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Teaching and Learning Mathematics and Statistics

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A mathematics degree aims “to develop in students the capacity for learning and for clear logical thinking” and “will develop [the students’] skills of abstract, logical thinking and reasoning”.

These quotations come from the publicly stated aims for mathematics degree courses from two universities which serve very different communities: the first takes students with very high qualifications and emphasises developing the next generation of researchers, the second takes students with much lower entry qualifications and has an emphasis on the development of employability skills. While degree programmes may have very different ‘inputs’ and aim for quite different ‘outputs’, an examination of the stated aims across the sector suggests some level of commonality: there appears to be a core of agreement that the aims of a mathematics degree involve developing certain types of analytic thinking skills.

Much of the research evidence examining teaching and learning in higher education is generic, despite concerns that many (particularly in the sciences) raise about the applicability to their own area. Joughin (2010) argues that too little account can be taken of subject context in interpreting research evidence and mathematicians argue that the nature of knowledge in mathematics is different even from other sciences and that the teaching and assessment of mathematics may need to be considered separately (LMS, 2010).

Clearly there are generic issues which may apply across all (or large parts) of higher education, but these are amply dealt with in other chapters. In this chapter we concentrate on the non-generic aspects of undergraduate mathematics emphasised again and again in different universities’ aims: abstraction and analytic thinking.

Moreover, the research literature tends to have explored these more closely in pure mathematics so we will not say much that is specific about the teaching of applied mathematics or statistics: we believe the ideas of this chapter will be relevant across many sub-domains of mathematics. We should also note that we do not discuss mathematics taught in or for other disciplines.

This chapter is divided in three sections. We first explore what we mean by learning to think mathematically and the research evidence for particular types of mathematical thought. We then examine mathematics teaching that might take account of these different ways of thinking mathematically. Finally we look at assessment and the profound influence this can have on learners and teachers of mathematics.

In doing so, we recognise that teaching is a craft. There is no evidence to suggest that there is only one correct method of teaching to develop even these core analytic skills. It is more likely that the quality of teaching depends on a complex combination of teacher intention, learner preference, subject matter, institutional opportunities and constraints, assessment choices and a wide range of other, often implicit, factors.

Like others who look at education in universities, we will use the word “innovative” occasionally – but we do so with caution. Often “traditional” and “innovative” are seen as code for “bad” and “good” by some in the education community and, occasionally, as code for “good” and “bad” by some in the mathematics community. Neither is right. Traditional lectures and closed book examinations are seen by many as well adapted for teaching and assessing mathematics and, to date, the evidence base for alternative forms of teaching as substantially better in achieving desired learning outcomes is lacking. Moreover, in some contexts, it is clear that both staff and students prefer traditional teaching and assessment methods.

Learning Mathematics

There is a concern that the transition from school mathematics to a degree course in mathematics is particularly difficult. The A-level Mathematics programme has to fulfil many roles which no other A-level needs to address: in addition to acting as preparation for further study in the subject, it is also a service subject giving support for other A-level programmes (such as physics or economics) and is also often a pre-requisite for degree level study in those, and other, subjects. To balance all of these aims, among other reasons, the A level mathematics curriculum has tended towards breadth of topic and fluency of calculation. While fluency is certainly desirable on entry to a mathematics degree, many mathematicians might prefer depth and understanding of fewer, but more targeted topics such as algebra and calculus. To some extent, the unique position of A-level Further Mathematics helps address this, but issues with the provision of this programme mean that, for many, the gap between a calculation-based mathematics preparation at school and a concept-based mathematics degree at university is too large and students struggle to develop the new ways of thinking required.

Research in student thinking tends to emphasise dichotomies: simple splits between the ways in which people think. Provided that one keeps in mind that it is likely that individual learners are more complicated, such dichotomies can be useful in understanding the broad issues associated with learning mathematics.

One way of thinking about mathematical learning has been particularly attractive to practitioners for many years: *concept image* and *concept definition*. Tall and Vinner (1981) describe a *concept image* as an individual’s set of mental pictures, processes and properties which they associate with a concept and a *concept definition* as the form of words which specifies the concept (e.g. its formal definition).

While not meant to be a realistic model of cognitive functions, the idea allows us to account for some of the learning issues we see in undergraduate mathematics. Figure 1 suggests some crude distinctions between thinking which uses only informal intuition and imagery, thinking which uses only a definition and thinking which combines both.

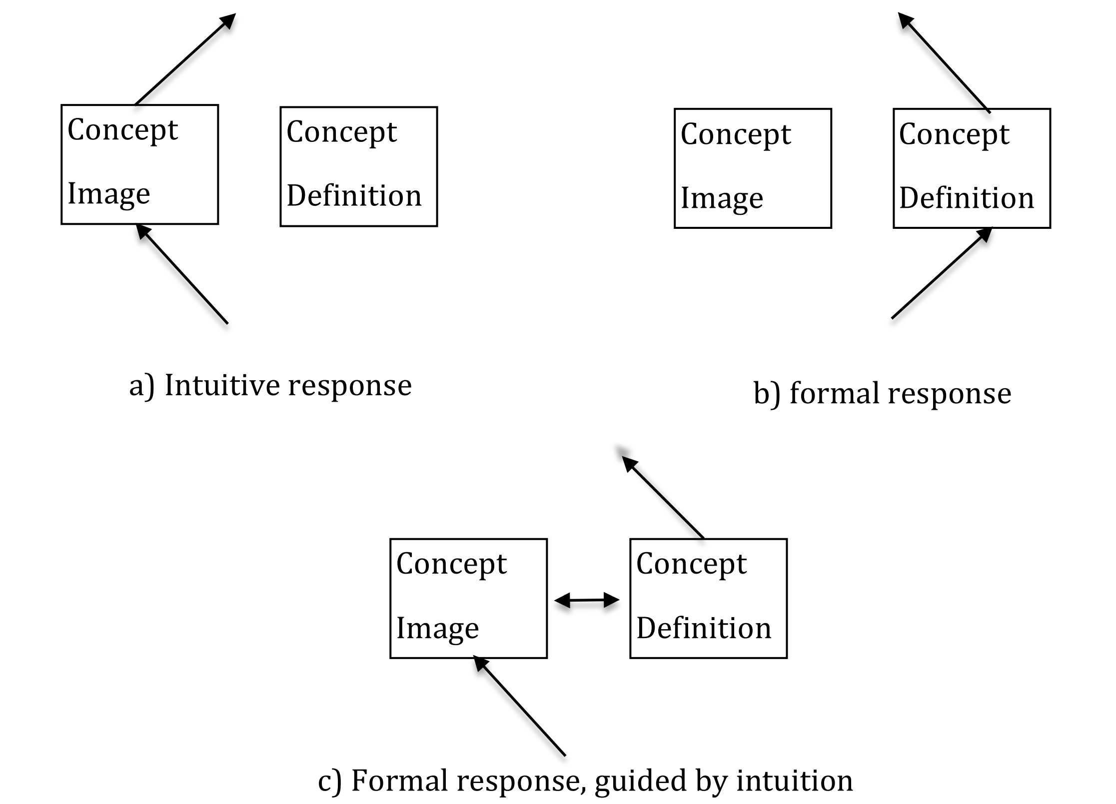


Figure 1: Different forms of response using concept image and definition

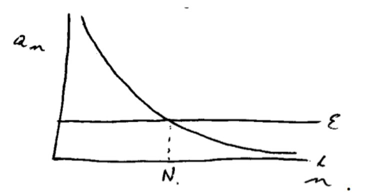
In the case study below (adapted from Pinto, 1998), we see students with quite different approaches to learning real analysis. It contrasts two approaches to learning we might call "intuitive" (using a concept image) and "formal" (using – or trying to use – a concept definition). It indicates the need for these two approaches to be co-ordinated and the problems of students taking a formal approach without intuition which might follow from a calculation-based pre-university experience. It further suggests that intuition without the ability to access the rules of "pushing symbols" can also be restricting (albeit that we might expect this to be less common amongst students).

Case Study One – Giving and Extracting Meanings in Real Analysis

In a traditional, lecture-based first course in real analysis, students were regularly interviewed and the researcher uncovered consistent ways in which students linked their knowledge and understanding of definitions with the construction of arguments, particularly in relation to their use of images.

The research noted two distinct strategies to developing arguments: "giving meaning" (starting from informal ideas – their concept images – and constructing arguments from that basis) and "extracting meaning" (starting from formal theory - the concept definition - and developing arguments as a form of calculation). Moreover, they found successful and unsuccessful examples of both of these strategies.

Students using a "giving meaning" strategy who were unsuccessful tended to have some form of imagery from which they would try to reconstruct a definition, but such a reconstruction *might well end up being a description of their imagery, given in some mathematical language. For example, the picture in figure 2 becomes the "definition" given as the student tries to describe it. Their imagery apparently admits only a few examples of convergent sequences – strictly decreasing ones – and their attempt to translate this into a formal definition loses the important relationship between the status of the quantifiers for ε and N.*



If $a_n \to l$, then there exists $\varepsilon > 0$, such that $|a_n - l| < \varepsilon$ for all $n \geq N$, where N is a large positive integer.

Figure 2: A less successful "giving meaning" learner.

In contrast, students "extracting meaning" who are unsuccessful have to rely on memory and, if they forget parts or misremember them, they cannot reliably reconstruct them. For example, the student giving the definition in figure 3 may seem to have a very good grasp, but further exploration seems to imply that the lack of universal quantification led the student to think that the definition refers to a particular value of epsilon.

A sequence (a_n) tends to a limit L for $\varepsilon > 0$ if there exists $N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$ provided $n \geq N$.

Figure 3: A less successful "extracting meaning" learner.

Successful learners in this study seemed to have some level of co-ordination between concept image and concept definition, the difference appearing to be only one of precedence. One student (whose definition and picture are given in figure 4) talked of the need to write the definition down "over and over again" first, learning to draw the picture some time afterwards: their emphasis was on the concept definition, but they could co-ordinate this with an image.

A sequence (a_n) tends to limit L if, $\forall\, \varepsilon > 0$, $\exists\, N(\varepsilon)$, s.t. $\forall\, n \geq N$;

$|a_n - L| < \varepsilon$.

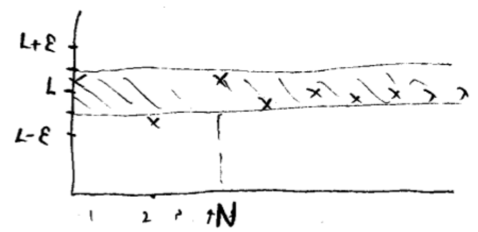


Figure 4: A more successful "extracting meaning" learner.

Another successful student, though, had a picture in mind and explicitly stated that he did not memorise the definition, but reconstructed it from the picture.

One can argue that the less successful learners in the case study were focussed on one of the boxes of figure 1, while the more successful perhaps had a balance between both (albeit still with a perceivable bias).

This study belongs to a long tradition of research into students understanding of proving (particularly in real analysis). As other studies before, it notes the contrast between the *intuitive and formal and highlights that neither, entirely on its own, is likely to be successful for many students.*

Clearly expert mathematicians use both their intuition and their knowledge of results and formal rules to guide their thinking – no doubt with different mathematicians, in different subfields and at different times, using a different balance between these parts of their thinking. However, our case study shows that not having access to the intuitions which come from a rich range of examples, counterexamples, prototypes and representations can result in students only being able to work on mathematical problems by manipulating formal language without ascribing meaning to this language (e.g. by pushing symbols). Similarly, students who are unaware of the status of the formal and reliant on pictures and metaphors will struggle with the precision required of a mathematics degree and may fall back on trying to merely mimic the formal language or, in the words of one colleague, “write nonsense in mathematical style”.

Of course this balance between intuitive and formal extends into applied mathematics and statistics as well. Applied mathematics relies on a high degree of fluency with a wide range of calculation tools, but it also requires intuition in the form of modelling and understanding the nature and applicability of models. Again, there appears to be a large gap between school and university, with little if any emphasis on the creation or critiquing of models or understanding of the modelling process in most A-level programmes. Moreover, one can see a clear distinction in the literature between those who see modelling and applications as an area to which mathematics can be applied and those who see modelling as a way of thinking about (all) mathematics (that is, using real world situations, in all their complexity, to allow students to encounter the need for particular types of mathematics). It is not clear that learners necessarily see this distinction.

Similarly in statistics there is a distinction between the application of statistics to describe or draw inferences from real world data and the development and understanding of the theoretical background of statistical techniques. The balance between these aims may well depend on the stated aims of the degree programme in which these modules sit.

So, across all domains, there is a need for students to develop particular, co-ordinated ways of thinking and thus, for teaching to support that development.

Interrogating practice

Can you give examples of how your intuition and formal understanding interact when you do mathematics? Look at students’ written work: can you see how formal understanding and intuition interact when they do mathematics?

Teaching Mathematics

The case study above comes from research undertaken on a very traditional mathematics module: a “definition-theorem-proof” lecture course. As mentioned earlier, we do not equate “traditional” with “bad”: there is much to commend the lecture. Done well, it provides clear information. In mathematics, it defines the syllabus, in that a student might feel rightly aggrieved if substantial assessed material was omitted (something which may not be true in many other subjects). It can help students obtain a good set of notes. Of course, lecturers can provide notes (or gapped notes with sections to be completed in lectures) or lectures can be recorded and accessed online, but many become worried that, in doing so, important learning and teaching processes are lost.

We suggest that some of these important aspects which can be lost, and which may need to be emphasised in the traditional lecture, are *attention, modelling mathematical thinking, engagement* and *contingency*.

The often unfair stereotype of a lecture is one of very low *attention*, with students engaged in mindless note-taking. In fact, a good lecturer can go beyond the mere delivery of notes: they can draw attention to specific items (for example, pointing out how a part of an earlier definition appears in the middle of a later proof). The lecturer can use diagrams alongside formal derivations and point out explicitly both the links between them and the status that each holds. Some have been known to go further and develop a “two board” system in which one board holds the formal derivations and a second board holds ideas, images, suggestions and working. This may allow the lecturer to draw explicit attention to the difference between a concept image and a concept definition and the importance of developing and integrating both.

The traditional lecture also allows an element of *modelling mathematical thinking*. Derivations are done ‘live’ (albeit in a time frame which, of necessity, ignores most of the deep thought processes which went into their construction). In deriving a result, a lecturer can explain how they think about the key ideas, which parts are the “clever tricks” - unique to the situation and which simply need to be remembered - and which are applicable strategies we see in the subject again and again. For example, to show that the identity in a group is unique, one might start by imagining there are two (e and e') and using the group properties to show they are equal; then note a few minutes later than an almost identical technique appears in the proof that the inverse of a given element is unique.

Traditional lectures are also stereotyped as places of low *engagement*. There are few questions asked by the students and few questions asked to the students. The experience of many lecturers who do try to ask questions in classes is of few hands going up and those always coming from a few, generally more successful students. But this need not be the case: setting an environment in which it is not acceptable to opt out of answering can be relatively easy. In schools, mini-whiteboards are commonly used to require an answer to a question from every pupil in a class (and, if organised appropriately, allow answers to be seen by the teacher without easily being seen by other pupils). Technology is now making possible something similar in lectures with the use of “clickers” or other form of audience response system (Rowlett, 2010). Even though these allow for only a restricted form of response, they do enhance engagement and reduce the opportunity to opt out. With the increased expectation that students come to lectures with smartphones and tablets we may soon be in the position of having the flexibility of response of a mini-whiteboard with the ease of use of a pre-installed clicker system.

Of course, audience response leads to the need for *contingency*. The stereotype is of a lecturer following a set of notes unwaveringly. However, getting responses back from students suggesting they have significant misconceptions means that the lecturer needs to be prepared to explore an area in more depth than they allowed for, invent new counter-examples which might expose the misconception for what it is or even unpick previous ideas to uncover the possible cause of the misconception.

To be fair, while a lecturer may be able to quickly invent a new example, a lecture is not a good place to unpick larger or deeper misconceptions, so there is a need for more opportunities for students to engage with the material and have teaching contingent on their needs. This can be achieved with seminar groups which are often part of many modules in mathematics. The frequency, size and nature of those seminars vary, but it is common for seminar groups to have around 20 students who go through an exercise sheet with a seminar leader (either a lecturer or a PhD student). Such seminars can be useful to complement the lecture as they can promote group work, help students exchange ideas and allow them to get help from lecturers. However these seminars have been criticised for the lack of structure and the variation in mode and content even across groups on the same module.

Clearly attention, modelling mathematical practice, engagement and contingency are not exclusive to the traditional lecture/seminar model; nor is the traditional lecture or seminar always the best place to exhibit these: there are a number of innovative forms of teaching mathematics at degree level.

The second case study - the Problem Solving Class – can appear as a good environment to develop all four of these factors in certain circumstances.

Iannone and Simpson (2012) found that 11 universities have a module with a title like “Problem Solving” on their undergraduate mathematics degree. Badger, Sangwin and Hawkes (2012) provide detailed case studies of six of these, as well as an analysis of the nature of such classes and the variety of activities taking place in them. In particular, they note one issue of concern about such classes: the nature of their mathematical content. Some classes are designed to teach students to be better mathematical problem solvers (so de-emphasise specific mathematical topics) while others have an explicit mathematical topic which they approach through a sequence of problems to solve. The case study below is of the former type, but there are a number of carefully developed sequences of problems of the latter type (notably for number theory, Burn 1997, and real analysis, Burn, 2000).

Case Study 2: Problem Solving Classes

The University of East Anglia introduced a problem solving module in 2012. It spans eight weeks in the first term of the first year and it is compulsory for all first year mathematics students. The module is divided into two parts of four weeks each. The first is dedicated to developing problem solving techniques and the second to proof writing. Teaching is in small seminar groups (of around 20 students) where students are asked to work together on given problems coordinated by a member of faculty. There are also weekly smaller seminar sessions with "Peer Guides": second and third year students who have been trained working in this environment. The problems in this module include many from Mason, Burton and Stacey (1982) but also have some developed by the module leaders. Such problems have been grouped into different categories:

Word Problems: *a problem given in narrative which needs to be translated in mathematical language. For example:*

> *What number exceeds its square by the greatest amount? What is that amount?*

Proof Production: *a problem for which the students need to find an appropriate statement to prove and prove it. For example:*

> *A number like 12321 is called a palindrome because it reads the same backwards as forwards. A friend of mine claims that all palindromes with four digits are exactly divisible by eleven. Are they?*

Proof Refinement: *the statement is given and the student needs to produce a proof written in formal mathematical language.*

Open Problems: *a problem which does not necessarily have only one solution or could be solved at different levels. For example*

> *You are looking for a set of points in the plane satisfying the following two conditions: (i) the distance between any two points is an integer; (ii) the points are not collinear. Can you find a set of three points satisfying these conditions? How about a set of five points? Seven points? Just how large a set can you find? Could it even be infinite? Extend: Points in space not all coplanar?*

The module has only two lectures, one at the beginning of each section. The first lecture is an introduction to the module, its structure and assessment. The second lecture consists partly of feedback on the first coursework task and partly as an introduction to writing proofs.

Assessment is 100% coursework in two parts. The first part is handed in at the end of the first section of the module and consists of a problem students have to solve. They hand in the solution to the problem and their working. The second part is a more complex problem where students are not only asked to solve the problem, but also write a proof in as polished a form and in as precise formal mathematical language as they can.

The rationale of these module comes from the idea that the only way to learn about problem solving strategies is by "doing": by experiencing the strategies and proofs with the support of peers. Indeed, one factor on which the success of these modules depends is that the lecturer needs to resist the temptation to lecture! The role of the lecturer is that of a facilitator, giving minimal advice in the problem solving stage and facilitating discussion at the group discussion stage.

Students so far have had mixed reactions to this module. For some, this becomes a much appreciated opportunity to engage in depth with problem and interact with fellow students, while others are puzzled by the lack of direct instruction in a module that does not resemble *the standard lecture/seminar modules they expect.*

While the problem solving class tends to focus on engagement and attention, there are other teaching innovations which focus on the formal. Notably, both Houston (2009) and McConlogue, Mitchell and Vivaldi (2010) discuss materials designed to lead students to attend to the precision of mathematical expression. These again often emphasise the link between and status of the formal and the informal. Consider the task

You are given four distinct complex numbers. How do you decide whether or not these numbers lie at the vertices of a square in the complex plane? (Do not use symbols or mathematical notation in your answer)

(McConlogue et al., 2010, p13)

This requires the translation of an answer probably obtained with extensive use of mathematical symbolism into language without that symbolism (but retaining precision).

Such mathematical writing, combining rigorous symbolism with visual representations, metaphors and other forms of expression often comes to the fore in the final year project. Most universities provide some form of project module, often contributing a large portion of the final year mark and the teaching on this is different yet again. Typically a student will be provided with some one-to-one or small group supervision and be expected to study otherwise independently; write up a report on the project, perhaps present a summary verbally or as a poster and perhaps be expected to respond to questions about it.

However, we may need to be careful with introducing pockets of very different teaching. The case study suggests that while an encounter with a very different form of teaching from that expected can be relished as a challenge by some, it can be disconcerting and difficult to adapt to for others. Suddenly encountering a self-study module or project after years of lecture courses may require providing clearer support for some students if it is not to be too great a shock.

Interrogating practice

Have you thought about what you can do in your module to help first year student to successfully move from school mathematics to university mathematics? At departmental level, can you think of strategies to help this transition in the design of your first year provision?

Assessment

Assessment is often thought of as the end of the teaching and learning process: a simple evaluation of how much of the latter took place in the context of the former. However, assessment may also be thought of as part of the process of learning and teaching. The phrase “assessment for learning” is widely used in schools and is intended to mean the use of assessment evidence (drawn broadly) to help teachers and pupils plan forthcoming learning (Black, Harrison, Lee, Marshall & Wiliam 2007).

However, one might also talk of "assessment as learning": in purely cognitive terms learning involves the structuring of knowledge in memory and as Karpicke and Blunt (2011) put it "a retrieval event may actually represent a more powerful learning activity than an encoding event". That is, being assessed on something may help fix it in your mind more clearly than being taught it again. So assessment, stereotypically seen by both students and lecturers as a necessary inconvenience to tell them how much the students have or haven't learned, may actually be very valuable in strengthening learning.

A recent survey on assessment of mathematics at universities in the UK (Iannone and Simpson, 2012) suggested that by far the most common summative assessment method in mathematics is the closed book examination. Data for 43 degree courses showed that the median contribution of closed book examinations towards the final degree was 72% and few departments have closed book examinations accounting for less than 50% of the final degree (when averaged across all their modules). When the final year project (often representing a large portion of a final degree classification) is removed, the median contribution of closed book examinations to the final degree classification was 80%.

However, there were clear variations between institutions. To some extent this may be because, as suggested earlier, different institutions have different aims and take students with different backgrounds. But this does not account for all the variance: for example, Iannone and Simpson (2012) noted two universities with similar entry requirements and similar views of themselves as research intensive departments had very different patterns of assessment. The first was disproportionately dominated by examinations: after the first year, every module delivered by the department (with the exception of the final year project) was assessed exclusively by closed book examination. The second had a disproportionately low number of closed book examinations, with at least 20% of most modules across all years coming from other forms of assessment and some modules even in the final year with no closed book examinations at all. There was no evidence of conservatism on the part of the first university – indeed, the pattern of assessment was the result of a reasoned and agreed policy in the department with the main drivers being concerns about plagiarism and the lack of validity of weekly coursework sheets. The second university had taken just as reasoned an approach to its assessment pattern, in this case wishing to emphasise the importance of developing, through assessment, skills of direct value to the workplace.

Of course, the second university still had a large proportion of closed book examinations in absolute terms (just under 50% averaged across all modules) as did all the universities sampled. The evidence from the general assessment literature suggests that the prevalence of the closed book examination would be something which conflicts with student preference. However, it may be that this is another area where we need to take care when applying the results of the general literature to mathematics. Iannone and

Simpson (in press) indicate that the sources of data in the generic research literature rarely include students on degrees in hard sciences. When mathematics students were asked – albeit in a study looking exclusively at research intensive universities – they appeared to see the closed book examination as the "gold standard": highly valued for fairness, discriminating on the grounds of ability and by far their preferred method for being assessed. This is quite at odds with the suggestions of the general literature.

That noted, the study also suggested that students would appreciate some further diversification of assessment. Despite the prevalence of closed book examinations there are a wide range of methods in use and available. Open book examinations tend to be used in statistics, programming projects in computing courses and most universities use some form of 'homework' (variously called example sheets, weekly coursework, tutorial sheets etc.) which may contribute a small amount to a module mark, particularly in the early years. To mark regular coursework can be onerous and, as indicated above, some mathematicians express concerns over its validity if there is widespread collusion or copying.

However, if we take the notion of "assessment as learning" seriously, it would suggest the need for some regular assessment like these example sheets. Computer aided assessment has been used in some areas of mathematics for many years. It has the advantages, if properly designed, of avoiding copying by providing individualised questions, improving feedback times and radically decreasing workload. The main issue is the difficulties associated with representing and evaluating answers when the intended solution is a mathematical expression. Packages like STACK (Sangwin, 2008) overcome many of these issues with the clever use of an underlying computer algebra system and a flexible question and answer design method.

The case study given here shows another way in which computer aided assessment (as well as assessment in general) can be cleverly reconceptualised – in this case, the students have to set the questions and design both the correct and distractor answers for online multiple choice tests. This fits the notion of "assessment as learning" rather well.

Case Study 3:The use of the platform PeerWise for assessing geometry and statistics

A team of researchers at Auckland University has constructed an open access platform which allows students "to create and to explain their understanding of course related assessment questions, and to answer and discuss questions created by their peers." (http://peerwise.cs.auckland.ac.nz). This platform is intended for use in any academic subject to implement continuous formative assessment which not only is not onerous on staff time but also allows students to assess each others' work.

In the UK, the Universities of Edinburgh and Glasgow use this platform for first year physics modules and Liverpool University use it for a first year chemistry module. In this case study we describe an adaptation of this platform in first year Geometry and second year Statistics modules at Leicester University.

Both modules involve large groups of students and opportunities for continuous assessment have in the past been restricted due to the demand they place on staff time. With the use of

PeerWise the fortnightly homework for these modules asks students to construct their own multiple choice questions to pose to their peers on selected sections of the syllabus and to critique (e.g. answer but also critically assess) their peer's questions. The lecturer of the modules acts as a moderator when the need arises and can monitor students' progress on the platform. The assessment patterns of the two modules is slight different:

Geometry – Year 1	
45% course test *50% project* *5% participation in PeerWise*	*Participation in PeerWise consists of students submitting at least 2 multiple choice questions every two weeks and for providing feedback and comments on between 6 and 8 questions produced by peers.*
Statistics – Year 2	
20% course test *20% course test* *50% open book examination* *10% participation in PeerWise*	*Participation in PeerWise consists of students submitting at least one multiple choice question every two weeks, and commenting on 4 submitted by peers.*

Both modules retain a large proportion of assessment by examination (in the form of a course test), but this reversed use of PeerWise includes more opportunities for continuous assessment.

The lecturer who introduced the use of this platform believes that asking the students to both provide the answers and design the questions helps them to think more deeply about the material. Part of the requirement of the fortnightly homework is that students take the questions designed by their peers to assess their own understanding and leave feedback on the questions they have answered. In this way students also engage with peer assessment and peer learning. Moreover as the lecturer can monitor students' activity, this is also an ideal tool to quickly flag up common problems and misunderstandings which can then be addressed in the lectures if needed.

This assessment method is relatively new but from initial participation data it appears that general engagement with the platform is very good (though high achieving students engage with the platform more than struggling students) and the students' feedback is generally very positive.

Interrogating practice

Have you thought about the way in which you decided about assessment of your modules? Could you introduce, given the constraints on your module, a component of "assessment as learning"?

Conclusions

Our section on learning suggests that, to achieve the aim of students with improved logical and analytical thinking, students need to develop intuitive understanding of concepts (which may involve rich sets of examples, counter-examples, representations and properties), the formal abilities to manipulate those concepts and a robust and reliable link between the intuitive and the formal. The section on teaching suggests ways in which different types of teaching might achieve this. Traditional lecturing can draw attention to both the intuitive and the formal, model how mathematicians integrate the two and need not be as un-engaging as the stereotype suggests. Other forms of teaching, such as problem solving classes, can put students in the position of modelling some aspects of creative mathematical processes in exploring problems to help develop their intuition and then, with a focus on proving and accuracy of expression, tie that intuition to the formal symbolism.

Our final section reconceives mathematics assessment as also a form of learning, rather than simply an evaluation or certification process and in doing so, shows that it can help students with the process of tying the intuitive and formal together. As with teaching, the evidence suggests strong reliance on the traditional (in this case, closed book examinations) across all university mathematics departments and notes that both staff and students can see these as entirely appropriate. Even where students might value a wider range of assessment methods, the “gold standard” remains the formal examination. But other innovative forms of assessment, such assessing knowledge through students’ construction of questions, may help us address students’ interest in a more varied assessment diet.

Clearly different mathematics departments can have very different aims, but at the core they have a common interest in developing particular types of analytic thought which constitutes a mathematical habit of mind. We argue that understanding how to teach and assess mathematics comes from a careful consideration of how students learn mathematics, which may be different in many ways from how students learning in other subject areas.

References

Badger, M., Sangwin, C. and Hawkes, T. (2012) *Teaching Problem Solving in Undergraduate Mathematics*, University of Birmingham.

Black, P. P. J., Harrison, C., Lee, C., Marshall, B., & Wiliam, D. (2007). *Assessment for learning*. New York: Open University Press.

Burn, R. P. (1997). *A pathway into number theory*. Cambridge University Press.

Burn, R. P. (2000). *Numbers and Functions: Steps to Analysis*. Cambridge University Press.

Houston, K. (2009). *How to think like a mathematician: a companion to undergraduate mathematics*. Cambridge University Press.

Iannone, P. and Simpson, A. (in press) “Students’ preferences in undergraduate mathematics assessment”, *Studies in Higher Education*.

Iannone P. and Simpson A. (Eds.) (2012) *Mapping university mathematics assessment practices*. University of East Anglia, Norwich.

Joughin, G. (2010). The hidden curriculum revisited: a critical review of research into the influence of summative assessment on learning. *Assessment and Evaluation in Higher Education* 35(3): 335–345.

Karpicke, J. D., & Blunt, J. R. (2011). Retrieval practice produces more learning than elaborative studying with concept mapping. *Science*, *331*(6018), 772-775.

London Mathematical Society. (LMS) (2010) Mathematics Degrees, their Teaching and Assessment. Accessed via http://futurelms.wordpress.com/2010/ 04/08/teaching- position-statement/

Mason, J., Burton, L. and Stacey, K. (1982) *Thinking mathematically.* Pearson Education Limited.

McConlogue, T., Mitchell, S., & Vivaldi, F. (2010). Beyond templates 2: exploring students' approaches to learning on a mathematical writing course. *MSOR Connections*, *10*(2), 10-13.

Pinto, M. (1998). *Students' Understanding of Real Analysis*, Unpublished PhD Thesis, Warwick University. Available at http://pqdtopen.proquest.com/#abstract?dispub=3491803

Rowlett, P. (2010). Ask the audience (yes, all of them). *MSOR Connections.* 10(1), 3-5.

Sangwin, C. J. (2008) Assessing Elementary Algebra with STACK. *International Journal of Mathematical Education in Science and Technology*, 38(8), 987-1002.

Tall, D., and Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics.* 12(2), 151-169.