Two-dimensional breaking wave impact on a vertical wall

by

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1 Introduction

We investigate the violent slamming of a steep wave onto a vertical wall. The novelty of our work relies on the assumption of a breaking type impact. With reference to the plane impermeable wall of Fig. 1, it is assumed that a thin air pocket is formed adjacent to the wall, between the lower and the upper impacted sections of the wall. The solution process, as outlined in the sequel, could be extended to accommodate an uneven bottom configuration. However, in the present study it is explicitly assumed that the bottom is horizontal.

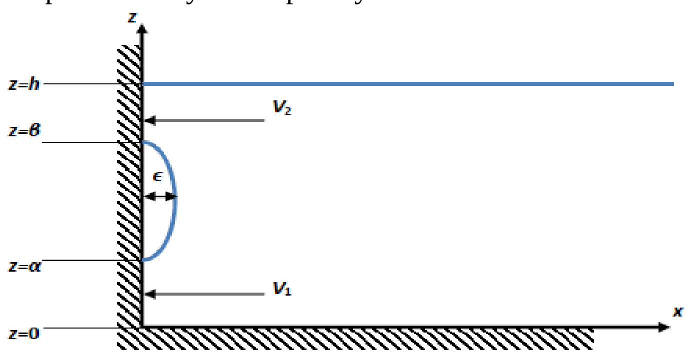

Figure 1 Definition sketch; The curves are free surfaces on which $\phi = 0$.

The boundary conditions at x=0 in the intervals $0 < z < \alpha$ and $\beta < z < h$ are defined by default through the normal components of the velocities of the wave before impact, which are assumed to be the constants V_1 and V_2, respectively. In the general case we assume that $V_1 \neq V_2$. In the intermediate region $\alpha < z < \beta$ the associated boundary condition is governed by the pressure which can be a function of z. The present analysis assumes that the pressure impulse (and accordingly the potential) in the air pocket is zero. Nevertheless the outlined methodology can be extended to accommodate non zero constant potential using the same procedure. It is also assumed that at the time of impact the width of the intermediate section between the wave front and the wall $\varepsilon \to 0$.

2 The mixed boundary value problem

In the realm of potential theory, the mixed boundary value problem in terms of $\phi(x,z)$, which is the sudden change in the velocity potential is:

$$\nabla^2\phi = 0 , \qquad (x > 0, \quad 0 < z < h) , \tag{1}$$

$$\partial\phi / \partial z = 0 , \qquad (x > 0, \quad z = 0) , \tag{2}$$

$$\phi = 0 , \qquad (x > 0, \quad z = h) , \tag{3}$$

$$\partial\phi / \partial x = V_1 , \qquad (x = 0, \quad 0 < z < \alpha) , \tag{4}$$

$$\phi = 0 , \qquad (x = 0, \quad \alpha < z < \beta) , \tag{5}$$

$$\partial\phi / \partial x = V_2 , \qquad (x = 0, \quad \beta < z < h) , \tag{6}$$

$$\phi \to 0 , \qquad (x \to \infty, \quad 0 < z < h) . \tag{7}$$

The form of the solution that satisfies eqs. (1)-(3) and the far-field behaviour (7) is

$$\phi(x,z)=\sum_{n=1}^{\infty} C_n \cos(\lambda_n z)e^{-\lambda_n x} \tag{8}$$

where $\lambda_n=(n-1/2)\pi/h$, $n \in \mathbb{N}$ and C_n are coefficients to be found from the remaining boundary conditions (4)-(6). Note that the present analysis is applied at the instant of the impact and hence C_n are constants, independent of time.

3 Triple trigonometrical series

Introducing eq. (8) into the boundary conditions (4)-(6) we find the following problem that involves triple trigonometrical series

$$\sum_{n=1}^{\infty} \lambda_n C_n \cos(\lambda_n z)=-V_1, \qquad (0<z<\alpha), \tag{9}$$

$$\sum_{n=1}^{\infty} C_n \cos(\lambda_n z)=0, \qquad (\alpha<z<\beta), \tag{10}$$

$$\sum_{n=1}^{\infty} \lambda_n C_n \cos(\lambda_n z)=-V_2, \qquad (\beta<z<h). \tag{11}$$

Most literature on mixed boundary value problems that involve trigonometrical series, concern dual – not triple – relations. Relevant examples are the studies of Tranter [1-3]. For a review of dual trigonometrical series the reader is referred to the classical book of Sneddon [4], which cites nearly all studies prior to the time it was published. Triple trigonometrical series concern more complicated mixed-boundary value problems and relevant examples are the works of Tranter [5] and Kerr et al. [6]. However, in both papers the boundary data in two of the three boundary sets are identically zero.

The required analysis to solve the mixed problem governed by eqs. (9)-(11) is quite complicated and therefore only the basic steps are given in the sequel. After several mathematical manipulations, the triple series of eqs. (9)-(11) are transformed into the following three conditions on new coefficients B_n

$$\sum_{n=1}^{\infty} B_n J_\nu(\mu_n y)=\sqrt{2/(\pi y)}F(\beta y), \qquad (0<y<b), \tag{12}$$

$$\sum_{n=1}^{\infty} \mu_n^{-1} B_n J_\nu(\mu_n y)=\sqrt{2/(\pi y\beta^2)}G(\beta y), \qquad (b<y<1), \tag{13}$$

$$\sum_{n=1}^{\infty} B_n J_\nu(\mu_n y)=0, \qquad (1<y<c). \tag{14}$$

where $B_n=\mu_n^{1/2}\left(\lambda_n C_n+(-1)^{n-1}2V_2/(h\lambda_n)\right)$, $F(\beta y)=-V_1+V_2$, $G(\beta y)=(2V_2/h)\sum_{n=1}^{\infty}(-1)^{n-1}\cos(\mu_n y)/\lambda_n^2$, $\nu=-1/2$, $b=\alpha/\beta$, $\mu_n=\lambda_n\beta$, $c=h/\beta$ $\mu_n=\lambda_n\beta$, $y=z/\beta$. The next step is to reduce the triple trigonometrical series to dual by satisfying the last one (14). This is accomplished assuming an alternative form of the unknown constants B_n according to which

$$B_n=\frac{1}{\mu_n^{1+p/2}J_{\nu+1}^2(\mu_n c)}\sum_{m=0}^{\infty} E_m J_{\nu+2m+1+p/2}(\mu_n), \quad -1\le p\le 1. \tag{15}$$

Eq. (15) satisfies eq. (14) as it holds that [7]

$$\sum_{n=1}^{\infty}\frac{J_{\nu+2m+1+p/2}(\mu_n)J_\nu(\mu_n y)}{\mu_n^{1+p/2}J_{\nu+1}^2(\mu_n c)}=0, \qquad y>1. \tag{16}$$

Introducing eq. (15) into eqs. (12) and (13) the problem is reduced to the dual trigonometrical series

$$E_0 a_1(y)+\sum_{m=1}^{\infty} E_m \sum_{n=1}^{\infty}\frac{J_{\nu+2m+1+p/2}(\mu_n)J_\nu(\mu_n y)}{\mu_n^{1+p/2}J_{\nu+1}^2(\mu_n c)}=\sqrt{2/(\pi y)}F(\beta y), \qquad (0<y<b), \tag{17}$$

$$E_0 a_2(y) + \sum_{m=1}^{\infty} E_m \sum_{n=1}^{\infty} \frac{J_{\nu+2m+1+p/2}(\mu_n) J_\nu(\mu_n y)}{\mu_n^{2+p/2} J_{\nu+1}^2(\mu_n c)} = \sqrt{2/(\pi y \beta^2)} G(\beta y), \qquad (b < y < 1), \tag{18}$$

where

$$a_j(y) = \sum_{n=1}^{\infty} \frac{J_{\nu+1+p/2}(\mu_n) J_\nu(\mu_n y)}{\mu_n^{j+p/2} J_{\nu+1}^2(\mu_n c)} \text{ for } j=1,2. \tag{19}$$

The reduced model of eqs. (17) and (18) is processed further using specific expressions that relate the infinite series of Bessel functions which appear in eqs. (17) and (18) and the Sonine-Schafheitlin integral. In particular it can be shown that

$$\int_0^\infty u^{1/2} J_{2m}(u) J_{-1/2}(yu)\, du = \sqrt{2/y}\; {}_2F_1(m+1/2, -m+1/2; 1/2; y^2) / \Gamma(2m+1), \tag{20}$$

$$\int_0^\infty u^{-1/2} J_{2m}(u) J_{-1/2}(yu)\, du = \sqrt{1/2y}\, m^{-1}\; {}_2F_1(m, -m; 1/2; y^2) / \Gamma(2m+1), \tag{21}$$

where ${}_2F_1$ is the hypergeometric function with single variable. Finally, after further mathematical manipulations the dual trigonometrical series of eqs. (17) and (18) are reduced to the more compact forms

$$\chi(\psi)\zeta_0/2 + \sum_{m=1}^{\infty} m^{-1} \zeta_m \cos(m\psi) = F^*(\psi), \qquad (0 < \psi < d^*), \tag{22}$$

$$\zeta_0/2 + \sum_{m=1}^{\infty} \zeta_m \cos(m\psi) = G^*(\psi), \qquad (d^* < \psi < \pi). \tag{23}$$

The new unknowns of the problem are now the coefficients ζ_m, whilst the parameters $\psi, \chi(\psi), F^*(\psi), G^*(\psi)$ depend only on the given data of the problem (via expressions omitted for brevity). Dual trigonometrical series of the form of eqs. (22) and (23) have been treated by several authors (see Sneddon [4]). The coefficients ζ_m are

$$\zeta_0 = \frac{2}{\pi}\left[\frac{1}{\sqrt{2}}\int_0^{d^*} h_1(t)dt + \int_{d^*}^{\pi} G^*(t)dt\right] \tag{24}$$

$$\zeta_m = \frac{2}{\pi}\left[\frac{1}{2\sqrt{2}}\int_0^{d^*} h_1(t)\left[P_n\cos(t) + P_{n-1}\cos(t)\right]dt + \int_{d^*}^{\pi} G^*(t)\cos(mt)dt\right],\ m = 1,2,3,\ldots \tag{25}$$

where P_n denotes the Legendre polynomial of degree n and

$$h_1(t) = 2\frac{d}{dt}\int_0^t \frac{\sin(u/2)}{\sqrt{\cos(u)-\cos(t)}}\left[-\frac{dF^*(u)}{du} + \frac{1}{2}\zeta_0\frac{d\chi(u)}{du} - \frac{2}{\pi}\sum_{m=1}^{\infty}\left(\int_{d^*}^{\pi} G^*(v)\cos(mv)dv\right)\sin(mu)\right]du\,. \tag{26}$$

4 Some results

Although eqs. (24)-(26) appear in a compact set of expressions, the coefficient ζ_0 lies within the definition of the function $h_1(t)$. The right-hand side of eq. (24) depends on ζ_0, so this coefficient can be computed after rearrangement of terms or by an iterative scheme. Once ζ_0 is accurately evaluated, expression (26) is wholly determined and can be used in eq. (25) to compute the other coefficients. Also, the truncation of the infinite series expressions has to be handled sensitively to ensure that eq. (16) is satisfied accurately and at least 200 modes were found necessary to achieve an accuracy of 4 significant digits for the coefficients ζ_m. The rapid decrease in magnitude of ζ_m, with increasing m, is shown in Fig. 2.

In the following we present some results for wave impacts for which we hold fixed h=1m, β=0.8m, $V_2=(gh)^{1/2}\approx$3m/s and vary α and V_1. The solution process indeed converges quite fast, not only for the leading term ζ_0 but also for the other expansion coefficients ζ_m, $m>0$. Figs. 3 and 4 show the pressure impulse $p=-\rho\phi$ [see eq. (8)] for unit density ρ, along the lower portion of the wall for several α (Fig. 3) and several values of the velocity V_1 (Fig. 4). As expected the pressure impulse is reduced for lower velocities. An interesting result is that the pressure impulse obtains its maximum value at the bottom. Also, for the same velocities the pressure impulse is decreased for longer wetted sections (Fig. 3). Further fluid mechanical consequences of the results will be presented in the talk.

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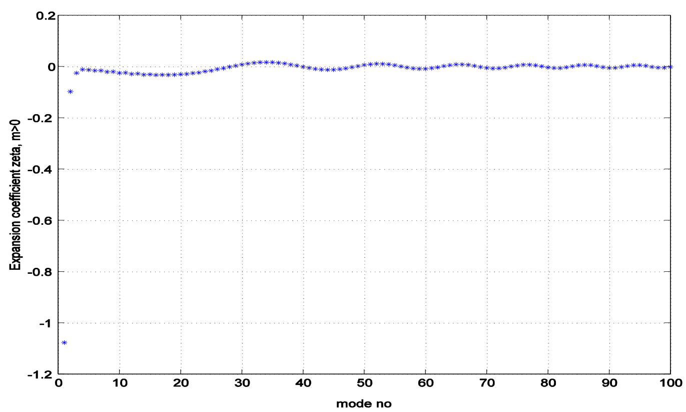

Figure 2 The expansion coefficients ζ_m, $m>0$ for h=1, α=0.1, β=0.8 and V_1=V_2=3 where $\zeta_0=-1.9744$.

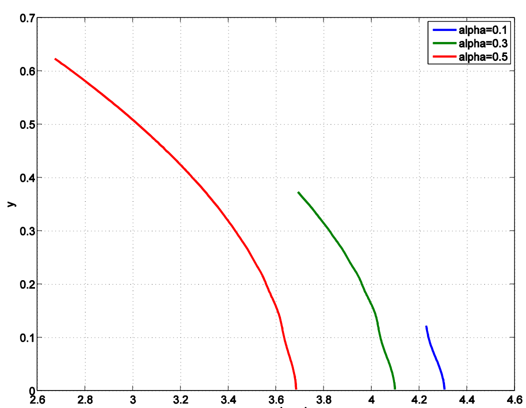

Figure 3 Normalized pressure impulse for equal velocities V_1=V_2=3 and variable α=0.1, 0.3 and 0.5.

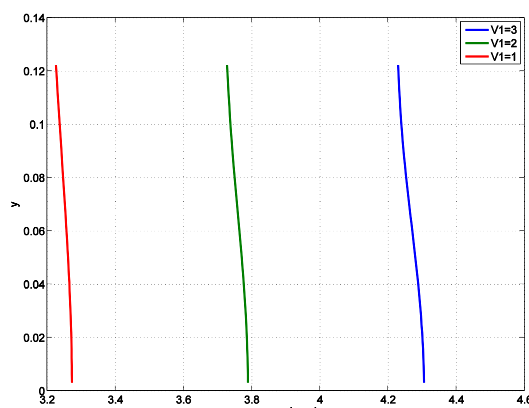

Figure 4 Normalized pressure impulse for α=0.1 and variable velocity V_1=3, 2 and 1.