There are only finitely many distance-regular graphs of fixed valency greater than two

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Abstract

In this paper we prove the Bannai-Ito conjecture, namely that there are only finitely many distance-regular graphs of fixed valency greater than two.

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Contents

1 Introduction

A finite, connected graph Γ with vertex set $V(\Gamma)$ and path-length distance d is said to be *distance-regular* if, for any vertices $x, y \in V(\Gamma)$ and any integers $1 \leq i, j \leq \max\{d(z,w) : z, w \in V(\Gamma)\}$, the number of vertices at distance i from x and distance j from y depends only on i, j and $d(x,y)$, independent of the choice of x and y. Many distance-regular graphs arise from classical objects, such as the Hamming graphs, the Johnson graphs, the Grassmann graphs, the bilinear forms graphs, and the dual polar graphs amongst others. In particular, distance-regular graphs give a framework to study these classical objects from a combinatorial point of view. In addition, distance-regular graphs and association schemes give an algebraic-combinatorial framework to study, for example, codes and designs [12, 18].

In their 1984 book, E. Bannai and T. Ito conjectured that there are only finitely many distance-regular graphs of fixed valency greater than two (cf. [5, p.237]). In this paper we prove that their conjecture holds:

Theorem 1.1 *There are only finitely many distance-regular graphs of fixed valency greater than two.*

History
A distance-transitive graph is a connected graph Γ such that for every four (not necessarily distinct) vertices x, y, u, v in $V(\Gamma)$ with $d(x,y) = d(u,v)$, there exists an automorphism τ of Γ such that $\tau(x) = u$ and $\tau(y) = v$ both hold. It is straight-forward to see that distance-transitive graphs are distance-regular graphs. In [14, 15], P. J. Cameron, C. E. Praeger, J. Saxl and G. M. Seitz proved that there are only finitely many finite distance-transitive graphs of fixed valency greater than two. They did this by applying Sims' conjecture [33] for finite permutation groups (i.e. that there exists an integral function f such that $|G_x| \leq f(d_{G_x})$ holds, where, for G a primitive permutation group acting on a finite set Ω, G_x denotes the stabilizer of x, $x \in \Omega$, and d_{G_x} denotes the length of any G_x-orbit in $\Omega \setminus \{x\}$), which they also showed to hold by using the classification of the finite simple groups (in [15] they gave a proof without many details, and in [14] Cameron worked out a detailed proof with an explicit diameter bound).

Note that for small diameter there are many distance-regular graphs which are not distance-transitive. On the other hand there are only five families of distance-regular but not distance-transitive graphs known with unbounded diameter, namely the Doob graphs [19] (see also [12, p.262]), the quadratic forms graphs [20] (see also [12, p.290]), the Hemmeter graphs [13] and the Ustimenko graphs [37] (for both, see also [12, p.279]) and the twisted Grassmann graphs [17]. Any member of the first four families is vertex-transitive, whereas the twisted Grassmann graphs have exactly two orbits under the full automorphism group [17].

The first class of distance-regular graphs for which the Bannai-Ito conjecture was shown is the class of regular generalized n-gons. Feit and Higman [21] (cf. [12, Theorem 6.5.1]) showed that a regular generalized n-gon has either valency 2 or $n \in \{3, 4, 6, 8, 12\}$. In addition, R. M. Damerell, and E. Bannai and T. Ito have independently shown that there are only finitely many Moore graphs with valency at least three [4, 16].

In the series of papers [6, 7, 8, 9], E. Bannai and T. Ito showed that their conjecture holds for valencies $k = 3, 4$, as well as for the special class of bipartite distance-regular graphs. In [27] and [28], J. H. Koolen and V. Moulton also showed that the conjecture holds for distance-regular graphs of fixed valency $k = 5, 6$ or 7, and for triangle-free distance-regular graphs of fixed valency $k = 8, 9$ or 10. More recently, in [3], together with S. Bang, they showed that the Bannai-Ito conjecture holds for regular near polygons and geodetic distance-regular graphs.

The proof for the Bannai-Ito conjecture that we present in this paper builds upon many of the concepts and ideas developed in [2, 3, 27, 28].

Structure of the paper
In Section 2, we present some definitions and previous results concerning distance-regular graphs and associated sequences and related structures, and in Section 3 we present some properties of certain generalizations of these sequences. In Section 4, we state without proof the key result of the paper (Theorem 4.2) and used this to prove Theorem 1.1. We also present an outline proof of Theorem 4.2, before proving it in Sections 5 to 9. In Section 10, we will present an application of Theorem 1.1 to distance-regular graphs of order (s, t). We conclude in Section 11 by discussing some possible future directions.

2 Preliminaries

In this section, we review some of the well-known theory of Christoffel numbers for orthogonal polynomials, interlacing and distance-regular graphs that will be used in this paper. We refer the reader to [5], [12] and [36] for more details.

2.1 Christoffel Numbers

Let L_1 be the arbitrary tridiagonal matrix defined by

$$L_1 := \begin{pmatrix} \alpha_0 & \beta_0 & & & & & & \\ \gamma_1 & \alpha_1 & \beta_1 & & & & & \\ & \gamma_2 & \alpha_2 & \beta_2 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \gamma_i & \alpha_i & \beta_i & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \gamma_{n-1} & \alpha_{n-1} & \beta_{n-1} \\ & & & & & & \gamma_n & \alpha_n \end{pmatrix}, \tag{1}$$

where $\alpha_i \geq 0$, $\beta_{i-1}, \gamma_i > 0$ are real numbers with $\alpha_0 = \gamma_0 = \beta_n = 0$ and $\gamma_1 = 1$, and $\alpha_i + \beta_i + \gamma_i = \beta_0$ holding for all $1 \leq i \leq n$. Let $v_i(x)$ $(0 \leq i \leq n+1)$ be the polynomials defined recursively by the equations

$$v_0(x) := 1, \; v_1(x) := x, \tag{2}$$

$$x v_i(x) = \beta_{i-1} v_{i-1}(x) + \alpha_i v_i(x) + \gamma_{i+1} v_{i+1}(x) \quad (1 \leq i \leq n-1), \tag{3}$$

$$v_{n+1}(x) = (x - \alpha_n) v_n(x) - \beta_{n-1} v_{n-1}(x), \tag{4}$$

and $F_i(x)$ $(0 \le i \le n)$ be the monic polynomials defined by setting $F_0(x) := 1$, $F_1(x) := x+1$ and

$$F_i(x) := \gamma_2 \cdots \gamma_i (v_0(x) + v_1(x) + \cdots + v_i(x)) \quad (2 \le i \le n).$$

Note that for each $2 \le i \le n$, the polynomial $F_i(x)$ satisfies the recurrence relation

$$F_i(x) = (x - \beta_0 + \beta_{i-1} + \gamma_i) F_{i-1}(x) - \beta_{i-1}\gamma_{i-1} F_{i-2}(x). \tag{5}$$

Moreover, by (2)–(5), for each $0 \le i \le n$, the polynomials $v_i(x)$ and $F_i(x)$ have degree i and have exactly i distinct real roots in the closed interval $[-\beta_0, \beta_0]$ (cf. [36, Theorem 3.3.1]). Note that the polynomial $(x - \beta_0)F_n(x)$ is the minimal polynomial of the matrix L_1.

Now, let $\kappa := \beta_0$ and define

$$\kappa_i \;:=\; v_i(\kappa) \quad (0 \le i \le n), \text{ and} \tag{6}$$

$$u_i(x) \;:=\; \frac{v_i(x)}{\kappa_i} \quad (0 \le i \le n). \tag{7}$$

Put $\kappa := \kappa_1$. Then the polynomials $u_i(x)$ $(0 \le i \le n)$ satisfy

$$u_i(\kappa) = 1 \ (0 \le i \le n); \tag{8}$$

$$u_0(x) = 1,\ u_1(x) = \frac{x}{\kappa},\ xu_i(x) = \gamma_i u_{i-1}(x) + \alpha_i u_i(x) + \beta_i u_{i+1}(x) \ (1 \le i \le n). \tag{9}$$

The sequence $(u_i(x))_{i=0}^n$ is called the *standard sequence* of L_1, and if θ is an eigenvalue of L_1, then the column vector $(u_0(\theta), u_1(\theta), \ldots, u_n(\theta))^T$ is a right eigenvector of L_1 associated to θ, by (9).

Note also that it follows by (7) that, for each eigenvalue θ of the matrix L_1, the equation

$$\sum_{i=0}^{n} \frac{v_i^2(\theta)}{\kappa_i} = \sum_{i=0}^{n} \kappa_i u_i^2(\theta) \tag{10}$$

holds.

Now, let $\beta_0 = \theta_0 > \theta_1 > \theta_2 > \cdots > \theta_n$ be the eigenvalues of L_1 and, for $i = 0, 1, \ldots, n$, define

$$m_i := \left(\frac{\sum_{j=0}^{n} \kappa_j}{\sum_{j=0}^{n} \frac{v_j^2(\theta_i)}{\kappa_j}} \right) \tag{11}$$

as well as the symmetric bilinear form $(\cdot,\cdot)$ on the polynomial ring $\mathbb{R}[x]$ by

$$(f, g) := \sum_{i=0}^{n} m_i f(\theta_i) g(\theta_i).$$

Then, $(v_i, v_i) \ne 0$ holds for all $0 \le i \le n$, and $(v_i, v_j) = (v_i, v_i)\delta_{i,j}$ holds for all $0 \le i, j \le n$, where $\delta_{i,j}$ is the Kronecker delta function on $\mathbb{N}_0 \times \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of non-negative integers. In particular, it follows that $(v_i)_{i=0}^n$ is a sequence of orthogonal polynomials with respect to $(\cdot,\cdot)$. Note that within the theory of orthogonal polynomials, the numbers m_i are referred to as the *Christoffel numbers* of the sequence $(v_i)_{i=0}^n$ ([36, Theorem 3.4.1], [5, p.201]). Analogously, we call the number m_i as defined in (11), the *Christoffel number* of L_1 associated with θ_i.

2.2 Interlacing

We now recall two results stated in [2] that provide us with some interrelationships between the eigenvalues of the matrix L_1 as defined in (1). The first generalizes the well-known Interlacing Theorem [12, Theorem 3.3.1], from which it immediately follows.

Lemma 2.1 *Suppose that A is a real $n \times n$ matrix for which there exists a non-singular diagonal matrix Q such that the matrix $Q^{-1}AQ$ is real and symmetric. If $\eta_1 \le \ldots \le \eta_n$ are the eigenvalues of A and $\theta_1 \le \ldots \le \theta_{n-1}$ are the eigenvalues of the matrix obtained by removing the ith row and ith column of A, with $i \in \{1, \ldots, n\}$, then*

$$\eta_1 \le \theta_1 \le \eta_2 \le \ldots \le \eta_{n-1} \le \theta_{n-1} \le \eta_n.$$

Note that in [2, Lemma 3.1] the condition that Q has to be a diagonal matrix was omitted. Without this condition the lemma is not true.

In particular, since $\beta_i\gamma_{i+1} > 0$ $(0 \le i \le n-1)$ and L_1 is tridiagonal, it follows that L_1 satisfies the conditions on A given in Lemma 2.1, and therefore the eigenvalues of L_1 must satisfy the inequalities given in this lemma.

The second result guarantees the existence of eigenvalues of L_1 lying within certain limits.

Lemma 2.2 *([2, Theorem 3.2])*
Let $\alpha_i, \beta_i, \gamma_i$ $(0 \le i \le n)$ be non-negative integers satisfying $\alpha_0 = \gamma_0 = \beta_n = 0$, $\beta_{i-1}, \gamma_i > 0$, $\alpha_i + \beta_i + \gamma_i = \beta_0$, $\beta_{i-1} \ge \beta_i$ and $\gamma_i \ge \gamma_{i-1}$ for all $1 \le i \le n$, and let L_1 be the tridiagonal matrix as defined in (1). For each $1 \le i \le n-1$, let $\ell(i) := |\{j : (\gamma_j, \alpha_j, \beta_j) = (\gamma_i, \alpha_i, \beta_i), 1 \le j \le n-1\}|$. Then the following statements hold.
(i) If $\ell(i) \ge 2$ then there is an eigenvalue θ of L_1 with

$$\alpha_i + 2\sqrt{\beta_i\gamma_i}\cos\left(\frac{2\pi}{\ell(i)+1}\right) \le \theta < k.$$

(ii) If $\ell(i) \ge 3$ then there is an eigenvalue θ of L_1 with

$$\alpha_i + 2\sqrt{\beta_i\gamma_i}\cos\left(\frac{j\pi}{\ell(i)+1}\right) \le \theta \le \alpha_i + 2\sqrt{\beta_i\gamma_i}\cos\left(\frac{(j-2)\pi}{\ell(i)+1}\right),$$

for all $j = 3, \ldots, \ell(i)$.

2.3 Distance-Regular Graphs

We now review some basic definitions and results concerning distance-regular graphs.

For Γ a finite, connected graph, denote by $d(x, y)$ the path-length distance between any two vertices x, y in the vertex set $V(\Gamma)$ of Γ (i.e. the length of a shortest path), and by $D = D_\Gamma$ the diameter of Γ (i.e. the maximum distance between any two vertices of Γ). For any $y \in V(\Gamma)$, let $\Gamma_i(y)$ be the set of vertices in Γ at distance precisely i from y, where $i \in \mathbb{N}_0$ is a non-negative integer not exceeding D. In addition, define $\Gamma_{-1}(y) = \Gamma_{D+1}(y) := \emptyset$.

Following [12, p.126], a finite, connected graph Γ is called a *distance-regular graph* if there are integers b_i, c_i, $i = 0, 1, \ldots, D$, such that, for any two vertices $x, y \in V(\Gamma)$ at distance $i = d(x, y)$, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$. In particular, Γ is regular with valency $k := b_0$. The numbers c_i, b_i and

$$a_i := k - b_i - c_i \quad (0 \le i \le D)$$

(i.e. the number of neighbors of y in $\Gamma_i(x)$ for $d(x, y) = i$) are called the *intersection numbers* of Γ. Note that $b_D = c_0 = a_0 := 0$ and $c_1 = 1$. In addition, we define $k_i := |\Gamma_i(y)|$ for any vertex $y \in V(\Gamma)$, $i = 0, 1, \ldots, D$. This definition for distance-regular graphs is easily seen to be equivalent to the one given in the introduction.

For Γ a distance-regular graph as above, we define

$$\mathcal{T}_\Gamma := \Big((c_i, a_i, b_i)\Big)_{i=1}^{D} \tag{12}$$

and we let

$$\mathcal{G}_\Gamma := \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1} \tag{13}$$

denote the (necessarily unique) maximal length subsequence of $\mathcal{T}_\Gamma$ for which the i th term of $\mathcal{G}_\Gamma$ is not equal to the $(i+1)$ th term of $\mathcal{G}_\Gamma$ for all $1 \le i \le D-1$. In addition, we define the numbers

$$\begin{aligned} \mathtt{h} = \mathtt{h}_\Gamma &:= |\{j \,:\, (c_j, a_j, b_j) = (c_1, a_1, b_1),\ 1 \le j \le D-1\}|\,, \text{ and} && (14)\\ \mathtt{t} = \mathtt{t}_\Gamma &:= |\{j \,:\, (c_j, a_j, b_j) = (b_1, a_1, c_1),\ \mathtt{h} < j \le D-1\}| && (15) \end{aligned}$$

which are called the *head* and the *tail* of Γ, respectively. Note that by [2, Lemma 2.1], it follows that tail $\mathtt{t}$ satisfies the following :

$$\mathtt{t} \le \mathtt{h} \text{ and, if } \mathtt{t} \ge 1 \text{ then } (c_{D-\mathtt{t}}, a_{D-\mathtt{t}}, b_{D-\mathtt{t}}) = \cdots = (c_{D-1}, a_{D-1}, b_{D-1}) = (b_1, a_1, 1). \tag{16}$$

2.3.1 Intersection Numbers

For the rest of Section 2, we suppose that Γ is a distance-regular graph with valency $k \ge 3$, diameter $D \ge 2$, intersection numbers a_i, b_i, c_i, $0 \le i \le D$ and $\mathcal{G}_\Gamma = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$.

In [12, Proposition 4.1.6] and [2, Lemma 2.1 (ii)], it is shown that the following inequalities always hold :

$$k = b_0 > b_1 \ge b_2 \ge \cdots \ge b_{D-1} > b_D = 0 \text{ and } 1 = c_1 \le c_2 \le \cdots \le c_D \le k, \tag{17}$$

$$a_i \ge a_1 + 1 - \min\{b_i, c_i\} \;\; (1 \le i \le D-1). \tag{18}$$

In particular, it follows that for every term $(\gamma_i, \alpha_i, \beta_i)$ in $\mathcal{G}_\Gamma$, $\beta_i \ge \beta_{i+1}$ and $\gamma_i \le \gamma_{i+1}$ hold. For each $1 \le i \le g$, define

$$\begin{aligned} s(i) = s_\Gamma(i) &:= \min\{j \,:\, (c_j, a_j, b_j) = (\gamma_i, \alpha_i, \beta_i)\,,\, 1 \le j \le D-1\}, && (19)\\ \ell(i) = \ell_\Gamma(i) &:= |\{j \,:\, (c_j, a_j, b_j) = (\gamma_i, \alpha_i, \beta_i),\ 1 \le j \le D-1\}|, && (20) \end{aligned}$$

and define $s(g+1) = D$. Note that $s(1) = 1$ $\ell(1) = \mathtt{h}_\Gamma$, $\ell(g+1) = 1$, and that $s(i+1) - s(i) = \ell(i)$ holds for all $1 \leq i \leq g$.

2.3.2 Diameter Bounds

The following result is originally due to A. A. Ivanov [26] (cf. [12, Theorem 5.9.8]). Note that $\mathbb{N}$ denotes the set of positive integers.

Theorem 2.3 *(A. A. Ivanov's Diameter Bound)*
Let $k \geq 3$ be an integer. Then there is a function $F : \mathbb{N} \to \mathbb{N}$ so that, for all distance-regular graphs Γ with valency k, diameter D_Γ, and head $\mathtt{h}_\Gamma$, the inequality

$$D_\Gamma \leq F(k)\ \mathtt{h}_\Gamma$$

holds.

Note that it was also shown in [26] (cf. [12, Theorem 5.9.8]) that one can in fact take $F(k) = 4^k$ in the last theorem.

Now, in order to show that there are only finitely many of distance-regular graphs Γ with fixed valency $k \geq 3$, it suffices to show that the diameter D_Γ of any such graph is bounded above by some function $f : \mathbb{N} \to \mathbb{N}$ depending only on k, since $|V(\Gamma)| \leq 1 + \sum_{i=1}^{D_\Gamma} k(k-1)^{i-1}$. Thus, in view of Theorem 2.3, it also suffices to show that the head $\mathtt{h}_\Gamma$ is bounded above by some function g in k. In particular, the following result also holds (as we can take $g(k)$ to be a constant function).

Corollary 2.4 *Suppose that $k \geq 3$ and $C \geq 1$ are positive integers. Then there are only finitely many distance-regular graphs Γ with valency k and head $\mathtt{h}_\Gamma \leq C$.*

2.3.3 Eigenvalues of Distance-Regular Graphs

The tridiagonal matrix $L_1 = L_1(\Gamma)$ associated to Γ is defined by

$$L_1 := \begin{pmatrix} 0 & b_0 & & & & & & \\ c_1 & a_1 & b_1 & & & & & \\ & c_2 & a_2 & b_2 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & c_i & a_i & b_i & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & c_{D-1} & a_{D-1} & b_{D-1} \\ & & & & & & c_D & a_D \end{pmatrix},$$

and $\theta \in \mathbb{R}$ is an eigenvalue of Γ if θ is an eigenvalue of $L_1(\Gamma)$ ([12, p.129]). Note that any distance-regular graph Γ with diameter $D = D_\Gamma$ has exactly $D+1$ distinct eigenvalues ([12, p.128]).

Moreover, if θ is an eigenvalue of Γ, then $(u_0, u_1, \ldots, u_D)^T$ is called the standard sequence of Γ associated with θ, which is a right eigenvector of $L_1(\Gamma)$ associated with eigenvalue θ, and the multiplicity $m(\theta)$ of θ is given by

$$m(\theta) = \frac{|V(\Gamma)|}{\sum_{i=0}^{D} k_i u_i^2(\theta)}. \tag{21}$$

This equation is known as *Biggs' formula* ([10, Theorem 21.4]). Note that in view of Equations (10) and (11) it follows by this last formula that the multiplicity of eigenvalue θ_i of Γ is equal to the Christoffel number m_i of $L_1(\Gamma)$.

3 Graphical Sequences

In this section, we define graphical sequences and tridiagonal sequences. Note that these are similar (but not identical) to the ones presented in [3]. The definition for these sequences is motivated by the sequences $\mathcal{G}_\Gamma$ and $\mathcal{T}_\Gamma$ associated to Γ a distance-regular graph that were presented in the last section.

For integers $\kappa \geq 3$ and $\lambda \geq 0$ with $\lambda \leq \kappa - 2$, define

$$V_{\kappa,\lambda} := \{(\gamma, \alpha, \beta) \in \mathbb{N}_0^3 \,:\, \beta, \gamma \geq 1,\ \gamma + \alpha + \beta = \kappa \text{ and } \alpha \geq \max\{\lambda + 1 - \beta, \lambda + 1 - \gamma\}\}.$$

Definition 3.1 *With κ, λ and $V_{\kappa,\lambda}$ as just defined above, a sequence $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ of distinct terms in $\mathbb{N}_0^3$ is called a (κ, λ)-*graphical sequence *if it satisfies the following conditions:*

(G0) $(\gamma_i, \alpha_i, \beta_i) \in V_{\kappa,\lambda}$ $(1 \leq i \leq g)$,

(G1) $(\gamma_1, \alpha_1, \beta_1) = (1, \lambda, \kappa - \lambda - 1)$,

(G2) $\beta_i \geq \beta_{i+1}$ $(1 \leq i \leq g - 1)$ *and* $\gamma_i \leq \gamma_{i+1}$ $(1 \leq i \leq g)$,

(G3) $\beta_{g+1} = 0$ *and* $\gamma_{g+1} + \alpha_{g+1} = \kappa$.

Let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence and let $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ be a function with $\ell(g + 1) = 1$. For each $1 \leq i \leq g + 1$, define $s_\ell(i) = s(i)$ by

$$\begin{aligned} s(1) &:= 1, \\ s(i) &:= 1 + \sum_{j=1}^{i-1} \ell(j) \quad (2 \leq i \leq g + 1). \end{aligned} \tag{22}$$

Definition 3.2 *With $\mathcal{G}$, ℓ and s as just defined above, the sequence of triples $\mathcal{T} = \mathcal{T}(\mathcal{G}, \ell) := \Big((c_m, a_m, b_m)\Big)_{m=1}^{s(g+1)}$ given by putting, for each $1 \leq i \leq g + 1$,*

$$(c_{s(i)+j}, a_{s(i)+j}, b_{s(i)+j}) = (\gamma_i, \alpha_i, \beta_i) \quad (0 \leq j \leq \ell(i) - 1)$$

is called the (κ, λ)-tridiagonal sequence (associated with $\mathcal{G}$ and ℓ).

Given $\mathcal{T} = \mathcal{T}(\mathcal{G}, \ell)$ as in this last definition, we define the *head* $\mathtt{h} = \mathtt{h}_{\mathcal{T}}$, the *tail* $\mathtt{t} = \mathtt{t}_{\mathcal{T}}$ and the *diameter* $D = D_{\mathcal{T}}$ of $\mathcal{T}$ to be

$$\begin{aligned} \mathtt{h}_{\mathcal{T}} &:= \ell(1), && (23)\\ \mathtt{t}_{\mathcal{T}} &:= \left|\{\mathtt{h}_{\mathcal{T}} < i \leq s(g+1) \, : \, (c_i, a_i, b_i) = (\kappa - \lambda - 1, \lambda, 1)\}\right|, \text{ and} && (24)\\ D_{\mathcal{T}} &:= s(g+1), && (25) \end{aligned}$$

respectively. Note that $\mathtt{h}$ and $\mathtt{t}$ satisfy

$$\mathtt{h} \geq \mathtt{t} \text{ and, if } \mathtt{t} \geq 1 \text{ then } (c_{D-\mathtt{t}}, a_{D-\mathtt{t}}, b_{D-\mathtt{t}}) = \cdots = (c_{D-1}, a_{D-1}, b_{D-1}) = (b_1, a_1, 1)$$

(see (16)).

Note that if Γ is a distance-regular graph, with diameter D_Γ and $\mathcal{T}_\Gamma = \Big((c_m, a_m, b_m)\Big)_{m=1}^{D_\Gamma}$, then, referring to (13) and (20), it follows that the sequence $\mathcal{G}_\Gamma$ is a (b_0, a_1)-graphical sequence and that $\mathcal{T}_\Gamma$ is the (b_0, a_1)-tridiagonal sequence $\mathcal{T}(\mathcal{G}_\Gamma, \ell_\Gamma)$.

Now, given a (κ, λ)-graphical sequence $\mathcal{G}$, function ℓ and the (κ, λ)-tridiagonal sequence $\mathcal{T} = \mathcal{T}(\mathcal{G}, \ell) = \Big((c_m, a_m, b_m)\Big)_{m=1}^{D}$ as in Definitions 3.1 and 3.2, we define the tridiagonal matrix $L_1(\mathcal{T})$ associated to $\mathcal{T}$ by

$$L_1(\mathcal{T}) := \begin{pmatrix} 0 & \kappa & & & & & & \\ c_1 & a_1 & b_1 & & & & & \\ & c_2 & a_2 & b_2 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & c_i & a_i & b_i & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & c_{D-1} & a_{D-1} & b_{D-1} \\ & & & & & & c_D & a_D \end{pmatrix}.$$

It follows by the results in Section 2.1, that the tridiagonal matrix $L_1(\mathcal{T})$ has exactly $D+1$ distinct eigenvalues, $\kappa = \theta_0 > \theta_1 > \theta_2 > \cdots > \theta_D$, say, which we call *the eigenvalues of* $\mathcal{T}$ and denote by

$$\mathcal{E}_{\mathcal{T}} := \{\theta_i \, : \, 0 \leq i \leq D\}. \tag{26}$$

Note that applying formulae (3) and (6) to the matrix $L_1(\mathcal{T})$ we obtain, for each $1 \leq i \leq g+1$,

$$\kappa_{s(i)+j} = \frac{\kappa}{\beta_i}\left(\frac{\beta_i}{\gamma_i}\right)^j \prod_{t=1}^{i-1}\left(\frac{\beta_t}{\gamma_t}\right)^{\ell(t)} = \frac{\kappa\, b_1 \cdots b_{s(i)+j-1}}{c_1 c_2 \cdots c_{s(i)+j}} \qquad (j = 0, \ldots, \ell(i) - 1). \tag{27}$$

We define the *Christoffel numbers* of $\mathcal{T}$ to be the Christoffel numbers associated with $L_1(\mathcal{T})$ (cf. Section 2.1).

Now, in case $L_1 = L_1(\mathcal{T}_\Gamma)$ for a distance-regular graph Γ then, for any θ, θ' distinct algebraic conjugate eigenvalues of Γ, the multiplicities of θ and θ' are equal ([5, Proposition III.1.5]). Hence so are the corresponding Christoffel numbers, which implies that $\sum_{i=0}^{D} \kappa_i u_i^2(\theta) = \sum_{i=0}^{D} \kappa_i u_i^2(\theta')$ holds.
Motivated by this fact, we will be interested in (κ, λ)-tridiagonal sequences $\mathcal{T}$ that satisfy the following key property:

(AC) Any two eigenvalues of $\mathcal{T}$ which are algebraically conjugate (over $\mathbb{Q}$) have the same Christoffel numbers.

We conclude this section with a useful result concerning graphical sequences. Suppose that $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ is a (κ, λ)-graphical sequence for some integers $\kappa \geq 3$ and $0 \leq \lambda \leq \kappa - 2$, as in Definition 3.1. For each $1 \leq i \leq g$ we define the *i th right* and *i th left guide point* by

$$\mathfrak{R}_i = \mathfrak{R}_i(\mathcal{G}) := \alpha_i + 2\sqrt{\beta_i\gamma_i} \quad \text{and} \quad \mathfrak{L}_i = \mathfrak{L}_i(\mathcal{G}) := \alpha_i - 2\sqrt{\beta_i\gamma_i} \quad (1 \leq i \leq g) \tag{28}$$

respectively. In addition, we put $\mathfrak{R}_{\max} = \mathfrak{R}_{\max}(\mathcal{G}) := \max\{\mathfrak{R}_i : 1 \leq i \leq g\}$.
Moreover, for each $1 \leq i \leq g$, we define the *i th guide interval* to be the open interval

$$I_i = I_i(\mathcal{G}) := (\mathfrak{L}_i, \mathfrak{R}_i). \tag{29}$$

The following lemma is a slight extension of Lemma 3.1 in [3]. We provide a proof of it for the sake of completeness. Note that a sequence $r_1, \ldots, r_n$ of real numbers is called *unimodal* if there exists some $1 \leq t \leq n$ satisfying $r_1 \leq r_2 \leq \cdots \leq r_t$ and $r_t \geq r_{t+1} \geq \cdots \geq r_n$.

Lemma 3.3 *Suppose that $\kappa \geq 3$ and $\lambda \geq 0$ are integers with $\lambda \leq \kappa - 2$, and that $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ is a (κ, λ)-graphical sequence. Then the following hold.*

(i) The inequality $\mathfrak{R}_i \geq \mathfrak{R}_1$ holds for all $1 \leq i \leq g$, with equality holding if and only if $(\gamma_i, \alpha_i, \beta_i) \in \{(1, \lambda, \kappa - \lambda - 1), (\kappa - \lambda - 1, \lambda, 1)\}$.

(ii) For any $2 \leq i \leq g$, if $\beta_i \geq \gamma_i$ then $\mathfrak{R}_{i-1} < \mathfrak{R}_i$.

(iii) For any $2 \leq i \leq g - 1$, if $\beta_i \leq \gamma_i$ then $\mathfrak{R}_{i+1} < \mathfrak{R}_i$.

In particular, by (ii) and (iii), it follows that the sequence $(\mathfrak{R}_i)_{i=1}^{g}$ is unimodal.

Proof: First note that by (G0) and (G1) in Definition 3.1, for each $1 \leq i, j \leq g$, we have

$$\mathfrak{R}_i - \mathfrak{R}_j = (\sqrt{\beta_j} - \sqrt{\gamma_j})^2 - (\sqrt{\beta_i} - \sqrt{\gamma_i})^2, \text{ and} \tag{30}$$
$$\beta_1 \geq \gamma_i. \tag{31}$$

Now, to see that (i) holds, note that by (G0),(G2) and (31), $\sqrt{\beta_1} - 1 \geq \left|\sqrt{\beta_i} - \sqrt{\gamma_i}\right|$ holds. Hence $\mathfrak{R}_i \geq \mathfrak{R}_1$ holds in view of (30) with $j = 1$. Moreover, equality holds if and only if $(\gamma_i, \alpha_i, \beta_i) = (\gamma_1, \alpha_1, \beta_1)$ if $\beta_i \geq \gamma_i$ and $(\gamma_i, \alpha_i, \beta_i) = (\beta_1, \alpha_1, \gamma_1)$ if $\gamma_i \geq \beta_i$.
To complete the proof of the lemma, note that (ii) and (iii) follow from (30) and (G2), since $\beta_{i-1} \geq \beta_i \geq \gamma_i \geq \gamma_{i-1}$ and $\gamma_{i+1} \geq \gamma_i \geq \beta_i \geq \beta_{i+1}$ hold for (ii) and (iii), respectively. ∎

4 A Key Result

In this section we will state without proof a key result (Theorem 4.2) that we will then use to prove the main result of this paper (Theorem 1.1). We will then give a sketch a proof of this key result which we will prove in Sections 5 to 9, inclusive.

For $w = (w_i)_{i=1}^n$ any sequence, we put

$$\underline{w} := \{w_i : 1 \leq i \leq n\}, \tag{32}$$

i.e. the set consisting of all distinct terms in W. To state Theorem 4.2, we will require the following key definition:

Definition 4.1 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$. A (κ, λ)-*quadruple *is a quadruple $(\mathcal{G}, \Delta; L, \ell)$ such that*
(i) $\mathcal{G} := \Big(\delta_i := (\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ is a (κ, λ)-graphical sequence (cf. Definition 3.1),
(ii) $\Delta = (\delta_{i_p})_{p=1}^{\tau}$ is a subsequence of $\mathcal{G}$ in which $(1, \lambda, \kappa - \lambda - 1) \in \underline{\Delta}$ (i.e., $i_1 = 1$) and $(\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1}) \notin \underline{\Delta}$, and
(iii) $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ and $L : \{1, \ldots, g+1\} \setminus \{i_1, \ldots, i_\tau\} \to \mathbb{N}$ are functions with $\ell(g+1) = 1$ and $L(i) = \ell(i)$ for all $i \in \{1, \ldots, g+1\} \setminus \{i_1, \ldots, i_\tau\}$.

Theorem 4.2 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$, and let $\mathcal{G} = \Big(\delta_i := (\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence. Suppose that $\Delta = (\delta_{i_p})_{p=1}^{\tau}$ is a subsequence of $\mathcal{G}$ with $(1, \lambda, \kappa - \lambda - 1) \in \underline{\Delta}$ and $(\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1}) \notin \underline{\Delta}$, and $L : \{1, \ldots, g+1\} \setminus \{i_1, \ldots, i_\tau\} \to \mathbb{N}$ is a function. Suppose $\epsilon > 0$ is a real number, $C := C(\kappa) > 0$ is a constant, and $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ is any function for which $(\mathcal{G}, \Delta; L, \ell)$ is a (κ, λ)-quadruple and the associated (κ, λ)-tridiagonal sequence $\mathcal{T} = \mathcal{T}(\mathcal{G}, \ell)$ satisfies*
(i) Property (AC),
(ii) $D_{\mathcal{T}} \leq C\mathtt{h}_{\mathcal{T}}$, and
(iii) $D_{\mathcal{T}} - (\mathtt{h}_{\mathcal{T}} + \mathtt{t}_{\mathcal{T}}) > \epsilon \mathtt{h}_{\mathcal{T}}$,
where $\mathtt{h}_{\mathcal{T}}$, $\mathtt{t}_{\mathcal{T}}$ and $D_{\mathcal{T}}$ are as defined in (23)–(25), respectively.
Then, there exist positive constants $F := F(\kappa, \mathcal{G}, \Delta, L)$ and $H := H(\kappa, \lambda, \epsilon, \mathcal{G}, \Delta, L)$ such that if $\ell(i_p) > F$ holds for all $1 \leq p \leq \tau$, then $\mathtt{h}_{\mathcal{T}} \leq H$ holds.

We will now use Theorem 4.2 to prove Theorem 1.1, the main theorem of this paper. To do this, we will make use of the following result:

Proposition 4.3 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$, and let $\mathcal{G} = \Big(\delta_i := (\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence. Suppose $\epsilon > 0$ is a real number, $C := C(\kappa) > 0$ is a constant, and $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ is any function with $\ell(g+1) = 1$, such that the associated (κ, λ)-tridiagonal sequence $\mathcal{T} = \mathcal{T}(\mathcal{G}, \ell)$ satisfies*
(i) Property (AC),

(ii) $D_{\mathcal{T}} \leq C\, \mathtt{h}_{\mathcal{T}}$*, and*
(iii) $D_{\mathcal{T}} - (\mathtt{h}_{\mathcal{T}} + \mathtt{t}_{\mathcal{T}}) > \epsilon\, \mathtt{h}_{\mathcal{T}}$*.*
Then there exists a positive constant $H := H(\kappa, \lambda, \epsilon, \mathcal{G})$ *such that* $\mathtt{h}_{\mathcal{T}} \leq H$ *holds.*

Proof: Suppose that κ, λ, $\mathcal{G}$, ϵ, C, ℓ are as in the statement of the proposition. First, we show that the following statement holds:

(‡) For each $i = 0, \dots, g-1$, there exists a subsequence $\mathcal{G}_i$ of $\mathcal{G}$ with precisely $(i+1)$-terms satisfying $(1, \lambda, \kappa - \lambda - 1) \notin \underline{\mathcal{G}_i}$ and $(\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1}) \in \underline{\mathcal{G}_i}$ for which there is a positive constant $\mathbb{L}_i := \mathbb{L}_i(\kappa, \lambda, \epsilon, \mathcal{G}, \mathcal{G}_i)$ such that

$$\ell(j) \leq \mathbb{L}_i$$

holds for all $(\gamma_j, \alpha_j, \beta_j) \in \underline{\mathcal{G}_i}$.

Proof of (‡)*:* We use induction on i. In case $i = 0$, (‡) holds for the subsequence $\mathcal{G}_0 := \Big((\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1})\Big)$ and constant $\mathbb{L}_0 := 1$.

So, assume that (‡) holds for all $i = s$, with $0 \leq s \leq g-2$, i.e. there is a subsequence $\mathcal{G}_s = \Big((\gamma_{i_p}, \alpha_{i_p}, \beta_{i_p})\Big)_{p=1}^{s+1}$ of $\mathcal{G}$ with $(1, \lambda, \kappa - \lambda - 1) \notin \underline{\mathcal{G}_s}$ and $(\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1}) \in \underline{\mathcal{G}_s}$ for which there is a positive constant $\mathbb{L}_s := \mathbb{L}_s(\kappa, \lambda, \epsilon, \mathcal{G}, \mathcal{G}_s) > 0$ such that $\ell(i_p) \leq \mathbb{L}_s$ holds for all $1 \leq p \leq s+1$.

Let $\mathcal{L}(\{i_1, \dots, i_{s+1}\})$ denote the set consisting of those functions $L : \{i_1, \dots, i_{s+1}\} \to \mathbb{N}$ satisfying $L(i_p) \leq \mathbb{L}_s$ for all $1 \leq p \leq s+1$. Note that the set $\mathcal{L}(\{i_1, \dots, i_{s+1}\})$ depends only on $\kappa, \lambda, \epsilon, \mathcal{G}$ and $\mathcal{G}_s$. Let Δ_s denote the subsequence of $\mathcal{G}$ obtained by removing the terms in $\mathcal{G}_s$ from $\mathcal{G}$. Put $m = m(\mathcal{G}, \mathcal{G}_s) := \min\{2 \leq i \leq g \,:\, (\gamma_i, \alpha_i, \beta_i) \in \underline{\Delta_s}\}$.

Define positive constants $\widetilde{F}_s = \widetilde{F}_s(\kappa, \lambda, \epsilon, \mathcal{G}, \mathcal{G}_s)$ and $\widetilde{H}_s = \widetilde{H}_s(\kappa, \lambda, \epsilon, \mathcal{G}, \mathcal{G}_s)$ by

$$\begin{aligned} \widetilde{F}_s &:= \max\{F(\kappa, \mathcal{G}, \Delta_s, L) \,:\, L \in \mathcal{L}(\{i_1, \dots, i_{s+1}\})\}, \\ \widetilde{H}_s &:= \max\{H(\kappa, \lambda, \epsilon, \mathcal{G}, \Delta_s, L) \,:\, L \in \mathcal{L}(\{i_1, \dots, i_{s+1}\})\}, \end{aligned}$$

where $F(\kappa, \mathcal{G}, \Delta_s, L)$ and $H(\kappa, \lambda, \epsilon, \mathcal{G}, \Delta_s, L)$ are the constants given by applying Theorem 4.2 to the (κ, λ)-quadruple $(\mathcal{G}, \Delta_s; L, \ell)$.

Then, by Theorem 4.2, either (a) $\ell(i) > \widetilde{F}_s$ holds for all $(\gamma_i, \alpha_i, \beta_i) \in \underline{\Delta_s}$, in which case we can let $\mathcal{G}_{s+1}$ be the sequence defined by adding the term $(\gamma_m, \alpha_m, \beta_m)$ to the beginning of $\mathcal{G}_s$ and put $\mathbb{L}_{s+1} := \max\{\mathbb{L}_s, C(\kappa)\, \widetilde{H}_s\}$, or (b) there exists $(\gamma_n, \alpha_n, \beta_n) \in \underline{\Delta_s}$ such that $\ell(n) \leq \widetilde{F}_s$ holds, in which case we can let $\mathcal{G}_{s+1}$ be the sequence defined by inserting the term $(\gamma_j, \alpha_j, \beta_j)$ with $j := \max\{m, n\}$ into the sequence $\mathcal{G}_s$ (according to its place in $\mathcal{G}$) and put $\mathbb{L}_{s+1} := \max\{\mathbb{L}_s, C(\kappa)\, \widetilde{F}_s\}$. This completes the proof that statement (‡) holds. ∎

To complete the proof of the proposition, we apply (‡) for $i = g-1$. In particular, for this choice of i, $\mathcal{G}_{g-1} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=2}^{g+1}$, and constant $\mathbb{L}_{g-1}$ depends only on $\kappa, \lambda, \epsilon$ and $\mathcal{G}$, and hence the set

$\mathcal{L}(\{i_1, \ldots, i_g\}) = \mathcal{L}(\{2, \ldots, g+1\})$ of those functions $L : \{i_1, \ldots, i_g\} \to \mathbb{N}$ satisfying $L(i_p) \leq \mathbb{L}_{g-1}$ for all $1 \leq p \leq g$ depends only on $\kappa, \lambda, \epsilon$ and $\mathcal{G}$. Since $(\mathcal{G}, \Delta; L, \ell)$ is a (κ, λ)-quadruple for the subsequence $\Delta = \Big((\gamma_1, \alpha_1, \beta_1) = (1, \lambda, \kappa - \lambda - 1)\Big)$ of $\mathcal{G}$ and L any function in $\mathcal{L}(\{2, \ldots, g+1\})$, it follows by applying Theorem 4.2 to $(\mathcal{G}, \Delta; L, \ell)$ that there exists a constant

$$C = C(\kappa, \lambda, \epsilon, \mathcal{G}) := \max\{F(\kappa, \mathcal{G}, \Delta, L), H(\kappa, \lambda, \epsilon, \mathcal{G}, \Delta, L) \,:\, L \in \mathcal{L}(\{2, \ldots, g+1\})\},$$

where $F(\kappa, \mathcal{G}, \Delta, L), H(\kappa, \lambda, \epsilon, \mathcal{G}, \Delta, L)$ are the constants given by Theorem 4.2 so that

$$\mathtt{h}_{\mathcal{T}} \leq C$$

holds. This completes the proof of the proposition. ∎

In order to prove Theorem 1.1, we will also make use of the following result from [2], which generalizes results of Bannai and Ito [8, 9] and Suzuki [35]:

Theorem 4.4 *([2, Theorem 1.2])*
Suppose that $k \geq 3$ is a fixed integer. Then there exists a positive number $\epsilon_0 = \epsilon_0(k)$, depending only on k, so that there are only finitely many distance-regular graphs with valency k, head $\mathtt{h}_\Gamma$, tail $\mathtt{t}_\Gamma$, and diameter D_Γ that satisfy

$$D_\Gamma - (\mathtt{h}_\Gamma + \mathtt{t}_\Gamma) \leq \epsilon_0 \, \mathtt{h}_\Gamma.$$

Proof of Theorem 1.1: Let $k \geq 3$ be a fixed integer. By Theorem 4.4, there exists a constant $\epsilon_0 = \epsilon_0(k) > 0$ (which depends only on k) such that there are only finitely many distance-regular graphs Γ with valency k, head $\mathtt{h}_\Gamma$, tail $\mathtt{t}_\Gamma$ and diameter D_Γ that satisfy

$$D_\Gamma - (\mathtt{h}_\Gamma + \mathtt{t}_\Gamma) \leq \epsilon_0 \mathtt{h}_\Gamma.$$

Now, suppose that Γ is any distance-regular graph with valency k that satisfies

$$D_\Gamma - (\mathtt{h}_\Gamma + \mathtt{t}_\Gamma) > \epsilon_0 \mathtt{h}_\Gamma. \tag{33}$$

Then, by Theorem 2.3 and (33), the (k, a_1)-tridiagonal sequence $\mathcal{T}_\Gamma = \mathcal{T}(\mathcal{G}_\Gamma, \ell_\Gamma)$ (cf. (12)) satisfies all of conditions (i)–(iii) in Proposition 4.3, where a_1 is an intersection number of Γ.

Therefore, for any distance-regular graph Γ with valency k that satisfies (33), it follows that

$$\mathtt{h}_\Gamma \leq C(k) := \max\{H(k, a_1, \epsilon_0(k), \mathcal{G}) : 0 \leq a_1 \leq k-2,\ \mathcal{G} \text{ is a } (k, a_1)\text{-graphical sequence}\}$$

where $H(k, a_1, \epsilon_0(k), \mathcal{G})$ is the constant given by Proposition 4.3 (note that in the formula for $C(k)$, taking a maximum is appropriate since the number of integers a_1 with $0 \leq a_1 \leq k-2$ is finite, and so is the number of (k, a_1)-graphical sequences). Theorem 1.1 now follows by applying Corollary 2.4 with the constant $C(k)$. ∎

The strategy that we use to prove Theorem 4.2 (whose proof will be presented in Section 9) is quite involved, and so we will now provide a brief overview of the proof before continuing.

Let $(\mathcal{G}, \Delta; L, \ell)$ be any (κ, λ)-quadruple as in the statement of Theorem 4.2, and put $\mathcal{G} := \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ and $\mathcal{T} := \mathcal{T}(\mathcal{G}, \ell)$.

By Lemma 3.3 (i), for each $2 \leq i \leq g$ satisfying $(\gamma_i, \alpha_i, \beta_i) \in \underline{\mathcal{G}} \setminus \{(1, \lambda, \kappa - \lambda - 1), (\kappa - \lambda - 1, \lambda, 1), (\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1})\}$, there exists a closed interval $\mathcal{I} = [\mathcal{I}_{\min}, \mathcal{I}_{\max}]$ with $\mathcal{I}_{\min} < \mathcal{I}_{\max}$, which we shall call a "well-placed interval" (see Section 7), such that
(W1) $\mathcal{I} \subseteq (\mathfrak{R}_1, \mathfrak{R}_{\max})$;
(W2) If $\mathcal{I} \cap I_i \neq \emptyset$ then $\mathcal{I} \subseteq I_i$ holds, $1 \leq i \leq g$;
(W3) $\mathcal{I} \subseteq I_i$
all hold (cf. (28), (29)).

In the first step of the proof of Theorem 4.2, we will approximate the Christoffel numbers of the eigenvalues of $\mathcal{T}$ inside a well-placed interval $\mathcal{I}$. To do this, we define the quantities

$$\begin{aligned} \mathfrak{c} = \mathfrak{c}(\mathcal{G}, \mathcal{I}) &:= \min\Big\{\{2 \leq i \leq g \,:\, \mathcal{I}_{\max} < \mathfrak{L}_i\} \cup \{g+1\}\Big\}; \\ \mathfrak{d} = \mathfrak{d}(\mathcal{G}, \mathcal{I}) &:= \max\Big\{\{2 \leq i \leq g \,:\, \mathcal{I}_{\max} < \mathfrak{L}_i\} \cup \{\mathfrak{c}\}\Big\}; \\ \mathtt{Gap}(\mathcal{I}) = \mathtt{Gap}_{\mathcal{G},\ell}(\mathcal{I}) &:= \begin{cases} \sum_{\mathfrak{c} \leq j \leq \mathfrak{d}} \ell(j) & \text{if } \mathfrak{c} \leq g \\ 0 & \text{if } \mathfrak{c} = g+1, \end{cases} \end{aligned}$$

(cf. (69), (70), (72)) and, for any eigenvalue $\theta \in \mathcal{I}$ of $\mathcal{T}$, we approximate the sum $\sum_{i=0}^{D_{\mathcal{T}}} \kappa_i u_i^2(\theta)$ (see Theorem 8.1) by bounding the following three subsums (cf. (22), (25), (27), (74)):

(1) Head sum: $\sum_{i=0}^{s(\mathfrak{a})-2} \kappa_i u_i^2(\theta)$;
(2) Gap sum: $\sum_{i=s(\mathfrak{a})-1}^{s(\mathfrak{b}+1)} \kappa_i u_i^2(\theta)$;
(3) Tail sum: $\sum_{s(\mathfrak{b}+1)+1}^{D_{\mathcal{T}}} \kappa_i u_i^2(\theta)$.

We can use the theory of three-term recurrence relations, to bound the Head sum and the Gap sum (see Theorem 8.7 and Corollary 8.8). However, for the Tail sum, there may exist some real numbers near to which we are unable to find good bounds for the Tail sum. Let $\mathcal{B}$ denote the set of these real numbers (cf. (52)). In Theorem 6.2, we show that $\mathcal{B}$ is finite and depends only on $\mathcal{G}, \Delta$ and L. In particular, for each $(\gamma_i, \alpha_i, \beta_i) \in \underline{\mathcal{G}} \setminus \{(1, \lambda, \kappa-\lambda-1), (\kappa-\lambda-1, \lambda, 1), (\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1})\}$, there always exists a well-placed interval $\mathcal{J} \subseteq I_i$ such that $\mathcal{J} \cap \mathcal{B} = \emptyset$ (cf. Corollary 7.3). Note that such a well-placed interval $\mathcal{J}$ depends only on $\mathcal{G}, \Delta$ and L. We strengthen the condition on the interval $\mathcal{I}$ by requiring that in addition to (W1)–(W3), it also satisfies $\mathcal{I} \cap \mathcal{B} = \emptyset$. Then for any such a well-placed interval, we can approximate the Tail sum as long as we require that $\ell(i) > F$ holds for all $(\gamma_i, \alpha_i, \beta_i) \in \underline{\Delta}$, where F is a positive constant depending only on $\kappa, \mathcal{G}, \Delta$ and L (cf. Theorem 8.9).

Now, by Condition (iii) of Theorem 4.2, we can find an element $(\gamma_i, \alpha_i, \beta_i) \in \underline{\mathcal{G}} \setminus \{(1, \lambda, \kappa - \lambda - 1), (\kappa - \lambda - 1, \lambda, 1), (\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1})\}$ satisfying $\ell(i) > \frac{\epsilon \mathbf{h}_{\mathcal{T}}}{g}$, and we can find a well-placed interval $\mathcal{I} \subseteq I_i$ such that $\mathcal{I} \cap \mathcal{B} = \emptyset$ and $\mathtt{Len}(\mathcal{I}) > \frac{\epsilon \mathbf{h}_{\mathcal{T}}}{g}$ both hold (cf. (23), (71)).

By the approximation given in Theorem 8.1 and Property (AC), it follows that for any real number $\delta > 0$, there exist two positive constants $C_1 = C_1(\kappa, \lambda, \epsilon, \delta, \mathcal{G}, \Delta, L)$ and $C_2 = C_2(\kappa, \lambda, \delta)$ such that any two eigenvalues $\theta, \eta \in \mathcal{I}$ of $\mathcal{T}$ which are conjugate algebraic numbers must satisfy $|\theta - \eta| \leq \delta$

if $\mathtt{h}_\mathcal{T} \geq C_1$ and $\mathtt{Gap}(\mathcal{I}) \leq C_2\mathtt{h}_\mathcal{T}$ all hold (cf. Theorem 9.1). In Claim 9.3, we show, by using interlacing, that the number of eigenvalues in $\mathcal{I}$ is at least $C_3\mathtt{h}_\mathcal{T}$, where C_3 is a positive constant depending only on $\mathcal{I}_{\max} - \mathcal{I}_{\min}, \epsilon$ and $\mathcal{G}$.

Now, we have to consider two cases: either $\mathtt{Gap}(\mathcal{I}) \leq C_2\mathtt{h}_\mathcal{T}$ or $\mathtt{Gap}(\mathcal{I}) > C_2\mathtt{h}_\mathcal{T}$. In the first case, $\mathtt{Gap}(\mathcal{I}) \leq C_2\mathtt{h}_\mathcal{T}$, we show by using Theorem 9.1, Claim 9.3 and Theorem 5.5, a result in number theory, that

$$\lim_{\mathtt{h}_\mathcal{T}\to\infty} \frac{|\{\eta : \text{eigenvalues of } \mathcal{T} \text{ that have an algebraic conjugate in } \mathcal{I}\}|}{\mathtt{h}_\mathcal{T}} = \infty$$

holds (cf. Proposition 9.2). Since the number of eigenvalues of $\mathcal{T}$ is exactly $D_\mathcal{T} + 1$ (cf. (26)) and $D_\mathcal{T} + 1 \leq (C(\kappa) + 1)\mathtt{h}_\mathcal{T}$ holds by condition (ii) of Theorem 4.2, there exists a constant $H > 0$ depending only on $\kappa, \lambda, \epsilon, \mathcal{G}, \Delta, L$ so that $\mathtt{h}_\mathcal{T} \leq H$ holds, as required.

In the second case, $\mathtt{Gap}(\mathcal{I}) > C_2\mathtt{h}_\mathcal{T}$, by the unimodality of the sequence $(\mathfrak{R}_i)_{i=1}^g$, we can find another well-placed interval $\mathcal{I}' := [\mathcal{I}'_{\min}, \mathcal{I}'_{\max}]$ which depends only on $\mathcal{G}, \Delta, L$ such that
(1) $\mathcal{I}'_{\min} > \mathcal{I}_{\max}$;
(2) $\mathcal{I}' \cap \mathcal{B} = \emptyset$;
(3) $\mathtt{Len}(\mathcal{I}') > \frac{\mathtt{Gap}(\mathcal{I})}{g}$,
all hold (see Proposition 7.4). So we can repeat the same process with $\mathcal{I}'$ instead of $\mathcal{I}$. Using the unimodality of the sequence $(\mathfrak{R}_i)_{i=1}^g$, the condition $\mathcal{I}'_{\min} > \mathcal{I}_{\max}$ implies that $\mathfrak{c}(\mathcal{G}, \mathcal{I}) \leq \mathfrak{c}(\mathcal{G}, \mathcal{I}')$, $\mathfrak{d}(\mathcal{G}, \mathcal{I}') \leq \mathfrak{d}(\mathcal{G}, \mathcal{I})$ and $\mathtt{Gap}(\mathcal{I}') < \mathtt{Gap}(\mathcal{I})$ all hold. Hence, the second case can be repeated at most g times so that, finally, the first case must be satisfied, from which Theorem 4.2 again follows.

5 Two Useful Results for Polynomials

In this section, we prove two useful results concerning roots of polynomials. The first one, Theorem 5.1, will be used in Theorem 6.2 to show that the set $\mathcal{B}$ (as we introduced in Section 4) is finite. The second result, Theorem 5.5, analyzes the polynomials having all roots in an interval. It will be used to bound the number of eigenvalues of a distance-regular graph in the proof of Proposition 9.2.

We denote the degree of any polynomial $p(x)$ by $\deg(p(x))$. The polynomial $p(x) = 0$ is called the *zero polynomial* and, for technical reasons, we define the degree of this polynomial to be -1 (cf. [25, p.158]). Two polynomials $p_1(x)$ and $p_2(x)$ are identical if their difference $p_1(x) - p_2(x)$ is the zero polynomial. Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers, respectively, and let $\mathbb{R}[x]$ denote the ring of polynomials in one variable x with real coefficients.

Theorem 5.1 *Let $q_1(x), q_2(x) \in \mathbb{R}[x]$ be two monic quadratic polynomials which are not squares of linear polynomials, and let $I \subseteq \mathbb{R}$ be the largest (infinite) interval on which both $q_1(x)$ and $q_2(x)$ are non-negative. Suppose $P_j(x) \in \mathbb{R}[x]$ $(1 \leq j \leq 4)$ are such that $C := \max\{\deg(P_j(x)) : 1 \leq j \leq 4\} \geq 0$. Put*

$$P(x) := P_1(x) + P_2(x)\sqrt{q_1(x)} + P_3(x)\sqrt{q_2(x)} + P_4(x)\sqrt{q_1(x)q_2(x)}.$$

Then the equation $P(x) = 0$ has at most $4(C + 2)$ roots in I, unless $q_1(x)$ is identical to $q_2(x)$ and $P_2(x) + P_3(x)$, $P_1(x) + q_1(x)P_4(x)$ are the zero polynomials, in which case $P(x) = 0$ for every $x \in I$.

Proof: For each $i, j \in \{0,1\}$, we define

$$P^{(ij)}(x) := P_1(x) + (-1)^i P_2(x)\sqrt{q_1(x)} + (-1)^{i+j} P_3(x)\sqrt{q_2(x)} + (-1)^j P_4(x)\sqrt{q_1(x)q_2(x)}\ ,$$

and put

$$P^*(x) := P^{(00)}(x) \times P^{(01)}(x) \times P^{(10)}(x) \times P^{(11)}(x). \tag{34}$$

Note that

$$P^{(00)}(x)P^{(01)}(x) = \Big(P_1(x) + P_2(x)\sqrt{q_1(x)}\Big)^2 - q_2(x)\Big(P_3(x) + P_4(x)\sqrt{q_1(x)}\Big)^2$$

has the form $U(x) + V(x)\sqrt{q_1(x)}$ with $U(x), V(x) \in \mathbb{R}[x]$ satisfying $\deg(U(x)) \le 2C + 4$ and $\deg(V(x)) \le 2C + 2$. Similarly,

$$P^{(10)}(x)P^{(11)}(x) = \Big(P_1(x) - P_2(x)\sqrt{q_1(x)}\Big)^2 - q_2(x)\Big(P_3(x) - P_4(x)\sqrt{q_1(x)}\Big)^2 = U(x) - V(x)\sqrt{q_1(x)}.$$

Hence, by (34), $P^*(x) = U(x)^2 - V(x)^2 q_1(x)$ is a real polynomial of degree at most $4C + 8$. This proves the theorem in the non-degenerate case when $P^*(x)$ is not the zero polynomial.

Assume now that $P^*(x)$ is the zero polynomial. We need to prove that this happens only if $q_1(x)$ is identical to $q_2(x)$ and $P_2(x) + P_3(x)$, $P_1(x) + q_1(x)P_4(x)$ are the zero polynomials. We first prove the following.

Claim 5.2 *If $q_1(x) - q_2(x)$ is not the zero polynomial then $P^*(x)$ is also not the zero polynomial.*

Proof of Claim 5.2 We first show that $P_1^2(x) - q_1(x)q_2(x)P_4(x)^2$ is not the zero polynomial if at least one of the polynomials $P_1(x), P_4(x)$ is not the zero polynomial. Take a root $\gamma \in \mathbb{C}$ of $q_1(x)$ which is not a root of $q_2(x)$. By the condition of the theorem, γ is the root of $q_1(x)q_2(x)$ of multiplicity 1. Assume that $P_1^2(x) - q_1(x)q_2(x)P_4(x)^2$ is the zero polynomial. Then γ is the root of $q_1(x)q_2(x)P_4(x)^2$ of odd multiplicity but it is either not the root of $P_1(x)^2$ or it is its root of even multiplicity, a contradiction. By the same argument, $P_2(x)^2 q_1(x) - P_3(x)^2 q_2(x)$ is not the zero polynomial if at least one of the polynomials $P_2(x), P_3(x)$ is not the zero polynomial. Since $C := \max\{\deg(P_j(x)) \,:\, 1 \le j \le 4\} \ge 0$, we always have either $P_1^2(x) \ne q_1(x)q_2(x)P_4(x)^2$ (if $P_1(x)P_4(x)$ is not the zero polynomial) or $P_2(x)^2 q_1(x) \ne P_3(x)^2 q_2(x)$ (if $P_2(x)P_3(x)$ is not the zero polynomial) for infinitely many $x \in I$.

Suppose $P^*(x)$ is the zero polynomial. Then one of the functions $P^{(ij)}(x)$, where $i, j \in \{0,1\}$, must be zero identically on $x \in I$. Hence

$$P_1(x) + (-1)^i P_2(x)\sqrt{q_1(x)} + (-1)^{i+j} P_3(x)\sqrt{q_2(x)} + (-1)^j P_4(x)\sqrt{q_1(x)q_2(x)} = 0. \tag{35}$$

Our aim is to show that this is only possible if all $P_j(x)$, $j = 1, 2, 3, 4$, are the zero polynomials which is not the case by the condition of the theorem.

We first claim that

$$P_1(x)P_2(x) = q_2(x)P_3(x)P_4(x). \tag{36}$$

Indeed, putting first two terms of (35) into the right hand side and squaring we obtain

$$\Big(P_1(x) + (-1)^i P_2(x)\sqrt{q_1(x)}\Big)^2 = q_2(x)\Big((-1)^i P_3(x) + P_4(x)\sqrt{q_1(x)}\Big)^2. \tag{37}$$

Since $q_1(x)$ is not the square of a linear polynomial, by the same argument for roots multiplicity as above, the function $S(x) + T(x)\sqrt{q_1(x)}$, where $S(x), T(x) \in \mathbb{R}[x]$, is zero identically on I if and only if $S(x)$ and $T(x)$ are the zero polynomials. Therefore, collecting terms for $\sqrt{q_1(x)}$ in (37) we obtain (36).

Similarly, putting the first and the third term of (35) to the right hand side, squaring and then using the same argument for the ring $\mathbb{R}[x] + \mathbb{R}[x]\sqrt{q_2(x)}$, we deduce that

$$P_1(x)P_3(x) = q_1(x)P_2(x)P_4(x). \tag{38}$$

Suppose first that $P_1(x)$ is the zero polynomial. Then, by (36) and (38), $P_2(x), P_3(x)$ or $P_4(x)$ is zero identically. If either $P_2(x)$ or $P_3(x)$ is the zero polynomial then, by (35), all four $P_j(x)$ must be the zero polynomials, a contradiction. If $P_4(x)$ is the zero polynomial then $P_2(x)\sqrt{q_1(x)} + (-1)^j P_3(x)\sqrt{q_2(x)} = 0$. But this yields $P_2(x)^2 q_1(x) = P_3(x)^2 q_2(x)$, a contradiction again. By the same argument, if any of the polynomials $P_2(x), P_3(x), P_4(x)$ is the zero polynomial, then by (36) and (38) one more polynomial must be a zero polynomial. One then concludes as above that all four polynomials are the zero polynomials.

Finally, if none of the polynomials $P_j(x)$ is the zero polynomial then multiplying (36) and (38) gives $P_1(x)^2 P_2(x)P_3(x) = q_1(x)q_2(x)P_2(x)P_3(x)P_4(x)^2$. Hence $P_1(x)^2 = q_1(x)q_2(x)P_4(x)^2$, which is a contradiction again. ∎

Now, to complete the proof of the theorem, suppose that $q_1(x)$ is identical to $q_2(x)$. Then $P(x) = P_1(x) + q_1(x)P_4(x) + (P_2(x) + P_3(x))\sqrt{q_1(x)}$ for all $x \in I$. If $P_1(x) + q_1(x)P_4(x)$ and $P_2(x) + P_3(x)$ are the zero polynomials then $P(x)$ is zero identically. Otherwise,

$$P(x)(P_1(x) + q_1(x)P_4(x) - (P_2(x) + P_3(x))\sqrt{q_1(x)}) = (P_1(x) + q_1(x)P_4(x))^2 - q_1(x)(P_2(x) + P_3(x))^2$$

is not the zero polynomial. So P has at most $2C + 4$ roots in I, which is better than required. ∎

In the remainder of this section, we will show the second useful result, Theorem 5.5.
For any real number $\kappa \geq 2$, we denote by $\mathcal{P}_\kappa$ the set of all irreducible monic polynomials $p(x) \in \mathbb{Z}[x]$ such that all of the roots of $p(x)$ are contained in the closed interval $[-\kappa, \kappa]$. Note $\mathcal{P}_\kappa \subseteq \mathcal{P}_{\kappa'}$ if $\kappa \leq \kappa'$.

Lemma 5.3 *Let $\kappa \geq 2$ be a real number and let n be a positive integer. Then the following holds.*
(i) The set consisting of all polynomials $p(x) \in \mathcal{P}_\kappa$ of degree at most n is finite.
(ii) $\mathcal{P}_\kappa$ is an infinite set.

Proof: (i) Obviously, any coefficient of each $p(x) \in \mathcal{P}_\kappa$ of degree at most n is in $[-(2\kappa)^n, (2\kappa)^n]$, so $\mathcal{P}_\kappa$ contains at most $(2(2\kappa)^n + 1)^{n+1}$ of such polynomials. See also [27, Lemma 7.1].

(ii) Let $P_n(1,0,1)(x)$ be the characteristic polynomial of the tridiagonal $(n \times n)$-matrix with zeroes on the diagonal and ones on the subdiagonals and superdiagonals. Then $P_n(1,0,1)(x)$ is a polynomial of degree n and has n distinct roots, $2\cos(\frac{i\pi}{n+1})$, $i = 1, \ldots, n$ ([10, p.11]). Thus, if we factorize $P_n(1,0,1)(x)$ into irreducible factors, say $q_1(x), \ldots, q_t(x)$, then $q_i(x) \neq q_j(x)$ if $1 \leq i < j \leq t$ and $q_i(x) \in \mathcal{P}_2$ for all $1 \leq i \leq t$. (ii) now follows immediately from (i). ∎

In fact, an old result of R. M. Robinson [32] asserts that if J is an interval of length strictly greater than 4 then there are infinitely many irreducible monic polynomials whose roots all lie in J. Moreover, none of them has a root of the form $2\cos(\pi r)$ with $r \in \mathbb{Q}$ as those lying in $\mathcal{P}_2$.

Now, for any real number $\zeta > 0$, let $\mathbf{I}_{\kappa,\zeta}$ be the set of all closed intervals of length ζ which are contained in the closed interval $[-\kappa, \kappa]$. For each $p \in \mathcal{P}_\kappa$ and $I \in \mathbf{I}_{\kappa,\zeta}$, we define

$$\Upsilon_\kappa(p, I) := \frac{|\{\theta \in I \,:\, p(\theta) = 0\}| - 1}{\deg(p(x))}\,, \text{ and} \tag{39}$$

$$\Upsilon_{\kappa,\zeta} := \sup\{\Upsilon_\kappa(p, I) \,:\, p \in \mathcal{P}_\kappa,\ I \in \mathbf{I}_{\kappa,\zeta}\}\,. \tag{40}$$

Remark 5.4 *Note that $\Upsilon_{\kappa,\zeta}$ is positive for all $\zeta > 0$ since by Lemma 5.3 (ii) there exists a polynomial $p(x) \in \mathcal{P}_2$ with degree $n > \frac{8\kappa}{\zeta}$ and so, by the pigeon hole principle, there exists an interval $I \in \mathbf{I}_{\kappa,\zeta}$ of length ζ such that $p(x)$ has at least $\frac{n\zeta}{4\kappa}$ roots in I. Even so, we now show that the limit of $\Upsilon_{\kappa,\zeta}$ as ζ tends to ∞ is zero.*

Theorem 5.5 *Let $\kappa \geq 2$ be a real number. Then*

$$\lim_{\zeta \to 0} \Upsilon_{\kappa,\zeta} = 0.$$

Proof: Fix $\kappa \geq 2$ and $\zeta \in (0,1)$. Let $p(x) \in \mathcal{P}_\kappa$ be of degree n, say, and let $I \in \mathbf{I}_{\kappa,\zeta}$. Since $p(x)$ is irreducible in $\mathbb{Z}[x]$, it has n distinct roots $\alpha_1, \ldots, \alpha_n \in [-\kappa, \kappa]$. Consider the discriminant $\Delta(p)$ of p given by

$$\Delta(p) := \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2.$$

Since $p(x)$ is a monic polynomial with integral coefficients, its discriminant $\Delta(p)$ is an integer. Moreover, $\Delta(p)$ is not zero as the roots of $p(x)$ are distinct and $\Delta(p) > 0$, so $\Delta(p) \geq 1$.

Without loss of generality, assume that $\{\alpha_1, \ldots, \alpha_t\}$ is the set of roots of $p(x)$ contained in I, for some $0 \leq t \leq n$. Let $\tau = \tau(p, I) := \frac{t}{n}$.

Claim 5.6 *If $t \geq 2$ then $\tau^2 \leq -\frac{2\ln(2\kappa)}{\ln \zeta}$.*

Proof of Claim: We have

$$1 \leq \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$$

$$
\begin{aligned}
&= \left(\prod_{1\le i<j\le t}(\alpha_i-\alpha_j)^2\right)\left(\prod_{1\le i<j\le n \text{ and } j>t}(\alpha_i-\alpha_j)^2\right)\\
&\le \zeta^{\tau n(\tau n-1)}(2\kappa)^{n(n-1)},
\end{aligned}
$$

since $t = \tau n \geq 2$, $|\alpha_i - \alpha_j| \leq \zeta$ for $1 \leq i < j \leq t$ and $|\alpha_i - \alpha_j| \leq 2\kappa$ for $1 \leq i < j \leq n$. Using $\tau n - 1 \geq \tau(n-1)/2$ and $0 < \zeta < 1$ we find that $1 \leq \zeta^{\frac{\tau n \tau(n-1)}{2}}(2\kappa)^{n(n-1)}$, so $1 \leq \zeta^{\tau^2/2} 2\kappa$. The claim follows by taking the logarithms of both sides of the last inequality. ∎

Now, let $q(x) \in \mathcal{P}_\kappa$ and $I \in \mathbf{I}_{\kappa,\zeta}$ be such that

$$\Upsilon_\kappa(q, I) \geq \frac{1}{2}\Upsilon_{\kappa,\zeta} > 0.$$

Such a $q(x)$ exists, since, as remarked before the statement of the theorem, $\Upsilon_{\kappa,\zeta}$ is positive. Since $\Upsilon_\kappa(q, I) > 0$, the polynomial $q(x)$ has at least 2 roots in I. Hence, by Claim 5.6 and (39), we have

$$\sqrt{-\frac{2\ln(2\kappa)}{\ln\zeta}} \geq \frac{|\{x \in I \,:\, q(x) = 0\}|}{\deg(q(x))} > \frac{|\{x \in I \,:\, q(x) = 0\}| - 1}{\deg(q(x))} = \Upsilon_\kappa(q, I) \geq \frac{1}{2}\Upsilon_{\kappa,\zeta} > 0,$$

from which the theorem immediately follows. ∎

6 Preliminary Results for the Christoffel Numbers

In this section, we will prove some results which we will use later in Section 8 for the approximation of Christoffel numbers.

Suppose that $\mathcal{G} := \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ is a (κ, λ)-graphical sequence, that $(\mathcal{G}, \Delta; L, \ell)$ is a (κ, λ)-quadruple as in Definition 4.1, and that $\Delta = (\delta_j)_{j=1}^{\tau}$.

Fix i with $0 \leq i \leq \tau - 1$. Let $0 \leq j_i < g$ be the integer for which

$$\delta_{\tau-i} = (\gamma_{g-j_i}, \alpha_{g-j_i}, \beta_{g-j_i}) \tag{41}$$

holds. We put $j_{-1} := -1$, and note that $j_i - j_{(i-1)} \geq 1$ necessarily holds.

Suppose $j_i - j_{(i-1)} \geq 2$. Then, for $n_i := j_i - j_{(i-1)} - 1$, we define the sequence $z^{(i)} = (z_s^{(i)})_{s=1}^{n_i}$ by putting

$$z_s^{(i)} := (\gamma_{g-j_{(i-1)}-s}, \alpha_{g-j_{(i-1)}-s}, \beta_{g-j_{(i-1)}-s}), \quad s = 1, \ldots, n_i.$$

In addition, for $N := \sum_{z \in \underline{z^{(i)}}} L(z)$, we let $w^{(i)} = (w_k^{(i)})_{k=1}^N$ be the sequence whose kth term $w_k^{(i)}$ is defined to be $z_j^{(i)}$ for the necessarily unique j for which

$$\sum_{s=1}^{j-1} L(z_s^{(i)}) < k \leq \sum_{s=1}^{j} L(z_s^{(i)}) \tag{42}$$

holds.

Now, suppose that θ is a real number, and that v_0 and v_1 are real numbers satisfying $(v_0, v_1) \neq (0,0)$. In addition, let $(v_j)_{j=0}^{N+1}$ be the sequence that is defined by the recurrence relations

$$\tilde{\beta}_j v_{j-1} + (\tilde{\alpha}_j - \theta) v_j + \tilde{\gamma}_j v_{j+1} = 0 \quad (j = 1, 2, \ldots, N) \,, \tag{43}$$

where $(\tilde{\gamma}_j, \tilde{\alpha}_j, \tilde{\beta}_j)$ denotes the jth term $w_j^{(i)}$ of the sequence $w^{(i)}$, and N is as above if $j_i - j_{(i-1)} \geq 2$ and $N := 0$ else. Then, in view of (43), for $j_i - j_{(i-1)} \geq 2$ there are polynomials $f_t^{(i)}(x)$, $g_t^{(i)}(x)$ in $\mathbb{Q}[x]$ (of degree $s-1$ and $s-2$, respectively) that, for $\theta \in [\mathfrak{R}_{g-j_i}, \kappa]$, satisfy $v_s = f_t^{(i)}(\theta) v_1 + g_t^{(i)}(\theta) v_0$ for each $s \geq 1$,

$$v_N = \begin{cases} f_1^{(i)}(\theta) v_1 + g_1^{(i)}(\theta) v_0 & \text{if } j_i - j_{(i-1)} \geq 2 \\ v_0 & \text{if } j_i - j_{(i-1)} = 1, \end{cases} \tag{44}$$

and

$$v_{N+1} = \begin{cases} f_2^{(i)}(\theta) v_1 + g_2^{(i)}(\theta) v_0 & \text{if } j_i - j_{(i-1)} \geq 2 \\ v_1 & \text{if } j_i - j_{(i-1)} = 1. \end{cases} \tag{45}$$

In addition, in case $j_i - j_{(i-1)} = 1$, we let $f_t^{(i)}(x)$ and $g_t^{(i)}(x)$ $(t = 1, 2)$ be the polynomials in $\mathbb{Q}[x]$ for which both $f_t^{(i)}(x) - t + 1$ and $g_t^{(i)}(x) + t - 2$ are the zero polynomials for $t = 1, 2$. Note that the degrees of the polynomials $f_t^{(i)}(x)$ and $g_t^{(i)}(x)$ are as follows:

$$\deg(f_1^{(i)}(x)) = \begin{cases} -1 + \sum_{z \in \underline{z^{(i)}}} L(z) & \text{if } j_i - j_{(i-1)} \geq 2 \\ -1 & \text{if } j_i - j_{(i-1)} = 1, \end{cases} ; \tag{46}$$

$$\deg(g_1^{(i)}(x)) = \begin{cases} \deg(f_1^{(i)}(x)) - 1 & \text{if } j_i - j_{(i-1)} \geq 2 \\ 0 & \text{if } j_i - j_{(i-1)} = 1 \end{cases} ; \tag{47}$$

$$\deg(f_2^{(i)}(x)) = \deg(f_1^{(i)}(x)) + 1; \tag{48}$$

$$\deg(g_2^{(i)}(x)) = \deg(f_1^{(i)}(x)). \tag{49}$$

Note also that $f_t^{(i)}(x)$ and $g_t^{(i)}(x)$ $(t = 1, 2,\ 0 \leq i \leq \tau - 1)$ depend only on the triple $(\mathcal{G}, \Delta, L)$ (and not on the function ℓ).

We now present the second key definition of this section. For the (κ, λ)-graphical sequence $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$, let $x_i = x_i(\theta)$ and $y_i = y_i(\theta)$ (where $|x_i| \geq |y_i|$) be the roots of the equation

$$\gamma_{g-i} x^2 + (\alpha_{g-i} - \theta) x + \beta_{g-i} = 0 \ (0 \leq i < g). \tag{50}$$

Definition 6.1 *For any integers $\kappa \geq 3$ and $\lambda \geq 0$ with $\lambda \leq \kappa - 2$, let $(\mathcal{G}, \Delta; L, \ell)$ be a (κ, λ)-quadruple with $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ and $\Delta = (\delta_i)_{i=1}^{\tau}$. With reference to (28), (44), (45) and (50), for $0 \leq i \leq \tau - 1$ and $\delta_{\tau - i} = (\gamma_{g-j_i}, \alpha_{g-j_i}, \beta_{g-j_i}) \in \underline{\Delta}$ satisfying $\beta_{g-j_i} \leq \gamma_{g-j_i}$, and for any real numbers v_0, v_1 satisfying $(v_0, v_1) \neq (0,0)$ we define the set $\mathcal{B}_i = \mathcal{B}_i(\mathcal{G}, \Delta, L)$ by*

$$\mathcal{B}_i(\mathcal{G}, \Delta, L) := \{\theta \in [\mathfrak{R}_{g-j_i}(\mathcal{G}), \mathfrak{R}_{\max}(\mathcal{G})] \,:\, F_i(\theta) = 0\} \,, \tag{51}$$

where $F_i(x)$ is the polynomial in $\mathbb{R}[x]$ given by

$$F_i(x) := \begin{cases} \prod_{\xi\in\{x_{j_i},y_{j_i}\}} \Big((x-\alpha_{g+1})(f_1^{(i)}(x)\xi - f_2^{(i)}(x)) + \gamma_{g+1}(g_1^{(i)}(x)\xi - g_2^{(i)}(x))\Big) & \text{if } i=0 \\ \prod_{(\xi,\chi)\in\{x_{j_i},y_{j_i}\}\times\{x_{j_{(i-1)}},y_{j_{(i-1)}}\}} \Big((f_1^{(i)}(x)\xi - f_2^{(i)}(x))\chi + g_1^{(i)}(x)\xi - g_2^{(i)}(x)\Big) & \text{if } i\neq 0. \end{cases}$$

With reference to (51), we also define the set $\mathcal{B} = \mathcal{B}(\mathcal{G},\Delta,L)$ by

$$\mathcal{B}(\mathcal{G},\Delta,L) := \bigcup_{0\le i\le \tau-1 \text{ and } \beta_{g-j_i}\le\gamma_{g-j_i}} \mathcal{B}_i(\mathcal{G},\Delta,L). \tag{52}$$

Note that since the polynomials $f_t^{(i)}(x)$ and $g_t^{(i)}(x)$ ($t=1,2,\ 0\le i\le\tau-1$) depend only on the triple $(\mathcal{G},\Delta,L)$, the polynomial $F_i(x)$ and the sets $\mathcal{B}_i$ and $\mathcal{B}$ in the last definition all also depend only on $(\mathcal{G},\Delta,L)$. Note that, if $\mathcal{R}_{\max}(\mathcal{G}) = \kappa$, then $F_i(\kappa) = 0$ for all i, $0\le i\le\tau-1$ (as the standard eigenvector for κ is the all-one vector and $x_i = 1$ for all $0\le i\le\tau-1$), and hence in this case $\kappa\in\mathcal{B}$.

Theorem 6.2 *Let $\kappa\ge 3$ and $\lambda\ge 0$ be integers with $\lambda\le\kappa-2$. Suppose that $(\mathcal{G},\Delta;L,\ell)$ is a (κ,λ)-quadruple. Then there exists a constant $C = C(\mathcal{G},\Delta,L) > 0$ such that*

$$|\mathcal{B}| \le C$$

holds, for $\mathcal{B} = \mathcal{B}(\mathcal{G},\Delta,L)$ as defined in Definition 6.1.

Proof: Let $(\mathcal{G},\Delta;L,\ell)$ be a (κ,λ)-quadruple, put $\mathcal{G} = \Big((\gamma_i,\alpha_i,\beta_i)\Big)_{i=1}^{g+1}$ and let $\mathcal{T} = \mathcal{T}(\mathcal{G},\ell)$ be the associated (κ,λ)-tridiagonal sequence. In addition, put $\Delta = (\delta_j)_{j=1}^{\tau}$ and let $\delta_{\tau-i} = (\gamma_{g-j_i},\alpha_{g-j_i},\beta_{g-j_i}) \in \underline{\Delta}$ with $\beta_{g-j_i}\le\gamma_{g-j_i}$ be as defined in (41).

To prove the theorem, we will use Theorem 5.1 to bound $|\mathcal{B}_i|$ by some constant depending only on $\mathcal{G}$, Δ and L for each $0\le i\le\tau-1$. To do this, we first define polynomials $q_s(x)$, $s=1,2$, and $P_j(x)$, $1\le j\le 4$ as in the statement of that theorem, breaking this definition into cases depending on i:

(a) $i=0$: Let $q_s(x) = (x-\alpha_{g-j_i})^2 - 4\beta_{g-j_i}\gamma_{g-j_i}$ ($s=1,2$). For each $\xi\in\{x_{j_i},y_{j_i}\}$, put

$$\begin{aligned} P_1(x) &:= \frac{(x-\alpha_{g-j_i})((x-\alpha_{g+1})f_1^{(i)} + \gamma_{g+1}g_1^{(i)})}{2\gamma_{g-j_i}} - (x-\alpha_{g+1})f_2^{(i)} - \gamma_{g+1}g_2^{(i)}, \\ P_2^{\xi}(x) &:= (-1)^{\delta_{\xi,y_{j_i}}}\left(\frac{(x-\alpha_{g+1})f_1^{(i)} + \gamma_{g+1}g_1^{(i)}}{2\gamma_{g-j_i}}\right), \\ P_3(x) &= P_4(x) = 0, \end{aligned}$$

where $\delta_{\xi,y_{j_i}}$ is the Kronecker delta function, and let $P_2(x) = P_2^{\xi}(x)$. Then, for this specific choice of polynomials, the polynomial $P(x) = P^{\xi}(x)$ in Theorem 5.1 becomes

$$P(x) = (x-\alpha_{g+1})(f_1^{(i)}(x)\xi - f_2^{(i)}(x)) + \gamma_{g+1}(g_1^{(i)}(x)\xi - g_2^{(i)}(x)),$$

which is precisely the factor that appears in the definition of the polynomial $F_0(x)$ in Definition 6.1.

Note that if $P_1(x)$ and $P_2(x)$ are the zero polynomials, then $(x-\alpha_{g+1})f_s^{(i)}+\gamma_{g+1}g_s^{(i)}$ are also the zero polynomials for $s=1,2$. This contradicts (46)–(49). Hence $\max\{\deg(P_j(x)) : 1\le j\le 4\}\ge 0$.

(b) $i\ge 1$: Let $q_1(x)=(x-\alpha_{g-j_i})^2-4\beta_{g-j_i}\gamma_{g-j_i}$ and $q_2(x)=(x-\alpha_{g-j_{(i-1)}})^2-4\beta_{g-j_{(i-1)}}\gamma_{g-j_{(i-1)}}$. Note that as $\beta_{g-j_{(i-1)}}\le\beta_{g-j_i}\le\gamma_{g-j_i}\le\gamma_{g-j_{(i-1)}}$ holds, $q_1(y)\neq q_2(y)$ for some $y\in\mathbb{R}$. For each $(\xi,\chi)\in\{x_{j_i},y_{j_i}\}\times\{x_{j_{(i-1)}},y_{j_{(i-1)}}\}$, let $P_s(x)\in\mathbb{Q}[x]$, $1\le s\le 4$, be polynomials such that $P(x):=(f_1^{(i)}(x)\xi-f_2^{(i)}(x))\chi+g_1^{(i)}(x)\xi-g_2^{(i)}(x)$, i.e. the factor appearing in the definition of the polynomial $F_i(x)$ in Definition 6.1, $i\ge 1$. Note that if $P_s(x)=0$ $(1\le s\le 4)$ are all the zero polynomials, then so are the polynomials $f_t^{(i)}(x)$ and $g_t^{(i)}(x)$, $t=1,2$. Thus $v_N(\theta)=v_{N+1}(\theta)=0$ hold for any real number θ, which is impossible as $(v_0,v_1)\neq(0,0)$. Hence $\max\{\deg(P_s(x)) : 1\le s\le 4\}\ge 0$.

With these definitions in hand we can now apply Theorem 5.1 simultaneously to cases (a) and (b). (Clearly, $q_1(x)$ and $q_2(x)$ are not squares of linear polynomials.) In particular, in view of (46)–(49),

$$|\mathcal{B}_i|\le\begin{cases}8\left(4+\deg(f_1^{(i)}(x))\right) & \text{if } i=0\\ 16\left(4+\deg(f_1^{(i)}(x))\right) & \text{if } i\neq 0\end{cases}$$

holds, from which the proof of the theorem now follows by taking

$$C(\mathcal{G},\Delta,L):=16\,|\mathcal{G}|\left(3+\sum_{(\gamma_i,\alpha_i,\beta_i)\in\underline{\mathcal{G}}\setminus\underline{\Delta}}L(i)\right).$$

∎

Now, for the (κ,λ)-quadruple $(\mathcal{G},\Delta;L,\ell)$, let $\mathcal{T}=\mathcal{T}(\mathcal{G},\ell)$ be the associated tridiagonal sequence. Let θ be a real number. Then, for each $0\le i<g$ satisfying $\theta>\mathfrak{R}_{g-i}$, there exist complex numbers $\nu_1^{(i)}(\theta)$ and $\nu_2^{(i)}(\theta)$ such that the terms in the standard sequence $\left(u_j=u_j(\theta)\right)_{j=0}^{D_{\mathcal{T}}}$ satisfy

$$u_{s(g-i+1)-j}(\theta)=\nu_1^{(i)}(\theta)x_i^j(\theta)+\nu_2^{(i)}(\theta)y_i^j(\theta)\quad(0\le j\le\ell(g-i)+1), \tag{53}$$

where $s(g-i+1)$, $x_i(\theta)$ and $y_i(\theta)$ are as defined in (22) and (50). Note that $x_i(\theta)-y_i(\theta)\neq 0$ holds, and that $(\nu_1^{(i)}(\theta),\,\nu_2^{(i)}(\theta))\neq(0,0)$ holds as $(u_0,u_1)\neq(0,0)$. Taking $j=0,1$ in (53) we obtain:

$$\nu_1^{(i)}(\theta)=\left(\frac{-y_i(\theta)}{x_i(\theta)-y_i(\theta)}\right)u_{s(g-i+1)}(\theta)+\left(\frac{1}{x_i(\theta)-y_i(\theta)}\right)u_{s(g-i+1)-1}(\theta); \tag{54}$$

$$\nu_2^{(i)}(\theta)=\left(\frac{x_i(\theta)}{x_i(\theta)-y_i(\theta)}\right)u_{s(g-i+1)}(\theta)+\left(\frac{-1}{x_i(\theta)-y_i(\theta)}\right)u_{s(g-i+1)-1}(\theta). \tag{55}$$

In particular, in view of (44), (45) and (53), for each $\delta_{\tau-i}=(\gamma_{g-j_i},\alpha_{g-j_i},\beta_{g-j_i})\in\underline{\Delta}$, there exist polynomials $f_t^{(i)}=f_t^{(i)}(x)$, $g_t^{(i)}=g_t^{(i)}(x)$ $(t=1,2)$ in $\mathbb{Q}[x]$ such that

$$u_{s(g-j_i+1)}(\theta)=\begin{cases}f_1^{(i)}(\theta)u_{s(g-j_{(i-1)})-1}(\theta)+g_1^{(i)}(\theta)u_{s(g-j_{(i-1)})}(\theta) & \text{if } i\neq 0\\ f_1^{(i)}(\theta)u_{D-1}(\theta)+g_1^{(i)}(\theta)u_D(\theta) & \text{if } i=0\end{cases} \tag{56}$$

and

$$u_{s(g-j_i+1)-1}(\theta) = \begin{cases} f_2^{(i)}(\theta)u_{s(g-j_{(i-1)})-1}(\theta) + g_2^{(i)}(\theta)u_{s(g-j_{(i-1)})}(\theta) & \text{if } i \neq 0 \\ f_2^{(i)}(\theta)u_{D-1}(\theta) + g_2^{(i)}(\theta)u_D(\theta) & \text{if } i = 0 \end{cases} \tag{57}$$

hold, where $f_t^{(i)}(x) - t + 1$ and $g_t^{(i)}(x) + t - 2$ are the zero polynomials if $j_i - j_{(i-1)} = 1$.

The last theorem of this section will play an important role later on in obtaining an upper bound for the Christoffel numbers of any eigenvalue of $\mathcal{T}(\mathcal{G}, \ell)$ within some closed interval not intersecting $\mathcal{B}$. For any non-empty closed real interval I, we define $I_{\min}$ and $I_{\max}$ to be the real numbers for which $I = [I_{\min}, I_{\max}]$ holds.

Theorem 6.3 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$. Suppose that $(\mathcal{G}, \Delta; L, \ell)$ is a (κ, λ)-quadruple and let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$. Suppose that I is a non-empty, closed subinterval of $(\mathfrak{R}_1, \mathfrak{R}_{\max})$ such that*

$$I \cap \Big(\mathcal{B} \cup \{\mathfrak{R}_i \,:\, 1 \leq i \leq g\}\Big) = \emptyset \text{ and } 2 \leq \mathfrak{b} < g \tag{58}$$

both hold, where $\mathfrak{b} = \mathfrak{b}(\mathcal{G}, I) := \max\{2 \leq i \leq g \,:\, I_{\max} < \mathfrak{R}_i\}$. Then for each $(\gamma_{g-i}, \alpha_{g-i}, \beta_{g-i}) \in \underline{\Delta}$ with $\mathfrak{b} + 1 \leq g - i \leq g$, there exist positive constants $C_i = C_i(\kappa, \mathcal{G}, \Delta, L, I) \geq 1$ and $M_i = M_i(\kappa, \mathcal{G}, \Delta, L, I) > 1$ such that, if $\ell(g-j) > C_i$ hold for all $j < i$ with $(\gamma_{g-j}, \alpha_{g-j}, \beta_{g-j}) \in \underline{\Delta}$, then

$$\left|\frac{\nu_1^{(i)}(\theta)}{\nu_2^{(i)}(\theta)}\right| > M_i \left(\frac{y_i(\theta)}{x_i(\theta)}\right)^{C_i} \tag{59}$$

holds for any real number $\theta \in I$, where $\mathfrak{R}_i$, $\underline{\Delta}$, $x_i(\theta)$, $y_i(\theta)$, $\mathcal{B} = \mathcal{B}(\mathcal{G}, \Delta, L)$ and $\nu_j^{(i)}(\theta)$ $(j = 1, 2)$ are as defined in (28), (32), (50), (52) and (53), respectively.

Proof: Let $\mathcal{T} = \mathcal{T}(\mathcal{G}, \ell)$ be the (κ, λ)-tridiagonal sequence associated to $(\mathcal{G}, \Delta; L, \ell)$, let $D := D_{\mathcal{T}}$ and let $\Delta = (\delta_i)_{i=1}^{\tau}$. Note that for each $0 \leq j < g - \mathfrak{b}$, $0 < \frac{y_j(\theta)}{x_j(\theta)} < 1$ holds for any $\theta \in I$, and also that $\frac{y_j(\theta)}{x_j(\theta)}$ is a non-zero continuous function (in θ) on the closed interval I. Hence, there exists a constant $0 < P = P(\mathcal{G}, I) < 1$ such that

$$\frac{y_j(\theta)}{x_j(\theta)} = \frac{\theta - \alpha_{g-j} - \sqrt{(\theta - \alpha_{g-j})^2 - 4\beta_{g-j}\gamma_{g-j}}}{\theta - \alpha_{g-j} + \sqrt{(\theta - \alpha_{g-j})^2 - 4\beta_{g-j}\gamma_{g-j}}} \leq P < 1 \tag{60}$$

holds for any $0 \leq j < g - \mathfrak{b}$ and for any $\theta \in I$. Note also that for each $0 \leq j < g - \mathfrak{b}$, $\beta_{g-j} \leq \gamma_{g-j}$ holds by Lemma 3.3.

Now, for each $\delta_{\tau-s} \in \underline{\Delta}$, let

$$\delta_{\tau-s} := (\gamma_{g-j_s}, \alpha_{g-j_s}, \beta_{g-j_s})$$

for some $0 \leq j_s < g$. We prove the theorem by induction on $0 \leq s \leq \mathfrak{s}$, where

$$\mathfrak{s} := \max\{i \,:\, \delta_{\tau-i} \in \underline{\Delta} \text{ and } \mathfrak{R}_{g-j_i} < I_{\min}\}.$$

First, suppose $s=0$. Let $\delta_\tau = (\gamma_{g-j_0}, \alpha_{g-j_0}, \beta_{g-j_0}) \in \underline{\Delta}$ for some $0 \le j_0 \le g - \mathfrak{b} - 1$.
By (9), $\gamma_{g+1} u_{D-1}(\theta) + (\alpha_{g+1} - \theta) u_D(\theta) = 0$ and $u_D(\theta) \neq 0$ for any $\theta \in I$. By (54)–(57), there exist polynomials $f_t^{(0)}(x)$, $g_t^{(0)}(x) \in \mathbb{Q}[x]$ $(t = 1, 2)$ such that

$$\begin{aligned} \nu_1^{(j_0)}(\theta) &= \left(\frac{(\theta - \alpha_{g+1})(-f_1^{(0)}(\theta) y_{j_0}(\theta) + f_2^{(0)}(\theta)) + \gamma_{g+1}(-g_1^{(0)}(\theta) y_{j_0}(\theta) + g_2^{(0)}(\theta))}{\gamma_{g+1}(x_{j_0}(\theta) - y_{j_0}(\theta))} \right) u_D(\theta) \quad \text{and} \\ \nu_2^{(j_0)}(\theta) &= \left(\frac{(\theta - \alpha_{g+1})(f_1^{(0)}(\theta) x_{j_0}(\theta) - f_2^{(0)}(\theta)) + \gamma_{g+1}(g_1^{(0)}(\theta) x_{j_0}(\theta) - g_2^{(0)}(\theta))}{\gamma_{g+1}(x_{j_0}(\theta) - y_{j_0}(\theta))} \right) u_D(\theta) \end{aligned}$$

both hold for all $\theta \in I$. It follows by (58) and $u_D(\theta) \neq 0$ that $\nu_t^{(j_0)}(\theta) \neq 0$ $(t = 1, 2)$ for all $\theta \in I$. Since the function $\frac{\nu_1^{(j_0)}(\theta)}{\nu_2^{(j_0)}(\theta)}$ is a non-zero continuous function on the closed interval I, by (60) there exist constants $N_{j_0} := N_{j_0}(\kappa, \mathcal{G}, \Delta, L, I) > 0$ and $C_{j_0} := C_{j_0}(\kappa, \mathcal{G}, \Delta, L, I) \ge 1$ so that

$$\left| \frac{\nu_1^{(j_0)}(\theta)}{\nu_2^{(j_0)}(\theta)} \right| \ge N_{j_0} > P^{C_{j_0} - 1} \ge \frac{1}{P} \left(\frac{y_{j_0}(\theta)}{x_{j_0}(\theta)} \right)^{C_{j_0}} \tag{61}$$

holds for any $\theta \in I$. Hence there exist constants $C_{j_0} = C_{j_0}(\kappa, \mathcal{G}, \Delta, L, I) \ge 1$ and $M_{j_0} = M_{j_0}(\kappa, \mathcal{G}, \Delta, L, I) := 1/P$ such that (59) holds for all $\theta \in I$. This completes the proof of the base case.

Now, suppose $0 < s \le \mathfrak{s}$, and assume that the theorem holds for all $\delta_{\tau - j} \in \underline{\Delta}$ with $0 \le j < s$. Let $x := x_{j_{(s-1)}}(\theta)$, $y := y_{j_{(s-1)}}(\theta)$, $\nu_t := \nu_t^{(j_{(s-1)})}(\theta)$ $(t = 1, 2)$ and $\ell := \ell(g - j_{(s-1)})$. In view of (53) and (56) and (57), there exist polynomials $f_t^{(s)}(x)$ and $g_t^{(s)}(x)$ $(t = 1, 2)$ such that, putting $f_1 := f_1^{(s)}(\theta)$, $f_2 := f_2^{(s)}(\theta)$, $g_1 := g_1^{(s)}(\theta)$ and $g_2 := g_2^{(s)}(\theta)$,

$$\begin{aligned} u_{s(g-j_s+1)} &= f_1 u_{s(g-j_{(s-1)})-1} + g_1 u_{s(g-j_{(s-1)})} = (f_1 x + g_1) \nu_1 x^\ell + (f_1 y + g_1) \nu_2 y^\ell; \quad (62) \\ u_{s(g-j_s+1)-1} &= f_2 u_{s(g-j_{(s-1)})-1} + g_2 u_{s(g-j_{(s-1)})} = (f_2 x + g_2) \nu_1 x^\ell + (f_2 y + g_2) \nu_2 y^\ell \quad (63) \end{aligned}$$

both hold for all $\theta \in I$. Let $x' := x_{j_s}(\theta)$ and $y' := y_{j_s}(\theta)$, and define $M_{j_s} = M_{j_s}(\kappa, \mathcal{G}, \Delta, L, I)$ by

$$M_{j_s}(\kappa, \mathcal{G}, \Delta, L, I) := \max \left\{ M_{j_{(s-1)}}, 2 \left| \frac{(f_1 y' - f_2) y + g_1 y' - g_2}{(f_1 y' - f_2) x + g_1 y' - g_2} \right|, 2 \left| \frac{(f_1 x' - f_2) y + g_1 x' - g_2}{(f_1 x' - f_2) x + g_1 x' - g_2} \right| : \theta \in I \right\}. \tag{64}$$

By the induction hypothesis, if $\ell(g - j_t) > C_{j_{(s-1)}}$ holds for all $0 \le t < s - 1$, then (59) holds for the case $i = j_{(s-1)}$. Moreover, there exists an integer $E_{j_s} := E_{j_s}(\kappa, \mathcal{G}, \Delta, L, I) \ge 1$ so that

$$\left| \frac{\nu_1}{\nu_2} \right| \left(\frac{x}{y} \right)^{E_{j_s}} > M_{j_s} \tag{65}$$

holds for any $\theta \in I$. Now take $C'_{j_s}(\kappa, \mathcal{G}, \Delta, L, I) := \max\{E_{j_s}, C_{j_{(s-1)}}\}$ and suppose that $\ell(g - j_t) > C'_{j_s}$ holds for all $0 \le t < s$. If $\nu_1^{(j_s)}(\eta) = 0$ holds for some $\eta \in I$, then $u_{s(g-j_s+1)-1}(\eta) = u_{s(g-j_s+1)}(\eta) y_{j_s}(\eta)$ holds and so, as $\eta \notin \mathcal{B}$,

$$\begin{aligned} &\bar{\nu}_1 \bar{x}^\ell \left((\bar{f}_1 \bar{y}' - \bar{f}_2) \bar{x} + \bar{g}_1 \bar{y}' - \bar{g}_2 \right) = \bar{\nu}_2 \bar{y}^\ell \left((-\bar{f}_1 \bar{y}' + \bar{f}_2) \bar{y} - \bar{g}_1 \bar{y}' + \bar{g}_2 \right) \quad \text{and} \\ &\left((\bar{f}_1 \bar{y}' - \bar{f}_2) \bar{x} + \bar{g}_1 \bar{y}' - \bar{g}_2 \right) \left((-\bar{f}_1 \bar{y}' + \bar{f}_2) \bar{y} - \bar{g}_1 \bar{y}' + \bar{g}_2 \right) \neq 0 \end{aligned}$$

both hold, where $\bar{\nu}_t = \nu_t^{(j_{(s-1)})}(\eta)$, $\bar{f}_t = f_t^{(s)}(\eta)$, $\bar{g}_t = g_t^{(s)}(\eta)$ $(t = 1, 2)$, $\bar{x} = x_{j_{(s-1)}}(\eta)$, $\bar{x}' = x_{j_s}(\eta)$, $\bar{y} = y_{j_{(s-1)}}(\eta)$, and $\bar{y}' = y_{j_{(s-1)}}(\eta)$. This contradicts (64) and (65) as

$$M_{j_s} < \left|\frac{\bar{\nu}_1}{\bar{\nu}_2}\right| \left(\frac{\bar{x}}{\bar{y}}\right)^{E_{j_s}} \leq \left|\frac{\bar{\nu}_1}{\bar{\nu}_2}\right| \left(\frac{\bar{x}}{\bar{y}}\right)^{\ell} = \left|\frac{(\bar{f}_1\bar{y}' - \bar{f}_2)\bar{y} + \bar{g}_1\bar{y}' - \bar{g}_2}{(\bar{f}_1\bar{y}' - \bar{f}_2)\bar{x} + \bar{g}_1\bar{y}' - \bar{g}_2}\right|.$$

Similarly, it follows that $\nu_t^{(j_s)}(\theta) \neq 0$ $(t = 1, 2)$ must hold for all $\theta \in I$. Moreover, by (53), (62) and (64), there exists a constant $N_{j_s} := N_{j_s}(\kappa, \mathcal{G}, \Delta, L, I) > 0$ such that

$$\begin{aligned}
\left|\frac{\nu_1^{(j_s)}(\theta)}{\nu_2^{(j_s)}(\theta)}\right| &= \left|\frac{\nu_1 x^\ell\Big((-f_1y' + f_2)x - g_1y' + g_2\Big) + \nu_2 y^\ell\Big((-f_1y' + f_2)y - g_1y' + g_2\Big)}{\nu_1 x^\ell\Big((f_1x' - f_2)x + g_1x' - g_2\Big) + \nu_2 y^\ell\Big((f_1x' - f_2)y + g_1x' - g_2\Big)}\right| \\
&\geq \frac{M_{j_s}\Big|(f_1y' - f_2)x + g_1y' - g_2\Big| - \Big|(f_1y' - f_2)y + g_1y' - g_2\Big|}{M_{j_s}\Big|(f_1x' - f_2)x + g_1x' - g_2\Big| + \Big|(f_1x' - f_2)y + g_1x' - g_2\Big|} \\
&\geq \frac{\Big|(f_1y' - f_2)y + g_1y' - g_2\Big|}{M_{j_s}\Big|(f_1x' - f_2)x + g_1x' - g_2\Big| + \Big|(f_1x' - f_2)y + g_1x' - g_2\Big|} \\
&\geq N_{j_s} \qquad (66)
\end{aligned}$$

holds for any $\theta \in I$. This implies that there exists a positive constant $C''_{j_s} = C''_{j_s}(\kappa, \mathcal{G}, \Delta, L, I) \geq 1$ such that $N_{j_s} > M_{j_s} P^{C''_{j_s}}$ holds. Hence by taking $C_{j_s} = C_{j_s}(\kappa, \mathcal{G}, \Delta, L, I) := \max\{C'_{j_s}, C''_{j_s}\}$, it follows that

$$\left|\frac{\nu_1^{(j_s)}(\theta)}{\nu_2^{(j_s)}(\theta)}\right| \geq N_{j_s} > M_{j_s} P^{C_{j_s}} \geq M_{j_s}\left(\frac{y_{j_s}(\theta)}{x_{j_s}(\theta)}\right)^{C_{j_s}}$$

holds for all $\theta \in I$. By applying the induction hypothesis, it follows that the desired result holds for each $(\gamma_{g-j_s}, \alpha_{g-j_s}, \beta_{g-j_s}) \in \underline{\Delta}$ satisfying $0 \leq j_s < g - \mathfrak{b}$. This completes the proof of the theorem. ∎

7 Well-Placed Intervals

In this section, we define the concept of a well-placed interval with respect to a graphical sequence, and derive some simple properties of such intervals that will be used later on. Note that our definition of a well-placed interval is similar (but not identical) to the one presented in [3].

Suppose that $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ is a (κ, λ)-graphical sequence, where $\kappa \geq 3$ and $0 \leq \lambda \leq \kappa - 2$ are integers.

For any closed subinterval $I = [I_{\min}, I_{\max}]$ of $(\mathfrak{R}_1, \mathfrak{R}_{\max})$ with positive length, define integers $\mathfrak{a} = \mathfrak{a}(\mathcal{G}, I)$, $\mathfrak{b} = \mathfrak{b}(\mathcal{G}, I)$, $\mathfrak{c} = \mathfrak{c}(\mathcal{G}, I)$ and $\mathfrak{d} = \mathfrak{d}(\mathcal{G}, I)$ (that depend only on $\mathcal{G}$ and I) by

$$\mathfrak{a}(\mathcal{G}, I) := \min\{2 \leq i \leq g \,:\, I_{\max} < \mathfrak{R}_i\}, \qquad (67)$$

$$\mathfrak{b}(\mathcal{G}, I) := \max\{2 \leq i \leq g \,:\, I_{\max} < \mathfrak{R}_i\}, \qquad (68)$$

$$\mathfrak{c}(\mathcal{G}, I) \quad := \quad \min\Big\{\{2 \le i \le g \,:\, I_{\max} < \mathfrak{L}_i\} \cup \{g+1\}\Big\}, \tag{69}$$

$$\mathfrak{d}(\mathcal{G}, I) \quad := \quad \max\Big\{\{2 \le i \le g \,:\, I_{\max} < \mathfrak{L}_i\} \cup \{\mathfrak{c}\}\Big\}. \tag{70}$$

The interval I is called a *well-placed interval with respect to* $\mathcal{G}$ if it satisfies the following conditions:

(W1) I is a closed subinterval of the open interval $(\mathfrak{R}_1, \mathfrak{R}_{\max})$ with positive length;

(W2) If $I \cap I_j \neq \emptyset$ then $I \subseteq I_j$ holds, $1 \le j \le g$;

(W3) $I \subseteq I_{\mathfrak{a}}$, where $\mathfrak{a} := \mathfrak{a}(\mathcal{G}, I)$.

From now on, we will denote well-placed intervals using calligraphic script (e.g. $\mathcal{I}$ instead of I) to help the reader follow the text.

In the rest of the section, we will derive some properties of well-placed intervals. We start with recording some simple properties of the numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$.

Lemma 7.1 *Let $\kappa \ge 3$ and $\lambda \ge 0$ be integers with $\lambda \le \kappa - 2$, and let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence. Let $\mathcal{I} = [\mathcal{I}_{\min}, \mathcal{I}_{\max}]$ be a well-placed interval with respect to $\mathcal{G}$. For the numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ as defined in (67)–(70), the following hold:*
(i) $2 \le \mathfrak{a} \le \mathfrak{b} \le g$.
(ii) $\mathfrak{c} \le \mathfrak{d}$.
(iii) If $\mathfrak{c} \le g$, then $2 \le \mathfrak{a} < \mathfrak{c} \le \mathfrak{d} \le \mathfrak{b} \le g$ holds.
(iv) $\{1 \le i \le g \,:\, 1 \le i < \mathfrak{a}$ or $\mathfrak{b} < i \le g\} \subseteq \{1 \le i \le g \,:\, \mathfrak{R}_i < \mathcal{I}_{\min}\}$.
(v) If $\mathfrak{c} \le g$, then $\{1 \le i \le g \,:\, \mathfrak{a} \le i < \mathfrak{c}$ or $\mathfrak{d} < i \le \mathfrak{b}\} \subseteq \{1 \le i \le g \,:\, \mathcal{I} \subseteq I_i\}$ holds.
(vi) If $\mathfrak{c} = g+1$, then $\{1 \le i \le g \,:\, \mathfrak{a} \le i \le \mathfrak{b}\} = \{1 \le i \le g \,:\, \mathcal{I} \subseteq I_i\}$ holds.

Proof: (i)–(iii) are simple consequences of the definitions of well-placed intervals and the numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$.

(iv)–(vi) are direct consequences of the following inequalities, which follow in view of the fact that the sequence $(\mathfrak{R}_i)_{i=1}^{g}$ is unimodal by Lemma 3.3:

$$\max\{\mathfrak{R}_i \,:\, 1 \le i < \mathfrak{a} \text{ or } \mathfrak{b} < i \le g\} < \mathcal{I}_{\min} < \mathcal{I}_{\max} < \min\{\mathfrak{R}_i \,:\, \mathfrak{a} \le i \le \mathfrak{b}\}$$

and

$$\max\{\mathfrak{L}_i \,:\, 1 \le i < \min\{\mathfrak{c},\ g+1\} \text{ or } \min\{\mathfrak{d},\ g+1\} < i \le g\} < \mathcal{I}_{\min}.$$

∎

We now present a result that ensures the existence of well-placed intervals.

Proposition 7.2 *Let $\kappa \ge 3$ and $\lambda \ge 0$ be integers with $\lambda \le \kappa - 2$, and let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence.*

(i) For each $(\gamma_i, \alpha_i, \beta_i) \in \underline{\mathcal{G}} \setminus \{(1, \lambda, \kappa - \lambda - 1), (\kappa - \lambda - 1, \lambda, 1), (\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1})\}$ *and for any closed subinterval* $I \subseteq (\mathfrak{R}_1, \mathfrak{R}_i)$ *with positive length, there exists a well-placed interval* $\mathcal{J}_i \subseteq I$ *with respect to* $\mathcal{G}$ *(cf. (28)).*

(ii) Let $\mathcal{I}$ *be a well-placed interval with respect to* $\mathcal{G}$*. Then any closed interval* $\mathcal{J} \subseteq \mathcal{I}$ *with positive length is also a well-placed interval with respect to* $\mathcal{G}$*. In particular,* $\mathfrak{a}(\mathcal{G}, \mathcal{J}) = \mathfrak{a}(\mathcal{G}, \mathcal{I})$, $\mathfrak{b}(\mathcal{G}, \mathcal{J}) = \mathfrak{b}(\mathcal{G}, \mathcal{I})$, $\mathfrak{c}(\mathcal{G}, \mathcal{J}) = \mathfrak{c}(\mathcal{G}, \mathcal{I})$, $\mathfrak{d}(\mathcal{G}, \mathcal{J}) = \mathfrak{d}(\mathcal{G}, \mathcal{I})$ *must all hold (cf. (67)–(70)).*

Proof: (i): Let $(\gamma_i, \alpha_i, \beta_i) \in \underline{\mathcal{G}} \setminus \{(1, \lambda, \kappa - \lambda - 1), (\kappa - \lambda - 1, \lambda, 1), (\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1})\}$ and suppose that $I = [I_{\min}, I_{\max}] \subseteq (\mathfrak{R}_1, \mathfrak{R}_i)$ is a subinterval with positive length. Define

$$\mathfrak{M}_i := \max\{I_{\min},\, y \;:\; y \in \{\mathfrak{R}_j,\, \mathfrak{L}_j \;:\; 1 \le j \le g\} \text{ and } I_{\min} \le y < I_{\max}\}.$$

Then $I_{\min} \le \mathfrak{M}_i < I_{\max}$, and the closed interval

$$\mathcal{J}_i := \left[\frac{I_{\max} + 2\,\mathfrak{M}_i}{3},\, \frac{2\,I_{\max} + \mathfrak{M}_i}{3}\right]$$

is a well-placed interval with respect to $\mathcal{G}$ satisfying $\mathcal{J}_i \subseteq I$.
(ii): This follows immediately from the definition of well-placed intervals. ▮

Now, suppose that $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ is a function with $\ell(g+1) = 1$. For $\mathcal{I}$ a well-placed interval with respect to $\mathcal{G}$, we define $\mathcal{C} = \mathcal{C}_{\mathcal{G},\mathcal{I}}$, $\mathtt{Len}(\mathcal{I}) = \mathtt{Len}_{\mathcal{G},\ell}(\mathcal{I})$ and $\mathtt{Gap}(\mathcal{I}) = \mathtt{Gap}_{\mathcal{G},\ell}(\mathcal{I})$ as follows:

$$\mathcal{C}_{\mathcal{G},\mathcal{I}} := \begin{cases} \{1 \le i \le g \,:\, \mathfrak{a} \le i < \mathfrak{c} \;\text{ or }\; \mathfrak{d} < i \le \mathfrak{b}\} & \text{if } \mathfrak{c} \le g \\ \{1 \le i \le g \,:\, \mathfrak{a} \le i \le \mathfrak{b}\} & \text{if } \mathfrak{c} = g+1 \end{cases},$$

$$\mathtt{Len}_{\mathcal{G},\ell}(\mathcal{I}) := \sum_{j \in \mathcal{C}} \ell(j)\,, \tag{71}$$

$$\mathtt{Gap}_{\mathcal{G},\ell}(\mathcal{I}) := \begin{cases} \sum_{\mathfrak{c} \le j \le \mathfrak{d}} \ell(j) & \text{if } \mathfrak{c} \le g \\ 0 & \text{if } \mathfrak{c} = g+1 \end{cases}. \tag{72}$$

Using Proposition 7.2, we now show that for any (κ, λ)-graphical sequence $\mathcal{G}$, there is a certain family of well-placed intervals with respect to $\mathcal{G}$ each of whose members avoid the set $\mathcal{B}(\mathcal{G}, \Delta, L)$ as defined in Definition 6.1.

Corollary 7.3 *Let* $\kappa \ge 3$ *and* $\lambda \ge 0$ *be integers with* $\lambda \le \kappa - 2$*. Suppose that* $(\mathcal{G}, \Delta; L, \ell)$ *is a* (κ, λ)*-quadruple and let* $\mathcal{G} = \left(\delta_i := (\gamma_i, \alpha_i, \beta_i)\right)_{i=1}^{g+1}$*. Then for any closed subinterval* $I \subseteq (\mathfrak{R}_1, \mathfrak{R}_i)$ *with positive length, there exists a well-placed interval* $\mathcal{J}_i$ *in* I *such that* $\mathcal{J}_i \cap \mathcal{B} = \emptyset$ *holds (cf. (28), (32) and (52)).*
In particular, $\mathtt{Len}(\mathcal{J}_i) \ge \ell(i)$ *also holds.*

Proof: Suppose that $(\gamma_i, \alpha_i, \beta_i)$ and I are as in the statement of the corollary. By Proposition 7.2 (i), there exists a well-placed interval $\mathcal{I} \subseteq I$ with respect to $\mathcal{G}$. By Theorem 6.2, the set $\mathcal{B}$ is finite.

Hence, by Proposition 7.2 (ii), we may take any closed subinterval $\mathcal{J}_i$ of $\mathcal{I} \setminus \mathcal{B}$ with positive length to give the desired well-placed interval. ∎

We conclude this section by showing that, in addition, well-placed intervals satisfying certain other properties also exist.

Proposition 7.4 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$. Let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence and $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ is a function with $\ell(g+1) = 1$. Suppose that $\mathcal{I}$ is a well-placed interval with respect to $\mathcal{G}$ such that $\texttt{Gap}(\mathcal{I}) \neq 0$ holds. Then there exists a well-placed interval $\mathcal{J}$ such that*
(i) $\mathcal{J}_{\min} > \mathcal{I}_{\max}$,
(ii) $\texttt{Gap}(\mathcal{J}) < \texttt{Gap}(\mathcal{I})$ and
(iii) $\texttt{Len}(\mathcal{J}) > \frac{\texttt{Gap}(\mathcal{I})}{g}$
all hold, where $\texttt{Len}(\mathcal{J}) := \texttt{Len}_{\mathcal{G},\ell}(\mathcal{J})$ and $\texttt{Gap}(\mathcal{I}) := \texttt{Gap}_{\mathcal{G},\ell}(\mathcal{I})$ are as defined in (71) and (72), respectively.

Proof: As the sequence $(\mathfrak{R}_i)_{i=1}^{g}$ is unimodal by Lemma 3.3 and since $\texttt{Gap}(\mathcal{I}) \neq 0$, there exists an integer j with $\mathfrak{c} \leq j \leq \mathfrak{d}$ such that $\ell(j) > \frac{\texttt{Gap}(\mathcal{I})}{g}$ and $\mathfrak{R}_j > \mathcal{I}_{\max}$ both hold, where $\mathfrak{c} = \mathfrak{c}(\mathcal{G}, \mathcal{I})$ and $\mathfrak{d} = \mathfrak{d}(\mathcal{G}, \mathcal{I})$ are as defined in (69) and (70). Hence, by Proposition 7.2, there exists such a well-placed interval $\mathcal{J} \subseteq (\mathcal{I}_{\max}, \mathfrak{R}_j) \subseteq I_j$ as $\texttt{Len}(\mathcal{J}) \geq \ell(j) > \frac{\texttt{Gap}(\mathcal{I})}{g}$. The result now follows. ∎

8 Christoffel Numbers

In this section, we prove a result that will allow us to bound the Christoffel numbers of the (κ, λ)-tridiagonal sequence associated to a (κ, λ)-quadruple. We will begin by stating the main theorem of this section, whose proof will be split into several steps. To state this result, we require some further definitions.

Let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence for some integers $\kappa \geq 3$ and $\lambda \geq 0$ with $\lambda \leq \kappa - 2$. Let x be a real number. For each $(\gamma_i, \alpha_i, \beta_i) \in \underline{\mathcal{G}} \setminus \{(\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1})\}$, define $\rho_i = \rho_i(x)$ and $\sigma_i = \sigma_i(x)$ to be the roots of the (auxiliary) equation

$$\beta_i z^2 + (\alpha_i - x)z + \gamma_i = 0 \ , \tag{73}$$

which, without loss of generality, we assume to satisfy $|\rho_i| \geq |\sigma_i|$ for all $1 \leq i \leq g$.

Theorem 8.1 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$. Suppose that $(\mathcal{G}, \Delta; L, \ell)$ is a (κ, λ)-quadruple and let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$. Suppose that $\mathcal{I}$ is a well-placed interval with respect to $\mathcal{G}$ satisfying $\mathcal{I} \cap \mathcal{B}(\mathcal{G}, \Delta, L) = \emptyset$, with $\mathcal{B}(\mathcal{G}, \Delta, L)$ as defined in Definition 6.1. Then there exist positive constants $F := F(\kappa, \mathcal{G}, \Delta, L, \mathcal{I})$, $C_1 := C_1(\kappa, \mathcal{G}, \mathcal{I})$ and $C_2 := C_2(\kappa, \mathcal{G}, \Delta, L, \mathcal{I})$ so that if*

$\ell(i) > F$ holds for all $(\gamma_i, \alpha_i, \beta_i) \in \underline{\Delta}$ then, for any $\theta \in \mathcal{I}$, the following holds :

$$\begin{aligned} C_1 \left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})} \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right)\rho_i^2\right)^{\ell(i)} &\leq \sum_{i=0}^{D} \kappa_i u_i^2 \\ &\leq C_2 (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right)\rho_i^2\right)^{\ell(i)} \quad , \end{aligned}$$

where κ_i and $u_i := u_i(\theta)$ are as defined in (6) and (7) for the matrix $L_1(\mathcal{T}(\mathcal{G},\ell))$, and $D := D_{\mathcal{T}(\mathcal{G},\ell)}$, $\mathfrak{a} := \mathfrak{a}(\mathcal{G},\mathcal{I})$, $\mathtt{Len}(\mathcal{I}) := \mathtt{Len}_{\mathcal{G},\ell}(\mathcal{I})$, $\mathtt{Gap}(\mathcal{I}) := \mathtt{Gap}_{\mathcal{G},\ell}(\mathcal{I})$ and $\rho_i := \rho_i(\theta)$ are as defined in (25), (67), (71), (72) and (73), respectively.

To prove Theorem 8.1, we will divide the sum $\sum_{i=0}^{D} \kappa_i u_i^2$ into three parts: The *Head sum* $\sum_{i=0}^{s(\mathfrak{a})-2} \kappa_i u_i^2$, the *Gap sum* $\sum_{i=s(\mathfrak{a})-1}^{s(\mathfrak{b}+1)} \kappa_i u_i^2$, and the *Tail sum* $\sum_{i=s(\mathfrak{b}+1)+1}^{D} \kappa_i u_i^2$. In particular, in Section 8.1 we will prove a preliminary result concerning three-term recurrence relations and, for completeness, recall some additional results on such recursions from previous papers. We will then use these results in Section 8.2 to derive bounds for the Head and the Gap sums (as well as to prove some results in Section 9). Then, in Section 8.3, we will derive an upper bound for the Tail sum which, together with the previous bounds, will be used to prove Theorem 8.1.

8.1 Three-Term Recurrence Relations

Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$. Suppose that $\mathcal{T} = \mathcal{T}(\mathcal{G},\ell)$ is a (κ,λ)-tridiagonal sequence and let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$. Let x be a real number, and let $\rho_i := \rho_i(x)$ and $\sigma_i := \sigma_i(x)$ be as defined in (73), noting that without loss of generality we are assuming $|\rho_i| \geq |\sigma_i|$ for all $1 \leq i \leq g$. If $x \notin \{\mathfrak{R}_i, \mathfrak{L}_i \,:\, 1 \leq i \leq g\}$, with $\mathfrak{R}_i$ and $\mathfrak{L}_i$ as defined in (28), then the roots ρ_i and σ_i are distinct, and so, by standard theory of recurrence relations, it follows that

$$u_{s(i)-1+j} = \omega_1^{(i)} \rho_i^j + \omega_2^{(i)} \sigma_i^j \quad (0 \leq j \leq \ell(i)+1) \tag{74}$$

holds for some complex numbers $\omega_1^{(i)} := \omega_1^{(i)}(x)$ and $\omega_2^{(i)} := \omega_2^{(i)}(x)$, where $u_i = u_i(x)$ are the numbers associated to the matrix $L_1(\mathcal{T})$ given by (7) and $s(i)$ is defined in (22). In this situation, note also that (1) if $|x - \alpha_i| > 2\sqrt{\beta_i\gamma_i}$ holds then the roots ρ_i and σ_i are real numbers with $|\rho_i| > \sqrt{\frac{\gamma_i}{\beta_i}} > |\sigma_i|$, and $\omega_1^{(i)}, \omega_2^{(i)}$ are real, and (2) if $|x - \alpha_i| < 2\sqrt{\beta_i\gamma_i}$ holds then the roots ρ_i and σ_i are complex numbers with $\sigma_i = \overline{\rho_i}$ and $|\rho_i| = |\sigma_i| = \sqrt{\frac{\gamma_i}{\beta_i}}$, and $\omega_1^{(i)}$, $\omega_2^{(i)}$ are complex numbers with $\omega_2^{(i)} = \overline{\omega_1^{(i)}}$.

We now prove a result that is analogous with the result [1, Proposition 3.1] that was proven to hold for distance-regular graphs.

Proposition 8.2
Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$. Let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ,λ)-graphical

sequence, $\ell : \{1, \dots, g+1\} \to \mathbb{N}$ be a function with $\ell(g+1) = 1$, and $\mathcal{T} = \mathcal{T}(\mathcal{G}, \ell)$ be the (κ, λ)-tridiagonal sequence associated to $\mathcal{G}$ and ℓ. Suppose that $\mathcal{I}$ is a well-placed interval with respect to $\mathcal{G}$. Then, for all $\theta \in I$ the following hold (cf. (67), (74)):
(i) $0 < \sigma_i(\theta) < \rho_i(\theta) < 1$, for all $1 \le i \le \mathfrak{a} - 1$.
(ii) $u_{s(i)-1}(\theta) > \prod_{j=1}^{i-1} \rho_j(\theta)^{\ell(j)}$, for all $2 \le i \le \mathfrak{a}$.
(iii) $-\omega_1^{(i)}(\theta) < \omega_2^{(i)}(\theta) < 0 < \omega_1^{(i)}(\theta)$, for all $1 \le i \le \mathfrak{a}$.

Proof: Suppose $\theta \in \mathcal{I}$, and put $\rho_i := \rho_i(\theta)$, $\sigma_i := \sigma_i(\theta)$, $(1 \le i \le g)$, $u_j := u_j(\theta)$ $(1 \le j \le D_{\mathcal{T}})$ and $\omega_j^{(i)} := \omega_j^{(i)}(\theta)$ $(j = 1, 2)$ as in (74).

(i): Since $\theta > \mathfrak{R}_i$ holds for all $1 \le i < \mathfrak{a}$ by (67), $0 < \sigma_i < \rho_i$ holds for all $1 \le i < \mathfrak{a}$. By (G0) in Definition 3.1 and by Lemma 3.3, $0 < \theta < \mathfrak{R}_{\max} = \max\{\kappa - (\sqrt{\beta_i} - \sqrt{\gamma_i})^2 \,:\, 1 \le i \le g\} \le \kappa$ and $\beta_i > \gamma_i$ $(1 \le i < \mathfrak{a})$ both hold. Hence

$$
\begin{aligned}
&2\beta_i - (\theta - \alpha_i) = (\kappa - \theta) + (\beta_i - \gamma_i) > 0 \quad \text{and} \\
&(2\beta_i - (\theta - \alpha_i))^2 - ((\theta - \alpha_i)^2 - 4\beta_i\gamma_i) = 4\beta_i(\kappa - \theta) > 0
\end{aligned}
$$

follow. Thus, (i) holds by (73) and the fact that $\rho_i = \frac{\theta - \alpha_i + \sqrt{(\theta-\alpha_i)^2 - 4\beta_i\gamma_i}}{2\beta_i}$ holds.

To prove that (ii) and (iii) hold, we will use the following claim.

Claim 8.3 *(a) $\rho_{i+1} < \rho_i$ $\;(1 \le i \le \mathfrak{a} - 1)$.*
(b) $u_{s(i)} > \rho_i\, u_{s(i)-1}$ $(1 \le i \le \mathfrak{a})$.
(c) $\omega_1^{(i)} > u_{s(i)-1}$ $(1 \le i \le \mathfrak{a})$.

Proof of Claim 8.3: In view of Proposition 8.2 (i) and $\beta_j\rho_j^2 + (\alpha_j - \theta)\rho_j + \gamma_j = 0$ $(1 \le j \le g)$, it follows that

$$
\begin{aligned}
&(\beta_i - 1)\rho_i^2 + (\alpha_i + 1 - \theta)\rho_i + \gamma_i = \rho_i(1 - \rho_i) > 0 \;\; \text{and} \\
&\beta_i\rho_i^2 + (\alpha_i - 1 - \theta)\rho_i + (\gamma_i + 1) = 1 - \rho_i > 0
\end{aligned}
$$

for all $1 \le i \le \mathfrak{a} - 1$. Hence, by (G2) in Definition 3.1, statement (a) in the claim holds.

We now prove statements (b) and (c) by using induction on i. Suppose $i = 1$. By

$$
\beta_1 \left(\frac{\theta}{\kappa}\right)^2 + (\lambda - \theta)\left(\frac{\theta}{\kappa}\right) + 1 = \left(1 - \frac{\theta}{\kappa}\right)\left(1 + (1 + \lambda)\left(\frac{\theta}{\kappa}\right)\right) > 0 \;,
$$

$\rho_1 u_0 = \rho_1 < \frac{\theta}{\kappa} = u_1$ hold. Thus, by Proposition 8.2 (i) and $\rho_1 u_0 = \rho_1(\omega_1^{(1)} + \omega_2^{(1)}) < \omega_1^{(1)}\rho_1 + \omega_2^{(1)}\sigma_1 = u_1$, (b) and (c) hold for $i = 1$.

Now let $2 \le i < \mathfrak{a}$, and suppose that (b) and (c) hold for all $2 \le j \le i$. By the induction hypothesis, it follows $u_{s(i+1)} - \rho_i u_{s(i+1)-1} = \omega_2^{(i)}\sigma_i^{\ell(i)}(\sigma_i - \rho_i) > 0$ by (i) of the proposition, and

$u_{s(i+1)} > \rho_i u_{s(i+1)-1} > \rho_{i+1} u_{s(i+1)-1}$ by statement (a) of the claim. Thus, by (74), (b) and (c) hold for all $1 \le i \le \mathfrak{a}$, which completes the proof of the claim. ∎

(ii): We prove this using induction on i. Suppose $i = 2$. Then by applying (b) and (c) of Claim 8.3 and statement (i) (with $i = 1$),

$$u_{s(2)-1} - \rho_1^{\ell(1)} = \rho_1^{\ell(1)}(\omega_1^{(1)} - 1) + \omega_2^{(1)}\sigma_1^{\ell(1)} > \rho_1(\omega_1^{(1)} - 1)(\rho_1^{\ell(1)-1} - \sigma_1^{\ell(1)-1}) > 0.$$

Therefore (ii) holds for $i = 2$.

Now let $2 \le i < \mathfrak{a}$, and suppose that (ii) holds for all $2 \le j \le i$. Using (i) and Claim 8.3 (c), it follows that

$$u_{s(i+1)-1} - u_{s(i)-1}\rho_i^{\ell(i)} = \omega_2^{(i)}(\sigma_i^{\ell(i)} - \rho_i^{\ell(i)}) > 0$$

holds. Hence, by induction, $u_{s(i+1)-1} > \prod_{j=1}^{i} \rho_j^{\ell(j)}$.

(iii): Using (ii) and Claim 8.3 (c), it follows $0 < u_{s(i)-1} = \omega_1^{(i)} + \omega_2^{(i)} < \omega_1^{(i)}$ for all $1 \le i \le \mathfrak{a}$. Now, (iii) follows immediately. ∎

We now recall a result that was originally stated using different terminology in [2] and [8].

Lemma 8.4 *(cf. [2, Lemma 5.1], [8, Proposition 7])*
Let $\kappa \ge 3$ and $\lambda \ge 0$ be integers with $\lambda \le \kappa - 2$. Let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence, $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ be a function with $\ell(g+1) = 1$, and $\mathcal{T} := \mathcal{T}(\mathcal{G}, \ell)$ be the (κ, λ)-tridiagonal sequence associated to $\mathcal{G}$ and ℓ with diameter $D_{\mathcal{T}}$ (cf. (25)). Let θ be any real number with $|\theta| \le \kappa$. Then for each $i = 1, \ldots, D_{\mathcal{T}} - 1$,
(i)

$$\frac{1}{3\kappa}\max\{|u_i(\theta)|, |u_{i+1}(\theta)|\} \le \max\{|u_{i-1}(\theta)|, |u_i(\theta)|\} \le 3\kappa \max\{|u_i(\theta)|, |u_{i+1}(\theta)|\}$$

and
(ii)

$$\left(\frac{1}{9\kappa^4}\right)\max\{\kappa_{i-1}u_{i-1}^2(\theta), \kappa_i u_i^2(\theta)\} \le \max\{\kappa_i u_i^2(\theta), \kappa_{i+1}u_{i+1}^2(\theta)\} \le 9\kappa^4 \max\{\kappa_{i-1}u_{i-1}^2(\theta), \kappa_i u_i^2(\theta)\}$$

hold, where κ_i and $u_i(\theta)$ are as defined in (6) and (7) for the matrix $L_1(\mathcal{T})$.

Proof: (i): Since $|\theta| \le \kappa$ and $0 < \beta_i$, $\gamma_i < \kappa$ $(1 \le i \le g)$ hold, it follows by (9) that

$$|u_{i+1}(\theta)| = \left|\left(\frac{\theta - \alpha_i}{\beta_i}\right)u_i(\theta) - \left(\frac{\gamma_i}{\beta_i}\right)u_{i-1}(\theta)\right| \le 2\,\kappa\,|u_i(\theta)| + \kappa\,|u_{i-1}(\theta)| \le 3\kappa\ \max\{|u_{i-1}(\theta)|,\ |u_i(\theta)|\}$$

and

$$|u_{i-1}(\theta)| = \left|\left(\frac{\theta - \alpha_i}{\gamma_i}\right)u_i(\theta) - \left(\frac{\beta_i}{\gamma_i}\right)u_{i+1}(\theta)\right| \le 2\,\kappa\,|u_i(\theta)| + \kappa\,|u_{i+1}(\theta)| \le 3\kappa\ \max\{|u_i(\theta)|,\ |u_{i+1}(\theta)|\}$$

all hold. Statement (i) now follows immediately.

(ii): Since $\frac{1}{\kappa}\kappa_{i+1} \le \kappa_i \le \kappa\,\kappa_{i-1},\ \ i = 1, \ldots, D_{\mathcal{T}}-1$, holds by (27), statement (ii) follows immediately from (i). ∎

For completeness, we now recall two results from [2].

Corollary 8.5 *([2, Corollary 4.2])*
Suppose $N \ge 2$ is an integer, and $\alpha \ge 0$, $\beta > 0$, $\gamma > 0$, x_0 and x_1 are real numbers satisfying $(x_0, x_1) \neq (0,0)$. Let ϵ be a real number with $0 < \epsilon < 2\sqrt{\beta\gamma}$. Then there exist positive real numbers $C_s := C_s(\beta, \gamma, \epsilon)$, $s = 1, 2, 3, 4$ such that for every real number θ with $|\theta - \alpha| \le 2\sqrt{\beta\gamma} - \epsilon$, and for all real numbers $x_2, \ldots, x_N$ satisfying $\gamma x_{i-1} + (\alpha - \theta)x_i + \beta x_{i+1} = 0$ $(i = 1, \ldots, N-1)$, we have

$$C_1 \max\left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\} \le \max\left\{\left(\frac{\beta}{\gamma}\right)^{i-1} x_{i-1}^2, \left(\frac{\beta}{\gamma}\right)^{i} x_i^2\right\} \le C_2 \max\left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\}$$

for $i = 1, 2, \ldots, N$, and

$$C_3 N \max\left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\} \le \sum_{i=0}^{N} \left(\frac{\beta}{\gamma}\right)^{i} x_i^2 \le C_4 N \max\left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\}.$$

Proposition 8.6 *([2, Proposition 4.3])*
Suppose $N \ge 2$ is an integer, and $\alpha \ge 0$, $\beta > 0$, $\gamma > 0$, x_0 and x_1 are real numbers satisfying $(x_0, x_1) \neq (0,0)$. Let κ, ϵ and ϵ' be positive real numbers. Then there exist constants $C_1 = C_1(\kappa, \alpha, \beta, \gamma, \epsilon) > 0$ and $C_2 = C_2(\beta, \gamma, \epsilon) > 1$ such that, for every real number θ with $|\theta - \alpha| \ge 2\sqrt{\beta\gamma} + \epsilon$, $|\theta| \le \kappa$, and

$$|x_1 - x_0\sigma| > \epsilon' \max\left\{|x_0|, \sqrt{\frac{\beta}{\gamma}}|x_1|\right\}$$

(with $\rho = \rho(\theta)$ and $\sigma = \sigma(\theta)$ the roots of $\beta x^2 + (\alpha - \theta)x + \gamma = 0$ with $|\rho| \ge |\sigma|$), and for all real numbers $x_2, \ldots, x_N$ satisfying $\gamma x_{i-1} + (\alpha - \theta)x_i + \beta x_{i+1} = 0$ $(i = 1, \ldots, N-1)$, we have

$$\sum_{i=0}^{N} \left(\frac{\beta}{\gamma}\right)^{i} x_i^2 \le C_1 \left(\left(\frac{\beta}{\gamma}\right)\rho^2\right)^{N} \max\left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\}$$

and, for all $n \le N$,

$$x_n^2 \le C_2\, \rho^{2n} \max\left\{x_0^2, \left(\frac{\beta}{\gamma}\right) x_1^2\right\}.$$

8.2 Bounding Head and Gap Sums

In this subsection, we obtain bounds for Head sum $\sum_{i=0}^{s(\mathfrak{a})-2} \kappa_i u_i^2$ and Gap sum $\sum_{i=s(\mathfrak{a})-1}^{s(\mathfrak{b}+1)} \kappa_i u_i^2$. In more detail, we prove the following:

Theorem 8.7 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$. Let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence, $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ be a function with $\ell(g+1) = 1$, and $\mathcal{T} := \mathcal{T}(\mathcal{G}, \ell)$ be the (κ, λ)-tridiagonal sequence associated to $\mathcal{G}$ and ℓ. Suppose that $\mathcal{I}$ is a well-placed interval with respect to $\mathcal{G}$. Then there exist positive constants $C_i := C_i(\kappa, \mathcal{G}, \mathcal{I})$ $(1 \leq i \leq 11)$ such that for any element θ in $\mathcal{I}$, the following all hold:*

(i) $\sum_{i=0}^{s(\mathfrak{a})-2} \kappa_i u_i^2 \leq C_1 \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)}$.

(ii) $\prod_{i=1}^{\mathfrak{a}-1} \rho_i^{2\ell(i)} < \max\left\{u_{s(\mathfrak{a})-1}^2, \left(\frac{\beta_\mathfrak{a}}{\gamma_\mathfrak{a}}\right) u_{s(\mathfrak{a})}^2\right\} \leq C_2 \prod_{i=1}^{\mathfrak{a}-1} \rho_i^{2\ell(i)}$.

(iii) Let $\widehat{\mathfrak{c}} := \min\{\mathfrak{c}, \mathfrak{b}+1\}$. *Then*

$$C_3 \left(\sum_{i=\mathfrak{a}}^{\widehat{\mathfrak{c}}-1} \ell(i)\right) \prod_{j=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_j}{\gamma_j}\right) \rho_j^2\right)^{\ell(j)} \leq \sum_{i=s(\mathfrak{a})-1}^{s(\widehat{\mathfrak{c}})} \kappa_i u_i^2 \leq C_4 \left(\sum_{i=\mathfrak{a}}^{\widehat{\mathfrak{c}}-1} \ell(i)\right) \prod_{j=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_j}{\gamma_j}\right) \rho_j^2\right)^{\ell(j)}.$$

(iv) Let $\widehat{\mathfrak{c}} := \min\{\mathfrak{c}, \mathfrak{b}+1\}$. *Then*

$$C_5 \prod_{i=1}^{\mathfrak{a}-1} \rho_i^{2\ell(i)} \leq \max\left\{u_{s(\widehat{\mathfrak{c}})-1}^2, \left(\frac{\beta_{\widehat{\mathfrak{c}}-1}}{\gamma_{\widehat{\mathfrak{c}}-1}}\right) u_{s(\widehat{\mathfrak{c}})}^2\right\} \prod_{i=\mathfrak{a}}^{\widehat{\mathfrak{c}}-1} \left(\frac{\beta_i}{\gamma_i}\right)^{\ell(i)} \leq C_6 \prod_{i=1}^{\mathfrak{a}-1} \rho_i^{2\ell(i)}.$$

(v) If $\mathfrak{c} \leq g$, *then* $\sum_{i=s(\mathfrak{c})}^{s(\mathfrak{d}+1)} \kappa_i u_i^2 < C_7 \left(9\kappa^4\right)^{\mathtt{Gap}(\mathcal{I})} \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)}$.

(vi) If $\mathfrak{c} \leq g$ *and* $\mathfrak{d} < \mathfrak{b}$ *both hold, then*

$$\begin{aligned} & C_8 \left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})} \left(\sum_{i=\mathfrak{d}+1}^{\mathfrak{b}} \ell(i)\right) \prod_{j=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_j}{\gamma_j}\right) \rho_j^2\right)^{\ell(j)} \\ \leq & \sum_{i=s(\mathfrak{d}+1)-1}^{s(\mathfrak{b}+1)} \kappa_i u_i^2 \leq C_9 (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \left(\sum_{i=\mathfrak{d}+1}^{\mathfrak{b}} \ell(i)\right) \prod_{j=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_j}{\gamma_j}\right) \rho_j^2\right)^{\ell(j)}. \end{aligned}$$

(vii) If $\mathfrak{c} \leq g$, *then*

$$\begin{aligned} C_{10} \left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})} \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)} & \leq \kappa_{s(\mathfrak{b}+1)} \max\left\{u_{s(\mathfrak{b}+1)-1}^2, \left(\frac{\beta_\mathfrak{b}}{\gamma_\mathfrak{b}}\right) u_{s(\mathfrak{b}+1)}^2\right\} \\ & \leq C_{11} (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)}, \end{aligned}$$

where κ_i and $u_i := u_i(\theta)$ are as defined in (6) and (7) relative to the matrix $L_1(\mathcal{T})$, and $s(i)$, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathtt{Gap}(\mathcal{I}) := \mathtt{Gap}_{\mathcal{G},\ell}(\mathcal{I})$ and $\rho_i := \rho_i(\theta)$ are as defined in (22), (67)–(70), (72) and (73), respectively.

Proof: Suppose that $\mathcal{G}$, ℓ, $\mathcal{T}$, $\mathcal{I}$ and θ are as in the statement of the theorem.

(i) and (ii): In order to apply Proposition 8.6, we first prove that there are positive constants $\epsilon_1 := \epsilon_1(\mathcal{G},\mathcal{I})$ and $\epsilon_2 := \epsilon_2(\mathcal{G},\mathcal{I})$ such that, for all $1 \le i \le \mathfrak{a}-1$,
(a) $|\theta-\alpha_i| \ge 2\sqrt{\beta_i\gamma_i}+\epsilon_1$, and
(b) $|u_{s(i)}-\sigma_i u_{s(i)-1}| > \epsilon_2 \max\left\{|u_{s(i)-1}|, \sqrt{\frac{\beta_i}{\gamma_i}}|u_{s(i)}|\right\}$
both hold.

For statement (a), we can take $\epsilon_1 = \epsilon_1(\mathcal{G},\mathcal{I}) := \min\{\mathcal{I}_{\min}-(\alpha_i+2\sqrt{\beta_i\gamma_i}) \,:\, 1\le i\le \mathfrak{a}-1\}$, in view of (67) and (W2).

By (67) and (73), inequalities $M_1 \ge \rho_i > 0$ and $\frac{\rho_i-\sigma_i}{\sqrt{\beta_i/\gamma_i}} \ge M_2 > 0$ all hold for any $\theta\in\mathcal{I}$ and for any $1\le i\le \mathfrak{a}-1$, where

$$
\begin{aligned}
M_1 = M_1(\mathcal{G},\mathcal{I}) &:= \max\left\{\frac{(\mathcal{I}_{\max}-\alpha_i)+\sqrt{(\mathcal{I}_{\max}-\alpha_i)^2-4\beta_i\gamma_i}}{2\beta_i} \,:\, 1\le i\le \mathfrak{a}-1\right\} \quad \text{and}\\
M_2 = M_2(\mathcal{G},\mathcal{I}) &:= \min\left\{\frac{\sqrt{\gamma_i}\sqrt{(\mathcal{I}_{\min}-\alpha_i)^2-4\beta_i\gamma_i}}{\beta_i\sqrt{\beta_i}} \,:\, 1\le i\le \mathfrak{a}-1\right\}.
\end{aligned}
$$

By (i) and (iii) of Proposition 8.2,

$$
\frac{|u_{s(i)}-\sigma_i u_{s(i)-1}|}{\max\left\{|u_{s(i)-1}|, \sqrt{\frac{\beta_i}{\gamma_i}}|u_{s(i)}|\right\}} = \left|\frac{\omega_1^{(i)}}{\omega_1^{(i)}\rho_i+\omega_2^{(i)}\sigma_i}\right| \frac{|\rho_i-\sigma_i|}{\sqrt{\frac{\beta_i}{\gamma_i}}} > \frac{1}{|\rho_i|}\frac{|\rho_i-\sigma_i|}{\sqrt{\frac{\beta_i}{\gamma_i}}} \ge \frac{M_2}{M_1} > 0
$$

holds, and hence (b) holds for $\epsilon_2 = \epsilon_2(\mathcal{G},\mathcal{I}) = \frac{M_2}{M_1}$.
Now by Proposition 8.6, there exist constants $M_3 = M_3(\kappa,\mathcal{G},\mathcal{I}) > 1$ and $M_4 = M_4(\kappa,\mathcal{G},\mathcal{I}) > 0$ such that for all $1\le i\le \mathfrak{a}-1$,

$$
\max\left\{u^2_{s(i+1)-1}, \left(\frac{\beta_{i+1}}{\gamma_{i+1}}\right)u^2_{s(i+1)}\right\} \le M_3\,\rho_i^{2\ell(i)}\max\left\{u^2_{s(i)-1}, \left(\frac{\beta_i}{\gamma_i}\right)u^2_{s(i)}\right\} \tag{75}
$$

and

$$
\sum_{j=0}^{\ell(i)+1}\kappa_{s(i)-1+j}u^2_{s(i)-1+j} \le M_4\,\kappa_{s(i)-1}\left(\left(\frac{\beta_i}{\gamma_i}\right)\rho_i^2\right)^{\ell(i)}\max\left\{u^2_{s(i)-1}, \left(\frac{\beta_i}{\gamma_i}\right)u^2_{s(i)}\right\} \tag{76}
$$

both hold. By applying (75) inductively and also using (76), it follows that, for each $1\le i\le \mathfrak{a}-1$,

$$
\begin{aligned}
\max\left\{u^2_{s(i+1)-1}, \left(\frac{\beta_{i+1}}{\gamma_{i+1}}\right)u^2_{s(i+1)}\right\} &\le M_3^i\prod_{j=1}^{i}\rho_j^{2\ell(j)}\max\left\{u_0^2, \left(\frac{\beta_1}{\gamma_1}\right)u_1^2\right\} < \kappa\, M_3^i\prod_{j=1}^{i}\rho_j^{2\ell(j)} \quad \text{and}\\
\sum_{j=0}^{\ell(i)+1}\kappa_{s(i)-1+j}u^2_{s(i)-1+j} &\le \kappa^2\,M_4\,M_3^{i-1}\prod_{j=1}^{i}\left(\left(\frac{\beta_j}{\gamma_j}\right)\rho_j^2\right)^{\ell(j)} \le \kappa^2\,M_4\,M_3^{\mathfrak{a}-2}\prod_{j=1}^{\mathfrak{a}-1}\left(\left(\frac{\beta_j}{\gamma_j}\right)\rho_j^2\right)^{\ell(j)}
\end{aligned}
$$

all hold. Statements (i) and (ii) now follow by taking

$$
C_1 := (\mathfrak{a}-1)\,\kappa^2\,M_4\,M_3^{\mathfrak{a}-2}\,, \quad C_2 := \kappa\,M_3^{\mathfrak{a}-1}
$$

and noting that $u^2_{s(\mathfrak{a})-1} > \prod_{j=1}^{\mathfrak{a}-1} \rho_j^{2\ell(j)}$ holds by Proposition 8.2 (ii).

(iii): By Lemma 7.1, $\mathcal{I} \subseteq I_i$ holds for each $\mathfrak{a} \le i \le \widehat{\mathfrak{c}}-1$. Let $\epsilon = \epsilon(\mathcal{G},\mathcal{I}) := \min\{|\alpha_i + 2\sqrt{\beta_i\gamma_i} - \mathcal{I}_{\max}|, |\mathcal{I}_{\min} - (\alpha_i - 2\sqrt{\beta_i\gamma_i})| \,:\, \mathfrak{a} \le i \le \widehat{\mathfrak{c}}-1\}$. Then $|\theta - \alpha_i| \le 2\sqrt{\beta_i\gamma_i} - \epsilon$ and $0 < \epsilon < 2\sqrt{\beta_i\gamma_i}$ both hold for all $\mathfrak{a} \le i \le \widehat{\mathfrak{c}}-1$. Hence by Corollary 8.5, there exist constants $M_j := M_j(\mathcal{G},\mathcal{I}) > 0$ ($5 \le j \le 8$) such that, for any $\mathfrak{a} \le i \le \widehat{\mathfrak{c}}-1$,

$$M_5 \max\left\{u^2_{s(i)-1}, \left(\frac{\beta_i}{\gamma_i}\right) u^2_{s(i)}\right\} \le \left(\frac{\beta_i}{\gamma_i}\right)^{\ell(i)} \max\left\{u^2_{s(i+1)-1}, \left(\frac{\beta_i}{\gamma_i}\right) u^2_{s(i+1)}\right\} \le M_6 \max\left\{u^2_{s(i)-1}, \left(\frac{\beta_i}{\gamma_i}\right) u^2_{s(i)}\right\} \tag{77}$$

and

$$M_7\ \ell(i) \max\left\{u^2_{s(i)-1}, \left(\frac{\beta_i}{\gamma_i}\right) u^2_{s(i)}\right\} \le \sum_{j=0}^{\ell(i)+1} \left(\frac{\beta_i}{\gamma_i}\right)^j u^2_{s(i)-1+j} \le M_8\ \ell(i) \max\left\{u^2_{s(i)-1}, \left(\frac{\beta_i}{\gamma_i}\right) u^2_{s(i)}\right\} \tag{78}$$

both hold. Note that for each $1 \le i \le g$, the following hold for all $0 \le j \le \ell(i)+1$:

$$\frac{1}{\kappa}\left(\frac{\beta_i}{\gamma_i}\right)^j \prod_{m=1}^{i-1}\left(\frac{\beta_m}{\gamma_m}\right)^{\ell(m)} \le \kappa_{s(i)-1+j} \le \kappa\left(\frac{\beta_i}{\gamma_i}\right)^j \prod_{m=1}^{i-1}\left(\frac{\beta_m}{\gamma_m}\right)^{\ell(m)}. \tag{79}$$

Hence, by applying (79) to (78) and by using (77) and statement (ii) of the theorem, it follows that, for each $\mathfrak{a} \le i \le \widehat{\mathfrak{c}}-1$, there exists a constant $M_9 := M_9(\kappa,\mathcal{G},\mathcal{I}) > 0$ such that

$$\begin{aligned}\sum_{j=0}^{\ell(i)+1} \kappa_{s(i)-1+j} u^2_{s(i)-1+j} &\le \kappa\ M_8\, \ell(i) \prod_{j=1}^{i-1}\left(\frac{\beta_j}{\gamma_j}\right)^{\ell(j)} \max\left\{u^2_{s(i)-1}, \left(\frac{\beta_i}{\gamma_i}\right) u^2_{s(i)}\right\}\\ &\le \kappa M_8 M_6^{i-\mathfrak{a}}\, \ell(i) \prod_{j=1}^{\mathfrak{a}-1}\left(\frac{\beta_j}{\gamma_j}\right)^{\ell(j)} \max\left\{u^2_{s(\mathfrak{a})-1}, \left(\frac{\beta_\mathfrak{a}}{\gamma_\mathfrak{a}}\right) u^2_{s(\mathfrak{a})}\right\}\\ &\le \kappa \max\{1, M_6^{\widehat{\mathfrak{c}}}\} M_8 M_9\, \ell(i) \prod_{j=1}^{\mathfrak{a}-1}\left(\left(\frac{\beta_j}{\gamma_j}\right)\rho_j^2\right)^{\ell(j)}\end{aligned}$$

holds and, similarly,

$$\sum_{j=0}^{\ell(i)+1} \kappa_{s(i)-1+j} u^2_{s(i)-1+j} \ge \frac{M_7}{\kappa} \min\{1, M_5^{\widehat{\mathfrak{c}}}\}\, \ell(i) \prod_{j=1}^{\mathfrak{a}-1}\left(\left(\frac{\beta_j}{\gamma_j}\right)\rho_j^2\right)^{\ell(j)}.$$

Hence (iii) follows by taking

$$C_3(\kappa,\mathcal{G},\mathcal{I}) := \frac{M_7}{3\kappa}\min\{1, M_5^{\widehat{\mathfrak{c}}}\},\ C_4(\kappa,\mathcal{G},\mathcal{I}) := \kappa\max\{1, M_6^{\widehat{\mathfrak{c}}}\} M_8 M_9,$$

in light of that fact that each element $\kappa_{s(i)-1+j} u^2_{s(i)-1+j}$ appears in the sum $\sum_{i=\mathfrak{a}}^{\widehat{\mathfrak{c}}-1}\sum_{m=0}^{\ell(i)+1} \kappa_{s(i)-1+m} u^2_{s(i)-1+m}$ at most three times.

(iv): This follows from (77) and statement (ii) of the theorem.

(v): By Lemma 8.4 (ii), statement (iv) of the theorem, (72) and (79), there exists a constant $C_7 = C_7(\kappa, \mathcal{G}, \mathcal{I}) > 0$ so that

$$\begin{aligned}\sum_{i=s(\mathfrak{d})}^{s(\mathfrak{d}+1)} \kappa_i u_i^2 &< 2\left(9\kappa^4\right)^{s(\mathfrak{d}+1)-s(\mathfrak{c})+1} \max\{\kappa_{s(\mathfrak{c})-1} u_{s(\mathfrak{c})-1}^2, \kappa_{s(\mathfrak{c})} u_{s(\mathfrak{c})}^2\} \\ &\leq C_7 \left(9\kappa^4\right)^{\mathtt{Gap}(\mathcal{I})} \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)}\end{aligned}$$

holds. (v) follows immediately.

(vi) and (vii): Using the same proof as for statement (iii), it can be seen that if $\mathfrak{d} < \mathfrak{b}$ then (77) and (78) both hold for all $\mathfrak{d}+1 \leq i \leq \mathfrak{b}$.
By (77)–(79), Lemma 8.4 (ii) and statement (iv) of the theorem, there exist constants $M_j = M_j(\kappa, \mathcal{G}, \mathcal{I}) > 0$ $(10 \leq j \leq 15)$ such that

$$\begin{aligned}\kappa_{s(\mathfrak{b}+1)} \max\left\{u_{s(\mathfrak{b}+1)-1}^2, \left(\frac{\beta_{\mathfrak{b}}}{\gamma_{\mathfrak{b}}}\right) u_{s(\mathfrak{b}+1)}^2\right\} &\leq M_{10}\, \kappa_{s(\mathfrak{d}+1)} \max\left\{u_{s(\mathfrak{d}+1)-1}^2, \left(\frac{\beta_{\mathfrak{d}}}{\gamma_{\mathfrak{d}}}\right) u_{s(\mathfrak{d}+1)}^2\right\} \\ &\leq M_{11}\, \kappa_{s(\mathfrak{c})}\, (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \max\left\{u_{s(\mathfrak{c})-1}^2, \left(\frac{\beta_{\mathfrak{c}-1}}{\gamma_{\mathfrak{c}-1}}\right) u_{s(\mathfrak{c})}^2\right\} \\ &\leq M_{12}\, (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \prod_{j=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell_i}\end{aligned}$$

holds, and moreover, if $\mathfrak{d} < \mathfrak{b}$ then for each $\mathfrak{d}+1 \leq i \leq \mathfrak{b}$,

$$\begin{aligned}\sum_{j=0}^{\ell(i)+1} \kappa_{s(i)-1+j} u_{s(i)-1+j}^2 &\leq M_{13}\, \ell(i)\, (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \prod_{j=1}^{\mathfrak{c}-1} \left(\frac{\beta_j}{\gamma_j}\right)^{\ell(j)} \max\left\{u_{s(\mathfrak{c})-1}^2, \left(\frac{\beta_{\mathfrak{c}}}{\gamma_{\mathfrak{c}}}\right) u_{s(\mathfrak{c})}^2\right\} \\ &\leq M_{14}\, \ell(i)\, (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \prod_{j=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_j}{\gamma_j}\right) \rho_j^2\right)^{\ell(j)}\end{aligned}$$

and

$$\sum_{j=0}^{\ell(i)+1} \kappa_{s(i)-1+j} u_{s(i)-1+j}^2 \geq M_{15}\, \ell(i) \left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})} \prod_{j=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_j}{\gamma_j}\right) \rho_j^2\right)^{\ell(j)}$$

all hold. By taking $C_8(\kappa, \mathcal{G}, \mathcal{I}) := \frac{M_{15}}{3}$, $C_9(\kappa, \mathcal{G}, \mathcal{I}) := M_{14}$ and $C_{11}(\kappa, \mathcal{G}, \mathcal{I}) := M_{12}$, it can be seen that the inequalities in (vi) and (vii) involving these constants all hold. It can also be seen in a similar fashion that there exists a constant $C_{10} = C_{10}(\kappa, \mathcal{G}, \mathcal{I}) > 0$ such that the left-hand inequality in (vii) holds. ∎

By using the previous theorem, we now obtain bounds for Gap sum $\sum_{i=s(\mathfrak{a})-1}^{s(\mathfrak{b}+1)} \kappa_i u_i^2$.

Corollary 8.8 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$. Let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i)\Big)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence, $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ be a function with $\ell(g+1) = 1$, and $\mathcal{T} := \mathcal{T}(\mathcal{G}, \ell)$ be the (κ, λ)-tridiagonal sequence associated to $\mathcal{G}$ and ℓ. Suppose that $\mathcal{I}$ is a well-placed interval with respect to $\mathcal{G}$. Then there exist positive constants $C := C(\kappa, \mathcal{G}, \mathcal{I})$ and $C' := C'(\kappa, \mathcal{G}, \mathcal{I})$ such that for any element θ in $\mathcal{I}$,*

$$
\begin{aligned}
C \left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})} \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)} &\leq \sum_{i=s(\mathfrak{a})-1}^{s(\mathfrak{b}+1)} \kappa_i u_i^2 \\
&\leq C'(9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)},
\end{aligned}
$$

where κ_i and $u_i := u_i(\theta)$ are as defined in (6) and (7) for the matrix $L_1(\mathcal{T}(\mathcal{G}, \ell))$, and $s(i)$, $\mathfrak{a} := \mathfrak{a}(\mathcal{G}, \mathcal{I})$, $\mathfrak{b} := \mathfrak{b}(\mathcal{G}, \mathcal{I})$, $\mathtt{Len}(\mathcal{I}) := \mathtt{Len}_{\mathcal{G},\ell}(\mathcal{I})$, $\mathtt{Gap}(\mathcal{I}) := \mathtt{Gap}_{\mathcal{G},\ell}(\mathcal{I})$ and $\rho_i := \rho_i(\theta)$ are as defined in (22), (67), (68), (71)-(73), respectively.

Proof: Constants C_i ($i = 3, 4, 7, 8, 9$) in this proof are the constants in Theorem 8.7. Note that $\mathtt{Len}(\mathcal{I}) \geq 1$ and $\left(9\kappa^4\right)^{\mathtt{Gap}(\mathcal{I})} \geq 1$. We break the proof into three cases:

(1) $\mathfrak{c} = g+1$: By (71) and (72), $\mathtt{Len}(\mathcal{I}) = \sum_{i=\mathfrak{a}}^{\mathfrak{b}} \ell(i) \geq 1$ and $\mathtt{Gap}(\mathcal{I}) = 0$. By applying Theorem 8.7 (iii) with $\widehat{\mathfrak{c}} = \mathfrak{b}+1$, Corollary 8.8 holds for $C := C_3$ and $C' := C_4$.

(2) $\mathfrak{c} \leq g$ and $\mathfrak{d} = \mathfrak{b}$: Then $\mathtt{Len}(\mathcal{I}) = \sum_{i=\mathfrak{a}}^{\mathfrak{c}-1} \ell(i) \geq 1$, and by applying Theorem 8.7 (iii) and (v) for $\widehat{\mathfrak{c}} = \mathfrak{c}$, the result follows for $C := C_3$ and $C' := C_4 + C_7$ as

$$
\begin{aligned}
& C_3 \left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})} \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)} \leq C_3 \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)} \\
\leq\ & \sum_{i=s(\mathfrak{a})-1}^{s(\mathfrak{c})} \kappa_i u_i^2 \\
\leq\ & C_4 \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)} \leq C_4 (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)}
\end{aligned}
$$

and

$$
\sum_{i=s(\mathfrak{c})}^{s(\mathfrak{b}+1)} \kappa_i u_i^2 < C_7 (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)} \leq C_7 (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)}
$$

all hold.

(3) $\mathfrak{c} < g$ and $\mathfrak{d} < \mathfrak{b}$: In this case, $\mathtt{Len}(\mathcal{I}) = \sum_{i=\mathfrak{a}}^{\mathfrak{c}-1} \ell(i) + \sum_{i=\mathfrak{d}+1}^{\mathfrak{b}} \ell(i)$ and by Theorem 8.7 (iii), (v) and (vi), the following all hold:

$$
\min\{C_3, C_8\} \left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})} \sum_{i=\mathfrak{a}}^{\mathfrak{c}-1} \ell(i) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)} \leq C_3 \sum_{i=\mathfrak{a}}^{\mathfrak{c}-1} \ell(i) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i}\right) \rho_i^2\right)^{\ell(i)}
$$

$$\leq \sum_{i=s(\mathfrak{a})-1}^{s(\mathfrak{c})} \kappa_i u_i^2$$

$$\leq C_4 \sum_{i=\mathfrak{a}}^{\mathfrak{c}-1} \ell(i) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i} \right) \rho_i^2 \right)^{\ell(i)} \leq \max\{C_4, C_7, C_9\} (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \, \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i} \right) \rho_i^2 \right)^{\ell(i)},$$

$$\sum_{i=s(\mathfrak{c})}^{s(\mathfrak{d}+1)} \kappa_i u_i^2 \leq C_7 (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i} \right) \rho_i^2 \right)^{\ell(i)} \leq \max\{C_4, C_7, C_9\} (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \, \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i} \right) \rho_i^2 \right)^{\ell(i)}$$

and

$$\min\{C_3, C_8\} \left(\frac{1}{9\kappa^4} \right)^{\mathtt{Gap}(\mathcal{I})} \sum_{i=\mathfrak{d}+1}^{\mathfrak{b}} \ell(i) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i} \right) \rho_i^2 \right)^{\ell(i)} \leq \sum_{i=s(\mathfrak{d}+1)-1}^{s(\mathfrak{b}+1)} \kappa_i u_i^2$$

$$\leq C_9 (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \sum_{i=\mathfrak{d}+1}^{\mathfrak{b}} \ell(i) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i} \right) \rho_i^2 \right)^{\ell(i)} \leq \max\{C_4, C_7, C_9\} (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \mathtt{Len}(\mathcal{I}) \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i} \right) \rho_i^2 \right)^{\ell(i)}.$$

Hence, the result now follows by taking

$$C := \frac{\min\{C_3, C_8\}}{2}, \quad C' := 3 \max\{C_4, C_7, C_9\},$$

in light of the fact $s(\mathfrak{c}) \leq s(\mathfrak{d}+1) - 1$. The corollary now follows. ∎

8.3 Bounding Tail Sum

In this section, we obtain an upper bound for the Tail sum $\sum_{i=s(\mathfrak{b}+1)+1}^{D} \kappa_i u_i^2$. Namely:

Theorem 8.9 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa - 2$. Suppose that $(\mathcal{G}, \Delta; L, \ell)$ is a (κ, λ)-quadruple and let $\mathcal{G} = \Big((\gamma_i, \alpha_i, \beta_i) \Big)_{i=1}^{g+1}$. Suppose that $\mathcal{I}$ is a well-placed interval with respect to $\mathcal{G}$ satisfying $\mathcal{I} \cap \mathcal{B} = \emptyset$ and $\mathfrak{b} < g$ (cf. (52) and (68)). Then there exist positive constants $F := F(\kappa, \mathcal{G}, \Delta, L, \mathcal{I})$ and $C := C(\kappa, \mathcal{G}, \Delta, L, \mathcal{I})$ so that if $\ell(i) > F$ holds for all $(\gamma_i, \alpha_i, \beta_i) \in \underline{\Delta}$ with $\mathfrak{b} < i \leq g$ then, for any $\theta \in \mathcal{I}$,*

$$\sum_{i=s(\mathfrak{b}+1)+1}^{D} \kappa_i u_i^2 \leq C (9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \prod_{i=1}^{\mathfrak{a}-1} \left(\left(\frac{\beta_i}{\gamma_i} \right) \rho_i^2 \right)^{\ell(i)}$$

holds, where κ_i and $u_i := u_i(\theta)$ are as defined in (6) and (7) for the matrix $L_1(\mathcal{T}(\mathcal{G}, \ell))$, and $s(i)$, $D := D_{\mathcal{T}(\mathcal{G},\ell)}$, $\mathfrak{a} := \mathfrak{a}(\mathcal{G}, \mathcal{I})$, $\mathtt{Gap}(\mathcal{I}) := \mathtt{Gap}_{\mathcal{G},\ell}(\mathcal{I})$ and $\rho_i := \rho_i(\theta)$ are as defined in (22), (25), (67), (72) and (73), respectively.

Proof: Suppose $(\mathcal{G}, \Delta; L, \ell)$ and $\mathcal{I}$ are as in the statement of the theorem. By Theorem 6.3, for each $0 \le i \le g - \mathfrak{b} - 1$ satisfying $(\gamma_{g-i}, \alpha_{g-i}, \beta_{g-i}) \in \underline{\Delta}$ there exist constants $C_i = C_i(\kappa, \mathcal{G}, \Delta, L, \mathcal{I}) \ge 1$ and $M_i = M_i(\kappa, \mathcal{G}, \Delta, L, \mathcal{I}) > 1$ such that if $\ell(g-j) > C_i$ holds for all $(\gamma_{g-j}, \alpha_{g-j}, \beta_{g-j}) \in \underline{\Delta}$ with $j < i$, then (59) holds for all $\theta \in \mathcal{I}$. Now put

$$\begin{aligned} F &= F(\kappa, \mathcal{G}, \Delta, L, \mathcal{I}) := \max\{C_i(\kappa, \mathcal{G}, \Delta, L, \mathcal{I}) \,:\, 0 \le i \le g - \mathfrak{b} - 1 \text{ and } (\gamma_{g-i}, \alpha_{g-i}, \beta_{g-i}) \in \underline{\Delta}\}; \\ M &= M(\kappa, \mathcal{G}, \Delta, L, \mathcal{I}) := \min\{M_i(\kappa, \mathcal{G}, \Delta, L, \mathcal{I}) \,:\, 0 \le i \le g - \mathfrak{b} - 1 \text{ and } (\gamma_{g-i}, \alpha_{g-i}, \beta_{g-i}) \in \underline{\Delta}\}. \end{aligned}$$

(†) Suppose that if $\{(\gamma_{g-i}, \alpha_{g-i}, \beta_{g-i}) \in \underline{\Delta} \,:\, 0 \le i \le g - \mathfrak{b} - 1\} \ne \emptyset$ then $\ell(g-i) > F$ holds for all $(\gamma_{g-i}, \alpha_{g-i}, \beta_{g-i}) \in \underline{\Delta}$ with $0 \le i \le g - \mathfrak{b} - 1$.

Let $\theta \in \mathcal{I}$. We will use the following:

Claim 8.10 *There exist constants $C_1 = C_1(\mathcal{G}, \mathcal{I}) > 0$ and $C_m = C_m(\kappa, \mathcal{G}, \Delta, L, \mathcal{I}) > 0$ $(m = 2, 3)$ such that, for all $0 \le i \le g - \mathfrak{b} - 1$, the following hold:*
(a)

$$|u_{s(g-i+1)-j}| \le C_1 \max\{|u_{s(g-i+1)-1}|, |u_{s(g-i+1)}|\}\; x_i^j \quad (0 \le j \le \ell(g-i) + 1).$$

(b)

$$\max\{|u_{s(g-i)-1}|, |u_{s(g-i)}|\} > C_2 \max\{|u_{s(g-i+1)-1}|, |u_{s(g-i+1)}|\}\; x_i^{\ell(g-i)}.$$

(c)

$$\prod_{j=0}^{g-\mathfrak{b}-1} {x_j}^{2\ell(g-j)} \max\{u_{D-1}^2, u_D^2\} < C_3 \max\{u_{s(\mathfrak{b}+1)-1}^2, u_{s(\mathfrak{b}+1)}^2\},$$

where x_j is defined in (50).

Proof of Claim 8.10: Let $0 \le i \le g - \mathfrak{b} - 1$. By (50) and Lemma 7.1 (iv), $x_i > y_i > 0$. Let $\nu_j^{(i)} = \nu_j^{(i)}(\theta)$ $(j = 1, 2)$ be as defined in (53).

(a): First suppose that $\nu_1^{(i)} \nu_2^{(i)} > 0$ holds. Then for all $0 \le j \le \ell(g-i) + 1$, (a) follows since

$$|u_{s(g-i+1)-j}| = |\nu_1^{(i)}|\, x_i^j + |\nu_2^{(i)}|\, y_i^j < (|\nu_1^{(i)}| + |\nu_2^{(i)}|)\, x_i^j = |u_{s(g-i+1)}|\, x_i^j.$$

Now suppose $\nu_1^{(i)} \nu_2^{(i)} < 0$. By (54) and (55),

$$\begin{aligned} \max\{|\nu_1^{(i)}|, |\nu_2^{(i)}|\} &\le 2 \max\left\{\frac{1}{x_i - y_i}, \frac{x_i}{x_i - y_i}\right\} \max\{|u_{s(g-i+1)-1}|, |u_{s(g-i+1)}|\} \\ &\le C_1 \max\{|u_{s(g-i+1)-1}|, |u_{s(g-i+1)}|\} \end{aligned} \tag{80}$$

holds, where

$$C_1 = C_1(\mathcal{G}, \mathcal{I}) := 2 \max\left\{\frac{\mathcal{I}_{\max} - \alpha_m}{\sqrt{(\mathcal{I}_{\min} - \alpha_m)^2 - 4\beta_m\gamma_m}}, \frac{\gamma_m}{\sqrt{(\mathcal{I}_{\min} - \alpha_m)^2 - 4\beta_m\gamma_m}} \,:\, 0 \le m \le g - \mathfrak{b} - 1\right\}.$$

Since $|u_{s(g-i+1)-j}| \le \max\{|\nu_1^{(i)}|, |\nu_2^{(i)}|\}\, x_i^j$ holds by $\nu_1^{(i)} \nu_2^{(i)} < 0$ and $x_i > y_i > 0$, (a) follows by (80).

(b): Suppose $(\gamma_{g-i}, \alpha_{g-i}, \beta_{g-i}) \in \underline{\mathcal{G}} \setminus \underline{\Delta}$. Then $\ell(g-i) = L(g-i)$ and, by Lemma 8.4 (i) and $0 < x_i < \frac{\mathcal{I}_{\max}-\alpha_{g-i}}{\gamma_{g-i}}$, it follows that

$$\begin{aligned} \max\{|u_{s(g-i)-1}|, |u_{s(g-i)}|\} &\geq \left(\frac{1}{3\kappa\, x_i}\right)^{\ell(g-i)} \max\{|u_{s(g-i+1)-1}|, |u_{s(g-i+1)}|\}\, x_i^{\ell(g-i)} \\ &\geq \left(\frac{\gamma_{g-i}}{3\kappa(\mathcal{I}_{\max}-\alpha_{g-i})}\right)^{L(g-i)} \max\{|u_{s(g-i+1)-1}|, |u_{s(g-i+1)}|\}\, x_i^{\ell(g-i)}, \end{aligned}$$

and thus (b) follows by taking $C_2 = C_2(\kappa, \mathcal{G}, \Delta, L, \mathcal{I})$, where

$$C_2 := \min\left\{\left(\frac{\gamma_{g-m}}{3\kappa(\mathcal{I}_{\max}-\alpha_{g-m})}\right)^{L(g-m)} : 0 \leq m \leq g-\mathfrak{b}-1 \text{ and } (\gamma_{g-m}, \alpha_{g-m}, \beta_{g-m}) \in \underline{\mathcal{G}} \setminus \underline{\Delta}\right\}.$$

Now suppose $(\gamma_{g-i}, \alpha_{g-i}, \beta_{g-i}) \in \underline{\Delta}$. By Theorem 6.3 with (†),

$$\begin{aligned} \max\{|u_{s(g-i+1)-1}|, |u_{s(g-i+1)}|\} &\leq \max\{1, x_i\}(|\nu_1^{(i)}| + |\nu_2^{(i)}|) \\ &< \left(1 + \frac{1}{M}\left(\frac{x_i}{y_i}\right)^F\right)(1+x_i)\,|\nu_1^{(i)}| \\ &< \left(1 + \frac{1}{M}\left(\frac{(\mathcal{I}_{\max}-\alpha_{g-i})^2}{\beta_{g-i}\gamma_{g-i}}\right)^F\right)\left(1 + \frac{\mathcal{I}_{\max}-\alpha_{g-i}}{\gamma_{g-i}}\right)|\nu_1^{(i)}| \end{aligned} \quad (81)$$

and

$$\begin{aligned} \max\{|u_{s(g-i)-1}|, |u_{s(g-i)}|\} &\geq |\nu_1^{(i)} x_i^{\ell(g-i)}| - |\nu_2^{(i)} y_i^{\ell(g-i)}| \\ &> |\nu_1^{(i)}|\, x_i^{\ell(g-i)}\left(1 - \frac{1}{M}\left(\frac{x_i}{y_i}\right)^F\left(\frac{y_i}{x_i}\right)^{\ell(g-i)}\right) \\ &> \left(1 - \frac{1}{M}\right)|\nu_1^{(i)}|\, x_i^{\ell(g-i)} \end{aligned} \quad (82)$$

all hold. By (81) and (82), statement (b) now follows by taking $C_2 = C_2(\kappa, \mathcal{G}, \Delta, L, \mathcal{I})$, where

$$C_2 := \frac{1 - \frac{1}{M}}{\max\left\{\left(1 + \frac{1}{M}\left(\frac{(\mathcal{I}_{\max}-\alpha_{g-m})^2}{\beta_{g-m}\gamma_{g-m}}\right)^F\right)\left(1 + \frac{\mathcal{I}_{\max}-\alpha_{g-m}}{\gamma_{g-m}}\right) : 0 \leq m \leq g-\mathfrak{b}-1 \text{ and } (\gamma_{g-m}, \alpha_{g-m}, \beta_{g-m}) \in \underline{\Delta}\right\}}.$$

(c): This follows by applying (b) inductively on i for $0 \leq i \leq g-\mathfrak{b}-1$. ∎

Let $0 \leq i \leq g-\mathfrak{b}-1$. By (a) and (c) of the claim, $\left(\frac{\gamma_{g-i}}{\beta_{g-i}}\right) x_i^2 > 1$ and Theorem 8.7 (vii), there exist constants $M_j = M_j(\kappa, \mathcal{G}, \mathcal{I}) > 0$ $(j = 1, 2)$ and $M_j = M_j(\kappa, \mathcal{G}, \Delta, L, \mathcal{I}) > 0$ $(j = 3, 4)$ such that

$$\begin{aligned} &\sum_{j=0}^{\ell(g-i)-1} \kappa_{s(g-i+1)-j} u_{s(g-i+1)-j}^2 \\ &< \kappa\kappa_D \prod_{j=0}^{i-1}\left(\frac{\gamma_{g-j}}{\beta_{g-j}}\right)^{\ell(g-j)} \sum_{m=0}^{\ell(g-i)-1}\left(\frac{\gamma_{g-i}}{\beta_{g-i}}\right)^m u_{s(g-i+1)-m}^2 \end{aligned}$$

$$
\begin{aligned}
&\le M_1\kappa_D \prod_{j=0}^{i-1}\left(\frac{\gamma_{g-j}}{\beta_{g-j}}\right)^{\ell(g-j)} \max\{u^2_{s(g-i+1)-1}, u^2_{s(g-i+1)}\} \sum_{m=0}^{\ell(g-i)-1}\left(\frac{\gamma_{g-i}}{\beta_{g-i}}\right)^m x_i^{2m}\\
&\le M_2\kappa_D \max\{u^2_{D-1}, u^2_D\} \prod_{j=0}^{i}\left(\left(\frac{\gamma_{g-j}}{\beta_{g-j}}\right)x_j^2\right)^{\ell(g-j)}\\
&\le M_2\kappa_D \max\{u^2_{D-1}, u^2_D\} \prod_{j=0}^{g-\mathfrak{b}-1}\left(\left(\frac{\gamma_{g-j}}{\beta_{g-j}}\right)x_j^2\right)^{\ell(g-j)}\\
&\le M_3\kappa_{s(\mathfrak{b}+1)} \max\{u^2_{s(\mathfrak{b}+1)-1}, u^2_{s(\mathfrak{b}+1)}\}\\
&\le M_4(9\kappa^4)^{\mathtt{Gap}(\mathcal{I})} \prod_{j=1}^{\mathfrak{a}-1}\left(\left(\frac{\beta_j}{\gamma_j}\right)\rho_j^2\right)^{\ell(j)} \qquad (83)
\end{aligned}
$$

holds. From (83) and

$$
\sum_{j=s(\mathfrak{b}+1)+1}^{D} \kappa_j u_j^2 = \sum_{i=0}^{g-\mathfrak{b}-1} \sum_{j=0}^{\ell(g-i)-1} \kappa_{s(g-i+1)-j} u^2_{s(g-i+1)-j} \ ,
$$

Theorem 8.9 now follows by taking $C(\kappa, \mathcal{G}, \Delta, L, \mathcal{I}) := (g - \mathfrak{b})M_4$. ∎

With these results in hand, we can now prove the main theorem of this section:

Proof of Theorem 8.1: Theorem 8.1 follows immediately by Theorem 8.7 (i), Corollary 8.8 and Theorem 8.9. ∎

9 Distribution of Eigenvalues and Proof of Theorem 4.2

In this section we prove Theorem 4.2 and thus complete the proof of the Bannai-Ito conjecture. To do this we will first prove two results concerning the distribution of the eigenvalues of a graphical sequence in a well-placed interval with respect to this sequence, using the results from the last four sections.

Theorem 9.1 *Let $\kappa \ge 3$ and $\lambda \ge 0$ be integers with $\lambda \le \kappa - 2$, and let $\mathcal{G} = \left(\delta_i := (\gamma_i, \alpha_i, \beta_i)\right)_{i=1}^{g+1}$ be a (κ, λ)-graphical sequence. Suppose that $\Delta = (\delta_{i_p})_{p=1}^{\tau}$ is a subsequence of $\mathcal{G}$ with $(1, \lambda, \kappa-\lambda-1) \in \underline{\Delta}$ and $(\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1}) \notin \underline{\Delta}$, $L : \{1, \ldots, g+1\} \setminus \{i_1, \ldots, i_\tau\} \to \mathbb{N}$ is a function, and $\mathcal{I}$ is a well-placed interval with respect to $\mathcal{G}$ satisfying $\mathcal{I} \cap \mathcal{B}(\mathcal{G}, \Delta, L) = \emptyset$ (cf. (52)). Suppose that $\epsilon > 0$ is a real number, $C := C(\kappa) > 0$ is a constant, and $\ell : \{1, \ldots, g+1\} \to \mathbb{N}$ is any function for which $(\mathcal{G}, \Delta; L, \ell)$ is a (κ, λ)-quadruple and the associated (κ, λ)-tridiagonal sequence $\mathcal{T} = \mathcal{T}(\mathcal{G}, \ell)$ satisfies*
(i) Property (AC),
(ii) $D_{\mathcal{T}} \le C\mathtt{h}_{\mathcal{T}}$, and
(iii) $\mathtt{Len}(\mathcal{I}) \ge \epsilon \mathtt{h}_{\mathcal{T}}$,
where $\mathtt{h}_{\mathcal{T}}$, $D_{\mathcal{T}}$ and $\mathtt{Len}(\mathcal{I}) := \mathtt{Len}_{\mathcal{G},\ell}(\mathcal{I})$ are as defined in (23), (25) and (71), respectively.
Then for any real number $\delta > 0$, there exist positive constants $F := F(\kappa, \mathcal{G}, \Delta, L, \mathcal{I})$, $C_1 :=$

$C_1(\kappa,\lambda,\epsilon,\delta,\mathcal{G},\Delta,L,\mathcal{I})$ and $C_2 := C_2(\kappa,\lambda,\delta)$ such that if $\ell(i_p) > F$ holds for all $1 \le p \le \tau$ and if there exist two conjugate algebraic numbers θ and η in $\mathcal{E}_{\mathcal{T}} \cap \mathcal{I}$ satisfying $|\theta-\eta| > \delta$ then

$$\textit{either} \quad \mathtt{h}_{\mathcal{T}} < C_1 \quad \textit{or} \quad \mathtt{Gap}(\mathcal{I}) > C_2 \mathtt{h}_{\mathcal{T}}$$

holds, where $\mathcal{E}_{\mathcal{T}}$ and $\mathtt{Gap}(\mathcal{I}) := \mathtt{Gap}_{\mathcal{G},\ell}(\mathcal{I})$ are as defined in (26) and (72), respectively.

Proof: Suppose that κ, λ, ϵ, C, $\mathcal{G}$, Δ, L, $\mathcal{I}$, ℓ and $\mathcal{T}$ are as in the statement of the theorem, and put $\mathtt{h} := \mathtt{h}_{\mathcal{T}}$ and $D := D_{\mathcal{T}}$. Let δ be any positive real number, and let θ and η be two conjugate algebraic numbers in $\mathcal{E}_{\mathcal{T}} \cap \mathcal{I}$ satisfying $|\theta-\eta| > \delta$. Without loss of generality, we assume $\eta - \theta > \delta$.

By applying Theorem 8.1 and the conditions $\epsilon\,\mathtt{h} \le \mathtt{Len}(\mathcal{I}) < D \le C\,\mathtt{h}$ given by (ii) and (iii) in the statement of the theorem, it follows that there exist positive constants $F := F(\kappa,\mathcal{G},\Delta,L,\mathcal{I})$, $M_1 := M_1(\kappa,\mathcal{G},\mathcal{I})$ and $M_2 := M_2(\kappa,\mathcal{G},\Delta,L,\mathcal{I})$ so that if $\ell(i_p) > F$ holds for all $1 \le p \le \tau$ then

$$\epsilon\,\mathtt{h}\,M_1\left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})}\prod_{i=1}^{\mathfrak{a}-1}\left(\left(\frac{\beta_i}{\gamma_i}\right)\rho_i^2(x)\right)^{\ell(i)} \le \sum_{i=0}^{D}\kappa_i u_i^2(x) \le \mathtt{h}\,M_2 C(9\kappa^4)^{\mathtt{Gap}(\mathcal{I})}\prod_{i=1}^{\mathfrak{a}-1}\left(\left(\frac{\beta_i}{\gamma_i}\right)\rho_i^2(x)\right)^{\ell(i)} \tag{84}$$

holds for any $x \in \mathcal{I}$, where κ_i and $u_i := u_i(x)$ are as defined in (6) and (7) for the matrix $L_1(\mathcal{T})$, and $\mathfrak{a}$, $\rho_i(x)$ are as defined in (67) and (73), respectively.
By Proposition 8.2 (i) and $\eta > \theta$, it follows that

$$0 < \rho_i(\theta) < \rho_i(\eta) < 1 \quad (i = 1,\ldots,\mathfrak{a}-1), \tag{85}$$

and moreover, by (85) and $\eta - \theta > \delta$,

$$\rho_1(\eta) > \rho_1(\theta) + \frac{\delta}{2(\kappa-\lambda-1)} > \left(1 + \frac{\delta}{2(\kappa-\lambda-1)}\right)\rho_1(\theta). \tag{86}$$

By applying (85) and (86) to (84), it follows that

$$\begin{aligned}\sum_{i=0}^{D}\kappa_i u_i^2(\eta) &\ge \epsilon\,\mathtt{h}\,M_1\left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})}\prod_{i=1}^{\mathfrak{a}-1}\left(\left(\frac{\beta_i}{\gamma_i}\right)\rho_i^2(\eta)\right)^{\ell(i)}\\ &> \epsilon\,\mathtt{h}\,M_1\left(\frac{1}{9\kappa^4}\right)^{\mathtt{Gap}(\mathcal{I})}\left(1+\frac{\delta}{2(\kappa-\lambda-1)}\right)^{2\mathtt{h}}\prod_{i=1}^{\mathfrak{a}-1}\left(\left(\frac{\beta_i}{\gamma_i}\right)\rho_i^2(\theta)\right)^{\ell(i)}\\ &\ge \frac{\epsilon M_1}{M_2 C}\left(\frac{1}{9\kappa^4}\right)^{2\mathtt{Gap}(\mathcal{I})}\left(1+\frac{\delta}{2(\kappa-\lambda-1)}\right)^{2\mathtt{h}}\sum_{i=0}^{D}\kappa_i u_i^2(\theta).\end{aligned} \tag{87}$$

Since θ and η are algebraic conjugates, $\sum_{i=0}^{D}\kappa_i u_i^2(\eta) = \sum_{i=0}^{D}\kappa_i u_i^2(\theta) > 0$ holds by Property (AC). Hence, by (87),

$$\ln\left(1+\frac{\delta}{2(\kappa-\lambda-1)}\right)^2 < \frac{\ln\left(\frac{M_2 C}{\epsilon M_1}\right)}{\mathtt{h}} + \frac{\mathtt{Gap}(\mathcal{I})}{\mathtt{h}}\ln(9\kappa^4)^2. \tag{88}$$

Now, put

$$C_1 := \frac{\ln\left(\frac{M_2 C}{\epsilon M_1}\right)}{\ln\left(1+\frac{\delta}{2(\kappa-\lambda-1)}\right)} \quad \text{and} \quad C_2 := \frac{\ln\left(1+\frac{\delta}{2(\kappa-\lambda-1)}\right)}{2\ln(9\kappa^4)}.$$

If $0 < \frac{M_2C}{\epsilon M_1} \leq 1$ then $\texttt{Gap}(\mathcal{I}) > C_2\texttt{h}$ holds as $\frac{\texttt{Gap}(\mathcal{I})}{\texttt{h}} > \frac{\ln\left(1+\frac{\delta}{2(\kappa-\lambda-1)}\right)}{\ln(9\kappa^4)} > \frac{\ln\left(1+\frac{\delta}{2(\kappa-\lambda-1)}\right)}{2\ln(9\kappa^4)}$ by (88). Moreover, if $\frac{M_2C}{\epsilon M_1} > 1$ and $\texttt{Gap}(\mathcal{I}) \leq C_2\texttt{h}$, then $\texttt{h} < \frac{\ln\left(\frac{M_2C}{\epsilon M_1}\right)}{\ln\left(1+\frac{\delta}{2(\kappa-\lambda-1)}\right)}$ holds by (88). Therefore Theorem 9.1 now follows for this choice of C_1 and C_2. ∎

Proposition 9.2 *Let $\kappa \geq 3$ and $\lambda \geq 0$ be integers with $\lambda \leq \kappa-2$, and let $\mathcal{G} = \left(\delta_i := (\gamma_i, \alpha_i, \beta_i)\right)_{i=1}^{g+1}$ be a (κ,λ)-graphical sequence. Suppose that $\Delta = (\delta_{i_p})_{p=1}^{\tau}$ is a subsequence of $\mathcal{G}$ with $(1,\lambda,\kappa-\lambda-1) \in \underline{\Delta}$ and $(\gamma_{g+1}, \alpha_{g+1}, \beta_{g+1}) \notin \underline{\Delta}$, $L : \{1,\dots,g+1\} \setminus \{i_1,\dots,i_\tau\} \to \mathbb{N}$ is a function, and $\mathcal{I}$ is a well-placed interval with respect to $\mathcal{G}$ satisfying $\mathcal{I} \cap \mathcal{B}(\mathcal{G},\Delta,L) = \emptyset$ (cf. (52)). Suppose that $\epsilon > 0$ is a real number, $C := C(\kappa) > 0$ is a constant, and $\ell : \{1,\dots,g+1\} \to \mathbb{N}$ is any function for which $(\mathcal{G},\Delta;L,\ell)$ is a (κ,λ)-quadruple and the associated (κ,λ)-tridiagonal sequence $\mathcal{T} = \mathcal{T}(\mathcal{G},\ell)$ satisfies*
(i) Property (AC),
(ii) $D_{\mathcal{T}} \leq C\texttt{h}_{\mathcal{T}}$, and
(iii) $\texttt{Len}(\mathcal{I}) \geq \epsilon\texttt{h}_{\mathcal{T}}$,
where $\texttt{h}_{\mathcal{T}}$, $D_{\mathcal{T}}$ and $\texttt{Len}(\mathcal{I}) := \texttt{Len}_{\mathcal{G},\ell}(\mathcal{I})$ are as defined in (23), (25) and (71), respectively.
Then for any real number $\mu > 0$, there exist positive constants $F := F(\kappa,\mathcal{G},\Delta,L,\mathcal{I})$, $G := G(\kappa,\lambda,\epsilon,\mu,\mathcal{G},\mathcal{I})$ and $H := H(\kappa,\lambda,\epsilon,\mu,\mathcal{G},\Delta,L,\mathcal{I})$ such that if $\ell(i_p) > F$ holds for all $1 \leq p \leq \tau$, and $\texttt{h}_{\mathcal{T}} \geq H$ and $\texttt{Gap}(\mathcal{I}) \leq G\,\texttt{h}_{\mathcal{T}}$ also hold, then the number of eigenvalues of $\mathcal{T}$ that have an algebraic conjugate in $\mathcal{I}$ is at least $\mu\texttt{h}$, where $\texttt{Gap}(\mathcal{I}) := \texttt{Gap}_{\mathcal{G},\ell}(\mathcal{I})$ is as defined in (72).

Proof: Suppose that κ, λ, ϵ, C, $\mathcal{G}$, Δ, L, $\mathcal{I}$, ℓ and $\mathcal{T}$ are as in the statement of the proposition, and put $\texttt{h} := \texttt{h}_{\mathcal{T}}$ and $D := D_{\mathcal{T}}$. Let μ be any positive real number.
In view of Theorem 5.5, there exists a constant $M_1 := M_1(\kappa,\epsilon,\mu,\mathcal{G},\mathcal{I}) > 0$ such that for any positive real number ζ satisfying $\zeta < M_1$,

$$\Upsilon_{\kappa,\zeta} \leq \frac{1}{2+\frac{48\pi\kappa\mu g}{\epsilon|\mathcal{I}|}} \tag{89}$$

holds (cf. (40)). Put

$$\zeta_0 = \zeta_0(\kappa,\epsilon,\mu,\mathcal{G},\mathcal{I}) := \min\left\{|\mathcal{I}|, \frac{M_1}{2}\right\} \quad \text{and } \Upsilon := \Upsilon_{\kappa,\zeta_0}. \tag{90}$$

Then by (89) and (90),

$$\Upsilon \leq \frac{1}{2+\frac{48\pi\kappa\mu g}{\epsilon|\mathcal{I}|}} < \frac{1}{2}. \tag{91}$$

By Lemma 5.3 (i) and Remark 5.4, there exists a constant $M_2 := M_2(\kappa,\epsilon,\mu,\mathcal{G},\mathcal{I}) > 0$ such that

$$\left|\left\{p(x) \in \mathcal{P}_\kappa \,:\, \deg(p(x)) \leq \frac{1}{\Upsilon}\right\}\right| \leq M_2$$

holds, and therefore

$$\left|\left\{x \in \mathcal{I} \cap \mathcal{E}_{\mathcal{T}} \,:\, \deg(x) \leq \frac{1}{\Upsilon}\right\}\right| \leq \frac{M_2}{\Upsilon}, \tag{92}$$

where $\deg(x)$ is the degree of the minimal polynomial of an algebraic number x (cf. (26)).

Now, let $F := F(\kappa, \mathcal{G}, \Delta, L, \mathcal{I})$, $C_1 := C_1(\kappa, \lambda, \epsilon, \mu, \mathcal{G}, \Delta, L, \mathcal{I})$ and $C_2 := C_2(\kappa, \lambda, \epsilon, \mu, \mathcal{G}, \mathcal{I})$ be the positive constants given by Theorem 9.1 by taking $\delta := \frac{\zeta_0}{2}$, and put

$$H := \max\left\{\frac{6\kappa\pi g}{\epsilon|\mathcal{I}|}, \frac{24\kappa\pi g M_2}{\epsilon\Upsilon|\mathcal{I}|}, C_1\right\} \text{ and } G := C_2.$$

We now show that for this choice of F, H and G, the proposition holds. To this end, let θ be any element in $\mathcal{E}_{\mathcal{T}} \cap \mathcal{I}$ satisfying $\deg(\theta) > \frac{1}{\Upsilon}$, and let $p_\theta(x) \in \mathcal{P}_\kappa$ be a minimal polynomial of θ. Then by Theorem 9.1, all roots of $p_\theta(x)$ must lie in the closed interval $[\theta - \frac{\zeta_0}{2}, \theta + \frac{\zeta_0}{2}]$. Hence, by (39), (40) and $\deg(\theta) = \deg(p_\theta) > \frac{1}{\Upsilon}$,

$$|\{x \in \mathcal{I} : p_\theta(x) = 0\}| \leq \left|\left\{x \in \left[\theta - \frac{\zeta_0}{2}, \theta + \frac{\zeta_0}{2}\right] : p_\theta(x) = 0\right\}\right| \leq \Upsilon \deg(p_\theta) + 1 < 2\Upsilon \deg(p_\theta). \tag{93}$$

Now, we prove the following claim.

Claim 9.3 *The number of eigenvalues of $\mathcal{T}$ in $\mathcal{I}$ is at least* $\left(\frac{\epsilon|\mathcal{I}|}{12\kappa\pi g}\right)\mathtt{h}$.

Proof of Claim 9.3 : As $\mathtt{h} \geq H$ and $|\mathcal{I}| < \kappa$ (by (W1)), $\mathtt{h} \geq \frac{6\kappa\pi g}{\epsilon|\mathcal{I}|} > \frac{1}{\epsilon}$ holds. Hence, as $\mathtt{Len}(\mathcal{I}) \geq \epsilon\mathtt{h} > 1$ (by statement (iii) of the proposition), there exists $m \in \{2, \ldots, g\}$ so that $\ell(m) > \frac{\epsilon\mathtt{h}}{g}$ and $\mathcal{I} \subseteq I_m$ hold, where I_m is the mth guide interval. Put $(\gamma, \alpha, \beta) := (\gamma_m, \alpha_m, \beta_m)$, $\ell := \ell(m)$ and $e := \left|\{j \in \{1, \ldots, \ell\} : \alpha + 2\sqrt{\beta\gamma}\cos\left(\frac{j\pi}{\ell+1}\right) \in \mathcal{I}\}\right|$. Note that $\mathcal{I} \subseteq I_m$ and $\alpha + 2\sqrt{\beta\gamma}\cos\left(\frac{j\pi}{\ell+1}\right) \in I_m$, for all $1 \leq j \leq \ell$. Since

$$\left(\alpha + 2\sqrt{\beta\gamma}\cos\left(\frac{(j-1)\pi}{\ell+1}\right)\right) - \left(\alpha + 2\sqrt{\beta\gamma}\cos\left(\frac{j\pi}{\ell+1}\right)\right) \leq \frac{2\pi\sqrt{\beta\gamma}}{\ell+1} \leq \frac{\kappa\pi}{\ell+1}$$

holds for all $2 \leq j \leq \ell$, it follows by $\ell > \frac{\epsilon\mathtt{h}}{g}$ and $\mathtt{h} \geq \frac{6\kappa\pi g}{\epsilon|\mathcal{I}|}$ that $e \geq \lfloor\frac{(\ell+1)|\mathcal{I}|}{\kappa\pi}\rfloor > \lfloor\frac{\epsilon\mathtt{h}|\mathcal{I}|}{\kappa\pi g}\rfloor \geq \frac{\epsilon\mathtt{h}|\mathcal{I}|}{2\kappa\pi g} \geq 3$. Hence by Lemma 2.2 (ii), there exists an eigenvalue $\theta \in \mathcal{E}_{\mathcal{T}} \cap \mathcal{I}$ and, moreover,

$$|\mathcal{E}_{\mathcal{T}} \cap \mathcal{I}| \geq \left\lfloor\frac{e}{3}\right\rfloor > \frac{e}{6} > \frac{\epsilon\mathtt{h}|\mathcal{I}|}{12\kappa\pi g}$$

holds. Claim 9.3 now follows immediately. ∎

By applying Claim 9.3, (92) and $\mathtt{h} \geq \frac{24\kappa\pi g M_2}{\epsilon\Upsilon|\mathcal{I}|}$ (by $\mathtt{h} \geq H$), it follows that

$$\left|\{x \in \mathcal{E}_{\mathcal{T}} \cap \mathcal{I} : \deg(x) > \frac{1}{\Upsilon}\}\right| \geq \frac{\epsilon|\mathcal{I}|\mathtt{h}}{12\kappa\pi g} - \left|\{x \in \mathcal{E}_{\mathcal{T}} \cap \mathcal{I} : \deg(x) \leq \frac{1}{\Upsilon}\}\right| \geq \frac{\epsilon|\mathcal{I}|\mathtt{h}}{24\kappa\pi g}. \tag{94}$$

Now, for each integer $i > \frac{1}{\Upsilon}$, let Δ_i be the set of those elements in $\mathcal{E}_{\mathcal{T}} \cap ([-\kappa, \kappa] \setminus \mathcal{I})$ of degree i that have an algebraic conjugate which is contained in $\mathcal{I}$, and let Θ_i be the set of those elements in $\mathcal{E}_{\mathcal{T}} \cap \mathcal{I}$ that have degree i. Then by (91) and (93), each element in Θ_i has an algebraic conjugate in $[-\kappa, \kappa] \setminus \mathcal{I}$. This implies that Δ_i is a non-empty set if and only if Θ_i is a non-empty set. Hence, for each integer $i > \frac{1}{\Upsilon}$ satisfying $\Theta_i \neq \emptyset$, the number of elements in the set

$$\Lambda_i := \{(\theta, \eta) \in \Theta_i \times \Delta_i : \theta \text{ and } \eta \text{ are conjugate algebraic numbers}\}$$

is bounded above and below as follows:

$$(1-2\,\Upsilon)\,i\;|\Theta_i| < |\Lambda_i| < 2\,i\,\Upsilon\;|\Delta_i|\,. \tag{95}$$

Hence, by (91), (94) and (95), the inequality

$$\frac{(1-2\,\Upsilon)\epsilon|\mathcal{I}|\,\mathtt{h}}{24\kappa\pi g} \le (1-2\,\Upsilon)\sum_{i>\frac{1}{\Upsilon},\ \Theta_i\neq\emptyset}|\Theta_i| < 2\,\Upsilon\sum_{i>\frac{1}{\Upsilon},\ \Delta_i\neq\emptyset}|\Delta_i| \tag{96}$$

holds, and therefore by (91) and (96), it follows that

$$\sum_{i>\frac{1}{\Upsilon},\,\Delta_i\neq\emptyset}|\Delta_i| > \frac{(1-2\,\Upsilon)\epsilon|\mathcal{I}|\,\mathtt{h}}{48\kappa\pi g\,\Upsilon} \ge \mu\,\mathtt{h} \tag{97}$$

holds. Since the number of eigenvalues of $\mathcal{T}$ which have an algebraic conjugate in $\mathcal{I}$ is at least $\sum_{i>\frac{1}{\Upsilon},\Delta_i\neq\emptyset}|\Delta_i|$, the proposition now follows immediately by (97). ∎

Proof of Theorem 4.2: Suppose that κ, λ, ϵ, C, $\mathcal{G}$, Δ, L, ℓ and $\mathcal{T}$ are as in the statement of the theorem, and put $\mathtt{h} := \mathtt{h}_{\mathcal{T}}$, $\mathtt{t} := \mathtt{t}_{\mathcal{T}}$, $D := D_{\mathcal{T}}$, $\mathtt{Len} := \mathtt{Len}_{\mathcal{G},\ell}$ and $\mathtt{Gap} := \mathtt{Gap}_{\mathcal{G},\ell}$ (cf. (71), (72)). By statement (iii) of the theorem and Lemma 3.3 (i), there exists an integer $s_0 \in \{2,\dots,g\}$ such that $\mathfrak{R}_{s_0} > \mathfrak{R}_1$ and $\ell(s_0) > \left(\frac{\epsilon}{g}\right)\mathtt{h}$ (cf. (28)). On the other hand, by Corollary 7.3, there exists a well-placed interval $\mathcal{J}_0$ in the s_0 th guide interval $I_{s_0} = (\mathfrak{L}_{s_0}, \mathfrak{R}_{s_0})$ (relative to $\mathcal{G}$) such that $\mathcal{J}_0 \cap \mathcal{B} = \emptyset$ and $\mathtt{Len}(\mathcal{J}_0) > \left(\frac{\epsilon}{g}\right)\mathtt{h}$ both hold as $\mathtt{Len}(\mathcal{J}_0) \ge \ell(s_0)$ (cf. (29), (52)). It follows by Proposition 9.2 for $(\epsilon,\mu) := \left(\frac{\epsilon}{g}, C(\kappa)+2\right)$ that there exist positive constants $F_0 := F_0(\kappa,\mathcal{G},\Delta,L)$, $G_0 := G_0(\kappa,\lambda,\epsilon,\mathcal{G})$ and $H_0 := H_0(\kappa,\lambda,\epsilon,\mathcal{G},\Delta,L)$ such that if $\ell(i) > F_0$ holds for all $(\gamma_i,\alpha_i,\beta_i) \in \underline{\Delta}$ then

$$\text{either } \ \mathtt{h} < H_0 \ \text{ or } \ \mathtt{Gap}(\mathcal{J}_0) > G_0\mathtt{h} \ \text{ holds,}$$

as $\mathcal{T}$ has exactly $D+1$ distinct eigenvalues (cf. (26)) and $D \le C\,\mathtt{h}$ holds by statement (ii) of the theorem.

Now, if $\mathtt{h} < H_0$, then the theorem follows by taking $H := H_0$ and $F := F_0$.

Otherwise, $\mathtt{h} \ge H_0$ and $\mathtt{Gap}(\mathcal{J}_0) > G_0\mathtt{h}$ both hold, so by Corollary 7.3 and Proposition 7.4 for $\mathcal{I} := \mathcal{J}_0$, there exists an integer s_1, $\mathfrak{c}(\mathcal{G},\mathcal{J}_0) \le s_1 \le \mathfrak{d}(\mathcal{G},\mathcal{J}_0)$, and a well-placed interval $\mathcal{J}_1$ in the s_1 th guide interval I_{s_1} such that

$$\mathtt{Len}(\mathcal{J}_1) > \frac{\mathtt{Gap}(\mathcal{J}_0)}{g} > \left(\frac{G_0}{g}\right)\mathtt{h}.$$

By applying Proposition 9.2 again for $(\epsilon,\mu) := (\frac{G_0}{g}, C(\kappa)+2)$, there exist positive constants $F_1 := F_1(\kappa,\mathcal{G},\Delta,L)$, $G_1 := G_1(\kappa,\lambda,\epsilon,\mathcal{G})$ and $H_1 := H_1(\kappa,\lambda,\epsilon,\mathcal{G},\Delta,L)$ such that if $\ell(i) > F_1$ holds for all $(\gamma_i,\alpha_i,\beta_i) \in \underline{\Delta}$ then

$$\text{either } \ \mathtt{h} < H_1 \ \text{ or } \ \mathtt{Gap}(\mathcal{J}_1) > G_1\mathtt{h} \ \text{ holds.}$$

Since $(\mathfrak{R}_i)_{i=1}^g$ is a finite unimodal sequence by Lemma 3.3, it follows by iteratively repeating this argument (if necessary) that there exist an integer m, $1 \le m \le g$, and positive constants $F_j :=$

$F_j(\kappa, \mathcal{G}, \Delta, L)$ and $H_j := H_j(\kappa, \lambda, \epsilon, \mathcal{G}, \Delta, L)$ $(0 \leq j \leq m)$ given by Proposition 9.2 such that if $\ell(i) > \max\{F_j : 0 \leq j \leq m\}$ holds for all $(\gamma_i, \alpha_i, \beta_i) \in \underline{\Delta}$ then

$$\max\{H_j : 0 \leq j \leq m-1\} \leq \mathtt{h} < H_m$$

holds. Theorem 4.2 now follows by taking

$$H := \max\{H_j : 0 \leq j \leq m\} \text{ and } F := \max\{F_j : 0 \leq j \leq m\}.$$

∎

10 Distance-Regular Graphs of Order (s, t)

In this section, we shall use our main result to show that, for fixed integer $t > 1$, there are only finitely many distance-regular graphs of order (s, t) whose smallest eigenvalue is different from $-t-1$. We begin by recalling the relevant definitions and some previous results.

Let Γ be a distance-regular graph. For any vertex x, the *local graph* of a vertex x is the subgraph of Γ induced by $\Gamma_1(x)$. For an integer $s \geq 1$, a *clique of size* s (or, s-clique) is a set of s vertices which are pairwise adjacent. Following H. Suzuki (see [34]), we say that a distance-regular graph Γ is of *order* (s, t) for some positive integers s, t, if the local graph of any vertex is the disjoint union of $t+1$ cliques of size s. In particular, a non-complete distance-regular graph with valency $k \geq 3$ and $c_2 = 1$ is of order (s, t) with $s = a_1 + 1$ and $t = \frac{k}{a_1+1}$.

Note that the Hamming graph $H(n, q)$ is a distance-regular graph of order $(n-1, q-1)$. Hence, for fixed positive integer t, there are infinitely many distance-regular graphs of order (s, t) where s is a positive integer. In addition, B. Mohar and J. Shawe-Taylor [30] (see also [12, Theorem 4.2.16]) showed that any distance-regular graph of order $(s, 1)$ with $s > 1$ is isomorphic to the line graph of a Moore graph or the point graph of some generalized $2D$-gon of order $(s, 1)$, where $D \in \{3, 4, 6\}$. Since the point graph of a generalized $2D$-gon of order $(s, 1)$ is exactly the same as the flag graph of a regular generalized D-gon of order (s, s), there are infinitely many distance-regular graphs of order $(s, 1)$ with $s > 1$.

The following proposition is well-known; we include its proof for completeness.

Proposition 10.1 *For s, t positive integers, let Γ be a distance-regular graph of order (s, t) with diameter $D \geq 2$. Then the smallest eigenvalue θ_D of Γ satisfies $\theta_D \geq -t-1$. Moreover, if $s > t$, then $\theta_D = -t-1$ holds.*

Proof: Let $\mathcal{C}$ be the set of $(s+1)$-cliques in Γ. Let M be the vertex-clique of size $s+1$ incidence matrix, that is, M is the $(|V(\Gamma)| \times |\mathcal{C}|)$-matrix such that the (x, C)-entry of M is 1 if $x \in C$ and 0 otherwise. Then $MM^T = A + (t+1)I$, where M^T is the transpose of M. As MM^T is positive semidefinite, it follows that all the eigenvalues of Γ are at least $-t-1$. Note that $|\mathcal{C}|(s+1) = |V(\Gamma)|(t+1)$ so that if $s > t$ then $|\mathcal{C}| < |V(\Gamma)|$, and, as the rank of M is at most $|\mathcal{C}|$, it follows that $A + (t+1)I$ is singular. This shows that A has $-t-1$ as its smallest eigenvalue. ∎

Corollary 10.2 *Let $t \geq 1$ be an integer. Then there are only finitely many distance-regular graphs of order (s, t) with $s \geq 1$ and $st \neq 1$ which have smallest eigenvalue not equal to $-t-1$.*

Proof: Let $t \geq 1$. If Γ is a distance-regular graph of order (s, t) such that its smallest eigenvalue is different from $-t-1$, then $s \leq t$ holds by Proposition 10.1. As the valency of Γ equals $s(t+1) \leq t(t+1)$, the corollary follows by Theorem 1.1 as long as $s(t+1) \neq 2$. ∎

Remark 10.3 *Not much is known concerning distance-regular graphs of order (s, t) with $t \geq 2$. The distance-regular graphs of order $(1, 2)$ and $(2, 2)$ were classified by N. L. Biggs, A. G. Boshier and J. Shawe-Taylor [11] and by A. Hiraki, K. Nomura and H. Suzuki [23], respectively. In [38], N. Yamazaki presented some strong results concerning distance-regular graphs of order $(s, 2)$ with $s > 2$. However, it is not known whether there are infinitely distance-regular graphs of order $(s, 2)$ with $s \geq 2$ and $c_2 = 1$.*

11 Concluding Remarks

In Section 1, we mentioned that Sims' conjecture on permutation groups could be used to prove that there are only finitely many finite, connected distance-transitive graphs of fixed valency greater than two. We conclude by recalling and discussing a combinatorial version of Sims' conjecture that is related to the Bannai-Ito conjecture.

To state this conjecture, we first recall the definition of association schemes (as defined by E. Bannai and T. Ito [5]). An *association scheme* $(X, \mathcal{R})$ is a finite set X together with a collection $\mathcal{R} = \{R_0, R_1, \ldots, R_r\}$ of non-empty binary relations on X satisfying the following conditions:
(i) $\mathcal{R}$ is a partition of $X \times X$;
(ii) $R_0 = \{(x, x) : x \in X\}$;
(iii) for each $R_i \in \mathcal{R}$, there exists i' such that $R_{i'} = \{(y, x) : (x, y) \in R_i\}$;
(iv) for any $0 \leq i, j, h \leq r$ and for any $(x, y) \in R_h$, the number $|\{z \in X : (x, z) \in R_i \text{ and } (z, y) \in R_j\}|$ is a constant p_{ij}^h which depends only on i, j, h not on the choice of (x, y).

Note that an association scheme in this sense is also called a *homogeneous coherent configuration* (see [22]). Also, an association scheme $(X, \mathcal{R})$ is called *primitive* if any non-trivial relation R_i $(i \neq 0)$ induces a directed connected graph on the vertex set X.

Let $(X, \mathcal{R})$ be a primitive association scheme. Then each non-trivial relation $R_i \in \mathcal{R}$ $(i \neq 0)$ induces a directed, connected, regular graph of valency $k_i := p_{ii'}^0$. L. Pyber [31, p.207] and M. Hirasaka [24, p.105] attribute the following conjecture to L. Babai.

Conjecture 11.1 (Babai's Conjecture)
There exists an integral function f such that for any primitive association scheme $(X, \{R_0, R_1, \ldots, R_r\})$,

$$k_{\max} \leq f(k_{\min})$$

holds, where $k_{\max} := \max\{k_i : 1 \leq i \leq r\}$ and $k_{\min} := \min\{k_i : 1 \leq i \leq r\}$.

For a primitive permutation group G on a finite set Ω, the orbits R_i of the induced action of G on $\Omega \times \Omega$ determine a primitive association scheme, denoted by $AS(G)$. Sims' conjecture follows from Conjecture 11.1 by considering the association scheme $AS(G)$ for a primitive permutation group G. Note also that the cyclotomic schemes (for a definition see [24, p.106]) provide examples of primitive association schemes with fixed smallest non-trivial valency and an unbounded number of classes. Therefore, in Conjecture 11.1 we cannot expect to provide a bound for r in terms of $k_{\min}$.

The main theorem of this paper, Theorem 1.1, implies that Conjecture 11.1 is true for primitive distance-regular graphs with diameter D as the sequence $(k_i)_{1\leq i\leq D}$ is unimodal by [12, Proposition 5.1.1 (i)] and $k_i \geq \sqrt{k}$ holds for all $i \geq 1$ by [12, Proposition 5.6.1].

One could also ask whether there exists an integral function f such that for any primitive commutative association scheme $(X, \{R_0, R_1, ..., R_r\})$ with multiplicities m_i $(i = 0, 1, \dots, r)$ with $m_0 = 1$,

$$m_{\max} \leq f(m_{\min})$$

holds, where $m_{\max} := \max\{m_i : 1 \leq i \leq r\}$ and $m_{\min} := \min\{m_i : 1 \leq i \leq r\}$. Such a function is not known to exist even for the class of Q-polynomial association schemes (for a definition see [12, p.58]), although the dual statement of Theorem 1.1 has been shown to be true by W. J. Martin and J. S. Williford [29]. In particular, they showed that for any $m_1 > 2$, there are only finitely many Q-polynomial association schemes with the property that the first idempotent in a Q-polynomial ordering has rank m_1.

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