \bigotimes_{\alpha<\kappa} X_\alpha$, then the Jordan algebra $\mathcal{J}_\mu(X)$ of μ is isomorphic to the Jordan algebra of the standard product measure λ on the generalized Cantor space $\mathcal{J}_\lambda(\{0,1\}^\kappa)$.

Proof. Firstly, let us prove when $\kappa = \omega$. It follows from Theorem 4.6.2 that μ is (strictly positive nonatomic) uniformly regular. By Theorem 4.3.6 and using the fact that $[0,1] \cong \{0,1\}^\omega$ (*cf.* Lemma 4.3.4), we obtain that $\mathcal{J}_\mu(X)$ is isomorphic to $\mathcal{J}_\lambda(\{0,1\}^\omega)$.

Let $\kappa > \omega$. It follows from Proposition 3.3.20 that it suffices to find two Boolean algebras of μ-Jordan measurable sets in X and λ-Jordan measurable sets in $\{0,1\}^\kappa$ that are sets of generators for $\mathcal{J}_\mu(X)$ and $\mathcal{J}_\lambda(\{0,1\}^\kappa)$ respectively such that these algebras are isomorphic. Since μ_α is uniformly regular for each $\alpha < \kappa$, by Proposition 4.3.2, the Boolean algebra $\mathfrak{B}_\alpha$ generated by the countable uniformly μ_α-dense family $\{U_n^\alpha : n \in \omega\}$

of Baire μ_α-Jordan measurable sets in X_α is a set of generators for the Jordan algebra $\mathcal{J}_{\mu_\alpha}(X_\alpha)$. By Remark 5.3.2, each Boolean algebra $\mathfrak{B}_\alpha$ is isomorphic to a Boolean algebra $f_\alpha(\mathfrak{B}_\alpha)$ (as a subalgebra of $\mathcal{J}_\lambda(\{0,1\}^\omega)$) that is a set of generators for $\mathcal{J}_\lambda(\{0,1\}^\omega)$ under the isomorphism f_α.

Let

$$\mathcal{G} = \bigcup_{\alpha<\kappa} \Big\{ B_n^\alpha \times \prod_{\substack{\gamma<\kappa \\ \gamma\neq\alpha}} X_\gamma : B_n^\alpha \in \mathfrak{B}_\alpha, n = 1, 2, 3, ... \Big\}.$$

Let F be the function given by

$$F : B_n^\alpha \times \prod_{\substack{\gamma<\kappa \\ \gamma\neq\alpha}} X_\gamma \longrightarrow f_\alpha(B_n^\alpha) \times \prod_{\substack{\gamma<\kappa \\ \gamma\neq\alpha}} \{0,1\},$$

and let $\mathfrak{B}$ be the Boolean algebra generated by $\mathcal{G}$. We claim the following:

(a) $\mathfrak{B}$ is a set of generators for $\mathcal{J}_\mu(X)$.

(b) The Boolean algebra $F(\mathfrak{B})$ generated by $\mathcal{H} = \{F(G) : G \in \mathcal{G}\}$ is a set of generators for $\mathcal{J}_\lambda(\{0,1\}^\kappa)$.

(c) F is an isomorphism between $\mathcal{G}$ and $\mathcal{H}$.

It is enough to prove (a) and (c), and then (b) will be followed.

Proof of (a) Let $\epsilon > 0$ and let A be a Jordan μ-measurable subset of X. So $\mu(\text{Int}(A)) = \mu(A) = \mu(\text{Cl}(A))$. Since $\text{Int}(A)$ is open, by Remark 4.2.11, there is a basic open set $H = H_{\alpha_1} \times H_{\alpha_2} \times \cdots \times H_{\alpha_n} \times \prod_{\alpha_i\neq\alpha} X_\alpha$ such that $H \subseteq \text{Int}(A)$ and $\mu(\text{Int}(A)\setminus H) < \epsilon/4$. As each H_{α_n} is open, by uniform regularity of μ_α, there is $B_\alpha^n \in \mathfrak{B}_\alpha$ such that $B_\alpha^n \subseteq H_{\alpha_n}$ and $\mu_\alpha(H_{\alpha_n} \setminus B_\alpha^n) < \epsilon/4n$. Set $B = B_\alpha^1 \times B_\alpha^2 \times \cdots \times B_\alpha^n \times \prod_{\alpha_i\neq\alpha} X_\alpha$. Then $B \subseteq H$ and

$$\begin{aligned}
\mu(H \setminus B) &= \mu(H_{\alpha_1} \times H_{\alpha_2} \times \cdots \times H_{\alpha_n} \times \prod_{\alpha_i\neq\alpha} X_\alpha \setminus B_\alpha^1 \times B_\alpha^2 \times \cdots \times B_\alpha^n \times \prod_{\alpha_i\neq\alpha} X_\alpha) \\
&= \mu\Big(\Big(\prod_{i=1}^n H_{\alpha_n} \setminus \prod_{i=1}^n B_\alpha^n\Big) \times \prod_{\alpha_i\neq\alpha} X_\alpha\Big) \\
&= \bigotimes_{i=1}^n \mu_{\alpha_i}(H_{\alpha_i} \setminus B_\alpha^i) \\
&< \epsilon/4.
\end{aligned}$$

Since $X \setminus \mathrm{Cl}(A)$ is open, by the same way above, we can find a basic open set $D = D_{\alpha_1} \times D_{\alpha_2} \times \cdots \times D_{\alpha_n} \times \prod_{\alpha_i \neq \alpha} X_\alpha$ such that $D \subseteq X \setminus \mathrm{Cl}(A)$ and $\mu([X \setminus \mathrm{Cl}(A)] \setminus D) < \epsilon/4$. This implies that $\mathrm{Cl}(A) \subseteq X \setminus D$ and $\mu([X \setminus D] \setminus \mathrm{Cl}(A)) < \epsilon/4$. By uniform regularity, for each coordinate $X \setminus D_{\alpha_n}$, there is $C_\alpha^n \in \mathfrak{B}_\alpha$ such that $X \setminus D_{\alpha_n} \subseteq C_\alpha^n$ and $\mu_\alpha(C_\alpha^n \setminus [X \setminus D_{\alpha_n}]) < \epsilon/4n$. Set $C = C_\alpha^1 \times C_\alpha^2 \times \cdots \times C_\alpha^n \times \prod_{\alpha_i \neq \alpha} X_\alpha$. Then $X \setminus D \subseteq C$ and

$$\mu(C \setminus [X \setminus D]) \quad < \quad \epsilon/4.$$

Summing all the above inequalities, we get $B \subseteq A \subseteq C$ and $\mu(C \setminus B) < \epsilon$. Since $B, C \in \mathfrak{B}$, the claim follows.

Proof of (c) Given $G \in \mathcal{G}$, $G = U_n^\alpha \times \prod_{\substack{\gamma<\kappa \\ \gamma \neq \alpha}} X_\gamma$, where $n \in \omega$, by the isomorphism f_α, $H = F(G) = f_\alpha(U_n^\alpha) \times \prod_{\substack{\gamma<\kappa \\ \gamma \neq \alpha}} \{0,1\}$ for each α. This implies that F is isomorphism.

Proof of (b) From (a) we obtained that $\mathfrak{B}$ is a set of generators for $\mathcal{J}_\mu(X)$, by (c) F will generate the Boolean algebra $F(\mathfrak{B})$ that is a set of generators for $\mathcal{J}_\lambda(\{0,1\}^\kappa)$ and isomorphic to $\mathfrak{B}$. Hence, by Proposition 3.3.20, $\mathcal{J}_\mu(X) \cong \mathcal{J}_\lambda(\{0,1\}^\kappa)$.

■

5.4 Conjectures on Uniform κ-Regularity

The main goal in this section is to prove an analogue to Theorem 4.4.2, Corollary 4.3.7 or Corollary 4.4.12 because obtaining one of them, the others can be obtained using the Stone duality. Namely:

(A) A Boolean algebra $\mathfrak{A}$ supports a uniformly κ-regular charge if and only if it is isomorphic to a subalgebra of the generalized Jordan algebra $\mathcal{J}^\kappa$ containing a dense copy of some fixed nice algebra $\mathfrak{A}_\kappa^*$.

(B) Any two strictly positive nonatomic uniformly κ-regular Radon measures on compact Hausdorff spaces have isomorphic Jordan algebras.

(C) Any two strictly positive (s-homogeneous) uniformly κ-regular charges on algebras of subsets of some topological spaces have isomorphic Jordanian algebras.

Regarding (A), surely that fixed algebra $\mathfrak{A}^*_\kappa$ cannot be $\mathrm{Clop}(2^\kappa)$ because a Boolean algebra that contains a copy of $\mathrm{Clop}(2^\kappa)$ has an independent sequence of length κ and hence it supports a measure of type κ. However, for $\kappa > \omega$, there might exist Boolean algebras $\mathfrak{A}$ that do not support a measure of type κ but all measures μ on $\mathfrak{A}$ satisfy $\mathrm{ur}(\mu) \geq \kappa$ (for more details see section 6 in [10]).

Borodulin-Nadzieja and Džamonja [10] found a partial characterization of higher uniformly regular cardinal (that stated above) which is a genralization of Proposition 4.4.1. Namely:

Proposition 5.4.1. [10, Proposition 5.2] If a Boolean algebra supports a uniformly κ-regular (nonatomic) charge, then it is isomorphic to a subalgebra of the completion of a quotient of $\mathrm{Free}(\kappa)$.

Regarding (B), we discover that this conjecture happens to be false for all infinite cardinals κ, as it is shown in the next theorem.

We show that (C) is only true for free algebras $\mathrm{Free}(\kappa)$ on κ generators.

We also found some results related to the above conjectures which will be shown hereafter.

Theorem 5.4.2. Assume that $\kappa^\omega = \kappa$. There exist two strictly positive nonatomic uniformly κ-regular Radon measures on a compact Hausdorff space such that their Jordan algebras are not isomorphic.

Proof. Let λ be the standard product measure on the compact Hausdorff space $X = \{0,1\}^\kappa$. Let $(\mathfrak{A}, \bar{\lambda})$ be the measure algebra of λ. Suppose that $Z = \mathrm{Stone}(\mathfrak{A})$ is the Stone space of the measure algebra $(\mathfrak{A}, \bar{\lambda})$ and let $\hat{\lambda}$ be the induced measure on Z. We now consider the following claims:

(I) $\lambda, \bar{\lambda}$ and $\hat{\lambda}$ are nonatomic.

Proof of (I). It is well-known that λ is nonatomic. Consequently, $\bar{\lambda}$ is also nonatomic. By Remark 4.2.13 (3), $\hat{\lambda}$ is nonatomic. ■

(II) λ is uniformly κ-regular on X in the sense of topology.

Proof of (II). This directly follows from Proposition 5.2.13 because the algebra Clop(X) of clopen sets in X is a pure base and is of size κ. ■

(III) λ is uniformly κ-regular on X in the sense of Boolean algebra.

Proof of (III). Given $\epsilon > 0$. Let E be any measurable set in X. Since λ is completion regular (Theorem 416U in [21]), there is a Baire set B such that $B \subseteq E$ and $\lambda(E \setminus B) = 0$. But for that B, there is a sequence of measurable rectangles $R_1, R_2, ...$ such that $\bigcup_{n<\omega} R_n \subseteq B$ such that $\lambda(B \setminus \bigcup_{n<\omega} R_n) < \epsilon$ (as λ is the extension of a measure on the algebra generated by those measurable rectangles). Therefore, $\lambda(E \setminus \bigcup_{n<\omega} R_n) < \epsilon$. We have only κ many measurable rectangles. If we assume $\mathcal{D}$ to be the set of countable unions of these measurable rectangles, then $\mathcal{D}$ would be uniformly λ-dense and of size κ^ω. Since $\kappa^\omega = \kappa$, so λ will be uniformly κ-regular. ■

(IV) $\bar{\lambda}$ is uniformly κ-regular on $\mathfrak{A}$ in the sense of Boolean algebra.

Proof of (IV). Since the Boolean algebra of λ-measurable sets supports uniform κ-regularity by (III), then $(\mathfrak{A}, \bar{\lambda})$ also supports uniform κ-regularity. ■

Remark 5.4.3. From (III) and (IV), we discover another difference between uniform regularity and uniform κ-regularity. Namely: both measure algebra $(\mathfrak{A}, \bar{\lambda})$ and the Boolean algebra of all λ-measurable sets in X are complete and support uniformly κ-regular measure. While no complete (Boolean) algebra supports uniformly regular measure (or charge), for more details see the comment below Proposition 4.6 in [10].

(V) $\hat{\lambda}$ is uniformly κ-regular on Z in the sense of topology.

Proof of (V). Follows from (III) and Proposition 5.2.18 (1). ■

(VI) λ is not a normal measure on X.

Proof of (VI). Note that in the proof we denote λ_κ to be λ for simplicity purpose. To show that λ_κ is not normal, we need to find a nowhere dense of positive measure λ_κ. Let μ be the Lebesgue measure on the unit interval $\mathbb{I}$. Consider the open set U given in page 68. That is, $U = \bigcup_{i=1}^{\infty}(q_i - \epsilon/2^i, q_i + \epsilon/2^i)$, where $\{q_1, q_2, q_3, ...\}$ is the set of rationals in $\mathbb{I}$. So U is open dense because it is the countable union of open intervals and contains all rationals in $\mathbb{I}$. We now explain that U is not Jordan measurable. We know that

$$\mu(\mathrm{Cl}(U)) = \mu(\mathbb{I}) = 1 \neq \mu(\mathrm{Int}(U)) = \mu(U) = \sum_{n=1}^{\infty} \mu(q_i - \epsilon/2^i, q_i + \epsilon/2^i) = \sum_{n=1}^{\infty} 2\epsilon/2^n = 2\epsilon.$$

For a fixed $\epsilon = 1/4$, we have $\mu(\partial(U)) = 1/2 > 0$. By Proposition 1.2.3.39, U can be identified with an open dense set, say V, in 2^ω. Assume that λ_ω is the Lebesgue measure on 2^ω. Thus, $\lambda_\omega(\partial(V)) = 1/2 > 0$. Let π_ω be the projection from 2^κ onto 2^ω and let λ_κ be the product measure on 2^κ. So

$$\lambda_\kappa(\pi_\omega^{-1}(E)) = \lambda(E)_\omega.\lambda_{\kappa\setminus\omega}(2^{\kappa\setminus\omega}) = \lambda_\omega(E) \quad \text{for all } E \in \mathrm{dom}(\lambda_\omega).$$

We claim that $\pi_\omega^{-1}(V)$ is a λ_κ-measurable subset of 2^κ but not Jordan λ_κ-measurable. Since $\pi_\omega^{-1}(V)$ can be identified as set $W = V \times 2^{\kappa\setminus\omega}$. W is open with measure $\lambda_\kappa(\mathrm{Int}(W)) = \lambda_\kappa(W) = \lambda_\omega(V) = 1/2$. On the other hand, $\mathrm{Cl}(W) = \mathrm{Cl}(V) \times 2^{\kappa\setminus\omega} = 2^\omega \times 2^{\kappa\setminus\omega} = 2^\kappa$. So $\lambda_\kappa(\mathrm{Cl}(W)) = \lambda_\kappa(2^\kappa) = \lambda_\omega(2^\omega) = \lambda_\omega(\mathrm{Cl}(V)) = 1$. Finally, $\lambda_\kappa(\partial(W)) = 1/2$. So W is not Jordan λ_κ-measurable but clearly is λ_κ-measurable. Since $\partial(W)$ is nowhere dense with positive measure, λ_κ is not normal. ■

(VII) The induced (Radon) measure $\hat{\lambda}$ on Z is normal.

Proof of VII. We know that Z is the Stone space of the measure algebra $(\mathfrak{A}, \bar{\lambda})$. Since every nowhere dense set is contained in a closed nowhere dense set, it suffices to show that every closed nowhere dense subset of Z has measure zero. Let N be a closed nowhere dense set in Z. Set $G = N^c$. G is open dense. Then $G = \bigcup_{\alpha \in I} \widehat{A}_\alpha$ for some $A_\alpha \in \mathfrak{A}$ and some indexed set I.

Since G is dense in Z, by Proposition 1.2.2.30, $\bigcup_{\alpha\in I} A_\alpha = X$, the unity of $\mathfrak{A}$.

Claim. There exists a countable subindex I_0 of I such that $\hat{\lambda}(G) = \hat{\lambda}(\bigcup_{\alpha\in I_0} \widehat{A}_\alpha)$

Proof of the Claim. We have $G = \bigcup_{\alpha\in I} \widehat{A}_\alpha$ for some index set I. Since $\hat{\lambda}$ is regular, there exists an F_σ-set F in Z such that $F \subseteq G$ and $\hat{\lambda}(G \setminus F) = 0$. F can be expressed as a countable union of compact sets K_n, i.e., $F = \bigcup_{n\in\omega} K_n$. Now, $\bigcup_{n\in\omega} K_n \subseteq \bigcup_{\alpha\in I} \widehat{A}_\alpha$ and so $K_n \subseteq \bigcup_{\alpha\in I} \widehat{A}_\alpha$ for every n. Since all $\hat{A}_\alpha$ are basic open sets, so for every n and every $x \in K_n$, there exists $\hat{A}(x)$ such that $x \in \hat{A}(x) \subseteq G$. Then $\bigcup_{\alpha\in I} \widehat{A}_\alpha$ forms a cover for K_n for every n. By compactness (of K_n), there is a finite subindex $I_n \subseteq I$ such that $K_n \subseteq \bigcup_{\alpha\in I_n} \widehat{A}_\alpha$. This implies that $\bigcup_{n\in\omega} K_n \subseteq \bigcup_{n\in\omega} \bigcup_{\alpha\in I_n} \widehat{A}_\alpha$. Set $I_0 = \bigcup_{n\in\omega} I_n$. Since every I_n is finite, so I_0 is countable and $\bigcup_{n\in\omega} K_n \subseteq \bigcup_{\alpha\in I_0} \widehat{A}_\alpha \subseteq \bigcup_{\alpha\in I} \widehat{A}_\alpha$. But $\hat{\lambda}(\bigcup_{n\in\omega} K_n) = \hat{\lambda}(\bigcup_{\alpha\in I} \widehat{A}_\alpha)$, hence $\hat{\lambda}\Big(G = \bigcup_{\alpha\in I} \widehat{A}_\alpha\Big) = \hat{\lambda}(\bigcup_{\alpha\in I_0} \widehat{A}_\alpha)$. The claim follows. ■

Now, we may suppose that there is a countable subindex $I_0 \subseteq I$ such that $\bar{\lambda}(\bigcup_{\alpha\in I_0} A_\alpha) = 1$. Without loss of generality replace A_α's by the finite unions, B_α for I_0. Then $\langle B_\alpha : \alpha \in I_0\rangle$ is an increasing family of elements of $\mathfrak{A}$ and so

$$\bar{\lambda}(X) = \sup_{\alpha\in I_0} \bar{\lambda}(B_\alpha) = 1.$$

Therefore,

$$\hat{\lambda}(G) = \hat{\lambda}(\bigcup_{\alpha\in I_0} \widehat{B}_\alpha) = \sup_{\alpha\in I_0} \hat{\lambda}(\hat{B}_\alpha) = \sup_{\alpha\in I_0} \bar{\lambda}(B_\alpha) = \bar{\lambda}(X) = \hat{\lambda}(Z) = 1$$

Therefore, $\hat{\lambda}(N) = 0$. Hence $\hat{\lambda}$ is normal. ■

(VIII) $\mathcal{J}_{\hat{\lambda}}(Z) = (\mathfrak{B}, \hat{\lambda}) \cong (\mathfrak{A}, \bar{\lambda}) \supset \mathcal{J}_\lambda(X)$, where $(\mathfrak{B}, \hat{\lambda})$ is the measure algebra of Z and $\mathfrak{B} = \text{dom}(\hat{\lambda})$ modulo $\hat{\lambda}$-null sets.

Proof of (VIII). The first equality i.e., $\mathcal{J}_{\hat{\lambda}}(Z) = (\mathfrak{B}, \hat{\lambda})$ obtain from Lemma 3.3.10. By Theorem 321J, [20], $(\mathfrak{B}, \hat{\lambda}) \cong (\mathfrak{A}, \bar{\lambda})$. The last one follows from case (V) and Lemma 3.3.12, and from the fact that all measure algebras are complete, hence $\mathcal{J}_\lambda(X) \subseteq (\mathfrak{A}, \bar{\lambda})$ and $\mathcal{J}_\lambda(X) \subset \mathcal{J}_{\hat{\lambda}}(Z)$ ■

Remark 5.4.4. From (VIII) we find out that λ and $\hat{\lambda}$ are both strictly positive nonatomic uniformly κ-regular on compact Hausdorff (zero-dimensional) spaces X and Z respectively, but their Jordan algebras are not isomorphic. Besides λ and $\hat{\lambda}$ are both homogeneous, complete and of Maharam type κ.

To make the result stronger, let us construct another measure on Z. Since Z is the Stone space of the measure algebra of a measure on X, there is a unique continuous surjective function $g : Z \longrightarrow X$ which maps each ultrafilter z to its unique limit point $x = g(z)$ (as X is compact Hausdorff, the limit point exists and is unique, by Proposition 4.4 and Exercise 4.1 in [31]). Since the measure algebra $(\mathfrak{A}, \bar{\lambda})$ is complete, by Theorem 1.2.2.31, Z is extremally disconnected. By Theorem 1.2.1.22 and Remark 1.2.1.23, g is irreducible. By Exercise 413Y(c) in [21], there is a Radon (probability) measure ν on Z of Maharam type κ such that $g(\nu) = \lambda$. We need to prove the following:

(i) ν is nonatomic.

Proof of (i). It follows from the fact that $\nu(\{z\}) = \lambda(g(\{z\})) = \lambda(\{x\}) = 0$. ∎

(ii) ν is not normal.

Proof of (ii). Suppose that ν is normal. By case (VI), λ is not normal. Then there is at least a nowhere dense set N in X such that $\lambda(N) > 0$. But $g^{-1}(N)$ is also nowhere dense in Z (see Lemma 1.2.1.24) and $\nu(g^{-1}(N)) = \lambda(N) > 0$. Hence ν cannot be normal.

∎

(iii) ν is uniformly κ-regular.

Proof of (iii). From (II) and (III) we notice that the family $\mathcal{R}$ of measurable rectangles in X, which are also open sets, determines the uniform κ-regularity of λ in the sense of Boolean algebra and topology, and ($|\mathcal{R}| = \kappa$). So $g^{-1}(\mathcal{R})$ is a family of open sets in Z of size κ because g is continuous. Given $\epsilon > 0$ and let U

be any open subset of Z. Since Z is zero-dimensional and ν is Radon, by Lemma 4.2.10, there is a clopen set $\widehat{A}$ with $\widehat{A} \subseteq U$ such that

$$\nu(U \setminus \widehat{A}) < \epsilon/2.$$

Since $g(\widehat{A}) = A$, and A is λ-measurable, by uniform κ-regularity of λ in the sense of Boolean algebra, there are $R_1, R_2, R_3, \cdots \in \mathcal{R}$ with $\bigcup_{n<\omega} R_n \subseteq A$ such that

$$\lambda(A \setminus \bigcup_{n<\omega} R_n) < \epsilon/4.$$

Since $\bigcup_{n<\omega} R_n$ is open (union of basic open sets), by uniform κ-regularity of λ in the sense of topology, there is $R \in \mathcal{R}$ such that $R \subseteq \bigcup_{n<\omega} R_n$ and $\lambda(\bigcup_{n<\omega} R_n \setminus R) < \epsilon/4$. Therefore,

$$R \subseteq A \text{ and } \lambda(A \setminus R) < \epsilon/2.$$

This implies

$$\nu(\widehat{A} \setminus g^{-1}(R)) = \nu(g^{-1}(A) \setminus g^{-1}(R)) = \lambda(A \setminus R) < \epsilon/2.$$

Now, we have

$$g^{-1}(R) \subset \widehat{A} \subseteq U \text{ and } \nu(U \setminus g^{-1}(R)) \leq \nu(U \setminus \widehat{A}) + \nu(\widehat{A} \setminus g^{-1}(R)) < \epsilon.$$

As U was taken arbitrarily, so $g^{-1}(\mathcal{R})$ witnesses that ν is uniformly κ-regular on Z. ■

We are still doubtful that whether ν is strictly positive or not. For fixing this problem, let us use $\sigma = \frac{1}{2}\nu + \frac{1}{2}\hat{\lambda}$. Clearly σ is a Radon measure on Z. We need to clarify some properties of this measure as well.

(a) σ is nonatomic. It is enough for σ to be nonatomic if either ν or $\hat{\lambda}$ is nonatomic, but both of them are nonatomic, so we are done.

(b) σ is strictly positive. This is because $\hat{\lambda}$ strictly positive, so any open set G with $\nu(G) = 0$ has $\hat{\lambda}(G) > 0 \Longrightarrow \sigma(G) = \frac{1}{2}\hat{\lambda}(G) > 0$.

(c) σ is not normal. That is, by case (VI) $\hat{\lambda}$ is normal, so $\sigma(N) = \frac{1}{2}\nu(N)$ for all nowhere dense subsets Z. By case (i) ν is normal, there is a nowhere dense of positive measure ν and consequently of positive measure σ.

(d) σ is uniformly κ-regular because it is the sum of two uniformly κ-regular measures.

In conclusion, we obtained two strictly positive nonatomic (homogeneous) uniformly κ-regular measures $\hat{\lambda}$ and σ on a compact Hausdorff zero-dimensional space Z such that the Jordan algebra $\mathcal{J}_{\hat{\lambda}}(Z)$ of $\hat{\lambda}$ is not isomorphic to the Jordan algebra $\mathcal{J}_\sigma(Z)$ of σ because $\hat{\lambda}$ is normal and so $\mathcal{J}_{\hat{\lambda}}(Z)$ is complete (by Lemma 3.3.10), while σ is not normal and so $\mathcal{J}_\sigma(Z)$ is not complete by Lemma 3.3.12. We are done. ∎

Note that we also provide some special cases in which the conjectures are true, for instance, the following results. Some other cases can be seen in the next section.

Proposition 5.4.5. Given any cardinal κ, any two homogeneous normal Radon measures of Maharam type κ on compact Hausdorff spaces have isomorphic Jordan algebras.

Proof. Since μ and ν are of Maharam type κ, then applying Maharam Theorem, we obtain that the measure algebras $(\mathfrak{A}, \bar{\mu})$ of μ is isomorphic to the measure algebra $(\mathfrak{B}, \bar{\nu})$ of ν. By Lemma 3.3.10, the measure algebra coincides with the Jordan algebra. This means that $(\mathfrak{A}, \bar{\mu}) = \mathcal{J}_\mu(X)$ and $(\mathfrak{B}, \bar{\nu}) = \mathcal{J}_\nu(Y)$. Hence,

$$\mathcal{J}_\mu(X) \cong \mathcal{J}_\nu(Y).$$

We are done. ∎

Proposition 5.4.6. Let μ be a strictly positive uniformly κ-regular Radon probability measure on a compact Hausdorff space X. Then the Jordan algebra $\mathcal{J}_\mu(X)$ of μ is isomorphic to the Jordan algebra $\mathcal{J}_{\nu'}(Y)$ of ν' on a closed subspace Y of $[0,1]^\kappa$, where ν' is the restriction of a Radon measure ν of type κ on $[0,1]^\kappa$.

Proof. Let $\mathcal{G}$ be a family of size κ consisting of open subsets of X that makes μ uniformly κ-regular. Let $\mathfrak{B}$ be a Boolean algebra generated by $\mathcal{G}$. Then $\mathfrak{B}$ has size κ and μ induces a charge μ_0 on $\mathfrak{B}$. If $Y = \text{Stone}(\mathfrak{B})$, by Remark 2.2.1, μ_0 extends to a Radon measure $\hat{\mu}$ on Y.

Since Y is completely regular and has $\omega_P(Y) = \kappa$, then $\hat{\mu}$ is uniformly κ-regular and so, by Tychonoff Theorem, Y is embeddable into $[0,1]^\kappa$ (i.e. Y is homeomorphic to a subspace, say Y' by a homeomorphism f). By Corollary 5.2.11, $\hat{\mu}$ induces a uniformly κ-regular measure ν' on Y' and so ν' is a Radon measure of Maharam type κ. Then ν' can be extended to a Radon measure ν of type κ on $[0,1]^\kappa$ such that $\nu(A) = \nu'(A \cap Y')$ for all $A \subseteq [0,1]^\kappa$. On the other hand, $f : Y \to Y'$ is the homeomorphism with $f(\hat{\mu}) = \nu|_{Y'} = \nu'$. By Example 3.3.19, f is Jordan isomorphism between $\mathcal{J}_{\hat{\mu}}(Y)$ and $\mathcal{J}_{\nu'}(Y')$. Hence $\mathcal{J}_{\hat{\mu}}(Y) \cong \mathcal{J}_{\nu'}(Y')$. But $\mathcal{J}_{\mu}(X) \cong \mathcal{J}_{\hat{\mu}}(Y)$ by Proposition 4.3.10. Thus, $\mathcal{J}_{\mu}(X) \cong \mathcal{J}_{\nu'}(Y')$. ■

For the above proposition we obtain the following:

Corollary 5.4.7. Let X be a closed subspace of $\{0,1\}^\kappa$ and μ a strictly positive uniformly κ-regular (finite) Radon measure on X (possibly $\mathrm{ur}(\mu) \leq \kappa$). Then there is a (finite) Radon measure ν on $\{0,1\}^\kappa$ of type κ such that $\nu_X = \mu$.

Proposition 5.4.8. Let μ be a uniformly κ-regular charge on an algebra $\mathcal{A}$ of subsets of a space X. Then the Jordanian algebra $J_\mu(X)$ of μ is isomorphic to the Jordan algebra $\mathcal{J}_{\nu'}(Y)$ of ν' on a closed subspace Y of $[0,1]^\kappa$, where ν' is the restriction of a Radon measure ν of type κ on $[0,1]^\kappa$.

Proof. Follows from Propositions 4.4.10 and 5.4.6. ■

From above Proposition, we conclude the following:

Corollary 5.4.9. Let μ be a uniformly κ-regular μ-completion charge on a Boolean algebra $\mathfrak{A}$. Then the charge algebra $(\mathfrak{A}, \mu)$ is isomorphic to the Jordan algebra $\mathcal{J}_{\nu'}(Y)$ of ν' on a closed subspace Y of $[0,1]^\kappa$, where ν' is the restriction of a Radon measure ν of type κ on $[0,1]^\kappa$.

5.5 Answer to an Open Problem Posed by Mercourakis & Grekas

We observe that our conjecture (B) is true for the following specific case. It is worth remarking that this answers an open problem posed by Grekas and Mercourakis [26] and [36].

Theorem 5.5.1. Given a cardinal κ and the product measure λ on $X = \{0,1\}^\kappa$. If μ is a strictly positive homogeneous Radon measure of Maharam type κ on X, then the Jordan algebra of μ is isomorphic to the Jordan algebra of λ. That is,

$$\mathcal{J}_\mu(X) \cong \mathcal{J}_\lambda(X).$$

Proof. It is known that λ is homogeneous of Maharam type κ and complete (by Theorem 254F (d), [19]). Let $\hat{\mu}$ be the completion of μ. Since μ is homogeneous of Maharam type κ, by Lemma 5.2.17 $\hat{\mu}$ is also homogeneous of Maharam type κ. By Maharam Theorem, the measure algebras of λ and $\hat{\mu}$ are isomorphic. By Remark 1.2.3.50, (X, Σ, λ) and $(X, \hat{T}, \hat{\mu})$ are isomorphic as measure spaces under an isomorphism $\psi : (X, \Sigma, \lambda) \longrightarrow (X, \hat{T}, \hat{\mu})$, say, where $\Sigma = \mathrm{dom}(\lambda)$ and $\hat{T} = \mathrm{dom}(\hat{\mu})$. But λ and $\hat{\mu}$ are complete measures, by Lemma 1.2.3.45 and Remark 1.2.3.46, we have

- $\psi^{-1}(C) \in \Sigma$ if and only if $C \in \hat{T}$;
- $\lambda(\psi^{-1}(C)) = \hat{\mu}(C)$; and
- $\hat{\mu}(\psi(C)) = \lambda(C)$ for every measurable rectangle $C \subseteq X$

In X measurable rectangles are the same as basic open sets, or equivalently clopen subsets of X. This implies that ψ preserves measures λ and $\hat{\mu}$ on the algebra $\mathcal{C}$ of clopen subsets of X. Since μ is strictly positive, by Remark 5.2.14, it is uniformly κ regular. Therefore, by Lemma 5.2.17, $\hat{\mu}$ is strictly positive uniformly κ-regular, and λ is uniformly κ-regular (by Theorem 5.4.2 (II)). Since $\mathcal{C}$ is a base for X in which all $C \in \mathcal{C}$ are open Jordan measurable sets with respect to both λ and μ (because they have boundary measure zero), by Proposition 5.2.19 or Lemma 5.3.3, $\mathcal{C}$ is the set of generators for both $\mathcal{J}_{\hat{\mu}}(X)$ and $\mathcal{J}_\lambda(X)$. Consequently, by Proposition 3.3.20, $\mathcal{J}_{\hat{\mu}}(X) \cong \mathcal{J}_\lambda(X)$. But $\mathcal{J}_\mu(X) = \mathcal{J}_{\hat{\mu}}(X)$ by Proposition 3.3.27. Hence $\mathcal{J}_\mu(X) \cong \mathcal{J}_\lambda(X)$. This completes the proof. ∎

Notice that the above theorem is only true for an arbitrary product of compact metric spaces ($\{0,1\}^\kappa$ and $[0,1]^\kappa$ are special cases of it). See Remark 5.4.4 for a counterexample when X is not a product of compact metric spaces.

From the above Theorem, we conclude the following:

Corollary 5.5.2. For any cardinal κ, any two strictly positive homogeneous Radon probability measures of Maharam type κ on $X = \{0,1\}^\kappa$ have isomorphic Jordan algebras.

Notice that for a given measure μ on a space X, the Jordan type (uniform density) of the Jordan algebra of μ might be greater than the Maharam type (density) of the measure algebra of μ despite of the Jordan algebra is a subalgebra of the measure algebra. As shown in the following example:

Example 5.5.3. Consider the product space $X = \{0,1\}^\mathfrak{c}$. Since each space $\{0,1\}$ is finite, so it is separable (it can be dense in itself). By Theorem 1.2.1.12, X is separable. Let $\mathcal{D} = \{d_n : n < \omega\}$ be a dense subset of X. Define a measure μ on the clopen algebra $\mathcal{C}$ of subsets of X by:

$$\mu(C) = \sum_{d_n \in C} 1/2^{n+1} \quad \text{for } C \in \mathcal{C}.$$

Then extend it to the Borel algebra, the σ-algebra generated by $\mathcal{C}$. By Theorem 7.3.11 in [7], this extension is a Radon measure on X, and it is strictly positive because every clopen set has a positive measure and it is basic open.

μ is separable because one can find a countable collection of these clopen sets that $\triangle$-approximates all μ-measurable sets. So the measure algebra $\mathrm{Ma}(X,\mu)$ of μ is of type ω (countable).

On the other hand, since $\mathcal{C}$ is a pure base of size $\mathfrak{c}$ and by Lemma 4.2.10, it approximates all open sets. Therefore, μ cannot be uniformly $< \mathfrak{c}$-regular. Hence, it should be uniformly $\mathfrak{c}$-regular. By Proposition 5.2.19, the Jordan algebra $\mathcal{J}_\mu(X)$ of is type $\mathfrak{c}$. Now we have

$$\mathcal{J}_\mu(X) \subset \mathrm{Ma}(X,\mu) \text{ and } \mathrm{ur}(\mu) = \mathfrak{c} > \omega = \mathrm{mt}(\mu).$$

The above example also indicates that uniformly κ-regular does not imply Maharam type κ of measures, in general, even on $\{0,1\}^\kappa$, where $\kappa > \omega$.

Definition 5.5.4. A charge μ on a Boolean algebra $\mathfrak{A}$ is said to be **s-homogeneous** if the induced measure $\hat{\mu}$ on the Stone space Z of $\mathfrak{A}$ is homogeneous.

Note that the Lebesgue charge λ on the free algebra on κ generators is s-homogeneous, (see Remark 1.2.3.40), because its extension is the standard product measure λ on $\{0,1\}^\kappa$ which is homogeneous.

Lemma 5.5.5. Let μ be a strictly positive homogeneous measure on a compact Hausdorff space X and let $\mathcal{G}$ be a uniformly μ-dense family of open subsets of X. If $\mathfrak{A}$ is a Boolean algebra generated by $\mathcal{G}$, then the restricted measure μ_0 to $\mathfrak{A}$ is an s-homogeneous charge.

Proof. We want to show that the induced measure $\hat{\mu}$ on the Stone space Y of $\mathfrak{A}$ is homogeneous. By Proposition 5.2.20, $\mathcal{J}_\mu(X) \cong \mathcal{J}_{\hat{\mu}}(Y)$ and so the measure algebras of μ and $\hat{\mu}$ are isomorphic, by Proposition 3.3.20. Then, the measure algebra of $\hat{\mu}$ has to be homogeneous because the measure algebra of μ is homogeneous. Therefore, $\hat{\mu}$ is homogeneous. Hence the charge μ_0 is s-homogeneous. ■

Remark 5.5.6. Since every strictly positive Radon measure μ on $\{0,1\}^\kappa$ is uniformly κ-regular (see Remark 5.2.14), so one might ask that Theorem 5.5.1 can be proved without strictly positive assumption. But it leads us to contradiction. Let us see how.
Assumption. Suppose it is true that for any cardinal κ, any two homogeneous Radon probability measures of Maharam type κ on $X = \{0,1\}^\kappa$ have isomorphic Jordan algebras. Then we claim the following:

Claim. For any cardinal κ, any two strictly positive homogeneous uniformly κ-regular Radon measures of Maharam type κ on all compact Hausdorff spaces have isomorphic Jordan algebras.

Proof of the claim. Since the product measure λ on $[0,1]^\kappa$ is a strictly positive homogeneous uniformly κ-regular Radon measure of Maharam type κ, it is enough to show that given any strictly positive homogeneous uniformly κ-regular Radon (probability) measure μ of Maharam type κ on some compact Hausdorff space X, its Jordan algebra $\mathcal{J}_\mu(X)$ is isomorphic to the Jordan algebra $\mathcal{J}_\lambda([0,1]^\kappa)$ of the product measure λ on the product space $[0,1]^\kappa$.

Let $\mathcal{G}$ be a family of open subsets of X that makes μ uniformly κ-regular. Clearly $|\mathcal{G}| = \kappa$. Let $\mathfrak{B}$ be a Boolean algebra generated by $\mathcal{G}$. Then $\mathfrak{B}$ has also size κ and

μ induces a strictly positive charge μ_0 on $\mathfrak{B}$ which is also s-homogeneous by Lemma 5.5.5, (Notice that if μ_0 is not strictly positive charge on $\mathfrak{B}$, we can work on the quotient algebra of $\mathfrak{B}$ modulo null sets which has the same size of $\mathfrak{B}$ because every G posses a positive measure and so it belongs to a class in the quotient algebra. Surely the charge on the quotient is strictly positive and its Stone space gives us the same Jordan algebra that Y gives, see Proposition 5.2.19). If $Y = \text{Stone}(\mathfrak{B})$, then Y is completely regular and has $\omega_P(Y) = \kappa$. By Remark 2.2.1, μ_0 extends to a Radon measure $\hat{\mu}$ on Y. By Propositions 4.2.12 and 5.2.13, $\hat{\mu}$ is strictly positive uniformly κ-regular, homogeneous (because μ_0 is s-homogeneous charge) and of Maharam type κ (because μ is of Maharam type κ). By Tychonoff Theorem, Y is homeomorphic to a closed subspace W of $[0,1]^\kappa$ under a homeomorphism h, say. This h induces a strictly positive homogeneous uniformly κ-regular measure ν of Maharam type κ on W. Therefore ν can be extended to a homogeneous Radon measure $\hat{\nu}$ of Maharam type κ. By Assumption (that we supposed it is true), $\mathcal{J}_{\hat{\nu}}([0,1]^\kappa) \cong \mathcal{J}_\lambda([0,1]^\kappa)$. But obviously $\mathcal{J}_\nu(W) = \mathcal{J}_{\hat{\nu}}([0,1]^\kappa)$ (because all subsets of $[0,1]^\kappa \setminus W$ have $\hat{\nu}$-measure zero). On the other hand, $\mathcal{J}_\mu(X) \cong \mathcal{J}_{\hat{\mu}}(Y)$ by Proposition 4.3.10 and clearly $\mathcal{J}_{\hat{\mu}}(Y) \cong \mathcal{J}_\nu(W)$. Summing up all these together yield, $\mathcal{J}_\mu(X) \cong \mathcal{J}_\lambda([0,1]^\kappa)$. ■

This shows that any two strictly positive homogeneous uniformly κ-regular Radon measures of Maharam type κ on any compact Hausdorff space have isomorphic Jordan algebras. But this is false in general, by Theorem 5.4.2, there are two strictly positive homogeneous uniformly κ-regular Radon measures of Maharam type κ on some compact Hausdorff zero-dimensional space which have non-isomorphic Jordan algebras.

From the above proof we find out that the strictly positive condition in Theorem 5.5.1 is necessary.

Lemma 5.5.7. Let μ be any charge on the free algebra $\text{Free}(\kappa)$ on κ generators. Then μ is continuous if and only if it is nonatomic.

Proof. Can be proved in the same way as in Lemma 2.1.15. ■

Remark 5.5.8. Note that nonatomicity of a measure on some compact Hausdorff space is implied by homogeneity. So, by Proposition 4.2.12, we obtain that every s-homogeneous charge on a Boolean algebra is always nonatomic.

section

The following result is a specific case for the conjectures (A) and (C).

Theorem 5.5.9. Any two strictly positive s-homogeneous charges of Maharam type κ on the free algebra Free(κ) on κ generators have isomorphic Jordanian algebras.

Proof. Let μ_0, ν_0 be two strictly positive s-homogeneous (nonatomic) charges of Maharam type κ on Free(κ). By Proposition 4.2.12, the induced Radon measures μ, ν on the Stone space Z of Free(κ) are strictly positive homogeneous of Maharam type κ. It is known that Z can be identified with the product space $X = \{0,1\}^\kappa$. By Corollary 5.5.2, $\mathcal{J}_\mu(Z) \cong \mathcal{J}_\nu(Z)$. But $J_{\mu_0}(\text{Free}(\kappa)) = \mathcal{J}_\mu(Z)$ and $J_{\nu_0}(\text{Free}(\kappa)) = \mathcal{J}_\nu(Z)$, by Proposition 4.4.10. Hence $J_{\mu_0}(\text{Free}(\kappa)) \cong J_{\nu_0}(\text{Free}(\kappa))$. ■

From Proposition 4.4.10 and Theorem 5.5.9 we conclude the following:

Corollary 5.5.10. Every strictly positive s-homogeneous charge of Maharam type κ on free algebra Free(κ) on κ generators has Jordanian algebra isomorphic to Jordan algebra of the product measure λ on 2^κ. In particular, for any strictly positive s-homogeneous charge μ of Maharam type κ on free algebra Free(κ) on κ generators , the charge algebra (Free(κ), μ) is isomorphic to a subalgebra of the Jordan algebra of the product measure λ on 2^κ

The above theorem is only true for free algebras, otherwise there is a counterexample, see Remark 5.4.4.

Bibliography

[1] C. D. Aliprantis. *Principles of real analysis.* Gulf Professional Publishing, 1998.

[2] A. Babiker. On uniformly regular topological measure spaces. *Duke Mathematical Journal*, 43(4):775–789, 1976.

[3] A. Babiker. Lebesgue measures on topological spaces. *Mathematika*, 24(01):52–59, 1977.

[4] H. Bauer. Über die beziehungen einer abstrakten theorie des Riemann-integrals zur theorie Radonscher masse. *Math. Z.*, 65:448–482, 1980.

[5] I. Berezanskii. Measures on uniform spaces. *Uspekhi Matematicheskikh Nauk*, 22(2):123–124, 1967.

[6] K. Bhaskara Rao and M. B. Rao. Existence of nonatomic charges. *Journal of the Australian Mathematical Society (Series A)*, 25(01):1–6, 1978.

[7] V. I. Bogachev. *Measure Theory*, volume 2. Springer, 2007.

[8] V. I. Bogachev. *Measure theory*, volume 1. Springer, 2007.

[9] P. Borodulin-Nadzieja. Measures on minimally generated Boolean algebras. *Topology and its Applications*, 154(18):3107–3124, 2007.

[10] P. Borodulin-Nadzieja and M. Džamonja. On the isomorphism problem for measures on Boolean algebras. *Journal of Mathematical Analysis and Applications*, 405(1):37–51, 2013.

[11] A. M. Bruckner, B. S. Thomson, and J. B. Bruckner. *Real analysis.* ClassicalRealAnalysis. com, 2008.

[12] J. Choksi and D. Fremlin. Completion regular measures on product spaces. *Mathematische Annalen*, 241(2):113–128, 1979.

[13] J. Dieudonné. *Treatise on analysis*, volume 2. New York: Academic Press, 1976.

[14] J. Dixmier. Sur certains espaces considérês par M. H. Stone. *Summa Brasil Math. J. (French)*, 2:151—-182, 1951.

[15] R. M. Dudley. *Real analysis and probability*, volume 74. Cambridge University Press, 2002.

[16] M. Džamonja and G. Plebanek. On Efimov spaces and Radon measures. *Topology and its Applications*, 154(10):2063–2072, 2007.

[17] R. Engelking. General topology. 1989. *Heldermann, Berlin*, 1989.

[18] D. Fremlin. Measure theory; the irreducible minimum, 2000.

[19] D. Fremlin. Measure theory volume 2. 2000.

[20] D. Fremlin. Measure theory volume 3: Measure algebras. *Torres Fremlin*, 25, 2002.

[21] D. H. Fremlin. *Measure theory*, volume 4. Torres Fremlin, 2000.

[22] D. H. Fremlin. Measure theory, vol. 5: Set-theoretic measure theory. *Torres Fremlin*, 2008.

[23] O. Frink. Jordan measure and Riemann integration. *The Annals of Mathematics*, 34(3):518–526, 1933.

[24] S. R. Givant and P. P. R. Halmos. *Introduction to Boolean algebras.* Springer, 2009.

[25] A. M. Gleason et al. Projective topological spaces. *Illinois Journal of Mathematics*, 2(4A):482–489, 1958.

[26] S. Grekas and S. Mercourakis. On the measure theoretic structure of compact groups. *Transactions of the American Mathematical Society*, 350(7):2779–2796, 1998.

[27] P. R. Halmos, P. R. Halmos, P. R. Halmos, P. R. Halmos, and H. Mathematician. *Measure theory*, volume 2. van Nostrand New York, 1950.

[28] P. R. Halmos and J. von Neumann. Operator methods in classical mechanics, ii. *Annals of Mathematics*, pages 332–350, 1942.

[29] N. R. Howes. *Modern analysis and topology.* Springer, 1995.

[30] A. (http://mathoverflow.net/users/41345/ameen). Uncountable atomless subalgebras of the boolean algebra of all jordan measurable sets in [0,1]. MathOverflow. URL:http://mathoverflow.net/q/156581 (version: 2014-02-03).

[31] K. D. Joshi. *Introduction to general topology.* New Age International, 1983.

[32] S. Kakutani. Notes on infinite product measure spaces, ii. *Proceedings of the Imperial Academy*, 19(4):184–188, 1943.

[33] V. Losert. Strong liftings for certain classes of compact spaces. In *Measure Theory Oberwolfach 1981*, pages 170–179. Springer, 1982.

[34] D. Maharam. Finitely additive measures on the integers. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 44–59, 1976.

[35] D. Maharam. Jordan fields and improper integrals. *Journal of mathematical analysis and applications*, 133(1):163–194, 1988.

[36] S. Mercourakis. Some remarks on countably determined measures and uniform distribution of sequences. *Monatshefte für Mathematik*, 121(1-2):79–111, 1996.

[37] J. D. Monk. Handbook of Boolean algebras. volumes 1-with the cooperation of Robert Bonnet. 1989.

[38] S. Okada. The product of pre-radon measures. *Bulletin of the Australian Mathematical Society*, 25(02):243–250, 1982.

[39] V. Olejček. Darboux property of finitely additive measure on *delta*-ring. *Mathematica Slovaca*, 27(2):195–201, 1977.

[40] D. Plachky. Darboux property of measures and contents. *Mathematica Slovaca*, 30(3):243–246, 1980.

[41] R. Pol. Note on the spaces p(s) of regular probability measures whose topology is determined by countable subsets. *Pacific J. Math*, 100:185–201, 1982.

[42] I. K. Rana. *An introduction to measure and integration*. American Mathematical Soc., 2005.

[43] K. B. Rao and M. B. Rao. *Theory of charges: a study of finitely additive measures*, volume 109. Access Online via Elsevier, 1983.

[44] M. B. Rao and K. Rao, B. Charges on Boolean algebras and almost discrete spaces. *Mathematika*, 20(02):214–223, 1973.

[45] R. Scozzafava. Completa additivita su opportune successioni di insiemi di una misura di probabilita semplicemente additiva fortemente non atomica. *Bolletino della Unione Matematica Italiana*, 16-B(5):639–648, 1979.

[46] Z. Semadeni. *Banach spaces of continuous functions*, volume 1. PWN Polish Scientific Publishers, 1971.

[47] R. Sikorski. *Boolean algebras*, volume 3. Springer-Verlag, 1969.

[48] A. Sobczyk and P. Hammer. A decomposition of additive set functions. *Duke Mathematical Journal*, 11(4):839–846, 1944.

[49] W. Stephen. *General topology*. Addison-Wesley, 1970.

[50] M. H. Stone. The theory of representation for Boolean algebras. *Transactions of the American Mathematical Society*, 40(1):37–111, 1936.

[51] T. Tao. *An introduction to measure theory*, volume 126. American Mathematical Soc., 2011.

[52] F. Topsøe. *Topology and measure.* Springer, 1970.

Index