ORBIT GROWTH FOR ALGEBRAIC FLIP SYSTEMS

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Abstract. An algebraic flip system is an action of the infinite dihedral group by automorphisms of a compact abelian group X. In this paper, a fundamental structure theorem is established for irreducible algebraic flip sytems, that is, systems for which the only closed invariant subgroups of X are finite. Using irreducible systems as a foundation, for expansive algebraic flip systems, periodic point counting estimates are obtained that lead to the orbit growth estimate

$$Ae^{hN} \leqslant \pi(N) \leqslant Be^{hN}$$

where $\pi(N)$ denotes the number of orbits of length at most N, A and B are positive constants and h is the topological entropy.

1. Background and main results

Stemming from the seminal works of Ja. Sinaĭ [25] and G. Margulis [16], periodic orbits in dynamical systems have been investigated using orbit growth functions, with entropy featuring as a constant controlling the exponential growth. An extensive body of work now exists for both flows and discrete time dynamical systems (see for example, [21], [20], [27], [8], [9]). The study of orbit growth functions for dynamical group actions in general was pursued in [18] and [19]. This context encompasses the case of a single invertible transformation which corresponds to an action of the group $G = \mathbb{Z}$.

Let G be a finitely generated group acting on some set X, with the action written as $x \mapsto g \cdot x$. The set $\mathcal{L} = \mathcal{L}(G)$ of finite index subgroups of G becomes a locally finite poset with the order arising from inclusion. For $L \in \mathcal{L}$, the number of L-periodic points in X is

$$\mathsf{F}(L) = |\{x \in X : g \cdot x = x \text{ for all } g \in L\}|. \tag{1}$$

An L-periodic orbit is the orbit of a point with stabilizer L, and the length of the orbit is denoted $[L] = [G : L]$, the index of L in G. Assuming that there are only finitely many orbits of length n for each $n \geqslant 1$, the number of L-periodic orbits is

$$\mathsf{O}(L) = \frac{1}{[L]} \left|\{x \in X : g \cdot x = x \Longleftrightarrow g \in L\}\right|, \tag{2}$$

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and the *orbit counting function* is defined by

$$\pi(N) = \sum_{L \in \mathcal{L} : [L] \leqslant N} \mathsf{O}(L).$$

It is well known that if $G = \mathbb{Z}$ acts, in the obvious way, via a hyperbolic automorphism on a torus, then

$$\pi(N) \sim C \frac{e^{hN}}{N},$$

where C is a positive constant and h is the topological entropy. Less well known is that for actions of $G = \mathbb{Z}^2$, some of the most natural analogues of these hyperbolic actions may have orbit asymptotics of differing shapes. In particular, [19] deals with examples where for all sufficiently large N,

$$Ae^{gN} \leqslant \pi(N) \leqslant Be^{gN},$$

where A and B are positive constants independent of N, and g is the upper growth rate of periodic points (which may or may not be equal to the entropy in this context [15]). We write this more briefly as $\pi(N) = \Theta(e^{gN})$. In contrast, [18, Ex. 1] is an example for which there is unlikely to be an estimate for $\pi(N)$ of a similarly simple form. This dichotomy is made more explicit by using an averaged orbit counting function (the dynamical Mertens' function) in place of $\pi(N)$ in [19], and is thought to depend on the relationship between entropy and the upper growth rate of periodic points.

The purpose of this article is to investigate orbit growth for another class of dynamical systems that also naturally generalizes the class of discrete time dynamical systems. Actions of the infinite dihedral group

$$D_\infty = \langle a, b : b^2 = 1, ab = ba^{-1} \rangle. \tag{3}$$

correspond to so-called flip systems [13] and arise as a result of time reversals in discrete time dynamical systems and flows (see for example [1], [5], [10], [24]). By an *algebraic flip system*, we mean an action of D_∞ by continuous automorphisms of a compact (metrizable) abelian group X. For convenience, write the generators of the action as α and β, where $\alpha(x) = a \cdot x$ and $\beta(x) = b \cdot x$, $x \in X$, and consider an algebraic flip system as a triple (X, α, β).

Since D_∞ is one of the simplest non-abelian examples of an infinite polycyclic group [28, Ch. 2], algebraic flip systems represent some of the most fundamental examples to which general treatments such as [22, Ch. 1] and [6] apply. Consequently, dynamical properties such as ergodicity and mixing may be interpreted easily using existing algebraic characterizations (see Remarks 2.5). However, questions concerning orbit growth for algebraic flip systems are not answered as readily, and this also motivates this work. The main result of this paper concerning orbit growth is the following.

Theorem 1.1. *If (X, α, β) is an expansive algebraic flip system, then*

$$\pi(N) = \Theta(e^{hN}),$$

where h is the topological entropy.

As is standard in the study of algebraic dynamical systems [22], the main structural results are developed in terms of the Pontryagin dual objects $\widehat{X}$, $\widehat{\alpha}$ and $\widehat{\beta}$. In general, $\widehat{X}$ is considered as a $\mathbb{Z}[t^{\pm 1}]$-module M, where t is an indeterminate, by identifying $\widehat{\alpha}$ with multiplication by t and extending in a natural way to polynomials. It is well known (see for example [23, Th. 2.6]) that if X is infinite and (X, α) is irreducible (that is, the only closed subgroups of X that are invariant under the action are finite), then M contains a finite index submodule of the form

$$R = \mathbb{Z}[t^{\pm 1}]/(f), \tag{4}$$

where $f \in \mathbb{Z}[t^{\pm 1}]$ is irreducible. For any $f \in \mathbb{Z}[t^{\pm 1}]$, let $f^*(t) = f(t^{-1})$ and $R^* = \mathbb{Z}[t^{\pm 1}]/(f^*)$. This defines an involutive automorphism (more briefly referred to throughout as an involution) of $\mathbb{Z}[t^{\pm 1}]$ and induces isomorphisms $R \to R^*$ and $R^* \to R$, whose composition is the identity. Fundamental to our analysis are $\mathbb{Z}[t^{\pm 1}]$-modules of the form $R \times R^*$ with an involution $\widehat{\beta} : R \times R^* \to R \times R^*$ given by

$$\widehat{\beta}(k, l) = (l^*, k^*), \tag{5}$$

for all $(k, l) \in R \times R^*$. Our structural starting point is the following.

Theorem 1.2. *Suppose X is an infinite compact abelian group and the algebraic flip system (X, α, β) is irreducible. Then there exists an irreducible polynomial $f \in \mathbb{Z}[t^{\pm 1}]$ such that precisely one of the following holds.*

(i) *$M = \widehat{X}$ contains a finite index submodule isomorphic to $R \times R^*$, where R is given by (4) with $(f) \neq (f^*)$, and the restriction of $\widehat{\beta}$ is given by (5).*

(ii) *$M = \widehat{X}$ contains a finite index submodule isomorphic to (4) with $(f) = (f^*)$, $R = R^*$, and the restriction of $\widehat{\beta}$ is given by either $k \mapsto k^*$ or $k \mapsto -k^*$, $k \in R$.*

When (X, α, β) is expansive and irreducible, in case (i) above, periodic point counting estimates may be obtained quite easily using well known results for a single automorphism. However, in case (ii) these results cannot be used directly; here we observe that $(f) = (f^*)$ implies $f = t^m g$ for some $m \in \mathbb{Z}$ and $g \in \mathbb{Z}[t^{\pm 1}]$ that is palindromic or antipalindromic. This allows key results from the theory of Toeplitz operators to be used, including Szegö's strong limit theorem, as a tool to obtain the periodic point counting estimate

$$\mathsf{F}(L) = \Theta(e^{h[L]}), \tag{6}$$

for subgroups of the form $L = \langle a^n, b \rangle \in \mathcal{L}$. Using irreducible systems as a foundation, this estimate is then extended to all expansive systems and finally to all $L \in \mathcal{L}$.

In Section 4, the orbit growth estimate $\pi(N) = \Theta(e^{hN})$ is obtained using (6) and following methods developed in [18]. Also given in Section 4 is an example for which $\pi(N) \sim \Phi(N)e^{hN}$, where Φ has prime period 2, showing that an asymptotic of the form $\pi(N) \sim Ce^{hN}$, for some constant $C > 0$, should not be expected for algebraic flip systems in general.

2. Irreducible algebraic flip systems and examples

Let D_∞ denote the infinite dihedral group, given by (3). Elements of the group ring $\mathbb{Z}D_\infty$ can be written uniquely in the form $f + gb$, where $f, g \in \mathbb{Z}[t^{\pm 1}]$, $b^2 = 1$ and $tb = bt^{-1}$. For any $f \in \mathbb{Z}[t^{\pm 1}]$, write $f^*(t) = f(t^{-1})$, so that $bf = f^*b$. If D_∞ acts on a compact abelian group X, write $\alpha(x) = a \cdot x$ and $\beta(x) = b \cdot x$, $x \in X$, for the generators of the action and (X, α, β) for the associated dynamical system (an *algebraic flip system*). For any such system, there is always a *complementary* algebraic flip system $(X, \alpha, -\beta)$, where $-\beta$ detotes the automorphism given by $x \mapsto -\beta(x)$.

The Pontryagin dual group $M = \widehat{X}$ may be considered as a left module over the group ring $\mathbb{Z}D_\infty$ as follows (standard results from duality theory may be found in [12]). Let $f = \sum_m c_m(f)t^m$ and $g = \sum_m c_m(g)t^m \in \mathbb{Z}[t^{\pm 1}]$, where $c_m(f)$ and $c_m(g)$ are both zero for all but finitely many $m \in \mathbb{Z}$. For any $x \in M$, set

$$(f + gb)x = \sum_m c_m(f)\widehat{\alpha}^m(x) + c_m(g)\widehat{\alpha}^m\widehat{\beta}(x).$$

By considering the subaction generated by α, one also obtains a $\mathbb{Z}$-action together with a realization of M as a module over the commutative ring $\mathbb{Z}\langle a\rangle \cong \mathbb{Z}[t^{\pm 1}]$. We call this the *canonical dual* $\mathbb{Z}[t^{\pm 1}]$*-module*. From a dual perspective, if M is a countable left $\mathbb{Z}D_\infty$-module, then an algebraic flip system (X_M, α_M, β_M) is obtained by setting $X_M = \widehat{M}$ and α_M and β_M to be the automorphisms dual to multiplication by t and b respectively.

The unique factorization domain $\mathbb{Z}[t^{\pm 1}]$ has a spectrum comprised of principal prime ideals and maximal ideals (necessarily non-principal). Whenever a prime ideal $\mathfrak{m}$ is maximal, $\mathbb{Z}[t^{\pm 1}]/\mathfrak{m}$ is a finite field. For any integral domain of the form $R = \mathbb{Z}[t^{\pm 1}]/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal, denote the field of fractions of R by $\mathbb{K}(\mathfrak{p})$. Using standard identifications, the localization $M_\mathfrak{p}$ may be considered as a $\mathbb{K}(\mathfrak{p})$ vector space and we denote its dimension by $\dim_{\mathbb{K}(\mathfrak{p})} M_\mathfrak{p}$. Standard machinery from commutative algebra also shows that if M is Noetherian, the set of associated primes $\mathrm{Ass}(M)$ is comprised entirely of maximal ideals if and only if M is finite. The reader is referred to [7] for further background on commutative algebra.

Proof of Theorem 1.2. Let $M = \widehat{X}$. Considering M as a $\mathbb{Z}D_\infty$-module, if L is any non-zero submodule, since $X_{M/L}$ dualizes to a D_∞-invariant subgroup of X_M, irreducibility forces $[M : L] < \infty$. Now consider M canonically as a $\mathbb{Z}[t^{\pm 1}]$-module. Since M is Noetherian, there exists $x \in M$ such that $\mathrm{ann}(x) = \mathfrak{p} \in \mathrm{Ass}(M)$. Let

$$L = \mathbb{Z}D_\infty x = \mathbb{Z}[t^{\pm 1}]x + \mathbb{Z}[t^{\pm 1}]bx.$$

If $\mathfrak{p}$ is non-principal, it is necessarily maximal and $\mathbb{Z}[t^{\pm1}]x \cong \mathbb{Z}[t^{\pm1}]/\mathfrak{p}$ is a finite field. Since $\mathbb{Z}[t^{\pm1}]bx \cong \mathbb{Z}[t^{\pm1}]x$, L is also finite and this contradicts our assumption that X is infinite, since $[M : L] < \infty$. Hence, we may assume that $\mathfrak{p} = (f)$ is principal, so $\mathfrak{p}^* = (f^*)$. Let $R = \mathbb{Z}[t^{\pm1}]/\mathfrak{p}$ and $R^* = \mathbb{Z}[t^{\pm1}]/\mathfrak{p}^*$. Note that since $\mathbb{Z}[t^{\pm1}]bx \cong \mathbb{Z}[t^{\pm1}]x$, $\operatorname{ann}(bx) = \mathfrak{p}^*$. If $rx = sbx$ and either $r \in \mathfrak{p}$ or $s \in \mathfrak{p}^*$, then $r \in \mathfrak{p}$ and $s \in \mathfrak{p}^*$. There are now two cases to consider, corresponding to the two cases of the theorem.

Case (i). If $rx \neq sbx$ for all $r \in \mathbb{Z}[t^{\pm1}] \setminus \mathfrak{p}$ and all $s \in \mathbb{Z}[t^{\pm1}] \setminus \mathfrak{p}^*$, then

$$\mathbb{Z}[t^{\pm1}]x \cap \mathbb{Z}[t^{\pm1}]bx = \{0\},$$

and so $L \cong R \times R^*$ and the restriction of $\widehat{\beta}$ is given by (5). To conclude this case, suppose for a contradiction that $\mathfrak{p} = \mathfrak{p}^*$. Then $K = \mathbb{Z}[t^{\pm1}](b+1)x \subset L$, is a $\mathbb{Z}D_\infty$-module with annihilator $\mathfrak{p}$ as a $\mathbb{Z}[t^{\pm1}]$-module. However,

$$\dim_{\mathbb{K}(\mathfrak{p})} K_{\mathfrak{p}} = 1 \text{ and } \dim_{\mathbb{K}(\mathfrak{p})} L_{\mathfrak{p}} = 2.$$

In particular, L/K is infinite, which contradicts the irreducibility assumption. Note that this also means $f \neq 0$ in case (i).

Case (ii). Now suppose there exist $r \in \mathbb{Z}[t^{\pm1}] \setminus \mathfrak{p}$ and $s \in \mathbb{Z}[t^{\pm1}] \setminus \mathfrak{p}^*$ such that $rx = sbx$. If $f = 0$, then $L' = 2ss^*L$ is a non-trivial submodule of $\mathbb{Z}[t^{\pm1}]x \cong \mathbb{Z}[t^{\pm1}]$ for which $bL' = L'$, yet L' has infinite index in $\mathbb{Z}[t^{\pm1}]x$, so this contradicts irreduciblity. Therefore, $f \neq 0$. Furthermore, since $fx = 0$,

$$0 = sbfx = sf^*bx = f^*sbx = f^*rx,$$

giving $f^*r \in \mathfrak{p}$, which implies $f^* \in \mathfrak{p}$, as $r \notin \mathfrak{p}$. As both f and f^* are irreducible, this forces $f = uf^*$ for some unit $u \in \mathbb{Z}[t^{\pm1}]$. Hence, $\mathfrak{p}^* = \mathfrak{p}$, and $\operatorname{ann}(x) = \operatorname{ann}(bx) = \mathfrak{p} = (f)$. Since $sL \subset \mathbb{Z}[t^{\pm1}]x$, all associated primes of $L/\mathbb{Z}[t^{\pm1}]x$ contain $\mathfrak{p} + (s) \supsetneq \mathfrak{p}$ and are therefore non-principal. Consequently, $[L : \mathbb{Z}[t^{\pm1}]x] < \infty$.

Suppose that $gx = gbx$ for some $g \in \mathbb{Z}[t^{\pm1}] \setminus \mathfrak{p}$. Then $g^* \notin \mathfrak{p}$, as $\mathfrak{p}^* = \mathfrak{p}$. Let $y = gg^*x$, so that $L' = \mathbb{Z}[t^{\pm1}]y$ is a non-trivial submodule of $\mathbb{Z}[t^{\pm1}]x$ with $bL' = L'$, $[\mathbb{Z}[t^{\pm1}]x : L'] < \infty$, $\operatorname{ann}(y) = \mathfrak{p}$ and $L' \cong R$. Furthermore, for any $h \in \mathbb{Z}[t^{\pm1}]$,

$$\widehat{\beta}(hy) = bhy = h^*g^*gbx = h^*g^*gx = h^*y.$$

Therefore, the restriction of $\widehat{\beta}$ to L' identifies with the map $k \mapsto k^*$, $k \in R$, and $[M : L'] < \infty$.

The remaining possibility is that $gx \neq gbx$ for all $g \in \mathbb{Z}[t^{\pm1}] \setminus \mathfrak{p}$. Then $y = (b-1)x \neq 0$ and $gy \neq 0$ for all $g \in \mathbb{Z}[t^{\pm1}] \setminus \mathfrak{p}$, so $\operatorname{ann}(y) = \mathfrak{p}$. Let $L' = \mathbb{Z}[t^{\pm1}]y$, so $bL' = L'$, $[\mathbb{Z}[t^{\pm1}]x : L'] < \infty$ and $L' \cong R$. Furthermore, for any $h \in \mathbb{Z}[t^{\pm1}]$,

$$\widehat{\beta}(hy) = bhy = h^*by = h^*b(b-1)x = -h^*(b-1)x = -h^*y,$$

and so the restriction of $\widehat{\beta}$ to L' identifies with the map $k \mapsto -k^*$, $k \in R$, and $[M : L'] < \infty$. □

Remark 2.1. If M is a $\mathbb{Z}D_\infty$-module for which there exists a principal $\mathbb{Z}[t^{\pm 1}]$-associated prime $\mathfrak{p} \neq 0$, then there is a $\mathbb{Z}D_\infty$-module $L \subset M$ that corresponds to an irreducible algebraic flip system. To see this, first select $x \in M$ such that $\operatorname{ann}(x) = \mathfrak{p}$ (so $\operatorname{ann}(bx) = \mathfrak{p}^*$).

If $\mathbb{Z}[t^{\pm 1}]x \cap \mathbb{Z}[t^{\pm 1}]bx = \{0\}$ and $\mathfrak{p}^* \neq \mathfrak{p}$, then $L = \mathbb{Z}D_\infty x \cong R \times R^*$, where $R = \mathbb{Z}[t^{\pm 1}]/\mathfrak{p}$. Identifying L with $R \times R^*$, since $b(k,l) = (l^*, k^*)$, $(k,l) \in L$, it follows that any non-trivial $\mathbb{Z}D_\infty$-submodule $K \subset L$ contains an element (k,l), where both k and l are non-zero, and $\operatorname{ann}((k,l)) = \mathfrak{p}\mathfrak{p}^*$. Therefore, $\mathfrak{p}, \mathfrak{p}^* \in \operatorname{Ass}(K)$, and since both

$$\dim_{\mathbb{K}(\mathfrak{p})} K_{\mathfrak{p}} = \dim_{\mathbb{K}(\mathfrak{p})} L_{\mathfrak{p}} = 1 \text{ and } \dim_{\mathbb{K}(\mathfrak{p}^*)} K_{\mathfrak{p}^*} = \dim_{\mathbb{K}(\mathfrak{p}^*)} L_{\mathfrak{p}^*} = 1,$$

all associated primes of L/K must be non-principal, and hence L/K is finite.

If $\mathbb{Z}[t^{\pm 1}]x \cap \mathbb{Z}[t^{\pm 1}]bx = \{0\}$ and $\mathfrak{p}^* = \mathfrak{p}$, we replace x by $y = (b+1)x \neq 0$, noting that $\operatorname{ann}(y) = \mathfrak{p}$ and $by = y$. Then, $L = \mathbb{Z}D_\infty y = \mathbb{Z}[t^{\pm 1}]y \cong \mathbb{Z}[t^{\pm 1}]/\mathfrak{p}$, and L is irreducible.

If $\mathbb{Z}[t^{\pm 1}]x \cap \mathbb{Z}[t^{\pm 1}]bx \neq \{0\}$, then we proceed exactly as in the proof of Theorem 1.2(ii) to find that $\mathfrak{p}^* = \mathfrak{p}$, and that there exists $y \in \mathbb{Z}D_\infty x$ such that $\operatorname{ann}(y) = \mathfrak{p}$ and $by = \pm y$. Then, $L = \mathbb{Z}D_\infty y = \mathbb{Z}[t^{\pm 1}]y \cong \mathbb{Z}[t^{\pm 1}]/\mathfrak{p}$, and once again L is irreducible.

Before considering periodic points and orbit growth in general, some familiar examples are considered.

Example 2.2 (An irreducible flip system on $\mathbb{T}^2$)**.** Consider the automorphisms of $\mathbb{T}^2$ given by the matrices

$$A = \begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}$$

Then, $B^2 = \mathrm{Id}$, $AB = BA^{-1}$ and so $(\mathbb{T}^2, A, B)$ is an algebraic flip system. Following a standard approach using $\mathbb{Z}[t^{\pm 1}]$-modules, $\widehat{X} \cong \mathbb{Z}[t^{\pm 1}]/\mathfrak{p}$, where $\mathfrak{p} = (1 + 4t + t^2)$. Under the identification of this module with the integral domain $\mathbb{Z}[\sqrt{3}]$, via the map $f(t) + \mathfrak{p} \mapsto f(-2 + \sqrt{3})$, the map $\widehat{\alpha}$ dual to multiplication by A corresponds to multiplication by $-2 + \sqrt{3}$. However, under this identification, the automorphism $\widehat{\beta}$ dual to multiplication by B is not the involution of $\mathbb{Z}[\sqrt{3}]$ induced by $f(t) \mapsto f(t^{-1})$ on $\mathbb{Z}[t^{\pm 1}]$, which is in fact the restriction of the non-trivial element $\tau \in \operatorname{Gal}(\mathbb{Q}(\sqrt{3})|\mathbb{Q})$ to $\mathbb{Z}[\sqrt{3}]$. Instead, the map dual to multiplication by B identifies with $\widehat{\beta} : \mathbb{Z}[\sqrt{3}] \to \mathbb{Z}[\sqrt{3}]$, given by $\widehat{\beta}(k) = (7 + 4\sqrt{3})\tau(k)$. Furthermore, for any $k \in \mathbb{Z}[\sqrt{3}]$,

$$\widehat{\alpha}\widehat{\beta}(k) = (-2 + \sqrt{3})(7 + 4\sqrt{3})\tau(k) = (7 + 4\sqrt{3})\tau((-2 - \sqrt{3})k) = \widehat{\beta}\widehat{\alpha}^{-1}(k).$$

To complement our later estimates for periodic point counts that are developed in terms of the more general perspective of Theorem 1.2, it is helpful to consider $\operatorname{Fix}(A^n, B)$ using the explicit description available in this example. Let $\lambda = -2 - \sqrt{3}$ and $\delta = -2 + \sqrt{3}$. The group of points fixed by A^n is the kernel of

$$A^n - \mathrm{Id} = \frac{1}{2}\begin{pmatrix} \delta^n + \lambda^n - 2 & \sqrt{3}(\delta^n - \lambda^n) \\ \frac{\sqrt{3}}{3}(\delta^n - \lambda^n) & \delta^n + \lambda^n - 2 \end{pmatrix}$$

and any point of the form $x(y) = (2y, y) \in \mathbb{T}^2$ is fixed by B. Note, for all $n \in \mathbb{N}$, both $\frac{1}{2}(\delta^n + \lambda^n)$ and $\frac{\sqrt{3}}{6}(\delta^n - \lambda^n)$ are integers. Let $p(n) = (n + \chi(n))/2$ and $q(n) = (n - \chi(n))/2$, where χ is the non-trivial Dirichlet character modulo 2. Then we have the following integer factor pairs:

$$\frac{\sqrt{3}}{6}(\delta^n - \lambda^n) = \begin{cases} \frac{\sqrt{3}}{6}(\delta^{p(n)} - \lambda^{q(n)}) \cdot (\delta^{q(n)} + \lambda^{p(n)}) & n \text{ even} \\ \frac{3+\sqrt{3}}{6}(\delta^{p(n)} - \lambda^{q(n)}) \cdot \frac{-1+\sqrt{3}}{2}(\delta^{q(n)} + \lambda^{p(n)}) & n \text{ odd} \end{cases}$$

$$\frac{1}{2}(\delta^n + \lambda^n - 2) = \begin{cases} \frac{\sqrt{3}}{6}(\delta^{p(n)} - \lambda^{q(n)}) \cdot \sqrt{3}(\delta^{q(n)} - \lambda^{p(n)}) & n \text{ even} \\ \frac{3+\sqrt{3}}{6}(\delta^{p(n)} - \lambda^{q(n)}) \cdot \frac{3-\sqrt{3}}{2}(\delta^{q(n)} - \lambda^{p(n)}) & n \text{ odd} \end{cases}$$

In particular, $\frac{1}{2}(\delta^n + \lambda^n - 2)$ and $\frac{\sqrt{3}}{6}(\delta^n - \lambda^n)$ share the integer factor

$$k(n) = \left(\tfrac{\chi(n)}{2} + \tfrac{\sqrt{3}}{6}\right)(\delta^{p(n)} - \lambda^{q(n)}).$$

Therefore, for any $j \in \mathbb{Z}$, if $y = j/k(n)$, then $(A^n - \mathrm{Id})x(y) = 0$. Consequently, for $j = 1, \ldots, k(n)$, we obtain distinct points in $\mathrm{Fix}(A^n, B)$. Hence, there exists $C > 0$ such that, for sufficiently large n, $|\mathrm{Fix}(A^n, B)| \geqslant C|\lambda|^{n/2}$. Since a similar estimate holds for $|\mathrm{Fix}(A^n, -B)|$ and since

$$|\mathrm{Fix}(A^n)| = \Theta(|\lambda|^n),$$

using the product estimate (see Proposition 3.3),

$$|\mathrm{Fix}(A^n, B)||\mathrm{Fix}(A^n, -B)| = \Theta(|\mathrm{Fix}(A^n)|),$$

it follows that

$$|\mathrm{Fix}(A^n, B)| = \Theta(|\lambda|^{n/2}).$$

To describe this example as in Theorem 1.2(ii), consider the finite index submodule $L \subset \mathbb{Z}[\sqrt{3}]$ generated by $(\widehat{\beta} - 1)(1) = 6 + 4\sqrt{3}$, and notice that for all $k \in \mathbb{Z}[\sqrt{3}]$,

$$\widehat{\beta}(k(6 + 4\sqrt{3})) = k^*(-6 - 4\sqrt{3}) = -k^*(6 + 4\sqrt{3}),$$

where $k^* = \tau(k)$. Therefore, in concordance with the statement of the theorem, the module L is $\widehat{\beta}$-invariant and the $\mathbb{Z}[t^{\pm 1}]$-module isomorphism $L \to \mathbb{Z}[\sqrt{3}]$ given by $k(6+4\sqrt{3}) \mapsto k$ conjugates the restriction of $\widehat{\beta}$ to L with the map $k \mapsto -k^*$. Dually, there is a finite to one projection of $(\mathbb{T}^2, A, B)$ induced by the inclusion $L \hookrightarrow \mathbb{Z}[\sqrt{3}]$ that conjugates multiplication by B with multiplication by the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Example 2.3 (The doubling map)**.** The invertible extension of the circle doubling map is given by $\alpha_X(x) = 2x$ on the one-dimensional solenoid X dual to the integral domain $\mathbb{Z}[\frac{1}{2}]$. The only involutive automorphisms of $\mathbb{Z}[\frac{1}{2}]$ correspond to multiplication by 1 or -1, so there is no possible algebraic flip system (X, α_X, β), as the only candidates for β do not satisfy $\alpha_X\beta = \beta\alpha_X^{-1}$. If (Y, α_Y, β) is an irreducible algebraic flip system and there is a continuous surjective homomorphism $\phi : Y \to X$ with $\phi\alpha_Y = \alpha_X\phi$, then Y must have topological dimension at least 2. The most natural example is with $Y = X^2$, $\alpha_Y(x, y) = (\alpha_X(x), \alpha_X^{-1}(y))$ and $\beta(x, y) = (\alpha_X^{-1}(y), \alpha_X(x))$. Then,

$$\alpha_Y\beta(x, y) = (y, x) = \beta\alpha_Y^{-1}(x, y).$$

Examples of algebraic flip systems on zero-dimensional groups are closely related to Bernoulli shifts and their time reversals.

Example 2.4 (A Bernoulli shift with a time reversal)**.** Suppose that H is a cyclic group with $|H| \geqslant 2$. Let $X = H^{\mathbb{Z}}$ and define the full shift α and the flip β on X by

$$\alpha(x)_j = x_{j+1} \text{ and } \beta(x)_j = x_{-j},$$

where $x = (x_j) \in X$. Then (X, α, β) is an algebraic flip system. Although the topological entropy of (X, α) is $\log |H|$, the entropy of (X, α, β) is $h = \frac{1}{2}\log |H|$. A straightforward calculation using n-periodic blocks for α gives

$$|\operatorname{Fix}(\alpha^n, \beta)| = |H|^{(n+2-\chi(n))/2},$$

where χ is the non-trivial Dirichlet character modulo 2. If $|H| \neq 2$, another simple calculation gives

$$|\operatorname{Fix}(\alpha^n, -\beta)| = C^{2-\chi(n)}|H|^{(n-2+\chi(n))/2},$$

where $C = |\ker(h \mapsto 2h)|$, $h \in H$. Hence,

$$|\operatorname{Fix}(\alpha^n, \beta)| = \Theta(e^{hn}) \text{ and } |\operatorname{Fix}(\alpha^n, -\beta)| = \Theta(e^{hn}).$$

Remarks 2.5. Theorem 1.2 shows that if A is an $n \times n$ integer matrix with $|\det(A)| = 1$ that has an irreducible characteristic polynomial f and $(\mathbb{T}^n, A)$ is the corresponding algebraic dynamical system, then there exists an irreducible algebraic flip system $(\mathbb{T}^n, A, B)$ only if f is palindromic or antipalindromic, as is the case in Example 2.2.

The construction in Example 2.3 may be used to produce an algebraic flip system from any algebraic dynamical system (X, α). However, this flip system may or may not be irreducible, even if (X, α) is irreducible. In terms of (4), Theorem 1.2 shows that this construction produces an irreducible flip system if and only if f is irreducible and $(f) \neq (f^*)$. If f is non-constant, the latter condition is the same as saying there exists $\xi \in \mathbb{C}$ with $f(\xi) = 0$ and $f(\xi^{-1}) \neq 0$.

Example 2.3 also subtly raises the issue of ergodicity. An algebraic flip system (X, α, β) is ergodic if and only if (X, α) is ergodic. This follows immediately from [22, Rmk. 1.7(3)] and the fact any finite index subgroup of D_∞ contains $\langle a^n \rangle$ for some $n \in \mathbb{N}$. In terms of dual modules, [22, Rmk. 1.7(3)] also shows that (X, α, β) is ergodic if and only if multiplication by $t^n - 1$ is injective on $M = \widehat{X}$ for all $n \in \mathbb{Z} \setminus \{0\}$. Ergodicity therefore dictates that β cannot commute with α (since if α and β commute, α is also an involution, whereby $(t^2 - 1)M = \{0\}$). It is also worth noting that since a subgroup of D_∞ is infinite if and only if it has finite index in D_∞, [22, Th. 1.6] shows that ergodicity and strong mixing are equivalent conditions for algebraic flip systems.

Theorem 1.2 facilitates a convenient description of irreducible algebraic flip systems. Let L denote the finite index submodule of M identified in the theorem, so that there is a finite to one projection $\pi : X_M \to X_L$ dual to the inclusion of L into M, and $\pi\alpha_M = \alpha_L\pi$ and $\pi\beta_M = \beta_L\pi$. As in [22, Ex. 5.2], the dual of $R = \mathbb{Z}[t^{\pm1}]/(f)$ may be identified with the shift invariant subgroup of $\mathbb{T}^{\mathbb{Z}}$ given by

$$X_f = \{(x_m) \in \mathbb{T}^{\mathbb{Z}} : T_f x = 0\},$$

where, for $f = \sum_m c_m t^m \in \mathbb{Z}[t^{\pm1}]$, $T_f : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ is defined by

$$(T_f x)_j = \textstyle\sum_m c_m x_{m+j}, \ j \in \mathbb{Z}.$$

Let $x = (x_j) \in \mathbb{T}^{\mathbb{Z}}$. Multiplication by the image of t in $\mathbb{Z}[t^{\pm1}]/(f)$ dualizes to the restriction of the shift $\sigma : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$, given by

$$\sigma(x)_j = x_{j+1},$$

to X_f. The automorphism dual to $g \mapsto g^*$, $g \in \mathbb{Z}[t^{\pm1}]$, is the flip, and is also denoted using the $*$-notation,

$$(x^*)_j = x_{-j}.$$

Clearly, $(-x^*)_j = -x_{-j}$. Whenever $(f) = (f^*)$, X_f is invariant under both $*$ and $-*$. Where no confusion will arise, the restrictions of the shift and the flip to X_f will also be denoted by σ and $*$ respectively. Hence, we now have a description of (X_L, α_L, β_L) in case (ii) of Theorem 1.2.

In case (i) of Theorem 1.2, $L \cong R \times R^*$, and the projected automorphism α_L on $X_L \cong X_f \times X_{f^*}$ is given by

$$\alpha_L(x, y) = (\sigma(x), \sigma(y)), \tag{7}$$

and the projected automorphism β_L dual to (5) is given by

$$\beta_L(x, y) = (y^*, x^*). \tag{8}$$

3. Periodic points

Let X be a compact abelian group and denote the continuous automorphisms of X by $\mathrm{Aut}(X)$. Throughout this section, assume that $n \in \mathbb{Z}$ is non-zero. For any $\Lambda \subset \mathrm{Aut}(X)$, let

$$\mathrm{Fix}(\Lambda) = \{x \in X : \lambda(x) = x \text{ for all } \lambda \in \Lambda\},$$

and note that

$$\mathrm{Fix}(\Lambda) = \bigcap_{\lambda\in\Lambda} \ker(\lambda - 1).$$

For any $\alpha \in \mathrm{Aut}(X)$,

$$\mathrm{Fix}(\alpha^n) = \mathrm{Fix}(\alpha^{-n}),$$

so if $\beta \in \mathrm{Aut}(X)$ satisfies $\alpha\beta = \beta\alpha^{-1}$, then $\mathrm{Fix}(\alpha^n)$ is β-invariant.

It is also useful to observe that for any involution $\beta \in \operatorname{Aut}(X)$,

$$(\beta+1)(\beta-1) = 0.$$

Therefore, since $\ker(\beta-1) = \operatorname{Fix}(\beta)$ and $\ker(\beta+1) = \operatorname{Fix}(-\beta)$,

$$(\beta+1)X \subset \operatorname{Fix}(\beta) \text{ and } (\beta-1)X \subset \operatorname{Fix}(-\beta). \tag{9}$$

We now introduce several useful preliminary results which pertain to algebraic flip systems quite generally.

Lemma 3.1. *Let* (X, α, γ) *be an algebraic flip system such that* $\operatorname{Fix}(\alpha^n)$ *is finite for all* $n \neq 0$. *If* $\delta, \phi \in \operatorname{Aut}(X)$ *are such that* $\delta^2 = 1$, $\alpha\phi = \phi\alpha$ *and* $\gamma = \phi\delta$ *then, provided* $\operatorname{Fix}(-\phi)$ *is finite,*

$$|\operatorname{Fix}(\alpha^n, \gamma)| = \Theta(|\operatorname{Fix}(\alpha^n, \delta)|)$$

Proof. Firstly, if $x \in \operatorname{Fix}(\delta)$, then

$$(\phi+1)(x) = \phi\delta(x) + x = (\gamma+1)(x).$$

Therefore, since $(\phi+1)\operatorname{Fix}(\alpha^n) \subset \operatorname{Fix}(\alpha^n)$,

$$(\phi+1)\operatorname{Fix}(\alpha^n, \delta) = (\gamma+1)\operatorname{Fix}(\alpha^n, \delta) \subset \operatorname{Fix}(\alpha^n, \gamma),$$

by (9). Hence,

$$\frac{|\operatorname{Fix}(\alpha^n, \delta)|}{|\operatorname{Fix}(-\phi)|} = \frac{|\operatorname{Fix}(\alpha^n, \delta)|}{|\ker(\phi+1)|} \leqslant |(\phi+1)\operatorname{Fix}(\alpha^n, \delta)| \leqslant |\operatorname{Fix}(\alpha^n, \gamma)|.$$

On the other hand, $\alpha\delta = \delta\alpha^{-1}$, so (X, α, δ) is also an algebraic flip system and $\delta = \phi^{-1}\gamma$. Therefore, we can repeat the argument above with the roles of γ and δ interchanged, and with ϕ^{-1} in place of ϕ to obtain

$$\frac{|\operatorname{Fix}(\alpha^n, \gamma)|}{|\operatorname{Fix}(-\phi)|} = \frac{|\operatorname{Fix}(\alpha^n, \gamma)|}{|\operatorname{Fix}(-\phi^{-1})|} \leqslant |\operatorname{Fix}(\alpha^n, \delta)|.$$

Since $|\operatorname{Fix}(-\phi)|$ does not depend on n, the result follows. □

Lemma 3.2. *Suppose* (X, α, β) *is an algebraic flip system such that* $\operatorname{Fix}(\alpha^n)$ *is finite for all* $n \neq 0$. *If* $|\operatorname{Fix}(\alpha^n, \beta)| \ll |\operatorname{Fix}(\alpha^n)|^{1/2}$ *and* $|\operatorname{Fix}(\alpha^n, -\beta)| \ll |\operatorname{Fix}(\alpha^n)|^{1/2}$, *then*

$$|\operatorname{Fix}(\alpha^n, \beta)| = \Theta(|\operatorname{Fix}(\alpha^n)|^{1/2}).$$

Proof. Since

$$|(\beta-1)\operatorname{Fix}(\alpha^n)| = \frac{|\operatorname{Fix}(\alpha^n)|}{|\operatorname{Fix}(\alpha^n, \beta)|},$$

it follows from (9) that

$$|\operatorname{Fix}(\alpha^n)| = |(\beta-1)\operatorname{Fix}(\alpha^n)||\operatorname{Fix}(\alpha^n, \beta)| \leqslant |\operatorname{Fix}(\alpha^n, -\beta)||\operatorname{Fix}(\alpha^n, \beta)|,$$

giving

$$|\operatorname{Fix}(\alpha^n, \beta)| \geqslant \frac{|\operatorname{Fix}(\alpha^n)|}{|\operatorname{Fix}(\alpha^n, -\beta)|} \gg |\operatorname{Fix}(\alpha^n)|^{1/2}.$$

□

In terms of periodic point counting estimates, the lemma above reveals a close relationship between an algebraic flip system and its complement. Although not necessary for our later estimates, the following result highlights the nature of this interdependence more concretely.

Proposition 3.3. *Suppose (X, α, β) is an algebraic flip system such that $\mathrm{Fix}(\alpha^n)$ is finite for all $n \neq 0$. If the 2-torsion subgroup of X is finite, then*

$$|\mathrm{Fix}(\alpha^n, \beta)||\mathrm{Fix}(\alpha^n, -\beta)| = \Theta(|\mathrm{Fix}(\alpha^n)|).$$

Proof. Let $K_2(X) = \ker(x \mapsto 2x)$. Since $\mathrm{Fix}(\beta) \cap \mathrm{Fix}(-\beta) \subset K_2(X)$,

$$\begin{aligned}
|\mathrm{Fix}(\alpha^n)| &\geqslant |\mathrm{Fix}(\alpha^n, \beta) + \mathrm{Fix}(\alpha^n, -\beta)| \\
&= \frac{|\mathrm{Fix}(\alpha^n, \beta)||\mathrm{Fix}(\alpha^n, -\beta)|}{|\mathrm{Fix}(\alpha^n, \beta) \cap \mathrm{Fix}(\alpha^n, -\beta)|} \\
&\geqslant |K_2(X)|^{-1}|\mathrm{Fix}(\alpha^n, \beta)||\mathrm{Fix}(\alpha^n, -\beta)|.
\end{aligned}$$

On the other hand,

$$2\,\mathrm{Fix}(\alpha^n) \subset \mathrm{Fix}(\alpha^n, \beta) + \mathrm{Fix}(\alpha^n, -\beta).$$

To see this, suppose $x \in 2\,\mathrm{Fix}(\alpha^n)$ and let $y \in \mathrm{Fix}(\alpha^n)$ be such that $2y = x$. Let $v = (\beta + 1)(y)$ and $w = x - v$. Then, by (9), $v \in \mathrm{Fix}(\alpha^n, \beta)$ and

$$w = 2y - \beta(y) - y = (\beta - 1)(-y) \in \mathrm{Fix}(\alpha^n, -\beta).$$

Hence,

$$\begin{aligned}
|\mathrm{Fix}(\alpha^n, \beta)||\mathrm{Fix}(\alpha^n, -\beta)| &\geqslant |\mathrm{Fix}(\alpha^n, \beta) + \mathrm{Fix}(\alpha^n, -\beta)| \\
&\geqslant |2\,\mathrm{Fix}(\alpha^n)| \\
&= \frac{|\mathrm{Fix}(\alpha^n)|}{|K_2(\mathrm{Fix}(\alpha^n))|} \\
&\geqslant |K_2(X)|^{-1}|\mathrm{Fix}(\alpha^n)|.
\end{aligned}$$

□

Remark 3.4. Suppose M is a $\mathbb{Z}D_\infty$-module that is 2-torsion free, and that M is Noetherian as a $\mathbb{Z}[t^{\pm 1}]$-module. Then multiplication by 2 is injective on M and [17, Lem. 4.3] shows that $M/2M$ is finite, provided $(0) \notin \mathrm{Ass}(M)$. Via duality, $|K_2(X)| = |M/2M|$, so $K_2(X)$ is also finite. In this situation, the proposition above therefore applies to (X_M, α_M, β_M), provided $|\mathrm{Fix}(\alpha_M^n)|$ is finite for all $n \neq 0$.

In order to proceed, some facts concerning topological entropy are required.

Let $f \in \mathbb{Z}[t^{\pm 1}]$. Considering f as a polynomial in $\mathbb{C}[t^{\pm 1}]$, write $f = ct^m \prod_{i=1}^{j}(t - \xi_i)$, where $c, m \in \mathbb{Z}$. The *Mahler measure* of f is given by

$$\mathsf{M}(f) = |c| \prod_{i=1}^{j} \max\{1, |\xi_i|\},$$

Note that if $f = c$, then the empty product means $\mathsf{M}(f) = |c|$. We also have $\mathsf{M}(f^*) = \mathsf{M}(f)$. The well known entropy formula for automorphisms of compact abelian groups [14] shows that in case (i) of Theorem 1.2, the toplogical entropy $h(\alpha)$ of (X, α) is $2 \log \mathsf{M}(f)$, and in case (ii), $h(\alpha) = \log \mathsf{M}(f)$. If f has no zeros on the complex unit circle, it is well known (see for example [3, Lem. 4]) that

$$|\operatorname{Fix}(\alpha^n)| = \Theta(e^{h(\alpha)n}). \tag{10}$$

Note that when α is ergodic much better estimates than (10) are known, which follow from [17], but these will not be needed here.

For any algebraic flip system (X, α, β), a standard argument using the definition of entropy shows that the topological entropy $h(\alpha, \beta)$ of (X, α, β) is precisely half that of (X, α), as $[D_\infty : \langle a \rangle] = 2$. The same is of course true for the complementary flip system.

Theorem 3.5. *Let (X, α, β) be as in Theorem 1.2. If f has no roots on the complex unit circle, then*

$$|\operatorname{Fix}(\alpha^n, \beta)| = \Theta(e^{hn}),$$

where $h = h(\alpha, \beta)$.

Proof. First assume that case (i) of Theorem 1.2 applies. Then M contains a finite index submodule of the form $L = R \times R^*$, the dual of which is $X_L = X_f \times X_{f^*}$, and α_L and β_L are given by (7) and (8). For all $x \in \mathbb{T}^{\mathbb{Z}}$,

$$\sigma^n x = x \Leftrightarrow \sigma^n x^* = x^*$$

and

$$x \in X_f \Leftrightarrow x^* \in X_{f^*}.$$

Hence,

$$\begin{aligned} |\operatorname{Fix}(\alpha_L, \beta_L)| &= |\{(x, y) \in X_f \times X_{f^*} : \sigma^n x = x, \sigma^n y = y \text{ and } y = x^*\}| \\ &= |\{(x, x^*) \in X_f : \sigma^n x = x\}| \\ &= |\{x \in X_f : \sigma^n x = x\}| \\ &= \Theta(e^{hn}), \end{aligned}$$

and the same estimate is true with $-\beta_L$ in place of β_L. Since the natural projection $X_M \to X_L$ has finite kernel, there is a constant $C > 0$ such that

$$|\operatorname{Fix}(\alpha_M^n, \beta_M)| \leqslant Ce^{hn} \text{ and } |\operatorname{Fix}(\alpha_M^n, -\beta_M)| \leqslant Ce^{hn}. \tag{11}$$

By (10) and Lemma 3.2, the required result follows.

Case (ii) in Theorem 1.2 is more delicate. In this case, the $\mathbb{Z}[t^{\pm 1}]$-module M contains a finite index submodule of the form $R = \mathbb{Z}[t^{\pm 1}]/(f)$, with $f = \sum_m c_m t^m$ and $(f) = (f^*)$. Because R is determined by the prime ideal (f), for ease, we are free to choose the generator f to be balanced around its constant term. In other words, $f^* = \mu f$ or $f^* = \mu t^{-1} f$, where $\mu = 1$ or $\mu = -1$. If $f^* = t^{-1} f$, then $f(-1) = f^*(-1) = -f(-1)$, and so $f(-1) = 0$. If $f^* = -t^{-1} f$, then similarly $f(1) = 0$. Therefore, the case $f^* = \mu t^{-1} f$ is precluded by the assumption that f has no roots on the complex unit circle.

Our aim now is to provide an upper estimate for the cardinality of $\mathrm{Fix}(\sigma^n, *)$ in X_f, which may be identified with the subgroup of $\mathbb{T}^{\mathbb{Z}/(n)}$ comprising points $x = (x_{\bar{l}})$ satisfying

$$x_{\bar{l}} = x_{\overline{-l}}, \tag{12}$$

and

$$\sum_m c_m x_{\overline{m+l}} = 0 \mod 1, \tag{13}$$

for all $0 \leqslant l \leqslant n-1$. Let χ be the non-trivial Dirichlet character modulo 2. For sufficiently large n, if (12) holds for all $1 \leqslant l \leqslant (n+\chi(n))/2$ and (13) holds for all $0 \leqslant l \leqslant (n-\chi(n))/2$, then

$$\sum_m c_m x_{\overline{m+n-l}} = \mu \sum_m c_m x_{\overline{m+l}}$$

for $1 \leqslant l \leqslant (n-\chi(n))/2$, as $f^* = \mu f$. That is, (12) and (13) hold for all $0 \leqslant l \leqslant n-1$. Hence, it follows that $\mathrm{Fix}(\sigma^n, *)$ is isomorphic to the kernel of the linear map $E_n : \mathbb{T}^n \to \mathbb{T}^n$ formed as follows. Set $k(n) = (n+2-\chi(n))/2$. Let the first $k(n)$ rows of the $n \times n$ matrix E_n be identical to those of the $n \times n$ circulant matrix corresponding to f, that is

$$\begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{-3} & c_{-2} & c_{-1} \\ c_{-1} & c_0 & c_1 & \dots & c_{-4} & c_{-3} & c_{-2} \\ c_{-2} & c_{-1} & c_0 & \dots & c_{-5} & c_{-4} & c_{-3} \\ & \dots & & \dots & & \dots & \end{pmatrix}$$

Let the remaining rows of E_n be

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & \dots & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & \dots & -1 & 0 & 0 \\ & \dots & & & \dots & & \dots & \end{pmatrix}$$

Since $\mathrm{Fix}(\sigma^n, *)$ is finite, so too is $\ker(E_n)$. Via duality,

$$|\mathrm{Fix}(\sigma^n, *)| = |\ker(E_n)| = |\mathbb{Z}^n / E_n \mathbb{Z}^n| = |\det E_n|.$$

Dividing E_n into a block matrix,

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$$

where A_n is $k(n) \times k(n)$, A_n is the $k(n) \times k(n)$ banded Toeplitz matrix corresponding to the Laurent polynomial f, and is invertible by [2, Th. 3.7] for sufficiently large n (as f does not have roots on the complex unit circle, and f has an equal number of roots inside and outside the complex unit circle as $(f) = (f^*)$). Hence, the Schur determinant identity gives

$$|\det E_n| = |\det A_n| |\det(D_n - C_n A_n^{-1} B_n)|, \tag{14}$$

and [2, Th. 3.7] also shows that there is a constant $\kappa_1 > 0$, independent of n, bounding the entries of A_n^{-1}. Let $r = \max\{|m| + 1 : c_m \neq 0\}$. The $k(n)\times(n-k(n))$ matrix $A_n^{-1}B_n$ has all entries zero in columns r to $n-k(n)-r$ and each entry is bounded by

$$\kappa_2 = \kappa_1 r \max\{|c_m| : m \leqslant r\}.$$

Furthermore, the $(n - k(n)) \times (n - k(n))$ matrix $C_n A_n^{-1} B_n$ is identical to $A_n^{-1}B_n$ with the top row removed (and also bottom row removed if n is even). Therefore, by elementary row and column switching, it follows that

$$|\det(D_n - C_n A_n^{-1} B_n)| = \left| \det \begin{pmatrix} A'_n & B'_n \\ C'_n & D'_n \end{pmatrix} \right|,$$

where A'_n is a $2r\times 2r$ matrix with entries bounded by $1+\kappa_2$, B'_n has all entries zero, D'_n has all entries zero except for an antidiagonal line of entries, all -1. Since the determinant of the block matrix on the right is $\det(A'_n)\det(D'_n)$, it follows that $|\det(D_n - C_n A_n^{-1} B_n)|$ is bounded. Furthermore, as f does not have roots on the complex unit circle and since f has an equal number of roots inside and outside the complex unit circle, Szegö's strong limit theorem [2, Th. 2.11] may be applied to yield $|\det A_n| = \Theta(\mathsf{M}(f)^{k(n)})$. So, by (14),

$$|\operatorname{Fix}(\sigma^n, *)| \ll \mathsf{M}(f)^{n/2} = e^{hn}.$$

All of these arguments can be repeated with $-*$ in place of $*$, and hence it also follows that

$$|\operatorname{Fix}(\sigma^n, -*)| \ll e^{hn}.$$

As in case (i), since the natural projection $\pi : X_M \to X_L$ has finite kernel, there is a constant $C > 0$ such that (11) holds, and so by (10) and Lemma 3.2, $|\operatorname{Fix}(\alpha_M^n, \beta_M)| = \Theta(e^{hn})$. □

The objective now is to establish the estimate (6) in the generality required for Theorem 1.1. If γ is an action of a countable group G by continuous automorphisms γ^g of a compact abelian group X, then γ is *expansive* if and only if there is a neighbourhood U of 0 such that $\bigcap_{g\in G} \gamma^g U = \{0\}$. Using the definition, it can be shown that an algebraic flip system (X, α, β) is expansive if and only if (X, α) is expansive. When (X, α) is expansive, by [22, Th. 6.5 and Cor. 6.15], the $\mathbb{Z}[t^{\pm 1}]$-module $M = \widehat{X}$ is Noetherian and any generator of a principal ideal in $\operatorname{Ass}(M)$ does not have roots on the complex unit circle. As before, $h(\alpha, \beta) = \frac{1}{2}h(\alpha)$ and (10) also applies in this more general setting (see [3]).

Theorem 3.6. *If (X, α, β) is an expansive algebraic flip system, then*

$$|\operatorname{Fix}(\alpha^n, \beta)| = \Theta(e^{hn}),$$

where $h = h(\alpha, \beta)$.

Proof. First note that for any Noetherian $\mathbb{Z}[t^{\pm 1}]$-module M, if M is infinite, the set of principal associated primes $\mathrm{Ass}^1(M)$ is non-empty. For a non-trivial case, assume X is infinite and let $M = \widehat{X}$ be the canonical dual $\mathbb{Z}[t^{\pm 1}]$-module. By the assumption of expansiveness, $(0) \notin \mathrm{Ass}^1(M)$. Also, for any submodule $L \subset M$,

$$\mathrm{Ass}^1(M) = \mathrm{Ass}^1(L) \cup \mathrm{Ass}^1(M/L).$$

Set $M_0 = \{0\}$. By Remark 2.1, we may choose $\mathfrak{p}_1 \in \mathrm{Ass}^1(M)$ and $x \in M$ with $\mathrm{ann}(x) = \mathfrak{p}_1$, so that $M_1 = \mathbb{Z}D_\infty x$ corresponds to an irreducible algebraic flip system. Now consider M/M_1 and note that $\mathrm{Ass}^1(M/M_1) \subset \mathrm{Ass}^1(M)$. If M/M_1 is infinite, then $\mathrm{Ass}^1(M/M_1) \neq \varnothing$ and there exists $y \in M \setminus M_1$ such that $\mathrm{ann}(y + M_1) = \mathfrak{p}_2$ for some $\mathfrak{p}_2 \in \mathrm{Ass}^1(M)$. Set $M_2 = \mathbb{Z}D_\infty y + M_1$ and note that y can once again be chosen so that M_2/M_1 is irreducible by Remark 2.1. Continuing in this way, we obtain a chain of submodules

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k$$

terminating with M/M_k finite and $\mathrm{Ass}^1(M/M_k) = \varnothing$. The process terminates because M is Noetherian. Each factor M_i/M_{i-1}, $1 \leqslant i \leqslant k$, corresponds to an irreducible algebraic flip system.

For any $\mathbb{Z}D_\infty$-module L, $h(\alpha_L, \beta_L) = \frac{1}{2}h(\alpha_L)$ and the entropy addition formula for single automorphisms [22, Sec. 14] may be applied to show that for $1 \leqslant i \leqslant k$,

$$h(\alpha_{M_i}, \beta_{M_i}) = h(\alpha_{M_i/M_{i-1}}, \beta_{M_i/M_{i-1}}) + h(\alpha_{M_{i-1}}, \beta_{M_{i-1}}). \tag{15}$$

We aim to show that for $1 \leqslant i \leqslant k$,

$$|\operatorname{Fix}(\alpha^n_{M_i}, \beta_{M_i})| \leqslant |\operatorname{Fix}(\alpha^n_{M_i/M_{i-1}}, \beta_{M_i/M_{i-1}})||\operatorname{Fix}(\alpha^n_{M_{i-1}}, \beta_{M_{i-1}})|. \tag{16}$$

Then, combining (15) and (16) with Theorem 3.5 and using induction, it will follow that

$$|\operatorname{Fix}(\alpha^n_{M_k}, \beta_{M_k})| \ll e^{hN},$$

and hence

$$|\operatorname{Fix}(\alpha^n_{M}, \beta_{M})| \ll e^{hN},$$

because there is a finite to one projection of M onto M_k. Since the same approach works for the complementary system $(X, \alpha, -\beta)$, as before, (10) and Lemma 3.2 may be applied to obtain the required estimate for $|\operatorname{Fix}(\alpha^n_M, \beta_M)|$.

Hence, it remains to prove (16). First note that $\operatorname{Fix}(\alpha^n_{M_i}, \beta_{M_i})$ and $\operatorname{Fix}(\alpha^n_{M_i/M_{i-1}}, \beta_{M_i/M_{i-1}})$ are finite for all $1 \leqslant i \leqslant k$, as these are subgroups of $\operatorname{Fix}(\alpha^n_{M_i})$ and $\operatorname{Fix}(\alpha^n_{M_i/M_{i-1}})$ respectively. For ease of notation, fix i, let $K = M_{i-1}$, $L = M_i$, $\eta = b - 1$, and $\tau = t^n - 1$. Since

$$\frac{L}{K + \eta L + \tau L} \cong \frac{L/(\eta L + \tau L)}{(K + \eta L + \tau L)/(\eta L + \tau L)},$$

it follows via duality (see for example [14, Lem. 7.2]) that

$$\begin{aligned}
|\operatorname{Fix}(\alpha_L^n,\beta_L)| &= |L/(\eta L+\tau L)| \\
&= \left|\frac{L/K}{(K+\eta L+\tau L)/K}\right|\left|\frac{K+\eta L+\tau L}{\eta L+\tau L}\right| \\
&= \left|\frac{L/K}{\eta(L/K)+\tau(L/K)}\right|\left|\frac{K}{K\cap(\eta L+\tau L)}\right| \\
&\leqslant \left|\frac{L/K}{\eta(L/K)+\tau(L/K)}\right|\left|\frac{K}{\eta K+\tau K}\right| \\
&= |\operatorname{Fix}(\alpha_{L/K}^n,\beta_{L/K})||\operatorname{Fix}(\alpha_K^n,\beta_K)|,
\end{aligned}$$

thus giving (16). □

For any finite index subgroup L of D_∞, let $\mathsf{F}(L)$ be given by (1) and a and b by (3). Following [13, Lem. 2.1], for each odd index n, there are exactly n distinct subgroups of index n,

$$\langle a^n,b\rangle,\langle a^n,ab\rangle,\langle a^n,a^2b\rangle,\dots,\langle a^n,a^{n-1}b\rangle, \tag{17}$$

and for each even index n, in addition to the subgroups (17), there is one additional subgroup $\langle a^{n/2}\rangle$ of index n. Building on Theorem 3.6, we now consider the number of periodic points for any given finite index subgroup of D_∞.

Theorem 3.7. *Let (X,α,β) be an expansive algebraic flip system. Then for finite index subgroups $L\subset D_\infty$,*

$$\mathsf{F}(L)=\Theta(e^{h[L]})$$

where $h=h(\alpha,\beta)$.

Proof. Let $n\in\mathbb{N}$. Firstly, (10) shows

$$\mathsf{F}(\langle a^{n/2}\rangle)=\Theta(e^{hn}),$$

and Theorem 3.6 shows

$$\mathsf{F}(\langle a^n,b\rangle)=\Theta(e^{hn}). \tag{18}$$

By placing $\delta=\beta$ and $\phi=\alpha$ in Lemma 3.1, Theorem 3.6 also shows that

$$\mathsf{F}(\langle a^n,ab\rangle)=\Theta(e^{hn}). \tag{19}$$

Finally, since

$$\langle a^n,a^{i+2}b\rangle=a\langle a^n,a^ib\rangle a^{-1},$$

for all $0\leqslant i\leqslant n-3$, we have

$$\mathsf{F}(\langle a^n,a^{i+2}b\rangle)=\mathsf{F}(\langle a^n,a^ib\rangle), \tag{20}$$

and so the remaining subgroups of index n are dealt with by (18) and (19). □

Corollary 3.8. *Let (X, α, β) be an expansive algebraic flip system. Then the upper and lower growth rates of periodic points coincide and are equal to the topological entropy. That is,*

$$h(\alpha, \beta) = \lim_{n\to\infty} \inf_{L\in\mathcal{L}:[L]\geqslant n} \frac{1}{[L]} \log \mathsf{F}(L) = \lim_{n\to\infty} \sup_{L\in\mathcal{L}:[L]\geqslant n} \frac{1}{[L]} \log \mathsf{F}(L),$$

where $\mathcal{L}$ denotes the set of finite index subgroups of D_∞.

The dynamical zeta function of a flip system is introduced in [13] (see also [15]). This is defined formally by

$$\zeta_{\alpha,\beta}(z) = \exp\left(\sum_{L\in\mathcal{L}} \frac{\mathsf{F}(L)z^{[L]}}{[L]}\right), \tag{21}$$

where $\mathcal{L}$ denotes the set of finite index subgroups of D_∞. Corollary 3.8 shows that the associated zeta function $\zeta_{\alpha,\beta}$ has radius of convergence e^{-h} (in agreement with [13, Cor. 2.3]). This mirrors the well known result for expansive compact abelian group automorphisms.

4. Orbit growth estimates

For any $L \in \mathcal{L}$, define $\mathsf{F}(L)$ and $\mathsf{O}(L)$ using (1) and (2). With the partial order on $\mathcal{L}$ defined by inclusion (that is, $L' \preccurlyeq L$ if and only if $L \leqslant L'$), $\mathcal{L}$ becomes a locally finite poset and there is a Möbius function μ defined on the incidence algebra of $\mathcal{L}$ (see Stanley [26, Sec. 3.7] for the details and examples). Moreover, since

$$\mathsf{F}(L) = \sum_{L'\in\mathcal{L}:\ L'\geqslant L} [L']\mathsf{O}(L'),$$

using Möbius inversion, it follows that

$$\mathsf{O}(L) = \frac{1}{[L]} \sum_{L'\in\mathcal{L}:\ L'\geqslant L} \mu(L', L)\mathsf{F}(L').$$

Thus,

$$\begin{aligned}\pi(N) &= \sum_{L\in\mathcal{L}:[L]\leqslant N} \frac{1}{[L]} \sum_{L'\in\mathcal{L}:\ L'\geqslant L} \mu(L', L)\mathsf{F}(L') \\ &= \underbrace{\sum_{L\in\mathcal{L}:[L]\leqslant N} \frac{\mathsf{F}(L)}{[L]}}_{\Psi(N)} + \underbrace{\sum_{L\in\mathcal{L}:[L]\leqslant N} \frac{1}{[L]} \sum_{L'\in\mathcal{L}:\ L'>L} \mu(L', L)\mathsf{F}(L')}_{\Delta(N)}.\end{aligned}$$

To obtain an asymptotic for $\pi(N)$, the main strategy is to concentrate on $\Psi(N)$ and show that $\Delta(N)$ does not make a significant contribution comparitively. To begin, we require a bound on the Möbius function for D_∞.

Lemma 4.1. *For any $L, L' \in \mathcal{L}$ with $L \leqslant L'$, $|\mu(L', L)| \leqslant [L]$.*

Proof. The following gives an outline of how to obtain an explicit formula for μ, and from this the required bound follows immediately. First note that if L is not normal in L', then by a standard application of Crapo's Theorem [4, Th. 1], $\mu(L',L)=0$. Hence, assume L is normal in L', then we can simply consider $\mu(L'/L,\{1\})$ for the finite group $K=L'/L$, which is either a cyclic or dihedral group of order $n\leqslant[L]$. If K is cyclic, then $\mu(K,\{1\})=\mu_{\mathbb{N}}(n)$, where $\mu_{\mathbb{N}}$ is the standard Möbius function on $\mathbb{N}$. If K is dihedral,

$$\mu(K,\{1\})=\frac{n\mu_{\mathbb{N}}(n/2)}{2}.$$

This is proved by considering abelian minimal normal subgroups of K and their complements and applying the reduction formula

$$\mu(K,\{1\})=-j\mu(K/J,\{1\}),$$

where J is any abelian minimal normal subgroup of K with j complements in K (see [11, Cor. 3.3]). $\square$

Proof of Theorem 1.1. First, we establish the order of magnitude of $\Delta(N)$. Applying Lemma 4.1 and Theorem 3.7,

$$\begin{aligned}\Delta(N) &= \sum_{L\in\mathcal{L}:[L]\leqslant N}\frac{1}{[L]}\sum_{L'\in\mathcal{L}:\ L'>L}\mu(L',L)\mathsf{F}(L')\\ &= \sum_{n\leqslant N}\frac{1}{n}\sum_{L\in\mathcal{L}:[L]=n}\ \sum_{L'\in\mathcal{L}:L'>L}\mu(L',L)\mathsf{F}(L')\\ &\ll e^{hN/2}\sum_{n\leqslant N}\ \sum_{L\in\mathcal{L}:[L]=n}\frac{1}{n}\cdot n\sum_{L'\in\mathcal{L}:L'>L}1\\ &\ll N^4e^{hN/2}.\end{aligned}$$

Therefore, it remains to show that $\Psi(N)=\Theta(e^{hN})$. Using Theorem 3.7, there exist constants $0<A<B$ such that $Ae^{h[L]}\leqslant\mathsf{F}(L)\leqslant Be^{h[L]}$, and since

$$\Psi(N)=\sum_{n\leqslant N}\frac{1}{n}\sum_{L\in\mathcal{L}:[L]=n}\mathsf{F}(L),$$

it follows that

$$A\sum_{n\leqslant N}\frac{1}{n}\cdot ne^{hn}\leqslant\Psi(N)\leqslant B\sum_{n\leqslant N}\frac{1}{n}\cdot(n+1)e^{hn}.$$

Hence, the required result follows. $\square$

Given that we have established $\Psi(N)=\Theta(e^{hn})$, it is interesting to note that the proof above also gives the following.

Corollary 4.2. *If (X,α,β) is an expansive algebraic flip system, then*

$$\pi(N)=\Psi(N)+\mathrm{O}(N^4e^{hN/2}).$$

For simple examples, $\Psi(N)$ can be described much more precisely.

Example 4.3. Consider again the Bernoulli shift with a time reversal introduced in Example 2.4, now with an alphabet comprised of p symbols, where p is a rational prime. We have

$$\mathsf{F}(\langle a,b\rangle)=p^{(n+2-\chi(n))/2}\text{ and }\mathsf{F}(\langle a,ab\rangle)=p^{(n+\chi(n))/2}.$$

where χ is the non-trivial Dirichlet character modulo 2. Since $\mathsf{F}(\langle a^{n/2}\rangle)=p^{n/2}$ when n is even, together with (20), these formulae provide precise periodic point counts for all $L\in\mathcal{L}$ with index n.

Using (20) and the fact that $\mathsf{F}(\langle a^i,a^jb\rangle)=\mathsf{F}(\langle a^i,a^{i+j}b\rangle)$ for all $i,j\in\mathbb{N}$, it follows that

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1}\mathsf{F}(\langle a^n,a^ib\rangle) &= 2^{\chi(n)-1}\left(\mathsf{F}(a^n,b)+(1-\chi(n))\mathsf{F}(a^n,ab)\right)\\ &= p^{n/2}(1+p+\chi(n)(\sqrt{p}-p-1)))\end{aligned}$$

Hence,

$$\begin{aligned}\Psi(N) &= \sum_{n\leqslant N}\frac{1}{n}\sum_{i=0}^{n-1}\mathsf{F}(\langle a^n,a^ib\rangle)+\sum_{n\leqslant N/2}\frac{1}{n}\mathsf{F}(\langle a^n\rangle)\\ &= \sum_{n\leqslant N}p^{n/2}(1+p+\chi(n)(\sqrt{p}-p-1)))+\sum_{n\leqslant N/2}\frac{p^n}{n}\\ &= (1+p-\sqrt{p})\sum_{n\leqslant N/2}p^n+\sqrt{p}\sum_{n\leqslant N}p^{n/2}+\sum_{n\leqslant N/2}\frac{p^n}{n}\\ &= \underbrace{\left(\frac{p(1+p-\sqrt{p})}{(p-1)p^{\chi(N)/2}}+\frac{p}{\sqrt{p}-1}\right)}_{\Phi(N)}p^{N/2}+\mathrm{O}\left(\frac{p^{N/2}}{N}\right).\end{aligned}$$

In particular, the function $\Phi(N)$ has prime period 2, and Corollary 4.2 gives

$$\pi(N)=\Phi(N)p^{N/2}+\mathrm{O}\left(\frac{p^{N/2}}{N}\right).$$

The results of this section show that since $\Psi(N)$ is a partial sum of the coefficients in $\log\zeta_{\alpha,\beta}$, where $\zeta_{\alpha,\beta}$ is the dynamical zeta function given by (21), $\zeta_{\alpha,\beta}$ carries appropriate information to describe the dominant growth in $\pi(N)$. However, this is only apparent once the orders of magnitude of $\Psi(N)$ and $\Delta(N)$ have been established, so $\zeta_{\alpha,\beta}$ does not appear to be particularly useful in this context.

Simple examples suggest that we might tentatively expect an asymptotic of the form $\pi(N)\sim\Phi(N)e^{hN}$, where Φ is periodic, for any expansive algebraic flip system. However, even if the periodic point counting estimate for irreducible systems (Theorem 3.5) were improved, the methods used in

Section 3 do not suffice to extend any such improvement to all expansive systems. On the other hand, the method used in Example 4.3 could be applied more generally if more precise estimates for $\mathsf{F}(L)$, $L \in \mathcal{L}$, were obtained.

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