A GENERALIZATION OF MARTIN'S AXIOM

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ABSTRACT. We define the $\aleph_{1.5}$–chain condition. The corresponding forcing axiom is a generalization of Martin's Axiom; in fact, $\mathsf{MA}^{1.5}_{<\kappa}$ implies $\mathsf{MA}_{<\kappa}$. Also, $\mathsf{MA}^{1.5}_{<\kappa}$ implies certain uniform failures of club–guessing on ω_1 that do not seem to have been considered in the literature before. We show, assuming CH and given any regular cardinal $\kappa \geq \omega_2$ such that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ and such that $\diamondsuit(\{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \omega_1\})$ holds, that there is a proper $\aleph_2$–c.c. partial order of size κ forcing $2^{\aleph_0} = \kappa$ together with $\mathsf{MA}^{1.5}_{<\kappa}$.

1. A GENERALIZATION OF MARTIN'S AXIOM. AND SOME OF ITS APPLICATIONS.

Martin's Axiom, often denoted by MA, is the following very well–known and very classical forcing axiom: If $\mathbb{P}$ is a partial order (poset, for short) with the countable chain condition[1] and $\mathcal{D}$ is a collection of size less than $2^{\aleph_0}$ consisting of dense subsets of $\mathbb{P}$, then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Martin's Axiom is obviously a weakening of the Continuum Hypothesis. Given a cardinal λ, MA_λ is obtained from considering, in the above formulation of MA, collections $\mathcal{D}$ of size at most λ rather than of size less than $2^{\aleph_0}$. Martin's Axiom becomes interesting when $2^{\aleph_0} > \aleph_1$.

$\mathsf{MA}_{\aleph_1}$ was the first forcing axiom ever considered ([9]). As observed by D. Martin, the consistency of MA together with $2^{\aleph_0} > \aleph_1$ follows from generalizing the Solovay–Tennenbaum construction of a model of Suslin's Hypothesis by iterated forcing using finite supports ([14]). Since then, a plethora of applications of MA ($+ 2^{\aleph_0} > \aleph_1$) have been

[1]A partial order has the countable chain condition (is c.c.c.) if and only if it has no uncountable antichains. More generally, given a cardinal κ, a partial order has the κ–chain condition (is κ–c.c.) if it has no antichains of size κ.

discovered in set theory, topology, measure theory, group theory, and other areas ([3] is a classical reference).

In the present paper we generalize Martin's Axiom to the class of posets satisfying what we call here the $\aleph_{1.5}$–*chain condition* ($\aleph_{1.5}$–c.c.). This terminology is intended to highlight the fact that every poset with the countable chain condition (i.e., with the $\aleph_1$–c.c.) is in our class and that every poset in our class has the $\aleph_2$–c.c. Of course it follows from the first inclusion that for every cardinal λ, the forcing axiom $\mathsf{MA}_\lambda^{1.5}$ for the class of all $\aleph_{1.5}$–c.c. posets relative to collections of λ-many dense sets implies MA_λ. Furthermore, as to the consistency of $\mathsf{MA}_\lambda^{1.5}$, there is no restriction on λ other than $\lambda < 2^{\aleph_0}$. More precisely, the same construction shows that $\mathsf{MA}_{\aleph_1}^{1.5}$, $\mathsf{MA}_{\aleph_{727}}^{1.5}$, $\mathsf{MA}_{\aleph_{\omega_2+\omega+3}}^{1.5}$, and so on are all consistent.[2] This construction will take the form of a forcing iteration, in a broad sense of the expression, involving certain partial symmetric systems of countable submodels as side conditions (cf. our earlier work in [1]).

Note that the collapse of ω_1 to ω with finite conditions has size $\aleph_1$ and therefore has the $\aleph_2$–chain condition. The forcing axiom for collections of $\aleph_1$–many dense subsets of this poset is obviously false. This shows of course that some restriction is necessary in order to obtain a consistent forcing axiom for posets with the $\aleph_2$–c.c., even relative to collections of $\aleph_1$–many dense sets. On the other hand, the definition of our class of posets will be wide enough to contain all c.c.c. posets and in fact to make the corresponding forcing axiom $\mathsf{MA}_\lambda^{1.5}$ (for $\lambda \geq \aleph_1$) strictly stronger than MA_λ. In fact, we will show for example that $\mathsf{MA}_\lambda^{1.5}$ implies certain 'uniform' failures of Club Guessing on ω_1 that do not seem to have been considered before in the literature, and which do not follow from MA_λ. As a matter of fact we do not know how to show the consistency of these statements by any method other than ours.

Before going on we will fix a setting for our notation. For the most part we will follow set–theoretic standards as set forth for example in [6] and in [8]. In particular, if $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is a forcing iteration and $\alpha \leq \kappa$, $\dot{G}_\alpha$ denotes the canonical $\mathcal{P}_\alpha$–name for the generic filter added by $\mathcal{P}_\alpha$, and $\leq_\alpha$ typically denotes the extension relation on $\mathcal{P}_\alpha$. We will also make use of less standard pieces of notation. For example, if N is a set for which $N \cap \omega_1$ is an ordinal, we will use δ_N to denote this intersection and will call δ_N the *h*eight of N. Other pieces of notation will be introduced when needed.

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[2]The same is true for the Solovay–Tenennbaum construction, i.e., the same construction shows the consistency of Martin's Axiom together with $2^{\aleph_0}$ being $\aleph_2$, $\aleph_{728}$, $\aleph_{\omega_2+\omega+4}$, and so on.

Let $\mathbb{P}$ be a poset and let N be a submodel of $\langle H(\theta), \in \rangle$ such that $\mathbb{P} \in N$ and $\theta \geq |\mathsf{TC}(\mathbb{P})|^+$.[3] Recall that a $\mathbb{P}$–condition p is $(N, \mathbb{P})$–generic if for every extension p' of p and every dense subset D of $\mathbb{P}$ belonging to N (equivalently, every maximal antichain D of $\mathbb{P}$ belonging to N) there is some condition in $D \cap N$ compatible with p'. Also, $\mathbb{P}$ is proper ([16]) if for every cardinal $\theta \geq |\mathsf{TC}(\mathbb{P})|^+$ it holds, for club–many countable $N \preccurlyeq H(\theta)$, that for every $p \in N \cap \mathbb{P}$ there is a condition q in $\mathbb{P}$ extending p and such that q is $(N, \mathbb{P})$–generic. Every poset $\mathbb{P}$ with the countable chain condition is proper as every condition is $(N, \mathbb{P})$–generic for every N as above.

Now we may proceed to the definition of the $\aleph_{1.5}$–c.c.

Definition 1.1. Given a poset $\mathbb{P}$, we will say that $\mathbb{P}$ *has the $\aleph_{1.5}$–chain condition* ($\mathbb{P}$ is $\aleph_{1.5}$–c.c. for short) if and only if for every regular cardinal $\lambda \geq |\mathsf{TC}(\mathbb{P})|^+$ there is a club $D \subseteq [H(\lambda)]^{\aleph_0}$ such that for every finite $\{N_i : i < n\} \subseteq D$ and every $p \in \mathbb{P}$, if $p \in N_j$ for some j such that $\delta_{N_j} = \min\{\delta_{N_i} : i < n\}$, then there is some condition extending p and $(N_i, \mathbb{P})$–generic for all i.[4]

Since the intersection of two clubs is again a club, we may assume that all members of D in the above definition are countable elementary substructures of $\langle H(\lambda), \in \rangle$. Also note that a poset $\mathbb{P}$ has the $\aleph_{1.5}$–chain condition if and only if it satisfies the above definition for some $\lambda \geq |\mathsf{TC}(\mathbb{P})|^+$ (this is true because the projection of a club is again a club). Finally note that every c.c.c. poset is $\aleph_{1.5}$–c.c. and that every $\aleph_{1.5}$–c.c. poset is proper.

Proposition 1.2. *If $\mathbb{P}$ has the $\aleph_{1.5}$–c.c., then $\mathbb{P}$ has the $\aleph_2$–c.c.*

Proof. Let A be a maximal antichain of $\mathbb{P}$ such that $|A| \geq \aleph_2$. Let λ and D be as in Definition 1.1 and let N_p be, for every $p \in A$, a member of D such that $A, p \in N_p$. Let $\delta < \omega_1$ be such that the set A' of $p \in A$ such that $\delta_{N_p} = \delta$ is uncountable. Pick some $p \in A'$ and some $p' \in A' \setminus N_p$. Then, since $\delta_{N_{p'}} = \delta_{N_p} = \delta$ and $p' \in N_{p'}$, there is a condition q extending p' such that q is $(N_p, \mathbb{P})$–generic. In particular q is compatible with some $\bar{p} \in A \cap N_p$. This is a contradiction since A is an antichain and $\bar{p} \neq p'$. $\square$

It is clear that the intersection of any c.c.c. poset $\mathcal{Q}$ with any $N \preccurlyeq H(\theta)$ such that $\mathcal{Q} \in N$ (for any $\theta > |\mathsf{TC}(\mathcal{Q})|$) is itself c.c.c. In fact, for every cardinal κ the κ–c.c. is hereditary, in the sense that if $\mathcal{Q}$ has the κ–c.c., θ is a cardinal such that $|\mathsf{TC}(\mathcal{Q})| < \theta$, and $N \preccurlyeq H(\theta)$ is such that

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[3]Where $\mathsf{TC}(x)$ denotes the transitive closure of x.
[4]Note that we are not assuming that $p \in \bigcap\{N_i : i < m\}$.

$\mathcal{Q} \in N$, then $\mathcal{Q} \cap N$ is κ–c.c. The reason is simply that two condition in $\mathcal{Q} \cap N$ are compatible in $\mathcal{Q}$ if and only if they have a common extension in $\mathcal{Q} \cap N$. This fact, for $\kappa = \aleph_1$, is a crucial ingredient in the Solovay–Tenenbaum consistency proof of MA with $2^{\aleph_0} = \lambda$, where one argues that if $\mathcal{Q}$ is a c.c.c. poset in the final extension $\mathbf{W}$ and $(A_i \mid i < \mu) \in \mathbf{W}$ is a sequence of maximal antichains of $\mathcal{Q}$ for $\mu < \lambda$, then there is a c.c.c. poset $\mathcal{R} \subseteq \mathcal{Q}$ such that each A_i is a maximal antichain of $\mathcal{R}$ and such that moreover $|\mathcal{R}| < \lambda$.[5] It is not clear whether or not the $\aleph_{1.5}$–c.c. is hereditary in the above sense, but in any case we can prove the following weak form of this property, which will suffice for our consistency proof (Theorem 2.1).

Fact 1.3. *Suppose $\mathcal{Q}$ has the $\aleph_{1.5}$–c.c., θ is a cardinal such that there is some strong limit cardinal χ with $\theta > \chi > |\mathsf{TC}(\mathcal{Q})|$, and M is an elementary substructure of $H(\theta)$ such that $\mathcal{Q} \in M$ and $^{\omega_1}M \subseteq M$. Then $\mathcal{R} = \mathcal{Q} \cap M$ is a complete suborder of $\mathcal{Q}$ with the $\aleph_{1.5}$–c.c.*

Proof. To start with, notice that $\mathcal{R}$ has the $\aleph_2$–chain condition and is a complete suborder of $\mathcal{Q}$ since M is closed under ω_1–sequences. To see that $\mathcal{R}$ has the $\aleph_{1.5}$–c.c., let $\chi \geq |\mathsf{TC}(\mathcal{Q})|^+$ be a strong limit cardinal in M and W a bijection between χ and $H(\chi)$ also living in M. Let $D \subseteq [H(\chi)]^{\aleph_0}$ be a club witnessing the $\aleph_{1.5}$–c.c. of $\mathcal{Q}$, let $p \in \mathcal{R}$, and let $N_0, \ldots, N_m$ be a finite set of countable elementary substructures of $\langle H(\chi), \in, W \rangle$ in D containing $\mathcal{Q}$ and such that $p \in N_j$ for some j with δ_{N_j} minimal.

Subclaim 1.3.1. *In M there are countable elementary substructures $M_0, \ldots, M_m$ of $(H(\chi), \in, W)$ such that for all i,*

    (a) $M_i \in M$,
    (b) $M_i \cap N_i = N_i \cap M$,[6] *and*
    (c) *there is an isomorphism $\varphi_i : (N_i, \in, W) \longrightarrow (M_i, \in, W)$ fixing $N_i \cap M_i$.*

Proof. By elementarity and since M contains all the relevant parameters (all the real numbers, $M \cap N_i$ and W), M also contains a set $\{M_0, \ldots, M_m\}$ such that for each i,

    • $M \cap N_i \subseteq M_i$ and
    • there is an isomorphism φ_i between $(N_i, \in, W, M_i \cap N_i \cap \chi)$ and $(M_i, \in, W, M_i \cap M_i \cap \chi)$.

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[5]One then argues that there is some stage of the iteration at which both $\mathcal{R}$ and $(A_i \mid i < \mu)$ were defined and at which we forced with $\mathcal{R}$.

[6]In particular, $\mathcal{Q} \in M_i$.

Note that φ_i has to be the identity on $M_i \cap N_i \cap \chi$. Finally, since there is a bijection $W : \chi \longrightarrow H(\chi)$ definable in $(H(\chi), \in, W)$, we have that φ_i fixes $M \cap N_i \cap \chi$ if and only if it fixes $M \cap N_i$.

$\square$

Fix $M_0, \ldots, M_m$ as in the above claim. Since $p \in M_j$ and $\delta_{M_j} = \min\{\delta_{M_0}, \ldots, \delta_{M_m}\}$, there is a condition $\overline{p} \in \mathcal{R}$ such that $\overline{p} \leq_{\mathcal{Q}} p$ and such that $\overline{p}$ is $(M_i, \mathcal{Q})$–generic for every i. Suppose towards a contradiction that there is $p' \leq_{\mathcal{R}} \overline{p}$, $i_0 < m + 1$, and some maximal antichain A of $\mathcal{R}$ in N_{i_0} such that no condition in $A \cap N_{i_0}$ is compatible with p'. Let $p^* \in \mathcal{R}$ be a common extension of p' and of some $p'' \in \varphi_i(A) \cap M_{i_0}$. To see that there are such p^* and p'', note that $\varphi_{i_0}(A) \in M_i$ is a maximal antichain of $\mathcal{Q}$. This is true since $\mathcal{Q} \in N_{i_0} \cap M_{i_0}$ and, therefore, φ_{i_0} fixes $\mathcal{Q}$.

Now note that $A \in M$ since M is closed under ω_1–sequences and $|A| \leq \aleph_1$. It follows that $\varphi_{i_0}(A) = A$ since φ_{i_0} is the identity on $M \cap N_{i_0}$. Also, $p'' \in N_{i_0}$. To see this, take a surjection $f : \omega_1 \longrightarrow A$ in $N_{i_0} \cap M$ (take for example the W–first surjection $f : \omega_1 \longrightarrow A$). Then $\varphi_{i_0}(f) = f \in M_{i_0}$ is a surjection from ω_1 onto $\varphi_{i_0}(A) = A$. Let $\zeta \in M_{i_0} \cap \omega_1$ such that $f(\zeta) = p''$. Then $\zeta \in N_{i_0} \cap \omega_1$ and so $p'' = f(\zeta) \in N_{i_0}$. This contradiction finishes the proof. $\square$

Corollary 1.4. *If $\mathcal{Q}$ has the $\aleph_{1.5}$–chain condition and X is a subset of $\mathcal{Q}$, then there is $\mathcal{R}$ such that*

(i) *$\mathcal{R}$ is a complete suborder of $\mathcal{Q}$,*
(ii) *$\mathcal{R}$ has the $\aleph_{1.5}$–chain condition,*
(iii) *$X \subseteq \mathcal{R}$, and*
(iv) *$|\mathcal{R}| \leq |X|^{\aleph_1}$*

We are ready now to define our generalisation of MA.

Definition 1.5. Given a cardinal λ, let $\mathsf{MA}_\lambda^{1.5}$ be the following statement: For every $\aleph_{1.5}$–c.c. poset $\mathbb{P}$ and every collection $\mathcal{D}$ of size λ consisting of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

In the remainder of this section we present more consequences of $\mathsf{MA}_\lambda^{1.5}$.

Proposition 1.6. *$\mathsf{MA}_{\aleph_2}^{1.5}$ implies that if $\mathbb{P}$ is $\aleph_{1.5}$–c.c. and $X \in [\mathbb{P}]^{\aleph_2}$, then there is $Y \in [X]^{\aleph_2}$ such that every nonempty $\sigma \in [Y]^{<\omega}$ has a lower bound in $\mathbb{P}$. In particular, $\mathsf{MA}_{\aleph_2}^{1.5}$ implies that any finite support product of $\aleph_{1.5}$–c.c. posets has the $\aleph_2$–c.c.*

Note that there are ZFC examples showing that the $\aleph_2$–c.c. is not productive ([15]). The proof of Proposition 1.6 is essentially the same

as the classical proof that the c.c.c. is productive under $\mathsf{MA}_{\aleph_1}$ (see for example [8], Lemma 2.23 and Theorem 2.24). For the reader's convenience we include the argument here.

Say that a topological space is $\aleph_{1.5}$–c.c. if its topology, ordered by inclusion, is an $\aleph_{1.5}$–c.c. partial order. Of course, a partial order $\mathbb{P}$ is $\aleph_{1.5}$–c.c. if and only if the topology on $(\mathbb{P}, \leq)$ generated by the collection of cones $\{p \in \mathbb{P} : p \leq q\}$ (for $q \in \mathbb{P}$) is $\aleph_{1.5}$–c.c. Hence, in order to prove the first assertion of Proposition 1.6 it suffices to prove the following proposition. The second assertion follows then immediately from the first by a standard Δ–system argument.

Proposition 1.7. $\mathsf{MA}^{1.5}_{\aleph_2}$ *implies that if X is an $\aleph_{1.5}$–c.c. topological space and $\{U_\alpha : \alpha < \omega_2\}$ is a collection of nonempty open subsets of X, then there is $I \in [\omega_2]^{\aleph_2}$ such that $\{U_\alpha : \alpha \in I\}$ has the finite intersection property.*

Proof. Let $(V_\alpha)_{\alpha < \omega_2}$ be the decreasing sequence of open sets given by $V_\alpha = \bigcup_{\beta > \alpha} U_\beta$. Note that there must be $\alpha < \omega_2$ such that $\bar{V}_\alpha = \bar{V}_\beta$ for all $\beta > \alpha$ (where $\bar{V}$ denotes topological closure). Otherwise there would be $Z \subseteq \omega_2$ unbounded in ω_2 such that $V_\alpha \setminus \bar{V}_\beta \neq \emptyset$ for all $\alpha < \beta$ in Z, which would yield the existence of $\aleph_2$–many pairwise disjoint open subsets of X, a contradiction since X is $\aleph_{1.5}$–c.c.

Let now α be as above, and let $\mathbb{P}$ be the partial order of all nonempty open subsets of V_α ordered by inclusion. For every $\beta > \alpha$ let D_β be the set of $U \in \mathbb{P}$ such that $U \subseteq U_\gamma$ for some $\gamma > \beta$. Since $\bar{V}_\alpha \subseteq \bar{V}_\beta$, it follows that D_β is a dense subset of $\mathbb{P}$. Hence, since $\mathbb{P}$ is $\aleph_{1.5}$–c.c., by $\mathsf{MA}^{1.5}_{\aleph_2}$ we may find a filter $G \subseteq \mathbb{P}$ meeting D_β for all $\beta > \alpha$, which gives the desired conclusion. $\square$

Note that the above (weak) productiveness result, as well as the productiveness of c.c.c. under $\mathsf{MA}_{\aleph_1}$, are instances of the following general result, which is proved exactly as above.

Proposition 1.8. *If Γ is a class of posets with the κ–c.c., then the forcing axiom for Γ and for collections of κ–many dense sets implies that for every $\mathbb{P} \in \Gamma$ and every $X \in [\mathbb{P}]^\kappa$ there is some $Y \in [X]^\kappa$ such that every nonempty $\sigma \in [Y]^{<\omega}$ has a lower bound in $\mathbb{P}$; in particular, this forcing axiom implies that any finite support product of members of Γ has the κ–c.c.*

Next we will show that $\mathsf{MA}^{1.5}_\lambda$ implies certain uniform failures of Club Guessing on ω_1, as advertised in the abstract. It will be convenient to consider the following natural notion of rank (see for example [1]).

Definition 1.9. Given a set X and an ordinal δ, we define the *Cantor–Bendixson rank of δ with respect to X*, $\mathrm{rank}(X, \delta)$, by specifying that

- $\mathrm{rank}(X, \delta) \geq 1$ if and only if δ is a limit point of ordinals in X, and that
- if $\mu > 1$, $\mathrm{rank}(X, \delta) \geq \mu$ if and only and for every $\eta < \mu$, δ is a limit of ordinals ϵ with $\mathrm{rank}(X, \epsilon) \geq \eta$.

Given an ordinal τ, we will say that a set X of ordinals is τ–*thin* in case $\mathrm{rank}(X, \delta) \leq \tau$ for all ordinals δ.

Definition 1.10. Given ordinals τ and λ, $\tau < \omega_1$, $(\cdot)_\lambda^\tau$ is the following statement: For every sequence $(A_i)_{i<\lambda}$, if A_i is a τ–thin subset of ω_1 for all $i < \lambda$, then there is a club $C \subseteq \omega_1$ such that $|C \cap X_i| < \aleph_0$ for all i.

Clearly, if $\tau \leq \tau'$ and $\lambda \leq \lambda'$, then $(\cdot)_{\lambda'}^{\tau'}$ implies $(\cdot)_\lambda^\tau$. In particular, for every infinite τ and every $\lambda \geq \omega_1$, $(\cdot)_\lambda^\tau$ implies $\neg\mathsf{WCG}$, where WCG denotes Weak Club Guessing on ω_1 (cf. [16]).

Fact 1.11. *For every cardinal $\lambda \geq \omega_1$, the following weakening of $(\cdot)_\lambda^\omega$ implies $2^{\aleph_0} > \lambda$: For every sequence $(A_\delta^i)_{i<\lambda,\, \delta\in\mathrm{Lim}(\omega_1)}$, if each A_δ^i is a cofinal subset of δ of order type ω, then there is a club $C \subseteq \omega_1$ such that $A_\delta^i \nsubseteq C$ for all $i < \lambda$ and $\delta \in \mathrm{Lim}(\omega_1)$.*

Proof. Suppose $2^{\aleph_0} \leq \lambda$ and let $(A_\delta^i)_{i<\lambda,\, \delta\in\mathrm{Lim}(\omega_1)}$ be such that for each δ, $\{A_\delta^i : i < \lambda\}$ contains all cofinal subsets of δ of order type ω. If $C \subseteq \omega_1$ is a club and $\delta \in C$ is a limit point of C, then there is $i < \lambda$ such that $A_\delta^i \subseteq C$. $\square$

Furthermore, it is not hard to see that full $(\cdot)_\lambda^\omega$ implies $\mathfrak{b} > \lambda$.

Fact 1.12. *For every cardinal $\lambda \geq \omega_1$, $\mathsf{MA}_\lambda^{1.5}$ implies $(\cdot)_\lambda^\tau$ for every $\tau < \omega_1$.*

Proof. Let $(A_i)_{i<\lambda}$ be as in the definition of $(\cdot)_\lambda^\tau$. Let $\mathbb{P}$ consist of all pairs (f, X) such that

- (a) $f \subseteq \omega_1 \times \omega_1$ is a finite function such that $\mathrm{rank}(f(\nu), f(\nu)) \geq \nu$ for every $\nu \in \mathrm{dom}(f)$,
- (b) X is finite set of triples (i, ν, a) such that $i < \lambda$, $\nu \in \mathrm{dom}(f)$, $\mathrm{rank}(A_i, f(\nu)) < f(\nu)$, and a is a finite subset of $f(\nu)$, and
- (c) for every $(i, \nu, a) \in X$, $\mathrm{range}(f \restriction \nu) \cap A_i = a$.

Given $\mathbb{P}$–conditions (f_0, X_0) and (f_1, X_1), (f_1, X_1) extends (f_0, X_0) if $f_0 \subseteq f_1$ and $X_0 \subseteq X_1$.

Using the fact that there are only $\aleph_1$–many finite functions $f \subseteq \omega_1 \times \omega_1$, it is easy to check that $\mathbb{P}$ is $\aleph_{1.5}$–c.c. (for example by arguments

as in Section 5 of [1] for similar forcings). Also, there is a collection $\mathcal{D}$ of $\max\{\lambda, \omega_1\}$–many dense subsets of ω_1 such that if G is a filter of $\mathbb{P}$ meeting all members of $\mathcal{D}$, then range$(\bigcup\{f : (f, X) \in G$ for some $X\})$ is a club witnessing $(\cdot)_\lambda^+$ for $(A_i)_{i<\lambda}$. $\square$

In contrast with Fact 1.12, no forcing axiom MA_λ implies $(\cdot)_\lambda^\tau$ for any infinite $\tau < \omega_1$ and any $\lambda \geq \omega_1$. The reason is simply that MA_λ can always be forced by a c.c.c. forcing and c.c.c. forcing preserves WCG.

It seems that Theorem 2.1 below provides the first known construction of a model of MA_λ, for any $\lambda > \omega_1$, in which WCG fails. Indeed, any long enough c.c.c. iteration $\mathcal{P}$ with finite supports producing a model of MA will necessarily force WCG. The reason is that $\mathcal{P}$ will add a Cohen real at some stage by its being a c.c.c. iteration with finite supports (and in fact at the ω–th stage). This Cohen real will add a weak club–guessing sequence $\vec{C}$ at that stage (for example by results in [7]) and, letting $\mathbf{W}$ be the corresponding model, $\vec{C}$ will remain weak club–guessing in the end since the tail forcing has the c.c.c. in $\mathbf{W}$ and therefore every club of ω_1 in the final model contains a club from $\mathbf{W}$.

Definition 1.13. ([1]) Given a partial order $\mathbb{P}$, $\mathbb{P}$ *is finitely proper* if for every cardinal $\theta \geq |\mathbb{P}|^+$, every finite set $\{N_i : i \in m\}$ of countable elementary substructures of $H(\theta)$ containing $\mathbb{P}$, and every condition $p \in \bigcap\{N_i : i < m\} \cap \mathbb{P}$ there is some $q \in \mathbb{P}$ extending p and $(N_i, \mathbb{P})$–generic for all i.

Definition 1.14. ([1], essentially) Given a cardinal λ, $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)_\lambda$ is the forcing axiom for the class of finitely proper posets of size $\aleph_1$ and for collections of λ–many dense sets.

Fact 1.15. *For every cardinal $\lambda \geq \omega_1$, $\mathsf{PFA}^{fin}(\omega_1)_\lambda$ implies the following.*

(1) MA_{ω_1}
(2) $(\cdot)_{\omega_1}^\tau$ *for every $\tau < \omega_1$.*
(3) *For every set $\mathcal{F}$ of size λ consisting of functions from ω_1 into ω_1 there is a function $g : \omega_1 \longrightarrow \omega_1$ such that both $\{\nu < \omega_1 : f(\nu) < g(\nu)\}$ and $\{\nu < \omega_1 : f(\nu) = g(\nu)\}$ are unbounded for every $f \in \mathcal{F}$. In particular, $\mathfrak{d}(\omega_1) > \lambda$, where $\mathfrak{d}(\omega_1)$ is the dominating number for $^{\omega_1}\omega_1$ and where domination is understood relative to the ideal of countable sets.*
(4) *For every $\mathcal{R} \subseteq [\omega_1]^{\aleph_1}$ of size λ there is some $b \in [\omega_1]^{\aleph_1}$ such that $|a \cap b| = \aleph_1$ and $|a \setminus b| = \aleph_1$ for every $a \in \mathcal{R}$. In particular, $\mathfrak{r}(\omega_1) > \lambda$, where $\mathfrak{r}(\omega_1)$ is the reaping number for $[\omega_1]^{\aleph_1}$ and*

where the reaping relation is understood again relative to the ideal of countable sets.

Proof. The proofs of (1) and (2) are either immediate or as the corresponding proofs from $\mathsf{MA}_\lambda^{1.5}$. (3) and (4) follow from considering Baumgartner's forcing for adding a club $C \subseteq \omega_1$ by finite approximations. $\square$

The following result is straightforward.

Fact 1.16. *Every finitely proper poset of size $\aleph_1$ has the $\aleph_{1.5}$–chain condition. In particular, for every cardinal λ, $\mathsf{MA}_\lambda^{1.5}$ implies $\mathsf{PFA}^{fin}(\omega_1)_\lambda$.*

The proof of the main theorem in [1] essentially shows the consistency of $\mathsf{PFA}^{fin}(\omega_1)_\lambda$ for arbitrary λ.

2. The consistency of $\mathrm{MA}_\lambda^{1.5}$

Our main theorem is the following:

Theorem 2.1. *(CH) Let $\kappa \geq \omega_2$ be a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ and $\Diamond(\{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \omega_1\})$ holds. Then there is a proper forcing notion $\mathcal{P}$ of size κ with the $\aleph_2$–chain condition such that the following statements hold in the generic extension by $\mathcal{P}$:*

(1) $2^{\aleph_0} = \kappa$
(2) $\mathsf{MA}_{<2^{\aleph_0}}^{1.5}$

The proof of Theorem 2.1 is an elaboration of the proof of the main theorem in [1]. Our approach in that paper consisted in building a certain type of finite support forcing iteration $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ of length κ (in a broad sense of 'forcing iteration')[7] using what one may describe as finite "symmetric" systems of countable elementary substructures of a fixed $H(\kappa)$[8] as side conditions. These systems of structures were added at the first stage $\mathcal{P}_0$ of the iteration. Roughly speaking, the fact that the supports of the conditions in the iteration were finite ensured that the inductive proofs of the relevant facts – mainly the $\aleph_2$–c.c. of all $\mathcal{P}_\alpha$ and their properness – went through. The use of the sets of structures as side conditions was crucial in the proof of properness.[9]

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[7] In the sense that $\mathcal{P}_\beta$ is a regular extension of $\mathcal{P}_\alpha$ whenever $\alpha < \beta \leq \kappa$. It follows of course that $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is forcing–equivalent to a forcing iteration $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ in the ordinary sense (that is, such that $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ for all α, where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$–name for a poset), but such a presentation of $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is not really natural.

[8] This κ is exactly the value that $2^{\aleph_0}$ attains at the end of the construction.

[9] For more on the motivation of this type of construction see [1].

Here we change the proof from [1] in basically three ways. One of the changes is in the proof of the $\aleph_2$–chain condition of the iteration (Lemma 2.19). The induction now is a bit different from the one in the corresponding proof from [1] (and it has to be as the iterands we are now considering are no longer of size $\aleph_1$). In fact, our forcing has the $\aleph_2$–c.c. but it is not $\aleph_2$–Knaster (on the other hand, the forcing in [1] is $\aleph_2$–Knaster). Another, more crucial, difference is the presence of a diamond–sequence which ensures that all posets with the $\aleph_{1.5}$–c.c. (with no restrictions on their size) occurring in the final extension have been dealt with at κ–many stages during the iteration. A third difference between the present construction and the construction from [1] is the fact that our systems of structures now exhibit a weaker form of symmetry than the ones in [1]; we call the systems in the present paper *partial symmetric systems*, as opposed to *symmetric systems* as in [1] (see Subsection 2.1 for more on the difference between these two forms of symmetry). The need for the weaker form of symmetry comes from the part of the argument using a diamond–sequence where we infer that all $\aleph_{1.5}$–c.c. posets occurring in the final model are 'captured' along the iteration. It is perhaps worth pointing out that the proof of properness now is almost identical to the corresponding proof from [1] (modulo notational changes).

Theorem 2.1 shows that all forcing axioms of the form $\mathsf{MA}^{1.5}_{<\kappa}$, for a fixed reasonably defined cardinal κ, are consistent (relative to the consistency of ZFC). This theorem shows also that no axiom of the form $\mathsf{MA}^{1.5}_\lambda$ decides the size of the continuum and thus, by Fact 1.16, fits nicely within the ongoing project of showing whether or not weak fragments of BPFA imply $2^{\aleph_0} = \aleph_2$.[10] The problem whether (consequences of) forcing axioms for classes of posets with small chain condition decide the size of the continuum does not seem to have received much attention in the literature so far.[11] One place where the problem has been addressed is of course our [1]. Prior to [1], M. Foreman and P. Larson showed in an unpublished note ([4]) that the forcing axiom (relative to collections of $\aleph_1$–many dense sets) for the class of posets of size $\aleph_2$ preserving stationary subsets of ω_1 implies $2^{\aleph_0} = \aleph_2$. Several natural problems in this area remain open. For example it is not known whether the forcing axiom (relative to collections of $\aleph_1$–many dense sets) for the class of semi-proper posets of size $\aleph_2$ implies $2^{\aleph_0} = \aleph_2$,

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[10]BPFA is the assertion that $H(\omega_2)^{\mathbf{V}}$ is a Σ_1–elementary substructure of $H(\omega_2)^{\mathbf{V}^{\mathbb{P}}}$ for every proper poset $\mathbb{P}$ ([2]). J. Moore showed that BPFA implies $2^{\aleph_0} = \aleph_2$ ([11]).

[11]Usually the focus has been on deriving $2^{\aleph_0} = \aleph_2$ from bounded forms of forcing axioms, that is forms of forcing axioms in which one considers only small maximal antichains but where the posets are allowed to have large antichains as well.

and the same is open for the forcing axiom for the class of all posets of size $\aleph_1$ preserving stationary subsets of ω_1, as well as for the forcing axiom for the class of all proper posets of size $\aleph_1$. Let us denote these two forcing axioms by, respectively, $\mathsf{MM}(\omega_1)$ and $\mathsf{PFA}(\omega_1)$. It is open whether or not $\mathsf{MM}(\omega_1)$ is equivalent to $\mathsf{PFA}(\omega_1)$,[12] and even whether $\mathsf{MM}(\omega_1)$ has consistency strength above ZFC.[13]

As we mentioned before, Theorem 2.1 shows in particular that the forcing axiom $\mathsf{MA}^{1.5}_{<\aleph_2}$ has the same consistency strength as ZFC. Other articles dealing with the consistency strength of other (related) fragments of PFA are [10], [5], [13], and [12].

The rest of the paper is organised as follows. In the following subsection we introduce the central notion of partial symmetric system of submodels and prove the main amalgamation results for these systems. Subsection 2.2 contains the construction of our iteration $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ ($\mathcal{P}_\kappa$ will witness Theorem 2.1). Finally, Subsection 2.3 contains proofs of the main facts about $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$. Theorem 2.1 follows then easily from these facts.

2.1. Partial symmetric systems of submodels.

Throughout the paper, if N and N' are models for which that there is a (unique) isomorphism from $(N, \in)$ into $(N', \in)$, then we denote this isomorphism by $\Psi_{N,N'}$. For this subsection let us fix a regular cardinal θ and a predicate $T \subseteq H(\theta)$. In this situation we will tend to refer to a structure $(N, \in, T \cap N)$ by the simpler expression $(N, \in, T)$.

Definition 2.2. Let $\{N_i : i < m\}$ be a finite set of countable subsets of $H(\theta)$. We will say that $\{N_i : i < m\}$ *is a partial T–symmetric system (of elementary substructures of $(H(\theta), \in, T)$)* if and only if the following holds.

(A) For every $i < m$, $(N_i, \in, T)$ is an elementary substructure of $(H(\theta), \in, T)$.

(B) Given distinct i, i' in m, if $\delta_{N_i} = \delta_{N_{i'}}$, then there is a unique isomorphism

$$\Psi_{N_i, N_{i'}} : (N_i, \in, T) \longrightarrow (N_{i'}, \in, T)$$

Furthermore, we ask that $\Psi_{N_i, N_{i'}}$ be the identity on $N_i \cap N_{i'}$.

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[12]On the other hand, $\mathsf{PFA}(\omega_1)$ is trivially equivalent to the forcing axiom for the class of semi-proper posets of size $\aleph_1$.

[13]$\mathsf{PFA}(\omega_1)$ can be easily forced over ZFC model. Also, the existence of a non–proper poset of size $\aleph_1$ preserving stationary subsets of ω_1 is consistent. In fact, Hiroshi Sakai has constructed such a poset assuming a suitably strong version of $\diamondsuit_{\omega_1}$ which holds in **L** and which can always be forced.

(C) For all i, i', j in m, if $N_j \in N_i$ and $\delta_{N_i} = \delta_{N_{i'}}$, then there is some $j' < m$ such that $\Psi_{N_i, N_{i'}}(N_j) = N_{j'}$.

(D) For all i, $j < m$, if $\delta_{N_i} < \delta_{N_j}$, then there is some $i' < m$ such that
  (1) $\delta_{N_{i'}} = \delta_{N_i}$,
  (2) $N_{i'} \in N_j$, and
  (3) $N_i \cap N_j = N_{i'} \cap N_i$.

We will often talk about partial symmetric systems of elementary substructures, without mentioning T, or even just partial symmetric systems, in contexts where T is understood or not relevant.

The present definition of symmetry is weaker than the one in [1] in the following sense: In the definition of *symmetric system* from [1], instead of condition (D) we had the following:

$(D)^+$ For all i, $j < m$, if $\delta_{N_i} < \delta_{N_j}$, then there is some $j' < m$ such that
  (1) $\delta_{N_{j'}} = \delta_{N_j}$, and
  (2) $N_i \in N_{j'}$.

It is easy to verify that every finite system of structures satisfying (A), (B), (C) and $(D)^+$ (i.e., satisfying the original definition of symmetric system) satisfies (A)–(D): Given i, j as in the hypothesis of (D), if $N_{j'}$ is such that $\delta_{N_{j'}} = \delta_{N_j}$ and $N_i \in N_{j'}$, then $N_{i'} := \Psi_{N_{j'}, N_j}(N_i)$ witnesses (D) relative to N_i and N_j.

The following lemma is immediate.

Lemma 2.3. *Let N and N' be countable elementary substructures of $(H(\theta), \in, T)$. Suppose $\mathcal{N} \in N$ is a partial T–symmetric system of elementary substructures of $H(\theta)$ and $\Psi : (N, \in, T) \longrightarrow (N', \in, T)$ is an isomorphism. Then $\Psi(\mathcal{N}) = \Psi``\mathcal{N}$ is also a partial T–symmetric system of elementary substructures of $H(\theta)$.*

We will need three amalgamations lemmas for the proof of Theorem 2.1. The first of the lemmas is Lemma 2.4. This lemma will be used in case $\alpha = 0$ of the inductive proof of Lemma 2.22.

Lemma 2.4. *Let $\mathcal{N}$ be a partial T–symmetric system of elementary substructures of $H(\theta)$ and let $N \in \mathcal{N}$. Then the following holds.*

(1) *$\mathcal{N} \cap N$ is a partial T–symmetric system.*

(2) *Suppose $\mathcal{N}^* \in N$ is a partial T–symmetric system such that $\mathcal{N} \cap N \subseteq \mathcal{N}^*$.*
   Let
   $$\mathcal{M} = \mathcal{N} \cup \bigcup \{ \Psi_{N_0, N_1} ``(\mathcal{N}^* \cap N_0) \mid N_0, N_1 \in \mathcal{N},\ \delta_{N_0} = \delta_{N_1} \leq \delta_N \}$$
   Then,

(i) *for every $N_0 \in \mathcal{N}$ such that $\delta_{N_0} \leq \delta_N$,*
  (a) $\mathcal{M} \cap N_0 = \Psi_{N_1,N_0}``(\mathcal{N}^* \cap N_1)$ *for every $N_1 \in \mathcal{N} \cap (N \cup \{N\})$ such that $\delta_{N_1} = \delta_{N_0}$, and*
  (b) $\mathcal{M} \cap N_0$ *is a partial T–symmetric system,*
  and
(ii) $\mathcal{M}$ *is a partial T–symmetric system.*

Proof. It is immediate to check that all conditions in the definition of partial symmetric system are inherited by $\mathcal{N} \cap N$, which gives (1).

Let us start the proof of (2). Let N_0 be as in the hypotheses of (i). Let $N_1 \in \mathcal{N} \cap (N \cup \{N\})$ be of the same height as δ_{N_0} and let us check that $\mathcal{M} \cap N_0 = \Psi_{N_1,N_0}``(\mathcal{N}^* \cap N_1)$. Obviously, $\Psi_{N_1,N_0}``(\mathcal{N}^* \cap N_1) \subseteq N_0 \cap \mathcal{M}$ by the construction of $\mathcal{M}$. For the reverse inclusion, suppose N_2, $N_3 \in \mathcal{N}$ are such that $\delta_{N_2} = \delta_{N_3} \leq \delta_N$ and $M \in \mathcal{N}^* \cap N_2$ is such that $\Psi_{N_2,N_3}(M) \in N_0$. Since $M \in \mathcal{N}^* \subseteq N$, we may assume without loss of generality that $N_2 = N$ if $\delta_{N_2} = \delta_N$ by conditions (B)–(D) in Definition 2.2 for $\mathcal{N}$, and $N_2 \in N$ if $\delta_{N_2} < \delta_N$ by conditions (B) and (C). We want to see that $\Psi_{N_2,N_3}(M) = \Psi_{N_1,N_0}(M')$ for some $M' \in \mathcal{N}^* \cap N_1$. We have two cases for this.

Suppose first that $\delta_{N_1} \leq \delta_{N_2}$. Let $N_0' = N_3$ if $\delta_{N_0} = \delta_{N_3}$ and, if $\delta_{N_0} < \delta_{N_3}$, by condition (D) for $\mathcal{N}$ let $N_0' \in N_3 \cap \mathcal{N}$ be such that $\delta_{N_0'} = \delta_{N_0}$ and $N_3 \cap N_0 = N_0 \cap N_0'$. Then, if $\delta_{N_0} < \delta_{N_3}$, $\Psi_{N_3,N_2}(N_0') \in \mathcal{N}$ and $M \in \mathcal{N}^* \cap \Psi_{N_3,N_2}(N_0')$ as $\Psi_{N_2,N_3}(M) \in N_0$. In that case, $M' := \Psi_{\Psi_{N_3,N_2}(N_0'),N_1}(M)$ is as desired since $M' \in \mathcal{N}^*$, which is true because N_1 and $\Psi_{N_3,N_2}(N_0')$ are both in $\mathcal{N} \cap N \subseteq \mathcal{N}^*$, $M \in \mathcal{N}^*$, and $\mathcal{N}^*$ satisfies (C). In the case $\delta_{N_0} = \delta_{N_3}$ we can of course take $M' = \Psi_{N_2,N_1}(M)$.

Finally, suppose $\delta_{N_2} < \delta_{N_1}$. By (D) applied to $\mathcal{N}$, let $N_3' \in N_0 \cap \mathcal{N}$ be such that $\delta_{N_3'} = \delta_{N_3}$ and $N_0 \cap N_3 = N_3 \cap N_3'$. Then $\Psi_{N_2,N_3}(M) \in N_3 \cap N_3'$, and therefore $\Psi_{N_2,N_3}(M) = \Psi_{N_2,N_3'}(M) = \Psi_{\Psi_{N_0,N_1}(N_3'),N_3'}(M')$ for $M' = \Psi_{N_2,\Psi_{N_0,N_1}(N_3')}(M)$. But $M' \in \mathcal{N}^*$ since N_2 and $\Psi_{N_0,N_1}(N_3')$ are both in $N \cap \mathcal{N} \subseteq \mathcal{N}^*$, $M \in \mathcal{N}^*$, and $\mathcal{N}^*$ satisfies (C). Finally, $\Psi_{\Psi_{N_0,N_1}(N_3'),N_3'} = \Psi_{N_1,N_0} \upharpoonright \Psi_{N_0,N_1}(N_3')$, which gives what we wanted.

Now, since $\mathcal{M} \cap N_0 = \Psi_{N_1,N_0}``(\mathcal{N}^* \cap N_1)$ for some (equivalently, every) $N_1 \in \mathcal{N} \cap (N \cup \{N\})$ of the same height as N_0, and since $\mathcal{N}^* \cap N_1$ is a partial symmetric system for every such N_1 by (1), $\mathcal{M} \cap N_0$ is a partial symmetric system by Lemma 2.3.

Let us proceed to the proof of (ii). It is clear that $\mathcal{M}$ satisfies condition (A) in Definition 2.2 thanks to Lemma 2.3.

To see that it satisfies condition (B), it suffices to show that if Q_0, $Q_1 \in \mathcal{M}$ are such that $\delta_{Q_0} = \delta_{Q_1} \leq \delta_N$, then Q_0 and Q_1 are isomorphic and the unique isomorphism between these structures fixes $Q_0 \cap Q_1$. By symmetry of the situation we may assume that there are N_0, N_1,

M_0 and M_1 in $\mathcal{N}$ such that $\delta_{N_0} = \delta_{N_1} \leq \delta_{M_0} = \delta_{M_1} \leq \delta_N$ and such that $Q_0 = \Psi_{N_0,N_1}(Q)$ and $Q_1 = \Psi_{M_0,M_1}(Q^*)$ for some Q and Q^* in $\mathcal{N}^*$. By (B) for $\mathcal{N}^*$ there is a unique isomorphism Ψ_{Q,Q^*} between Q and Q^* and Ψ_{Q,Q^*} fixes $Q \cap Q^*$. Then $\Psi := \Psi_{M_0,M_1} \circ \Psi_{Q,Q^*} \circ (\Psi_{N_1,N_0} \upharpoonright Q_0)$ is of course an isomorphism between Q_0 and Q_1. To see that Ψ fixes $Q_0 \cap Q_1$, suppose $x \in Q_0 \cap Q_1$. Let $N_1' = M_1$ if N_1 and M_1 have the same height and, if $\delta_{N_1} < \delta_{M_1}$, let $N_1' \in M_1 \cap \mathcal{N}$ be such that $N_1' \cap N_1 = N_1 \cap M_1$. Then $Q' := \Psi_{N_0,N_1'}(Q) \in \mathcal{M} \cap N_1' \subseteq \mathcal{M} \cap M_1$. Since $x \in N_1' \cap N_1$, we have $\Psi_{N_1',N_1}(x) = x$. In particular $x \in Q'$ since $\Psi_{N_1',N_1}(Q') = Q_0$ and $x \in Q_0$. But then $x \in Q' \cap Q_1$, and since both Q' and Q_1 are in $\mathcal{M} \cap M_1$ and $\mathcal{M} \cap M_1$ is a partial symmetric system by (i), the unique isomorphism Ψ_{Q',Q_1} between Q' and Q_1 fixes x. But then $\Psi = \Psi_{Q',Q_1} \circ (\Psi_{N_1,N_1'} \upharpoonright Q_0)$ since $\Psi_{N_1',N_1}(Q') = Q_0$, and therefore $\Psi(x) = x$ since $\Psi_{N_1',N_1}(x) = x$.

Let us verify now (C) for $\mathcal{M}$. Let N_0, $N_1 \in \mathcal{N}$ be of the same height and let $M \in \mathcal{M} \cap N_0$. We want to see that $\Psi_{N_0,N_1}(M) \in \mathcal{M}$. Without loss of generality we may assume $\delta_M < \delta_N$. The case $\delta_{N_0} \leq \delta_N$ follows immediately from part (i) (a). For the case $\delta_{N_0} > \delta_N$, let $N' \in N_0 \cap \mathcal{N}$ be such that $\delta_{N'} \leq \delta_N$ and $M \in N'$ (since $\mathcal{N}$ satisfies (C), we may also assume that $M \in \mathcal{M} \setminus \mathcal{N}$). Then $\Psi_{N_0,N_1}(N') \in \mathcal{N} \cap N_1$ by (C) for $\mathcal{N}$, and we are done again by the previous case since $\Psi_{N',\Psi_{N_0,N_1}(N')}(M) = \Psi_{N_0,N_1}(M)$.

We still need to consider the situation when N_0 and N_1 are in $\mathcal{M} \setminus \mathcal{N}$, $\delta_{N_0} = \delta_{N_1}$, $M \in \mathcal{M} \cap N_0$, and at least one of N_0, N_1 is not in $\mathcal{N}$. Suppose first $N_0 \notin \mathcal{N}$ and $N_1 \notin \mathcal{N}$. Then there is some $M_0 \in \mathcal{N}$ such that $\delta_{M_0} \leq \delta_N$ and $N_0 \in M_0$ and such that $N_0 = \Psi_{\overline{M}_0,M_0}(\overline{N}_0)$ for some $\overline{M}_0 \in \mathcal{N} \cap (N \cup \{N\})$ of the same height as M_0 and some $\overline{N}_0 \in \mathcal{N}^* \cap \overline{M}_0 \in \overline{N}_0$. We can also find $M_1 \in \mathcal{N}$ such that $\delta_{M_1} \leq \delta_N$ and $N_1 \in M_1$ and such that $N_1 = \Psi_{\overline{M}_1,M_1}(\overline{N}_1)$ for some $\overline{M}_1 \in \mathcal{N} \cap (N \cup \{N\})$ of the same height as M_1 and some $\overline{N}_1 \in \mathcal{N}^* \cap \overline{M}_1$. Let $\epsilon \in \{0,1\}$ with δ_{M_ϵ} minimal, let $M_\epsilon' = M_{1-\epsilon}$ if $\delta_{M_0} = \delta_{M_1}$ and, if $\delta_{M_\epsilon} < \delta_{M_{1-\epsilon}}$, let $M_\epsilon' \in \mathcal{N} \cap M_{1-\epsilon}$ be of the same height as M_ϵ and such that $M_{1-\epsilon} \cap M_\epsilon = M_\epsilon \cap M_\epsilon'$. Then $\Psi_{M_\epsilon,M_\epsilon'}(N_\epsilon) \in M_{1-\epsilon} \cap \mathcal{M}$, and we can finish thanks to part (i) (b) by looking at $M_{1-\epsilon} \cap \mathcal{M}$ and taking the appropriate isomorphic copies. Finally, in the cases when $N_0 \notin \mathcal{N}$ and $N_1 \in \mathcal{N}$ and when $N_0 \in \mathcal{N}$ and $N_1 \notin \mathcal{N}$, we can run an easier variant of the argument above.

Finally let us verify (D) for $\mathcal{M}$. Suppose M_0, M_1 are both in $\mathcal{M}$ and $\delta_{M_0} < \delta_{M_1}$. We want to see that there is some $M_0' \in \mathcal{M} \cap M_1$ of the same height as M_0 and such that $M_0' \cap M_0 = M_0 \cap M_1$. We may of course assume that M_0 and M_1 are not both in $\mathcal{N}$.

Suppose $M_0 \notin \mathcal{N}$ and $M_1 \in \mathcal{N}$. Let $N_0 \in \mathcal{N}$ be such that $\delta_{N_0} \leq \delta_N$ and $M_0 \in N_0$. If $\delta_{N_0} = \delta_{M_1}$, then we may take $M_0' = \Psi_{N_0,M_1}(M_0)$, and if $\delta_{N_0} < \delta_{M_1}$ we may take $M_0' = \Psi_{N_0,N_0'}(M_0)$, where $N_0' \in \mathcal{N} \cap M_1$ is of the same height as N_0 and such that $N_0' \cap N_0 = N_0 \cap M_1$. If $\delta_{N_0} > \delta_{M_1}$, then there is some $M_1' \in \mathcal{N} \cap N_0$ of the same height as M_1 and such that $M_1 \cap N_0 = M_1' \cap M_1$. By part (i) (b) for $\mathcal{M} \cap N_0$ there is some $M_0^* \in \mathcal{M} \cap M_1'$ of the same height as M_0 and such that $M_0^* \cap M_0 = M_1' \cap M_0$. Now it is easy to check that $\Psi_{M_1',M_1}(M_0^*) \in \mathcal{M}$ satisfies the desired conclusion.

Next, suppose $M_0 \in \mathcal{N}$ and $M_1 \notin \mathcal{N}$. Let $N_1 \in \mathcal{N}$ be such that $\delta_{N_1} \leq \delta_N$ and $M_1 \in N_1$. Since $\delta_{M_0} < \delta_{N_1}$, there is some $M_0^* \in \mathcal{N} \cap N_1$ of the same height as M_0 and such that $M_0^* \cap M_0 = M_0 \cap N_1$. By part (i) (b) for $\mathcal{M} \cap N_1$, there is some $M_0' \in \mathcal{M} \cap M_1$ of the same height as M_0^* and such that $M_0^* \cap M_0' = M_0^* \cap M_1$. Now it is easy to check that M_0' is as desired.

Finally, suppose that none of M_0 and M_1 is in $\mathcal{N}$. Let $N_0, N_1 \in \mathcal{N}$ be such that $\delta_{N_0} \leq \delta_N$, $\delta_{N_1} \leq \delta_N$, $M_0 \in N_0$ and $M_1 \in N_1$. Suppose $\delta_{N_0} \leq \delta_{N_1}$. Let $N_0' = N_1$ if N_0 and N_1 have the same height and, if $\delta_{N_0} < \delta_{N_1}$, let $N_0' \in \mathcal{N} \cap N_1$ be of the same height as N_0 and such that $N_0 \cap N_1 = N_0 \cap N_0'$. Then, $M_0^* := \Psi_{N_0,N_0'}(M_0) \in \mathcal{M} \cap N_1$. By (i) (b) for $\mathcal{M} \cap N_1$ we may now find $M_0' \in M_1$ of the same height as M_0^* and such that $M_1 \cap M_0^* = M_0^* \cap M_0'$. It is then easy to check that M_0' is as desired. The other case is when $\delta_{N_1} < \delta_{N_0}$. Let $N_1' \in \mathcal{N} \cap N_0$ be such that $\delta_{N_1'} = \delta_{N_1}$ and $N_0 \cap N_1 = N_1 \cap N_1'$. By part (i) (b) for $\mathcal{M} \cap N_0$ we may pick $M_0^* \in N_1' \cap \mathcal{M}$ of the same height as M_0 and such that $N_1' \cap M_0 = M_0^* \cap M_0$. Let $M_0^{**} = \Psi_{N_1',N_1}(M_0^*) \in \mathcal{M}$. Now, by (i) (b) for $\mathcal{M} \cap N_1$ we may find $M_0' \in \mathcal{M} \cap M_1$ of the same height as M_0^{**} and such that $M_0' \cap M_0^{**} = M_0^{**} \cap M_1$, and it is easy to check that M_0' is as desired. $\square$

Next comes our second amalgamation lemma for partial symmetric systems. It will be used in the proof of Lemma 2.19.

Lemma 2.5. *Let $\mathcal{N}_0$ and $\mathcal{N}_1$ be partial T–symmetric systems of elementary substructures of $H(\theta)$. Suppose that $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = R$ and that, for some $m < \omega$, there are enumerations $(N_i^0)_{i<m}$ and $(N_i^1)_{i<m}$ of $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively, together with an isomorphism Ψ between $\langle \bigcup \mathcal{N}_0, \in, T, R, N_i^0 \rangle_{i<m}$ and $\langle \bigcup \mathcal{N}_1, \in, T, R, N_i^1 \rangle_{i<m}$ which is the identity on R. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a partial T–symmetric system.*

Proof. $\mathcal{N}_0 \cup \mathcal{N}_1$ obviously satisfies condition (A) in Definition 2.2.

For condition (B), we must show that if i_0, $i_1 < m$ are such that $\delta_{N_{i_0}^0} = \delta_{N_{i_1}^1}$, then the isomorphism $\Psi_{N_{i_0}^0,N_{i_1}^1} := \Psi \circ \Psi_{N_{i_0}^0,N_{i_1}^0}$ fixes $N_{i_0}^0 \cap$

$N_{i_1}^1$. Now, if $x \in N_{i_0}^0 \cap N_{i_1}^1$, then $x \in R \cap N_{i_0}^0$, which implies that $\Psi(x) = x \in N_{i_0}^1 \cap N_{i_1}^1$ as Ψ is an isomorphism between the structures $\langle \bigcup \mathcal{N}_0, \in, N_i^0 \rangle_{i<m}$ and $\langle \bigcup \mathcal{N}_1, \in, N_i^1 \rangle_{i<m}$. But then $x \in N_{i_0}^0 \cap N_{i_1}^0$ again by the fact that Ψ is an isomorphism between $\langle \bigcup \mathcal{N}_0, \in, N_i^0 \rangle_{i<m}$ and $\langle \bigcup \mathcal{N}_1, \in, N_i^1 \rangle_{i<m}$, which implies that $\Psi_{N_{i_0}^0, N_{i_1}^0}(x) = x$ and hence that $((\Psi \upharpoonright N_{i_1}^0) \circ \Psi_{N_{i_0}^0, N_{i_1}^0})(x) = \Psi_{N_{i_0}^0, N_{i_1}^1}(x) = x$.

A similar argument establishes condition (C) for $\mathcal{N}_0 \cup \mathcal{N}_1$: Let M_1, M_2, $M_3 \in \mathcal{N}_0 \cup \mathcal{N}_1$ be such that $M_2 \in M_1$ and $\delta_{M_1} = \delta_{M_3}$. We must prove that $\Psi_{M_1, M_3}(M_2)$ is also in $\mathcal{N}_0 \cup \mathcal{N}_1$. We may clearly assume that there are indices $i, j \in \{1, 2, 3\}$ such that $M_i \in \mathcal{N}_0$ and $M_j \in \mathcal{N}_1$. The case when M_1 and M_3 are both in $\mathcal{N}_1$ and M_2 is in $\mathcal{N}_0$ can be treated as follows. First note that there are i_1 and i_2 such that $M_1 = N_{i_1}^1$ and $M_2 = N_{i_2}^0$. As $M_2 \in M_1$ and Ψ is an isomorphism fixing R (in particular, M_2), $N_{i_2}^0 \in N_{i_1}^0$. But $\Psi_{N_{i_2}^0, N_{i_2}^1} = \Psi_{N_{i_1}^0, N_{i_1}^1} \upharpoonright N_{i_2}^0$ and this isomorphism also fixes M_2. So M_1, $M_2 = N_{i_2}^1$ and M_3 are in $\mathcal{N}_1$. The last case that needs to be considered is when $M_3 \in \mathcal{N}_0$ and $M_1 \in \mathcal{N}_1$. Just as before, we can ensure the existence of i_1, i_2 and i_3 such that $M_3 = N_{i_3}^0$, $M_1 = N_{i_1}^1$ and $M_2 = N_{i_2}^1$. Finally, let i_4 be such that $N_{i_4}^1 = \Psi_{N_{i_1}^1, N_{i_3}^1}(N_{i_2}^1)$ and note that $N_{i_4}^0 = \Psi_{M_1, M_3}(M_2)$.

Finally, as to condition (D) for $\mathcal{N}_0 \cap \mathcal{N}_1$, it suffices to show that if i_0, $i_1 < m$ are such that $\delta_{N_{i_0}^0} < \delta_{N_{i_1}^1}$, then there is some i such that $\delta_{N_i^1} = \delta_{N_{i_0}^0}$, $N_i^1 \in N_{i_1}^1$, and $N_{i_1}^1 \cap N_{i_0}^0 = N_{i_0}^0 \cap N_i^1$. For this, by condition (D) for $\mathcal{N}_0$ we may find $i < m$ such that $\delta_{N_i^0} = \delta_{N_{i_0}^0}$, $N_i^0 \in N_{i_1}^0$, and $N_i^0 \cap N_{i_0}^0 = N_i^0 \cap N_{i_0}^0$. Then, since Ψ is an isomorphism, N_i^1, N_i^0 and $N_{i_0}^0$ have the same height and $N_i^1 \in N_{i_1}^1$. To see that $N_{i_1}^1 \cap N_{i_0}^0 = N_i^1 \cap N_{i_0}^0$, note that any set in either $N_{i_1}^1 \cap N_{i_0}^0$ or $N_i^1 \cap N_{i_0}^0$ is in R, and use the fact that Ψ is the identity on R together with the fact that $N_{i_1}^0 \cap N_{i_0}^0 = N_i^0 \cap N_{i_0}^0$.   □

Our final amalgamation lemma is the following. This lemma will be used in the proof of Lemma 2.31.

Lemma 2.6. *Let $\mathcal{N}_0$, $\mathcal{N}_1$ and $\mathcal{N}_2$ be partial T–symmetric systems of elementary substructures of $H(\theta)$ such that $\mathcal{N}_0 \subseteq \mathcal{N}_2$. Suppose that*

$$\left(\bigcup \mathcal{N}_0\right) \cap \left(\bigcup \mathcal{N}_1\right) = R$$

and that, for some $m < \omega$, there are enumerations $(N_i^0)_{i<m}$ and $(N_i^1)_{i<m}$ of $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively, together with an isomorphism Ψ between $\langle \bigcup \mathcal{N}_0, \in, T, R, N_i^0 \rangle_{i<m}$ and $\langle \bigcup \mathcal{N}_1, \in, T, R, N_i^1 \rangle_{i<m}$ which is the identity

on R. Suppose, in addition, that

$$\left(\bigcup \mathcal{N}_2\right) \cap \left(\bigcup \mathcal{N}_1\right) = R$$

Then $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}$, where

$$\mathcal{N} = \{\Psi_{N_0,N_1}(M) \mid M \in \mathcal{N}_2, M \in N_0, N_0 \in \mathcal{N}_0, N_1 \in \mathcal{N}_1, \delta_{N_0} = \delta_{N_1}\},$$

is a partial T–symmetric system.

Proof. Note that $\mathcal{M} = \mathcal{N}_0 \cup \mathcal{N}_1$ is a partial symmetric system by Lemma 2.5. Note also that $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}$ can be written as

$$\mathcal{M}_m \cup \{M \in \mathcal{N}_2 \mid \text{ there is no } N \in \mathcal{N}_0 \text{ such that } M \in N\}$$

for some finite $\subseteq$–increasing sequence $(\mathcal{M}_i)_{i \leq m}$ of partial symmetric systems such that $\mathcal{M}_0 = \mathcal{M}$ and such that for all $i < m$ there is some $N \in \mathcal{N}_0$ such that

$$\mathcal{M}_{i+1} = \mathcal{M}_i \cup \bigcup \{\Psi_{N_0,N_1}``(\mathcal{N}_2 \cap N) \mid N_0, N_1 \in \mathcal{M}_i, \delta_{N_0} = \delta_{N_1} \leq \delta_N\}$$

By Lemma 2.4, each $\mathcal{M}_i$ (for $i \leq m$) is a partial symmetric system. Also note that in fact $(\mathcal{M}_i)_{i \leq m}$ can be set up in such a way that there are $N^0, \ldots, N^{m-1} \in \mathcal{N}_0$ such that for every $i < m$, $\mathcal{M}_{i+1}$ is

$$\mathcal{M}_i \cup \bigcup_{k \leq i} \{\Psi_{N_0,N_1}``(\mathcal{N}_2 \cap N^k) \mid N_0 \in \mathcal{N}_0, N_1 \in \mathcal{N}_0 \cup \mathcal{N}_1, \delta_{N_0} = \delta_{N_1} \leq \delta_{N^k}\}$$

Indeed, start with any $N^0 \in \mathcal{N}_0$ such that $N^0 \cap (\mathcal{N}_2 \setminus \mathcal{N}_0) \neq \emptyset$ and build the corresponding $\mathcal{M}_1$. Note that $\mathcal{M}_1 \cap N \subseteq \mathcal{N}_2 \cap N$ for every $N \in \mathcal{N}_0$ since condition (C) in Definition 2.2 holds for $\mathcal{N}_2 \supseteq \mathcal{N}_0$ and since Ψ_{N_0,N_1} is the identity on R whenever $N_0 \in \mathcal{N}_0$ and $N_1 \in \mathcal{N}_1$ are such that $\delta_{N_0} = \delta_{N_1}$. Now we pick, if possible, any $N^1 \in \mathcal{N}_0$ such that $N^1 \cap (\mathcal{N}_2 \setminus \mathcal{M}_1) \neq \emptyset$, and build the corresponding $\mathcal{M}_2$. We iterate this process until we get nothing new. Let us verify now that $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}$ satisfies conditions (A)–(D) in Definition 2.2.

Condition (A) is obviously true. For condition (B), since it holds for $\mathcal{N}_2 \supseteq \mathcal{N}_0$ it suffices to show, by the above representation of $(\mathcal{M}_i)_{i \leq m}$, that if $M \in \mathcal{N}_2$, $\overline{M} \in \mathcal{N}_2$ is such that $\delta_{\overline{M}} = \delta_M$, and $N_0 \in \mathcal{N}_0$, $N_1 \in \mathcal{N}_1$ are such that $\overline{M} \in N_0$ and $\delta_{N_0} = \delta_{N_1}$, then $\Psi := \Psi_{N_0,N_1} \circ \Psi_{M,\overline{M}}$ fixes $M \cap \Psi_{N_0,N_1}(\overline{M})$ (all other cases are easily proved using our hypothesis on $\mathcal{N}_0$ and $\mathcal{N}_1$ and the fact that $\mathcal{N}_2$ is a partial symmetric system extending $\mathcal{N}_0$). Let $x \in M \cap \Psi_{N_0,N_1}(M')$ and note that $x \in M \cap R$. By our assumption on $\mathcal{N}_0$ and $\mathcal{N}_1$ there is some $N_1' \in \mathcal{N}_0$ such that $N_1' \cap R = N_1 \cap R$. By condition (D) in Definition 2.2 for $\mathcal{N}_2$ there is some $M' \in \mathcal{N}_2 \cap N_1'$ such that $\delta_{M'} = \delta_M$ and $M \cap N_1' = M \cap M'$. But then $\Psi(x) = (\Psi_{N_1',N_1} \circ \Psi_{M,M'})(x) = x$ since $x \in M \cap M' \cap N_1 \cap N_1'$.

As to condition (C) it suffices to argue, again thanks to the above representation of $(\mathcal{M}_i)_{i \leq m}$, that if $M \in \mathcal{N}_2$, $\overline{M} \in \mathcal{N}_2$ is such that $\delta_{\overline{M}} = \delta_M$, $N_0 \in \mathcal{N}_0$, $N_1 \in \mathcal{N}_1$ are such that $\overline{M} \in N_0$ and $\delta_{N_0} = \delta_{N_1}$, and $Q \in M \cap \mathcal{N}_2$, then $\Psi(Q) \in \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}$ for $\Psi = \Psi_{N_0,N_1} \circ \Psi_{M,\overline{M}}$ (again, all remaining cases can be proved easily using our hypotheses). But $\Psi_{M,\overline{M}}(Q) \in \mathcal{N}_2$ by (C) for $\mathcal{N}_2$, and therefore $\Psi_{N_0,N_1}(\Psi_{M,\overline{M}}(Q)) \in \mathcal{N}$.

Finally we verify condition (D). Suppose M and Q are both in $\mathcal{N}_2$ and $\delta_M < \delta_Q$. Suppose first that there are $N_0 \in \mathcal{N}_0$, $N_1 \in \mathcal{N}_1$ of the same height such that $M \in N_0$ and let us show that there is some $\tilde{M} \in Q \cap \mathcal{N}_2$ of the same height as M such that $\Psi_{N_0,N_1}(M) \cap Q = \tilde{M} \cap \Psi_{N_0,N_1}(M)$. For this, note that by our hypothesis on $\mathcal{N}_0$ and $\mathcal{N}_1$ there is some $N_1' \in \mathcal{N}_0$ such that $N_1' \cap R = N_1 \cap R$. By (D) for $\mathcal{N}_2 \supseteq \mathcal{N}_0$ there is some $\tilde{M} \in \mathcal{N}_2 \cap Q$ of the same height as M and such that $\Psi_{N_0,N_1'}(M) \cap Q = \Psi_{N_0,N_1'}(M) \cap \tilde{M}$. But now it is easy to check that $\Psi_{N_0,N_1}(M) \cap Q = \Psi_{N_0,N_1}(M) \cap \tilde{M}$ using the fact that if $x \in \Psi_{N_0,N_1}(M) \cap Q$, then $x \in R$ (which is true since $(\bigcup \mathcal{N}_2) \cap (\bigcup \mathcal{N}_1) = R$). Finally suppose there are $N_0 \in \mathcal{N}_0$, $N_1 \in \mathcal{N}_1$ of the same height such that $Q \in N_0$ and let us show that there is some $M' \in \Psi_{N_0,N_1}(Q) \cap \mathcal{N}$ such that $M \cap \Psi_{N_0,N_1}(Q) = M \cap M'$: By (D) for $\mathcal{N}_2$ there is some $\tilde{M} \in \mathcal{N}_2 \cap Q$ of the same height as M and such that $M \cap Q = M \cap \tilde{M}$. Then it is easy to see that $M' = \Psi_{N_0,N_1}(\tilde{M}) \in \mathcal{N} \cap \Psi_{N_0,N_1}(Q)$ is as desired, again using the fact that $(\bigcup \mathcal{N}_2) \cap (\bigcup \mathcal{N}_1) = R$. The remaining cases in the verification of (D) are easier as usual. $\square$

2.2. **The forcing construction.** We start our forcing preparations fixing $\vec{X} = \langle X_\alpha : \alpha \in \kappa, \mathrm{cf}(\alpha) \geq \omega_1 \rangle$, a $\Diamond(\{\alpha \in \kappa, \mathrm{cf}(\alpha) \geq \omega_1\})$–sequence. Note that the existence of such a diamond sequence implies $2^{<\kappa} = \kappa$. Let Φ be a bijection between κ and $H(\kappa)$. Let also $\lhd$ be a well–order of $H(\kappa^+)$ in order type 2^κ. Φ exists since $|H(\kappa)| = \kappa$ by $2^{<\kappa} = \kappa$, and $\lhd$ exists since $|H(\kappa^+)| = 2^\kappa$.

We will define an iteration $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$, together with a function $\Upsilon : \kappa \longrightarrow H(\kappa)$ such that each $\Upsilon(\alpha)$ is a $\mathcal{P}_\alpha$–name in $H(\kappa)$.

Let $\langle \theta_\alpha : \alpha \leq \kappa \rangle$ be the strictly increasing sequence of regular cardinals defined as $\theta_0 = |2^\kappa|^+$ and $\theta_\alpha = |2^{\sup\{\theta_\beta : \beta < \alpha\}}|^+$ if $\alpha > 0$. For each $\alpha \leq \kappa$ let $\mathfrak{N}_\alpha^*$ be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing Φ, $\lhd$ and $\langle \theta_\beta : \beta < \alpha \rangle$. Let also $\mathfrak{N}_\alpha = \{N^* \cap H(\kappa) : N^* \in \mathfrak{N}_\alpha^*\}$ and note that if $\alpha < \beta$, then $\mathfrak{N}_\alpha^*$ belongs to all members of $\mathfrak{N}_\beta^*$ containing the ordinal α.

The properness of each $\mathcal{P}_\alpha$ will be witnessed by the club $\mathfrak{N}_\alpha^*$ (see Lemma 2.22). The proof of this will be by induction on α.

Let us proceed to the definition of $\langle \mathcal{P}_\alpha : \alpha \le \kappa \rangle$ now. Conditions in $\mathcal{P}_0$ are pairs of the form $(\emptyset, \Delta)$, where

(A) Δ is a finite set of ordered pairs of the form $(N, 0)$ such that $\mathrm{dom}(\Delta)$ is a partial Φ–symmetric system.

Given $\mathcal{P}_0$–conditions $q_\epsilon = (\emptyset, \Delta_\epsilon)$ for $\epsilon \in \{0, 1\}$, q_1 extends q_0 if and only if

(B) $\mathrm{dom}(\Delta_0) \subseteq \mathrm{dom}(\Delta_1)$

Notation 2.7. If q is an ordered pair (F, Δ) such that F is a function and Δ is a relation, we denote F by F_q and Δ by Δ_q. In this context, we will use $\mathrm{supp}(q)$ to denote the domain of F_q ($\mathrm{supp}(q)$ stands for the *support* of q). Also, if q is as above and ξ is an ordinal, *the restriction of q to ξ*, denoted by $q|_\xi$, is defined as the pair

$$q|_\xi := (F_q \restriction \xi, \{(N, \min\{\beta, \xi\}) : (N, \beta) \in \Delta_q\})$$

Let $\alpha \le \kappa$, $\alpha > 0$, and suppose that we have defined $\mathcal{P}_\xi$ for all $\xi < \alpha$. Suppose also that if $\xi < \alpha$, then $\mathcal{P}_\xi \subseteq H(\kappa)$, and if $q \in \mathcal{P}_\xi$, then q is an ordered pair of the form (F_q, Δ_q), where

- F_q is a finite function with domain included in ξ, and
- Δ_q is a finite relation $\{(N_i, \gamma_i) \mid i \in n\}$ such that $\mathrm{dom}(\Delta_q) = \{N_i \mid i < n\}$ is a partial Φ–symmetric system and, for all i, γ_i is an ordinal such that $\gamma_i \le \xi$.

It will be convenient to consider the following technical variant of the $\aleph_{1.5}$–c.c. in the context of our construction. In this definition, and throughout the paper, if $\mathbb{P} \subseteq H(\kappa)$ is a poset, $N \subseteq H(\kappa)$, and $G \subseteq \mathbb{P}$ is a filter, then by $N[G]$ we mean $\{\tau_G : \tau \in N, \tau$ a $\mathbb{P}$–name$\}$. $\mathbb{P}$ might certainly fail to be a member of N and might not even be definable there.

Definition 2.8. (In $\mathbf{V}[G]$ for a $\mathcal{P}_\xi$–generic filter G, $\xi < \alpha$) A poset $\mathcal{Q} \subseteq H(\kappa)^{\mathbf{V}}$ is $\aleph_{1.5}$–*c.c. relative to* $\mathbf{V}$ *and* G if and only if there is a club D of $[H(\kappa)^{\mathbf{V}}]^{\aleph_0}$, D belonging to $\mathbf{V}$, with the following property:
If

(1) $p \in \mathcal{Q}$,
(2) $\{N_i : i \in n\} \subseteq D$ is finite,
(3) $p \in N_j[G]$ for some j such that $\delta_{N_j} = \min\{\delta_{N_i} : i < n\}$, and
(4) $\{(N_i, \xi) : i < m\} \subseteq \Delta_u$ for some $u \in G_\xi$,

then there is an extension of p which is $(N_i[G], \mathcal{Q})$–generic for all i.

If $\alpha = \alpha_0 + 1$, then let $\Upsilon(\alpha_0)$ be a $\mathcal{P}_{\alpha_0}$–name in $H(\kappa)$ for an $\aleph_{1.5}$–c.c. poset relative to $\mathbf{V}$ and $\dot{G}_{\alpha_0}$ such that $\Upsilon(\alpha_0)$ is (say) the canonical $\mathcal{P}_{\alpha_0}$–name for trivial forcing on $\{0\}$ unless X_{α_0} is defined and codes

(via some fixed reasonable translating function π)[14] a $\mathcal{P}_{\alpha_0}$–name $\dot{X}$. In that case, $\Upsilon(\alpha_0)$ is a $\mathcal{P}_{\alpha_0}$–name in $H(\kappa)$ for an $\aleph_{1.5}$–c.c. poset relative to $\mathbf{V}$ and $\dot{G}_{\alpha_0}$ such that $\Upsilon(\alpha_0)$ is $\dot{X}$ if $\dot{X}$ is such a poset and such that $\Upsilon(\alpha_0)$ is trivial forcing on $\{0\}$ if $\dot{X}$ is not such a poset.

Let $0 < \alpha \leq \kappa$. The definition of $\mathcal{P}_\alpha$ is as follows (regardless of whether α is a successor or a limit ordinal). Conditions in $\mathcal{P}_\alpha$ are pairs of the form $q = (F_q, \Delta_q)$ with the following properties.

(C0) F_q is a finite function whose domain is included in α.

(C1) Δ_q is a finite set of pairs (N, γ) with $\gamma \leq \min\{\alpha, \mathsf{sup}(N \cap \kappa)\}$.

(C2) For every $\xi < \alpha$, the restriction $q|_\xi$ of q to ξ is a condition in $\mathcal{P}_\xi$.

(C3) If $\xi \in \mathrm{dom}(F_q)$, then $F_q(\xi) \in H(\kappa)$ and $q|_\xi \Vdash_{\mathcal{P}_\xi} \check{F}_q(\xi) \in \Upsilon(\xi)$.[15]

(C4) If $\xi \in \mathrm{dom}(F_q)$, $N \in \mathfrak{N}_{\xi+1}$, and $(N, \beta) \in \Delta_q$ for some $\beta \geq \xi + 1$, then $q|_\xi$ forces in $\mathcal{P}_\xi$ that $F_q(\xi)$ is $(N[\dot{G}_\xi], \Upsilon(\xi))$–generic.

Given conditions $q_\epsilon = (F_\epsilon, \Delta_e)$ (for $\epsilon \in \{0, 1\}$) in $\mathcal{P}_\alpha$, q_1 extends q_0 if and only if the following holds:

(D1) $q_1|_\xi \leq_\alpha q_0|_\xi$ for all $\xi < \alpha$.

(D2) $\mathrm{dom}(F_0) \subseteq \mathrm{dom}(F_1)$ and if $\xi \in \mathrm{dom}(F_0)$, then $q_1|_\xi$ forces in $\mathcal{P}_\xi$ that $F_1(\xi)$ $\Upsilon(\xi)$–extends $F_0(\xi)$.

(D3) If $(N, \beta) \in \Delta_0$, then there exists $\widetilde{\beta} \geq \beta$ such that $(N, \widetilde{\beta}) \in \Delta_1$.

We define also the notion of being an $\aleph_{1.5}$–c.c. poset relative to $\mathbf{V}$ and G for any $\mathcal{P}_\kappa$–generic filter G in the natural way: A poset $\mathcal{Q} \subseteq H(\kappa)^{\mathbf{V}}$ is $\aleph_{1.5}$–c.c. *relative to* $\mathbf{V}$ *and* G if and only if there is a club D of $[H(\kappa)^{\mathbf{V}}]^{\aleph_0}$, D belonging to $\mathbf{V}$, with the following property:

If

(1) $p \in \mathcal{Q}$,

(2) $\{N_i : i \in n\} \subseteq D$ is finite,

(3) $p \in N_j[G]$ for some j such that $\delta_{N_j} = \min\{\delta_{N_i} : i < n\}$, and

(4) $\{(N_i, \mathsf{sup}(N_i \cap \kappa)) : i < m\} \subseteq \Delta_u$ for some $u \in G_\xi$,

then there is an extension of p which is $(N_i[G], \mathcal{Q})$–generic for all i.

2.3. The Main Facts.

We are going to prove the relevant properties of the forcings $\mathcal{P}_\alpha$. Theorem 2.1 will follow immediately from them.

Our first lemma is immediate from the definitions.

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[14]π can be taken to be for example the following surjection $\pi : \mathcal{P}(\kappa) \longrightarrow H(\kappa^+)$: if $a \in H(\kappa^+)$, then $\pi(X) = a$ if and only if $X \subseteq \kappa$ codes a structure (κ', E) isomorphic to $(\mathsf{TC}(\{a\}), \in)$ (for some unique cardinal $\kappa' \leq \kappa$). We will denote this isomorphism by π_X.

[15]As a convenient simplification, we will tend to omit inverted circumflexes (i.e., we will tend to write x rather than $\check{x}$) in cases where we clearly deal with canonical names.

Lemma 2.9. $\mathcal{P}_\kappa = \bigcup_{\beta<\kappa} \mathcal{P}_\beta$, and $\emptyset \neq \mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ for all $\alpha \leq \beta \leq \kappa$.

An inductive verification shows that if q is a condition in $\mathcal{P}_\alpha$ ($\alpha \leq \kappa$), $(N, \varrho) \in \Delta_q$, and ρ is an ordinal below ϱ, then $(F_q, \Delta_q \cup \{(N, \rho)\})$ is also a condition in $\mathcal{P}_\alpha$. In fact, the original condition q is clearly equivalent to this condition. Also note that if r is a condition in $\mathcal{P}_\beta$ and $\alpha \leq \beta \leq \kappa$, then the relations Δ_r and $\Delta_{r|_\alpha}$ have the same domain. In particular, if $q \in \mathcal{P}_\alpha$, $\{q^i : i \in I\}$ is a finite subset of $\mathcal{P}_\beta$, and $q \leq_\alpha q^i|_\alpha$ for all $i \in I$, then $\mathrm{dom}(\Delta_{q^i}) = \mathrm{dom}(\Delta_{q^i|_\alpha}) \subseteq \mathrm{dom}(\Delta_q) = \mathrm{dom}(\Delta_{q|_0})$ (the inclusion follows from clauses (B) and $(D3)$ in the definition of our forcing). To sum up, $\Delta_q \cup \bigcup\{\Delta_{q^i|_\alpha} : i \in I\}$ and Δ_q have he same domain, and this domain is of course a partial Φ–symmetric system. Again by $(D3)$, if $q \leq_\alpha q^i|_\alpha$ and (N, ρ) is in $\Delta_{q^i|_\alpha}$, then there is $\varrho \geq \rho$ such that (N, ϱ) is in Δ_q. So, after repeating a finite number of times the first observation of this paragraph we have the following result:

Lemma 2.10. *Let α and β be ordinals such that $\alpha \leq \beta \leq \kappa$. If q is in P_α, $\{q^i : i \in I\}$ is a finite subset of $\mathcal{P}_\beta$ and $q \leq_\alpha q^i|_\alpha$ for all $i \in I$, then q and $(F_q, \Delta_q \cup \bigcup\{\Delta_{q^i|_\alpha} : i \in I\})$ are two equivalent conditions in the forcing $\mathcal{P}_\alpha$.*

Lemma 2.11 shows that $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is a forcing iteration in a broad sense.

Lemma 2.11. *If $q = (F_q, \Delta_q) \in \mathcal{P}_\alpha$, $r = (F_r, \Delta_r) \in \mathcal{P}_\beta$, and $q \leq_\alpha r|_\alpha$, then*

$$r \wedge_\alpha q := (F_q \cup (F_r \restriction [\alpha, \beta)), \Delta_q \cup \Delta_r)$$

is a condition in $\mathcal{P}_\beta$ extending r. In particular, every maximal antichain in $\mathcal{P}_\alpha$ is a maximal antichain in $\mathcal{P}_\beta$, and therefore $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$.

Proof. The proof proceeds by induction on $\beta \geq \alpha$. The case $\beta = \alpha$ follows from Lemma 2.10, since it implies that $r \wedge_\alpha q$ is equivalent to q. Let us assume now that $\beta = \sigma + 1$ with $\sigma \geq \alpha$. Clearly, $r \wedge_\alpha q$ satisfies clauses $(C0)$ and $(C1)$ in the definition of $\mathcal{P}_{\sigma+1}$. By the induction hypothesis we know that the restriction of $r \wedge_\alpha q$ to σ, that is,

$$(r \wedge_\alpha q)|_\sigma = (F_q \cup (F_r \restriction [\alpha, \sigma)), \Delta_q \cup \Delta_{r|_\sigma})$$

is a condition in $\mathcal{P}_\sigma$ extending $r|_\sigma$. Therefore, $r \wedge_\alpha q$ also satisfies $(C2)$. If $\sigma \notin dom(F_r)$, then $r \wedge_\alpha q$ is a condition in $\mathcal{P}_{\sigma+1}$, since clause $(C4)$ is automatically satisfied. If $\sigma \in dom(F_r)$, then $(r \wedge_\alpha q)|_\sigma$ forces in $\mathcal{P}_\sigma$ that $F_r(\sigma)$ is in $\Upsilon(\sigma)$ (since $r|_\sigma$ forces this and $(r \wedge_\alpha q)|_\sigma$ extends $r|_\sigma$). This concludes the verification of $(C3)$ for $q \wedge_\alpha r$. Now we check that $(r \wedge_\alpha q)|_\sigma$ forces that $F_r(\sigma)$ is $(N[\dot{G}_\sigma], \Upsilon(\sigma))$–generic for all those

N in $\mathfrak{N}_{\sigma+1}$ such that $(N, \sigma + 1) \in \Delta_q \cup \Delta_r$. But such an N is such that $(N, \sigma + 1) \in \Delta_r$ (by definition of $\mathcal{P}_\alpha$, $\mathrm{range}(\Delta_q) \subseteq \alpha + 1$). Since r satisfies $(C4)$, $r|_\sigma$ (and hence, the restriction of $r \wedge_\alpha q$ to σ) forces that $F_r(\sigma)$ is $(N[\dot{G}_\sigma], \Upsilon(\sigma))$–generic. Finally note that the induction hypothesis and the inclusion $\Delta_r \subseteq \Delta_{r \wedge_\alpha q}$ together imply that $r \wedge_\alpha q$ extends r. The case when β is a nonzero limit ordinal follows directly from the induction hypothesis.

<div align="right">□</div>

The following four amalgamation results are, essentially, Lemmas 3.5–3.8 from [1]. The proofs of those lemmas are easy mechanical verifications, not particularly illuminating, and translate almost word by word to proofs of the present lemmas, so the familiarized reader may skip the present proofs.

Lemma 2.12. *Let $\alpha < \kappa$ and let $q_0 = (F_0, \Delta_0)$ and $q_1 = (F_1, \Delta_1)$ be conditions in $\mathcal{P}_{\alpha+1}$ such that there is $v \in H(\kappa)$, a condition $r = (F_r, \Delta_r)$ in $\mathcal{P}_\alpha$, and a finite set $\{M_j : j \in n\}$ with the following properties:*

(a) *$\alpha + 1 \leq \mathsf{sup}(M_j \cap \kappa)$ and $(M_j, \alpha) \in \Delta_r$ for all $j < n$,*
(b) *r extends both $q_0|_\alpha$ and $q_1|_\alpha$,*
(c) *$\alpha \in \mathrm{dom}(F_0) \cap \mathrm{dom}(F_1)$ and r forces in $\mathcal{P}_\alpha$ that v extends both $F_0(\alpha)$ and $F_1(\alpha)$ in $\Upsilon(\alpha)$, and*
(d) *r forces in $\mathcal{P}_\alpha$ that v is $(M_j[G_\alpha], \Upsilon(\alpha))$–generic for all $j < n$ such that $M_j \in \mathfrak{N}_{\alpha+1}$.*

Then,

$$q_2 = (F_r \cup \{\langle \alpha, v \rangle\}, \Delta_r \cup \Delta_0 \cup \Delta_1 \cup \{(M_j, \alpha + 1) : j \in n\})$$

is a condition in $\mathcal{P}_{\alpha+1}$ extending both q_0 and q_1.

Proof. First we check that q_2 is in $\mathcal{P}_{\alpha+1}$. It follows from (a) and (b) together with Lemma 2.10 that the restriction of q_2 to α is equivalent to r, and hence that q_2 satisfies clauses $(C0)$–$(C2)$. Condition $(C3)$ for q_2 follows from (c). Finally we must show that r forces that v is $(N[\dot{G}_\alpha], \Upsilon(\alpha))$–generic for all those N in $\mathfrak{N}_{\alpha+1}$ such that $(N, \alpha+1)$ is in $\Delta_0 \cup \Delta_1 \cup \{(M_j, \alpha + 1) : j \in n\}$ (recall that if $(N, \gamma) \in \Delta_r$, then $\gamma \leq \alpha$). But for such an N, if $(N, \alpha+1) \in \Delta_i$ ($i \in \{0, 1\}$), it suffices to recall that $r \leq_\alpha q_i|_\alpha$, that r forces that v extends both $F_0(\alpha)$ and $F_1(\alpha)$, and that (by clause $(C4)$ applied to q_i) $q_i|_\alpha$ forces that $F_i(\alpha)$ is $(N[\dot{G}_\alpha], \Upsilon(\alpha))$–generic. Hence, r forces that v is $(N[\dot{G}_\alpha], \Upsilon(\alpha))$–generic. The case when $(N, \alpha + 1) \in \{(M_j, \alpha + 1) : j \in n\}$ follows from (d). Finally note that (b), (c) and the inclusion $\Delta_i \subseteq \Delta_0 \cup \Delta_1 \cup \{(M_j, \alpha + 1) : j \in n\}$ imply together that q_2 extends q_i for $i \in \{0, 1\}$. □

Exactly the same proof establishes the following variant of Lemma 2.12.

Lemma 2.13. *Let $\alpha < \kappa$ and let $q_0 = (F_0, \Delta_0)$ and $q_1 = (F_1, \Delta_1)$ be conditions in $\mathcal{P}_{\alpha+1}$, $r = (F_r, \Delta_r)$ a condition in $\mathcal{P}_\alpha$, and $\{M_j : j \in n\}$ a finite set with the following properties:*

(a) *$\alpha + 1 \leq \mathsf{sup}(M_j \cap \kappa)$ and $(M_j, \alpha) \in \Delta_r$ for all $j < n$,*
(b) *r extends both $q_0|_\alpha$ and $q_1|_\alpha$, and*
(c) *$\alpha \notin \mathrm{dom}(F_0) \cup \mathrm{dom}(F_1)$.*

Then,
$$q_2 = (F_r,\ \Delta_r \cup \Delta_0 \cup \Delta_1 \cup \{(M_j, \alpha + 1) : j \in n\})$$
is a condition in $\mathcal{P}_{\alpha+1}$ extending both q_0 and q_1.

Suppose, in addition, that $v \in H(\kappa)$ is such that

(d) *r forces in $\mathcal{P}_\alpha$ that v is $(M_j[G_\alpha], \Upsilon(\alpha))$–generic for all $j < n$ such that $M_j \in \mathfrak{N}_{\alpha+1}$, and*
(e) *r forces in $\mathcal{P}_\alpha$ that v is $(N[G_\alpha], \Upsilon(\alpha))$–generic for all N such that $(N, \alpha + 1) \in \Delta_0 \cup \Delta_1$ and $N \in \mathfrak{N}_{\alpha+1}$.*

Then,
$$q_2' = (F_r \cup \{\langle \alpha, v \rangle\},\ \Delta_r \cup \Delta_0 \cup \Delta_1 \cup \{(M_j, \alpha + 1) : j \in n\})$$
is a condition in $\mathcal{P}_{\alpha+1}$ extending both q_0 and q_1.

Lemma 2.14. *Assume that $0 \leq \sigma < \alpha \leq \kappa$. Let $q_\xi = (F_\xi, \Delta_\xi)$ ($\xi \in \{0, 1\}$) be conditions in $\mathcal{P}_\alpha$ such that $\mathrm{supp}(q_0) \cup \mathrm{supp}(q_1) \subseteq \sigma$ and such that there exists a condition $r = (F_r, \Delta_r) \in \mathcal{P}_\sigma$ extending both $q_0|_\sigma$ and $q_1|_\sigma$. Then q_0 and q_1 are compatible in $\mathcal{P}_\alpha$.*

Proof. Define $q_2 = (F_r, \Delta_r \cup \Delta_0 \cup \Delta_1)$. We prove by induction on β, $\sigma \leq \beta \leq \alpha$, that $q_2|_\beta$ is a condition in $\mathcal{P}_\beta$ extending $q_0|_\beta$ and $q_1|_\beta$. The successor step follows from Lemma 2.13. $\square$

Lemma 2.15. *Given $\beta \leq \kappa$ and given conditions $q_\xi = (F_\xi, \Delta_\xi)$ (for $\xi \in \{0, 1\}$) in $\mathcal{P}_\beta$, let $Z_\xi = \mathrm{supp}(q_\xi) \cup (\beta \cap \bigcup \mathrm{dom}(\Delta_{q_\xi}))$. Let $\alpha \leq \beta$ be such that $Z_0 \cap Z_1 \subseteq \alpha$, and assume there is a condition $r = (F_r, \Delta_r)$ in $\mathcal{P}_\alpha$ extending $q_0|_\alpha$ and $q_1|_\alpha$. Let $F_r^{0,1} = F_r \cup (F_0 \restriction [\alpha, \beta)) \cup (F_1 \restriction [\alpha, \beta))$.*
Then the natural amalgamation of r, q_0 and q_1, i.e.,
$$(q_0 \wedge q_1) \wedge_\alpha r := (F_r^{0,1}, \Delta_r \cup \Delta_0 \cup \Delta_1)$$
is a $\mathcal{P}_\beta$–condition extending q_0 and q_1.

Proof. The proof is by induction on $\beta \geq \alpha$. The case $\beta = \alpha$ follows from Lemma 2.10, since this lemma implies that $(q_0 \wedge q_1) \wedge_\alpha r$ is equivalent to r. We consider now the case when $\beta = \sigma + 1$ with $\sigma \geq \alpha$. Clearly, $(q_0 \wedge q_1) \wedge_\alpha r$ satisfies clauses $(C0)$ and $(C1)$. Using the induction

hypothesis we know that the restriction of the amalgamation to σ is a condition in $\mathcal{P}_\sigma$ which extends both $q_0|_\sigma$ and $q_1|_\sigma$. In particular, if $\sigma \in dom(F_r^{0,1})$, then $((q_0 \wedge q_1) \wedge_\alpha r)|_\sigma$ forces that $F_r^{0,1}(\sigma)$ is in $\Upsilon(\sigma)$. Therefore, $(q_0 \wedge q_1) \wedge_\alpha r$ also satisfies $(C2)$ and $(C3)$. Let us assume that $\sigma \in dom(F_r^{0,1})$. In fact, since $supp(q_0) \cap supp(q_1) \subseteq \alpha$ and $\sigma \geq \alpha$, we may assume that $\sigma \in supp(q_0) \setminus supp(q_1)$ (the proof when $\sigma \in supp(q_1)$ is identical). We must show that the condition $((q_0 \wedge q_1) \wedge_\alpha r)|_\sigma$ forces that $F_r^{0,1}(\sigma)$ is $(N[\dot{G}_\sigma], \Upsilon(\sigma))$–generic for all those N in $\mathfrak{N}_{\sigma+1}$ such that $(N, \sigma+1) \in \Delta_0 \cup \Delta_1$. Since $\sigma \geq \alpha$ and $Z_0 \cap Z_1 \subseteq \alpha$, it follows for such an N that $(N, \sigma+1) \in \Delta_0$. Since q_0 satisfies $(C4)$ and $F_r^{0,1}(\sigma) = F_0(\sigma)$, $q_0|_\sigma$ (and hence, $((q_0 \wedge q_1) \wedge_\alpha r)|_\sigma$) forces what we want. Finally note that the induction hypothesis, the choice of $F_r^{0,1}$ and the inclusion $\Delta_0 \cup \Delta_1 \subseteq \Delta_{(q_0 \wedge q_1) \wedge_\alpha r}$ together imply that $(q_0 \wedge q_1) \wedge_\alpha r$ extends q_0 and q_1. The case when β is a nonzero limit ordinal follows directly from the induction hypothesis. $\square$

The following lemma gives a representation of $\mathcal{P}_{\alpha+1}$ as a certain dense subset of an iteration of the form $\mathcal{P}_\alpha * \dot{\mathcal{Q}}_\alpha$.

Lemma 2.16. *For all $\alpha < \kappa$, $\mathcal{P}_{\alpha+1}$ is isomorphic to a dense suborder of $\mathcal{P}_\alpha * \dot{\mathcal{Q}}_\alpha$, where $\dot{\mathcal{Q}}_\alpha$ is, in $\mathbf{V}^{\mathcal{P}_\alpha}$, the collection of all pairs (W, Q) such that W has cardinality at most 1 and*

(∘) *if $v \in W$, then there is some $r = (F_r, \Delta_r) \in \dot{G}_\alpha$ such that*

$$(F_r \cup \{\langle \alpha, v \rangle\}, \Delta_r \cup \{(N, \alpha+1) : N \in Q\}) \in \mathcal{P}_{\alpha+1},$$

(∘) *if $W = \emptyset$, then there is some $r = (F_r, \Delta_r) \in \dot{G}_\alpha$ such that*

$$(F_r, \Delta_r \cup \{(N, \alpha+1) : N \in Q\}) \in \mathcal{P}_{\alpha+1},$$

ordered by $(W_1, Q_1) \leq_{\dot{\mathcal{Q}}_\alpha} (W_0, Q_0)$ if and only if

(i) $Q_0 \subseteq Q_1$, *and*

(ii) *if $W_0 = \{v\}$, then $W_1 = \{v'\}$ for some $v' \in \Upsilon(\alpha)$ which extends v in $\Upsilon(\alpha)$.*

Proof. Let $\widetilde{\mathcal{P}}_{\alpha+1}$ consist of all $(r, \check{x})$, where $r \in \mathcal{P}_\alpha$ and $r \Vdash_{\mathcal{P}_\alpha} \check{x} \in \dot{\mathcal{Q}}_\alpha$. Then $\psi : \mathcal{P}_{\alpha+1} \longrightarrow \widetilde{\mathcal{P}}_{\alpha+1}$ is an isomorphism, where $\psi(q) = (q|_\alpha, \check{x})$, $x = (\{F_q(\alpha)\}, \Delta_q^{-1}(\alpha+1))$ if $\alpha \in supp(q)$, and $x = (\emptyset, \Delta_q^{-1}(\alpha+1))$ if $\alpha \notin supp(q)$. $\square$

Lemma 2.17. *If α and $\dot{\mathcal{Q}}_\alpha$ are as in the statement of Lemma 2.16, then $\dot{\mathcal{Q}}_\alpha$ is forced by $\mathcal{P}_\alpha$ to have the $\aleph_2$–chain condition.*

Proof. Let G be $\mathcal{P}_\alpha$–generic filter over $\mathbf{V}$ and let $\mathcal{Q}_\alpha$ be the interpretation of $\dot{\mathcal{Q}}_\alpha$ by G. Working in $\mathbf{V}[G]$ suppose $\{(W_i, Q_i) : i < \omega_2\} \subseteq \mathcal{Q}_\alpha$ and $W_i = \{v_i\}$ is nonempty for each i. Since $\Upsilon(\alpha)$ is forced by $\mathcal{P}_\alpha$ to

have the $\aleph_2$–c.c., there are $v \in \Upsilon(\alpha)$, i_0, $i_1 < \omega_2$, and $r \in G$ such that r forces that v extends both v_{i_0} and v_{i_1}. The proof will be finished in this case once we show $(F_r \cup \{\langle \alpha, v \rangle\}, \Delta_r \cup \{(N, \alpha + 1) \: : \: N \in Q_{i_0} \cup Q_{i_1}\})$ is a condition in $\mathcal{P}_{\alpha+1}$. Note that, by definition of $\mathcal{Q}_\alpha$, there are r_0 and r_1 in G such that $(F_{r_0} \cup \{\langle \alpha, v_{i_0} \rangle\}, \Delta_{r_0} \cup \{(N, \alpha + 1) \: : \: N \in Q_{i_0}\})$ and $(F_{r_1} \cup \{\langle \alpha, v_{i_1} \rangle\}, \Delta_{r_1} \cup \{(N, \alpha + 1) \: : \: N \in Q_{i_1}\})$ are conditions in $\mathcal{P}_{\alpha+1}$. Since G is a filter and by extending r if necessary, we can assume that r extends both r_0 and r_1. Now we are done by a simple application of Lemma 2.12.

The proof for the case when $\{(W_i, Q_i) \: : \: i < \omega_2\} \subseteq \mathcal{Q}_\alpha$ is such that $W_i = \emptyset$ for cofinally many i in ω_2 is very similar (the only difference is that we use the first part of Lemma 2.13 instead of Lemma 2.12).   □

The next step in the proof of Theorem 2.1 will be to show that all $\mathcal{P}_\alpha$ (for $\alpha \leq \kappa$) have the $\aleph_2$–chain condition. We first prove that $\mathcal{P}_0$ is $\aleph_2$–Knaster.[16]

Lemma 2.18. $\mathcal{P}_0$ *is* $\aleph_2$*–Knaster.*

Proof. Suppose $m < \omega$ and $q_\xi = \{N_i^\xi \: : \: i < m\}$ is a $\mathcal{P}_0$–condition for each $\xi < \omega_2$. For notational convenience we are identifying a $\mathcal{P}_0$–condition q with $\mathrm{dom}(\Delta_q)$, which is fine for this proof. By CH we may assume that $\{\bigcup_{i<m} N_i^\xi \: : \: \xi < \omega_2\}$ forms a Δ–system with root X. Furthermore, by CH we may assume, for all ξ, $\xi' < \omega_2$, that the structures $\langle \bigcup_{i<m} N_i^\xi, \in, \Phi, X, N_i^\xi \rangle_{i<m}$ and $\langle \bigcup_{i<m} N_i^{\xi'}, \in, \Phi, X, N_i^{\xi'} \rangle_{i<m}$ are isomorphic and that the corresponding isomorphism fixes X. The first assertion follows from the fact that there are only $\aleph_1$–many isomorphism types for such structures. For the second assertion note that, if Ψ is the unique isomorphism between $\langle \bigcup_{i<m} N_i^\xi, \in, \Phi, X, N_i^\xi \rangle_{i<m}$ and $\langle \bigcup_{i<m} N_i^{\xi'}, \in, \Phi, X, N_i^{\xi'} \rangle_{i<m}$, then the restriction of Ψ to $X \cap \kappa$ has to be the identity on $X \cap \kappa$. Since there is a bijection $\Phi \: : \: H(\kappa) \longrightarrow \kappa$ definable in $(H(\kappa), \in, \Phi)$, we have that Ψ fixes X if and only if it fixes $X \cap \kappa$. It follows that Ψ fixes X. Hence, by Lemma 2.5 we have, for all ξ, $\xi' < \omega_2$, that $q_\xi \cup q_{\xi'}$ extends both q_ξ and $q_{\xi'}$.   □

Lemma 2.19. *For every* $\alpha \leq \kappa$*,* $\mathcal{P}_\alpha$ *has the* $\aleph_2$*–chain condition.*

Proof. The proof is by induction on α, being the conclusion for $\alpha = 0$ a consequence of the above lemma.

For $\alpha = \sigma + 1$ the conclusion follows immediately from Lemmas 2.16 and 2.17 together with the induction hypothesis for σ and the fact that the $\aleph_2$–c.c. is preserved under two–step iterations.

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[16] Given a cardinal μ, a partial order $\mathbb{P}$ is μ–Knaster if for every $X \in [\mathbb{P}]^\mu$ there is $Y \in [X]^\mu$ consisting of pairwise compatible conditions.

For α a nonzero limit ordinal, suppose q_ξ is a $\mathcal{P}_\alpha$–condition for all $\xi < \omega_2$. Suppose first that $\mathrm{cf}(\alpha) \neq \omega_2$. There is then some $\sigma < \alpha$ such that $I = \{\xi < \omega_2 : \mathrm{supp}(q_\xi) \subseteq \sigma\}$ has size $\aleph_2$. By induction hypothesis we may find distinct ξ and ξ' in I such that $q_\xi|_\sigma$ and $q_{\xi'}|_\sigma$ are compatible in $\mathcal{P}_\sigma$. But now it follows from Lemma 2.14 that q_ξ and $q_{\xi'}$ are compatible.

Finally, suppose $\mathrm{cf}(\alpha) = \omega_2$. For each $\xi < \omega_2$, let Z_ξ be the union of the sets $\mathrm{supp}(q_\xi)$ and $\alpha \cap \bigcup \mathrm{dom}(\Delta_{q_\xi})$. By CH we may find $I \subseteq \omega_2$ of size $\aleph_2$ such that $\{Z_\xi : \xi \in I\}$ forms a Δ–system with root X.

Let now $\sigma < \alpha$ be such that $X \subseteq \sigma$ (σ exists by $\mathrm{cf}(\alpha) \geq \omega_1$). By induction hypothesis we may find distinct ξ and ξ' in I such that $q_\xi|_\sigma$ and $q_{\xi'}|_\sigma$ are compatible in $\mathcal{P}_\sigma$. Let $r_{\xi,\xi'}$ be a condition in $\mathcal{P}_\sigma$ extending $q_\xi|_\sigma$ and $q_{\xi'}|_\sigma$. By Lemma 2.15 it follows that the natural amalgamation of $r_{\xi,\xi'}$, q_ξ and $q_{\xi'}$ is a $\mathcal{P}_\alpha$–condition extending q_ξ and $q_{\xi'}$.

$\square$

Corollary 2.20. *For every $\alpha \leq \kappa$ and every cardinal $\kappa' \geq \kappa$, $\mathcal{P}_\alpha$ forces $H(\kappa')^{\mathbf{V}[\dot{G}_\alpha]} = H(\kappa')^{\mathbf{V}}[\dot{G}_\alpha]$ and forces $(N^* \cap H(\kappa'))[\dot{G}_\alpha] = N^*[\dot{G}_\alpha] \cap H(\kappa')^{\mathbf{V}[\dot{G}_\alpha]}$ whenever θ is regular and N^* is a countable elementary substructure of $H(\theta)$ such that $\mathcal{P}_\alpha \in N^*$ and $\kappa' \in N^*$.*

Definition 2.21. Given $\alpha \leq \kappa$, a condition $q \in \mathcal{P}_\alpha$, and a countable elementary substructure $N \prec H(\kappa)$, we will say that q is $(N, \mathcal{P}_\alpha)$–pre-generic in case

- (◦) $\alpha < \kappa$ and the pair (N, α) is in Δ_q, or else
- (◦) $\alpha = \kappa$ and the pair $(N, \mathsf{sup}(N \cap \kappa))$ is in Δ_q.

The properness of all $\mathcal{P}_\alpha$ is an immediate consequence of Lemma 2.22. Much of the machinery in our definition of $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is there precisely to make the proof of the lemma work. This lemma is proved by reciting very much the same mantra as in the proof of a corresponding lemma in [1]. We will repeat this mantra for the reader's benefit, of course adapted to the present situation.

Lemma 2.22. *Suppose $\alpha \leq \kappa$ and $N^* \in \mathfrak{N}_\alpha^*$. Let $N = N^* \cap H(\kappa)$. Then the following conditions hold.*

- $(1)_\alpha$ *For every $q \in N$ there is $q' \leq_\alpha q$ such that q' is $(N, \mathcal{P}_\alpha)$–pre-generic.*
- $(2)_\alpha$ *If $\mathcal{P}_\alpha \in N^*$ and $q \in \mathcal{P}_\alpha$ is $(N, \mathcal{P}_\alpha)$–pre-generic, then q is $(N^*, \mathcal{P}_\alpha)$–generic.*

Proof. The proof will be by induction on α. We start with the case $\alpha = 0$. For simplicity we are going to identify a $\mathcal{P}_0$–condition $q =$

$(\emptyset, \Delta_q)$ with $\mathrm{dom}(\Delta_q)$. The proof of $(1)_0$ is trivial: It suffices to set $q' = q \cup \{N\}$.

The proof of $(2)_0$ is also easy: Let E be a dense subset of $\mathcal{P}_0$ in N^*. It suffices to show that there is some condition in $E \cap N^*$ compatible with q. Notice that $q \cap N^* \in \mathcal{P}_0$ by Lemma 2.4 (i). Hence, we may find a condition $q^\circ \in E \cap N^*$ extending $q \cap N^*$. Now, by Lemma 2.4 there is a partial Φ–symmetric system $\mathcal{M}$ extending $q \cup q^\circ$. It follows that $\mathcal{M}$ is a condition in $\mathcal{P}_0$ extending q and q°.

Let us proceed to the case $\alpha = \sigma + 1$. We start by proving $(1)_\alpha$. Assume first that $\sigma \in \mathrm{supp}(q)$. By $(1)_\sigma$ we may assume, by extending $q|_\sigma$, that $q|_\sigma$ is $(N, \mathcal{P}_\sigma)$–pre-generic. Let $\dot{D}$ be the $\lhd$–first $\mathcal{P}_\sigma$–name for a club D of $[H(\kappa)]^{\aleph_0}$ in $\mathbf{V}$ such that D witnesses that $\Upsilon(\sigma)$ is $\aleph_{1.5}$–c.c. relative to $\mathbf{V}$ and $\dot{G}_\sigma$. Note that $q|_\sigma$ forces $N \in \dot{D}$ since $\dot{D} \in N^*$ and since $q|_\sigma$ forces $H(\kappa)^{N^*[\dot{G}_\sigma]} = N$ by $(2)_\sigma$. But then there is some $t \in \mathcal{P}_\sigma$ extending $q|_\sigma$ and some $y \in H(\kappa)$ such that t forces that y is an $(N[\dot{G}_\sigma], \Upsilon(\sigma))$–generic condition extending $F_q(\sigma)$. It suffices to define q' as the condition $(F_t \cup \{\langle \sigma, y \rangle\}, \Delta_q \cup \Delta_t \cup \{(N, \alpha)\})$ (Lemma 2.12 ensures that q' is indeed a condition extending q).

The proof in the case that $q = (F, \Delta)$ with $\mathrm{dom}(F) \subseteq \sigma$ can be reduced to the previous case by the following Claim.

Claim 2.23. *If $q = (F, \Delta)$ and $\sigma \notin \mathrm{dom}(F)$, then we can find a condition $q' = (F', \Delta')$ extending q and such that $\sigma \in \mathrm{dom}(F')$.*

Proof. This is true, using Lemma 2.13, by essentially the same argument as above since $(2)_\sigma$ guarantees that $q|_\sigma$ is also $(M^*, \mathcal{P}_\sigma)$–generic for all $M^* \in \mathfrak{N}_{\sigma+1}^*$ such that $(M, \sigma + 1) \in \Delta_q$ for $M = M^* \cap H(\kappa)$, which implies that a condition forcing that all these M are in $\dot{D}$ can be found as in that argument. Also note that, by its being definable, the weakest condition of $\Upsilon(\sigma)$ is forced to be in $H(\kappa)^{M^*[\dot{G}_\sigma]} = M[\dot{G}_\sigma]$ for all these M's, where the equality holds by Corollary 2.20. $\square$

Now let us prove $(2)_\alpha$. Let A be a maximal antichain of $\mathcal{P}_\alpha$ in N^*, and assume without loss of generality that $q = (F_q, \Delta_q)$ extends some condition q^* in A. We must show $q^* \in N$. Note that A is in N by the $\aleph_2$–c.c. of $\mathcal{P}_\alpha$. By Claim 2.23 we may also assume that $\sigma \in \mathrm{dom}(F_q)$. Let G_σ be a $\mathcal{P}_\sigma$-generic filter over $\mathbf{V}$ with $q|_\sigma \in G_\sigma$. By $(2)_\sigma$ we have that G_σ is also generic over N^*.

Let E be the set of $\Upsilon(\sigma)$–conditions v such that either

  (i) there exists some $a \in \mathcal{P}_\alpha$ extending some member of A such that $a|_\sigma \in G_\sigma$, $\sigma \in \mathrm{dom}(F_a)$, and such that $F_a(\sigma) = v$, or else
  (ii) there is no $a \in \mathcal{P}_\alpha$ extending any member of A such that $a|_\sigma \in \dot{G}_\sigma$, $\sigma \in \mathrm{dom}(F_a)$, and such that $F_a(\sigma) \leq_{\Phi(\sigma)} v$.

E is a dense subset of $\Upsilon(\sigma)$, and $E \in N^*[G_\sigma]$ by E being definable in $H(\kappa^+)^{\mathbf{V}}[\dot{G}_\sigma]$ from $\mathcal{P}_\alpha$ together with the fact that $H(\kappa^+)^{N^*[G_\sigma]} \preccurlyeq H(\kappa^+)^{\mathbf{V}}[\dot{G}_\sigma]$ and N^* contains $\mathcal{P}_\alpha$ and A. Note that E is in fact in $N[\dot{G}_\sigma]$ by Corollary 2.20. Suppose $F_q(\sigma) = \overline{v}$. Since $\overline{v}$ is $(N[G_\sigma], \Upsilon(\sigma))$–generic, we may find some $v' \in E \cap N[G_\sigma]$ and some $v^* \ \Upsilon(\sigma)$–extending both v' and $\overline{v}$.

Claim 2.24. *Condition (i) above holds for v'.*

Proof. Let r be a condition in G_σ extending $q|_\sigma$ and deciding v^*, and let $u = (F_r \cup \{\langle \sigma, v^* \rangle\}, \Delta_r \cup \Delta_q)$. By Lemma 2.12, u is a $\mathcal{P}_\alpha$–condition extending q. In particular, u extends a condition in A, and therefore it witnesses the negation of condition (ii) for v', so condition (i) must hold for v'.

$\square$

By the above claim and by $H(\kappa^+)^{N^*[G_\sigma]} \preccurlyeq H(\kappa^+)^{\mathbf{V}}[G_\sigma]$ there is a in $N^*[G_\sigma]$ witnessing that condition (i) holds for v', and actually $a \in N$ since $N^*[G_\sigma] \cap \mathbf{V} = N^*$ by $(2)_\sigma$. Now we extend $q|_\sigma$ to a condition r deciding a, and deciding also some common extension $v^* \in \Upsilon(\sigma)$ of $\overline{v}$ and v'. We may also assume that r extends $a|_\sigma$. It follows that $(F_r \cup \{\langle \sigma, v^* \rangle\}, \Delta_r \cup \Delta_a \cup \Delta_q)$ is a common extension of q and a by Lemma 2.12. Hence $q^* = a$.

It remains to prove the lemma for the case when α is a nonzero limit ordinal. The proof of $(1)_\alpha$ is easy. Let $\sigma \in N \cap \alpha$ be above supp(q). By induction hypothesis we may find $r \in \mathcal{P}_\sigma$ extending $q|_\sigma$ and such that $(N, \sigma) \in \Delta_r$. One can now easily check that the result of stretching the marker σ in (N, σ) up to α if $\alpha < \kappa$ and up to $\mathsf{sup}(N \cap \kappa)$ if $\alpha = \kappa$ is a condition in $\mathcal{P}_\alpha$ extending q with the desired property.

For $(2)_\alpha$, let A be a maximal antichain of $\mathcal{P}_\alpha$ in N^*, and assume without loss of generality that $q = (F_q, \Delta_q)$ extends some condition q^* in A. We must show $q^* \in N$. Suppose first that $\mathrm{cf}(\alpha) = \omega$. In this case we may take $\sigma \in N^* \cap \alpha$ above supp(q). Let G_σ be $\mathcal{P}_\sigma$–generic with $q|_\sigma \in G_\sigma$. In $N^*[G_\sigma]$ it is true that there is a condition $q^\circ \in \mathcal{P}_\alpha$ such that

(a) $q^\circ \in A$ and $q^\circ|_\sigma \in G_\sigma$, and
(b) supp$(q^\circ) \subseteq \sigma$.

(the existence of such a q° is witnessed in $\mathbf{V}[G_\sigma]$ by q^*.)

Since $q|_\sigma$ is $(N^*, \mathcal{P}_\sigma)$–generic by induction hypothesis, $q^\circ \in N^*$. By extending q below σ if necessary, we may assume that $q|_\sigma$ decides q° and extends $q^\circ|_\sigma$. But now, if $q = (F_q, \Delta_q)$, the natural amalgamation $(F_q, \Delta_q \cup \Delta_{q^\circ})$ of q and q° is a $\mathcal{P}_\alpha$–condition extending them by Lemma 2.14. It follows that $q^* = q^\circ$.

Finally, suppose $\mathrm{cf}(\alpha) \geq \omega_1$. This will be the only place where we use the (partial) symmetry of $\mathrm{dom}(\Delta_q)$. The crucial observation in this case is that if $N' \in \mathrm{dom}(\Delta_q)$ is such that $\delta_{N'} < \delta_N$ and $N'' \in N \cap \mathrm{dom}(\Delta_q)$ is such that $\delta_{N''} = \delta_{N'}$ and $N'' \cap N' = N' \cap N$, then $\mathsf{sup}(N' \cap N \cap \alpha) \leq \mathsf{sup}(N'' \cap \alpha)$. Hence, since such an N'' is countable, we may fix $\sigma \in N \cap \alpha$ above $\mathrm{supp}(q) \cap N$ and above $\mathsf{sup}(N' \cap N \cap \alpha)$ for all $N' \in \mathrm{dom}(\Delta_q)$ with $\delta_{N'} < \delta_N$.

As in the above case, if G_σ is $\mathcal{P}_\sigma$–generic with $q|_\sigma \in G_\sigma$, then in $N^*[G_\sigma]$ we can find a condition $q^\circ \in A$ such that $q^\circ|_\sigma \in G_\sigma$ (again, the existence of such a condition is witnessed in $\mathbf{V}[G_\sigma]$ by q), and such a q° will necessarily be in N^*. By extending q below σ we may assume that $q|_\sigma$ decides q° and extends $q^\circ|_\sigma$. The proof of $(2)_\alpha$ in this case will be finished if we can show that there is a condition $\overline{q}$ extending q and q°.

The condition $\overline{q}$ can be built by recursion on $\mathrm{supp}(q^\circ) \setminus \sigma$ (note that by the choice of σ, $\min(\mathrm{supp}(q) \setminus \sigma) \geq \mathsf{sup}(N \cap \alpha)$, and therefore $\min(\mathrm{supp}(q) \setminus \sigma) > \max(\mathrm{supp}(q^\circ)))$. This finite construction mimics the proof of $(1)_\alpha$ for successor α. The details are as follows.

Let $(\xi_i)_{i<r}$ be the strictly increasing enumeration of $\mathrm{supp}(q^\circ) \setminus \sigma$. We may assume that $r > 0$, and therefore $r - 1 \geq 0$. We build a sequence $(q_i)_{i<r}$ of conditions as follows:

For $i = 0$, we first extend $q|_\sigma$ to a $\mathcal{P}_{\xi_0}$–condition r extending $q|_{\xi_0}$ and $q^\circ|_{\xi_0}$. r can be found by appealing to Lemma 2.14 if $\sigma < \xi_0$, and if $\sigma = \xi_0$ it is enough of course to set $r = q|_\sigma$.

If $\overline{N}$ is a relevant structure – meaning that $(\overline{N}, \xi_0 + 1) \in \Delta_{q|_{\xi_0+1}}$ and $\overline{N} \in \mathfrak{N}_{\xi_0+1}$ –, then r forces also that the first club $D \subseteq [H(\kappa)^{\mathbf{V}}]^{\aleph_0}$ in $\mathbf{V}$ (in the well–order of $H(\kappa^+)[\dot{G}_{\xi_0}]$ induced by $\lhd$) witnessing that the poset $\Upsilon(\xi_0)$ is $\aleph_{1.5}$–c.c. relative to $\mathbf{V}$ and $\dot{G}_{\xi_0}$ is such that every relevant $\overline{N}$ is in D. (For such an $\overline{N}$ there is some $\overline{N}^* \in \mathfrak{N}^*_{\xi_0+1}$ containing $\lhd$ and such that $\overline{N} = \overline{N}^* \cap H(\kappa)$ which, since $\lhd \in \overline{N}^*$, implies that $\overline{N}^*$ contains a name $\dot{D}$ for that club. Applying this fact and $(2)_{\xi_0}$ we conclude that $\overline{q}|_{\xi_0}$ forces $\overline{N} \in \dot{D}$.) Finally, r forces that $F_{q^\circ}(\xi_0)$ is in $N[\dot{G}_{\xi_0}]$ and hence in $N'[\dot{G}_{\xi_0}]$ for some relevant N' with $\delta_{N'[\dot{G}_{\xi_0}]}$ minimal. The reason is that $\delta_{N'[\dot{G}_{\xi_0}]} = \delta_{N'}$ for every relevant N' by $(2)_{\xi_0}$ and that by our choice of σ we know that if $\overline{N} \in \mathrm{dom}(\Delta_q)$ and $\xi_0 \in \overline{N}$, then $\delta_{\overline{N}} \geq \delta_N$. Putting these facts together we get that there is some $v^* \in H(\kappa)$ and some extension of r forcing that v^* $\Upsilon(\xi_0)$–extends $F_{q^\circ}(\xi_0)$ and is $(\overline{N}[\dot{G}_{\xi_0}], \Upsilon(\xi_0))$–generic for all relevant $\overline{N}$. It follows now from Lemma 2.12 that there is a $\mathcal{P}_{\xi_0+1}$–condition q_0 extending r, $q|_{\xi_0+1}$ and $q^\circ|_{\xi_0+1}$.

For i such that $i + 1 < r$, we assume inductively that $q_i \in \mathcal{P}_{\xi_i+1}$ extends $q|_{\xi_i+1}$ and $q^\circ|_{\xi_i+1}$, and obtain $q_{i+1} \in \mathcal{P}_{\xi_{i+1}+1}$ from q_i by arguing exactly as in the case $i = 0$ with ξ_{i+1} instead of ξ_0 and starting with q_i rather than $q|_\sigma$. In the end we obtain $q_{i+1} \in \mathcal{P}_{\xi_{i+1}+1}$ extending both $q|_{\xi_{i+1}+1}$ and $q^\circ|_{\xi_{i+1}+1}$. Let $\mu = \xi_{r-1} = \max(\mathrm{supp}(q^\circ))$ and let

$$\bar{q} = (F_{q_{r-1}} \cup (F_q \upharpoonright [\mu + 1,\, \alpha)), \Delta_{q_{r-1}} \cup \Delta_{q^\circ} \cup \Delta_q)$$

Claim 2.25. $\bar{q}$ *is a condition in* $\mathcal{P}_\alpha$ *extending both* q *and* q°.

Proof. We prove by induction that if $\mu + 1 \leq \xi \leq \alpha$, then $\bar{q}|_\xi$ is in $\mathcal{P}_\xi$ and $\bar{q}|_\xi \leq_\xi q^\circ|_\xi$, $q|_\xi$. The case $\xi = \mu + 1$ follows from Lemma 2.10, since it implies that $\bar{q}|_{\mu+1}$ is equivalent to q_{r-1} (recall that $q_{r-1} \leq_{\mu+1} q^\circ|_{\mu+1}$, $q|_{\mu+1}$). Assume now that $\xi = \eta + 1$ for $\eta \geq \mu + 1$. It suffices to show that $\bar{q}|_{\eta+1}$ satisfies condition (C4) in the definition of $\mathcal{P}_{\eta+1}$ since it clearly satisfies the other conditions. In other words, we must show that if $\eta \in dom(F_q)$, then $\bar{q}|_\eta$ forces that $F_q(\eta)$ is $(M[\dot{G}_\eta], \Upsilon(\eta))$–generic for all those $M \in \mathfrak{N}_{\eta+1}$ for which there exists an ordinal $\beta \geq \eta + 1$ such that $(M, \beta) \in \Delta_{q_{r-1}} \cup \Delta_{q^\circ} \cup \Delta_q$. But such a pair (M, β) cannot be in $\Delta_{q_{r-1}}$, since all markers occurring in side conditions in $q_{r-1} \in \mathcal{P}_{\mu+1}$ are at most $\mu + 1 < \eta + 1$. On the other hand, we know that $\eta \in \mathrm{supp}(q) \setminus \sigma = \mathrm{supp}(q) \setminus (N \cap \alpha)$. It follows that there is no $M \in \mathrm{dom}(\Delta_{q^\circ})$ such that $M \in \mathfrak{N}_{\eta+1}$ (such a countable M is in N, and therefore $M \cap \alpha \subseteq N \cap \alpha$), and hence (M, β) is neither in Δ_{q°. We conclude that such a pair (M, β) is in Δ_q. By (C4) applied to $q|_{\eta+1}$, we have that $q|_\eta$ (and hence, $\bar{q}|_\eta$) forces what we want. Finally note that the induction hypothesis $\bar{q}|_\eta \leq_\eta q^\circ|_\eta$, $q|_\eta$, the definition of $\bar{q}$, and the fact that the maximum of the support of q° is $\mu < \eta$ together imply that $\bar{q}|_{\eta+1} \leq_{\eta+1} q^\circ|_{\eta+1}$, $q|_{\eta+1}$. The case when ξ is limit follows from the induction hypothesis. $\square$

The above claim finishes the proof of $(2)_\alpha$ for limit α and the proof of the lemma. $\square$

Corollary 2.26. *For all* $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ *is proper.*

The following lemma is a straightforward application of Claim 2.23 and Lemma 2.11.

Lemma 2.27. *For every* $\alpha < \kappa$ *and every condition* $q \in \mathcal{P}_\kappa$, q *forces that the collection of all* y *such that there is some* $(F, \Delta) \in \dot{G}_\kappa$ *with* $F(\alpha) = y$ *generates a* $\mathbf{V}[\dot{G}_\alpha]$–*generic filter on* $\Upsilon(\alpha)$.

Lemma 2.28 follows from the usual counting of nice names for subsets of κ using $(\kappa^{<\kappa})^\mathbf{V} = \kappa$ and Lemma 2.19.

Lemma 2.28. $\mathcal{P}_\kappa$ *forces* $\kappa^{<\kappa} = \kappa$.

Lemma 2.29. *If $\dot{\mathcal{R}}$ is a $\mathcal{P}_\kappa$–name for an $\aleph_{1.5}$–c.c. poset included in $H(\kappa)^{\mathbf{V}}$, then $\mathcal{P}_\kappa$ forces that $\dot{\mathcal{R}}$ is $\aleph_{1.5}$–c.c. relative to $\mathbf{V}$ and $\dot{G}_\kappa$.*

Proof. Let G be generic for $\mathcal{P}_\kappa$, let $\mathcal{R}$ be the interpretation of $\dot{\mathcal{R}}$ by G, and let $\dot{\mathcal{D}}$ be a $\mathcal{P}_\kappa$–name for a club of $[H(\kappa^+)[\dot{G}_\kappa]]^{\aleph_0}$ witnessing that $\mathcal{R}$ has the $\aleph_{1.5}$–chain condition.[17] Fix a sufficiently large cardinal θ such that $\dot{\mathcal{R}}$ and $\dot{\mathcal{D}}$ are in $H(\theta)$. Define $\mathfrak{N}^*$ as the set of all countable elementary substructures of $H(\theta)$ containing all the relevant parameters. Let also $\mathfrak{N} = \{N^* \cap H(\kappa) : N^* \in \mathfrak{N}^*\}$ and note that, by properness together with Lemma 2.22, the club $\mathfrak{N}$ witnesses that $\dot{\mathcal{R}}$ is $\aleph_{1.5}$–c.c. relative to $\mathbf{V}$ and $\dot{G}_\kappa$:

If $u \in \mathcal{P}_\kappa$ and $\{N_i : i \in m\} \subseteq \mathfrak{N}$ are such that Δ_u includes $\{(N_i, \mathsf{sup}(N_i \cap \kappa)) : i < m\}$, then u forces that $(N_i^* \cap H(\kappa^+))[\dot{G}_\kappa] = H(\kappa^+)^{N_i^*[\dot{G}_\kappa]}$ is in $\dot{\mathcal{D}}$ whenever $N_i^* \in \mathfrak{N}^*$ is such that $N_i^* = N_i \cap H(\kappa)$. Also note that, by the assumption $\dot{\mathcal{R}} \subseteq H(\kappa)$ and the $\aleph_2$–chain condition of this poset, u forces that $(N_i^* \cap H(\kappa^+))[\dot{G}_\kappa]$ and $N_i[\dot{G}_\kappa]$ have the same maximal antichains of $\dot{\mathcal{R}}$. It follows that u forces that a condition in $\dot{\mathcal{R}}$ is $((N_i^* \cap H(\kappa^+))[\dot{G}_\kappa], \dot{\mathcal{R}})$–generic iff it is $(N_i[\dot{G}_\kappa], \dot{\mathcal{R}})$–generic. Thus u forces that if $y \in \dot{\mathcal{R}}$ belongs to some $N_i[\dot{G}_\kappa]$ with δ_{N_i} minimal among $\{\delta_{N_j} : j < m\}$, then there is an extension of y in $\dot{\mathcal{R}}$ which is $(N_i[\dot{G}_\kappa], \dot{\mathcal{R}})$–generic for all i. This is true since u forces $\delta_{(N_i^* \cap H(\kappa^+))[\dot{G}_\kappa]} = \delta_{N_i[\dot{G}_\kappa]} = \delta_{N_i}$ by Lemma 2.22. $\square$

Definition 2.30. Let Q be an elementary substructure of $H(\theta)$, for some regular $\theta > \theta_\kappa$, and suppose Q is closed under ω–sequences and contains Φ, $\lhd$ and $\vec{X}$. Suppose $Q \cap \kappa$ is an ordinal δ in κ. Let $\alpha \leq \delta$, let q be a condition in $\mathcal{P}_\alpha$, and let $q^* \in \mathcal{P}_\alpha \cap Q$ be a condition with the following properties.

(1) $F_{q^*} = F_q$
(2) There is an enumeration $((N_i, \gamma_i) : i < n)$ of $\mathrm{dom}(\Delta_q)$ and an enumeration $((M_i, \tilde{\gamma}_i) : i < n)$ of $\mathrm{dom}(\Delta_{q^*})$ such that
   - there is a set R with
   $$\left(\bigcup \mathrm{dom}(\Delta_q)\right) \cap \left(\bigcup \mathrm{dom}(\Delta_{q^*})\right) = R = \left(\bigcup \mathrm{dom}(\Delta_q)\right) \cap Q$$
   and an isomorphism Ψ between
   $$\left\langle \bigcup(\mathrm{dom}(\Delta_{q^*}), \in, R, \Phi, M_i\right\rangle_{i<n}$$

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[17]Note that there is such a $\dot{\mathcal{D}}$ since $|\mathcal{R}|^{\mathbf{V}[G]} \leq \kappa$ by Lemma 2.28 and therefore, by definition of $\aleph_{1.5}$–c.c., there must be a club of $[H(\kappa^+)^{\mathbf{V}[G]}]^{\aleph_0}$ in $\mathbf{V}[G]$ witnessing that $\mathcal{R}$ is $\aleph_{1.5}$–c.c. Also, remember that $\mathcal{P}_\kappa$ forces $H(\kappa^+)[\dot{G}_\kappa] = H(\kappa^+)^{\mathbf{V}[\dot{G}_\kappa]}$ (Corollary 2.20).

and

$$\langle \bigcup(\mathrm{dom}(\Delta_q)), \in, R, \Phi, N_i \rangle_{i<n}$$

which is the identity on R, and such that

- for all $i < n$, $\tilde{\gamma}_i = \gamma_i$ if $\gamma_i < \delta$ and $\tilde{\gamma}_i = \mathsf{sup}(N_i \cap \delta)$ if $\gamma_i = \delta$, and for all $\xi \in R \cap N_i \cap \alpha$,
    - (a) $M_i \in \mathfrak{N}_{\xi+1}$ iff $N_i \in \mathfrak{N}_{\xi+1}$, and
    - (b) if $M_i \in \mathfrak{N}_{\xi+1}$, then $\Psi \restriction M_i$ is an isomorphism between $(M_i, \in, \mathcal{P}_\xi)$ and $(N_i, \in, \mathcal{P}_\xi)$.

Then we say that q^* is *a reflection of q to Q*.

Lemma 2.32 will make use of the following reflection result.

Lemma 2.31. *Let Q be an elementary substructure of $H(\theta)$, for some regular $\theta > \theta_\kappa$, and suppose Q is closed under ω–sequences and contains Φ, $\lhd$ and $\vec{X}$. Suppose $Q \cap \kappa$ is an ordinal δ in κ. Let $\alpha \leq \delta$, let $q_0 \in \mathcal{P}_\alpha \cap Q$ and let $q = (F_q, \{(N_i, \gamma_i) \ : \ i < n\})$ be a $\mathcal{P}_\alpha$–condition extending q_0. Then*

- (1) *there is a reflection of q to Q extending q_0, and*
- (2) *for every q^*, if $q^* = (F_q, \{(M_i, \tilde{\gamma}_i)\})$ is a reflection of q to Q as witnessed by the isomorphism*

$$\Psi : \langle \bigcup(\mathrm{dom}(\Delta_{q^*})), \in, R, \Phi, M_i \rangle_{i<n} \longrightarrow \langle \bigcup(\mathrm{dom}(\Delta_q)), \in, R, \Phi, N_i \rangle_{i<n}$$

and $q^\dagger \in Q \cap \mathcal{P}_\alpha$ extends q^*, then

$$\overline{q} = (F_{q^\dagger}, \Delta_{q^\dagger} \cup \Delta_q \cup \{(M, 0) \ : \ M \in \mathcal{N}\} \cup \Delta)$$

is a condition in $\mathcal{P}_\alpha$, where $\mathcal{N}$ is the collection of all $\Psi_{N^0, N^1}(M)$ such that $M \in \mathrm{dom}(\Delta_{q^\dagger}) \cap N^0$, $N^0 \in \mathrm{dom}(\Delta_{q^*})$, $N^1 \in \mathrm{dom}(\Delta_q)$, and $\delta_{N^0} = \delta_{N^1}$, and where Δ is the collection of all pairs $(\Psi(M), \gamma)$ such that there is some i with

- (a) $(M, \gamma) \in M_i \cap \Delta_{q^\dagger}$,
- (b) $\gamma \leq \tilde{\gamma}_i$, *and*
- (c) $M_i \in \mathfrak{N}_{\gamma+1}$

Proof. We prove the first part first. For all i let $\tilde{\gamma}_i$ be obtained from γ_i as in the definition of reflection of q to Q. Of course $\tilde{\gamma}_i = \gamma_i$ unless $\alpha = \delta$. In any case $\tilde{\gamma}_i \in Q$ since Q is ω–closed. It is straightforward to see that $\tilde{q} = (F_q, \{(N_i, \tilde{\gamma}_i) \ : \ i < n\})$ is a condition in $\mathcal{P}_\alpha$ extending q_0. Note also that F_q is in Q since for every $\xi < \alpha$, $X_\xi \in Q$, $X_\xi \subseteq Q$, and $\mathcal{P}_\xi$ has the $\aleph_2$–c.c. (and therefore for every $\xi < \alpha$ there is a set $Y_\xi \subseteq H(\kappa)$, $Y_\xi \in Q$, such that for every $v \in H(\kappa)$ and every $\mathcal{P}_\xi$–condition r, if $r \Vdash_\xi \check{v} \in \Upsilon(\xi)$, then $v \in Y_\xi$). Since $Q \preccurlyeq H(\theta)$ contains all reals, q_0, F_q, $(\tilde{\gamma}_i)_{i<n}$ and $\{(N_i \cap Q, \tilde{\gamma}_i) \ : \ i < n\}$, we may find in Q a reflection

of q to Q extending q_0: The existence of such a q^* can be expressed by a sentence with parameters in Q and this sentence is true in $H(\theta)$ as witnessed by $\tilde{q}$.

We will prove (2) by induction on α. Let $q^\dagger \in Q \cap \mathcal{P}_\alpha$ extend q^* and let $\bar{q}$ be obtained from $q^\dagger$ as in (2). We prove by induction on $\beta \leq \alpha$ that $\bar{q}$ is a $\mathcal{P}_\beta$–condition. For $\beta = 0$ this follows immediately from Lemma 2.6. The present use of partial symmetric systems rather than fully symmetric systems in the sense of [1] is needed precisely to make this point of the argument go through. With the stronger notion of symmetry from [1] it might not be possible to amalgamate even the restriction to 0 of $\bar{q}$ and q into a $\mathcal{P}_0$–condition. The limit case follows from the induction hypothesis. For the successor case, let β be such that $\beta + 1 < \alpha$ and suppose $\bar{q}|_\beta \in \mathcal{P}_\beta$. It suffices to show that if β is in the support of q, then $\bar{q}|_\beta$ forces that $F_{q^\dagger}(\beta)$ is

(1) $(N_i[\dot{G}_\beta], \Upsilon(\beta))$–generic for every $i < n$ such that $\gamma_i > \beta$ and $N_i \in \mathfrak{N}_{\beta+1}$, and

(2) $(\Psi(M)[\dot{G}_\beta], \Upsilon(\beta))$–generic whenever $M \in \mathcal{N}_{\beta+1}$, $(M, \gamma) \in \Delta_{q^\dagger}$ for some $\gamma \geq \beta + 1$, and there is some i such that $M \in M_i$, $\gamma \leq \tilde{\gamma}_i$ and $M_i \in \mathfrak{N}_{\beta+1}$.

We prove the first part only (the proof of the second part is essentially the same). Note that for every i as in (1), $M_i \in \mathfrak{N}_{\beta+1}$ and $\tilde{\gamma}_i > \beta$. Hence, $\bar{q}|_\beta$ forces that $F_{q^\dagger}(\beta)$ is $(M_i[\dot{G}_\beta], \Upsilon(\beta))$–generic since $\bar{q}|_\beta$ extends $q^\dagger|_\beta$ and $q^\dagger|_\beta$ forces this. Suppose towards a contradiction that there is a $\mathcal{P}_\beta$–condition t extending $\bar{q}|_\beta$, a $\mathcal{P}_\beta$–name $\dot{A} \in N_i$ for a maximal antichain of $\Upsilon(\beta)$, and some $y \in H(\kappa) \cap Q$ such that $t \Vdash_\beta \check{y} \in \Upsilon(\beta)$ and such that t forces that $\check{y}$ extends $F_{q^\dagger}(\beta)$ and that no condition in $\dot{A} \cap N_i[\dot{G}_\beta]$ is compatible with y. We may assume that $\dot{A}$ consists of pairs $(r, \check{x})$, with $r \in \mathcal{P}_\beta$ and $\check{x}$ the canonical $\mathcal{P}_\beta$–name of some $x \in H(\kappa)^{\mathbf{V}}$. Then $\Psi^{-1}(\dot{A}) \in M_i$ is a $\mathcal{P}_\beta$–name for a maximal antichain of $\Upsilon(\beta)$ by the choice of Ψ since $\Psi^{-1} \upharpoonright N_i$ preserves the predicate $\mathcal{P}_\beta$. Note that, by the $\aleph_2$–c.c. of $\mathcal{P}_\beta$ together with the $\aleph_2$–c.c. of $\Upsilon(\beta)$ in $\mathbf{V}[G_\beta]$, there is a surjection $\varphi : \omega_1 \longrightarrow \dot{A}$ in N_i. Let now t^* be a reflection of t to Q extending $q^\dagger|_\beta$ (which exists by (1) for β) and let $t' \leq_\beta t^*$, $t' \in Q$, and $y' \in H(\kappa) \cap Q$ be such that t' forces that y' is a common extension in $\Upsilon(\beta)$ of y and of some $x \in \Psi^{-1}(\dot{A}) \cap M_i$. These objects exist since $q^\dagger|_\beta$ is $(M_i, \mathcal{P}_\beta)$–generic and forces $F_{q^\dagger}(\beta)$ to be $(M_i[\dot{G}_\beta], \Upsilon(\beta))$–generic and since t^* extends $q^\dagger|_\beta$. By extending t' within Q if necessary we may of course assume that t' decides x, that $(r, \check{x}) \in \Psi^{-1}(\dot{A}) \cap M_i$ for some r and that t' extends r. There is some $\nu < \delta_{M_i}$ such that $\Psi^{-1}(\varphi)(\nu) = (r, \check{x})$. But then $\varphi(\nu) = (\Psi(r), \check{x}) \in$

$\dot{A} \cap N_i$. Now, by induction hypothesis there is a condition $\tilde{t}$ in $\mathcal{P}_\beta$ extending t, t' and $\Psi(r)$. This is a contradiction since $\tilde{t}$ forces that $x \in \dot{A} \cap N_i$ is compatible with y.   □

Lemma 2.32. $\mathcal{P}_\kappa$ *forces* $\mathsf{MA}^{1.5}_{<\kappa}$.

Proof. Let q be a $\mathcal{P}_\kappa$–condition, let $\chi < \kappa$, and let $\dot{\mathcal{R}} \subseteq H(\kappa)$ and $\dot{A}_i$ (for $i < \chi$) be $\mathcal{P}_\kappa$–names such that q forces that $\dot{\mathcal{R}}$ is an $\aleph_{1.5}$–c.c. poset relative to $\mathbf{V}$ and $\dot{G}_\kappa$ and that each $\dot{A}_i$ is a maximal antichain of $\dot{\mathcal{R}}$. By Lemma 2.28, Corollary 1.4 and Lemma 2.29 we will be done if we show that there is some condition extending q and forcing that there is a filter on $\dot{\mathcal{R}}$ intersecting all $\dot{A}_i$. Let X be a subset of κ coding the $\mathcal{P}_\kappa$–name $\dot{\mathcal{R}}$ via our fixed translating function π.

Now, using the fact that $\vec{X}$ is a $\diamondsuit(\{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \omega_2\})$–sequence, we may fix an elementary substructure Q of some large enough $H(\theta^*)$ containing Φ, q, $\mathcal{P}_\kappa$, $\dot{\mathcal{R}}$, $(\dot{A}_i)_{i \in \chi}$, $\vec{X}$, X and a well–order W of $H(\theta)$ for some strong limit $\theta > \kappa$, closed under ω–sequences, and such that $\delta = Q \cap \kappa$ is an ordinal such that $X_\delta = X \cap \delta$ (since $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$, the set of $\delta \in \kappa$ for which there is a Q as above contains a λ–club for every regular cardinal $\lambda < \kappa$, $\lambda \geq \omega_1$). Let $\dot{\mathcal{R}}_0$ be the $\mathcal{P}_\delta$–name coded by X_δ. By taking δ from within a suitable club we may further assume that q forces in $\mathcal{P}_\kappa$ for all ξ, ξ' in δ that if $\pi_{X_\delta}(\xi)$ and $\pi_{X_\delta}(\xi')$ are compatible conditions in $\dot{\mathcal{R}}$, then there is an ordinal below δ coding a common extension in $\dot{\mathcal{R}}$ of $\pi_{X_\delta}(\xi)$ and $\pi_{X_\delta}(\xi')$ (since this is true for a club of κ in the extension which, by the κ–c.c. of $\mathcal{P}_\kappa$, includes a club of κ in $\mathbf{V}$). Using this, we may assume as well that δ is taken so that q forces in $\mathcal{P}_\kappa$, for every $\mathcal{P}_\delta$–name for a maximal antichain of $\dot{\mathcal{R}}_0$, that $\dot{A}$ is a maximal antichain of $\dot{\mathcal{R}}$. This is true since every $\dot{A}$ as above can be taken to be in $H(\kappa)$ and since $\Phi : \kappa \longrightarrow H(\kappa)$ is a surjection.

The following claim follows from the closure of Q under ω–sequences together with the above choice of δ.

Claim 2.33. q *forces in* $\mathcal{P}_\delta$ *that* $\dot{\mathcal{R}}_0$ *is an* $\aleph_{1.5}$*–c.c. poset relative to* $\mathbf{V}$ *and* $\dot{G}_\delta$.

Proof. Let $q' \leq_\delta q$ and let $\{N_0^*, \ldots, N_{n-1}^*\}$ be a set of countable elementary substructures of $\langle H(\theta), \in, W \rangle$ containing everything relevant (this includes a name $\dot{D}$ for a club witnessing the $\aleph_{1.5}$–c.c. of $\dot{\mathcal{R}}$ relative to $\mathbf{V}$ and $\dot{G}_\kappa$) and such that $(N_i, \delta) \in \Delta_{q'}$ for all i, where $N_i := N_i^* \cap H(\kappa)$. Suppose towards a contradiction that there is some $y \in Q$ such that $q' \Vdash_\delta \check{y} \in \dot{\mathcal{R}}_0$ and such that $y \in N_0$, where $\delta_{N_0} = \min\{\delta_{N_0}, \ldots, \delta_{N_{n-1}}\}$, and that q' forces that there is no extension of y which is $(N_i[\dot{G}_\delta], \dot{\mathcal{R}}_0)$– generic for all $i < n$.

By the proof of Lemma 2.31 (1) together with the proof of Subclaim 1.3.1 we may find a reflection q^* of q' to Q so that, in addition, if Ψ is the isomorphism witnessing that q^* is a reflection of q' to Q and $M_i = \Psi^{-1}\text{``}N_i$ for all i, then there are $M_i^* \in Q$ (for $i < n$) such that each M_i^* is a countable elementary substructure of $\langle H(\theta), \in, W \rangle$ containing everything relevant and such that $M_i = M_i^* \cap H(\kappa)$ and $(M_i^*, \in, W)$ is isomorphic to $(N_i^*, \in, W)$ by an isomorphism fixing $\dot{D}$ and $\dot{R}$ whose restriction to M_i is $\Psi \upharpoonright M_i$. We may also assume that $\Psi \upharpoonright M_i$ preserves $\mathcal{P}_\beta$ for every i and every $\beta \in N_i \cap Q$. Clearly we may assume that q^* also forces $y \in \dot{\mathcal{R}}$. Let $q^\dagger \in Q$ be an extension of q^* and let $y' \in H(\kappa) \cap Q$ be such that $q^\dagger$ forces that y' is a condition in $\dot{\mathcal{R}}$ extending y and $(M_i[\dot{G}_\kappa], \dot{\mathcal{R}})$–generic for all i. By Lemma 2.31 (2), $q^\dagger$ and q' can be amalgamated into a $\mathcal{P}_\delta$ condition $\bar{q}$. Now we argue that $\bar{q}$ forces that y' is $(N_i[\dot{G}_\delta], \dot{\mathcal{R}}_0)$–generic for all i.

For this, suppose towards a contradiction that there is a $\mathcal{P}_\delta$–condition $\bar{q}'$ extending $\bar{q}$, some i, and some $\dot{A} \in N_i$ such that $\dot{A}$ is a $\mathcal{P}_\delta$–name such that $\bar{q}'$ forces that $\dot{A}$ is a maximal antichain of $\dot{\mathcal{R}}_0$ and that there is no condition in $\dot{A} \cap N_i$ compatible with y'. Note that, by our choice of δ, $\dot{A}$ is a $\mathcal{P}_\kappa$–name such that $\bar{q}'$ forces in $\mathcal{P}_\kappa$ that $\dot{A}$ is a maximal antichain of $\dot{\mathcal{R}}$. We may therefore assume that $\dot{A}$ is actually a $\mathcal{P}_\kappa$–name for a maximal antichain of $\dot{\mathcal{R}}$. Let q^{**} be a reflection of $\bar{q}'$ to Q. The rest of the argument is again as in the proof of Lemma 2.31 (2): We find an extension $q^\ddagger \in Q$ of q^{**} together with some y'' and some $x \in M_i$ such that $q^\ddagger$ forces that $\check{x} \in \Psi^{-1}(\dot{A})$ and that y'' is a common extension in $\dot{\mathcal{R}}$ of x and y'. We then find by (the proof of) Lemma 2.31 (2) an extension of $q^\ddagger$ and $\bar{q}'$ forcing $\check{x} \in \dot{A} \cap N_i$, which is a contradiction. Exactly as in the proof of Lemma 2.31 (2), the argument uses the fact that $\dot{\mathcal{R}}$ has the $\aleph_2$–c.c. and so, since $\mathcal{P}_\delta$ has also the $\aleph_2$–c.c., there is an enumeration of $\dot{A}$ in N_i in length ω_1. $\quad\square$

It follows from the above claim that q forces $\Upsilon(\delta) = \dot{\mathcal{R}}_0$. Finally, we may extend q to a condition q' such that $\delta \in \text{supp}(q')$. Then, by Lemma 2.27, q' forces that there is a filter H on $\dot{\mathcal{R}}_0$ meeting all $\dot{A}_i$, and of course H generates a filter on $\dot{\mathcal{R}}$. $\quad\square$

Lemma 2.34. $\mathcal{P}_\kappa$ *forces* $2^{\aleph_0} = \kappa$.

Proof. $\mathbf{V}^{\mathcal{P}_\kappa} \models 2^{\aleph_0} \geq \kappa$ follows for example from the fact that $\mathcal{P}_\kappa$ forces $\mathsf{MA}_{<\kappa}^{1.5}$. $\mathbf{V}^{\mathcal{P}_\kappa} \models 2^{\aleph_0} \leq \kappa$ follows from Lemma 2.28. $\quad\square$

Lemma 2.34 finishes the proof of Theorem 2.1.

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