Economic Inquiry

THE ATTACK-AND-DEFENSE GROUP CONTESTS: BEST SHOT VERSUS WEAKEST LINK

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This study analyzes a group contest in which one group (defenders) follows a weakest link, whereas the other group (attackers) follows a best shot impact function (IF). We fully characterize the Nash and coalition-proof equilibria and show that with symmetric valuation the coalition-proof equilibrium is unique up to the permutation of the identity of the active player in the attacker group. With asymmetric valuation, it is always an equilibrium for one of the highest valuation players to be active; it may also be the case that the highest valuation players in the attacker group free-ride completely on a group member with a lower valuation. However, in any equilibrium, only one player in the attacker group is active, whereas all the players in the defender group are active and exert the same effort. We also characterize the Nash and coalition-proof equilibria for the case in which one group follows either a best shot or a weakest link but the other group follows an additive IF. (*JEL* C72, D70, D72, D74, H41)

I. INTRODUCTION

Consider a situation in which a group of firms is engaged in illegal price fixing. Their businesses are spread over several countries and each of them exerts irreversible resources on hiding their activities and on legal experts to avoid possible prosecution (Malik 1990). Antitrust authorities in those countries (the Competition and Market Authority in the United Kingdom, and the antitrust division of the Department of Justice in the United States, for example) also exert costly resources on investigation to detect possible cartels. Thus, we can depict the antitrust authorities as a group of "attackers" and the colluding firms as a group of "defenders." For the antitrust authorities, the efforts have the nature of a "best shot," that is, if any of the authorities can detect the cartel, then it will solve the problem for all others. Hence, essentially the best effort exerted among the authorities represents the strength of the investigation. However, for the colluding firms, the resources exerted have the nature of a "weakest link," that is, if one of them gets detected, then the whole cartel will be detected. Therefore, the lowest avoidance effort determines the strength of hiding the collusion.[1]

The above-mentioned situation can be structured as a group contest, in which members of groups exert irreversible efforts that translate into "group effort." Then a group contest success function (CSF; a function that maps the group efforts into the probabilities of winning) determines which group is going to win. A function that translates the individual group member efforts into a group effort is called an impact function (IF; Wärneryd 1998). In the case above, the colluding firms follow a weakest link IF, and the antitrust authorities follow a best shot IF. We term this family of games the "Attack-and-Defense Group Contest."

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1. In some particular circumstances, the strength of either the attackers or the defenders can arguably be viewed also as perfectly substitutable or additive in nature. We discuss them in detail in Section 3.

ABBREVIATIONS

CES: Constant Elasticity of Substitution
CIA: Central Intelligence Agency
CSF: Contest Success Function
FBI: Federal Bureau of Investigation
IF: Impact Function

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This type of structure is quite common in the field. In addition to the case of avoidance efforts in collusion discussed above, there are very many situations in which groups are engaged in Attack-and-Defense Contests. One prominent example is from defense economics (Arce, Kovenock, and Roberson 2012; Conybeare, Murdoch, and Sandler 1994). The siege game between different intelligence agencies and terrorist organizations follows this structure. Intelligence agencies such as the Central Intelligence Agency (CIA) or the Federal Bureau of Investigation (FBI) trying to stop terrorists follow a best shot IF, as if any of them can capture or uncover a terrorist ploy, it will solve the issue. However, terrorist organizations such as the Al-Quida or the Lashkar-e-Taiba follow a weakest link IF, as if any of their ploys get detected, then the terror links will be exposed.[2] Another example comes from the system reliability (Hausken 2008; Varian 2004) literature. A system of software operations follows a weakest link structure, as if one of them is captured by viruses, then the whole system will get infected. However, for virus coders, it is a best shot situation, as if any of the viruses gets in through the system, then it can infect and capture the whole system. Furthermore, in corrupt societies the members of a political party exert efforts to conceal information regarding corruption in the party, whereas members of the civil society exert efforts to uncover it (OECD 2003). Understandably, the strength of the party will be as strong as the weakest member, but for the civil society any successful member will bring in the required result for all.

Of course, this structure can also capture situations beyond the nature of attack and defense. For example, a firm in a patent race may run parallel R&D teams, but another firm may run a big R&D team that works sequentially by specialized team members (Abernathy and Rosenbloom 1968; Nelson 1961). Hence, the resultant R&D of the first firm in the patent race will have the nature of a best shot as the best product will represent the firm. However, the resultant R&D of the second firm will depend on the strength of the weakest member.

Individual attack-and-defense game is explored extensively in the literature. Clark and Konrad (2007) analyze a game in which multiple battlefields are linked and two players, with limited resources, allocate the resources into the battlefields. The resources allocated by both players in a particular battlefield determine the probability of a player to win the battlefield. One player, the attacker, will have to win at least one battlefield to win the game; but the other player, the defender, will have to win every battlefield. This type of structure is employed in other studies (Arce, Kovenock, and Roberson 2012; Hausken 2008; Kovenock and Roberson 2012) and is generalized in a model by Kovenock and Roberson (2010). Each of these studies, however, analyzes individual conflict and does not consider any group dynamics in this context.

The literature on group contests starts with the work by Katz, Nitzan, and Rosenberg (1990) who consider symmetric valuation and a lottery (Tullock 1980) CSF with a perfectly substitute (or additive) IF. Baik (2008) extends this to asymmetric valuation and shows that only the highest valuation player in a group exerts positive effort in equilibrium, whereas other players free-ride. Similar conclusions are derived by Baik (1993) for the case of an additive IF and a CSF satisfying certain general regularity conditions. Lee (2012) employs the weakest link IF for all groups and characterizes possible equilibria. He shows that in any equilibrium all group members exert the same effort. Multiple equilibria exist, but the coalition-proof equilibrium is unique. Kolmar and Rommeswinkel (2013) use a constant elasticity of substitution IF (ranging from perfect substitute to weakest link) and characterize the set of equilibria. Chowdhury, Lee, and Sheremeta (2013a) employ the best shot IF and show that only one player in each active group exerts positive effort in equilibrium. However, the active player need not be a highest valuation group member. All these studies employ a stochastic (lottery, *à la* Tullock 1980) CSF. Another stream of research, instead, uses a deterministic (all-pay auction, *à la* Baye, Kovenock, and de Vries 1996) CSF. Among them, Baik, Kim, and Na (2001) and Topolyan (2014) use additive; Chowdhury, Lee, and Topolyan (2013b) use weakest link; Barbieri, Malueg, and Topolyan (2014) use best shot; and Chowdhury and Topolyan (2015) use hybrid IFs.

These studies make several important contributions to the literature. However, none except the last explore a situation in which different groups can follow different IFs. Table 1 summarizes the fit of the current study in this area of literature. The columns in Table 1 show the IF,

2. On the other hand, when the roles reverse, i.e., terrorist groups attack a country, then any successful attack serves their purposes and hence they follow a best shot technology. However, the intelligence agencies now are in the defense positions and they lose if any successful attack occurs. Hence they follow a weakest link technology.

TABLE 1
Fit of the Current Study

CSF\IF	Best Shot	Perfect Substitute	Weakest Link
Stochastic (Tullock)	Chowdhury, Lee, and Sheremeta (2013a)	Katz, Nitzan, and Rosenberg (1990) and Baik (2008)	Lee (2012)
		Kolmar and Rommeswinkel (2013): CES from perfect substitute to weakest link	
	Current study: Allows the possibility of different IFs for different groups		
Deterministic (all-pay auction)	Barbieri, Malueg, and Topolyan (2014)	Baik, Kim, and Na (2001) and Topolyan (2014)	Chowdhury, Lee, and Topolyan (2013b)
	Chowdhury and Topolyan (2015)		

and the rows show the CSF implemented in the group contests employed in each of the studies.

To summarize, in this study we make a threefold contribution. First, we provide a better understanding of situations in which groups are engaged in attack-and-defense conflicts. For the first time in the literature, we introduce a theoretical underpinning of attack-and-defense contests in which groups, rather than individuals, are involved. Second, we fill in a gap in the group contest literature by providing contests in which different groups follow different IFs. Third, we consider the prize to have the nature of group-specific public good (e.g., if the colluding firms are not detected, then every firm benefits; and if they are detected, then every antitrust authority does). Hence, we also contribute to the literature on collective action (Olson 1965) and public-good games (Barbieri and Malueg 2008; Bergstrom, Blume, and Varian 1986; Bliss and Nalebuff 1984) with weakest link or best shot network externalities (Cornes 1993; Hirshleifer 1983, 1985).

II. THE MODEL

A. Model Set-Up

We structure the model along similar lines to Chowdhury, Lee, and Sheremeta (2013a). Consider a contest in which two groups compete to win a group-specific public-good prize. Group g consists of $m_g \geq 2$ risk-neutral players who exert costly efforts to win the prize. The individual group members' valuation for the prize may differ across groups; however, it is the same within a group. Let $v_g > 0$ represent the valuation for the prize of any player in group g, and $x_{gi} \geq 0$, measured in the same unit as the prize values, represent the effort level exerted by player i in group g.

Next, we specify the *group IF* as $f_g : \mathbb{R}_+^{m_g} \rightarrow \mathbb{R}_+$, such that the group effort of group g is given by $X_g = f_g(x_{g1}, x_{g2}, \ldots, x_{gm_g})$. The following assumptions define the best shot technology for group 1 and the weakest link technology for group 2.

ASSUMPTION 1. *The group effort of group 1 is represented by the maximum effort level exerted by the players in group 1, that is,* $X_1 = \max\{x_{11}, x_{12}, \ldots, x_{1m_1}\}$.

ASSUMPTION 2. *The group effort of group 2 is represented by the minimum effort level exerted by the players in group 2, that is,* $X_2 = \min\{x_{21}, x_{22}, \ldots, x_{2m_2}\}$.

To specify the winning probability of group g, denote $p_g(X_1, X_2) : \mathbb{R}_+^2 \rightarrow [0, 1]$ as a *contest success function* (CSF). We assume a logit form group CSF (Münster 2009).

ASSUMPTION 3. *The probability of winning the prize for group* g *is*

$$p_g(X_1, X_2) = \begin{cases} X_g / (X_1 + X_2) & \text{if } X_1 + X_2 > 0 \\ 1/2 & \text{if } X_1 + X_2 = 0. \end{cases}$$

We assume all players forgo their efforts and they have a common cost function with unit marginal cost as described by *Assumption 4*.

ASSUMPTION 4. *The common cost function is* $c(x_{gi}) = x_{gi}$.

Only the members of the winning group receive the prize. Let u_{gi} represent the payoff for player i in group g. Under the above assumptions, the payoff for player i in group g is

$$u_{gi} = v_g X_g / (X_1 + X_2) - x_{gi}. \quad (1)$$

Equation (1) along with the four assumptions represents the *attack-and-defense group contest*. To close the structure, we assume that all players in the contest choose their effort levels independently and simultaneously, and that all of the above (including the valuations, group compositions, IFs, and the CSF) is common knowledge. We employ Nash equilibrium as our solution concept.

We use the following definitions throughout the paper.

DEFINITION 1. *If player* i *in group* g *exerts strictly positive effort, that is,* $x_{gi} > 0$*, then the player is called* active. *Otherwise (when* $x_{gi} = 0$*) the player is called* inactive.

DEFINITION 2. *If the group effort of group* g *is strictly positive, that is,* $X_g > 0$*, then group* g *is called* active. *Otherwise (when* $X_g = 0$*) the group is called* inactive.

Let I_g (I_{-g}) denote the index set of all players in group g (other than g, respectively), and $\mathscr{P}_g$ denote the set of all nonempty subsets of I_g. Given a strategy profile $\boldsymbol{x} = \left(x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}\right)$ and player $i \in I_{-g}$, let x_{-gi} represent the effort of player i in group other than g.

DEFINITION 3. *We say that a coalition of players* $C \in \mathscr{P}_g$ *of group* g blocks *a strategy profile* **x** *if there exists a strategy profile* **y** *such that* $u_{gi}(\mathbf{y}) \geq u_{gi}(\mathbf{x})$ *for all* $i \in C$, $u_{gj}(\mathbf{y}) > u_{gj}(\mathbf{x})$ *for some* $j \in C$, $x_{-gi} = y_{-gi}$ *for all* $i \in I_{-g}$, *and* $x_{gk} = y_{gk}$ *for all* $k \notin C$.

In other words, a coalition of players blocks a strategy profile if the members of the coalition have an incentive to deviate *altogether*.

DEFINITION 4. *A strategy profile* **x** *is called a* coalition-proof equilibrium *if no coalition* $C \in \mathscr{P}_g$, $g = 1, 2$, *blocks* **x**.

It follows from *Definition 4* that any coalition-proof equilibrium is a Nash equilibrium.

B. Solution with Symmetric Valuation

We begin by stating *Lemma 1*. This lemma points out (from *Assumption 3*) that both groups actively participate in the contest.

LEMMA 1. In any equilibrium both groups are active.

Assumption 1 gives rise to *Lemma 2*. This result is analogous to *Lemma 2* of Chowdhury, Lee, and Sheremeta (2013a), and the proof follows similar lines.

LEMMA 2. In any equilibrium only one player in group 1 is active.

The following result in *Lemma 3* is analogous to *Lemma 1* of Lee (2012) and holds due to the weakest link technology in group 2.

LEMMA 3. In any equilibrium all players of group 2 are active and choose the same effort level.

Lemmata 1, *2*, and *3* simplify the group contest into a seemingly individual contest in which each group behaves like an individual. Consider a situation in which only player i of group 1 is active and puts positive effort x_1 ("attacks") and all players of group 2 "defend," exerting the same effort level x_2. Note that because of the best shot technology in group 1 and the weakest link technology in group 2, we have $X_1 = x_1$ and $X_2 = x_2$. Equation (1) yields the following first-order conditions for an interior Nash equilibrium.

$$v_1 x_2 / \left(x_1 + x_2\right)^2 = 1, \tag{2}$$

$$v_2 x_1 / \left(x_1 + x_2\right)^2 \geq 1. \tag{3}$$

Equation (2) results from the fact that the active player of group 1 is competing individually against group 2. Inequality (3) ensures that no player in group 2 wants to decrease her effort level (due to the weakest link technology no player in group 2 wants to deviate to a higher effort level). Equation (2) implies $x_1 = \sqrt{v_1 x_2} - x_2$; plug this to (3) to obtain

$$x_2 \leq \left(v_1 v_2^2\right) / \left(v_1 + v_2\right)^2 \equiv \bar{x}_2.$$

Therefore, there exists a continuum of Nash equilibria such that only one player in group 1 exerts effort $x_1 = \sqrt{v_1 x_2} - x_2$ and every player in group 2 exerts effort $x_2 \in (0, \bar{x}_2]$.

As the weakest effort determines the survival of the defenders, all members put forth the same effort in a Nash equilibrium. However, as the highest effort determines the success of the attacking group, its members have an incentive to free-ride—and in equilibrium only one of the group members exerts effort while all others free-ride on him. Note that although there exists a continuum of Nash equilibria, there is a unique coalition-proof equilibrium where all players in group 2 choose the highest effort level $\bar{x}_2$. *Theorem 1* summarizes this result.

Theorem 1. The attack-and-defense contest with symmetric valuation has a unique coalition-proof equilibrium (up to the permutation of the identity of the active player in group 1), where one player in group 1 exerts effort $x_1 = \left(v_1^2 v_2\right) / \left(v_1 + v_2\right)^2$, every other player in group 1 puts zero effort, and every player in group 2 exerts effort $x_2 = \left(v_1 v_2^2\right) / \left(v_1 + v_2\right)^2$. There is a continuum of Nash equilibria such that every player in group 2 exerts effort $0 < x_2 \leq \left(v_1 v_2^2\right) / \left(v_1 + v_2\right)^2$, one player in group 1 exerts effort $x_1 = \sqrt{v_1 x_2} - x_2$, and all other players in group 1 exert zero effort.

Observe that $GM \equiv \sqrt{v_1 v_2}$ and $AM \equiv (v_1 + v_2)/2$ are the geometric and arithmetic means of the valuations, respectively, and the $GM - AM$ ratio represents the relative dispersion in valuation. Remarkably, the equilibrium efforts of active players are directly proportional to their own valuation and $(GM/AM)^2$. Thus, *Theorem 1* implies that the more dispersed the valuations, the greater the effort levels in the equilibria.[3]

C. Extension to Asymmetric Valuation

Assume now that the individual group members' valuation for the prize may differ within and across groups. This asymmetry in values can reflect player asymmetry, or an exogenous sharing rule of the group-specific prize, in which the prize-shares among the members of a group are different. Also, note that if we relax Assumption 4 and consider player asymmetry in marginal costs then, due to risk neutrality, the model can again be transformed into an asymmetric valuation model after appropriate rescaling. Let $v_{gi} > 0$ represent the valuation for the prize of player i in group g. Without loss of generality, assume $v_{g(t-1)} \geq v_{gt}$ for $1 < t \leq m_g$.

Here we cannot apply the method we used in the previous section to find Nash equilibria because, given the heterogeneity of valuations in the defender team, it is not clear which valuation to use to derive the first-order conditions. We can, however, simplify the game as follows. Assume all players in group 2 choose the same effort level (either in or off equilibrium). Clearly, no player wants to deviate to a higher effort due to the weakest link technology. It may be the case, however, that some player wants to exert a lower effort. Thus, a strategy $\left(x_{21}, \ldots, x_{2m_2}\right)$ is a part of equilibrium if and only if $x_{21} = x_{22} = \cdots = x_{2m_2} = x_2$ and no player in group 2 wants to deviate to some lower effort.

Clearly, only one player in group 1 is active due to the best shot IF. Suppose one of the highest valuation players (say, player 1) is active and exerts effort x_{11}. Due to the weakest link technology, all players in group 2 exert the same effort level x_2. As in the previous section, player 1 in group 1 is competing individually against group 2, which yields the following first-order condition.

$$v_{11} x_2 / \left(x_{11} + x_2\right)^2 = 1. \tag{4}$$

To ensure that no player in group 2 wants to deviate, the following system of inequalities must be satisfied.

$$\begin{cases} v_{21} x_{11} / \left(x_{11} + x_2\right)^2 \geq 1 \\ \vdots \\ v_{2m_2} x_{11} / \left(x_{11} + x_2\right)^2 \geq 1. \end{cases}$$

As the valuations are descending within the group, this system is equivalent to

$$v_{2m_2} x_{11} / \left(x_{11} + x_2\right)^2 \geq 1. \tag{5}$$

Denote by x_{1k}^b the best response of player k of group 1 under the condition that player k puts the highest effort in her group. As $v_{11} x_2 / \left(x_{11}^b + x_2\right)^2 = 1$ and the valuations are descending, the first-order conditions for the payoff maximization imply that $x_{11}^b \geq x_{1k}^b$ for all $k > 1$, which shows that when *only* player 1 in group 1 is active, no other (inactive) player in group 1 wants to deviate.

Hence Equations (4) and (5) characterize all Nash equilibria in which one of the highest valuation players of group 1 is active. This system of equations has a continuum of solutions which are of the following form: $x_{11} = \sqrt{v_{11} x_2} - x_2$ and $x_2 \in (0, \bar{x}_2]$, where

$$\bar{x}_2 = \left(v_{11} v_{2m_2}^2\right) / \left(v_{11} + v_{2m_2}\right)^2. \tag{6}$$

We are now ready to characterize all Nash equilibria for the case of asymmetric valuation.

Theorem 2. The Nash equilibria of the attack-and-defense contest with asymmetric valuation are as follows.

3. We thank an anonymous referee for pointing us to this observation.

There exists a continuum of Nash equilibria such that all players in group 2 are active and exert effort $x_2 \in (0, \bar{x}_2]$ while only one of the highest valuation players in group 1 is active and exerts effort $x_1 = \sqrt{v_{11} x_2} - x_2$, where $\bar{x}_2 = \left(v_{11} v_{2m_2}^2\right) / \left(v_{11} + v_{2m_2}\right)^2$.

If $4v_{1k} \geq v_{11}$ for some k such that $v_{1k} < v_{11}$, then in addition there exists a continuum of equilibria such that: all players in group 2 are active and exert effort $x_2 \in (0, r_k]$, where $r_k = \min_{1 \leq j < k} \sup \left\{0 \leq y \leq \bar{x}_2 : 2\sqrt{v_{1j} v_{1k}} - v_{1j} \geq \sqrt{y v_{1k}}\right\}$, $\bar{x}_2 = \left(v_{1k} v_{2m_2}^2\right) / \left(v_{1k} + v_{2m_2}\right)^2$; player k in group 1 is active and exerts effort $x_{1k} = \sqrt{v_{1k} x_2} - x_2$; all other players in group 1 put no effort.

Proof. Let us investigate the possibility that the highest valuation players free-ride on a player with a lower valuation. Suppose only player k (such that $v_{1k} < v_{11}$) in group 1 is active, then he chooses effort level $x_{1k} = \sqrt{v_{1k} x_2} - x_2$. Therefore, an inactive player j earns payoff

$$\pi_{1j}^{\text{free-ride}} = \left(\sqrt{v_{1k}} - \sqrt{x_2}\right) v_{1j} / \sqrt{v_{1k}}.$$

Let us investigate whether player j has a profitable deviation. As player j is trying to maximize her payoff, she would deviate, if at all, to an effort level x_{1j} such that $x_{1j} = \sqrt{v_{1j} x_2} - x_2$, which is implied by the corresponding first-order condition. As $v_{1(t-1)} \geq v_{1t}$ for $1 < t \leq m_g$, we have $x_{1(t-1)} \geq x_{1t}$. Consequently, no player t in group 1 such that $t > k$ has an incentive to deviate, for player t will not be the best shot in her group if she exerts effort level x_{1t}. Fix player $j < k$ who exerts effort $x_{1j} = \sqrt{v_{1j} x_2} - x_2$, then player j's payoff is

$$\pi_{1j}^{\text{active}} = \left(\sqrt{v_{1j}} - \sqrt{x_2}\right) v_{1j} / \sqrt{v_{1j}} - \sqrt{v_{1j} x_2} + x_2.$$

Hence player j has no incentive to become active if and only if $\pi_{1j}^{\text{free-ride}} \geq \pi_{1j}^{\text{active}}$, that is,

$$\left(\sqrt{v_{1k}} - \sqrt{x_2}\right) v_{1j} / \sqrt{v_{1k}} \geq \left(\sqrt{v_{1j}} - \sqrt{x_2}\right) v_{1j} / \sqrt{v_{1j}} - \sqrt{v_{1j} x_2} + x_2,$$

which is equivalent to

$$(7) \qquad 2\sqrt{v_{1j} v_{1k}} - v_{1j} \geq \sqrt{x_2 v_{1k}}.$$

Note that for any fixed v_{1k}, the right-hand side of Equation (7) approaches zero as $x_2 \to 0$. Thus, condition (7) is satisfied for some $x_2 > 0$ if and only if $2\sqrt{v_{1j} v_{1k}} - v_{1j} > 0$, which is equivalent to

$$(8) \qquad 4v_{1k} \geq v_{1j}.$$

Note also that if condition (7) holds for some $x_2 = r$, then it holds for all $0 < x_2 \leq r$. Fix player k in group 1 such that $v_{1k} < v_{11}$, and for each player $j < k$ define

$$r_{jk} = sup \left\{0 \leq y \leq \bar{x}_2 : 2\sqrt{v_{1j} v_{1k}} - v_{1j} \geq \sqrt{y v_{1k}}\right\},$$

where $\bar{x}_2 = \left(v_{1k} v_{2m_2}^2\right) / \left(v_{1k} + v_{2m_2}\right)^2$. By convention we let $sup\{\emptyset\} = -\infty$. Next, define

$$(9) \qquad r_k = \min_{1 \leq j < k} r_{jk}.$$

By construction, $r_k > 0$ if and only if $4v_{1k} \geq v_{1j}$ for all $j < k$, that is, $4v_{1k} \geq v_{11}$. □

Theorem 2 states that there always exist equilibria in which only one of the highest valuation players in the attacker group is active. In addition, if the highest valuation is not too far from the rest, the highest valuation players may completely free-ride on one of the lower valuation players. Similarly to the case of symmetric valuations, equilibrium efforts of active players are proportional to the dispersion of their valuations.

Finally, let us investigate the existence of coalition-proof equilibria in which the highest valuation players may be inactive. Suppose only player k in group 1 is active (where $v_{1k} < v_{11}$), then there is a unique candidate for the coalition-proof equilibrium ($x_{1j} = 0$ for all $j \neq k$).

$$(10) \qquad x_{1k} = \left(v_{1k}^2 v_{2m_2}\right) / \left(v_{1k} + v_{2m_2}\right)^2,$$

$$(11) \qquad x_2 = \left(v_{1k} v_{2m_2}^2\right) / \left(v_{1k} + v_{2m_2}\right)^2.$$

Consequently player j in group 1, who puts no effort, earns payoff

$$\pi_{1j}^{\text{free-ride}} = \frac{v_{1k}^2 v_{2m_2}}{\left[v_{1k}^2 v_{2m_2} + v_{1k} v_{2m_2}^2\right]} v_{1j} = \frac{v_{1k}}{\left[v_{1k} + v_{2m_2}\right]} v_{1j}.$$

As the valuations are descending within the group, we again conclude that no player $j > k$ wants to become active. Fix player $j < k$; her best option for a deviation is

$$x_{1j} = \left(v_{1j}^2 v_{2m_2}\right) / \left(v_{1j} + v_{2m_2}\right)^2,$$

in which case her payoff is

$$\pi_{1j}^{\text{active}} = \frac{v_{1j}^2 v_{2m_2}}{\left[v_{1j}^2 v_{2m_2} + v_{1j} v_{2m_2}^2\right]} v_{1j} - \frac{v_{1j}^2 v_{2m_2}}{\left(v_{1j} + v_{2m_2}\right)^2}.$$

Hence, player j will free-ride on player k if and only if $\pi_{1j}^{\text{free-ride}} \geq \pi_{1j}^{\text{active}}$, which is equivalent to

$$(12) \qquad v_{1k} \geq v_{1j}^2 / \left(v_{2m_2} + v_{1j}\right).$$

Rewrite condition (12) as $v_{1j}(v_{1k} - v_{1j}) + v_{2m_2} v_{1k} \geq 0$, from which it is evident (as $v_{1j} \geq v_{1k}$) that if condition (12) is satisfied for player 1 in group 1, then it is satisfied for every $j < k$. Thus, all players in group 1, except player k, are better off exerting no effort if and only if

$$(13) \qquad v_{1k} \geq v_{11}^2 / \left(v_{2m_2} + v_{11}\right).$$

Therefore, a coalition-proof equilibrium where only player k in group 1 is active exists if and only if condition (13) is satisfied; such equilibrium is uniquely determined by Equations (10) and (11). Note that there always exists a coalition-proof equilibrium in which a highest valuation player is active. If at least two players in group 1 tie for the highest valuation, then the coalition-proof equilibrium effort levels where a highest valuation player in group 1 is active are unique while any one of the highest valuation players is active. These results are summarized in *Theorem 3*. Figure 1 provides a diagrammatic representation of the Nash and coalition-proof equilibria.

Theorem 3. The coalition-proof equilibria of the attack-and-defense contest with asymmetric valuation are as follows.

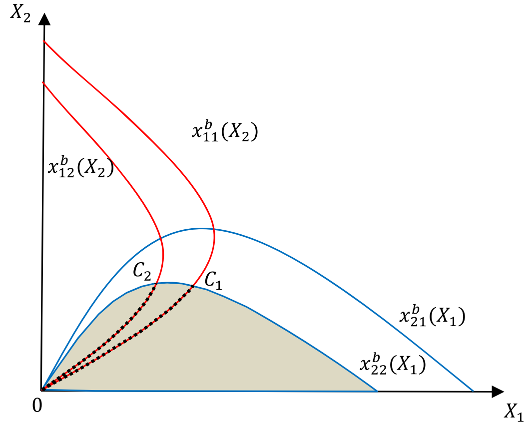
One of the highest valuation players in group 1 is active and exerts effort $x_1 = \left(v_{11}^2 v_{2m_2}\right) / \left(v_{11} + v_{2m_2}\right)^2$, while every player in group 2 exerts effort $x_2 = \left(v_{11} v_{2m_2}^2\right) / \left(v_{11} + v_{2m_2}\right)^2$.

If $v_{1k} \geq v_{11}^2 / \left(v_{2m_2} + v_{11}\right)$ for some k > 1, then, in addition, there exists a coalition-proof equilibrium in which only player k in group 1 is active and exerts effort $x_{1k} = \left(v_{1k}^2 v_{2m_2}\right) / \left(v_{1k} + v_{2m_2}\right)^2$, while every player in group 2 exerts effort $x_2 = \left(v_{1k} v_{2m_2}^2\right) / \left(v_{1k} + v_{2m_2}\right)^2$.

The economic intuition that we provided in Section 2.2 is similar for the case of asymmetric valuations. The equilibrium effort of the active player in group 1 is directly proportional to her own valuation and the dispersion in valuation, when the lowest valuation in group 2 is taken into account. A similar conclusion holds for group 2.

Following Lee (2012) and Chowdhury, Lee, and Sheremeta (2013a), we consider a 2 × 2 case in Figure 1, in which there are two group members in the attacker group and two in the defender group. The curved lines are the best response functions and the shadowed area depicts the equilibria specific to the defender group (due to the weakest link technology). Hence, the intersection of the best response of the attacker group and the shadowed area is the set of Nash equilibria. The dotted lines OC_1 and OC_2 represent exactly that. When player $i(=1, 2)$ is the active player in the attacker group, then OC_i shows the set of Nash equilibria, and C_i turns out to be the unique coalition-proof equilibrium.

FIGURE 1
Equilibria of the Attack-and-Defense Contest



III. CASES WITH AN ADDITIVE IF

It can be argued that the attack effort does not necessarily need to follow a best shot function. To be precise, it may be possible that the effort of one member of the attacker group can add up to (or substitute) another member's effort. Examples of this may be situations in which the attackers such as the CIA or the FBI make mutually exclusive geographical or jurisdictional restrictions. In such a case the attacker group follows a perfectly substitute IF. To incorporate this structure, we retain all the other assumptions in the model the same but replace *Assumption 1* with

ASSUMPTION 1′. *The group effort of group 1 is represented by the sum of effort levels exerted by the players in group 1, that is,* $X_1 = \sum_{i=1}^{m_1} x_{1i}$.

Similar to the analyses above, all the players in group 2 exert the same effort. Following

Baik (2008), the equilibrium effort of group 1 is unique. Only the highest value player(s) in group 1 exert positive effort, whereas all other group members exert zero effort. If there is more than one player with the highest valuation, then the total equilibrium effort of group 1 can be unambiguously determined, but not the individual efforts. This is summarized in *Theorem 4*.

Theorem 4. The attack-and-defense contest under Assumptions 1′, 2, 3, and 4 has a continuum of equilibria. There may be multiple equilibria corresponding to the same effort level of group 2, depending on which players in group 1 are active. In equilibrium every player in group 2 exerts effort $0 < x_2 \le \left(v_{11}v_{2m_2}^2\right) / \left(v_{11} + v_{2m_2}\right)^2$, the highest valuation players in group 1 exert collective effort of $x_1 = \left(v_{11}^2 v_{2m_2}\right) / \left(v_{11} + v_{2m_2}\right)^2$, while all other players in group 1 exert zero effort.

There may exist multiple coalition-proof equilibria. All of them, however, induce the same group efforts. In any coalition-proof equilibrium the highest valuation players in group 1 exert collective effort $x_1 = \left(v_{11}^2 v_{2m_2}\right) / \left(v_{11} + v_{2m_2}\right)^2$, each player in group 2 exerts effort $x_2 = \left(v_{11}v_{2m_2}^2\right) / \left(v_{11} + v_{2m_2}\right)^2$, and all other players in group 1 put no effort.

Proof. Clearly, in any equilibrium, all players in group 2 exert the same effort level x_2 due to the weakest link technology. Let I_1 and I_{1H} denote the index sets of all players in group 1 and the highest valuation players in group 1, respectively. Fix player $k \in I_{1H}$, and suppose by contradiction there exists a player $j \in I_1 \backslash I_{1H}$ whose equilibrium effort level is $x_{1j} > 0$. As before, denote by X_1 equilibrium effort of group 1. Then the first-order conditions imply

$$v_{1k}x_2 / \left(X_1 + x_2\right)^2 = 1,$$

$$v_{1j}x_2 / \left(X_1 + x_2\right)^2 = 1.$$

This leads to a contradiction as $v_{1k} > v_{1j}$. Therefore, only the highest valuation players in group 1 exert a positive effort and $v_{1j}x_2/(X_1 + x_2)^2 < 1$ for all $j \in I_1 \backslash I_{1H}$. The following system characterizes all equilibria (along with $x_{1j} = 0$ for all $j \in I_1 \backslash I_{1H}$).

$$v_{11}x_2 / \left(X_1 + x_2\right)^2 = 1,$$

$$v_{2m_2}X_1 / \left(X_1 + x_2\right)^2 \ge 1.$$

Therefore any strategy profile, such that every player in group 2 exerts effort $0 < x_2 \le \left(v_{11}v_{2m_2}^2\right) / \left(v_{11} + v_{2m_2}\right)^2$, $\sum_{i \in I_{1H}} x_{1i} = \sqrt{v_{11}x_2} - x_2$ and $x_{1j} = 0$ for all $j \in I_1 \backslash I_{1H}$ in group 1, is an equilibrium. To derive coalition-proof equilibria, one needs to solve the system

$$v_{11}x_2 / \left(X_1 + x_2\right)^2 = 1,$$

$$v_{2m_2}X_1 / \left(X_1 + x_2\right)^2 = 1.$$

Note that all the coalition-proof equilibria result in the same outcome with respect to the group efforts: $\left(v_{11}^2 v_{2m_2}\right) / \left(v_{11} + v_{2m_2}\right)^2$ and $\left(v_{11}v_{2m_2}^2\right) / \left(v_{11} + v_{2m_2}\right)^2$ in groups 1 and 2, respectively. This completes the proof. □

Next, it may also be argued that instead of following a weakest link IF, the defenders follow an additive IF. To incorporate this, we retain all the other assumptions from Section 2 unaltered, but *Assumption 2* is replaced with:

ASSUMPTION 2′. *The group effort of group 2 is represented by the sum of effort levels exerted by the players in group 2, that is,* $X_2 = \sum_{i=1}^{m_2} x_{2i}$.

Again, following the analysis in *Theorem 2*, it is easy to show that only one player in group 1 is active in equilibrium. Multiple equilibria exist, and the efforts depend on the identity of the active player in group 1. Once again, similar to Baik (2008), the equilibrium effort of group 2 is unique. Only the highest value player(s) in group 2 exert positive effort and all other group members exert zero effort. If there is more than one player with the highest valuation, then the total equilibrium effort of group 2 can be unambiguously determined, but not the individual efforts. This result is summarized in *Theorem 5*; the proof is similar to the ones of *Theorems 3* and *4*.

Theorem 5. The attack-and-defense contest under Assumptions 1, 2′, 3, and 4 has up to m_1 equilibria. In any equilibrium there is only one active player in group 1. The condition for only player k in group 1 to be active in equilibrium is $v_{1k} \ge v_{11}^2 / \left(v_{21} + v_{11}\right)$. If player k in group 1 is active, then the highest valuation players in group 2 exert collective effort $x_2 = \left(v_{1k}v_{21}^2\right) / \left(v_{1k} + v_{21}\right)^2$, and player k in group 1 exerts $x_{1k} = \left(v_{1k}^2 v_{21}\right) / \left(v_{1k} + v_{21}\right)^2$

while all other players from either group exert zero effort. Coalition-proof equilibria are the same as the Nash equilibria.

IV. DISCUSSION

We analyze a group contest in which one group follows a best shot and the other group follows a weakest link IF. This setting may be viewed as a stylized representation of situations in which one group attacks and the best effort out of the group members determines the strength of the attack, whereas the other group defends and the weakest effort among the group members represents the strength of the defense. This study adds to the attack-and-defense literature as it introduces a group setting in this area of literature for the first time. It also introduces different groups with different IFs in the group contest literature for the first time.

We fully characterize Nash and coalition-proof equilibria and show that under symmetric valuation, the game has a unique coalition-proof equilibrium up to the permutation of the identity of the active player in the attacker group. When the valuations are asymmetric, a wider variety of equilibria is possible. It is always an equilibrium for one of the highest valuation players to be active, but it may also be possible that the active player does not have the highest valuation. In any equilibrium, only one player in the attacker group is active, whereas all the players in the defender group are active and exert the same effort. We also characterize Nash and coalition-proof equilibria for the case in which one group follows a perfectly substitute IF, whereas the other group follows either a best shot or a weakest link IF. A remarkable feature of the coalition-proof equilibria is that the equilibrium effort levels are proportional to the dispersion in valuation.

The results suggest that all the defenders (the colluding firms or the terrorist group members), due to the perfectly complementary nature of their actions, participate in defense activity. However, they exert resources only according to the strength of the weakest member of the group. If the attackers (the antitrust or security authorities) follow a perfectly substitute collective action, then all the weaker or less efficient group members free-ride on the strongest one. However, if the collective action has the nature of a best shot, and the valuation or efficiency of the strongest group member is not too high, then other group members may exert resources instead.

It can be shown that if one introduces budget constraint as in Baik (2008), it does not affect participation in the contest: only one player in group 1 is active, while all players in group 2 exert the same effort (the details are available in the *Online Appendix*). This is in contrast with Baik (2008), who finds that wider participation is possible with budget constraints. There are several other ways—both in theory and in application—to extend the current analysis. First, it may be possible to employ a generic CES IF and vary the elasticity of substitution across groups to achieve a very general solution in this area of investigation. The analyses can be extended to more than two groups and with the employment of more than two IFs, a first attempt of which can be observed in Lee and Song (2014). Finally, it is also possible to test the theoretical predictions and investigate whether any particular equilibrium is focal (Sheremeta 2011), or whether within and across groups design tools such as punishment (Abbink et al. 2010) or communication (Cason, Sheremeta, and Zhang 2012) affect subject behavior by implementing them in the laboratory. However, each of these issues is beyond the scope of this study and we leave them for future research.

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SUPPORTING INFORMATION

Additional Supporting Information may be found in the online version of this article:

Appendix S1. Budget Constraints.