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# Dynamic Multilateral Markets

## Proof of Theorem 1

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**Theorem 1** *There exists a unique SE payoff profile in any multilateral bargaining game.*

**Lemma 1** *For any $n \times n$ matrix $A$ and diagonal $n \times n$ matrices $D_\alpha$ and $D_\delta$ with positive determinants, let $\Sigma = D_\alpha^{1/2} D_\delta^{1/2} A D_\alpha^{1/2} D_\delta^{1/2}$. Then,*

*1.1* $\lambda(D_\alpha A D_\delta) = \lambda(\Sigma)$ *and*
*1.2* $\lambda(D_\delta - D_\alpha A D_\delta) = \lambda(D_\delta - \Sigma)$,

*where $\lambda(M)$ denotes an eigenvalue of the matrix $M$.*

**Proof 1** *Proof: Let $\widetilde{\mathbf{z}} = (D_\alpha^{1/2} D_\delta^{-1/2})\mathbf{z}$ for an eigenvector $\mathbf{z}$.*

$$1)\quad \Sigma = D_\alpha^{1/2} D_\delta^{1/2} A D_\alpha^{1/2} D_\delta^{1/2} = (D_\alpha^{-1/2} D_\delta^{1/2})(D_\alpha A D_\delta)(D_\alpha^{1/2} D_\delta^{-1/2}) \Rightarrow$$
$$(D_\alpha^{-1/2} D_\delta^{1/2})(D_\alpha A D_\delta)(D_\alpha^{1/2} D_\delta^{-1/2})\mathbf{z} = \lambda \mathbf{z} \Rightarrow (D_\alpha A D_\delta)\widetilde{\mathbf{z}} = \lambda \widetilde{\mathbf{z}}.$$

$$2)\quad D_\delta - \Sigma = D_\delta - (D_\alpha^{-1/2} D_\delta^{1/2})(D_\alpha A D_\delta)(D_\alpha^{1/2} D_\delta^{-1/2}) \Rightarrow$$
$$D_\delta \mathbf{z} - (D_\alpha^{-1/2} D_\delta^{1/2})(D_\alpha A D_\delta)(D_\alpha^{1/2} D_\delta^{-1/2})\mathbf{z} = \lambda \mathbf{z} \Rightarrow D_\delta \widetilde{\mathbf{z}} - (D_\alpha A D_\delta)\widetilde{\mathbf{z}} = \lambda \widetilde{\mathbf{z}}. \blacksquare$$

**Lemma 2** *The system,*

$$x_i = \delta_i x_i + \sum_{S \in \mathcal{S}_i} \theta_S \pi_S \frac{\alpha_i}{\alpha(S)} \left( v(S) - \sum_{i \in S} \delta_i x_i \right), \quad \forall i \in \mathcal{N}, \tag{1}$$

*has a unique solution $\mathbf{x}(\boldsymbol{\theta})$ for any vector of agreement probabilities $\boldsymbol{\theta} = (\theta_S)_{S \subseteq \mathcal{N}}$. This solution is continuous in $\boldsymbol{\theta}$.*

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**Proof 2** *Let* $\mathbf{I}_c = 1$ *if* $c$ *is true and* $\mathbf{I}_c = 0$ *otherwise and define* $f$ *as:*

$$f_i(x) = \delta_i x_i + \sum_{S\in\mathcal{S}_i} \theta_S \pi_S \frac{\alpha_i}{\alpha(S)}(v(S) - \sum_{i\in S}\delta_i x_i) = \Phi x + \phi, \tag{2}$$

$$\Phi = (I - D_\alpha \Pi)D_\delta, \quad I, D_\alpha, D_\delta, \Pi \in R_+^{N\times N}, \phi \in R_+^N,$$

$$I = \mathbf{I}_{i=j},\ (D_\alpha)_{ij} = \mathbf{I}_{i=j}\alpha_i,\ (D_\delta)_{ij} = \mathbf{I}_{i=j}\delta_i,\ \phi_i = \alpha_i \sum\nolimits_{S\in\mathcal{S}_i} \gamma_S v(S),$$

$$\Pi_{ij} = \sum\nolimits_{S\in\mathcal{S}_i\cap\mathcal{S}_j} \gamma_S, \quad \gamma_S = \theta_S\pi_S/\alpha(S) \geq 0, \quad \forall i,j \in \mathcal{N},$$

*Hence,* $D_\alpha$ *and* $D_\delta$ *are diagonal matrices and* $\Pi$ *is a non-negative symmetric matrix with all real eigenvalues.*

*First, we prove that all eigenvalues of* $\Phi$ *lie in the interval* $[-\widehat{\delta}, \widehat{\delta}]$*, where* $\widehat{\delta} = \max_i \delta_i < 1$*. We note that the eigenvalues of* $D_\alpha \Pi D_\delta$ *are bounded from above by* $\widehat{\delta}$*,*

$$\forall j \in \mathcal{N}, \quad \sum\nolimits_{i=1}^N (D_\alpha \Pi D_\delta)_{ij} = \sum\nolimits_{i=1}^N \alpha_i \delta_j \sum\nolimits_{S\in\mathcal{S}_i\cap\mathcal{S}_j} \gamma_S$$

$$\leq \delta_j \sum\nolimits_{i=1}^N \alpha_i \sum\nolimits_{S\in\mathcal{S}_i\cap\mathcal{S}_j} \frac{\pi_S}{\alpha(S)} = \delta_j \sum\nolimits_{S\in\mathcal{S}_j} \pi_S \leq \delta_j$$

$$\Rightarrow \lambda_{\max}(D_\alpha \Pi D_\delta) \leq ||D_\alpha \Pi D_\delta||_1 \leq \widehat{\delta}.$$

*In order to show the lower bound on the eigenvalues of* $D_\alpha \Pi D_\delta$*, we note that* $\Pi$ *is positive semidefinite,*

$$\forall \mathbf{z} \in R^N, \quad \mathbf{z}^T \Pi \mathbf{z} = \sum_{S\subseteq\mathcal{N}} \gamma_S \left(\sum_{i\in S}\sum_{j\in S} z_i z_j\right) = \sum_{S\subseteq\mathcal{N}} \gamma_S \left(\sum_{i\in S} z_i\right)^2 \geq 0,$$

*which implies that all eigenvalues of* $\Pi$ *are nonnegative. As* $\Pi$ *is symmetric, it can be diagonalized, i.e.* $\Pi = P\Lambda P^T$*, where the diagonal matrix* $\Lambda$ *contains the nonnegative eigenvalues of* $\Pi$*. Let*

$$\Sigma := D_\delta^{1/2} D_\alpha^{1/2} \Pi D_\alpha^{1/2} D_\delta^{1/2} = (D_\delta^{1/2} D_\alpha^{1/2} P)\Lambda(D_\delta^{1/2} D_\alpha^{1/2} P)^T.$$

*By Sylvester's law of inertia, the number of negative eigenvalues is the same for the symmetric matrix* $\Sigma$ *and for* $\Lambda$*. As the latter diagonal matrix has only nonnegative entries, the same must hold for the former. On the other hand,* $\Sigma$ *has the same eigenvalues as* $D_\alpha \Pi D_\delta$ *by Lemma 1.1. Hence, we conclude that all eigenvalues of* $D_\alpha \Pi D_\delta$ *and of* $\Sigma$ *lie between* $0$ *and* $\widehat{\delta}$*.*

*Then, applying Weyl's inequality to the symmetric matrix* $D_\delta - \Sigma$*, we obtain the bounds,*

$$\lambda_{\min}(D_\delta - \Sigma) \geq \lambda_{\min}(D_\delta) + \lambda_{\min}(-\Sigma) > -\widehat{\delta} > -1,$$

$$\lambda_{\max}(D_\delta - \Sigma) \leq \lambda_{\max}(D_\delta) + \lambda_{\max}(-\Sigma) \leq \widehat{\delta} < 1.$$

*By Lemma 1.2,* $D_\delta - \Sigma$ *and* $\Phi = D_\delta - D_\alpha \Pi^x D_\delta$ *have the same set of eigenvalues, which proves that all eigenvalues of* $\Phi$ *are less than one in modulus. By the Contraction Mapping Theorem and Lemma 2.1 in Bramoulle (2001), it follows that (1) has a unique solution* $\mathbf{x}(\boldsymbol{\theta}) \in R^N$*, i.e., the solution to (1) is nonsingular. As (1) is a linear system,* $x_i(\theta)$ *is given by the Cramer's rule as the ratio of two determinants that are continuous functions of* $\boldsymbol{\theta}$ *(with non-vanishing denominator for all* $\boldsymbol{\theta}$*). Therefore,* $\mathbf{x}(\boldsymbol{\theta})$ *is continuous in* $\boldsymbol{\theta}$*.* ■

**Lemma 3** *A SE* $\mathbf{x}$ *is continuous in* $v(S)$ *for any coalition* $S \in \Theta = \{S \subseteq \mathcal{N} : v(S) \geq 0\}$.

**Proof 3** *In what follows, we omit the discount factors* $\delta_1, ..., \delta_N$ *in all references to SE. We define the vectors,*

$$\mathbf{v} = (v(T))_{T\in\Theta}, \quad \widetilde{\mathbf{v}} = (v(T)_{T\in\Theta\setminus S}, \widetilde{v}(S)), \quad \widehat{\mathbf{v}} = (v(T)_{T\in\Theta\setminus S}, \widehat{v}(S)).$$

*Let* $\mathbf{x}^{\mathbf{v}}(\boldsymbol{\theta}) \in R^N$ *be the unique (by Lemma 2) solution to (1) for the vector of coalitional values* $\mathbf{v}$ *and agreement probabilities* $\boldsymbol{\theta} = (\theta_T)_{T\in\Theta}$.

*Let* $\boldsymbol{\theta}(x^v) = (\theta_T(\mathbf{x}^{\mathbf{v}}))_{T\in\Theta}$, *where* $\theta_T(\mathbf{x}^{\mathbf{v}}) = 1$ *if* $v(T) > \sum_{i\in T} \delta_i x_i^v$ *and, conversely,* $\theta_T(\mathbf{x}^{\mathbf{v}}) = 0$ *if* $(v(T) \leq \sum_{i\in T} \delta_i x_i^v)$.

*In what follows, we show that the SE* $\mathbf{x}^{\widetilde{\mathbf{v}}}$ *converges to the SE* $\mathbf{x}^{\mathbf{v}} = \mathbf{x}^{\mathbf{v}}(\boldsymbol{\theta}(\mathbf{x}^{\mathbf{v}}))$ *when* $\widetilde{\mathbf{v}}$ *approaches* $\mathbf{v}$.

*If* $v(T) \neq \sum_{i\in T} \delta_i x_i^v$ *for all* $T \in \Theta$, *then, due to the continuity of* $\mathbf{x}^{\widetilde{\mathbf{v}}}(\boldsymbol{\theta})$ *in* $\widetilde{\mathbf{v}}(\mathbf{S})$, *it holds for sufficiently small* $|\widetilde{v}(S) - v(S)|$ *that,*

$$\sum_{i\in T} \delta_i x_i^{\widetilde{\mathbf{v}}}(\boldsymbol{\theta}(\mathbf{x}^{\mathbf{v}})) \approx \sum_{i\in T} \delta_i x_i^v > v(T) \Rightarrow \sum_{i\in T} \delta_i x_i^{\widetilde{\mathbf{v}}}(\boldsymbol{\theta}(\mathbf{x}^{\mathbf{v}})) > v(T),$$
$$\sum_{i\in T} \delta_i x_i^{\widetilde{\mathbf{v}}}(\boldsymbol{\theta}(\mathbf{x}^{\mathbf{v}})) \approx \sum_{i\in T} \delta_i x_i^{\mathbf{v}} < v(T) \Rightarrow \sum_{i\in T} \delta_i x_i^{\widetilde{\mathbf{v}}}(\boldsymbol{\theta}(\mathbf{x}^{\mathbf{v}})) < v(T).$$

*As* $\mathbf{x}^{\mathbf{v}}$ *is a SE, then* $\mathbf{x}^{\widetilde{\mathbf{v}}}(\boldsymbol{\theta}(\mathbf{x}^{\mathbf{v}}))$ *is also a SE and*

$$\mathbf{x}^{\widetilde{\mathbf{v}}}(\boldsymbol{\theta}(\mathbf{x}^{\mathbf{v}})) \to \mathbf{x}^{\mathbf{v}}(\boldsymbol{\theta}(\mathbf{x}^{\mathbf{v}})) = \mathbf{x}^{\mathbf{v}} \quad when \quad \widetilde{v}(S) \to v(S).$$

*Consider now the case* $\Delta(v) := \{T \in \Theta : \sum_{i\in T} \delta_i x_i^{\mathbf{v}} = v(T)\} \neq \emptyset$. *In other words, there is at least one coalition that is indifferent between agreement and disagreement in the SE* $\mathbf{x}^{\mathbf{v}}$. *Take* $\widetilde{v}(S) > v(S)$ *such that for all* $\widehat{v}(S) \in \{(v(S), \widetilde{v}(S)\}$ *we have that* $\Delta(\widehat{v}) = \emptyset$. *Such* $\widetilde{v}(S)$ *exists as there is only a finite number of values* $\widehat{v}(S)$ *for which* $\Delta(\widehat{v}) \neq \emptyset$. *The proof of the last claim is similar to that of Lemma 1 in A Appendix in the main text and is omitted.*

*Then, by the same argument as in the SE existence proof, we can find a fixed point* $\widetilde{\boldsymbol{\theta}} \in \varphi_{\Delta(v)}(\widetilde{\boldsymbol{\theta}})$ *in the game with values* $\widetilde{\mathbf{v}}$, *where* $\widetilde{\theta}_S = \theta_S(\mathbf{x}^{\mathbf{v}})$ *for all* $S \in \Theta\setminus\Delta(v)$ *and* $\varphi_{\Delta(v)}$ *is defined in (4). Then,*

$$\widetilde{v}(S) \to v(S) \Rightarrow \mathbf{x}^{\widetilde{\mathbf{v}}}(\widetilde{\boldsymbol{\theta}}) \to \mathbf{x}^{\mathbf{v}}(\widetilde{\boldsymbol{\theta}}) = \mathbf{x}^{\mathbf{v}}(\boldsymbol{\theta}(\mathbf{x}^{\mathbf{v}})) = \mathbf{x}^{\mathbf{v}},$$

*where the last but one equality follows because* $\widetilde{\theta}_T = \theta_T(\mathbf{x}^{\mathbf{v}})$ *for all* $T \in \Theta\setminus\Delta(v)$ *and for all* $T \in \Delta(v)$,

$$\sum_{i\in T} \delta_i x_i^{\mathbf{v}} = v(T) \Rightarrow \theta_T(\mathbf{x}^{\mathbf{v}})(\sum_{i\in T} \delta_i x_i^{\mathbf{v}} - v(T)) = \widetilde{\theta}_T(\sum_{i\in T} \delta_i x_i^{\mathbf{v}} - v(T)).$$

*Moreover, as* $\mathbf{x}^{\mathbf{v}} = \mathbf{x}^{\mathbf{v}}(\widetilde{\boldsymbol{\theta}})$ *is a SE, then for all* $T \in \Theta\setminus\Delta(v)$,

$$\sum_{i\in T} \delta_i x_i^{\widetilde{\mathbf{v}}}(\widetilde{\boldsymbol{\theta}}) \approx \sum_{i\in T} \delta_i x_i^{\mathbf{v}}(\widetilde{\boldsymbol{\theta}}) > v(T) \Rightarrow \sum_{i\in T} \delta_i x_i^{\widetilde{\mathbf{v}}}(\widetilde{\boldsymbol{\theta}}) > v(T) \quad \& \quad \theta_T = \widetilde{\theta}_T = 0,$$
$$\sum_{i\in T} \delta_i x_i^{\widetilde{\mathbf{v}}}(\widetilde{\boldsymbol{\theta}}) \approx \sum_{i\in T} \delta_i x_i^{\mathbf{v}}(\widetilde{\boldsymbol{\theta}}) < v(T) \Rightarrow \sum_{i\in T} \delta_i x_i^{\widetilde{\mathbf{v}}}(\widetilde{\boldsymbol{\theta}}) < v(T) \quad \& \quad \theta_T = \widetilde{\theta}_T = 1.$$

*Hence,* $\mathbf{x}^{\widetilde{\mathbf{v}}}(\widetilde{\boldsymbol{\theta}})$ *is an SE.* ■

**Proof 4** *Theorem 1:*

***Existence:*** *Let $\Theta = \{S \subseteq \mathcal{N} : v(S) > 0\}$ be the set of productive coalitions and define the system of linear equations,*

$$z_i = \delta_i z_i + \sum_{S \in \mathcal{S}_i} \theta_S \pi_S \frac{\alpha_i}{\alpha(S)} (v(S) - \sum_{i \in S} \delta_i z_i), \quad \forall i \in \mathcal{N}. \tag{3}$$

*By Lemma 2, there is a unique solution $\mathbf{z}(\boldsymbol{\theta})$ to (3) for any $\boldsymbol{\theta} \in [0,1]^{\#\Theta}$ that is continuous in $\boldsymbol{\theta}$. For a set of coalitions $\Delta \subseteq \Theta$, define the correspondence $\varphi_\Delta(\boldsymbol{\theta}) : [0,1]^{\#\Theta} \rightrightarrows [0,1]^{\#\Theta}$ as follows,*

$$\begin{aligned} &\forall S \in \Delta, \quad \varphi_\Delta(\boldsymbol{\theta})_S \quad \begin{cases} = 1 & if \quad \sum_{i \in S} \delta_i z_i(\boldsymbol{\theta}) < v(S), \\ \in [0,1] & if \quad \sum_{i \in S} \delta_i z_i(\boldsymbol{\theta}) = v(S), \\ = 0 & if \quad \sum_{i \in S} \delta_i z_i(\boldsymbol{\theta}) > v(S), \end{cases} \\ &\forall S \in \Theta \backslash \Delta, \quad \varphi_\Delta(\theta)_S = \theta_S. \end{aligned} \tag{4}$$

*By definition, this correspondence has a closed graph and convex and non-empty images (i.e., $\forall \boldsymbol{\theta} \in [0,1]^{\#\Theta}$, $\varphi(\boldsymbol{\theta})$ is a nonempty and convex subset of $[0,1]^{\#\Theta}$). Then, by Kakutani's fixed point theorem, there exists some $\widetilde{\boldsymbol{\theta}}$ such that $\widetilde{\boldsymbol{\theta}} \in \varphi_\Delta(\widetilde{\boldsymbol{\theta}})$. By setting $\Delta = \Theta$, we obtain that $\mathbf{z}(\widetilde{\boldsymbol{\theta}})$ is a SE,*

$$\begin{aligned} (1-\delta_i) z_i(\widetilde{\theta}) &= \sum\nolimits_{S \in \mathcal{S}_i} \widetilde{\theta} \pi_S \frac{\alpha_i}{\alpha(S)} (v(S) - \sum\nolimits_{i \in S} \delta_i z_i(\widetilde{\theta})) \\ &= \sum\nolimits_{S \in \mathcal{S}_i} \pi_S \frac{\alpha_i}{\alpha(S)} \max(v(S) - \sum\nolimits_{i \in S} \delta_i z_i(\widetilde{\theta}), 0), \quad \forall i \in \mathcal{N}. \end{aligned}$$

***Uniqueness:*** *Let the set $\Theta = \{S \subseteq \mathcal{N} : v(S) > 0\}$ of productive coalitions be non-empty and, for the sake of contradiction, assume that there are two SE, $\mathbf{x}$ and $\mathbf{z}$, $\mathbf{x} \neq \mathbf{z}$, with the respective sets of agreeing coalitions $\mathcal{S}^{\mathbf{x}} \subseteq \Theta$ and $\mathcal{S}^{\mathbf{z}} \subseteq \Theta$,*

$$\begin{aligned} \mathcal{S}^{\mathbf{x}} &= \{C \in \Theta : \theta_C = 1(0) \Leftrightarrow v(C) > \sum_{i \in C} \delta_i x_i \quad (v(C) \leq \sum_{i \in C} \delta_i x_i)\}, \\ \mathcal{S}^{\mathbf{z}} &= \{C \in \Theta : \theta_C = 1(0) \Leftrightarrow v(C) > \sum_{i \in C} \delta_i z_i \quad (v(C) \leq \sum_{i \in C} \delta_i z_i)\}. \end{aligned}$$

*Note that the cardinalities $\#\mathcal{S}^{\mathbf{x}}$ and $\#\mathcal{S}^{\mathbf{z}}$ must be at least one and that $\mathcal{S}^{\mathbf{x}} \neq \mathcal{S}^{\mathbf{z}}$ as, otherwise, Lemma 2 implies $\mathbf{x} = \mathbf{z}$. Take a coalition $C \in \mathcal{S}^{\mathbf{x}} \setminus \mathcal{S}^{\mathbf{z}}$ (or $C \in \mathcal{S}^{\mathbf{z}} \setminus \mathcal{S}^{\mathbf{x}}$). By our existence results, for each value $\widetilde{v}(C) \in [0, v(C)]$, there is a SE $\mathbf{x}^{\widetilde{\mathbf{v}}}$. Due to the continuity of $\mathbf{x}^{\widetilde{\mathbf{v}}}$ in $\widetilde{v}(C)$ (Lemma 3) and the fact that $0 \leq \sum_{i \in C} \delta_i x_i < v(C)$ (as $C \in \mathcal{S}^{\mathbf{x}}$), there is a value $\widetilde{v}(C) < v(C)$ such that $\sum_{i \in C} \delta_i x_i^{\widetilde{\mathbf{v}}} = \widetilde{v}(C)$. Hence, if we set $\theta_C(x^{\widetilde{\mathbf{v}}}) = 0$, $\mathbf{x}^{\widetilde{\mathbf{v}}}$ will be a SE. Then, we can remove $C$ from $\Theta$, i.e., set $v(C) = 0$, without destroying the SE $\mathbf{x}^{\widetilde{\mathbf{v}}}$. On the other hand, the SE $\mathbf{z}$ does not change after removing $C$ because $C \notin \mathcal{S}^{\mathbf{z}}$,*

$$\sum_{i \in C} \delta_i z_i^{\widetilde{\mathbf{v}}} = \sum_{i \in C} \delta_i z_i \geq v(C) > \widetilde{v}(C).$$

*Thus, we have two different SE, $\mathbf{x}^{\widetilde{\mathbf{v}}}$ and $\mathbf{z}^{\widetilde{\mathbf{v}}} = \mathbf{z} \neq \mathbf{x}^{\widetilde{\mathbf{v}}}$ in the game with $\#\Theta - 1$ productive coalitions. We can keep removing productive coalitions one by one from $\Theta$ but keep their matching probabilities unchanged until either the set of agreeing coalitions is the same for the two different SE or the set of agreeing coalitions is empty for one (and only one) of the SE. The former case contradicts Lemma 2 while the latter is incompatible with $\#\Theta \geq 1$, i.e., with the existence of productive coalitions.* ■

## References

Bramoulle, Y.: 2001, Interdependent utilities, preference inde- terminacy, and social networks, *mimeo, University of Maryland* .