ON THE JACQUET CONJECTURE ON THE LOCAL CONVERSE PROBLEM FOR p-ADIC GL_N

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ABSTRACT. Based on previous results of Jiang, Nien and the third-named author, we prove that any two *minimax* unitarizable supercuspidals of p-adic GL_N that have the same depth and central character admit a *special pair* of Whittaker functions. As a corollary of our result, we prove Jacquet's conjecture on the local converse problem for GL_N, when N is prime.

1. INTRODUCTION

In the representation theory of a group G, one of the basic problems is to characterize its irreducible representations up to isomorphism. If G is the group of points of a reductive algebraic group defined over a non-archimedean local field F, there are many invariants that one can attach to a representation π of G, some of which are the central character and depth. Capturing all of these invariants, however, is (at least conjecturally) a family of complex functions, invariants themselves, called the *local gamma factors* of π.

Now let $\mathrm{G}_N := \mathrm{GL}_N(F)$ and let π be an irreducible generic representation of G_N. The family of local gamma factors $\gamma(s, \pi \times \tau, \psi)$, for τ an irreducible generic representation of G_r, ψ an additive character of F and $s \in \mathbb{C}$, can be defined using Rankin–Selberg convolution [JPSS83] or the Langlands–Shahidi method [S84]. The following is a local analogue of a conjecture of Jacquet on precisely which family of local gamma factors should uniquely determine π.

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Conjecture 1.1 (Local analogue of Jacquet's Conjecture)**.** *Let π_1, π_2 be irreducible generic representations of G_N. If*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable s, for all irreducible generic representations τ of G_r with $r = 1, \ldots, [\frac{N}{2}]$, then $\pi_1 \cong \pi_2$.

We refer to the introductions of [Ch06] and [JNS15] for more related discussions on the previous known results on this conjecture.

In [JNS15, Section 2.4], Conjecture 1.1 is shown to be equivalent to the same conjecture with the adjective "generic" replaced by "unitarizable supercuspidal" (recall that an irreducible representation is *supercuspidal* if it is not a subquotient of a properly parabolically induced representation, while all supercuspidal representations are generic). However, in the situation that π_1, π_2 are both supercuspidal, it may be that the upper bound $[\frac{N}{2}]$ is no longer sharp, at least within certain families of supercuspidals: for example, for *simple* supercuspidals (of depth $\frac{1}{N}$), the upper bound may be lowered to 1 (see [BH14, Proposition 2.2] and [AL15, Remark 3.18] in general, and [X13] in the tame case).

Thus, for $m \geq 1$ an integer, it is convenient for us to say that irreducible supercuspidal representations π_1, π_2 of G_N *satisfy hypothesis $\mathcal{H}_m$* if

($\mathcal{H}_m$) $\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$ as functions of the complex variable s, for all irreducible supercuspidal representations τ of G_m.

For $r \geq 1$, we say that π_1, π_2 *satisfy hypothesis $\mathcal{H}_{\leq r}$* if they satisfy hypothesis $\mathcal{H}_m$, for $1 \leq m \leq r$. Then we can state a family of "conjectures".

Conjecture $\mathcal{J}(N, r)$. *If π_1, π_2 are irreducible supercuspidal representations of G_N which satisfy hypothesis $\mathcal{H}_{\leq r}$, then $\pi_1 \simeq \pi_2$.*

Thus the local analogue of Jacquet's conjecture is (equivalent to) Conjecture $\mathcal{J}(N, [\frac{N}{2}])$, while Conjecture $\mathcal{J}(N, N-2)$ is a Theorem due to Chen [Ch96, Ch06] and to Cogdell and Piatetski-Shapiro [CPS99]. On the other hand, examples in [Ch96] show that Conjecture $\mathcal{J}(4, 1)$ is false, and similar examples show that $\mathcal{J}(N, 1)$ is false when N is composite.

Conjecture $\mathcal{J}(N, [\frac{N}{2}])$ is suggested by an analogous conjecture for automorphic representations as follows. Let k be a number field and set $\mathbb{A} = \mathbb{A}_k$, the ring of adeles of k.

Conjecture 1.2 (Jacquet, [CPS99, Conjecture 1]). *Suppose* Π *is an irreducible admissible generic representation of* $\mathrm{GL}_N(\mathbb{A})$ *whose central character* ω_Π *is trivial on* $k^\times$ *and whose L-function* $L(\Pi, s)$ *is convergent in some half-plane. Suppose moreover that, for any cuspidal automorphic representation* Σ *of* $\mathrm{GL}_r(\mathbb{A})$*, with* $1 \leq r \leq [\frac{N}{2}]$*, the corresponding L-functions* $L(s, \Pi \times \Sigma)$ *and* $L(s, \widetilde{\Pi} \times \widetilde{\Sigma})$ *are entire (for* $\widetilde{\Pi}$ *and* $\widetilde{\Sigma}$ *the contragredient representations of* Π *and* Σ *respectively), are bounded in vertical strips and satisfy the standard functional equation*

$$L(s, \Pi \times \Sigma) = \epsilon(s, \Pi \times \Sigma) L(1 - s, \widetilde{\Pi} \times \widetilde{\Sigma}).$$

Then Π *is a cuspidal automorphic representation of* $\mathrm{GL}_N(\mathbb{A})$.

The heuristics behind Conjecture 1.2 have been explained in [CPS99, Section 8]. Roughly speaking, if Π were automorphic but not cuspidal, then Π would be a constituent of an induced representation $\mathrm{Ind}(\Sigma_1 \otimes \cdots \otimes \Sigma_s)$, where Σ_i is a cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{A})$, and at least one m_i is less than or equal to $[\frac{N}{2}]$. Then, for the corresponding Σ_i, the L-function $L(s, \Pi \times \widetilde{\Sigma}_i)$ should have a pole. Arguments similar to those in the proof of [CPS99, Section 7, Theorem] imply that, if Conjecture 1.2 is true then so is Conjecture $\mathcal{J}(N, [\frac{N}{2}])$.

On the other hand, the literature does not seem to address the question about whether $\mathcal{J}(N, 1)$ is false when N is prime or, more generally, whether conjecture $\mathcal{J}(N, [\frac{N}{2}])$ is optimal. This is the subject of ongoing work of the first three named authors.

Finally we describe the contents of the paper and the scheme of the proof. In [JNS15], Jiang, Nien and the third-named author introduced the notion of a *special pair of Whittaker functions* for a pair of irreducible unitarizable supercuspidal representations π_1, π_2 of G_N (see Section 2). They proved that if there is such a pair, and π_1, π_2 satisfy hypothesis $\mathcal{H}_{\leq [N/2]}$, then π_1, π_2 are equivalent. They also found special pairs of Whittaker functions in many cases, in particular the case of depth zero representations. Here we prove another case (the so-called *minimax* case), the simplest case left open in [JNS15]; as we will see, this is sufficient to prove Conjecture $\mathcal{J}(N, [\frac{N}{2}])$ in the case that N is prime, and we hope that it will be a first step in an inductive proof allowing all N to be treated (in the same way that the supercuspidal representations of G_N are constructed inductively in [BK93]). It is worth noting that when N is prime, only representations which are twists of depth zero representations are treated in [JNS15].

Each supercuspidal representation π of G_N is irreducibly induced from a representation of a compact-mod-centre subgroup, which we call an *extended maximal simple type* [BK93, §6]; amongst the data from which this is built, is a *simple stratum* $[\mathfrak{A}, n, 0, \beta]$, and the degree of the field extension $F[\beta]/F$ is called the *degree* $\deg(\pi)$ of π. We say that the representation π is *max* if $\deg(\pi) = N$, and *minimax* if it is max and the element β is *minimal* in the sense of [BK93, (1.4.14)] (see Section 3 for recollections).

In [JNS15], the authors also prove that, if π_1, π_2 are not both max and satisfy hypothesis $\mathcal{H}_{\leq [N/2]}$, then they are equivalent. Thus, in order to settle Jacquet's conjecture, the only remaining case is that of max representations, and here we treat the case of minimax representations. The reduction of the general max case to the minimax case is the subject of future work.

After preparing the ground in Sections 5, 6, we prove that any pair of minimax unitarizable supercuspidal representations of G_N with the same (positive) depth and central character possesses a special pair of Whittaker functions (see Proposition 7.2).

Finally, when N is prime, any irreducible supercuspidal representation is a twist by some character of either a depth zero representation or of a minimax supercuspidal representation. Jacquet's conjecture then follows from the results in [JNS15] and a reduction to representations which are of minimal depth among their twists (see Section 4).

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1.1. **Notation.** Throughout, F is a locally compact nonarchimedean local field, with ring of integers $\mathfrak{o}_F$, maximal ideal $\mathfrak{p}_F$, and residue field k_F of cardinality q_F and characteristic p; we also write ν_F for the normalized valuation on F, with image $\mathbb{Z}$, and $|\cdot|$ for the normalized absolute value on F, with image $q_F^{\mathbb{Z}}$. We use similar notation for finite extensions of F. We fix once and for all an additive character ψ_F of F which is trivial on $\mathfrak{p}_F$ and nontrivial on $\mathfrak{o}_F$.

For $r \geq 1$, we set $\mathrm{G}_r = \mathrm{GL}_r(F)$, and denote by U_r the unipotent radical of the standard Borel subgroup B_r of G_r, consisting of upper-triangular matrices. We denote by ψ_r the standard nondegenerate character of U_r, given by

$$\psi_r(u) = \psi_F\left(\sum_{i=1}^{r-1} u_{i,i+1}\right),$$

where (u_{ij}) is the matrix of $u \in \mathrm{U}_r$. We also denote by Z_r the centre of G_r, and by P_r the standard mirabolic subgroup consisting of matrices with last row equal to $(0, \dots, 0, 1)$.

We fix an integer $N \geq 2$ and abbreviate $\mathrm{G} = \mathrm{G}_N$. We also put $V = F^N$ and $A = \mathrm{End}_F(V)$, and identify G with $\mathrm{Aut}_F(V)$ via the standard basis of F^N.

All representations considered are smooth representations with complex coefficients.

2. Special pairs of Whittaker functions

In this section, we recall the main results on special pairs of Whittaker functions. For further background, we refer to [JNS15] and the references therein.

Let π be an irreducible supercuspidal representation of G. By the existence and uniqueness of local Whittaker models, $\mathrm{Hom}_{\mathrm{G}}(\pi, \mathrm{Ind}_{\mathrm{U}_N}^{\mathrm{G}} \psi_N)$ is a one-dimensional space, and we write $\mathcal{W}(\pi, \psi_N)$ for the image of any non-zero map in this space. A *Whittaker function* for π is any function $W \in \mathcal{W}(\pi, \psi_N)$. In [JNS15], the following definitions were introduced.

Definition 2.1. *Let π be an irreducible unitarizable supercuspidal representation of* G *and let* $\mathbf{K}$ *be a compact-mod-centre open subgroup of* G*. A nonzero Whittaker function W for π is called* $\mathbf{K}$-special *if the support of W satisfies* $\mathrm{Supp}(W) \subseteq \mathrm{U}_N\mathbf{K}$*, and*

$$W(k^{-1}) = \overline{W(k)} \text{ for all } k \in \mathbf{K},$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Definition 2.2. *For $i = 1, 2$, let π_i be an irreducible unitarizable supercuspidal representation of* G *and let W_i be a nonzero Whittaker function for π_i. Suppose moreover that π_1, π_2 have the same central character. Then (W_1, W_2) is called a* special pair *of Whittaker functions for the pair (π_1, π_2) if there exists a compact-mod-centre open subgroup* $\mathbf{K}$ *of* G *such that W_1 and W_2 are both* $\mathbf{K}$*-special and*

$$W_1(p) = W_2(p), \text{ for all } p \in P_N.$$

The condition in this definition that the representations have the same central character is rather mild in our situation since, by [JNS15, Corollary 2.7], if π_1, π_2 are irreducible supercuspidal representations of G which satisfy hypothesis $\mathcal{H}_1$, then they have the same central character.

The following is one of the main results in [JNS15], which provides a general approach to proving Conjecture $\mathcal{J}(N, [\frac{N}{2}])$.

Proposition 2.3 ([JNS15, Theorem 1.5])**.** *Let π_1, π_2 be irreducible unitarizable supercuspidal representations of* G *which have a special pair of Whittaker functions and satisfy hypothesis $\mathcal{H}_{\leq[N/2]}$. Then $\pi_1 \simeq \pi_2$.*

3. Strata

In order to use Proposition 2.3, we need to recall some parts of the construction of supercuspidal representations in [BK93], in particular the notion of a stratum.

We begin with a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A = \mathrm{End}_F(V)$, with Jacobson radical $\mathfrak{P}$, and we denote by $e = e(\mathfrak{A}|\mathfrak{o}_F)$ the $\mathfrak{o}_F$-period of $\mathfrak{A}$, that is, the integer such that $\mathfrak{p}_F\mathfrak{A} = \mathfrak{P}^e$. For any such hereditary order $\mathfrak{A}$, there is an ordered basis with respect to which $\mathfrak{A}$ is in *standard form*, that is, it consists of matrices with coefficients in $\mathfrak{o}_F$ which are block upper-triangular modulo $\mathfrak{p}_F$. Such a basis can be found as follows.

Recall that an $\mathfrak{o}_F$-lattice chain in V is a set of $\mathfrak{o}_F$-lattices which is linearly ordered by inclusion and invariant under multiplication by scalars in $F^\times$. Then there is a unique $\mathfrak{o}_F$-lattice chain $\mathfrak{L} = \{L_i \mid i \in \mathbb{Z}\}$ in V such that $\mathfrak{A} = \{x \in A \mid xL_i \subseteq L_i \text{ for all } i \in \mathbb{Z}\}$. (The set $\mathfrak{L}$ is uniquely determined by $\mathfrak{A}$, though the base point L_0 for the indexing is arbitrary.) For $i = 0, \ldots, e-1$, we choose an ordered set $\mathcal{B}_i$ of vectors in L_i whose image in L_i/L_{i+1} is a basis and then the ordered basis obtained by concatenating $\mathcal{B}_0, \ldots, \mathcal{B}_{e-1}$ is as required.

A hereditary order $\mathfrak{A}$ gives rise to a parahoric subgroup $U(\mathfrak{A}) = U^0(\mathfrak{A}) = \mathfrak{A}^\times$ of G, together with a filtration by normal open subgroups $U^n(\mathfrak{A}) = 1 + \mathfrak{P}^n$, for $n \geq 1$, as well as a compact-mod-centre subgroup $\mathfrak{K}(\mathfrak{A}) = \{g \in \mathrm{G} \mid g\mathfrak{A}g^{-1} = \mathfrak{A}\}$, the normalizer of $\mathfrak{A}$ in G. We also get a "valuation" $\nu_{\mathfrak{A}}$ on A by $\nu_{\mathfrak{A}}(x) = \sup\{n \in \mathbb{Z} \mid x \in \mathfrak{P}^n\}$, and, for $x \in F$, we have $\nu_{\mathfrak{A}}(x) = e(\mathfrak{A}|\mathfrak{o}_F)\nu_F(x)$.

A *stratum* in A is a quadruple $[\mathfrak{A}, n, r, \beta]$, where $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order, $n \geq r \geq 0$ are integers, and $\beta \in \mathfrak{P}^{-n}$. Strata $[\mathfrak{A}, n, r, \beta_i]$, with $i = 1, 2$, are called *equivalent* if $\beta_1 - \beta_2 \in \mathfrak{P}^{-r}$. Thus, when $r \geq \left[\frac{n}{2}\right]$, the equivalence class of a stratum $[\mathfrak{A}, n, r, \beta]$ corresponds to a character ψ_β of $U^{r+1}(\mathfrak{A})$, trivial on $U^{n+1}(\mathfrak{A})$, via

$$\psi_\beta(x) = \psi_F \circ \mathrm{tr}_{A/F}(\beta(x-1)), \quad \text{for } x \in U^{r+1}(\mathfrak{A}).$$

The stratum $[\mathfrak{A}, n, r, \beta]$ is called *pure* if $E = F[\beta]$ is a field with $E^\times \subseteq \mathfrak{K}(\mathfrak{A})$, and $\nu_{\mathfrak{A}}(\beta) = -n$. A pure stratum $[\mathfrak{A}, n, r, \beta]$ is called *simple* if $r < -k_0(\beta, \mathfrak{A})$, where $k_0(\beta, \mathfrak{A})$ is an invariant whose definition we do not recall here (see [BK93, Definition 1.4.5]). In particular, a stratum of the form $[\mathfrak{A}, n, n-1, \beta]$ is simple if and only if β is *minimal over* F in the sense of [BK93, (1.4.14)], that is:

(i) $\nu_E(\beta)$ is coprime to the ramification index $e = e(E/F)$; and
(ii) if ϖ_F is any uniformizer of F, then the image of $\varpi_F^{-\nu_E(\beta)}\beta^e + \mathfrak{p}_E$ in k_E generates the residue field extension k_E/k_F.

We call a simple stratum of the form $[\mathfrak{A}, n, n-1, \beta]$ a *minimal* stratum; a special case is when $\beta \in F$, in which case we say the stratum is *scalar*.

Finally, we call a simple stratum of the form $[\mathfrak{A}, n, n-1, \beta]$ *minimax*, if the extension $E = F[\beta]/F$ is maximal in A (that is, of degree N), in which case $n = -\nu_E(\beta)$ and $e(\mathfrak{A}|\mathfrak{o}_F) = e(E/F)$ so they are coprime.

The first step in the construction and classification of the positive depth supercuspidal representations of G in [BK93] is to prove that any such representation π *contains* a simple stratum $[\mathfrak{A}, n, n-1, \beta]$, in the sense that $\mathrm{Hom}_{U^n(\mathfrak{A})}(\psi_\beta, \pi) \neq 0$. The *depth* $\ell(\pi)$ of π is then the depth $n/e(\mathfrak{A}|\mathfrak{o}_F)$ of any such simple stratum; this is independent of any choices, as is the degree of the extension $E = F[\beta]/F$. In particular, if π contains a minimax stratum $[\mathfrak{A}, n, n-1, \beta]$, then n and $e = e(E/F)$ are determined by the depth $\ell(\pi) = n/e$, since they are coprime, as is the residue class degree $f = f(E/F) = N/e$. For more details, see for example [KM90, Proposition 1.14].

Remark 3.1. *If π is an* essentially tame *positive depth supercuspidal representation, then it arises from an admissible pair $(K/F, \xi)$ of degree N, where ξ is a character of $K^\times$, non-trivial on U_K^r but trivial on U_K^{r+1}, for some $r \geq 1$. In this case, π contains a minimax stratum if and only if the restriction of ξ to U_K^r does not factor through any proper norm map.*

4. Twisting by characters

In this section, we give a modest reduction of Conjecture $\mathcal{J}(N, r)$ to supercuspidal representations which are of minimal depth amongst all representations obtained from them by twisting by a character.

For π a representation of G and χ a character of $F^\times$, we write $\pi\chi$ for the representation $\pi \otimes \chi \circ \det$ of G. We say that an irreducible supercuspidal representation π of G is *of minimal depth in its twist class* if

$$\ell(\pi\chi) \geq \ell(\pi), \text{ for all characters } \chi \text{ of } F^\times.$$

The following lemma is well-known but we include its proof for lack of a *precise* reference. (The case $N = 2$ can be found in [BH06, 13.3 Theorem] while the general case can be extracted from [C84, §4].)

Lemma 4.1. *An irreducible supercuspidal representation π of* G *is of minimal depth in its twist class if and only if it does* not *contain a scalar stratum.*

In the case that N is prime, this implies that any minimal stratum contained in an irreducible supercuspidal representation of minimal depth in its twist class is a minimax stratum. (One can also see this directly from the classification of Carayol [C84] – see also [B87, p209].)

Proof. Let χ be a character of $F^\times$ of depth r, so that $r \geq 0$ is minimal such that χ is trivial on $U^{r+1}(\mathfrak{o}_F)$. For $\mathfrak{A}$ a hereditary $\mathfrak{o}_F$-order of period $e = e(\mathfrak{A}|\mathfrak{o}_F)$, the determinant maps $U^{n+1}(\mathfrak{A})$ surjectively onto $U^{[\frac{n}{e}]+1}(\mathfrak{o}_F)$; thus $\chi \circ \det|U^{re}(\mathfrak{A})$ is a non-trivial character, which is trivial on restriction to $U^{re+1}(\mathfrak{A})$. Moreover, if $r > 0$ then this character takes the form ψ_α, for a scalar stratum $[\mathfrak{A}, re, re-1, \alpha]$ and, conversely, any scalar stratum arises in this way as the restriction of a character of $F^\times$ composed with the determinant.

Suppose now that π is an irreducible representation of G containing a scalar stratum $[\mathfrak{A}, n, n-1, \alpha]$ and let χ be a character of $F^\times$ such that $\chi \circ \det|U^n(\mathfrak{A}) = \psi_\alpha$. Then $\pi\chi^{-1}$ contains the trivial character of $U^n(\mathfrak{A})$ so has depth strictly smaller than $n/e(\mathfrak{A}) = \ell(\pi)$. Thus π is not of minimal depth in its twist class.

Conversely, suppose that π is not of minimal depth in its twist class and let χ be a character of $F^\times$ such that $\pi' = \pi\chi^{-1}$ is of minimal depth in the twist class; in particular, $\ell(\pi') < \ell(\pi)$.

Suppose first that $\ell(\pi') > 0$ and let $[\mathfrak{A}, n, n-1, \beta]$ be a minimal stratum contained in π', which we now know cannot be scalar. Denote by r the level of χ and, if $r > 0$, by $[\mathfrak{A}, re, re-1, \alpha]$ a scalar stratum such that $\chi \circ \det|U^{re}(\mathfrak{A}) = \psi_\alpha$, as above. If $re < n$ then $\pi = \pi'\chi$ also contains the character $\psi_\beta|U^n(\mathfrak{A})$, so that $\ell(\pi) = \ell(\pi')$, which is absurd. Similarly, if $re = n$ then $r > 0$ and π contains the character $\psi_\beta\psi_\alpha$ of $U^n(\mathfrak{A})$, which corresponds to the minimal stratum $[\mathfrak{A}, n, n-1, \beta+\alpha]$; we again get the contradiction that $\ell(\pi) = \ell(\pi')$. Thus $re > n$ and π contains the character ψ_α of $U^{re}(\mathfrak{A})$; that is, π contains the scalar stratum $[\mathfrak{A}, re, re-1, \alpha]$, as required.

The case $\ell(\pi') = 0$ is similar, using the fact that π' then contains the trivial character of $U^1(\mathfrak{A})$, with $\mathfrak{A}$ a maximal $\mathfrak{o}_F$-order in A. □

We will not recall here the definitions of local factors of pairs of supercuspidal representations from [JPSS83]. However, from the definitions (see also [JPSS83, Theorem 2.7]), a straightforward check shows the following, which is surely well-known.

Lemma 4.2. *Let r be a natural number with $r < N$, let π, τ be generic irreducible representations of* G, G_r *respectively, let χ be a character of $F^\times$, and let $s \in \mathbb{C}$. Then*

$$\begin{aligned} L(s, \pi\chi \times \tau) &= L(s, \pi \times \tau\chi), \\ \varepsilon(s, \pi\chi \times \tau, \psi) &= \varepsilon(s, \pi \times \tau\chi, \psi), \\ \gamma(s, \pi\chi \times \tau, \psi) &= \gamma(s, \pi \times \tau\chi, \psi). \end{aligned}$$

We also recall that the depth $\ell(\pi)$ of an irreducible supercuspidal representation can be determined from the conductor of the standard epsilon factor $\varepsilon(s, \pi, \psi) = \varepsilon(s, \pi \times \mathbf{1}, \psi)$, where $\mathbf{1}$ is the trivial representation of G_1 (see [B87]); indeed the same is true for an arbitrary discrete series representation π, by [LR03, Theorem 3.1].

Now we can reduce Conjecture $\mathcal{J}(N, r)$ to the following special case:

Conjecture $\boldsymbol{\mathcal{J}_0(N, r)}$. *If π_1, π_2 are irreducible supercuspidal representations of* G *of minimal depth in their twist class which satisfy hypothesis $\mathcal{H}_{\leq r}$, then $\pi_1 \simeq \pi_2$.*

Proposition 4.3. *For $1 \leq r < N$, Conjecture $\mathcal{J}_0(N, r)$ is equivalent to Conjecture $\mathcal{J}(N, r)$.*

Proof. It is clear that Conjecture $\mathcal{J}(N, r)$ implies Conjecture $\mathcal{J}_0(N, r)$. For the converse, we assume that Conjecture $\mathcal{J}_0(N, r)$ is true, and let π_1, π_2 be irreducible supercuspidal representations of G which satisfy hypothesis $\mathcal{H}_{\leq r}$. Then, for χ any character of $F^\times$ and τ any supercuspidal representation of G_m with $1 \leq m \leq r$, Lemma 4.2 and property $\mathcal{H}_{\leq r}$ imply that

$$\begin{aligned} \gamma(s, \pi_1\chi \times \tau, \psi) &= \gamma(s, \pi_1 \times \tau\chi, \psi) \\ &= \gamma(s, \pi_2 \times \tau\chi, \psi) = \gamma(s, \pi_2\chi \times \tau, \psi). \end{aligned} \tag{4.4}$$

In particular, using the case $m = 1$ with $\tau = \mathbf{1}$ the trivial representation, this implies that $\ell(\pi_1\chi) = \ell(\pi_2\chi)$.

Now we pick a character χ of $F^\times$ such that $\pi_1\chi$ is of minimal depth in its twist class. Then $\pi_2\chi$ is also of minimal depth in its twist class and (4.4) now implies that the representations $\pi_1\chi, \pi_2\chi$ satisfy hypothesis $\mathcal{H}_{\leq r}$. Thus, by the assumption that Conjecture $\mathcal{J}_0(N, r)$ is true, we deduce that $\pi_1\chi \simeq \pi_2\chi$, whence $\pi_1 \simeq \pi_2$ as required. □

5. Unipotent and mirabolic subgroups

Although we have fixed standard mirabolic and maximal unipotent subgroups, it will be convenient in the sequel to allow these to vary, working in the basis-free setting of Section 3. Thus, in this section, we gather some notation for arbitrary mirabolic and maximal unipotent subgroups.

A maximal flag in V

$$\mathcal{F} = \{0 = V_0 \subset V_1 \subset \cdots \subset V_{N-1} \subset V_N = V\},$$

with $\dim_F(V_i) = i$, determines both a maximal unipotent subgroup $U_{\mathcal{F}}$ and a mirabolic subgroup $P_{\mathcal{F}}$ by

$$\begin{aligned} U_{\mathcal{F}} &= \{g \in \mathrm{G} \mid (g-1)V_i \subseteq V_{i-1}, \text{ for } 1 \leq i \leq N\}, \\ P_{\mathcal{F}} &= \{g \in \mathrm{G} \mid (g-1)V \subseteq V_{N-1}\}. \end{aligned}$$

Of course, $P_{\mathcal{F}}$ does not depend on the whole flag $\mathcal{F}$, but $U_{\mathcal{F}}$ does: there is a bijection between maximal flags in V and maximal unipotent subgroups of G.

Given now an ordered basis $\mathcal{B} = (v_1, \dots, v_N)$ of V we get a decomposition $V = \bigoplus_{i=1}^N Y_i$, where $Y_i = \langle v_i \rangle_F$ is the F-linear span of v_i. We set $A_{ij} = \mathrm{Hom}_F(Y_j, Y_i)$ so that $A = \bigoplus_{1\leq i,j\leq N} A_{ij}$, and define $\mathbf{1}_{ij} \in A_{ij}$ by $\mathbf{1}_{ij}(v_j) = v_i$. Thus saying that $a = (a_{ij})$ is the matrix of some $a \in A$ with respect to $\mathcal{B}$, is the same as saying

$$a = \sum_{1\leq i,j\leq N} a_{ij}\mathbf{1}_{ij}.$$

We also get a maximal flag $\mathcal{F}_{\mathcal{B}}$ by setting

$$V_i = \bigoplus_{j=1}^{i} Y_j = \langle v_1, \dots, v_i \rangle_F.$$

We denote the corresponding unipotent subgroup and mirabolic by $U_{\mathcal{B}}$ and $P_{\mathcal{B}}$ respectively. Finally, we get a nondegenerate character $\psi_{\mathcal{B}}$ of $U_{\mathcal{B}}$, given by

$$\psi_{\mathcal{B}}(u) = \psi_F\left(\sum_{i=1}^{N-1} u_{i,i+1}\right),$$

where $u \in U_{\mathcal{B}}$ and (u_{ij}) is the matrix of u with respect to the basis $\mathcal{B}$. The same formula also defines a function $\psi_{\mathcal{B}}$ on $P_{\mathcal{B}}$ (though it is not a character).

The standard mirabolic subgroup, maximal unipotent subgroup, and nondegenerate character, are given by choosing $\mathcal{B}$ to be the standard basis of $V = F^N$.

6. Minimax strata

In this section, for a minimax stratum $[\mathfrak{A}, n, n-1, \beta]$, we examine the relationship between the basis with respect to which β is in companion form and the order $\mathfrak{A}$.

We first recall some material from [BH98, Section 2]. Suppose $\beta \in A$ is such that $E = F[\beta]$ is a field extension of F of maximal degree N. We define the *function* ψ_β of A by

$$\psi_\beta(x) = \psi_F \circ \mathrm{tr}_{A/F}(\beta(x-1)), \quad \text{for } x \in A.$$

There is an ordered basis $\mathcal{B} = (v_1, \ldots, v_N)$ for V with respect to which

$$\psi_\beta | U_{\mathcal{B}} = \psi_{\mathcal{B}}$$

is the nondegenerate character associated to $\mathcal{B}$. Indeed, there is, up to $E^\times$-conjugacy, a unique maximal unipotent subgroup U such that ψ_β is trivial on the derived group U^{der}, and we have $U_{\mathcal{B}} = U$. More explicitly, if $v_1 \in V$ is arbitrary, then putting

$$v_j = \beta^{j-1} v_1, \text{ for } 2 \le j \le N, \tag{6.1}$$

gives a basis as required, and every such basis arises in this way. With respect to the basis $\mathcal{B}$, the matrix of β is the companion matrix of the minimum polynomial of β.

The crucial (though trivial) observation is that we also have an equality of functions (not characters)

$$\psi_\beta | P_{\mathcal{B}} = \psi_{\mathcal{B}}. \tag{6.2}$$

Since E/F is maximal, there is a unique hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A normalized by $E^\times$; more precisely, identifying V with E, it is given by the $\mathfrak{o}_F$-lattice chain $\{\mathfrak{p}_E^i \mid i \in \mathbb{Z}\}$ so we have

$$\mathfrak{A} = \{x \in A \mid x\mathfrak{p}_E^i \subseteq \mathfrak{p}_E^i, \text{ for all } i \in \mathbb{Z}\}.$$

It has $\mathfrak{o}_F$-period $e(E/F)$, and consequently $\nu_{\mathfrak{A}}(\beta) = \nu_E(\beta)$. We assume $n = -\nu_{\mathfrak{A}}(\beta) > 0$ so that $[\mathfrak{A}, n, 0, \beta]$ is a pure stratum, and then the restriction of the function ψ_β defines a character of $U^{[\frac{n}{2}]+1}(\mathfrak{A})$; moreover, by (6.2), we have an equality of characters

$$\psi_\beta | U^{[\frac{n}{2}]+1}(\mathfrak{A}) \cap P_{\mathcal{B}} = \psi_{\mathcal{B}}. \tag{6.3}$$

Now we have the following:

Lemma 6.4. *Suppose $[\mathfrak{A}, n, 0, \beta]$ is a pure stratum such that the field $E = F[\beta]$ is of maximal degree in A. Write B for the centralizer of E in A and $\mathfrak{B} = \mathfrak{A} \cap B$, and let P be* any *mirabolic subgroup of* G. *Then, for any integers $k > m \ge 1$, we have*

$$\big(U^m(\mathfrak{B})U^k(\mathfrak{A})\big) \cap P = U^k(\mathfrak{A}) \cap P.$$

Proof. Notice that actually $B = E$ and $\mathfrak{B} = \mathfrak{o}_E$ in this situation. We pick an arbitrary uniformizer ϖ_E for E. We prove that, for any $m \ge 1$,

$$\big(U^m(\mathfrak{B})U^{m+1}(\mathfrak{A})\big) \cap P = U^{m+1}(\mathfrak{A}) \cap P,$$

and the result follows by iteration. This claim is equivalent to the following additive statement: setting $\mathcal{P} = \{p - 1 \mid p \in P\}$, we have

$$\left(\mathfrak{p}_E^m + \mathfrak{P}^{m+1}\right) \cap \mathcal{P} = \mathfrak{P}^{m+1} \cap \mathcal{P},$$

where $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$ as usual. So suppose $x \in \mathfrak{p}_E^m$ and $y \in \mathfrak{P}^{m+1}$ are such that $z := x + y \in \mathcal{P}$. In particular, z has eigenvalue 0, and the same is then true of $\varpi_E^{-m} z \in \mathfrak{o}_E + \mathfrak{P}$, and of its image in $\mathfrak{A}/\mathfrak{P}$. However, this image is in $k_E \hookrightarrow \mathfrak{A}/\mathfrak{P}$, and the only element of k_E with eigenvalue 0 is 0 itself. Thus $\varpi_E^{-m} z \in \mathfrak{P}$ and $z \in \mathfrak{P}^{m+1}$, as required. □

If $\mathcal{B} = (v_1, \ldots, v_N)$ is an ordered basis, we put $Y_i = \langle v_i \rangle_F$ and $A_{ij} = \mathrm{End}_F(Y_j, Y_i)$, as before. We say that $\mathcal{B}$ is a *splitting basis for* $\mathfrak{A}$ if

$$\mathfrak{A} = \bigoplus_{1 \le i,j \le N} \left(\mathfrak{A} \cap A_{ij}\right).$$

In particular, any basis with respect to which $\mathfrak{A}$ is in standard form is a splitting basis. Any permutation of a splitting basis $\mathcal{B}$ is also a splitting basis; more generally, any basis obtained by a monomial change of basis from $\mathcal{B}$ is a splitting basis.

Now we specialize to the case of a minimax stratum $[\mathfrak{A}, n, n-1, \beta]$; in this case, we prove that any basis $\mathcal{B}$ with respect to which ψ_β defines the nondegenerate character $\psi_{\mathcal{B}}$ of $U_{\mathcal{B}}$ is also a splitting basis for $\mathfrak{A}$.

Lemma 6.5. *Suppose $[\mathfrak{A}, n, n-1, \beta]$ is a minimax stratum and let $\mathcal{B} = (v_1, \ldots, v_N)$ be an ordered basis for V such that $\psi_\beta | U_{\mathcal{B}}$ is the nondegenerate character $\psi_{\mathcal{B}}$. Then $\mathcal{B}$ is a splitting basis for $\mathfrak{A}$. Moreover, writing $Y_i = \langle v_i \rangle_F$ and $A_{ij} = \mathrm{End}_F(Y_j, Y_i)$, for each $1 \le i, j \le N$ the lattice $\mathfrak{A} \cap A_{ij}$ depends only on i, j and the depth $n/e(\mathfrak{A}|\mathfrak{o}_F)$ of the stratum.*

Proof. The basis $\mathcal{B}$ is given as in (6.1) so, identifying V with E via $v_1 \mapsto 1$, we may assume $\mathcal{B} = (1, \beta, \ldots, \beta^{N-1})$. We put $n = -\nu_E(\beta)$, $e = e(E/F) = e(\mathfrak{A}|\mathfrak{o}_F)$ and $f = f(E/F)$. Multiplication by n induces a bijection $\mathbb{Z}/e\mathbb{Z} \to \mathbb{Z}/e\mathbb{Z}$. Thus, for each $i = 0, \ldots, e-1$, there is a unique integer r_i, with $0 \le r_i < e$, such that $nr_i \equiv -i \pmod{e}$. We write $nr_i = d_i e - i$; then the fact that β is minimal implies that, for each $i = 0, \ldots, e-1$, the set

$$\mathcal{B}_i' := \{\varpi_F^{nk+d_i} \beta^{ek+r_i} \mid 0 \le k \le f-1\}$$

reduces to a basis for $\mathfrak{p}_E^i/\mathfrak{p}_E^{i+1}$. Thus the ordered basis $\mathcal{B}'$, obtained by ordering each $\mathcal{B}_i'$ arbitrarily and then concatenating $\mathcal{B}_0', \ldots, \mathcal{B}_{e-1}'$, is a splitting basis for $\mathfrak{A}$ with respect to which $\mathfrak{A}$ is in standard form. The change of basis matrix from $\mathcal{B}'$ to $\mathcal{B} = (1, \ldots, \beta^{ef-1})$ is monomial with entries from $\varpi_F^{\mathbb{Z}}$. Moreover, the (i, j) entry depends only on (i, j, n, e), and the result follows since n, e are determined by n/e as they are coprime. □

7. Jacquet's conjecture

Finally, we prove that if π_1, π_2 are irreducible supercuspidal representations of G with the same depth and central character containing minimax strata, then they have a special pair of Whittaker functions. (It is somewhat remarkable that the equalities of depth and central character are sufficient here.) When N is prime, any positive depth supercuspidal of minimal depth in its twist class contains a minimax stratum, so Jacquet's conjecture will follow from Propositions 2.3 and 4.3, together with the depth zero case from [JNS15, Corollary 1.7].

At this point, we need to recall a little more on the construction of the supercuspidal representations of G. Since it is all we will need, we only give definitions for representations which contain a minimax stratum. Thus, in the slightly different language of [BK93], we are recounting the constructions of Carayol [C84].

Let $[\mathfrak{A}, n, n-1, \beta]$ be a minimax stratum, with $E = F[\beta]$. Associated to the simple stratum $[\mathfrak{A}, n, 0, \beta]$, we have the following compact open subgroups of G, contained in the normalizer $\mathfrak{K}(\mathfrak{A})$:

$$\begin{aligned} H^1 &= H^1(\beta, \mathfrak{A}) = U^1(\mathfrak{o}_E)U^{\left[\frac{n}{2}\right]+1}(\mathfrak{A}), \\ J^1 &= J^1(\beta, \mathfrak{A}) = U^1(\mathfrak{o}_E)U^{\left[\frac{n+1}{2}\right]}(\mathfrak{A}), \\ J &= J(\beta, \mathfrak{A}) = U^0(\mathfrak{o}_E)U^{\left[\frac{n+1}{2}\right]}(\mathfrak{A}), \\ \mathbf{J} &= \mathbf{J}(\beta, \mathfrak{A}) = E^\times U^{\left[\frac{n+1}{2}\right]}(\mathfrak{A}). \end{aligned}$$

A *simple character* is then a character of H^1 which extends the character ψ_β of $U^{\left[\frac{n}{2}\right]+1}(\mathfrak{A})$. Given such a simple character θ, there is a unique irreducible representation η of J^1 which contains θ on restriction to H^1 (indeed, it is a multiple). An *extended maximal simple type* is then an irreducible representation Λ of $\mathbf{J}$ which extends η. Given such a maximal simple type, the representation

$$\text{c-ind}_{\mathbf{J}}^{\mathrm{G}} \Lambda$$

is irreducible and supercuspidal and, moreover, every irreducible supercuspidal representation containing θ arises in this way.

Any irreducible supercuspidal representation π containing a minimax stratum $[\mathfrak{A}, n, n-1, \beta]$ contains some simple character θ associated to a simple stratum $[\mathfrak{A}, n, 0, \beta']$, with $[\mathfrak{A}, n, n-1, \beta']$ minimax and equivalent to $[\mathfrak{A}, n, n-1, \beta]$. Thus π also contains $[\mathfrak{A}, n, n-1, \beta']$ and we may assume $\beta' = \beta$.

Recall that we need to exhibit a special pair of Whittaker functions. In [PS08], Whittaker functions are constructed which carry the properties that we need. We record the result in a form that does not require additional background. Recall first (see §6) that to β we associate a basis $\mathcal{B}$ and a unipotent subgroup $U_{\mathcal{B}}$ such that

$$\psi_\beta | U_{\mathcal{B}} = \psi_{\mathcal{B}}$$

is the nondegenerate character associated to $\mathcal{B}$.

Proposition 7.1 ([PS08, Theorem 5.8], [JNS15, §4.3])**.** *There exists a Whittaker function W for π such that* $\mathrm{Supp}(W) \subseteq U_{\mathcal{B}}\mathbf{J}$ *and such that, for $g \in P_{\mathcal{B}}$,*

$$W(g) = \begin{cases} \psi_{\mathcal{B}}(u)\theta(h) & \textit{if } g = uh \in (J^1 \cap U_{\mathcal{B}})H^1, \\ 0 & \textit{otherwise.} \end{cases}$$

Our main result is:

Proposition 7.2. *For $i = 1, 2$, let π_i be a (positive depth) unitarizable supercuspidal representation of* G *containing a minimax stratum. Suppose that π_1, π_2 have the same depth and the same central character. Then π_1, π_2 have a special pair of Whittaker functions.*

We remark that a special case of this result, in the case of *simple* supercuspidal representations, was proved in [AL15, Theorem 4.6].

Proof. For $i = 1, 2$, let $[\mathfrak{A}_i, n_i, n_i - 1, \beta_i]$ be a minimax stratum of period $e_i = e(\mathfrak{A}_i|\mathfrak{o}_F)$ contained in π_i, with the property that π_i also contains a simple character θ_i of $H_i^1 = H^1(\beta_i, \mathfrak{A}_i)$. Since the representations have the same depth we have $n_1/e_1 = n_2/e_2$; since they are minimax, we have $\gcd(n_i, e_i) = 1$ and we may write $n = n_1 = n_2$ and $e = e_1 = e_2$.

Fix $v \in V$ and let $g \in \mathrm{G}$ be the change of basis matrix from the basis $\mathcal{B} = (v, \beta_1 v, \ldots, \beta_1^{N-1} v)$ to $(v, \beta_2 v, \ldots, \beta_2^{N-1} v)$. Then, replacing the stratum $[\mathfrak{A}_2, n, n-1, \beta_2]$ and the simple character θ_2 by their conjugates by g, we can assume that *both* β_i are in companion matrix form with respect to $\mathcal{B}$, i.e. $\beta_1^{j-1} v = \beta_2^{j-1} v$, for $1 \le j \le N$. By Lemma 6.5, the hereditary orders coincide: $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}$. By Lemma 6.4, we have $H_i^1 \cap P_{\mathcal{B}} = U^{\left[\frac{n}{2}\right]+1}(\mathfrak{A}) \cap P_{\mathcal{B}}$ so, by (6.3), we have

$$\theta_i | H_i^1 \cap P_{\mathcal{B}} = \psi_{\mathcal{B}},$$

independent of i. Moreover, we then have $\mathrm{Hom}_{H_i^1 \cap U_{\mathcal{B}}}(\theta_i, \psi_{\mathcal{B}}) \neq 0$.

We abbreviate $J_i^1 = J^1(\beta_i, \mathfrak{A})$ and $\mathbf{J}_i = \mathbf{J}(\beta_i, \mathfrak{A})$, and denote by η_i the unique irreducible representation of J_i^1 containing θ_i. Then we also have $\mathrm{Hom}_{J_i^1 \cap U_{\mathcal{B}}}(\eta_i, \psi_{\mathcal{B}}) \neq 0$, by [PS08, Theorem 2.6]. Writing $\pi_i =$

c-ind$_{\mathbf{J}_i}^{\mathrm{G}}\Lambda_i$, with Λ_i extending η_i, we have $\mathbf{J}_i \cap U_{\mathcal{B}} = J_i^1 \cap U_{\mathcal{B}}$ and we see that $\mathrm{Hom}_{\mathbf{J}_i \cap U_{\mathcal{B}}}(\Lambda_i, \psi_{\mathcal{B}}) \neq 0$. Thus we have a Whittaker function W_i as constructed in Proposition 7.1 relative to the pair $(U_{\mathcal{B}}, \psi_{\mathcal{B}})$ and these coincide on $P_{\mathcal{B}}$ since, for $g \in P_{\mathcal{B}}$,

$$W_i(g) = \begin{cases} \psi_{\mathcal{B}}(g) & \text{if } g \in (J_i^1 \cap U_{\mathcal{B}})(H_i^1 \cap P_{\mathcal{B}}), \\ 0 & \text{otherwise.} \end{cases}$$

Putting $\mathbf{K} = \mathfrak{K}(\mathfrak{A})$, both W_i are $\mathbf{K}$-special (see [JNS15, Lemma 4.2]) so we have found a special pair of Whittaker functions. □

Corollary 7.3. *For N prime, Conjecture $\mathcal{J}(N, [\frac{N}{2}])$ is true.*

Proof. Let π_1, π_2 be unitarizable supercuspidal representations of G satisfying hypothesis $\mathcal{H}_{\leq [N/2]}$, and of minimal depth in their twist classes. In particular, from hypothesis $\mathcal{H}_1$, the representations π_1, π_2 have the same central character and depth. If both have depth zero then they are equivalent by [JNS15, Corollary 1.7], so we assume they are of positive depth. Since N is prime, π_1, π_2 both contain minimax strata, so Proposition 7.2 implies that they have a special pair of Whittaker functions, and Proposition 2.3 implies that $\pi_1 \simeq \pi_2$. Thus Conjecture $\mathcal{J}_0(N, [\frac{N}{2}])$ is true, and the result now follows from Proposition 4.3. □

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