

DYNAMICS ON HOMOGENEOUS SPACES AND
APPLICATIONS TO SIMULTANEOUS DIOPHANTINE
APPROXIMATION

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Abstract

We give some new results about diophantine simultaneous inequalities involving one quadratic form and one linear generalising the Oppenheim conjecture. In the first part we compute an exact lower asymptotic estimate of the number of integral values taken by such pairs, by using uniform distribution of unipotents flows. In the second part, we prove an S -adic version of the Oppenheim type problem for pairs. The proof uses S -adic dynamics and strong approximation. We also discuss a conjecture due to A. Gorodnik about finding optimal conditions which ensure density for pairs in dimension greater than three. This conjecture was partially the motivation of this thesis and is still open at this time.

Introduction

The purpose of this thesis is to address some issues of the theory of Diophantine inequalities involving one quadratic form and one linear form in light of the theory of unipotent actions on homogeneous flows. For instance a first problem is to look, given any $\varepsilon > 0$, whether the system of inequalities given by

$$|Q(x)| < \varepsilon \text{ and } |L(x)| < \varepsilon \quad (A)$$

has integer solutions $x \in \mathbb{Z}^n$ where Q and L are respectively a indefinite nondegenerate quadratic form and respectively a non zero linear form in $n \geq 3$ variables. A positive answer to this problem is already known for pairs $F = (Q, L)$ satisfying both nice geometric and arithmetic conditions. The result is due to S.G. Dani and G.A. Margulis ([DM90]) for the case $n = 3$ and to A. Gorodnik for the case $n \geq 4$ ([Gor04]). We pursue this study by giving a proof of a *quantitative version* of Dani and Margulis' solution of (A) in dimension three, more precisely we compute an exact asymptotic lower bound of the number of integral solutions of (A) lying in growing subsets of \mathbb{R}^3 . The second contribution is a generalisation of Gordonik's result to the S -adic setting. The proof requires an interplay between two mutually complementary methods based on ergodic theory and strong approximation property.

Historically the case of a single quadratic form has been a great challenge during the last century in the theory of diophantine approximation. In 1929, Oppenheim conjectured that given an indefinite nondegenerate quadratic form in $n \geq 3$ ¹ which is not proportional to a rational form and for any $\varepsilon > 0$, there exists $x \in \mathbb{Z}^n$ such that

$$0 < |Q(x)| < \varepsilon$$

It is easy to see that this statement of the conjecture is equivalent to the density of $Q(\mathbb{Z}^n)$ in \mathbb{R} . The Oppenheim conjecture remained an open problem for more than fifty years. A complete solution was finally given by G.A. Margulis in 1986 using ergodic theory. The first attempts towards a solution to the Oppenheim conjecture relies on geometry of numbers to the count lattice points in a large ellipsoid due to Jarník and Walfisz. Using this

¹Originally Oppenheim stated the conjecture for $n \geq 5$ in analogy of Meyer's theorem which states that given a *rational* indefinite quadratic form in $n \geq 5$ variables then $m(Q) = \inf_{x \in \mathbb{Z}^n, z \neq 0} |Q(x)| = 0$. The extension of the conjecture to the case of $n \geq 3$ variables is due to Davenport and Heilbronn in 1946.

method, Chowla proved in 1934, the Oppenheim conjecture for indefinite quadratic forms in $n \geq 9$ variables. Later, by using a modification of Hardy-Littlewood's Circle method, Davenport and Heilbronn proved the same result for $n \geq 5$ in 1946. The same method allowed Birch, Davenport and Ridout to prove in 1959 the conjecture for general quadratic form for $n \geq 21$. Some progress has been made by R.C. Baker and Schlickewaei in 1986 by solving the general conjecture for $n \geq 18$ with some restrictions on the signature of Q . This was the best known result before Margulis achieved to give a complete solution of the Oppenheim conjecture.

In parallel to these developments, a remark made by Raghunathan in the late seventies about orbit closures of unipotent actions on the space of lattices introduced new insights on the Oppenheim conjecture (this was already formulated in a implicit way in a paper of Cassels and Swinnerton-Dyer in 1955). This new point of view consists to exploit *à la* Felix Klein the action of the symmetry group $\text{SO}(Q)$ of the cone defined by Q on the homogenous space of lattices in \mathbb{R}^n , namely $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$. Therefore in order to establish the Oppenheim conjecture, we are led to prove that the orbit of \mathbb{Z}^n (viewed as a lattice) under the action of $\text{SO}(Q)$ is dense in the space of lattices in \mathbb{R}^n . The Raghunathan conjecture states more generally that the closure of an orbit Ux under any unipotent subgroup U and any point x of a homogeneous space should be an orbit of the same point under a subgroup L containing U , i.e. $\overline{Ux} = Lx$.

The hidden dynamics of unipotent actions becomes more visible if one deals with the action of a one-parameter unipotent flows $u(t)_{t \in \mathbb{R}}$ on the space of lattices in \mathbb{R}^n . Indeed such flows have a polynomial growth property, that is, given any lattice Λ , the function $t \mapsto \|u(t)\Lambda\|$ is a polynomial of degree n . This fundamental property implies that unipotent flows show to satisfy the so-called *nondivergence* property which means roughly that orbits do not spend in average most of their life outside compact subsets.

Since the proof of the Oppenheim conjecture reduces to the case $n = 3$, Margulis showed using highly nontrivial properties of one-parameter subgroups of $\text{SL}(3, \mathbb{R})$ that Raghunathan's conjecture is true for $\text{SO}(2, 1)$ acting on $\text{SL}(3, \mathbb{R})/\text{SL}(3, \mathbb{Z})$ ². Shortly afterwards in the early nineties, M.Ratner successfully solved the Raghunathan conjecture in full generality, and thus yields a direct proof of the Oppenheim conjecture. Fortunately this was not the end of the story, and the influence of the theory of unipotent flows is increasing in various directions. Another example of application in the theory of diophantine approximation on manifolds is the proof of Sprindžuk's conjecture made by Kleinbock and Margulis. At the opposite for non-unipotent actions, the famous Littlewood's conjecture concerning values of products of linear forms at integral points can be seen as an analog of the Oppenheim conjecture³. This conjecture is still open and it is a consequence of

²It is interesting to note that when dealing with dynamical methods working in low dimensions is clearly advantageous while dealing with Circle methods is definitely not. As it was remarked before the natural limit for the Circle method is $n = 5$.

³The Littlewood conjecture (1930) states that for any pair of reals (α, β) , we have that $\liminf_n n \|n\alpha\| \|n\beta\| = 0$ where $\|\cdot\|$ is the distance to the set of integers, that is,

a conjecture of Margulis which is an analog of Raghunathan's conjecture in the context of diagonal actions. The best known result is that the set of exception to the Littlewood conjecture has null Hausdorff dimension (due to Einsiedler, Lindenstrauss and Katok).

The Oppenheim conjecture itself was subject to several generalisations. A first question concerns the distribution of the values of $Q(\mathbb{Z}^n)$ in the real line. Since the proof of the Oppenheim conjecture uses ergodic techniques, it is natural to expect that uniform distribution property of the values of $Q(\mathbb{Z}^n)$ is satisfied. This part is known as the *quantitative version* of the Oppenheim conjecture and the main contributions are due to Dani, Eskin, Margulis and Mozes. Another kind of generalisation concerns the S -adic version of the Oppenheim conjecture, more precisely one can consider families of quadratic forms $(Q_s)_{s \in S}$ over the product of both archimedean and non-archimedean completions instead of a single real quadratic form. The validity of the S -adic version of the Oppenheim conjecture has been proved by Borel and Prasad.

Our contribution is to proceed such similar generalisations to the Oppenheim conjecture for pairs $F = (Q, L)$ which satisfies conditions so that the density set $F(\mathbb{Z}^n)$ holds. Let us state the main results obtained in this thesis.

Quantitative Oppenheim conjecture for pairs.

Let $0 \leq a < b$ be given. We denote by $\|\cdot\|$ the euclidian norm and by Vol the Lebesgue measure on \mathbb{R}^n . Let ν be a continuous positive function on the sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$ and let $\Omega = \{x \in \mathbb{R}^n : \|x\| < \nu(\frac{x}{\|x\|})\}$ a star-shaped open set. The quantitative Oppenheim conjecture amounts to give an estimate of the following quantity

$$\rho_{a,b}(T) = \frac{\#\{x \in \mathbb{Z}^n \cap T\Omega : a < Q(x) < b\}}{\text{Vol}(x \in T\Omega : a < Q(x) < b)} \text{ when } T \rightarrow \infty$$

Equidistribution is satisfied if $\rho_{a,b}(T) \rightarrow 1$ when $T \rightarrow \infty$.

The equidistribution was proved in two steps, Dani-Margulis provided the lower bound (1993) and Eskin-Margulis-Mozes (1998) provided the upper bound, the second step has shown to be more intricate than the first one.

Under the hypothesis above, we have the following

- $\liminf_T \rho_{a,b}(T) \geq 1$
- $\lim_T \rho_{a,b}(T) = 1$ for signature $(p, q) \neq (3, 1)$ or $(2, 1)$.
- $\lim_T \rho_{a,b}(T)(\log T)^{-k} = 1$ with $k = 2$ or 1 when $(p, q) = (3, 1)$, resp. $(2, 1)$.

$$\inf_{(x,y,z) \in \mathbb{Z}^3 - \{0\}} x(x\alpha - y)(x\beta - z) = 0$$

As mentioned before Dani-Margulis and Gorodnik were successful in finding sufficient conditions which guarantee density for pairs (Q, L) respectively for $n = 3$ and $n \geq 4$. A natural question is to check if the values of pairs at integral points are **equidistributed**, more precisely let introduce the following quantity

$$\rho_{a,b,\alpha}(T) = \frac{\#\{x \in \mathbb{Z}^n \cap T\Omega : a < Q(x) < b, 0 < L(x) < \alpha\}}{\text{Vol}(x \in T\Omega : a < Q(x) < b, 0 < L(x) < \alpha)}$$

Equidistribution for pairs is satisfied if $\rho_{a,b,\alpha}(T) \rightarrow 1$ when $T \rightarrow \infty$.

Unfortunately the existence of *non-generic*⁽⁴⁾ points in dimension $n \geq 4$ does not allow us to obtain equidistribution in this case. However in dimension 3, the situation is better and a first step towards uniform distribution would be to obtain an asymptotic lower estimate for $\rho_{a,b,\alpha}(T)$ in dimension 3. We obtain the following quantitative version for pairs (Q, L) in dimension 3.

Theorem. *Let $F = (Q, L)$ be a pair consisting of a quadratic form Q and a nonzero linear form L in dimension three satisfying the the following conditions*

1. Q is nondegenerate.
2. The cone $\{Q = 0\}$ intersects tangentially the plane $\{L = 0\}$.
3. No linear combination of Q and L^2 is rational.

Then for any $0 \leq a < b$ and $\alpha > 0$, we have

$$\liminf_{T \rightarrow \infty} \rho_{a,b,\alpha}(T) \geq 1$$

S -arithmetic version of Gorodnik's result.

Let $F = (Q, L)$ be a pair consisting of a quadratic form Q and a nonzero linear form L in $n \geq 4$ variables satisfying the the following conditions Q is nondegenerate and $Q|_{L=0}$ is indefinite and no linear combination of Q and L^2 is rational. Then it was proved by Gorodnik that the set $F(\mathbb{Z}^n)$ is dense in \mathbb{R}^2 . By using S -adic version of Ratner's orbit closure theorem we are able to generalises Gorodnik's result to the S -adic setting. Indeed, we get with analogous conditions that $\{(0, 0)\}$ is an accumulation point of $F(\mathbb{Z}^n)$.

Theorem. *Let $F = (Q, L)$ be a pair consisting of a quadratic form Q and a nonzero linear form L with coefficients in k_S in dimension $n \geq 4$ satisfying the the following conditions*

1. Q is nondegenerate

⁴A point $x \in G/\Gamma$ is said to be generic if the orbit of x under the stabilizer H of the pair is dense in G/Γ .

2. $Q|_{L=0}$ is isotropic i.e. $\{Q_s = 0\} \cap \{L_s = 0\} \neq \{0\}$ for all $s \in S$
3. For each $s \in S$, the forms $\alpha_s Q_s + \beta_s L_s^2$ are irrational for any α_s, β_s in k_s such that $(\alpha_s, \beta_s) \neq (0, 0)$

Then for any $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^n - \{0\}$ such that

$$|Q_s(x)|_s < \varepsilon \text{ and } |L_s(x)|_s < \varepsilon \text{ for each } s \in S$$

We also obtain the same result when we relax the condition (3) by allowing some linear combination to be rational but we need to add the condition that $Q|_{L=0}$ is nondegenerate so that strong approximation holds for the stabiliser of F .

Theorem. Let $Q = (Q_s)_{s \in S}$ be a quadratic form on k_S^n and $L = (L_s)_{s \in S}$ be a linear form on k_S^n with $n \geq 4$. Suppose that the pair $F = (Q, L)$ satisfies the following conditions,

1. Q is nondegenerate
2. $Q|_{L=0}$ is nondegenerate and isotropic
3. The quadratic form $\alpha Q + \beta L^2$ is irrational for any units α, β in k_S such that $(\alpha, \beta) \neq (0, 0)$

Then for any $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^n - \{0\}$ such that

$$|Q_s(x)|_s < \varepsilon \text{ and } |L_s(x)|_s < \varepsilon \text{ for each } s \in S$$

Unfortunately it is not possible to conclude that $F(\mathbb{Z}^n)$ is dense in k_S^2 due to the fact that square roots in nonarchimedean fields are not trivial contrary to the archimedean case.

In Chapter 1, we recall the foundation of the theory of unipotent flows on homogeneous spaces. In Chapter 2, we give an overview on Oppenheim conjecture by giving some history of the problem and by explaining the reasons of the success of the dynamical point of view. In Chapter 3, some generalisations on Oppenheim's conjecture are explained in detail. A quantitative Oppenheim type problem for pairs in dimension 3 is proved in Chapter 4. The S -adic version of Gorodnik's theorem for pairs in dimension 4 is proved in Chapter 5 by using both dynamics of unipotents actions and strong approximation. In the last chapter we discuss a conjecture of Gorodnik for pairs which is still open.

Chapter 1

Dynamics of unipotents flows on homogeneous spaces

In this chapter we give an overview of the most important results needed in the theory of unipotent flows on homogeneous spaces. The theory of unipotent dynamics was motivated in part by the Oppenheim conjecture, and one of the main result is the proof of the Raghunathan conjecture due to M.Ratner. We state all the results in the general framework of algebraic and Lie groups over any ground field k of characteristic zero. The homogeneous spaces in question are given either by the space of lattices in \mathbb{R}^n or by the space of lattices in S -adic Lie groups.

1.1 Preliminaries

Instead of looking at $G = \mathrm{SL}(n, \mathbb{R})$ as a Lie group, it can be useful to emphasise on the polynomial feature of the equation $\det(g) = 1$ which characterises G . This point of view allows one to forget the ground field and to speak intrinsically about the *algebraic group* SL_n .

1.1.1 Linear algebraic groups

Let n, r be integers with $n, r \geq 1$. For any field k of characteristic zero (e.g. $\mathbb{C}, \mathbb{R}, \mathbb{Q}$, a finite extension of \mathbb{Q}_p or a finite product of such groups), we denote by $k[x_1, \dots, x_n]$, the ring of polynomial in n variables with coefficients in k . Given any family P_1, \dots, P_r of polynomials in $k[x_1, \dots, x_n]$, we define the set of zeros of the family to be

$$Z(P_1, \dots, P_r) = \{x \in k^n \mid P_j(x) = 0 \text{ for every } 1 \leq j \leq r\}$$

Such subsets define the closed subsets of a topology on k^n called the *Zariski topology*. For any subset A of k^n , the *Zariski closure* of A is the smallest Zariski-closed subset which contains A , we denote it by $\mathrm{cl}(A)$. A subset A of k^n is said to be *Zariski-dense*

in B if $\text{cl}(A) = B$. The subsets X which are given by a Zariski-closed subset of the form $Z(P_1, \dots, P_r)$ are called the *affine varieties*. We begin to define the important notion of (linear) algebraic group.

Denote by K an algebraic closure of k and by $\text{Gal}(K/k)$ the absolute Galois group of k , that is, the group of (field) automorphisms of K which leaves the field k invariant. The absolute Galois obviously acts on the ring of polynomial with coefficients in K , by acting on the coefficients.

The group $\text{GL}_n(K)$ is an open set in K^{n^2} given by the non-vanishing of the determinant which is clearly a polynomial in the matrix entries and with coefficients in K .

Definition 1.1.1. *A subgroup G of $\text{GL}_n(K)$ is an algebraic group over K if it is a Zariski-closed subset given by a family of polynomials P_1, \dots, P_r of polynomials with matrix entries and coefficients in K , i.e.*

$$G = \{g = (g_{k,l}) \in \text{GL}_n(K) \mid P_j(g_{1,1}, \dots, g_{n,n}) = 0 \text{ for every } 1 \leq j \leq r\}$$

Moreover if the coefficient of the P_j 's can be chosen all in k then we say that G is an algebraic group defined over k .

It is easy to see that a algebraic group G over K is defined over K if and only if $G^\sigma = G$ for any $\sigma \in \text{Gal}(K/k)$ where G^σ is equal to $Z(P_1^\sigma, \dots, P_r^\sigma)$.

Definition 1.1.2. *For any ring A in K , put*

$$\text{GL}_n(A) := \{(a_{k,l}) \in \text{GL}_n(K) \mid a_{k,l} \in A \text{ and } \det(a_{k,l})^{-1} \in A \text{ for every } 1 \leq k, l \leq n\}$$

Given any algebraic group G , the set $G(A) := G \cap \text{GL}_n(A)$ is called the set of A -points of G .

The special linear group and its subgroups

The fundamental algebraic group which will be of constant use is the *special linear group*. It can be defined to be the Zariski closed subset associated with the single polynomial $P(g) = \det(g) - 1$ with coefficients in k

$$\text{SL}_{n|K} := Z(\det - 1) = \{g = (g_{k,l}) \in \text{GL}_n(K) \mid \det(g) = 1\}$$

It is easy to see that since \det has rational coefficients that $\text{SL}_{n|K}$ is defined over \mathbb{Q} .

The two following example of algebraic subgroups in SL_n will be of importance to us:

The diagonal subgroup $\mathbb{D}_n = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$ of diagonal matrices with determinant one.

The subgroup $\mathbb{U}_n = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ of upper triangular matrices with ones on the diagonal.

1.1.2 Lie groups and Lie algebras

Definition 1.1.3. A real (resp. complex) Lie group is a smooth (resp. complex) manifold G together with a structure such that both the multiplication map $G \times G \rightarrow G$ and the inverse map $G \rightarrow G$ are smooth (resp. holomorphic)

All our Lie groups (otherwise stated) are assumed to be real.

Definition 1.1.4. Let G a Lie group, then the tangent space at identity $T_e(G)$ of G is called the Lie algebra of G we denote it by $\text{Lie}(G)$ or \mathfrak{g} .

Definition 1.1.5. The adjoint representation $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$ is defined by:

$$\text{Ad}(g)x = \left. \frac{d}{dt} \right|_{t=0} (g \exp(tx) g^{-1}) \quad \forall g \in G \quad \forall x \in \mathfrak{g}$$

Definition 1.1.6. A Lie group G is said to be unimodular if its Haar measure is both left and right invariant i.e. $|\det(\text{Ad}(g))| = 1$ for all $g \in G$.

Definition 1.1.7. An element $g \in G$ is said

- . Unipotent or Ad-unipotent if $(\text{Ad}g - \text{Id})^k = 0$ for $k \in \mathbb{N}$ ie all the eigenvalues of $\text{Ad}g$ equal 1.
- . Quasi-unipotent if all the eigenvalues of $\text{Ad}g$ are of absolute value equal to 1.
- . Semisimple if the operator $\text{Ad}g$ is diagonalizable over \mathbb{C} .

Define by induction the ascending series of G by: $G^1 = G$ and $G^{k+1} = [G^k, G^k]$ where $[G, G]$ is the derived group of G ie the group generated by the commutators $[a, b] = aba^{-1}b^{-1}$ of elements of G .

Definition 1.1.8. A Lie group G is said to be solvable if there exists $n \in \mathbb{N}$ such that $G^n = \{e\}$. The maximal connected solvable normal subgroup R of G is called the radical of G denoted $\mathcal{R}(G)$

Definition 1.1.9. A connected Lie group G is said to be semisimple if its radical $\mathcal{R}(G)$ is trivial. Moreover if G has no nontrivial proper normal connected subgroups we say that G is simple.

Every connected semisimple Lie group G can be uniquely decomposed into an almost direct product $G = G_1 \dots G_n$ of its normal simple subgroups called the simple factors of G , that is, G_i and G_j commute and the intersection $G_i \cap G_j$ is discrete if $i \neq j$.

For example compact semisimple Lie groups (e.g. $\text{SO}(n), n \geq 3$) cannot contain a unipotent one-parameter subgroup. On the contrary noncompact semisimple connected Lie groups (e.g. $\text{SL}(n, \mathbb{R})$ for $n \geq 2$) are generated by unipotent one parameter subgroups.

Let G be a connected semisimple Lie group. Then as seen before G is an almost direct product of its simple factors. G can be written as $G = KS$ where K is the product of all the compact factors of G and S the noncompact ones.

Definition 1.1.10. Let G be a connected semisimple Lie group which admits the almost direct product $G = KS$ as above. Say that G is without compact factors or isotropic if K is trivial.

If a connected Lie group G is not semisimple, it is always possible to factorize it by its radical whenever the characteristic of the field is of characteristic zero (see [OV91], §4)

Proposition 1.1.11 (Levi decomposition). Let G be a connected Lie group, R the radical of G and $L \subset G$ a maximal connected semisimple subgroup called a Levi subgroup of G . Then $L \cap R$ is discrete and G is generated by L and R . The decomposition $G = LR$ is called the Levi decomposition of G . Moreover if G is simply connected then $G = L \times R$.

1.1.3 Quadratic forms and Orthogonal groups

Let k be a field of characteristic zero, a quadratic form Q over k in n variables is given by a homogeneous polynomial of degree two with coefficients in k . Such a quadratic form can be written

$$Q(x_1, \dots, x_n) = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j \text{ where } a_{ij} = a_{ji} \text{ in } k \text{ for any } 1 \leq i, j \leq n$$

or

$$Q(x) = x^t A x \text{ for any } x \in \mathcal{M}_{n,1}(k)$$

where $A = (a_{ij})$ is a symmetric matrix with respect to the canonical basis of \mathbb{R}^n (identified with $\mathcal{M}_{n,1}(k)$ the set of matrices with coefficients in k with n rows and one column). Then A is called the matrix associated of Q .

Some definitions and general properties of quadratic forms.

Let Q be a quadratic form over k with associated matrix A , then

Q is said to be *nondegenerate* if $\det(A) \neq 0$, if not we say that Q is *degenerate*.

Q is said to be *isotropic* if there exists $x \in k^n$, $x \neq 0$ such that $Q(x) = 0$, if not we say that Q is *anisotropic*.

To each quadratic form Q , we can associate a bilinear symmetric form $B : k^n \times k^n \rightarrow k$ with

$$B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y))$$

If V is a vector subspace in k^n , the *orthogonal* of V (w.r.t. B) is the subspace

$$V^\perp = \{x \in k^n : B(v, x) = 0 \text{ for every } v \in V\}$$

The *radical* of Q is defined to be equal to

$$\text{rad}(V) = \{x \in V : B(v, x) = 0 \text{ for every } v \in V\}$$

It is easy to verify that, Q is nondegenerate if $\text{rad}(V) = \{0\}$.

Equivalent classes and representative sets of quadratic forms

Two quadratic forms Q and Q' are said to be equivalent over k if there exists $g \in \text{GL}_n(k)$ such that $Q(x) = Q'(gx)$ for every $x \in k^n$. In this case, we write $Q \simeq Q'$. Let k^{*2} denote the set of nonzero squares of k^* , we have the fundamental following result of classification

Proposition 1.1.12 ([Ser73], Chap. IV, §1.6, Theorem 1'). *Any quadratic form in $n \geq 2$ variables with coefficients in k is equivalent (over k) to*

$$a_1x_1^2 + \dots + a_nx_n^2$$

where a_i lies in a set of representatives in k^*/k^{*2} ⁽¹⁾

The case when k is a local field (e.g. \mathbb{C} , \mathbb{R} , \mathbb{Q}_p) is of most importance in number theory. It is not difficult to verify that the set k^*/k^{*2} has respectively representatives given by

- $\text{Rep}(\mathbb{C}^*/\mathbb{C}^{*2}) = \{1\}$ and $\text{Rep}(\mathbb{R}^*/\mathbb{R}^{*2}) = \{\pm 1\}$
- $\text{Rep}(\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}) = \{1, p, u, up\}$ where $\left(\frac{u}{p}\right) = -1$ (when p odd) and $\text{Rep}(\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}) = \{\pm 1, \pm 2, \pm 5, \pm 10\}$ (for a complete discussion on squares in \mathbb{Q}_p^* , see [Ser73], Chap. I, §3.3).

Corollary 1.1.13. *Let Q be a quadratic form in $n \geq 2$ variables with coefficients in k , then if*

- $k = \mathbb{C}$, then $Q \simeq x_1^2 + \dots + x_k^2$, where k is the rank of Q
- $k = \mathbb{R}$, then $Q \simeq x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_n^2$ where $(p, k-p)$ is the signature of Q and k the rank of Q
- $k = \mathbb{Q}_p$, then $Q \simeq a_1x_1^2 + \dots + a_nx_n^2$, where $a_i \in \{0, 1, p, u, up\}$ with $\left(\frac{u}{p}\right) = -1$ if $p \geq 3$ and $a_i \in \{0, \pm 1, \pm 2, \pm 5, \pm 10\}$ if $p = 2$.

In particular if Q is nondegenerate then rank of Q is equal to n , i.e. $k = n$.

A *hyperbolic plane* P (w.r.t. Q) in $V = k^n$, is a two-dimensional vector subspace spanned by two nonzero vectors $u, v \in k^n$, say $Q(u) = 0$. The restriction of Q to P is of the form $Q(x) = x_1x_2$ if one take $\{u, v, e_3, \dots, e_n\}$ as a basis of k^n .

Assume Q be nondegenerate, we denote by $i(Q)$ the *isotropic index* of Q , that is, the maximum number of hyperbolic planes w.r.t. Q contained in k^n . Therefore if $i(Q) = r$, then there exist hyperbolic planes P_1, \dots, P_r such that

$$V \simeq P_1 \oplus \dots \oplus P_r \oplus V_{an} \quad \text{where the restriction of } Q \text{ to } V_{an} \text{ is anisotropic.}$$

We state an important result due to E. Witt,

¹It is sometimes denoted by $Q \simeq \langle a_1, \dots, a_n \rangle$ in literature in order to emphasize the importance of the field of definition.

Theorem 1.1.14 (Witt, e.g. see [Ger] Theorem 2.44). *Let Q be a nondegenerate quadratic in V and suppose there exists an isometry $\rho : F \rightarrow F'$ between two subspaces F, F' of V . Then there exists an isometry $\bar{\rho} : V \rightarrow V$ such that $\bar{\rho}|_F = \rho$.*

Corollary 1.1.15 (Witt decomposition). *Let Q be a quadratic form in V , then we get the decomposition*

$$V \simeq \text{rad}(Q) \oplus V_{\text{hyp}} \oplus V_{\text{an}}$$

where V_{hyp} is a orthogonal sum of hyperbolic planes and such that $Q|_{V_{\text{an}}}$ is anisotropic.

We come now to the most important geometric object in the study of quadratic forms

Definition 1.1.16. *Let Q be a quadratic form over a k -vector space V of dimension n and $\text{SL}(V)$ be the group of automorphisms on V with determinant equal to one. The subgroup of $\text{SL}(V)$ consisting of the isometries of Q is called the special orthogonal group of Q , and is denoted by*

$$\text{SO}(Q) = \{g \in \text{SL}(V) \mid Q(gx) = Q(x) \text{ for every } x \in V\}$$

We collect some useful and classical properties of orthogonal groups,

Proposition 1.1.17. *Let Q be a quadratic form over a k -vector space V of dimension $n \geq 3$, then*

1. $\text{SO}(Q)$ is a semisimple Lie group if Q is nondegenerate
2. $\text{SO}(Q)$ has no compact factors if Q is isotropic
3. $\text{SO}(Q)^\circ$ a connected maximal subgroup of $\text{SL}(V)$ if Q is isotropic
4. $\text{SO}(Q)$ acts transitively on the level sets $\{Q = m\}$

Proof. For property (1) (see [Kn69], §2.1, example 3). The second assertion is proved in ([B91] §23.4). The proof of property (3) in the real case generalises to any local field. Let $H = \text{SO}(Q)^\circ$ and $G = \text{SL}(n, \mathbb{R})$. Let L be a closed connected subgroup of G , such that $H \subseteq L \subseteq G$. Let $\mathfrak{h} \subseteq \mathfrak{l} \subseteq \mathfrak{g}$ be the correspondent Lie algebras. Then the complexified quotient $\mathfrak{l}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ is a subalgebra $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$. In particular it is an $\text{SO}(n, \mathbb{C})$ -module for the ad-representation. Since $\mathfrak{h}_{\mathbb{C}}$ consists of skew-symmetric matrices and $\mathfrak{g}_{\mathbb{C}}$ consists of matrices of trace zero, thus $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ consists of symmetric matrices with trace zero. But since $\text{SO}(n, \mathbb{C})$ acts irreducibly on $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ we deduce that $\mathfrak{l}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}$ or $\mathfrak{l}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$, therefore $L = \text{SL}(n, \mathbb{R})$ or $L = \text{SO}(Q)^\circ$.

We now prove property (4), let $x, y \in V$ such that $Q(x) = Q(y) = m$ for some $m \in k$. Denote by F the vector subspace spanned by x and y . Define the endomorphism $\rho : F \rightarrow F$ by $\rho(x) = y$ and if $\dim(F) = 2$, $\rho(y) = x$. Then ρ preserves $Q|_F$, hence by Witt's theorem we can find an isometry $\bar{\rho} : V \rightarrow V$ for Q which extends ρ . If $\bar{\rho} \notin \text{SO}(Q)$, it suffices to consider $\tilde{\rho} = \bar{\rho} \circ \sigma$ where σ is some isometry for Q of determinant -1 which fixes x .

Proposition 1.1.18. *Let Q be an isotropic nondegenerate quadratic form over V a k -vector space of dimension $n \geq 3$, then if $\mathrm{SO}(Q)$ is defined over \mathbb{Q} then Q is proportional with a quadratic form with rational coefficients.*

Proof. This follows easily from the fact that if $C(Q) = C(Q')$, then there exists a nonzero scalar $\lambda \in k$ such that $Q' = \lambda Q$.

1.2 Homogeneous space and lattices

Definition 1.2.1. *A homogeneous space X is a differentiable manifold given by the quotient G/H of a connected Lie group G by a closed subgroup H of G .*

A homogeneous space is always smooth as a manifold and therefore provides a fertile ground for the study of dynamical systems.

Any right-invariant Haar measure on G induces a smooth volume measure on G/H which is called a Haar measure as well. The space G/H is said to be finite volume if the Haar measure ν on G/H is G -invariant and finite.

Definition 1.2.2. *A discrete subgroup Γ of G is said to be a lattice if Γ has finite covolume in G , i.e. $\nu(G/\Gamma) < \infty$.*

Any lattice Γ in G gives rise to a homogeneous space $X = G/\Gamma$ of finite volume. A basic example is the d -dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, it is clearly compact, hence finite volume for the Haar measure induced by the Lebesgue measure therefore \mathbb{Z}^d is a lattice in \mathbb{R}^d .

Definition 1.2.3. *A lattice Γ in G is said to be uniform if G/Γ is compact.*

The question of the existence of uniform lattices is highly non-trivial,

Theorem 1.2.4 (Borel, Theorem C, [B63]). *Let G be a connected semisimple Lie group then G has uniform lattices.*

Some lattices can be realised as the integers points of some linear algebraic group over \mathbb{Q} ,

Theorem 1.2.5 (Borel, Harish-Chandra, [BH62]). *Let G be a semisimple \mathbb{Q} -algebraic group then $G(\mathbb{Z})$ is a lattice in $G(\mathbb{R})$.*

Since SL_d is a semisimple algebraic group defined over \mathbb{Q} , we get the following corollary

Corollary 1.2.6. *Let $d \geq 2$, then $\mathrm{SL}(d, \mathbb{Z})$ is a lattice in $\mathrm{SL}(d, \mathbb{R})$.*

1.2.1 Mahler's Compactness Criterion

Bounded subsets of $X_d = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ can be described using *Mahler's Compactness Criterion*. Roughly one can see that arbitrarily large elements for a suitable topology (see below) in X_d correspond to lattices with arbitrarily small vectors.

By definition of the quotient topology on X_d , a sequence $\{\Lambda_i\}$ of lattice in \mathbb{R}^d converges to some lattice Λ in \mathbb{R}^d if and only if there exists a basis $(f_{i,1}, \dots, f_{i,d})$ of $\{\Lambda_i\}$ which converges to a basis (f_1, \dots, f_d) of Λ . A sequence of lattices $\{\Lambda_i\}$ goes to infinity in X_d if for any compact subset Θ of X_d there exists some n large enough such that $\{\Lambda_i\}$ move away from Θ for any $i \geq n$. For $d \geq 2$, it is not difficult to show that X_d is not compact.

For any lattice Λ in \mathbb{R}^d , one denotes by $d(\Lambda)$ the volume of the torus \mathbb{R}^d/Λ . It is given by the formula $d(\Lambda) = |\det(f_1, \dots, f_d)|$ where (f_1, \dots, f_d) is any basis of Λ .

Let us state the following result on the topology on the space of lattices in \mathbb{R}^d ,

Theorem 1.2.7 (Mahler's Compactness criterion). *A subset Θ of X_d is relatively compact in X_d if and only if there exists constants $\alpha, \beta > 0$ such that for all $\Lambda \in \Theta$, one has:*

$$d(\Lambda) \leq \beta \text{ and } \inf_{\Lambda - \{0\}} \|v\| \geq \alpha.$$

Proof. (see e.g. [PR], Proposition 4.8)

The theorem just says that a set of lattices is relatively compact if and only if their volume are bounded and they avoid a small ball around the origin.

Siegel summation formula

Denote like above by ν the probability *Haar* measure on X_d arising from the *Haar* measure on G . The interpretation of the elements of X_d as lattices in \mathbb{R}^d gives rises to the following important connection between ν and the *Lebesgue* measure on \mathbb{R}^d .

Let φ a function on \mathbb{R}^d , denote by $\tilde{\varphi}$ the function on X_d given by $\tilde{\varphi}(\Lambda) = \sum_{\Lambda - \{0\}} \varphi(x)$.

Theorem 1.2.8 ([Sieg]). *For any $\varphi \in L^1(\mathbb{R}^d)$ one has $\tilde{\varphi}(\Lambda) < \infty$ for ν -almost all the lattices $\Lambda \in X_d$ and we have the formula*

$$\int_{\mathbb{R}^d} \varphi(x) dx = \int_{X_d} \tilde{\varphi}(\Lambda) d\nu(\Lambda)$$

This formula is very useful for counting lattices points inside regions in \mathbb{R}^d , indeed if φ is the characteristic function of a subset B of \mathbb{R}^d , the integrand in the right-hand side gives the cardinality of the finite set $\Lambda - \{0\} \cap B$.

1.2.2 Homogeneous actions

Let G be a Lie group and $H \subset G$ its closed subgroup. Then G acts transitively by left translations $x \mapsto gx$ on the right homogeneous space G/H , hence H can be viewed as the isotropy subgroup of some element in G/H .

Definition 1.2.9. Let F a subgroup of G , the left action $x \mapsto gx$ of F on G/H is said to be a homogeneous action and this action is denoted by $(G/H, F)$.

We assume in the following that G is a connected Lie group and $\nu(G/H) < \infty$, a important case concerns the action of a one-parameter group $F := \{\varphi_t : t \in \mathbb{R}\}$ with $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

Definition 1.2.10. If F a one-parameter subgroup $F := \{\varphi_t : t \in \mathbb{R}\}$ of G , then the homogeneous action $(G/H, F)$ is called a homogeneous flow.

There is an isomorphism between:

$$\{\text{The left } F\text{-action on } G/H\} \longleftrightarrow \{\text{The right } F\text{-action on } H \backslash G\}.$$

Hence it is equivalent to treat the case of right or left actions, for convenience we make the choice to deal with left actions only.

1.2.3 Ergodic theory of homogeneous flows

Suppose that $X = G/H$ is a homogeneous space and F a Lie subgroup of G which acts by left translations on X . We denote by μ the Haar measure on X and we say that a measurable subset A of X is F -invariant if $\rho_b(A) = A$ for any $b \in F$ where $\rho : b \mapsto \rho_b$ is the left translation by b . In the case where F is a one parameter subgroup $\{\varphi_t : t \in \mathbb{R}\}$, the F -invariance of A is equivalent to the fact that $\varphi_t(A) = A$ for every $t \in \mathbb{R}$. The definition can be extended to subsets up to a null measure set.

Definition 1.2.11. The action of F on X is said to be ergodic (with respect to μ) if every F -invariant measurable subset A of X is either null or full i.e. either $\mu(A) = 0$ or $\mu(X - A) = 0$. In this case, we say that the measure μ is a F -invariant ergodic measure on X .

The orbits of ergodic action have nice topological properties,

Proposition 1.2.12 (see e.g.[Zim84], Prop. 2.1.7 and 2.1.10). Let F a locally compact group separable which acts on X and measure-preserving. If μ is a F -invariant ergodic measure then the orbit Fx is dense in $\text{supp}(\mu)$. Moreover if F is a Lie subgroup of G , then $\text{supp}(\mu) = Fx$.

Proof. We can assume $X = \text{supp}(\mu)$. Let D a countable dense subset action of X and \mathcal{D} the set of all open balls with center points of D and positive rational radius. For all $B \in \mathcal{D}$, and $g \in F$ let ρ_g the left translation map by g , the set $A_B = \cup_{n \in \mathbb{Z}} \rho_g^{-n}(B)$ is the set of points x such that the orbit Fx pass by B . This set verifies $\rho_g^{-1}(A_B) = A_B$. By ergodicity of the action, $\mu(A_B) = 1$ thus $A = \cap_{B \in \mathcal{D}} A_B$ also verifies $\mu(A) = 1$. The first statement is proved since the orbits of points of A are dense in X . Since $F = \mathbf{F}(\mathbb{R})$ where \mathbf{F} is a \mathbb{R} -algebraic group acting on the \mathbb{R} -variety \mathbf{X} and that the $\mathbf{F}(\mathbb{R})$ -orbits in $\mathbf{X}(\mathbb{R})$ are locally closed, the F -ergodic measures on $\mathbf{X}(\mathbb{R})$ have their support carried by $\mathbf{F}(\mathbb{R})$ -orbits.

The reason we focus on ergodic measures is that any invariant measure by the flow is a geometrical combination of ergodic measures ⁽²⁾ (see [EW], §1.4)

Proposition 1.2.13 (Ergodic decomposition). *Let $\phi : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be a measure preserving map of a Borel preserving map. Then there is a Borel probability space (Y, \mathcal{B}_Y, ν) and a measurable map $y \mapsto \mu_y$ for which*

- μ_y is a ϕ -invariant ergodic probability measure on X for almost every y
- $\mu = \int_Y \mu_y d\nu(y)$

Pointwise Ergodic theorem

We state one of the most important result in Ergodic theory known as the pointwise convergence Ergodic theorem for the following reason:

Theorem 1.2.14 (Birkhoff). *Let μ a probability measure which is ergodic with respect to a flow $\{\phi_t : t \in \mathbb{R}\}$ on X . Then almost every $\langle \phi_t \rangle$ -orbit in X is uniformly distributed with respect to μ i.e.*

$$\frac{1}{T} \int_0^T f(\phi_t x) dt \longrightarrow \int_X f d\mu \quad \text{when } n \rightarrow \infty \text{ for a.e. } x \in X$$

Proof. See e.g. [EW]

1.2.4 Geodesic and horocycle flows

Let us define the *hyperbolic plane* by $\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im}z > 0\}$ which is know to be of constant curvature -1 relatively to the *Poincaré metric* $\frac{dx^2 + dy^2}{y^2}$. The group $SL_2(\mathbb{R})$ acts on \mathcal{H} by *Möbius transformations*:

$$g.z = \frac{az + b}{cz + d} \quad \text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

This action induces a transitive and isometric action of $PSL_2(\mathbb{R})$ on \mathcal{H} , the stabilizer of $z = i$ is $K = SO(2)$ which is compact and \mathcal{H} is identified with $PSL_2(\mathbb{R})/PSO_2(\mathbb{R})$.

Definition 1.2.15. *The unit tangent bundle on \mathcal{H} is the set defined by:*

$$\mathbb{T}\mathcal{H} = \{(z, v) \mid z \in \mathcal{H} \text{ and } v \text{ is a tangent vector of } \mathcal{H} \text{ at } z \text{ such that } |v| = \text{Im}z\}$$

Thus $PSL_2(\mathbb{R})$ acts on $\mathbb{T}\mathcal{H}$ by:

²The precise statement is that the convex hull of the set of ϕ -ergodic probability measures on X is equal to the set of ϕ -invariant probability measures on X

$$g.(z, v) = (g.z, g'(z)v) = \left(\frac{az + b}{cz + d}, \frac{v}{(cz + d)^2} \right)$$

This action allows to identify $\mathrm{PSL}_2(\mathbb{R})$ with $\mathbb{T}\mathcal{H}$ since the differential action is *free* and *transitive*.

The hyperbolic distance on \mathcal{H} is defined by:

$$d(x, y) = \inf \{l(\gamma) \mid \gamma: [0, 1] \rightarrow \mathcal{H} \text{ is } \mathcal{C}_p^1, \gamma(0) = x \text{ and } \gamma(1) = y\}$$

where $l(\gamma) = \int_0^1 \frac{|\dot{\gamma}(t)|}{\mathrm{Im}(\gamma(t))} dt$ is the length of the curve γ with respect to the Poincare metric on \mathcal{H} .

Proposition 1.2.16. *The geodesics of \mathcal{H} are the half circles which their center lies on the real axis and the lines orthogonal to the real axis.*

Geodesic flow

Let $(z, v) \in \mathbb{T}\mathcal{H}$ and γ the geodesic through g and of direction v . Let z_t the point on γ being at distance t from z in the direction given by v and let v_t the unit tangent vector of γ at z_t in the same direction.

Definition 1.2.17. *The one parameter group given by $g_t(z, v) = (z_t, v_t)$ is called the geodesic flow*

Using the identification between $\mathbb{T}\mathcal{H}$ and $\mathrm{PSL}_2(\mathbb{R})$, the realization of the geodesic flow has a very nice algebraic description in terms of matrix multiplication.

Proposition 1.2.18. *The geodesic flow on $\mathbb{T}\mathcal{H}$ correspond to the right multiplication in $\mathrm{PSL}_2(\mathbb{R})$ by the matrix $a_{t/2}$ where $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.*

Proof. Using the identification between $\mathbb{T}\mathcal{H}$ and $\mathrm{PSL}_2(\mathbb{R})$, for any $(z, v) \in \mathbb{T}\mathcal{H}$ there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that:

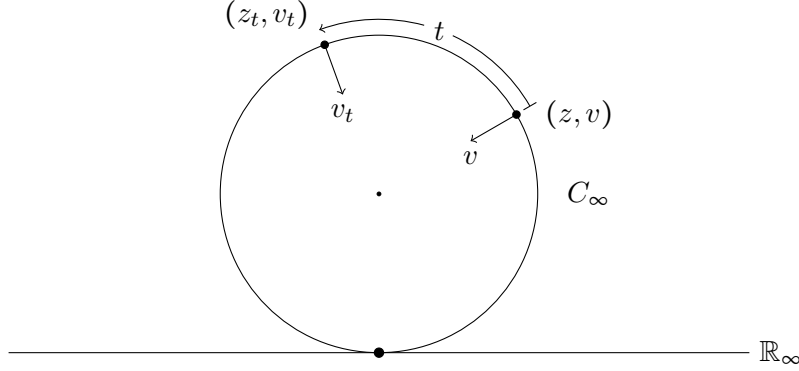
$$g.(z_0, v_0) = (z, v) \text{ where } z_0 = i \text{ and } v_0 \text{ relative to } z_0$$

because g acts isometrically ie g preserves geodesics we have

$$g_t(z, v) = g_t(g.(z_0, v_0)) = g.g_t(z_0, v_0) = ga_{t/2}.(z_0, v_0)$$

the last equality is obvious regarding the geodesic determined by z_0 and v_0 .

Figure 1.1: The horocycle flow



Horocycle flow

Let $(z, v) \in \mathbb{T}\mathcal{H}$ and γ the geodesic associated ie the geodesic in \mathcal{H} such that $\gamma(0) = z$ and $\gamma'(0) = v$. Let C_t the hyperbolic circle of center $\gamma(t)$ through z and let C_∞ the limit when $t \rightarrow \infty$ of C_t called the positive horocycle determined by (z, v) . Let z_t the unit normal vector to C_∞ which is at distance t from z (following the trigonometric sense) and v_t the unit normal vector to C_∞ at z_t directed in-ward.

Definition 1.2.19. *The one-parameter group $h_t(z, v) = (z_t, v_t)$ is called the horocyclic flow on $\mathbb{T}\mathcal{H}$*

The horocycle flow can be define more concretely in the following way: for all $(z, v) \in \mathbb{T}\mathcal{H}$, there exists a unique horocycle C_∞ (i.e. a circle tangent to the real axis) through z and having v as a normal (inward) vector, the horocycle flow is just obtain by pushing (z, v) along C_∞ with a distance t .

Using the identification again between $\mathbb{T}\mathcal{H}$ and $\text{PSL}_2(\mathbb{R})$, we obtain an analog of the last proposition for the horocycle flow.

Proposition 1.2.20. *The horocycle flow on $\mathbb{T}\mathcal{H}$ correspond to the right multiplication in $\text{PSL}_2(\mathbb{R})$ by the unipotent group $U = \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$.*

The fact that we can translate the action of the geodesic and horocycle flow in terms of matrix multiplication make the situation easier to deal with, for instance can deduces immediatly the following relations between the two flows:

Proposition 1.2.21. *Let $H = \left\{ h_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}$ the one parameter unipotent group then we have the fundamental relation of commutation:*

$$\begin{cases} u_s a_t = a_t u_s e^{-2t} \\ h_b a_t = a_t h_b e^{2t} \end{cases}$$

Remark. If we put $A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\}$ the proposition implies that right U -orbit on $\mathrm{PSL}_2(\mathbb{R})$ form the *contracting foliation* for the right A -action relative to the left-invariant metric on $\mathrm{PSL}_2(\mathbb{R})$ since $a_{-t} u_s a_t = u_s e^{-2t} \rightarrow Id$ when $t \rightarrow \infty$ while H -action form the *expanding foliation* since $a_{-t} h_b a_t = h_b e^{2t}$ diverges when $t \rightarrow \infty$. For any left-invariant metric d_G in G we have,

$$d_G(g a_t, g u_s a_t) = d_G(g a_t, g a_t a_{-t} u_s a_t) = d_G(I, a_{-t} u_s a_t) = d_G(I, u_{e^{-t} s}) \rightarrow 0 \text{ as } t \rightarrow +\infty$$

Ergodicity of geodesic and horocycle flows

A proof of the following theorem relies on the notion Mautner phenomenon (e.g. see [DWMo], Proposition 3.2.3)

Theorem 1.2.22 (see e.g. [DWMo], Corollary 3.2.4). *The geodesic flow h_t is ergodic on $\mathrm{SL}(2, \mathbb{R})/\Gamma$.*

The previous result is due to Hopf but follows from a more general statement,

Theorem 1.2.23 (Moore, [Mo66]). *Suppose G is a connected, simple Lie group with finite center, Γ a lattice in G and $\{a_t\}$ a one-parameter subgroup of G such that its closure $\overline{\{a_t\}}$ is not compact. Then $\{a_t\}$ is ergodic on G/Γ (w.r.t. the Haar measure on G/Γ).*

Corollary 1.2.24. *The horocycle flow h_t is ergodic on $\mathrm{SL}(2, \mathbb{R})/\Gamma$.*

Since the horocycle flow is ergodic on $X = G/\Gamma$, the pointwise ergodic theorem shows that almost every orbit of $\{h_t\}$ on X is uniformly distributed w.r.t. the G -invariant probability measure μ , in other words for any bounded continuous function f on G/Γ

$$\frac{1}{T} \int_0^T f(h_t x) dt \rightarrow \int_{G/\Gamma} f d\mu \text{ as } T \rightarrow \infty \text{ for } \mu\text{-almost every } x \in X$$

There is a fundamental question which confronts us: Can we replace *almost* by *all* in the previous convergence and more generally can we also replace the horocycle flow by any one-parameter unipotent subgroup. These are the issues we discuss now in the next section.

1.3 Ratner's classification theorems

1.3.1 The Raghunathan conjecture

A representative example of a unipotent flow is given by the horocycle flow on the unit tangent bundle of a surface of constant curvature -1 and finite area. In 1936, Hedlund [Hedl] proved that horocycle orbits are either closed or dense, more precisely he proved that

Theorem 1.3.1 (Hedlund). *If $G = \mathrm{SL}(2, \mathbb{R})$, Γ is a lattice in G and U is a unipotent one-parameter subgroup of G , then every U -orbit on G/Γ is either closed or dense in G/Γ . In particular if G/Γ is compact then all the U -orbits are dense.*

This result was strengthened by Furstenberg in 1972 [Furst72], who proved that under the same hypothesis, the action of U on the compact space G/Γ is uniquely ergodic, i.e. the G -invariant probability measure is the only U -invariant probability measure on G/Γ .

The Raghunathan conjecture is a far reaching generalisation of this fact to any homogeneous space and unipotent action. The conjecture is mentioned for the first time by Dani in [Dan81] and it was formulated in two different versions. The first one concerns the original version of a conjecture previously raised by Raghunathan in the mid seventies about the structure of the orbit closure under unipotent actions in homogeneous spaces. The second one deals with the classification of ergodic measures and is due to Dani himself.

Conjecture 1.3.2 (Raghunathan topological conjecture). *Let G be a connected Lie group, and let Γ be a lattice in G . Let U be an Ad-unipotent subgroup of G . Then for any $x \in G/\Gamma$, the closure of Ux is a homogeneous space and there exists a closed connected subgroup $L := L(x)$ which contains U such that $\overline{Ux} = Lx$.*

The second conjecture emphasises the classification of U -ergodic measures generalising the above-mentioned result of Furstenberg to arbitrary connected Lie groups.

Conjecture 1.3.3 (Dani measure classification conjecture). *Let G be a connected Lie group, and let Γ be a discrete subgroup in G . Let U be an Ad-unipotent subgroup of G . Then any ergodic U -invariant measure μ on G/Γ is algebraic, that is, there exists $x \in G/\Gamma$ and a subgroup $L \subset G$ containing U such that Lx is closed and μ is the L -invariant probability measure on Lx .*

In the same paper [Dan81], Dani proved his conjecture for G reductive and U horospherical w.r.t. some element $a \in G$ (i.e. $U = U_a = \{g \in G \mid a^n g a^{-n} \rightarrow e, n \rightarrow +\infty\}$) which is maximal. Later in 1986, Dani proved the Raghunathan conjecture for G reductive and U an arbitrary horospherical subgroup of G [Dan86a].

The first result on Raghunathan's conjecture for nonhorospherical subgroups of semisimple groups was obtained by Dani and Margulis in [DM90]. They proved the conjecture in the case when $G = \mathrm{SL}(3, \mathbb{R})$ and $U = \{u(t)\}$ is a one-parameter unipotent subgroup of G such that $u(t) - I$ has rank 2 for all $t \neq 0$.

The result of Furstenberg mentioned above implies for the case when $G = \mathrm{SL}(2, \mathbb{R})$, Γ is a uniform lattice and $U = \{u(t)\}$ is a one-parameter unipotent subgroup of G that every orbit of $\{u(t)\}$ on G/Γ is uniformly distributed w.r.t. the G -invariant probability measure, in other words for any bounded continuous function f on G/Γ ,

$$\frac{1}{T} \int_0^T f(u_t x) dt \rightarrow \int_{G/\Gamma} f d\mu \text{ as } T \rightarrow \infty$$

Dani and Smilie extended this result to nonuniform lattices Γ in $G = \mathrm{SL}(2, \mathbb{R})$ and every nonperiodic orbit $\{u(t)\}$ in [DanSm]. Such results had already been proved by Dani in [Dan82] for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. In this paper Dani formulated the uniform distribution conjecture for unipotent flows on $\mathrm{SL}(n, \mathbb{R})/\Gamma$. This was extended to arbitrary connected Lie groups and actions of closed subgroups generated by unipotents by Margulis at the ICM 1990 Kyoto (Japan),

Conjecture 1.3.4 (Dani uniform distribution conjecture). *Let G be a connected Lie group and Γ a lattice in G . Let $\{u(t)\}$ be a one-parameter unipotent subgroup of G . Then for any $x \in G/\Gamma$ there exists a closed subgroup of F of G (containing U) such that the orbit Fx is closed and the trajectory $\{u(t)x : t \geq 0\}$ is uniformly distributed with respect to a (unique) F -invariant probability measure μ_x supported on Fx . In other words, for any bounded continuous function f on G/Γ ,*

$$\frac{1}{T} \int_0^T f(u_t x) dt \rightarrow \int_{G/\Gamma} f d\mu_x \text{ when } T \rightarrow \infty$$

1.3.2 Ratner's results on the Raghunathan conjecture

In a series of papers, M. Ratner made a major breakthrough in the theory of dynamics on homogeneous spaces by solving in full generality the Dani-Raghunathan conjectures in all their variants. The first result concerns the proof of measure classification version of Raghunathan's conjecture which classifies ergodic measures invariant under unipotent subgroups,

Theorem 1.3.5 (Ratner's measure classification theorem, [Ratn91a]). *Let G be a connected Lie group, and let Γ be a lattice in G . Let U be a connected Lie subgroup of G generated by one-parameter unipotent groups. Then any ergodic U -invariant measure μ on G/Γ is algebraic; that is, there exists $x \in G/\Gamma$ and a subgroup $L \subset G$ containing U (and generated by unipotents) such that Lx is closed and μ is the L -invariant probability measure on Lx .*

The general philosophy in the theory of unipotent dynamics is that classifying ergodic invariant measures is easier than classifying intermediate subgroups, indeed the notion of entropy is a very useful tool for the classification of ergodic measures which has no equivalent in the side of the intermediate subgroups. We have the following correspondence (see proposition 1.2.12)

$$\begin{aligned} \text{Ergodic } U\text{-invariant measures on } G/\Gamma &\leftrightarrow \text{Intermediate subgroups } U \subset L \subset G \\ \mu &\leftrightarrow Lx = \text{supp}(\mu) \text{ with } x \in G/\Gamma \end{aligned}$$

Theorem 1.3.6 (Ratner's orbit closure theorem). *Let G be a connected Lie group, and let Γ be a lattice in G . Let H be a connected Lie subgroup of G generated by one-parameter unipotent groups. Then for any $x \in G/\Gamma$ there exists a closed connected subgroup $P \subset H$ such that $\overline{Hx} = Px$ and Px admits a P -invariant probability measure.*

The following theorem due to N. Shah gives extra information about the structure of the intermediate subgroups arising from Ratner's orbit closure theorem above,

Theorem 1.3.7 ([Sh91], Prop. 3.2). *Let G be a semisimple algebraic group defined over \mathbb{Q} and Γ be an arithmetic subgroup of G . Let E be a subgroup of $G(\mathbb{R})$ generated by unipotent elements and assume that*

$$\overline{E\Gamma} = R\Gamma \text{ where } R \text{ is a closed connected subgroup of } G(\mathbb{R})$$

such that $R \cap \Gamma$ has finite covolume in R . Then $R = \tilde{E}(\mathbb{R})^\circ$ where \tilde{E} is the smallest \mathbb{Q} -subgroup of G whose group of real points contains E .

1.4 Uniform distribution of unipotent trajectories

Let G be a connected Lie group and Γ a discrete subgroup of G . Let U be a connected closed subgroup of G . Ratner's uniform distribution theorem below says that if Γ is a lattice of a connected Lie group G and $U = \{u_t\}$ be a unipotent one-parameter subgroup of G then for any $x \in G/\Gamma$ the $\{u_t\}$ -orbit of x is uniformly distributed in its closure, that is for any bounded continuous function φ on G/Γ ,

$$\frac{1}{T} \int_0^T \varphi(u_t x) dt \longrightarrow \int_{G/\Gamma} \varphi d\mu_x$$

when $T \longrightarrow \infty$ and μ_x is the H -invariant probability measure supported by the orbit $Hx = \{u_t x \mid t \in \mathbb{R}\}$, in particular if $\{u_t x \mid t \in \mathbb{R}\}$ is dense in G/Γ , that is to say if x is U -generic, then one can take $\mu_x = \mu$ to be the G -invariant probability measure of G/Γ and the orbit of x is uniformly distributed with respect to μ .

Theorem 1.4.1 (Ratner's uniform distribution theorem, [Ratn91b]). *Let G be a connected Lie group and Γ a lattice in G . Let $\{u(t)\}$ be a one-parameter unipotent subgroup of G . Then for any $x \in G/\Gamma$ there exists a closed subgroup of F of G (containing U) such that the orbit Fx is closed and the trajectory $\{u(t)x : t \geq 0\}$ is uniformly distributed with respect to a (unique) F -invariant probability measure μ_x supported on Fx . In other words, for any bounded continuous function f on G/Γ ,*

$$\frac{1}{T} \int_0^T f(u_t x) dt \rightarrow \int_{G/\Gamma} f d\mu_x \text{ when } T \rightarrow \infty$$

1.4.1 Quantitative nondivergence

The set $\mathcal{S}(U)$ of U is defined to be the set of all $x \in G/\Gamma$ such that there exists a proper closed subgroup H containing U such that Hx is closed and has H -invariant finite measure. This set is called the singular set for U , we also put $\mathcal{G}(U) = G/\Gamma - \mathcal{S}(U)$ the set of U -generic points. It is clear that for any U -generic point x , the orbit Ux is dense in G/Γ . Let \mathcal{H} the class of all proper closed connected subgroups H of G such that $H \cap \Gamma$

is a lattice in H and $\text{Ad}(H \cap \Gamma)$ is Zariski dense in $\text{Ad}(H)$, where Ad denotes the adjoint representation of G . The class \mathcal{H} is countable ([DM93], Proposition 2.1) and we have $\mathcal{S}(U) = \bigcup_{H \in \mathcal{H}} X(H, U)\Gamma/\Gamma$ where $X(H, U) = \{g \in G \mid Ug \subseteq gH\}$ ([DM93], Proposition 2.3). These notions make sense when $U = \{u(t) \mid t \in \mathbb{R}\}$ is a one-parameter unipotent subgroup, in this case one can use the fundamental property shared by unipotent flows that is to have polynomial growth with respect to the parameter. In the linear context this property has the following application which has an elementary proof,

Proposition 1.4.2 ([DM93], Prop 4.2). *Let V a finite dimensional real vector space and let A be an algebraic variety of V . Then for any compact C of A and any $\varepsilon > 0$ there exists a compact subset D of A such that the following holds: for any neighbourhood Φ of D in V there exists neighbourhood Φ' of C in V such that for any unipotent one-parameter subgroup $\{u_t\}$ of $\text{GL}(V)$, any $v \in V - \Phi$ and any $T > 0$, we have*

$$\lambda(\{t \in [0, T] \mid u_tv \in \Phi'\}) < \varepsilon \lambda(\{t \in [0, T] \mid u_tv \in \Phi\})$$

If $C \subset D \subset V$ satisfy the property of the proposition, one says that the *relative size* of the compact subset C in D does not exceed ε . Now we state the fundamental *quantitative nondivergence* property, about the behaviour of unipotent flows near singular sets.

Theorem 1.4.3 ([DM93], Theorem 1). *Let G be a connected Lie group and Γ a discrete subgroup of G . Let U be a closed connected subgroup of G generated by its unipotent elements. Let F be a compact subset of $\mathcal{G}(U)$. Then for any $\varepsilon > 0$ there exists a neighbourhood Φ of $\mathcal{S}(U)$ such that for any unipotent one-parameter subgroup $\{u_t\}$ of G and any $x \in F$*

$$\lambda(\{t \in [0, T] \mid u_tx \in \Phi\}) \leq \varepsilon T \text{ for all } T \geq 0$$

In other words, the theorem says that the orbit under any unipotent flow of any U -generic point x lying in a compact set spends in average most of its life outside a neighbourhood of the singular set of U .

1.4.2 Linearisation of singular sets

In this section we discuss the breakthrough paper [DM93] which introduced one of the most important techniques in homogeneous dynamics. The linearisation technique consists of assigning to each $H \in \mathcal{H}$ a finite dimensional linear representation (ρ_H, V_H) of G , a map $\eta_H = \eta : G \rightarrow V_H$ and an algebraic variety A_H of V_H such that $\eta^{-1}(A_H) = X(H, U)$. Hence the singular set of U can be seen as $\mathcal{S}(U) = \bigcup_{H \in \mathcal{H}} \eta^{-1}(A_H)\Gamma/\Gamma$ after linearisation. Using proposition 1.4.2 one can prove the following result about the behaviour of unipotent flows near singular set for closed connected subgroups of G generated by its unipotent elements.

More precisely, let $\mathfrak{g} = \text{Lie}(G)$ and for every $H \in \mathcal{H}$ with dimension h we associate the vector space given by $V_H = \wedge^h \mathfrak{g}$ and the representation $\rho_H := \wedge^h \text{ad} : G \rightarrow \text{GL}(V_H)$. Let ξ_1, \dots, ξ_h be a basis of subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the subgroup $H \subset G$. Then $\wedge^h \mathfrak{h} = \mathbb{R} \cdot \rho_H$

where $p = p_H := \xi_1 \wedge \dots \wedge \xi_h \in V_H$. The following map is the main tool of the theory of linearisation

$$\begin{aligned} \eta = \eta_H : G &\rightarrow V_H \\ g &\mapsto \rho(g) \cdot p_H \end{aligned}$$

Clearly $\eta(H) \subset \mathbb{R}p$. For any $H \in \mathcal{H}$, let $\Gamma_H = N_\Gamma(H)$ the normaliser of H in Γ

Lemma 1.4.4 ([DM93], Lemma 3.1). $\Gamma_H = \{\gamma \in \Gamma : \rho(\gamma)p = \pm p\}$

Proof. If $\rho(\gamma)p = \pm p$ for $\gamma \in \Gamma$, then $\text{Ad}(\gamma)$ normalises \mathfrak{h} and hence $\gamma \in \Gamma_H$. Conversely, if $\gamma \in \Gamma_H$ then γ normalises the lattice $\Gamma \cap H \subset H$, and hence the conjugaison $\gamma \mapsto \gamma h \gamma^{-1}$ preserves the bi-invariant Haar measure on H and (up to a sign) the volume form on \mathfrak{h} .

Lemma 1.4.5 ([DM93], Proposition 3.2). *Let $H \in \mathcal{H}$ and let $A = A_H \subset V$ the Zariski closure of the set $\eta(X(H, U))$. Then*

$$X(H, U) = \eta^{-1}(A)$$

Proof. The inclusion $X(H, U) \subset \eta^{-1}(A)$ comes from the definition. Conversely, let $\mathfrak{u} = \text{Lie}(U)$ the one-dimensional subspace of \mathfrak{g} . Then,

$$g \in X(H, U) \iff \mathfrak{u} \subset \text{Ad}(g)(\mathfrak{h}) \iff \eta(g) \wedge \mathfrak{u} = 0 \iff \eta(g) \in A_0$$

where $A_0 = \{v \in V : v \wedge \mathfrak{u} = 0\}$ is Zariski closed and contains A .

Self-intersection points

Definition 1.4.6. *Let M be a subset of G . We say that $g \in M$ is a point of (H, Γ) -intersection if there exists an element $\gamma \in \Gamma - \Gamma_H$ such that $g\gamma \in M$.*

$$g \in M \text{ is not S.I. (self-intersection)} \iff \text{the map } g\Gamma \mapsto g\Gamma_H \text{ is injective}$$

We are interested in the self-intersection of the set $X(H, U)$. The normaliser $\Gamma_H = N_\Gamma(H)$ is contained in $X(H, U)$ and if the equality $N_\Gamma(H) = X(H, U)$ holds then the set $X(H, U)$ has no (H, Γ) -self-intersection.

Proposition 1.4.7 ([DM93], Proposition 3.3). *The set of (H, Γ) -intersection points of $X(H, U)$ is contained in the union of $X(H', U)$ where H' are in \mathcal{H} and have dimension less than h .*

Using the aforementioned proposition and induction, one can prove the following fundamental result,

Theorem 1.4.8 ([DM93], Theorem 3.4). *Let Γ a discrete subgroup of G and let $H \in \mathcal{H}$. Then the orbit*

$$\eta_H(\Gamma) = \rho_H(\Gamma)p \text{ is discrete in } V_H$$

From the above theorem it is not difficult to deduce,

Corollary 1.4.9 ([DM93], Corollary 3.5). *Let $H \in \mathcal{H}$, A the Zariski closure of $\eta(X(H, U))$ in V and D the compact set in A . Denote by Y_D the set of (H, Γ) -intersections of $\eta^{-1}(D)$ and let K be a compact set in $G/\Gamma - Y_D\Gamma$. Then there exists a neighbourhood ϕ of D in V such that $\eta^{-1}(\phi) \cap K\Gamma$ has self-intersection.*

1.4.3 Proof of Theorem 1.4.3

We present the proof as given in [DM93]. The singular set $\mathcal{S}(U)$ is equal to the countable unions of σ -compact subsets $X(H, U)$ for $H \in \mathcal{H}$. Therefore it suffices to construct, for any $\varepsilon > 0$, $x \in \mathcal{G}(U)$, $H \in \mathcal{H}$ and any compact sets $C \subset X(H, U)$, $K \subset G/\Gamma$, neighbourhood Ω of the compact set $C\Gamma \subset G/\Gamma$ such that

$$(*) \quad \lambda\{t \in [0, T] : u_t x \in \Omega \cap K\} \leq \varepsilon T \quad \text{for all } T \geq 0$$

We proceed by induction on $h = \dim(H)$, the case of dimension 1 is trivial. Assume $h \geq 2$. We fix an $\varepsilon > 0$, $x \in \mathcal{G}(U)$, $H \in \mathcal{H}$, $C \subset X(H, U)$ and a compact K in G/Γ . By proposition 1.4.7 and by induction hypothesis, there exists a neighbourhood Ω' of the set $Y_H\Gamma \subset G/\Gamma$ such that

$$(**) \quad \lambda\{t \in [0, T] : u_t x \in \Omega' \cap K\} \leq \varepsilon T/2 \quad \text{for all } T \geq 0$$

At this moment, we introduce the linearisation procedure. Let

$$V = V_H, \rho = \rho_H, \eta = \eta_H \text{ and } A = A_H$$

We are now in a linear setup, thus proposition 1.4.2 implies that there exists a compact subset $D \subset A$ containing $\eta(C)$ as a subcompact of relative size at most $\varepsilon/4$. One can write $x = g\Gamma$ for some $g \in G$, since $\eta(\Gamma)$ is discrete in V by theorem 1.4.8, the set $\eta(g\Gamma) = \rho(g)\eta(\Gamma)$ is discrete as well. We have $\eta(g\Gamma) \cap A = \emptyset$, indeed by proposition 1.4.5

$$X(H, U) = \eta^{-1}(A) \text{ and } g\gamma \notin X(H, U) \text{ for all } \gamma \in \Gamma$$

In particular there exists a neighbourhood $\Phi' = \Phi'(D) \subset V$ such that $\eta(g\Gamma) \cap \Phi' = \emptyset$. Let K' be a compact set in G such that $K\Gamma' = K_1 = K - \Omega' \subset G/\Gamma$. By corollary 1.4.9, there exists a neighbourhood $\Phi'' = \Phi''(D)$ in V so that the set $\eta^{-1}(\Phi'') \cap K'\Gamma$ has no (H, Γ) -intersections. We put $\Phi = \Phi' \cap \Phi''$. Then there exists a neighbourhood $\Psi = \Psi(\eta(C)) \subset \Phi \subset V$ so that for all $T \geq 0$

$$(* * *) \quad \lambda\{t \in [0, T] : \rho(u_t)v \in \Psi\} < \varepsilon \lambda\{t \in [0, T] : \rho(u_t)v \in \Phi\} \text{ for any } v \in V - \Phi$$

The end of the proof consists of showing that $\Omega = \eta^{-1}(\Psi)\Gamma$ is the required neighbourhood of $C\Gamma$. We fix a $T > 0$ and for every $q \in \eta(\Gamma) = \rho(\Gamma)p$ we define

$$\begin{aligned} I(q) &= \{t \in (0, T) : \rho(u_t g)q \in \Phi\} \\ J(q) &= \{t \in (0, T) : \rho(u_t g)q \in \Psi \text{ and } u_t g\Gamma \in K_1\} \\ I'(q) &= \{t \in (0, T) : \exists a \geq 0 : [t, t+a] \subset I(q) \text{ and } u_{t+a}g\Gamma \in K_1\} \end{aligned}$$

Clearly for any $q \in \eta(\Gamma)$, we have the following inclusions $J(q) \subset I'(q) \subset I(q)$.

Concerning I' , given any $q_1, q_2 \in \eta(\Gamma)$ we have either $q_1 = \pm q_2$ or $I'(q_1) \cap I'(q_2) = \emptyset$. Indeed if we write $q_1 = \eta(\gamma_1)$ and $q_2 = \eta(\gamma_2)$ then given any $t \in I'(q_1) \cap I'(q_2)$, there exists a $a \geq 0$ such that $[t, t+a] \subset I(q_1) \cap I(q_2)$ and $u_{t+a}g\Gamma \in K_1 = K'\Gamma$. We already know that $\eta^{-1}(\Phi) \cap K'\Gamma$ has no (H, Γ) -self-intersections, then $\gamma_1\Gamma_H = \gamma_2\Gamma_H$ and the lemma 1.4.4 we get that $q_1 = \rho(\gamma_1)p = \pm\rho(\gamma_2)p = \pm q_2$.

Let $q \in \eta(\Gamma)$ and let (a, b) a connected component of $I(q)$. Thus, $\rho(u_a g)q \notin \Phi$ and $J(q) \cap (a, b) \neq \emptyset$ then $I'(q)$ contains an interval (a, c) , $a < c \leq b$, such that $J(q) \cap (a, b) \subset (a, c]$. Hence by applying $(***)$ to $v = \rho(u_a g)q \in V - \Phi$ and $T = c - a > 0$, we obtain

$$\lambda(J(q) \cap (a, b)) \leq \frac{\varepsilon}{4} \lambda(I'(q) \cap (a, b))$$

In particular, $\lambda(J(q)) \leq \frac{\varepsilon}{4} \lambda(I'(q))$ therefore

$$\lambda\{t \in [0, T] : u_t g\Gamma \in \Omega \cap K_1\} = \lambda\left(\bigcup_{q \in \eta(\Gamma)} J(q)\right) \leq \frac{\varepsilon}{4} \sum_{q \in \eta(\Gamma)} \lambda(I'(q)) \leq \varepsilon T/2 \text{ for all } T \geq 0$$

The last inequality follows because $I'(q_1) \cap I'(q_2) = \emptyset$ if $q_1 \neq \pm q_2$. Hence $(*)$ follows from the inequality $(**)$.

Uniform equidistribution theorem

Now we can state the Uniform equidistribution theorem of Dani-Margulis which motivated the introduction of linearisation before, this result provided a uniform version with respect to x of Ratner's uniform distribution theorem provided x stays in compact subsets which avoid a finite number of singular subsets.

Theorem 1.4.10 ([DM93], Prop. 9.4). *Let G be a connected Lie group and Γ a lattice of G . Let μ be the G -invariant probability measure on G/Γ . Let $U = \{u_t\}$ be a unipotent one-parameter subgroup of G and φ be a bounded continuous function on G/Γ . Let K be a compact subset of G/Γ and let $\varepsilon > 0$ be given. Then there exist finitely many proper closed subgroups H_1, \dots, H_k such that each $H_i \cap \Gamma$ is a lattice in H_i and compact subsets C_1, \dots, C_k of $X(H_1, U), \dots, X(H_k, U)$ respectively, for which the following holds: For any compact subset F of $K - \cup C_i \Gamma/\Gamma$ there exists a $T_0 \geq 0$ such that for all $x \in F$ and $T > T_0$,*

$$\left| \frac{1}{T} \int_0^T \varphi(u_t x) dt - \int_{G/\Gamma} \varphi d\mu \right| < \varepsilon$$

1.5 Topological rigidity of products of real and p -adic Lie groups

1.5.1 Orbit closure theorem for regular p -adic Lie groups after Ratner

Let k a local field of characteristic zero (i.e. \mathbb{R} , \mathbb{C} or a finite extension of \mathbb{Q}_p or equivalently the fields k_s for variable number fields). If G is an algebraic group defined over k , then the group $\mathcal{G}(k)$ of rational points of \mathcal{G} is endowed with a structure of Lie group over k , and the Lie algebra of the Lie group $\mathcal{G}(k)$, is the space of rational points over k of the Lie algebra of \mathcal{G} , as an algebraic group, that is:

$$\text{Lie}_{Gr}(\mathcal{G}(k)) = \text{Lie}_{Al-Gr}(\mathcal{G})(k)$$

Assumptions. For each place $s \in S$, we are given a Lie group G_s over k_s and a closed subgroup H_s generated by Ad-unipotent one-dimensional subgroups over k_s . The product G of the G_s is then in a natural way a locally compact topological group, and the product H of the H_s is a closed subgroup of G . We identify $G_t (t \in S)$ to the subgroup of G consisting of the elements $g_s (s \in S)$ such that $g_s = 1$ for $s \neq t$. Two slight restrictions are imposed on G_s , if s is not archimedean.

1. *The kernel of the adjoint representation is the center $Z(G_s)$ of G .*
2. *The orders of the finite subgroups of G_s are bounded.*

If G is the group of rational points of an algebraic group which is connected in the Zariski topology, the first condition is fulfilled. The condition 2 is always verified if G_s is linear (see e.g. [B94] §7.3). Ratner introduced the following notion.

Definition 1.5.1. *Let $s \in S$ a non-archimedean place, then G_s is said to be Ad-regular if it satisfies the first condition and regular if it satisfies both conditions.*

In particular we see from the remarks above that if G is the group of rational points of a linear algebraic group which is connected in the Zariski topology defined over k_s , it is regular. This notion was exploited by Ratner to prove in [R93] the following Orbit closure Theorem valid for any p -adic regular Lie groups which is not necessarily the group of points of a linear algebraic groups,

Theorem 1.5.2 (Ratner). *Let G, H be as before. Let M be a closed subgroup of G containing H and Γ a discrete subgroup of finite covolume of M . Let $x \in M/\Gamma$. Then M contains a closed subgroup L such that $\overline{H \cdot x} = L \cdot x$ and $L \cap M_x$, where M_x is the stabilizer of x , has finite covolume in L .*

1.5.2 Rigidity of unipotent orbits in S -adic groups

Now we turn to the generalisation of the Raghunathan conjecture for S -arithmetic products. For simplicity we consider the problem of the topological rigidity under unipotent action within the group S -arithmetic group $G_S = \mathrm{SL}_n(k_S)$. Let Γ_S be the S -arithmetic subgroup of G_S given by $\Gamma_S = \mathrm{SL}_n(\mathcal{O}_S)$. The ring \mathcal{O}_S is a lattice in k_S . Define $\Omega_S = G_S/\Gamma_S$, it is the space of free \mathcal{O}_S -submodules of k_S^n of maximal rank and determinant one. Then Ω_S is the homogeneous space of unimodular lattices in k_S^n , by lattice we mean a discrete subgroup of G_S of finite covolume with respect to the Haar measure. For every $s \in S$, let \mathcal{U}_s be a unipotent k_s -algebraic subgroup of SL_{n/k_s} and denote by $\mathcal{U} = \prod_{s \in S} \mathcal{U}_s(k_s)$ the associated unipotent subgroup of G_S .

We are interested with the left action of \mathcal{U} on the homogeneous space Ω_S and more particularly with the closure of such orbits. If $x \in \Omega_S$ it turns out that the closure of the orbit $\mathcal{U}x$ is also an orbit of x . This result is the generalisation of Ratner's orbit closure theorem proven independently by Margulis-Tomanov and Ratner, we state it as it is in [MT94].

Theorem 1.5.3 ([MT94], Theorem 11.1). *Assume that \mathcal{U} is generated by its one-dimensional unipotent subgroups. Then for any $x \in \Omega_S$, there exists a closed subgroup $M = M(x) \subset G_S$ containing \mathcal{U} such that the closure of the orbit $\mathcal{U}x$ coincides with Mx and Mx admits a M -invariant probability measure.*

The following theorem generalises the quantitative nondivergence property (1.4.3) to the S -adic case,

Theorem 1.5.4 ([MT94], Theorem 11.6). *Let U be a subgroup of G_S generated by unipotent k_S -algebraic subgroups of G_S contained in U . Let F be a compact subset of $\mathcal{G}(U)$. Assume that Γ is a lattice in G_S . Then for any $\varepsilon > 0$, there exists a neighbourhood Ω of $\mathcal{S}(U)$ such that for any one-parameter unipotent subgroup $\{u_t\}$ of G_S , where $t \in k_v$, $v \in S$, any $x \in F$ and any $B \geq 0$*

$$\sigma_v\{t \in k_v \mid |t|_v < B, u_t x \in \Omega\} \leq \varepsilon B$$

where σ_v is the Haar probability in k_v .

Chapter 2

The Oppenheim conjecture on quadratic forms

2.1 The Oppenheim conjecture

Let Q denote a nondegenerate quadratic form on \mathbb{R}^n which is indefinite i.e. $Q(x) = 0$ for some $x \in \mathbb{R}^n - \{0\}$ or equivalently $Q(\mathbb{R}^n) = \mathbb{R}$. It may be written

$$Q(x) = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j \quad (a_{ij} = a_{ji} \in \mathbb{R}, \det(a_{ij}) \neq 0)$$

Unless otherwise stated we assume $n \geq 3$. We are concerned with the set $Q(\mathbb{Z}^n)$.

Definition 2.1.1. Q is said to be rational if Q is proportional to a rational quadratic form that is, if there exists $c \in \mathbb{R}^*$ such that $Q = c.Q_0$ for some Q_0 quadratic form with coefficients in \mathbb{Q} . If not Q is said to be irrational.

We can assume that Q_0 have integral coefficients. Therefore, if Q is rational one has that

$$Q(\mathbb{Z}^n) = c.Q_0(\mathbb{Z}^n) \subset c.\mathbb{Z} \text{ is discrete}$$

The Oppenheim conjecture states that, conversely, if Q is irrational, then $Q(\mathbb{Z}^n)$ is not discrete around the origin. More precisely, consider the three following conditions:

- (i) Q is irrational
- (ii) Given $\varepsilon > 0$, there exists $x \in \mathbb{Z}^n$ such that $0 < |Q(x)| < \varepsilon$.
- (ii)' Given $\varepsilon > 0$, there exists $x \in \mathbb{Z}^n$ such that $|Q(x)| < \varepsilon$.

The Oppenheim conjecture is that (i) \Rightarrow (ii), the condition (i) leaves open the possibility that $Q(\mathbb{Z}^n)$ accumulates to zero only on one side, but Oppenheim showed (see [Op3]) that this cannot happen for $n \geq 3$ but it can for $n = 2$. It follows that the condition (ii) implies:

(iii) $Q(\mathbb{Z}^n)$ is dense in \mathbb{R}

so that the conjectural dichotomy was in fact:

– Q rational $\Leftrightarrow Q(\mathbb{Z}^n)$ is discrete

– Q irrational $\Leftrightarrow Q(\mathbb{Z}^n)$ is dense

Counterexample of the Oppenheim conjecture for $n = 2$.

Let $Q(x, y) = y^2 - \beta^2 x^2$ where β is positive irrational quadratic integer such that $\beta^2 \notin \mathbb{Q}$. Thus Q is clearly irrational. For any $x, y \in \mathbb{Z}, x, y \neq 0$ there exists $c > 0$ such that:

$$|\beta - y/x| \geq \frac{c}{x^2}$$

For $x \neq 0$, we can write

$$Q(x, y) = x^2(y/x + \beta)(y/x - \beta)$$

We have to prove that $|Q(x, y)|$ has strictly positive lower bound for $x, y \in \mathbb{Z}$ not both zero. This is clear if one of them is equal to zero. So let x, y be both positive. Then $|y/x + \beta| \geq \beta$, hence $|Q(x, y)| \geq c\beta$. Thus Q does not assume arbitrarily small values at integer points.

Reduction of the Oppenheim conjecture to dimension 3

The first relevant case to examine for the Oppenheim conjecture is when the dimension is equal to three. Actually this case is the only we have to check because it suffices to prove the Oppenheim conjecture for dimension 3.

Lemma 2.1.2. *It suffices to prove the Oppenheim conjecture for $n = 3$*

Proof. We proceed by induction. In fact, for $n > 3$ we select a rational basis $v_1, \dots, v_n \in \mathbb{Q}^n$ in \mathbb{R}^n such that the restriction of Q to the linear hull $L = \langle v_1, \dots, v_{n-1} \rangle$ is an indefinite nondegenerate quadratic form and Q is nontrivial on the the plane $\langle v_1, v_2 \rangle$ (we can do this by reducing Q to principal axes, then perturbing the basis obtain to make it rational). Assume that the form is rational on the subspace L . We set

$$L_t = \langle v_1, \dots, v_{n-1}, v_{n-1} + tv_n \rangle$$

Then for a sufficiently small $|t| < \delta$, the restriction of Q to L_t is an indefinite nondegenerate form as well. Assume that for some rational $0 < t < \delta$ the form $Q|_{L_t}$ is rational. Since the restriction of Q to $L \cap L_t = \langle v_1, \dots, v_{n-2} \rangle$ is nontrivial for $n > 3$, it follows that there exists $\alpha > 0$ such that the form αQ has rational coefficients on L and L_t . But then αQ has rational coefficients on all of \mathbb{R}^n . This contradiction shows that the restriction of Q to some rational hyperplane $L_t, t \in \mathbb{Q}$, is a nondegenerate indefinite irrational form, and one can lower the dimension to $n - 1$.

2.2 History of the conjecture

In 1929, Oppenheim stated his conjecture on the values of indefinite irrational quadratic forms at integer points. This conjecture is the analog of the classical result due to Meyer for irrational forms. Indeed, given a quadratic form Q in n variables, let us set

$$m(Q) = \inf \{|Q(x)| : x \in \mathbb{Z}^n - \{0\}\}$$

Meyer's theorem says that if Q is rational and $n \geq 5$ then Q represents zero over \mathbb{Z} non-trivially i.e. there exists $x \in \mathbb{Z}^n - \{0\}$ such that $Q(x) = 0$ i.e. $m(Q) = 0$. Therefore the Oppenheim conjecture is equivalent to the statement that if Q is irrational and $n \geq 5$ then $m(Q) = 0$.

Later on it was realised that $m(Q)$ should be also equal to 0 under the weaker condition that $n \geq 3$, the formulation of the Oppenheim conjecture in its final form due to Davenport and Heilbronn who made the conjecture for the particular case of diagonal forms (i.e. given by $Q(x_1, \dots, x_n) = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2$). We summarise briefly some of the most important results of the history of the Oppenheim conjecture. We follow the presentation of Margulis as in [Mar97], see the same paper for more details.

Chowla's first result

The first significant result towards the Oppenheim conjecture after those made by Oppenheim himself was obtained in 1934 by Chowla (see [Ch34]) and concerns indefinite diagonal forms

$$Q(x_1, \dots, x_n) = \lambda_1 x_1^2 + \dots + \lambda_n x_r^2 + \lambda_{r+1} x_{r+1}^2 + \dots + \lambda_n x_n^2$$

such that $n \geq 9$ with all the ratios λ_i/λ_j ($i \neq j$) are irrational.

The proof uses a theorem of Jarník and Walfisz on the number of integer points lying in a large ellipsoid. Let Q be a positive definite quadratic form in n variables. Let $N_Q(X)$ be the number of integer solutions of the inequality $Q(x) \leq X$, in other words the number of points from the lattice \mathbb{Z}^n lying in the ellipsoid $\mathcal{E}_Q(X) := \{x \in \mathbb{R}^n : Q(x) \leq X\}$. Using a linear change of variables, one can easily see that

$$\text{Vol}(\mathcal{E}_Q(X)) = C_Q \cdot X^{n/2} \text{ where } C_Q = \frac{\pi^{n/2}}{\sqrt{D}\Gamma(\frac{n}{2} + 1)} \text{ with } D = \text{disc}(Q) > 0.$$

In 1930, Jarník and Walfisz proved the following result

Theorem 2.2.1 (Jarník-Walfisz). *Suppose Q is a positive definite diagonal irrational form in $n \geq 5$ variables then:*

$$N_Q(X) = \text{Vol}(\mathcal{E}_Q(X)) + o(X^{n/2-1}).$$

It follows from this theorem and the volume estimate of an ellipsoid given before that for any fixed $\varepsilon > 0$, the asymptotic of the number of integral solutions of the inequalities $X \leq Q(x) \leq X + \varepsilon$ is given by

$$N_Q(X + \varepsilon) - N_Q(X) \sim \varepsilon C_Q \frac{n}{2} X^{n/2-1} \text{ as } X \rightarrow \infty.$$

In particular, the gaps between successive values of Q must tend to 0, that is

$$N_Q(X + \varepsilon) - N_Q(X) > 0 \quad \text{for any } \varepsilon > 0 \text{ and for any } X \geq X_0(\varepsilon). \quad (2.1)$$

2.2.1 The Oppenheim conjecture from the analytic point of view

In 1972, Davenport and Heilbronn conjectured in [DavH] that the same result should be true for general positive definite irrational quadratic form in $n \geq 5$ variable, in the same paper they also got a partial result using the Hardy-Littlewood circle method. Now let us deduce Chowla's result, so let $n \geq 9$ and all the ratios λ_i/λ_j ($i \neq j$) be irrational numbers. The fact that Q is positive definite encouraged Chowla to make the following remark.

Chowla's argument: *Of nine positive or negative number, at least five must have the same sign.*

Hence we may assume that $\lambda_1, \dots, \lambda_5$ are positive and $\lambda_i < 0$ for some $i \in \{6, \dots, n\}$. Applying equality (2.1) to the positive definite quadratic form $Q(x_1, \dots, x_5) = \lambda_1 x_1^2 + \dots + \lambda_5 x_5^2$, we get that there exists $y \geq X_0(\varepsilon)$, $y > 0$ and integers m_1, \dots, m_n not all equal to zero such that

$$\sum_{r=6}^n \lambda_r m_r^2 = -y \quad \text{and} \quad y < \lambda_1 m_1^2 + \dots + \lambda_5 m_5^2 \leq y + \varepsilon$$

This concludes the proof since we found non trivial integer vector m satisfying $|Q(m)| < \varepsilon$.

The Oppenheim conjecture for diagonal forms for $n \geq 5$

The Oppenheim conjecture for diagonal forms in $n \geq 5$ variables was proved by Davenport and Heilbronn in 1946, clearly as we already saw before it is enough to prove it for $n = 5$. Actually, Davenport and Heilbronn proved a much stronger statement which gives a quantitative estimate on the distribution of the values of Q at integral points lying in a box.

Theorem 2.2.2 (Davenport-Heilbronn, [DavH]). *Let $\lambda_1, \dots, \lambda_5$ real numbers, not all of the same sign and none of them is zero, such that the ratios λ_i/λ_j ($i \neq j$) are irrational. Let Q the diagonal quadratic form given by $Q(x_1, \dots, x_5) = \lambda_1 x_1^2 + \dots + \lambda_5 x_5^2$. Then there exists arbitrarily large integers P such that the inequalities*

$$|Q(x_1, \dots, x_5)| < 1 \text{ and } 1 \leq x_i \leq P \text{ for each } i \in \{1, \dots, 5\}$$

have more than γP^3 integral solutions where $\gamma = \gamma(\lambda_1, \dots, \lambda_5) > 0$.

The Oppenheim conjecture for general quadratic forms

For general quadratic forms the best result towards the Oppenheim conjecture concerns the case when $n \geq 21$ and are due to Birch, Davenport and Ridout (see [DavR]). In 1986, Baker and Schlickewei proved a slight generalisation for $n \geq 18$ with restrictive signatures. At this stage, the methods coming from analytic number theory were not sufficient to prove the Oppenheim conjecture for small number of variables.

2.3 Dynamical interpretation of the Oppenheim conjecture

The turning point for the conjecture occurred in the eighties when M.S. Raghunathan formulated his conjecture on closures of orbits of unipotent actions in the space of lattices. In particular a special case, namely Theorem 2.3.1 below, would imply Oppenheim's conjecture. In 1986, using this strategy, Margulis proved the Oppenheim conjecture in full generality. Let us consider the three following conditions

- (i) Q is *irrational*
- (ii) Given $\varepsilon > 0$, there exists $x \in \mathbb{Z}^n$ such that $0 < |Q(x)| < \varepsilon$.
- (ii)' Given $\varepsilon > 0$, there exists $x \in \mathbb{Z}^n$ such that $|Q(x)| < \varepsilon$.

Actually Margulis first proved a *weak form* of the Oppenheim conjecture, namely the implication (i) \Rightarrow (ii)' ([Mar86], [Mar89a]). When informed by A.Borel of the fact that the Oppenheim conjecture was a slightly stronger one, he quickly completed his argument and finally established the full Oppenheim conjecture in [Mar89b]. The most difficult was to prove (i) \Rightarrow (ii)', following the strategy of Raghunathan it was sufficient for Margulis to prove the following statement on closures of orbits:

Theorem 2.3.1. *Any relatively compact orbit of $\mathrm{SO}(2,1)^\circ$ in $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$ is compact.*

This is a particular case of the Raghunathan conjecture which was proved in full generality by M.Ratner few years after. In the original proof of this theorem made by Margulis, the key point is to use the dynamics of some one-parameters subgroups of $\mathrm{SL}(3, \mathbb{R})$ acting on the homogeneous space of lattices, namely $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$. In a certain sense, Margulis' proof is elementary since it does not make any use of major results other than Mahler's compactness criterion. However the proof is tricky and requires some deep and technical topological results about one-parameter subgroups of $\mathrm{SL}(3, \mathbb{R})$ (see [Mar89a], [Mar89b]).

2.3.1 Deduction of the Oppenheim conjecture from Theorem 2.3.1

Let Q a non-degenerate indefinite quadratic form over \mathbb{R}^3 , and $S = \text{SO}(Q)$. Let as usual $G = \text{SL}(3, \mathbb{R})$, $\Gamma = \text{SL}(3, \mathbb{Z})$, $q(x) = 2x_1x_3 - x_2^2$ and $H = \text{SO}(q)^\circ$. Since the form Q is indefinite and nondegenerate over \mathbb{R}^3 there exists $g \in G$ such that $Q(x) = \lambda q(gx)$ for some $\lambda \in \mathbb{R}$. It follows that $H = gSg^{-1}$ and H is generated by the two unipotent subgroups U_1 and U_1^T .

Lemma 2.3.2. *If Theorem 2.3.1 is true then the Oppenheim conjecture holds.*

Proof. To prove the Oppenheim conjecture it suffices to prove that the set $q(g\mathbb{Z}^3)$ has 0 as an accumulation point provided that the form Q is irrational. Assume the opposite: 0 is an isolated point of the set. Then the orbit Hx is relatively compact in G/Γ where $x = g\Gamma \in G/\Gamma$. Indeed, suppose that $h_k g\Gamma \rightarrow \infty$ for some sequence $(h_k) \in H$. Then by Mahler's criterion, 0 is a limit point for the family of lattices $h_k g\Gamma, k \in \mathbb{N}$. But $q(h_k g\Gamma) = q(gx)$, and we arrive to a contradiction.

Hence the orbit $Hg\Gamma$ is relatively compact in G/Γ . By the hypothesis, it must be compact. Then $\Gamma \cap g^{-1}Hg$ is a lattice in $g^{-1}Hg$ which is Zariski dense by Borel's density theorem. It follows that $g^{-1}Hg$ is defined over \mathbb{Q} . But $H = \text{SO}(q)^\circ$, it implies that q is rational. Indeed every symmetric matrix B preserved by the subgroup H is proportional to the associated symmetric matrix C of q since H is generated by the two unipotent subgroups U_1 and U_1^T

$$u_1(1)Bu_1(1)^t = B \text{ and } u_1(1)^tBu_1(1) = B \text{ implies } B = \mu C \text{ for some } \mu \in \mathbb{R}.$$

Therefore $\mathbb{R}C$ consists of all symmetric matrices preserved by the \mathbb{Q} -defined subgroup $g^{-1}Hg$. Since the lattice $\Gamma \cap g^{-1}Hg$ is Zariski dense, hence $\mathbb{R}C$ is determined by a system of linear equations with integer coefficients. But such a system has a rational solution, hence for some $\mu \in \mathbb{R}$ the matrix $B = \mu C$ of Q has all its entries rational. This contradicts the fact that Q is irrational and the Oppenheim conjecture is proved.

2.3.2 The proof of Oppenheim conjecture using Ratner's theorem

The proof of the Oppenheim can be easily settled using Ratner's Orbit closure theorem. We prove the Oppenheim conjecture using the following version of Ratner's theorem.

Theorem 2.3.3. *Let G be a connected Lie group, and let Γ be a lattice in G . Let H be a connected Lie subgroup of G generated by one-parameter unipotent groups. Then for any $x \in G/\Gamma$ there exists a closed connected subgroup $H \subset L$ such that $\overline{Hx} = Lx$.*

In order to apply this theorem let $G = \text{SL}(n, \mathbb{R})$ and $\Gamma = \text{SL}(n, \mathbb{Z})$. It follows by Corollary 1.2.6 that Γ is a lattice of G . Let Q be an indefinite nondegenerate irrational quadratic form over \mathbb{R}^n of signature (r, s) with $r, s \geq 1$.

There exists $g \in G$ such that

$$Q(x) = \lambda q(gx) \text{ for some } \lambda \in \mathbb{R} \text{ and with } q(x) = x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_n^2.$$

Let $H = \text{SO}(Q)^\circ$ then $\text{SO}(q)^\circ = gHg^{-1}$, moreover H is generated by unipotents elements since $\text{SO}(q) = \text{SO}(r, s)$ is generated by unipotent elements for $n = r + s \geq 3$. We can apply Ratner's theorem to obtain the existence of a closed connected subgroup L of G which contains H such that

$$\overline{H.o} = L.o$$

where o is the origin of the space of lattices G/Γ . In other words,

$$\text{one has } \overline{HZ^n} = LZ^n \text{ for some closed connected } L \text{ such that } H \subset L \subset G$$

At first sight, it seems difficult to classify such intermediate subgroups, but using property (3) of Proposition 1.1.17 yields that H is maximal among all closed connected subgroups of G , hence $L = H$ or $L = G$. If $L = H$, since $L \cap \Gamma$ is a lattice in L using Borel's density theorem (see e.g. [Furst76]) we get that L is defined over \mathbb{Q} . Thus Q is proportional to a rational form, which leads to a contradiction. Hence $L = G$ and $\overline{H\Gamma} = G$,

$$\overline{Q(\mathbb{Z}^n)} = \overline{Q(H\Gamma\mathbb{Z}^n)} \supset Q(G\mathbb{Z}^n) = Q(\mathbb{R}^n - \{0\}) = \mathbb{R}$$

that is $\overline{Q(\mathbb{Z}^n)} = \mathbb{R}$ and the Oppenheim conjecture is proved.

2.3.3 The Littlewood conjecture

We introduce here a conjecture due to Littlewood (1930) which can be seen as an analog of the Oppenheim conjecture for cubic forms instead of quadratic forms. In particular, as the Oppenheim conjecture is deduced from Raghunathan's conjecture, Littlewood's conjecture can be deduced from a dynamical statement which is a conjecture due to Margulis which is the analog of the Raghunathan conjecture for diagonal actions.

Let $a \in \mathbb{R}^n$, denote by $\pi(a) = \prod_{i=1}^n |a_i|$ and $\pi_+(a) = \prod_{i=1}^n |a_i|_+$ with $|a|_+ = \max(1, a)$.

Let be given an arbitrary $y = (\alpha, \beta) \in \mathbb{R}^2$, the conjecture of Littlewood states that

$$\liminf_{n \rightarrow +\infty} n \|n\alpha\| \|n\beta\| = 0$$

with $\|\cdot\|$ be the distance to the nearest integer. Using the notations above, the Littlewood conjecture can be written in the following way

$$(\mathcal{L}) \quad \text{For any } \varepsilon > 0, \text{ there exists } q \in \mathbb{Z} - \{0\} \text{ and } p \in \mathbb{Z}^2 \text{ such that } |q|. \pi(qy + p) < \varepsilon.$$

Using multiplicative version of *Khinchine's transference principle*, we have the equivalent statement.

$$(\mathcal{L}') \quad \text{For any } \varepsilon > 0, \text{ there exists } q \in \mathbb{Z}^2 - \{0\} \text{ and } p \in \mathbb{Z} \text{ such that } \pi_+(q). |qy + p| < \varepsilon.$$

Dynamical interpretation of the Littlewood's conjecture

Given any $(\alpha, \beta) \in \mathbb{R}^2$, one can define a cubic form with real variables defined by

$$P(x, y, z) = x(x\alpha + y)(x\beta + z)$$

It is easy to see that the Littlewood's conjecture is equivalent to the following assertion

$$(\mathcal{L}'') \quad \inf_{\mathbb{Z}^3 - \{0\}} |P(x, y, z)| = 0 \text{ for any } (\alpha, \beta) \in \mathbb{R}^2.$$

In particular, (\mathcal{L}'') was partially proved by Cassels-Swinnerton-Dyer for pairs (α, β) of cubic irrationals in the same cubic fields. The best result known around the 2000's was due to Pollington and Velani [PV].

Recall that **Bad** := $\{x \in [0, 1] \text{ such that } \exists c > 0 \text{ with } \|qx\| > \frac{c}{q} \text{ for all } q \geq 1\}$.

Theorem 2.3.4 (Pollington-Velani). *Given $\alpha \in \mathbf{Bad}$ there exists a subset $G(\alpha)$ of \mathbf{Bad} with Hausdorff dimension one such that, for any $\beta \in G(\alpha)$, the pair (α, β) satisfies (\mathcal{L}) .*

The best known result toward Littlewood's conjecture is due to Einsiedler, Katok and Lindenstrauss and shows that the set of exceptions of (\mathcal{L}) is very small.

Theorem 2.3.5 ([EKL06]). *The set of pairs (α, β) for which the Littlewood's conjecture fails,*

$$\Xi := \{(\alpha, \beta) \in \mathbb{R}^2 : \liminf_{n \rightarrow +\infty} n \|n\alpha\| \|n\beta\| > 0\}$$

has Hausdorff dimension zero.

The analogy with Oppenheim's conjecture invites us to consider the strategy used by Margulis. Hence we are naturally led to the study of the stabiliser of P under the action of $\mathrm{SL}(3, \mathbb{R})$. By remarking that $P = P_0 \circ g_{\alpha, \beta}$, where

$$P_0(x, y, z) = xyz \text{ and } g_{\alpha, \beta} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3, \mathbb{R})$$

it suffices to consider the stabiliser of P_0 since the stabiliser of P is the image of the stabiliser of P_0 by the conjugation by $g_{\alpha, \beta}$. The stabiliser of P_0 is given by the subgroup A of full diagonal matrices in $\mathrm{SL}(3, \mathbb{R})$, for technical reasons it is more convenient to consider the two-parameter diagonal subgroup given by

$$A^+ = \left\{ a_{s,r} = \begin{pmatrix} e^{-s-r} & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & e^s \end{pmatrix} : r, s \in \mathbb{R}_+ \right\} \subset A.$$

By applying Mahler's compactness criterion in the homogeneous space $X_3 = G/\Gamma$ of lattices in \mathbb{R}^3 , it is not difficult to prove the following equivalence.

Proposition 2.3.6. *The pair (α, β) satisfies (\mathcal{L}'') if and only if the A^+ -orbit of $g_{\alpha, \beta}\Gamma$ is unbounded in G/Γ . Moreover, for any $\delta > 0$, there is a compact C_δ in X_3 so that if*

$$\liminf_{n \rightarrow +\infty} n \|n\alpha\| \|n\beta\| \geq \delta.$$

Then $A^+ \cdot g_{\alpha, \beta}\Gamma \subset C_\delta$.

The analogy with the Oppenheim conjecture ends here, the major issue is that A^+ is not generated by unipotent elements, thus Ratner's theorem does not apply. The rigidity of diagonal actions on G/Γ have been conjectured by Margulis.

Conjecture 2.3.7 (Margulis). *Let $d \geq 3$ and μ be an A -invariant ergodic measure on X_d . Then there is an intermediate subgroup $A \subset L \subset G$ such that μ is L -invariant and supported by a closed L -orbit Lx for some $x \in X_d$.*

In order to prove Theorem 2.3.5, Einsiedler, Katok and Lindenstrauss proved a particular case of Margulis's conjecture involving an additional entropy condition.

Theorem 2.3.8 ([EKL06]). *Let μ be an A -invariant ergodic measure on X_d such that there exists a one parameter subgroup of A acting with positive entropy, then there is an intermediate subgroup $A \subset L \subset G$ such that μ is L -invariant and supported by a closed L -orbit Lx for some $x \in X_d$. In particular if $d = 3$, μ is the Haar measure on X_3 .*

Following the analogy with Ratner's theorems, Margulis suggested a conjecture about the orbit closures under the action of the diagonal subgroup A . We state it in the case of dimension 3 as it is in [Shap11] (see Conjecture 6.1).

Conjecture 2.3.9 (Margulis). *For each $x \in X_3$, one of the following three options occur:*

1. Ax is dense
2. Ax is closed
3. Ax is contained in a closed orbit Hx of an intermediate subgroup H in $\mathrm{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$ which contains A , where H could be one the following three subgroup of $\mathrm{SL}(3, \mathbb{R})$:

$$H_1 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, H_2 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, H_3 = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}.$$

U.Shapira exhibited in ([Shap11], Theorem 6.2) a counterexample to this conjecture which proceeded a previous similar contradiction of the conjecture due to F. Maucourant who treats the case of the subgroups of A while Shapira deals with the maximal split torus ([Mau10]). In the same paper, Shapira proved that the set of exception of the inhomogeneous uniform version of the Littlewood conjecture is of null Lebesgue measure, that is, for almost any pair (α, β)

$$\forall \gamma, \delta \in \mathbb{R} \quad \liminf_{n \rightarrow +\infty} |n| \|n\alpha - \gamma\| \|n\beta - \delta\| = 0.$$

Chapter 3

Some generalisations of the Oppenheim conjecture

The validity of the different forms of Raghunathan conjecture, namely topological rigidity, uniform distribution and S -arithmetic version of it, have given rise in parallel to some extensions of the Oppenheim conjecture.

The first generalisation that we introduce in §4.1.1 concerns the density of the values taken by pairs $F = (Q, L)$ consisting of one quadratic form and one linear form at integral points. The first result in this direction deals with pairs in dimension 3 satisfying natural arithmetic and geometric conditions and is due to S.G. Dani and G.A. Margulis (see Theorem 3.1.1).

For dimension greater than 3, sufficient conditions to guarantee the density of $F(\mathbb{Z}^n)$ were given by A. Gorodnik (see Theorem 3.1.3). It is conjectured by A. Gorodnik that one can relax the geometric condition on the pair in order to obtain optimal conditions which ensure density (see conjecture 6.1.1).

Historically the first generalisation of the Oppenheim conjecture is due to A. Borel and G. Prasad who generalised the original proof of Margulis to the S -arithmetic case (see theorem 3.2.1). The proof rests essentially on the use of strong approximation property together with tools of geometry of numbers. Later, as soon as Raghunathan conjecture was proved in full generality for product of real and p -adic Lie groups, Borel gave a stronger result which proves density instead of non-discreteness around the origin (see Theorem 3.2.2). The advantage of his proof is that it does not require the use of strong approximation, however it involves technical group-theoretical arguments to eliminate superfluous contribution of the intermediate subgroups coming from Ratner's classification.

Finally in §3.3, we treat the problem of the distribution of the values of $F(\mathbb{Z}^n)$ which are known to be dense in the real line under Oppenheim's assumptions. The so-called quantitative Oppenheim conjecture states that the points of $F(\mathbb{Z}^n)$ are not only dense in the real line but also equidistributed with respect to Lebesgue measure when the signature of F is not $(2, 1)$.

These highly nontrivial results have been proved in three steps,

- a) The lower bound due to Dani and Margulis (see [DM93], corollary 5)
- b) The upper bound due to Eskin, Margulis and Mozes for signatures not equal to $(2, 1)$ or $(2, 2)$ (see [EMM98])
- c) The case of low signatures due to Eskin, Margulis and Mozes (see [EMM05])

3.1 Oppenheim type problem for pairs (Q, L)

A similar Oppenheim type problem concerns the existence of integral solutions of simultaneous diophantine inequalities involving one quadratic form and one linear form. More precisely given a pair (Q, L) and $(a, b) \in \mathbb{R}^2$ the problem is to find sufficient conditions which guarantee the existence of an integral vector in $x \in \mathbb{Z}^n$ such that

$$(A) \text{ For any } \varepsilon > 0 \text{ one has simultaneously } |Q(x) - a| < \varepsilon \text{ and } |L(x) - b| < \varepsilon.$$

That is to say that the set $\{(Q(x), L(x)) : x \in \mathbb{Z}^n\}$ is dense in \mathbb{R}^2 . This problem was first considered by S.G. Dani and G.Margulis [DM90] for a pair (Q, L) consisting of one nondegenerate indefinite quadratic form and a nonzero linear form in dimension 3.

Theorem 3.1.1 (Dani-Margulis). *Let Q be a non degenerate indefinite quadratic form and L be a nonzero linear form in real coefficients and in three variables. Suppose that*

1. *The plane $\{L = 0\}$ is tangential to the cone $\{Q = 0\}$*
2. *No linear combination (with real coefficients) of Q and L^2 is rational*

Then the set $\{(Q(x), L(x)) : x \in \mathbb{Z}^3\}$ is dense in \mathbb{R}^2 .

3.1.1 Case when density fails

The geometric condition (1) cannot be replaced by the weaker condition that the intersection of the plane and the cone is nonzero, in other words the condition $Q|_{L=0}$ is indefinite is not sufficient for density to hold in dimension three. Indeed, due to a result of Kleinbock-Margulis we can measure the size of the set of pairs for which density fails. Let assume that $F = (Q, L)$ is a pair in three variables such that $Q|_{L=0}$ is indefinite, hence there exists $g \in \text{SL}(3, \mathbb{R})$ such that $Q(x) = Q_0(gx)$ and $L(x) = L_0(gx)$ where $Q_0(x) = 2x_1x_3 - x_2^2$ and $L_0(x) = x_2$ (note that in the case of the assumptions of Theorem 3.1.1 we may have $L_0(x) = x_3$ instead). We denote it by $(Q, L) = (Q_0^g, L_0^g)$, this allows us to translate properties of such pairs into properties of the associated g 's in $\text{SL}(3, \mathbb{R})$. We have the following,

Theorem 3.1.2 (Dani). *There exists $g \in \mathrm{SL}(3, \mathbb{R})$ and $\varepsilon > 0$ such that the following conditions are satisfied for the pair $(Q, L) = (Q_0^g, L_0^g)$*

- (i) *No linear combination (with real coefficients) of Q and L^2 is rational.*
- (ii) *There does not exist any nonzero integral vector $x \in \mathbb{Z}^3$ such that*

$$|Q(x)| < \varepsilon \text{ and } |L(x)| < \varepsilon.$$

Moreover, the set

$$\mathcal{E} = \{g \in \mathrm{SL}(3, \mathbb{R}) \text{ such that (i) and (ii) holds for some } \varepsilon > 0\}$$

has Hausdorff dimension 8, in particular (i)–(ii) are satisfied for uncountably many pairs (Q, L) .

The proof of this theorem can be found in ([Dan00], Theorem 7.3), it essentially rests on the following result due to Kleinbock and Margulis. Let us define the diagonal one-parameter subgroup of $\mathrm{SL}(3, \mathbb{R})$ by

$$D = \{\mathrm{diag}(e^{-t}, 1, e^t) : t \in \mathbb{R}\}$$

and

$$\mathcal{B} = \{\Lambda \in \mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z}) \text{ such that } D.\Lambda \text{ is bounded}\}.$$

It was proved in [KM] that the set of $g \in \mathrm{SL}(3, \mathbb{R})$ such that $g\mathbb{Z}^3 \in \mathcal{B}$ intersects every nonempty set of $\mathrm{SL}(3, \mathbb{R})$ in a set of Hausdorff dimension 8. The proof of Theorem 3.1.2 consists to show that the exception set \mathcal{E} is an intersection of some nonempty subset of $\mathrm{SL}(3, \mathbb{R})$ with the previous set.

3.1.2 Density in higher dimension

When the dimension is greater than 3, in contrast with the three dimensional case, the assumption $Q|_{L=0}$ is indefinite is sufficient for density to hold, this result is due to A.Gorodnik [Gor04]

Theorem 3.1.3 (Gorodnik). *Let Q be a non degenerate quadratic form and L be a nonzero linear form in dimension $n \geq 4$. Suppose that*

1. $Q|_{L=0}$ is indefinite
2. No linear combination (with real coefficients) of Q and L^2 is rational

Then the set $\{(Q(x), L(x)) : x \in \mathbb{Z}^n\}$ is dense in \mathbb{R}^2 , that is, given any $(a, b) \in \mathbb{R}^2$ and any $\varepsilon > 0$ there exists a nonzero $x \in \mathbb{Z}^n$ such that

$$|Q(x) - a| < \varepsilon \text{ and } |L(x) - b| < \varepsilon.$$

The proof of this theorem reduces to the case of dimension 4, and it uses a refinement of Ratner's Orbit closure theorem due to N.Shah, namely Theorem 1.3.7. The condition (1) is a sufficient condition to ensure that we have $\{(Q(x), L(x)) : x \in \mathbb{R}^n\} = \mathbb{R}^2$. The most important obstruction to prove density for pairs is that the identity component of the stabilizer of a pair (Q, L) is no longer maximal among the connected Lie subgroups of $\mathrm{SL}(4, \mathbb{R})$ in contrast with the case of the isotropy groups $\mathrm{SO}(3, 1)^\circ$ or $\mathrm{SO}(2, 2)^\circ$. To overcome this difficulty, Gorodnik considers separately equivalent classes of pair (Q, L) into two types by the following classification:

Let $G = \mathrm{SL}(n, \mathbb{R})$ for any pair (Q, L) there exists a $g \in G$ such that for any $x \in \mathbb{R}^n$ $(Q(x), L(x)) = (Q_0(gx), L_0(gx))$ where the pairs (Q_0, L_0) are given by

$$(Q_0(x), L_0(x)) = (x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2, x_n) \text{ for } p = 1, \dots, n \text{ or}$$

$$(Q_0(x), L_0(x)) = (x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{n-2}^2 + x_{n-1}x_n, x_n) \text{ for } p = 1, \dots, \left[\frac{n-2}{2}\right].$$

The pairs (Q, L) equivalent to the first (resp.second) pair (Q_0, L_0) are said to be of type (I) (resp. type (II)). We define $Q_0^g(x) = Q_0(gx)$ and $L_0^g(x) = L_0(gx)$ for any $g \in G$ and $x \in \mathbb{R}^n$.

Let be given a pair (Q, L) consisting of a non degenerate quadratic form and a nonzero linear form. Then there exists $g \in G$ such that $(Q, L) = (Q_0^g, L_0^g)$ and the stabilizer of the pair (Q, L) is defined by the following subgroup of G ,

$$\mathrm{Stab}(Q, L) = \{h \in G \mid (Q^h, L^h) = (Q, L)\}.$$

Clearly one has $\mathrm{Stab}(Q, L) = g\mathrm{Stab}(Q_0, L_0)g^{-1}$ and we are reduced to study the stabilizer of canonical pairs (Q_0, L_0) . The proof of the Theorem 3.1.3 is divided in two parts following each type and consists of applying Ratner-Shah's theorems 1.3.7 and to study the action of $\mathrm{Stab}(Q_0, L_0)$ on the dual space of \mathbb{C}^4 . Despite the non maximality of the stabilizer, he has able to classify all the complex semisimple Lie algebras in $\mathfrak{sl}(4, \mathbb{C})$. The situation for pairs of type (II) is more complicated than with pairs of type (I) since the dual action of the stabilizer has three irreducible components for the pairs of type (II), instead of two for the pairs of type (I).

3.2 S -arithmetic Oppenheim conjecture after Borel-Prasad

S -arithmetic setting

Let k be a number field and \mathcal{O} the ring of integers of k . For every normalized absolute value $|\cdot|_v$ on k , let k_v be the completion of k at v . In the sequel S is a finite set of places of k containing the set S_∞ of the archimedean ones, k_S the direct sum of the fields $k_s (s \in S)$ and \mathcal{O}_S the ring of S -integers of k (i.e. of elements $x \in k$ such that $|x|_v \leq 1$ for $v \in S$). For s non-archimedean, the valuation ring of the local field k_s is denoted \mathcal{O}_s .

Let F be a quadratic form on k_S^n . Equivalently, F can be viewed as a family $(F_s)(s \in S)$, where F_s is a quadratic form on k_s^n . The form F is non-degenerate if and only if each F_s is non-degenerate. We say that F is isotropic if each F_s is so, i.e. if there exists for every $s \in S$ an element $x_s \in k_s^n - \{0\}$ such that $F_s(x_s) = 0$.

The form F is said to be rational (over k) if there exists a quadratic form F_o on k^n and a unit c of k_S such that $F = c.F_o$, irrational otherwise.

If we take $k = \mathbb{Q}$ and $S = S_\infty$, the quadratic forms are real and in this case we know that:

Any real non-degenerate quadratic form is isotropic if and only if it is indefinite.

Following this observation A.Borel and G.Prasad suggested and proved a generalization of the Oppenheim conjecture in the S -arithmetic setting in [BP92],

Theorem 3.2.1. *Let $n \geq 3$ and F be an isotropic non-degenerate quadratic form on k_S^n . Then the following two conditions are equivalent:*

(i) F is irrational.

(ii) Given $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^n$ such that $0 < |F_s(x)|_v < \varepsilon$ for all $s \in S$.

The proof in [BP92] is modeled on Margulis's original proof of the Oppenheim conjecture. Their proof is quite technical, it involves a lot of different methods and relies heavily on weak and strong approximation property. In contrast with the real case, if $F(\mathcal{O}_S^n)$ accumulates around zero it does not implies density in k_S^2 . This issue is due to the fact that the quotient k_s^*/k_s^{*2} has order 1 or 2 depending on whether s is a real or a complex place and order 4 or 8 when s is non-archimedean. The density of $F(\mathcal{O}_S^n)$ is characterised by the following equivalence:

* The set $F(\mathcal{O}_S^n)$ is dense in k_S if and only if for any $\varepsilon > 0$, and $\overline{c_s} \in k_s^*/k_s^{*2}$, there exist $x \in \mathcal{O}_S^n$ such that $0 < |F_s(x)|_s < \varepsilon$ and such that $F_s(x) \in \overline{c_s}$ for all $s \in S$.

This equivalence has been raised in ([BP92], §6) in order to find a strategy towards density in the absence of the proof of Raghunathan conjecture in the S -arithmetic case at this time. It is still an open problem to show density using the equivalence above i.e. without requiring the use of Ratner's theorems.

The density of $F(\mathcal{O}_S^n)$ in k_S was established in ([B94], §8), using Ratner's rigidity of unipotent S -products.

Theorem 3.2.2. *Let $n \geq 3$ and F be an isotropic non-degenerate quadratic form on k_S^n . Suppose F is irrational, then $F(\mathcal{O}_S^n)$ is dense in k_S .*

We will give the proof of Borel and Prasad and we begin by fixing some notations.

Let $G_s = \mathrm{SL}(n, k_s), H_s = \mathrm{SO}(F_s)$ and the associated S -products $G = \prod_{s \in S} G_s, H = \prod_{s \in S} H_s$. Let \mathcal{G}_s be SL_n viewed as an algebraic group over k_s and \mathcal{H}_s the algebraic group over k_s

such that $\mathcal{H}(k_s) = H_s$.

The proof reduces to the following statement which makes use of Ratner's Orbit Closure Theorem for S -products,

Theorem 3.2.3 ([B94],§8). *If F is irrational, the orbit $H.o$ is dense in Ω .*

The next lemma is useful in order to eliminate irrelevant intermediate subgroups coming from the possible orbit closures of $H.o$

Lemma 3.2.4. *Let E be a field of characteristic zero, \mathfrak{g} a simple Lie algebra over E , $\sigma \neq 1$ an involutive automorphism of \mathfrak{g} and \mathfrak{k} the fixed point set of σ . Assume that \mathfrak{k} is semi-simple. Then any \mathfrak{k} -invariant subspace of \mathfrak{g} containing \mathfrak{k} is equal to \mathfrak{k} or \mathfrak{g} . In particular, \mathfrak{k} is a maximal proper subalgebra of \mathfrak{g} .*

Following the notation of [BT73], we let H_s^+ denote the subgroup of H_s generated by one-dimensional unipotent (hence Ad-unipotent) subgroups.

Proposition 3.2.5. *H_s^+ is a closed and open normal subgroup of finite index of H_s .*

Proof. We divide the proof in three cases:

- If $k_s = \mathbb{C}$, this is immediate, since H_s is semisimple and connected in the usual topology, and in this case we just have $H_s^+ = H_s$.
- If $k_s = \mathbb{R}$, then H_s^+ is the topological identity component of H_s and has index two.
- If k_s be non-archimedean. Let $\tilde{\mathcal{H}}_s$ be the universal covering of H_s , i.e. the spinor group of F_s and $\mu : \tilde{\mathcal{H}}_s \rightarrow \mathcal{H}_s$ the central isogeny. Let $\tilde{H}_s = \tilde{\mathcal{H}}_s(k_s)$. It is known that $\tilde{H}_s = \tilde{H}_s^+$ is generated by one-dimensional unipotent subgroups ([BT73], §6.15), that $\mu(\tilde{H}_s^+) = H_s^+$ ([BT73], §6.3) and $\mu(\tilde{H}_s^+)$ is a normal open and closed subgroup of finite index of H_s ([BT73], §3.20), whence our assertion in that case.

We note that H_s^+ is not compact, since it is of finite index in H_s and the latter, being the orthogonal group of an isotropic form, is not compact. We let \mathfrak{h}_s be the Lie algebra of H_s and N_s the normalizer of \mathfrak{h}_s in G_s , i.e.

$$N_s = \{g \in G_s | \text{Ad}_g(\mathfrak{h}_s) = \mathfrak{h}_s\}$$

Claim. N_s is also the normalizer of H_s or of H_s^+ .

In fact, both groups, viewed as Lie subgroups of G_s , have \mathfrak{h}_s as their Lie algebra, therefore any element $g \in G_s$ normalizing H_s or H_s^+ belongs to N_s . Conversely, since \mathfrak{h}_s is the space of rational points of the Lie algebra of \mathcal{H}_s and is of course Zariski-dense in it, the automorphism $\text{Int}(g) : x \mapsto gxg^{-1}$ of \mathcal{G}_s leaves stable \mathcal{H}_s , hence g normalizes H_s and therefore also H_s^+ and the claim is proved.

Lemma 3.2.6 ([B94],§8). *Assume that H_s^+ has finite index in N_s . Let M be a subgroup of G_s containing H_s^+ . Then either $M = G_s$ or $M \subset N_s$.*

Proof of the lemma. (i) Since H_s^+ has finite index in H_s , it suffices to show that H_s has finite index in N_s . The only quadratic forms on k_s^n invariant under H_s are the multiples of F_s . If $x \in N_s$, then ${}^t x F_s x$ is invariant under H_s , hence of the form $c F_s$ ($c \in k_s^*$). It has the same determinant as F_s ; hence $c^n = 1$, and therefore N_s/H_s is isomorphic to a subgroup of the group of n -th roots of unity.

(ii) Identify F_s to a symmetric, invertible, matrix. Then the map

$$\sigma : x \mapsto F_s \cdot {}^t x^{-1} \cdot F_s^{-1} (x \in G_s)$$

is an automorphism of G_s , obviously of order two, and H_s is the fixed point set of σ .

The differential $d\sigma$ of σ at the origin is an involutive automorphism of \mathfrak{g}_s with fixed point set \mathfrak{h}_s . The group \mathcal{G}_s (resp. \mathcal{H}_s) is simple (resp. semisimple) as an algebraic group; therefore \mathfrak{g}_s (resp. \mathfrak{h}_s) is a simple (resp. semisimple) Lie algebra.

By lemma 3.2.4, any \mathfrak{h}_s -invariant subspace of \mathfrak{g}_s containing \mathfrak{h}_s equal to \mathfrak{h}_s or to \mathfrak{g}_s . Now let M be a subgroup of G_s containing H_s^+ but not contained in N_s . We have to show that $M = G_s$.

Let \mathfrak{g} be the subspace generated by the subalgebras $\text{Ad}(m)(\mathfrak{h}_s)$, ($m \in M$). It is normalized by M , obviously, and in particular by H_s^+ .

Therefore it is \mathfrak{h}_s -invariant. It is not equal to \mathfrak{h}_s , since M is not in N_s . By the previous remark, $\mathfrak{g} = \mathfrak{g}_s$.

There exists therefore a finite set of elements $m_i \in M$ ($1 \leq i \leq d$) such that $\mathfrak{g}_s = \oplus_i \mathfrak{h}_i$ with $\mathfrak{h}_i = \text{Ad}(m_i)(\mathfrak{h}_s)$. The Lie algebra \mathfrak{h}_i is the Lie algebra of $H_i = m_i \cdot H_s^+ \cdot m_i^{-1}$.

Let $Q = H_1 \times \dots \times H_d$ be the product of the H_i 's and

$$\mu : Q \longrightarrow G_s \tag{3.1}$$

$$(h_1, \dots, h_d) \mapsto h_1 \dots h_d. \tag{3.2}$$

It is a morphism of k_s -manifolds, whose image is contained in M . The tangent space at the identity of Q is the direct sum of the \mathfrak{h}_i 's. Therefore the differential $d\mu$ of μ at the identity maps the tangent space to Q onto \mathfrak{g}_s . This implies that $\mu(Q)$ contains an open neighborhood of the identity in G_s (see [Ser92], III, §10.2).

Since it belongs to M , the latter is an open subgroup of G_s . It contains H_s^+ , which is not compact, as noted before, hence is noncompact.

Moreover, it is elementary that $G_s = \text{SL}_n(k_s)$ is generated by the group of unipotent upper triangular matrices and its conjugates. It then follows from Theorem (T) in [Pra] that $M = G_s$.

Proof of the Theorem 3.2.2

Let $\Gamma = \text{SL}_n(\mathcal{O}_S)$. It is viewed as a discrete subgroup of G via the embeddings $\text{SL}_n(k) \hookrightarrow \text{SL}_n(k_s)$. The quotient $\Omega = G/\Gamma$ has finite volume. We let o be the coset Γ in Ω . Let

H^+ be the product of the groups H_s^+ . Since H_s^+ has finite index and is normal, open and closed in H_s , the same is true for H^+ in H , and it is equivalent to prove that $H^+.o$ is dense in Ω .

By the p -adic version of Ratner's closure orbit theorem 1.5.2, there exists a closed subgroup L of G such that $L.o$ is the closure of $H^+.o$ and $L \cap \Gamma$ has finite covolume in L . Let $M_s = L \cap G_s$. It is a closed normal subgroup of L which contains H_s^+ . By Lemma 3.2.6, we have either $M_s \subset N_s$ or $M_s = G_s$. Now let P_s be the projection of L into G_s . It normalizes M_s and contains it. Assume $M_s \subset N_s$. Then $\text{Ad}(g)(g \in P_s)$ leaves invariant the Lie algebra of M_s , which is the same as that of H_s , hence g belongs to N_s . In particular, P_s is closed and open in N_s . If $M_s = G_s$, then $P_s = G_s$. Therefore the product M of the M_s is normal, closed and open, of finite index in the product P of the P_s , and P is closed. We have of course $M \subset L \subset P$. As a consequence, M is normal, open and closed, of finite index, in L . Now define Q_s by the rule: $Q_s = H_s$ if $M_s \subset N_s$, and $Q_s = G_s$ if $M_s = G_s$, and let Q be the product of the Q_s . Then $Q \cap L$ is open and closed, of finite index, in both L and Q . Therefore $Q \cap \Gamma$ has finite covolume in Q . By Proposition 1.2 in [BP92], there exists a k -subgroup \mathcal{Q} of SL_n such that $\mathcal{Q}(k_s) = Q_s$ for every $s \in S$. This shows first of all that either $Q_s = G_s$ for all $s \in S$ or $Q_s = H_s$ for all $s \in S$. In the first case, $L = G$ and $H^+.o$ is dense. We have to rule out the second one.

In that case $H.o$ is closed, $L = H^+$ and $H \cap \Gamma$ has finite covolume in H . Moreover Q is the orthogonal group of a form F_o on k_n , and there exists a unit c of k_S such that $F = c.F_o$, i.e. F is rational over k , contradicting our assumption.

3.3 Quantitative Oppenheim conjecture

The dynamical nature of the proof of the Oppenheim conjecture leads naturally to the question of the uniform distribution of the values of indefinite irrational real quadratic forms at integral points. Let Q be a real irrational indefinite quadratic form in $n \geq 3$ variables, let $a < b$ and let $N_{(a,b)}^Q$ denotes the number of integral points v in a ball of radius T with $a < Q(v) < b$. The Oppenheim conjecture is equivalent to the statement $N_{(a,b)}^Q(T) \rightarrow \infty$ when $T \rightarrow \infty$. The well-known asymptotic

$$|\{v \in \mathbb{Z}^n : \|v\| < T\}| \sim \text{Vol}\{v \in \mathbb{R}^n : \|v\| < T\} \text{ as } T \rightarrow \infty$$

suggests the following estimate

$$\lim_{T \rightarrow \infty} \frac{N_{(a,b)}^Q(T)}{\text{Vol}\{v \in \mathbb{R}^n : a < Q(v) < b, \|v\| < T\}} = 1. \quad (3.3)$$

In fact this estimate is true if one replaces \lim by \liminf , this result is due to Dani and Margulis ([DM93], Corollary 5 (i)) and is known as the lower bound for quantitative version of the Oppenheim conjecture (see 3.3.1 for more precisions) and Eskin-Margulis-Mozes for the upper bounds (see [EMM98], [EMM05]). The result obtained are not only valid for balls B_T but more generally for $T\Omega$ where $\Omega = \{x \in \mathbb{R}^n : \|x\| < \nu(\frac{x}{\|x\|})\}$ a star-shaped open set with ν a continuous positive function on the sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$.

3.3.1 Lower bounds for quantitative Oppenheim conjecture

We review the method of Dani-Margulis which has been successful to prove that for any non degenerate indefinite irrational quadratic form Q , for any $\theta > 0$ there exists $T_0 > 0$ such that for all $T \geq T_0$

$$\#\{x \in \mathbb{Z}^n \cap T\Omega : a < Q(x) < b\} \geq (1 - \theta)\text{Vol}(x \in T\Omega : a < Q(x) < b).$$

More generally they proved a uniform version with respect to a compact subset of $\mathcal{O}(p, n)$ where $\mathcal{O}(p, n)$ is defined to be the set of indefinite quadratic form of signature $(p, n - p)$ with discriminant ± 1 . We give the result as it is in ([DM93], Corollary 5 (i)).

Theorem 3.3.1. *Let \mathcal{K} be a compact subset of $\mathcal{O}(p, n)$. For any $\theta > 0$ there exists a finite subset \mathcal{S} such that each $Q \in \mathcal{S}$ is a scalar multiple of a rational quadratic form and for any compact subset \mathcal{C} of $\mathcal{K} - \mathcal{S}$ there exists $T_0 \geq 0$ such that for all $Q \in \mathcal{C}$ and $T > T_0$,*

$$\#\{x \in \mathbb{Z}^n \cap T\Omega : a < Q(x) < b\} \geq (1 - \theta)\text{Vol}(x \in T\Omega : a < Q(x) < b).$$

The proof of this theorem requires a uniform version of Ratner's equidistribution (see Theorem 1.4.10). However for a single quadratic form, this theorem was proved previously by Dani and Mozes and independently by Ratner but with a positive constant by only using Ratner's equidistribution theorem.

3.3.2 Idea of the proof of Theorem 3.3.1

Let Q_0 be the quadratic form defined by

$$Q_0(x) = 2x_1x_n + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{n-1}^2 \text{ for any } x \in \mathbb{R}^n$$

with respect to the canonical basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n . Clearly $Q_0 \in \mathcal{O}(p, n)$, hence for any $Q \in \mathcal{O}(p, n)$ one can find $g \in G$ such that $Q = Q_0^g$. Let $H = \text{SO}(Q_0)$ the special orthogonal group of Q_0 , the quotient $H \backslash G$ is homeomorphic to $\mathcal{O}(p, n)$. Hence we can always parametrize the elements of $\mathcal{O}(p, n)$ by elements of G via the correspondence $g \in G \leftrightarrow Q = Q_0^g \in \mathcal{O}(p, n)$. Let U be the subgroup of H consisting of all $h \in H$ such that $he_i = e_i$ for all $1 \leq i \leq n-2$, $he_{n-1} \in \text{span}(e_1, e_{n-1})$ and $he_n \in \text{span}(e_1, e_{n-1}, e_n)$. It is straightforward to verify that U is a unipotent one-parameter subgroup in G . Let M be the subgroup of H leaving invariant the subspace $V^+ = \text{span}(e_1 + e_n, e_2, \dots, e_p)$, the latter is a maximal subspace on which Q_0 is positive. Therefore $M \simeq \text{SO}(p-1)$, then it is compact set equipped with a normalised Haar measure denoted by σ . Now let \mathcal{K} be a compact subset of $\mathcal{O}(p, n)$ so that it corresponds to a compact subset K of G such that for any $Q \in \mathcal{K}$ there exists a $g \in \text{Int}K$ such that $Q = Q_0^g$ and such that $MK = K$.

One can find explicitly a finite subset \mathcal{S} of \mathcal{K} such that any $Q \in \mathcal{S}$ is a scalar multiple of a rational form and for any compact subset \mathcal{C} of $\mathcal{K} - \mathcal{S}$ there exists a compact subset C of $K - \cup C_i \Gamma$ such that any $Q \in \mathcal{C}$ is of the form Q_0^g for some $g \in C$ ([DM93], Prop.9.3).

The proof of the theorem relies on the obvious equality, where any B Borel subset of M and any ψ continuous nonnegative function on G/Γ

$$\int_T^{lT} \int_B \sum_{v \in g\mathbb{Z}^n} \psi(u_t m v) d\sigma(m) dt = \int_T^{lT} \int_B \tilde{\psi}(u_t m g \Gamma) d\sigma(m) dt. \quad (3.4)$$

The number $N_{(a,b)}^Q(T)$ can be approximated by the left hand side of (3.4) for suitable choices for B and ψ . The right hand side of (3.4) can be estimated uniformly using the following inequality which is an easy consequence of the Uniform Equidistribution Theorem 1.4.10

Proposition 3.3.2 ([DM93]). *Let φ be a bounded continuous nonnegative function on G/Γ and B a Borel subset of M . Let $\delta > 0$ and $l > 1$ be given. Then there exists a $T' \geq 0$ such that for all $g \in C$ and $T > T'$ we have*

$$\frac{1}{(l-1)T} \int_T^{lT} \int_B \varphi(u_t m g \Gamma) d\sigma(m) dt \geq (1-\delta)\sigma(B) \int_{G/\Gamma} \varphi d\mu. \quad (3.5)$$

3.3.3 Method for the counting

Let $Q \in \mathcal{C}$ as above, hence $Q = Q_0^g$ for some $g \in C$. Denote,

$$N_{Q,\Omega}(a, b, T) = \#\{x \in \mathbb{Z}^n \cap T\Omega : a < Q(x) < b\}.$$

By the previous remark we get

$$N_{Q,\Omega}(a, b, T) = \#\left\{x \in g\mathbb{Z}^n \cap T(g\Omega) : a < Q_0(x) < b\right\}.$$

Put

$$E_{a,b}(T) = \left\{x \in T(g\Omega) : a < Q_0(x) < b\right\}$$

or in other words

$$N_{Q,\Omega}(a, b, T) = \#(E_{a,b}(T) \cap g\mathbb{Z}^n).$$

The strategy consists to cover the domain $E_{a,b}(T)$ using a flow-box Δ w.r.t. U which plays the role of a fundamental domain for the action of H . One has

$$\tilde{\chi}_\Delta(u_t m g \Gamma) = \sum_{v \in g\mathbb{Z}^n} \chi_\Delta(u_t m v) := \#\{(u_t m)^{-1} \Delta \cap g\mathbb{Z}^n\}$$

where χ_Δ is the characteristic function of Δ and $\tilde{\chi}_\Delta$ its Siegel transform.

Using the property of the flow $\{u_t\}$ and the fact that M acts transitively on the level sets of Q_0 , the domain $E_{a,b}(T)$ can be approximated by the following cover

$$E_{a,b}(T) \subset \bigcup_j \bigcup_{t \in [T, lT]} \bigcup_{m \in B_j} (u_t m)^{-1} \Delta$$

for some Borel B_j subsets of M . Consequently the quantity

$$N_{Q,\Omega}(a, b, T) = \#(E_{a,b}(T) \cap g\mathbb{Z}^n)$$

can be approximated by

$$\frac{1}{(l-1)T} \int_T^{lT} \int_B \tilde{\chi}_\Delta(u_t m g \Gamma) d\sigma(m) dt.$$

Finally using the inequality (3.5) and Siegel summation formula we obtain the following

$$\frac{1}{(l-1)T} \int_T^{lT} \int_{B_j} \tilde{\chi}_\Delta(u_t m g \Gamma) d\sigma(m) dt \geq (1-\delta) \sigma(B_j) \lambda(\Delta) \quad (3.6)$$

then taking the sum of the contribution of each of the B_j 's and comparing the right-hand side with the volume of the domain, we obtain lower bounds for a single quadratic form.

Chapter 4

Quantitative lower bounds for Oppenheim conjecture for pairs

4.1 Quantitative lower bounds for pairs in dimension three

The aim of this chapter is to give a quantitative version of density Theorem 3.1.1 for pairs (Q, L) in the spirit of the proof Dani and Margulis. Let us recall what we mean by this. Let $0 \leq a < b$ and $\alpha > 0$ be given. We denote by $\|\cdot\|$ the euclidian norm and by Vol or by λ the Lebesgue measure on \mathbb{R}^n . Let ν be a continuous positive function on the sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$ and let $\Omega = \{x \in \mathbb{R}^n : \|x\| < \nu(\frac{x}{\|x\|})\}$ a star-shaped open set. We are interested in giving a lower estimate for the following quantity

$$\#\{x \in \mathbb{Z}^n \cap T\Omega : a < Q(x) < b, \quad 0 < L(x) < \alpha\} \text{ when } T \longrightarrow \infty.$$

Our main result is to prove that it is possible to find asymptotically exact lower bounds for pairs (Q, L) under the conditions of Theorem 3.1.1 ([DM90], Corollary 2). Let us introduce the real and the integers points of the domain,

$$\mathcal{D}(a, b, \alpha)(\mathbb{R}) = \{x \in \mathbb{R}^n : a < Q(x) < b, \quad 0 < L(x) < \alpha\}$$

and

$$\mathcal{D}(a, b, \alpha)(\mathbb{Z}) = \{x \in \mathbb{Z}^n : a < Q(x) < b, \quad 0 < L(x) < \alpha\}.$$

Since there is no ambiguity one can omit the parameters and simply denote them $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{Z})$ respectively.

The main result of the chapter is the following theorem.

Theorem 4.1.1. *Let Q be a non degenerate indefinite quadratic form and L be a nonzero linear form in real coefficients in three variables. Suppose that the cone $\{Q = 0\}$ intersects the plane $\{L = 0\}$ tangentially and that no linear combination of Q and L^2 is a rational form. Let $0 \leq a < b$ and $\alpha, \theta > 0$ be given. Then there exists $T_0 > 0$ such that for any $T \geq T_0$,*

$$\#\{\mathcal{D}(\mathbb{Z}) \cap T\Omega\} \geq (1 - \theta) \text{Vol}(\mathcal{D}(\mathbb{R}) \cap T\Omega)$$

The proof of this theorem is similar to the proof of Dani-Margulis [DM93], this is essentially due to the fact that the geometry of the problem is similar and also because all the points corresponding to the pairs in the condition of ([DM90], Corollary 2) are generic. In dimension 4, the geometry of the problem is also similar but a new issue appears due to the fact that the stabilizer of the pair (Q, L) is no longer maximal contrary to the case $n = 3$. The existence of non-generic points leads to estimates integral along periodic orbits corresponding to the cusps of the homogeneous space G/Γ (see §6.2).

4.1.1 Remark on the upper bounds in dimension three

The problem of the upper estimate is much more delicate. For a single indefinite quadratic form in $n \geq 3$, it was proved in [EMM98] that for an irrational quadratic form of signature not equal to $(2, 2)$ or $(2, 1)$ that we have exact asymptotic :

$$\#\{\mathcal{D}(\mathbb{Z}) \cap T\Omega\} \sim \text{Vol}(\mathcal{D}(\mathbb{R}) \cap T\Omega) \text{ as } T \rightarrow +\infty.$$

This result fails for low signatures, in our case i.e. for signature $(2, 1)$, there exists irrational forms for which along a sequence T_j , we get the following upper bound,

$$\#\{\mathcal{D}(\mathbb{Z}) \cap T_j\Omega\} > c \text{Vol}(\mathcal{D}(\mathbb{R}) \cap T\Omega) \log(T_j)^{1-\varepsilon} \text{ for } \varepsilon > 0 \text{ small enough.}$$

Such irrational forms must be extremely well approximated by split rational forms (EWAS) as shown in [EMM05]. An example can be given by the irrational forms $Q_\beta(x) = x_1^2 + x_2^2 - \beta x_3^2$, with $\beta > 0$ an irrational number extremely well approximated by rational numbers.

4.2 Geometry and unipotent dynamics in $\mathcal{D}(\mathbb{R})$

Let (Q, L) be a pair in \mathbb{R}^3 equipped with the standard basis $\{e_1, e_2, e_3\}$. Let (Q_0, L_0) be the pair in \mathbb{R}^3 defined by $Q_0(x) = 2x_1x_3 - x_2^2$ and $L_0(x) = x_3$ with respect to this basis. If one has $(Q, L) = (Q_0^g, L_0^g)$ for some $g \in G$ then $\mathcal{D}(\mathbb{R}) = g^{-1}E_{a,b}^\alpha$ where

$$E_{a,b}^\alpha = \{x \in \mathbb{R}^n : a < Q_0(x) < b, 0 < L_0(x) < \alpha\}.$$

Hence we have $\#\{\mathcal{D}(\mathbb{Z}) \cap T\Omega\} = \#\{\mathcal{D}(\mathbb{R}) \cap T\Omega \cap \mathbb{Z}^3\} = \#\{E_{a,b}^\alpha \cap Tg\Omega \cap g\mathbb{Z}^3\}$ and $\text{Vol}(\mathcal{D}(\mathbb{R}) \cap T\Omega) = \text{Vol}(E_{a,b}^\alpha \cap Tg\Omega)$ since the Lebesgue measure is G -invariant.

Let S^2 denote the unit sphere in \mathbb{R}^3 for the euclidian norm, for each subset B of S^2 let $c(B)$ denote the cone over B with vertex at 0, namely $c(B) = \{\lambda x \mid \lambda \geq 0, x \in B\}$. We denote by S^+ the restriction of the unit sphere to the space $\mathbb{R}_+^3 = \{x \in \mathbb{R}^3 \mid x_i \geq 0\}$ equipped with the Riemannian metric induced by the unit sphere and by $A(a, b)$ the annular region of \mathbb{R}^3 given by the spherical shells $\{\lambda x \mid a \leq \lambda \leq b, x \in S^+\}$.

Recall that $\Omega = \{x \in \mathbb{R}^n : \|x\| < \nu(\frac{x}{\|x\|})\}$, then $g\Omega = \{x \in \mathbb{R}^n : \|x\| < \nu_g(\frac{x}{\|x\|})\}$ where the

function $\nu_g : S^+ \rightarrow \mathbb{R}^+$ is given by $\nu_g(x) = \frac{1}{\|g^{-1}x\|} \nu\left(\frac{g^{-1}x}{\|g^{-1}x\|}\right)$.

For each $d \geq 0$ let $E_d = \{x \in S^+ \mid Q_0(x) = d\}$ and let $v_0 = \frac{e_1 + e_3}{\sqrt{2}}$, then $Q_0(v_0) = \|v_0\| = 1$.

Let M be the subgroup of $\text{SO}(Q_0)$ which leaves invariant the axis of the level sets of Q_0 namely $\mathbb{R}v_0$, then $M \simeq \text{SO}(2)$. Hence M is compact, it also leaves invariant each E_d and it acts transitively on each of them. We denote by σ_d the M -invariant probability measure on E_d for each $d \geq 0$.

4.2.1 Cover of S^+

Let fix an arbitrary $0 < \omega < 1$ and choose subsets B_1, \dots, B_k of M in the following way: Firstly we cover S^+ with subsets Φ_0, \dots, Φ_k as follows:

1. $\bigcup_{j=0}^k \overline{\Phi_j} = S^+$ so that Φ_1, \dots, Φ_k are mutually disjoint and $\bigcup_{j=1}^k \overline{\Phi_j}$ contains a neighbourhood of E_0
2. Each Φ_j for $1 \leq j \leq k$ is bounded by a piecewise smooth curve which is transverse to E_0 at all points of intersection
3. For each $j \in \{1, \dots, k\}$ and for any $x, x' \in \Phi_j$ we assume that $\frac{\nu_g(x)}{\nu_g(x')} < \omega^{-1}$.

Transversality condition (2) implies that there exists $d_0 > 0$ and compacts subsets Ψ_1, \dots, Ψ_k of Φ_1, \dots, Φ_k respectively such that for all $1 \leq j \leq k$ and $d \in [0, d_0]$,

$$\sigma_d(\Psi_j) > \omega \sigma_d(\Phi_j). \quad (4.1)$$

It is clear that there exists β large enough so that for any $r \geq \beta$,

$$E_{a,b}^\alpha \cap A(r, \infty) \subseteq c\left(\bigcup_{0 < d \leq d_0} E_d\right).$$

Using inequality (4.1) we deduce that for any $r \geq \beta$, and for all $1 \leq j \leq k$, $\kappa > 1$

$$\lambda(E_{a,b}^\alpha \cap A(r, \kappa r) \cap c(\Psi_j)) > \omega \lambda(E_{a,b}^\alpha \cap A(r, \kappa r) \cap c(\Phi_j)). \quad (4.2)$$

Replacing the initial choice if necessary, we assume that

$$\Phi_0 \supseteq \bigcup_{d \geq d_0} E_d \text{ and that } \overline{\Phi_0} \cap E_0 = \emptyset.$$

Let $\varepsilon > 0$ be such that $\max_{1 \leq j \leq k} d(\Phi_j, \Psi_j) < 2\varepsilon$. We define B_1, \dots, B_k subsets of M by

$$B_j = \{m \in M \mid B(e_1, \varepsilon) \subseteq m\Phi_j\} \text{ for each } 1 \leq j \leq k$$

where the open neighbourhood $B(., \varepsilon)$ are taken with respect the metric d induced on S^+ .

4.2.2 Unipotent flows on level sets

Let $0 < a < b$, $0 < \omega < 1$ and $\varepsilon > 0$ be given as above. Let $I = [\sqrt{a}, \sqrt{b}]$, then $Q_0(Iv_0) \in (a, b)$, that is $Iv_0 \subseteq E_{a,b} \cap \mathcal{P}$ where $Iv_0 = \{\lambda v_0 \mid \lambda \in I\}$ and $\mathcal{P} = \{x \in \mathbb{R}^3 \mid x_2 = 0\}$. Let $U = \{u_t \mid t \in \mathbb{R}\}$ where

$$u_t = \exp(Nt) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \text{ with } N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $\{u_t\}$ is a unipotent one-parameter subgroup in G which leaves invariant the pair (Q_0, L_0) , that is $Q_0(u_t x) = Q_0(x)$ and $L_0(u_t x) = L_0(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$. Now let $0 < \tau < a/2$ and let $I(\tau)$ be the τ -neighbourhood of Iv_0 in $E_{a,b} \cap \mathcal{P}$. We define the twisted box generated by $I(\tau)$ under the flow $\{u_t\}$ in \mathbb{R}^3 of length 2τ to be

$$\Delta = \Delta(a, b; \tau) = \{u_s v \mid |s| \leq \tau, v \in I(\tau)\}$$

It is clear that Δ is a compact neighbourhood of Iv_0 in $E_{a,b} \cap A(\omega a, \omega^{-1}b)$ transverse to \mathcal{P} and that it does not contain any fixed point of U . Let $l > 1$ and $T > 0$, for each $1 \leq j \leq k$ we put

$$S_j(T) = \bigcup_{T \leq t \leq lT} \bigcup_{m \in B_j} (u_t m)^{-1} \Delta.$$

The following proposition is the analog of Proposition 9.7 ([DM93]) with $E_{a,b}^\alpha$ instead of $E_{a,b}$

Proposition 4.2.1. *There exists $T_0 > 0$ such that for all $T \geq T_0$, for all $l > 1$ and for each $1 \leq j \leq k$ the set $S_j(T)$ is contained in*

$$A((1-\varepsilon)a\omega T^2/2, l^2(1+\varepsilon)\omega^{-1}bT^2/2) \cap c(\Phi_j).$$

and contains

$$E_{a,b}^\alpha \cap A((1+\varepsilon)bT^2/2, l^2(1-\varepsilon)aT^2/2) \cap c(\Psi_j).$$

Proof. Suppose $w \in Iv_0$, for all $t \in \mathbb{R}$ we have

$$\frac{u_t w}{\|u_t w\|} = \frac{1}{\sqrt{(2+2t+t^4/2)}} \begin{pmatrix} 1+t^2/2 \\ t \\ 1 \end{pmatrix} \text{ then } \lim_{t \rightarrow \infty} \frac{u_t w}{\|u_t w\|} = e_1$$

Hence there exists $T_0 > 0$ such that for all $|t| \geq T_0$ and for any $w \in \Delta$,

$$\frac{u_t w}{\|u_t w\|} \in B(e_1, \varepsilon) \text{ and } (1-\varepsilon)\frac{t^2}{2}\|w\| \leq \|u_t w\| \leq (1+\varepsilon)\frac{t^2}{2}\|w\|.$$

Suppose $v \in S_j(T)$ some $T \geq T_0$, we have $\|v\| = \|mv\| = \|u_{-t}w\|$ and by the last inequality we get $(1 - \varepsilon)\frac{T^2}{2}\|w\| \leq \|u_{-t}w\| \leq (1 + \varepsilon)l^2\frac{T^2}{2}\|w\|$. Hence we obtain $(1 - \varepsilon)\frac{T^2}{2}a\omega \leq \|v\| \leq (1 + \varepsilon)l^2\frac{T^2}{2}b\omega^{-1}$. Moreover, $\frac{mv}{\|v\|} = \frac{mv}{\|mv\|} = \frac{u_{-t}w}{\|u_{-t}w\|} \in B(e_1, \varepsilon)$ hence $\frac{v}{\|v\|} \in m^{-1}B(e_1, \varepsilon) \subseteq \Phi_j$, that is $v \in c(\Phi_j)$.

Suppose also that $T_0 > 2/\sqrt{\varepsilon}$ and let $v \in E_{a,b}^\alpha \cap A((1 + \varepsilon)bT^2/2, l^2(1 - \varepsilon)aT^2/2) \cap c(\Psi_j)$. Let $w \in Iv_0$ be such that $Q_0(w) = Q_0(v)$. Then $\|v\| \geq (1 + \varepsilon)bT^2/2 \geq b \geq \|w\|$ since $\|u_{-t}w\|$ increases continuously to infinity there exists $t \geq 0$ such that $\|u_{-t}w\| = \|v\|$. Hence there exists $m \in M$ such that $u_{-t}w = mv$. Since $\|v\| \leq l^2(1 - \varepsilon)aT^2/2$ we have $t \leq lT$. Similarly $t \geq T$ since $\|u_{-t}w\|^2 \geq b^2(1 + (1 + t^2/2)^2)$ and $\|v\| \geq (1 + \varepsilon)bT^2/2$ using the condition $T_0 > 2/\sqrt{\varepsilon}$. For $t \geq T$, $\frac{mv}{\|v\|} = \frac{mv}{\|mv\|} = \frac{u_{-t}w}{\|u_{-t}w\|} \in B(e_1, \varepsilon)$ and $m^{-1}e_1 \in B(\frac{v}{\|v\|}, \varepsilon)$, since $v \in c(\Psi_j)$ we get $m^{-1}B(e_1, \varepsilon) = B(m^{-1}e_1, \varepsilon) \subseteq B(\frac{v}{\|v\|}, 2\varepsilon) \subseteq \Psi_j$ hence $m \in B_j$.

We have also the following lemma which is Lemma 9.8 in [DM93]

Lemma 4.2.2. *Let $l > 1$, $0 < \delta < l - 1$ and $0 < a < b$ such that $b/a < (1 + \delta)$ be given. Then there exists $0 < \tau < \delta$ and $T_0 \geq 0$ such that following holds: if for $T > 0$,*

$$E(T) = \{x \in \mathbb{R}^3 \mid u_t x \in \Delta(\tau) \text{ for some } t \in [T, lT]\}$$

then for all $x \in \mathbb{R}^3$ and $T \geq T_0$ we have:

1. $\sigma(\{m \in M \mid mx \in E(T)\}) \leq l^2(1 + 2\delta)\lambda(E(T))/\lambda(ME(T))$
2. For each $1 \leq j \leq k$, $\lambda(S_j(T)) \leq (1 + \delta)\sigma(B_j)\lambda(ME(T))$.

4.2.3 Cover of $E_{a,b}^\alpha \cap r(g\Omega)$

The condition (3) at section §4.2.1 allows us to find a finite cover of $g\Omega$ in terms of the Φ_j 's, there exists $\rho_0, \dots, \rho_k > 0$ such that:

$$\bigcup_{j=1}^k A(0, \rho_j) \cap c(\Phi_j) \subseteq g\Omega \subseteq \bigcup_{j=0}^k A(0, \omega^{-1}\rho_j) \cap c(\Phi_j).$$

Hence we obtain the following cover of $E_{a,b}^\alpha \cap r(g\Omega)$ for any $r \geq 0$,

$$\bigcup_{j=1}^k A(0, r\rho_j) \cap E_{a,b}^\alpha \cap c(\Phi_j) \subseteq E_{a,b}^\alpha \cap r(g\Omega) \subseteq \bigcup_{j=0}^k A(0, \omega^{-1}\rho_j) \cap E_{a,b}^\alpha \cap c(\Phi_j).$$

One can notice that the right hand-side has a extra term coming with the index $j = 0$, this obstruction is superficial because $\overline{\Phi_0}$ is chosen to be disjoint from E_0 and in this case $c(\Phi_0)$ meets $E_{a,b}^\alpha$ in a bounded volume area. Hence

$$E_{a,b}^\alpha \cap c(\Phi) \subseteq A(0, \rho') \quad (4.3)$$

for some ρ' large enough. We finally obtain a cover of $E_{a,b}^\alpha \cap r(g\Omega)$ in terms of the Φ_j 's for $1 \leq j \leq k$.

Let us finish this section with the following lemma, concerning the comparison of the volume of elements of the cover, such as in [DM93] mentioned as formula (9.6).

Lemma 4.2.3. *Let $l > 1$, there exists $0 < \zeta < 1$ such that for any open subset $\Phi \subseteq S^2$ there exists $r_0 \geq 0$ such that for all $r \geq r_0$*

$$l\lambda(E_{a,b}^\alpha \cap A(\zeta^{-1}r, \zeta l^2 r) \cap c(\Phi)) \geq \lambda(E_{a,b}^\alpha \cap A(r, l^2 r) \cap c(\Phi)) \geq l^{-1}\lambda(E_{a,b}^\alpha \cap A(\zeta r, \zeta^{-1} l^2 r) \cap c(\Phi)).$$

4.3 Proof of Theorem 4.1.1

Proof. We follow exactly the same strategy used in ([DM93], Cor. 5 (i)). Let $\theta > 0$ and $l > 1$ be such that $l^{-4} > (1 - \theta/4)$. We can assume that $b/a < 1 + \delta$ by taking the sum over intervals of size less than $1 + \delta$ if necessary. Let $\zeta \in (0, 1)$ as in the lemma above and let $0 < \omega < 1$ be chosen to verify $\omega^6 > \max(\zeta, 1 - \theta/4)$ so that the previous lemma apply for ω instead of ζ . Choose also δ to be such that $2\delta < 1 - \omega$ and $\delta < l - 1$. Let $\tau > 0$ such that Lemma 4.2.2 holds for l, δ for some T_0 and let $\Delta = \Delta(\tau)$. Let $\chi = \chi_\Delta$ be the characteristic function of Δ and denote by $\tilde{\chi}$ the Siegel transform of χ , that is the function defined in G/Γ by :

$$\tilde{\chi}(g\Gamma) = \sum_{x \in g\mathbb{Z}^3} \chi(x) \text{ for any } g \in G.$$

Since $\chi \in L^1(\mathbb{R}^3)$, following Siegel we have $\tilde{\chi} \in L^1(G/\Gamma)$ and $\int_{G/\Gamma} \tilde{\chi} d\mu = \int_{\mathbb{R}^3} \chi d\lambda$. Let φ be a real-valued bounded continuous function on G/Γ such that

$$0 \leq \varphi \leq \tilde{\chi} \text{ and } \int_{G/\Gamma} \varphi d\mu \geq \omega \int_{G/\Gamma} \tilde{\chi} d\mu = \omega \lambda(\Delta).$$

Since $g\Gamma$ is U -generic after theorem 3.1.1 (see [DM90], Corollary 2), we can apply Proposition 3.3.2 to φ , there exists $T_1 \geq T_0$ such that for all $1 \leq j \leq k$ and for all $T > T_1$:

$$\frac{1}{(l-1)T} \int_T^{lT} \int_{B_j} \varphi(u_t m g \Gamma) d\sigma(m) dt \geq (1 - \delta) \sigma(B_j) \int_{G/\Gamma} \varphi d\mu \geq \omega^2 \sigma(B_j) \lambda(\Delta).$$

We have the following immediate identity coming from the definition

$$\frac{1}{(l-1)T} \int_T^{lT} \int_{B_j} \tilde{\chi}_\Delta(u_t m g \Gamma) d\sigma(m) dt = \frac{1}{(l-1)T} \int_T^{lT} \int_{B_j} \sum_{x \in g\mathbb{Z}^3} \chi(u_t m x) d\sigma(m) dt.$$

Using the previous inequality and the fact that $\varphi \leq \tilde{\chi}$, we obtain for $T \geq T_1$

$$\frac{1}{(l-1)T} \int_T^{lT} \int_{B_j} \sum_{x \in g\mathbb{Z}^3} \chi(u_t m x) d\sigma(m) dt \geq \omega^2 \sigma(B_j) \lambda(\Delta). \quad (4.4)$$

In the other hand, by definition Δ is constituted by U -orbits of length 2τ , thus

$$\begin{aligned} \frac{1}{(l-1)T} \int_T^{lT} \int_{B_j} \sum_{x \in g\mathbb{Z}^3} \chi(u_t m x) d\sigma(m) dt &\leq \frac{2\tau}{(l-1)T} \int_{B_j} \sum_{x \in g\mathbb{Z}^3 \cap S_j(T)} \chi(u_t m x) d\sigma \\ &\leq \frac{2\tau}{(l-1)T} \sum_{x \in g\mathbb{Z}^3 \cap S_j(T)} \sigma\{m \in M \mid mx \in E(T)\} \\ &\leq \frac{2\tau}{(l-1)T} \#(S_j(T) \cap g\mathbb{Z}^3) \sup_{x \in S_j(T)} \sigma\{m \in M \mid mx \in E(T)\} \\ &\leq l^2 \omega^{-1} \#(S_j(T) \cap g\mathbb{Z}^3) \frac{2\tau}{(l-1)T} \frac{\lambda(E(T))}{\lambda(ME(T))} \text{ by inequality (1) of Lemma 4.2.2.} \end{aligned}$$

Using the fact that $\lambda(E(T)) = \frac{2\tau + (l-1)T}{2\tau} \lambda(\Delta)$, the last term above is at least

$$\leq l^2 \omega^{-1} (1 + 2\tau) \#(S_j(T) \cap g\mathbb{Z}^3) \frac{\lambda(\Delta)}{\lambda(ME(T))} \text{ for } T > \max((l-1)^{-1}, T_1).$$

Comparing with the inequality 4.4, using the fact that $\tau < \delta$, we obtain

$$\#(S_j(T) \cap g\mathbb{Z}^3) \geq l^{-2} \omega^4 \lambda(ME(T)) \sigma(B_j)$$

and using inequality (2) of Lemma 4.2.2 and the fact that $(1 + \delta) \geq \omega$ we arrive to

$$\#(S_j(T) \cap g\mathbb{Z}^3) \geq l^{-2} \omega^5 \lambda(S_j(T)).$$

By the double inclusion of Proposition 4.2.1, there exists $T_2 > T_1 + (l-1)^{-1}$ such that for all $T \geq T_2$ and each $1 \leq j \leq k$

$$\begin{aligned}
& \#(E_{a,b}^\alpha \cap A((1-\delta)a\omega T^2/2, l^2(1+\delta)\omega^{-1}bT^2/2) \cap c(\Phi_j) \cap g\mathbb{Z}^3) \\
& \geq \#(S_j(T) \cap g\mathbb{Z}^3) \geq l^{-2}\omega^5\lambda(S_j(T)) \\
& \geq l^{-2}\omega^5\lambda(E_{a,b}^\alpha \cap A((1+\delta)bT^2/2, l^2(1-\delta)aT^2/2) \cap c(\Psi_j)).
\end{aligned}$$

Using inequality (4.2) of the previous section, there exists $T_3 > T_2$ such that for all $T > T_3$ the last term above is at least:

$$l^{-2}\omega^6\lambda(E_{a,b}^\alpha \cap A((1+\delta)bT^2/2, l^2(1-\delta)aT^2/2) \cap c(\Phi_j)).$$

Using the Lemma 4.2.3 and the choice of δ there exists $T_4 \geq T_3$ such that for all $T \geq T_4$, the last term above is at least:

$$l^{-4}\omega^6\lambda(E_{a,b}^\alpha \cap A((1-\delta)\omega aT^2/2, l^2(1+\delta)w^{-2}bT^2/2) \cap c(\Phi_j)).$$

We decompose $A(0, r)$ for $r \geq r'$ where $r' = l^2(1+\delta)w^{-1}bT_4^2/2$ as follows, we construct a finite sequence r_1, \dots, r_p verifying $r_p < \dots < r_1 = r$ and $r_p \leq r'$ such that:

Each interval (r_i, r_{i+1}) is of the form $((1-\delta)\omega aT^2/2, l^2(1+\delta)w^{-1}bT^2/2)$ for $T \geq T_4$.

Using the fact that $l^{-4}\omega^6 \geq (1-\theta/2)$, we obtain for all $1 \leq i \leq p-1$

$$\#(E_{a,b}^\alpha \cap A(r_{i+1}, r_i) \cap c(\Phi_j) \cap g\mathbb{Z}^3) \geq (1-\theta/2)\lambda(E_{a,b}^\alpha \cap A(r_{i+1}, w^{-1}r_i) \cap c(\Phi_j)).$$

By summing for for all $1 \leq i \leq p-1$

$$\begin{aligned}
\#(E_{a,b}^\alpha \cap A(0, r) \cap c(\Phi_j) \cap g\mathbb{Z}^3) & \geq (1-\theta/2)\lambda(E_{a,b}^\alpha \cap A(r_p, w^{-1}r) \cap c(\Phi_j)) \\
& \geq (1-\theta/2)\lambda(E_{a,b}^\alpha \cap A(r', w^{-1}r) \cap c(\Phi_j)).
\end{aligned}$$

Using the cover of $r(g\Omega)$, we can replace r by $r\rho_j$ for each $1 \leq j \leq k$ in the previous inequality, hence for any $r \geq \max_{0 \leq j \leq k} \{r'\rho_j^{-1}\}$ we obtain

$$\#(E_{a,b}^\alpha \cap r(g\Omega) \cap g\mathbb{Z}^3) \geq (1-\theta/2) \sum_{j=1}^k \lambda(E_{a,b}^\alpha \cap A(0, r\rho_j) \cap c(\Phi_j))$$

$$\begin{aligned}
&\geq (1 - \theta/2) \sum_{j=1}^k \lambda(E_{a,b}^\alpha \cap A(r', \omega^{-1}r\rho_j) \cap c(\Phi_j)) \\
&\geq (1 - \theta/2) \left\{ \left(\sum_{j=0}^k \lambda(E_{a,b}^\alpha \cap A(r', \omega^{-1}r\rho_j) \cap c(\Phi_j)) \right) - \lambda(E_{a,b}^\alpha \cap A(r', \omega^{-1}r\rho_0) \cap c(\Phi_0)) \right\}.
\end{aligned}$$

Using the hypothesis on Φ_0 , we saw in Section §4.2.3 that there exists some $\rho' \geq r'$ such that $E_{a,b}^\alpha \cap c(\Phi_0) \subseteq A(0, \rho')$, hence the last term above is at least

$$(1 - \theta/2) \left\{ \lambda(E_{a,b}^\alpha \cap r(g\Omega)) - \lambda(E_{a,b}^\alpha \cap A(0, \rho')) \right\}.$$

Then there exists $r_0 \geq \max_{0 \leq j \leq k} \{r' \rho_j^{-1}\}$ such that

$$\lambda(E_{a,b}^\alpha \cap A(0, \rho')) \leq \frac{\theta}{2} \lambda(E_{a,b}^\alpha \cap r_0(g\Omega)).$$

Finally we obtain that for any $r \geq r_0$ we have

$$\#(E_{a,b}^\alpha \cap r(g\Omega) \cap g\mathbb{Z}^3) \geq (1 - \theta) \lambda(E_{a,b}^\alpha \cap r\Omega)$$

and using the fact that $\mathcal{D}(\mathbb{R}) = g^{-1}E_{a,b}^\alpha$ we arrive to

$$\#(\mathcal{D}(\mathbb{Z}) \cap r\Omega) \geq (1 - \theta) \lambda(\mathcal{D}(\mathbb{R}) \cap r\Omega).$$

This finishes the proof of the Theorem.

Chapter 5

Values of pairs involving one quadratic form and one linear form at S -integral points

5.1 Introduction

The aim of this chapter is to give a S -arithmetic generalisation of the result below is due to A.Gorodnik [Gor04], following the spirit of the proof of Borel and Prasad of Theorem 3.2.1. In dimension 3, the S -arithmetic generalisation of Theorem 3.1.1 is an analogue of Theorem 3.2.2 where the orthogonal group is replaced by the connected component stabiliser of the pair. Indeed, the classification of the intermediate subgroups leads to the only possibility that the stabiliser should be maximal among subgroups of $\mathrm{SL}(3, k_S)$. In dimension greater than 3, the situation is more complicated due to the non-maximality of the stabilizer, however by using topological rigidity of unipotent orbits and a subtle classification of intermediate subgroups, A.Gorodnik arrives to the following generalisation of the Oppenheim conjecture for pairs,

Theorem 5.1.1 (Gorodnik). *Let $F = (Q, L)$ be a pair of a quadratic form Q and L be a nonzero linear form in dimension $n \geq 4$ satisfying the the following conditions*

1. Q is nondegenerate
2. $Q|_{L=0}$ is indefinite
3. No linear combination of Q and L^2 is rational

Then the set $F(\mathbb{Z}^n)$ is dense in \mathbb{R}^2 .

The proof of this theorem reduced to the case of the dimension 4. The condition (1) is a sufficient condition to ensure that we have $F(\mathbb{R}^n) = \mathbb{R}^2$ and this is a conjecture that this

condition can be weakened in order to make it necessary. The most important obstruction to prove density for pairs is that the identity component of the stabilizer of a pair (Q, L) is no longer maximal among the connected Lie subgroups of $\mathrm{SL}(4, \mathbb{R})$ in contrast with the case of the isotropy groups $\mathrm{SO}(3, 1)^\circ$ or $\mathrm{SO}(2, 2)^\circ$.

Let (Q, L) be a pair consisting of a non degenerate quadratic form and a nonzero linear form there exists $g \in G$ such that $(Q, L) = (Q_0^g, L_0^g)$, the stabilizer of the pair (Q, L) is defined by the following subgroup of G ,

$$\mathrm{Stab}(Q, L) = \{h \in G \mid (Q^h, L^h) = (Q, L)\}$$

Clearly one has $\mathrm{Stab}(Q, L) = g\mathrm{Stab}(Q_0, L_0)g^{-1}$ and we are reduced to study the stabilizer of canonical pairs (Q_0, L_0) . The pairs such that $Q|_{L=0}$ is nondegenerate (resp. degenerate) are said be of type (I) (resp. II). The proof of the Theorem 5.1.1 is divided in two parts following each type and consists to apply Ratner's orbit closure theorem, and to study the action of $\mathrm{Stab}(Q_0, L_0)$ on the dual space of \mathbb{C}^4 . A remarkable fact is that the density is proved without showing the density of the orbit closure of the stabilizer in the homogeneous space, indeed if the intermediate subgroup has non trivial irreducible components the orbit of this subgroup is closed in G/Γ . However, we are hopefully able to classify all the complex semisimple Lie algebras in $\mathfrak{sl}(4, \mathbb{C})$, and check density case by case using the constrain on rationality given by the condition (3). The situation for pairs of type (II) is more complicated than with pairs of type (I) since the dual action of the stabilizer has three irreducible components for the pairs of type (II), instead of two for the pairs of type (I).

We are going to show an S -arithmetic generalisation of these results. Our proof is influenced by the work of Borel-Prasad on the generalised the Oppenheim conjecture for quadratic forms and by Gorodnik's proof of Theorem 5.1.1.

5.2 S -arithmetic Oppenheim type problem for pairs

5.2.1 S -arithmetic setting

Let us recall what we mean by S -arithmetic setting by fixing some notations. Let k be a number field, that is a finite extension of \mathbb{Q} and let \mathcal{O} be the ring of integers of k . For every normalised absolute value $|\cdot|_s$ on k , let k_s be the completion of k at s , we identify s with the specific absolute value $|\cdot|_s$ on k_s defined by the formula $\mu(a\Omega) = |a|_s \mu(\Omega)$, where μ is any Haar measure on the additive group k_s , $a \in k_s$ and Ω is a measurable subset of k_s of finite measure. We denote by Σ_k the set of places of k .

In the sequel S is a finite set of Σ_s which contains the set S_∞ of archimedean places¹, k_S the direct sum of the fields k_s ($s \in S$) and \mathcal{O}_S the ring of S -integers of k (i.e. of elements $x \in k$ such that $|x|_s \leq 1$ for $s \notin S$). For s non-archimedean, the valuation ring of the local

¹Note that if $s \notin S$, since $S \supset S_\infty$ s is necessarily nonarchimedean !

field k_s is defined to be $\mathcal{O}_s = \{x \in k \mid |x|_s \leq 1\}$.

In all the statements of the article, without loss out generality one can replace k by \mathbb{Q} but for sake of completeness we choose to work with number fields.

Let (Q, L) be a pair consisting on one quadratic form and one nonzero linear form on k_S^n . Equivalently, (Q, L) can be viewed as a family $(Q_s, L_s)(s \in S)$, where Q_s is a quadratic form on k_s^n and L_s a nonzero linear form on k_s^n . The form Q is non-degenerate if and only each Q_s is non-degenerate. We say that Q is isotropic if each Q_s is so, i.e. if there exists for every $s \in S$ an element $x_s \in k_s^n - \{0\}$ such that $Q_s(x_s) = 0$. For any quadratic form Q , we denote by $\text{rad}(Q)$ (resp. $c(Q)$) the radical (resp. the isotropy cone) of Q , by definition Q is nondegenerate (resp. isotropic) if and only if $\text{rad}(Q) \neq 0$ (resp. $c(Q) \neq 0$). The form Q is said to be rational (over k) if there exists a quadratic form Q_0 on k^n and a unit c of k_S such that $Q = c.Q_0$, irrational otherwise. For any $s \in S$ let $\overline{k_s}$ denotes an algebraic closure of k_s . If G is a locally compact group, G° denotes the connected component of the identity in G .

5.2.2 Main result

Let $F := F_S = (Q_s, L_s)_{s \in S}$ be a pair given on k_S^n and let $(a, b) \in k_S^2$. We are interested in finding sufficient conditions which guarantees the existence of nontrivial S -integral solutions $x \in \mathcal{O}_S^n$ of the following simultaneous diophantine problem

$$(A_S) \text{ For any } \varepsilon > 0, |Q_s(x) - a_s|_s < \varepsilon \text{ and } |L_s(x) - b_s|_s < \varepsilon \text{ for each } s \in S.$$

Obviously as in the real case, we need to find sufficient conditions on F so that the set $F(\mathcal{O}_S^n)$ would be dense in k_S^2 . One have to be careful since the condition (A_S) is not equivalent to density (see [BP92], §6).

Our main results give the required conditions for assertion (A_S) to hold for $(a, b) = (0, 0)$, namely that $F(\mathcal{O}_S^n)$ is not discrete around the origin in k_S^2 . It may be seen as an S -arithmetic version of Theorem 5.1.1

Theorem 5.2.1. *Let $F = (Q, L)$ be a pair consisting of a quadratic form Q and L be a nonzero linear form with coefficients in k_S in dimension $n \geq 4$ satisfying the the following conditions:*

1. Q is nondegenerate
2. $Q|_{L=0}$ is isotropic i.e. $\{Q_s = 0\} \cap \{L_s = 0\} \neq \{0\}$ for all $s \in S$
3. For each $s \in S$, the forms $\alpha_s Q_s + \beta_s L_s^2$ are irrational for any α_s, β_s in k_s such that $(\alpha_s, \beta_s) \neq (0, 0)$

then for any $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^n - \{0\}$ such that

$$|Q_s(x)|_s < \varepsilon \text{ and } |L_s(x)|_s < \varepsilon \text{ for each } s \in S$$

We also obtain the same result when we relax the condition (3) of the previous theorem by the following weaker one:

Theorem 5.2.2. *Let $Q = (Q_s)_{s \in S}$ be a quadratic form on k_S^n and $L = (L_s)_{s \in S}$ be a linear form on k_S^n with $n \geq 4$. Suppose that the pair $F = (Q, L)$ satisfies the following conditions,*

1. Q is nondegenerate
2. $Q|_{L=0}$ is nondegenerate and isotropic
3. The quadratic form $\alpha Q + \beta L^2$ is irrational for any units α, β in k_S such that $(\alpha, \beta) \neq (0, 0)$

then for any $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^n - \{0\}$ such that

$$|Q_s(x)|_s < \varepsilon \text{ and } |L_s(x)|_s < \varepsilon \text{ for each } s \in S$$

5.2.3 Remarks.

(a) These theorems reduce to dimension 4, (see § 5.3). The key is the use of the weak approximation in k_S following the idea of Borel-Prasad ([BP92], Proposition 1.3) and an argument of Ellenberg-Venkatesh ([EV08], Lemma 2).

(b) According to the *Lefschetz principle* all the results for Lie groups over complex numbers, are also valid for any algebraically closed field of characteristic zero. The following results are proven for the field of complex numbers in [Gor04], all the proofs are the same for the algebraic closures of nonarchimedean local fields. It is not difficult to see applying Ratner's rigidity, that the density of theorem 5.2.1 holds by using the analog of the original method of Gorodnik [Gor04] in the S -arithmetic setting.

(c) Even if we assume that $\alpha Q + \beta L^2$ is irrational, it can be possible that $\alpha_s Q_s + \beta_s L_s^2$ is rational for some place s , in this case applying Ratner's rigidity is not relevant anymore. To carry out this situation we adapt the strategy of Borel and Prasad ([BP92], § 5.6) to obtain the complete picture. The proof of theorem 5.2.2 (see § 5.6) uses strong approximation in algebraic groups (see § 5.4.2) and some elementary results of geometry of numbers.

(d) Unfortunately we are not able to show the density of $F(\mathcal{O}_S^n)$ under the conditions of Theorem 5.2.2 using our method. We are also even unable to show that $|Q_s(x)|_s$ and $|L_s(x)|_s$ are both nonzero for any $s \in S$ and $x \in \mathcal{O}_S^n$ as in the conclusion of Theorem 5.2.1. We discuss this issue in § 5.6.1.

(e) One can hope to relax condition (2) by only asking $\alpha Q + \beta L^2$ to be isotropic as it is conjectured by Gorodnik (see [Gor04], Conjecture 15]) and still open until now. The major issue is that reduction to lower dimension fails to hold (see § 5.6.1).

5.3 Weak approximation and reduction to the dimension 4

The number fields satisfy a nice *local-global principle* called the weak approximation which can be seen as a refinement of the Chinese remainder theorem.

Theorem 5.3.1. *Let S be a finite set of Σ_k . Let be given $\alpha_s \in k_s$ for each $s \in S$. Then there exists an $\alpha \in k$ which is arbitrarily close to α_s for all $s \in S$ with respect to the s -adic topology.*

Proof. (See e.g. [L], Theorem 1, p.35)

One can reformulate this theorem as follows: the diagonal embedding $k \rightarrow \prod_{s \in S} k_s$ is dense, the product being equipped with the product of the s -adic topologies.

Proposition 5.3.2. *Let $F = (Q, L)$ be a pair consisting with a quadratic form Q and a nonzero linear form L in k_S^n ($n \geq 5$) such that*

1. Q is nondegenerate
2. $Q|_{L=0}$ is isotropic
3. The quadratic form $\alpha Q + \beta L^2$ is irrational for any units α, β in k_S such that $(\alpha, \beta) \neq (0, 0)$.

Then there exists a k -rational subspace V of k^n of codimension 1 such that $F|_{V_S}$ satisfies the conditions (1)(2)(3), moreover V can be chosen such that $Q|_{\{L=0\} \cap V_S}$ is nondegenerate.

Proof. When s is an archimedean real place, it is proved in ([Gor04], Proposition 4) that there exists V_s a subspace of k^n of codimension 1 such that $F_s|_{V_s}$ verifies conditions (1)(2)(3). In the case of archimedean complex places and nonarchimedean places, one may replace the condition $Q_s|_{L_s=0}$ of type (I) which only valid for real places by equivalent condition that $Q_s|_{L_s=0}$ is nondegenerate which is valid for all $s \in S$. The rest of the proof is identical to the original proof. Hence for any $s \in S$ we may find V_s a subspace of k^n of codimension 1 so that the conditions (1)(2)(3) are satisfied by $F_s|_{V_s}$ and one can choose V_s such that $Q_s|_{\{L_s=0\} \cap V_s}$ is nondegenerate.

Assume that $n \geq 5$. For each $s \in S$, it is well known that the orbit of V_s under the orthogonal group $\text{SO}(Q_s)$ is open in the Grassmanian variety $\mathcal{G}_{n-1,n}(k_s)$ of the hyperplanes in k_s^n for the analytic topology. This can be seen using the fact that the map

$\mathrm{SO}(Q_s) \mapsto \mathcal{G}_{n-1,n}(k_s)$ given by $g \mapsto g(V_s)$ is submersive²

in particular the image contains a neighbourhood of the image of any g after the implicit function theorem. Moreover by weak approximation we can find a rational subspace in V' of codimension 1 in k^n such that $V' \otimes_k k_s$ is arbitrarily close to V_s for all $s \in S$, in particular they belong to the same open orbit. We have established that $F_{s|V_s}$ satisfies conditions (1) and (2), it is equivalent to say that

$$\mathrm{rad}(Q_s) \cap V_s = \{0\} \text{ and } c(Q_{s|L_s=0}) \cap V_s \neq \{0\}. \quad (*)$$

Since the subspace $\mathrm{rad}(Q_s)$ is invariant under the action of the orthogonal group $\mathrm{SO}(Q_s)$, the condition $(*)$ above is verified by any element of $\mathcal{G}_{n-1,n}(k_s)$ which lies in the orbit of V_s under $\mathrm{SO}(Q_s)$. In particular, $V' \otimes_k k_s$ satisfies $(*)$ for each $s \in S$. Hence we obtain a k -rational subspace V' of k^n such that $F_{|V_S}$ satisfies the conditions (1)(2). It remains to prove that $F_{|V_S}$ satisfies the condition (3)³. By weak approximation again, there exists $e \in V'(k) \cap \{L = 0\}$ such that $Q_{s|L_s=0}(e_s) \neq 0$ for all $s \in S$. Let $\tilde{Q} = \alpha Q + \beta L^2$ for an arbitrarily choice of $\alpha, \beta \in k_S$ with $(\alpha, \beta) \neq (0, 0)$. Applying e to it, we have that

$$\tilde{Q}_s(e_s) = \alpha_s Q_s(e_s) + \beta_s L_s(e_s)^2 = \alpha_s Q_s(e_s) \neq 0 \text{ for any } s \in S.$$

Hence multiplying by some unit in k_S we may assume that $\tilde{Q}_s(e_s) = 1$ for all $s \in S$. Now let

$$\mathcal{V} = \{V \in \mathcal{G}_{n-1,n}(k) \mid e \in V(k) \text{ and } F_{|V_S} \text{ satisfies conditions (1)(2)}\}.$$

It is nonempty because it contains V' . Suppose that there is no $V \in \mathcal{V}$ such that $F_{|V_S}$ satisfies the condition (3). Then $\tilde{Q}(x) \in k$ for any $x \in V(k), V \in \mathcal{V}$. For each $s \in S$, the map $x \mapsto \tilde{Q}_s(x)$ is a regular function on \bar{k}_s^n and it takes values in k on the union of $V(k), V \in \mathcal{V}$. The latter union is clearly Zariski-dense in \bar{k}_s^n and it is defined over k . Therefore $\tilde{Q}_s(x) \in k$ for all $x \in k^n$ and this implies that \tilde{Q}_s is rational over k . Since $e \in V$ for any $V \in \mathcal{V}$ and $\tilde{Q}_s(e) = 1$ for all $s \in S$, $\tilde{Q}_s(x)$ is independent of s for any $x \in \mathcal{V}(k), V \in \mathcal{V}$. This implies that \tilde{Q} is rational, this is a contradiction. Hence there exists some $V \in \mathcal{V}$ such that $F_{|V_S}$ satisfies the conditions (1)(2)(3).

Corollary 5.3.3. *It suffices to prove Theorem 5.2.2 for $n = 4$.*

Proof. It follows from the proposition by descending induction on n .

²i.e. the induced map on tangent space is surjective

³Remark that following the definition being irrational over k for a quadratic form is not equivalent to be irrational over all the $(k_s)_{s \in S}$!

5.4 Strong approximation

5.4.1 Adeles and strong approximation for number fields

The set of adeles \mathbb{A} of k is the subset of the direct product $\prod_{s \in \Sigma} k_s$ consisting of those $x = (x_s)$ such that $x \in \mathcal{O}_s$ for almost all $s \in \Sigma_f$. The set of adeles \mathcal{A} is a locally topological ring with respect to the adèle topology given by the base of open sets of the form $\prod_{s \in S} U_s \times \prod_{s \notin S} \mathcal{O}_s$ where $S \subset \Sigma$ is finite with $S \supset S_\infty$ and U_s are open subsets of k_s for each $s \in S$. For any subset $S \subset \Sigma$ finite with $S \supset S_\infty$, the ring of S -integral adeles is defined by:

$$\mathbb{A}(S) = \prod_{s \in S} k_s \times \prod_{s \notin S} \mathcal{O}_s, \text{ thus we can see that } \mathbb{A} = \bigcup_{S \supset S_\infty} \mathbb{A}(S)$$

We define also \mathbb{A}_S to be the image of \mathbb{A} onto $\prod_{s \notin S} k_s$, clearly $\mathbb{A} = k_S \times \mathbb{A}_S$.

Theorem 5.4.1 (Strong approximation). *If $S \neq \emptyset$ the image of k under the diagonal embedding is dense in \mathbb{A}_S .*

5.4.2 Strong approximation in algebraic groups

Let G be an algebraic group defined over k , we defined the group $G(\mathbb{A}_S)$ of S -adeles of G to be the image of $G(\mathbb{A})$ under the natural projection of $\prod_{s \in \Sigma} G(k_s)$ onto $\prod_{s \notin S} G(k_s)$. Thus,

$$G(\mathbb{A}_S) = \prod_{s \in S} G(k_s) \times \prod_{s \notin S} G(\mathcal{O}_s)$$

Definition 5.4.2. *A k -algebraic group G is said to satisfy the strong approximation property relative to S if the image under the diagonal embedding $G_k \hookrightarrow G(\mathbb{A}_S)$ is dense (equivalently in term of full adelic group $G(k)G(k_S)$ is dense in $G(\mathbb{A})$).*

Definition 5.4.3. *An isogeny between two algebraic groups G and H defined over k , is a surjective morphism $\mu : G \rightarrow H$ which has finite kernel. If μ is a k -morphism, we say that μ is a k -isogeny.*

Definition 5.4.4. *An algebraic group G is said to be simply connected, if for any connected H any isogeny $\mu : H \rightarrow G$ is an isomorphism.*

An important class of algebraic groups which satisfies the strong approximation property are given by unipotent subgroups (see e.g. [PR], §7.1). Thus, for a connected k -group G the Levi decomposition $G = LR_u(G)$ implies that it suffices to check the strong approximation property for reductive groups.

The following theorem is due to V. Platonov, the proof consists to reduce the problem of strong approximation to the Kesner-Tits conjecture which was also proved by itself in [Pra].

Theorem 5.4.5 (Strong approximation). *Let G be a reductive k -group and S a set of places of k . Then G has the strong approximation property with respect to S if and only if it satisfies the following conditions*

1. G is simply connected
2. G has no k -simple compact factors (i.e. G is isotropic)

The proof of this theorem (see e.g. [PR], §7.4) when $S \supset S_\infty$ is finite relies on the important observation that the closure of $G(\mathcal{O}_S)$ is open in $G(\mathbb{A}_S)$.

This observation can be generalised to any S -arithmetic subgroup in $G(\mathcal{O}_S)$. Recall that an S -arithmetic subgroup Γ is a subgroup of G_S which is commensurable with $G(\mathcal{O}_S)$, i.e. if $\Gamma \cap G(\mathcal{O}_S)$ has finite index both in $G(\mathcal{O}_S)$ and Γ_S (see e.g. [PR], §7.5).

Proposition 5.4.6. *If G satisfies condition of the theorem, any S -arithmetic subgroup Γ in $G(\mathcal{O}_S)$ is dense open subgroup in $G(\mathbb{A}_S)$.*

For an arbitrary algebraic group G , the strong approximation property should holds for the universal covering which by definition is simply connected. In general the existence of such universal cover is not always guaranteed, however for semisimple algebraic groups the existence is always satisfied.⁴

Proposition 5.4.7 ([PR], Thm 2.10). *For any semisimple k -group G , there exists a simply connected group \tilde{G} and an isogeny $\sigma : \tilde{G} \rightarrow G$ which is defined over k .*

Definition 5.4.8. *The isogeny $\sigma : \tilde{G} \rightarrow G$ is called the universal covering and the $\pi(G) = \ker(\sigma)$ is the fundamental group of G . Moreover if G is k -split, the universal cover is defined over k .*

Theorem 5.4.9 ([PR], Thm 4.1). *Let $\varphi : G_S \rightarrow H_S$ be a surjective k -morphism of algebraic groups. If Γ is an S -arithmetic subgroup of G_S , then $\varphi(\Gamma_S)$ is an S -arithmetic subgroup of H_S .*

Then every k -isogeny send any arithmetic subgroup to another one. In particular universal coverings do so providing the group is semisimple,

Corollary 5.4.10. *Suppose G_S is semisimple. If Γ_S is an S -arithmetic subgroup of \tilde{G}_S , then $\sigma(\Gamma_S)$ is an S -arithmetic subgroup of G_S .*

5.5 Stabilizer of pairs (Q, L)

For each $s \in S$, let $G_s = \mathrm{SL}_4(k_s)$, $G_S = \prod_{s \in S} \mathrm{SL}_4(k_s) = \mathrm{SL}_4(k_S)$.

Let $F = (Q, L)$ be a pair on k_S^4 satisfying the conditions (1)(2)(3) of Theorem 5.2.2.

⁴The situation is quite different in the category of topological groups, the universal cover always exists, this is due to the definition of the simple connectedness which is more restrictive in the category of algebraic groups than in the topological context.

5.5.1 Stabilizer of a pair over a local field

For every $s \in S$ we realize Q_s on a four-dimensional quadratic vector space (W_s, Q_s) over k_s equipped with the standard basis $\mathcal{B} = \{e_1, \dots, e_4\}$. For each $s \in S$, let define H_s the stabilizer of the pair F_s under the action of G_s , in other words

$$H_s = \{g \in G_s \mid Q_s \circ g = Q_s, L_s \circ g = L_s\}.$$

Equivalently one can write $H_s = \{g \in \text{SO}(Q_s) \mid L_s \circ g = L_s\}$. Clearly it is a linear algebraic group defined over k_s . Let us define $V_s = \{L_s = 0\}$, it is an hyperplane of W_s which induces a quadratic isotropic subspace $(V_s, Q_{s|V_s})$ of dimension 3 in W_s . We have two cases following $(V_s, Q_{s|V_s})$ is nondegenerate or not. If s is a real place the first (resp. second) case corresponds to pairs of type (I)(resp. type (II)) in the terminology of [Gor04].

Case 1. Assume $(V_s, Q_{s|V_s})$ is nondegenerate, then one can write the the following decomposition $W_s = V_s \oplus V_s^\perp$ where the orthogonal complement is a one-dimensional subspace of W_s . Let v a nonzero vector of W_s such that $V_s^\perp = \langle v \rangle$. Since $V_s^\perp = \{L_s \neq 0\}$ it is clearly H_s -invariant, therefore H_s acts by $x \mapsto \lambda x$ on the line $\langle v \rangle$. The linear form $L_{s|V_s^\perp}$ is nonzero and is H_s -invariant then H_s acts trivially on the line V_s^\perp . Let define $w_4 = v$ and complete to obtain a basis $\mathcal{B}' = \{w_1, \dots, w_4\}$ of W_s where $\langle w_1, w_2 \rangle$ is an hyperbolic plane in V_s . Hence we obtain for any $x \in W_s$, with respect to the basis $\mathcal{B}' = \{w_1, \dots, w_4\}$ that

$$Q_s(x) = x_1x_2 + a_3x_3^2 + a_4x_4^2 \text{ and } L_s(x) = x_4 \text{ with } a_3, a_4 \in k_s^*.$$

Therefore the stabiliser of $F_s = (Q_s, L_s)$ is given by

$$H_s = \left\{ \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array} \right) \mid M \in \text{SO}(Q_{s|L_s=0}) \right\} \subseteq \text{SL}_4(\overline{k_s}).$$

Case 2. Assume now the other case when $(V_s, Q_{s|V_s})$ is degenerate, this is equivalent to say that $\text{rk}(Q_{s|V_s}) \leq 2$ or that the discriminant of $Q_{s|V_s}$ is zero. Let us write the following decomposition $W_s = V_s \oplus k_s v$ where v is a nonzero vector of W_s such that $L_s(v) \neq 0$. Let define $w_4 = v$ and complete to obtain a basis $\mathcal{B}' = \{w_1, \dots, w_4\}$ of W_s where $\langle w_1, w_2 \rangle$ is an hyperbolic plane in V_s . Clearly one has $Q_s(x) = x_1x_2 + a_3x_3^2 + a_4x_4^2$ for some $a_3, a_4 \in k_s^*$. Suppose that $Q_{|\langle w_3, w_4 \rangle}$ is anisotropic then $Q_{s|V_s}(x) = x_1x_2 + a_3x_3^2$, since it is degenerate it has zero discriminant, that is, $a_3 = 0$, that is a contradiction. Therefore W_s is the direct sum of two hyperbolic plane, and with respect to the basis $\mathcal{B}' = \{w_1, \dots, w_4\}$ we obtain for any $x \in W_s$,

$$Q_s(x) = x_1x_2 - x_3x_4 \text{ and } L_s(x) = x_4.$$

Now let define U_s by the two-parameter unipotent subgroup in $\text{SL}(4, \overline{k_s})$

$$U_s = \left\{ \left(\begin{array}{cccc} 1 & a & b & ab \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{array} \right) \mid a, b \in \overline{k_s} \right\} \subseteq \text{SL}(4, \overline{k_s}).$$

By straightforward computation, one can check that U_s leaves invariant the pair F_s given by $Q_s(x) = x_1x_2 - x_3x_4$ and $L_s(x) = x_4$ in our case.

5.5.2 Stabilizer of pairs in k_S

We $F = (Q_s, L_s)_{s \in S}$ be a pair satisfying the conditions of the main theorem. We define two subsets of S corresponding to places following we are in the case 1 or 2 . Let us define S_1 (resp. S_2) to be the set of $s \in S$ such that $Q_{s|L_s=0}$ is nondegenerate (resp. degenerate). Thus we have the following partition of S given by $S = S_1 \cup S_2$.

For each $s \in S_1$, H_s is isomorphic to $\text{SO}(Q_{s|L_s=0})$ over k_s and since $Q_{s|L_s=0}$ is nondegenerate, H_s is a semisimple Lie subgroup of G_s which is noncompact since $Q_{s|L_s=0}$ is isotropic.

Let \mathcal{H}_s be the algebraic group defined over k_s such that $\mathcal{H}_s(k_s) = H_s$. Let H_s^+ be the subgroup of H_s generated by one-dimensional unipotent subgroups.

Let put

$$H_1 = \prod_{s \in S_1} H_s, H_1^+ = \prod_{s \in S_1} H_s^+ \text{ and } H_2 = \prod_{s \in S_2} U_s.$$

We also define

$$H_S = H_1 \times H_2 \text{ and } H_S^+ = H_1^+ \times H_2.$$

Therefore H_S is an algebraic subgroup of $\text{SL}_4(k_S)$ which leaves invariant the pair $F = (Q, L)$ with respect to the basis \mathcal{B}' . Obviously $H_2^+ = H_2$ thus H_S^+ is generated by one-dimensional unipotent subgroups.

5.5.3 Proof of the Theorem 5.2.1

Let F be a pair which satisfies the conditions of theorem 5.2.1. Let $g \in G_S$ the matrix of the basis \mathcal{B}'_S in the standard basis of k_S^4 . By definition gH_Sg^{-1} leaves invariant the pair $F = (Q_s, L_s)_{s \in S}$, and H_S^+ is generated by one-dimensional unipotent subgroups. Put $\Lambda = g\mathcal{O}_S^n$, it is clearly a lattice in k_S^4 in other words an element of the homogeneous space Ω_S . Applying theorem 6.1, one obtains

$$\overline{H_S^+ \Lambda} = \widetilde{M} \Lambda \tag{5.1}$$

where \widetilde{M} is an algebraic subgroup of G_S defined over k such that $\widetilde{M}(k_s)$ contains $g_s H_s^+ g_s^{-1}$ (resp. $g_s U_s g_s^{-1}$) for $s \in S_1$ (resp. $s \in S_2$).

Assume first that $S = S_\infty$. Using the equality (1) we can deduce that the set $F_s(\mathcal{O}^4)$ is dense in k_s^2 for all $s \in S_\infty$. Indeed, if $s \in S_1$ the density is satisfied after ([Gor04], proposition 10) for pairs of type (I) and if $s \in S_2$ the same is proved in ([Gor04], proposition 14) for pairs of type (II). In particular this proves the theorem when $S = S_\infty$.

Now let assume $S \neq S_\infty$ and let be given $s \in S_f$. The set \mathcal{O}^4 is bounded in k_s^4 , thus for any neighbourhood U of the origin in k_s^4 one can find an integer $a_s \in \mathcal{O}_s$ such that $a_s \cdot \mathcal{O}^4 \subset U$. In other words, given any $\varepsilon > 0$ one can find $a_s \in \mathcal{O}_s$ such that :

$$|Q_s(a_s x)|_s \leq \varepsilon \text{ and } |L_s(a_s x)|_s \leq \varepsilon \text{ for all } x \in \mathcal{O}^4.$$

Thus for each $s \in S_f$, we can associate an integer $a_s \in \mathcal{O}_s$ satisfying the previous inequalities. By strong approximation one can find $a \in \mathcal{O}$ such that $|a|_s = |a_s|_s$ for all $s \in S_f$. Put $\|a\|_\infty = \max_{s \in S_\infty} |a|_s$, by the previous case we can find $x \in \mathcal{O}^4$ such that:

$$|Q_s(x)|_s \leq \varepsilon / \|a\|_\infty^2 \text{ and } |L_s(x)|_s \leq \varepsilon / \|a\|_\infty \text{ for all } s \in S_\infty.$$

We immediately obtain for all $s \in S_\infty$

$$|Q_s(a_s x)|_s = |a_s|_s^2 |Q_s(x)|_s \leq \varepsilon \text{ and } |L_s(a_s x)|_s = |a_s|_s |L_s(x)|_s \leq \varepsilon.$$

Hence given any $\varepsilon > 0$, we get a nonzero element $y = a \cdot x \in \mathcal{O}_S^4$ satisfying the conclusion of the theorem, i.e.

$$|Q_s(y)|_s \leq \varepsilon \text{ and } |L_s(y)|_s \leq \varepsilon \text{ for all } s \in S.$$

5.6 Proof of the theorem 5.2.2

Let $F = (Q, L)$ be a pair in k_S^n which satisfies the conditions of theorem 5.2.2 . After § 5.3, we know that it suffices to show it for $n = 4$.

Let assume first that the form $\alpha Q_s + \beta L_s^2$ is irrational for any α, β in k_s such that $(\alpha, \beta) \neq (0, 0)$ for all $s \in S$, that is to say that F satisfies the conditions of Theorem 5.2.1. Hence the origin in k_S^4 is an accumulation point in $F(\mathcal{O}_S^4)$ and the conclusion of Theorem 5.2.2 holds in this case.

Now it remains to prove the theorem when some linear combination over k_s of Q_s and L_s^2 is rational for some $s \in S$. The proof is slight modification due of ([BP92], §4). The only difference is that we deal with pairs instead of quadratic forms, therefore with stabilisers which are not given by orthogonal groups in general. However since $Q_{L=0}$ is assumed to be nondegenerate the stabiliser of a pair is the orthogonal group of $Q_{L=0}$ and strong approximation holds.

Suppose that there exists $v \in S$ and a rational form Q_0 on k^v such that,

$$\alpha_v Q_v + \beta_v L_v^2 = \lambda_v Q_0$$

for some $\lambda_v \in k_v^*$ and α_v, β_v in k_v such that $(\alpha_v, \beta_v) \neq (0, 0)$. Let us choose α, β in k_S such that $(\alpha)_v = \alpha_v$ and $(\beta)_v = \beta_v$ and set $\tilde{Q} = \alpha Q + \beta L^2$, then $\tilde{Q}_v = \lambda_v Q_0$. Define the set S' to be the set of places $s \in S$ such that $\tilde{Q}_s = \lambda_s Q_0$ for some $\lambda_s \in k_s^*$. Obviously $v \in S'$ and $S \neq S'$ since \tilde{Q} is irrational, and one can put $T = S - S'$.

Let $Z_0 = \{Q_0 = L = 0\}$ be the k_S -algebraic variety defined by Q_0 and L and for each $s \in S$ put $Z_{0,s} = Z_0 \times k_s$. Let also for each $s \in S$ the k_s -algebraic variety defined by $Z_s = \{Q_s = L_s = 0\}$. Obviously Z_s characterises the form Q_s up to a multiple in k_s^* and $Z_{0,s} = Z_s$ for all $s \in S'$ but $Z_{0,t} \neq Z_t$ for any $t \in T$. Since $Z_t \neq \{0\}$ ($Q_t|_{L_t=0}$ is isotropic) and defined over k_t , $Z_t(k_t)$ is Zariski dense in Z_t and $Z_{0,t}(k_t) \neq Z_t(k_t)$ for any $t \in T$.

We define $H_0 = \text{SO}(Q_0) \cap \text{Stab}_G(L)$ and let V be the subspace $\text{Ker}L = \{L = 0\}$. By surjectivity of L we have immediately $\text{Stab}_G(L) = V^G$ i.e. $H_0 = \text{SO}(Q_0) \cap V^G$. Moreover by construction, for each $s \in S'$ one has $\lambda_s Q_0|_{V_s} = \tilde{Q}_s|_{V_s} = \alpha_s Q_s|_{V_s}$ thus $H_0(k_s) = \text{SO}(Q_s|_{V_s})$ is noncompact for each $s \in S'$ since $Q|_V$ is isotropic.

For each $t \in T$, suppose R_t is an open subgroup of H_0 and let

$$X_t = \{x \in k_t^4 - Z_0(k_t) \mid r_t \cdot x \in Z_t(k_t) \text{ for some } r_t \in R_t\}.$$

Clearly X_t is a nonempty open subset of k_t^4 defined up to multiple in k_s^* . The key of the proof of the theorem lies on the following result of geometry of numbers,

Lemma 5.6.1. *Given a polydisc $\Delta = \prod_{s \in S'} \Delta_s$ centred in the origin in $k_{S'}^4$, then there exists an integral vector $x \in \mathcal{O}_S^4$ such that $x_s \in \Delta_s$ for all $s \in S'$ and $x_t \in X_t$ for all $t \in T$.*

In other words we can find an integral vector x with T -component lying in X_T and arbitrarily small S -components. For sake of completeness, we give a more detailed proof of the lemma that the one given in ([BP92], §4)

Proof of the lemma. Let be given an arbitrary $t \in T$, since X_t is nonempty and defined up to a multiple in $k_t - \{0\}$, we can find a nonzero vector $e_{t,1} \in X_t$ such that the blunt line $k_t^* \cdot e_{t,1}$ is still contained in X_t . Let complete $e_{t,1}$ into a basis $\{e_{t,1}, \dots, e_{t,n}\}$ of k_t^n , let us define for any positive real r

$$\begin{aligned} D_{t,r} &= \{x \in k_t : |x|_t \leq r\} \\ B_{t,r} &= \bigoplus_{2 \leq j \leq n} D_{t,r} \cdot e_{t,j} \\ C_{t,r} &= D_{t,r} e_{t,1} \oplus B_{t,r} = \bigoplus_{1 \leq j \leq n} D_{t,r} \cdot e_{t,j} \end{aligned}$$

The fact that the line $k_t^* \cdot e_{t,1}$ is contained in X_t and the fact that R_t is an open subgroup of H_0 provide two reals $a \geq b > 0$ so small that

- (i) $e_{t,1} + C_{t,a}$ is contained in X_t and $(D_{t,m} - D_{t,1})e_{t,1} \oplus B_{t,a} \subset X_t$ for any real $m > 1$
- (ii) The sum of any $|T| - 1$ elements of $C_{t,b}$ is contained in $C_{t,a}$.

Now we consider $\Theta = \prod_{s \in S'} \Theta_s$ be a bounded polydisc centred on the origin such that the sum of any $2|T|$ elements of Θ is contained in Δ . If we are able to prove the claim $(\star)_t$ below the proof of the lemma will follows

$$\begin{aligned} (\star)_t \text{ There exists } x(t) \in \mathcal{O}_S^4 \text{ and a real } m \geq 2 \text{ such that } x(t)_s \in \Theta_s + \Theta_s, \text{ if } s \in S', \\ x(t)_t \in (D_{t,m} - D_{t,1})e_{t,1} \oplus B_{t,b} \text{ and } x(t)_{t'} \in C_{t',b} \text{ if } t' \in T - \{t\}. \end{aligned}$$

Indeed, assume $(\star)_t$ is fulfilled and let define $x = \sum_{t \in T} x(t) \in \mathcal{O}_S^4$.

For any $s \in S'$, $x_s = \sum_{t \in T} x(t)_s$ is the sum of $2|T|$ elements of Θ_s hence it is contained in Δ_s .

One the other hand, by condition (ii), for any $t' \in T$

$$x_{t'} = \sum_{t \in T} x(t)_{t'} = x(t')_{t'} + \sum_{t \in T-t'} x(t)_{t'} \in (D_{t,m} - D_{t,1})e_{t',1} + B_{t',b} + C_{t',a}$$

The fact that $B_{t',b} \subset B_{t',a}$ and the condition (i) imply that $x_{t'} \in X_{t'}$, hence $x \in \mathcal{O}_S^4$ satisfies all the condition of the lemma.

It remains to show that $(\star)_t$ for any given $t \in T$ which we fix from now. For a positive real number r , put

$$\Omega_r = \Theta \times (D_{t,r}e_{t,1} \oplus B_{t,b/2}). \quad \prod_{t' \in T-\{t\}} C_{t',b/2} \subset k_S^4$$

Recall that each completion k_s ($s \in S$) is equipped with a additive Haar measure ν_s , we denote by vol the product of the ν_s ($s \in S$) which gives a Haar measure on k_S^4 . Let μ be the pullback of the Haar measure on k_S^4 with respect to the natural projection $\pi : k_S^4 \rightarrow k_S^4/\mathcal{O}_S^4$. Also let denote by c the covolume of \mathcal{O}_S^4 in k_S^4 , that is, $c := \mu(k_S^4/\mathcal{O}_S^4)$. Since \mathcal{O}_S^4 is discrete in k_S^4 , the set $(\Omega_1 + \Omega_1) \cap \mathcal{O}_S^4$ is finite with cardinality, say $q \geq 1$. Now the map $r \mapsto \text{vol}(\Omega_r)$ is obviously increasing function on \mathbb{R}_+^* , thus we may find a real $m \geq 2$ such that:

$$\text{vol}(\Omega_{m/2}) > (q+1).c$$

Therefore,

$$(q+1).c < \text{vol}(\Omega_{m/2}) \leq \int_{k_S^4/\mathcal{O}_S^4} \left(\int_{\pi_{|\Omega_{m/2}}^{-1}(y)} d\text{vol} \right) d\mu(y) = \int_{k_S^4/\mathcal{O}_S^4} \text{vol}(\pi_{|\Omega_{m/2}}^{-1}(y)) d\mu(y) \quad (5.2)$$

Suppose that all the fibers $\pi_{|\Omega_{m/2}}^{-1}(y)$ have less or equal than $q+1$ elements, then

$$\int_{k_S^4/\mathcal{O}_S^4} \text{vol}(\pi_{|\Omega_{m/2}}^{-1}(y)) d\mu(y) \leq (q+1). \mu(k_S^4/\mathcal{O}_S^4) = (q+1).c$$

This contradicts inequality (5.2), hence it must exists at least one fiber, say $\pi_{|\Omega_{m/2}}^{-1}(y)$, which contains at least $q+2$ elements $\{y_0, y_1, \dots, y_{q+1}\}$ in $\Omega_{m/2}$. Clearly one can write

$$y_i = y_0 + \mathcal{O}_S^4 \quad \text{for } 1 \leq i \leq q+1$$

so that we get $q+1$ distincts elements in \mathcal{O}_S^4 by taking $x_i := y_0 - y_i$ for every $i = 1, \dots, q+1$. Obviously, since $(\Omega_1 + \Omega_1) \cap \mathcal{O}_S^4$ has q elements, one of the x_i 's, say x_1 , must lie outside $\Omega_1 + \Omega_1$.

Let define $x(t) := x_1$, then immediately $x(t) \in \mathcal{O}_S^4$. It is easy to check that $x(t)$ satisfies $(\star)_t$. Indeed, let $s \in S'$, then

$$x(t)_s = y_{0,s} - y_{1,s} \in \Omega_{m/2,s} + \Omega_{m/2,s} = \Theta_s + \Theta_s$$

At the place t , we have

$$x(t)_t \in (D_{t,m/2} - D_{t,1})e_{t,1} \oplus B_{t,b/2} + (D_{t,m/2} - D_{t,1})e_{t,1} \oplus B_{t,b/2} = (D_{t,m} - D_{t,1})e_{t,1} \oplus B_{t,b}$$

On the other hand it is clear that

$$x(t)_{t'} \in C_{t',b/2} + C_{t',b/2} = C_{t',b} \text{ if } t' \in T - \{t\}$$

and the lemma is proved.

Proof of the Theorem. In order to use strong approximation (see §5.4.2) let $\sigma : \widetilde{H}_0 \rightarrow H_0$ be the universal covering of H_0 , such isogeny exists because H_0 is semisimple (Prop. 5.4.7), indeed this comes from the fact that $Q|_V$ is nondegenerate implies

$$H_0 = \left\{ \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array} \right) \mid M \in \text{SO}(Q|_V) \right\} \simeq \text{SO}(Q|_V) \text{ is semisimple.}$$

Let Λ_S be the stabilizer of \mathcal{O}_S^4 in $H_0(k)$, via the diagonal embedding we realize Λ_S as a S -arithmetic subgroup of $H_0(k_S)$. Let Λ_T be the projection of Λ_S on $H_0(k_T)$. We know that H_0 and \widetilde{H}_0 are both isotropic over k_s ($s \in S'$) and it has no compact factors over k_s for $s \in S'$. Hence the strong approximation property applied to \widetilde{H}_0 with respect to S' yields a set of open subgroups of finite index R_t in $H_0(k_t)$ for $t \in T$ such that the product $R_T = \prod_{t \in T} R_t$ is contained in the closure of Λ_T .

Indeed, let $\widetilde{\Lambda}$ be an arbitrary S -arithmetic subgroup in $\widetilde{H}_0(k)$, which can be seen as a discrete subgroup in $\widetilde{H}_0(k_S)$. Let denote by $\widetilde{\Lambda}_T$ its projection on $\widetilde{H}_0(k_T)$. By strong approximation (Prop. 5.4.6) the closure of $\widetilde{\Lambda}$ is open in $\mathbb{A}_{S'}$, in particular its projection $\widetilde{\Lambda}_T$ is dense in a open subgroup of $H_0(k_T)$. Therefore, using corollary 5.4.10 the subgroup $\sigma(\widetilde{\Lambda}_T)$ is also an arithmetic subgroup which is dense in an open subgroup of $H_0(k_T)$. Hence the subgroup $R_T := \text{cl}(\sigma(\widetilde{\Lambda}_T)) \cap \text{cl}(\Lambda_T)$ is the good candidate since $\sigma(\widetilde{\Lambda}_T)$ and Λ_T are commensurable as arithmetic subgroups in $H_0(k_T)$.

Let $\varepsilon > 0$ arbitrary and choose a polydisc $(\Delta_s)_{s \in S'}$ in $k_{S'}$ centred at the origin and small enough so that the following simultaneous inequalities are satisfied:

$$\text{For each } s \in S', |Q_s(z)|_s \leq \varepsilon \text{ and } |L_s(z)|_s \leq \varepsilon \text{ for any } z \in \Delta_s.$$

As above we introduce the set X_t for $t \in T$ associated to with R_T . By the previous lemma we get $x \in \mathcal{O}_S^4$ such that $x_s \in \Delta_s$ for all $s \in S'$ and $x_t \in X_t$ for all $t \in T$. In one hand for each $t \in T$ there exists $r_t \in R_t$ such that $z_t = r_t x_t \in Z_t(k_t)$, hence since R_t is open we find $y_t \in R_t x_t$ close enough to z_t which satisfies the following inequalities:

$$0 < |Q_t(y_t)|_t \leq \varepsilon/2 \text{ and } 0 < |L_t(y_t)|_t \leq \varepsilon/2 .$$

Moreover since R_T is contained in the closure of Λ_T , we can find $\gamma \in \Lambda_S$ such that:

$$0 < |Q_t((\gamma x)_t)|_t \leq \varepsilon \text{ and } 0 < |L_t((\gamma x)_t)|_t \leq \varepsilon \text{ for each } t \in T.$$

It suffices to show that the element $\gamma x \in \mathcal{O}_S^4$ satisfies the last inequalities for $s \in S'$. Since $\gamma \in H_0$, it leaves invariant both \widetilde{Q}_s and L_s for all $s \in S'$ thus it leaves also invariant Q_s for all $s \in S'$. Therefore we get the following inequalities

$$|Q_s((\gamma x)_s)|_s \leq \varepsilon \text{ and } |L_s((\gamma x)_s)|_s \leq \varepsilon \text{ for each } s \in S'.$$

This finishes the proof.

5.6.1 Comments and open problems

The problem of null values

The argument used in [BP92] (Theorem A, (iii)) to prove that for any $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^n$ such $0 < |Q_s(x)|_s < \varepsilon$ does not easily generalise for pairs. In fact, we do not prove that the $x \in \mathcal{O}_S^n$ of theorem 5.2.1 and 5.2.2 satisfies the stronger condition that $0 < |Q_s(x)|_s < \varepsilon$ and $0 < |L_s(x)|_s < \varepsilon$ for finite places $s \in S_f$.

Towards density

It should be possible to obtain density of $F(\mathcal{O}_S^n)$ for a pair $F = (Q, L)$ over k_S under the assumption of theorem 5.2.1. For this we need an analog of the theorem 1 of [Gor04] for nonarchimedean field which has no clear reason to fail, a significant difference with the classical Oppenheim conjecture is that the stabilizer of such pairs is no more maximal. On the other hand, it seems difficult to prove density of $F(\mathcal{O}_S^n)$ under the assumption of theorem 5.2.2 when we allow some linear combination of Q_s and L_s^2 to be a multiple of a rational form. By the same method we can easily obtain the analog for a pair $F = (Q, L)$

* The set $F(\mathcal{O}_S^n)$ is dense in k_S^2 if and only if for any $\varepsilon > 0$, and $\overline{c_s} \in k_s^*/k_s^{*2}$, there exist $x \in \mathcal{O}_S^n$ such that $0 < |Q_s(x)|_s < \varepsilon$ and $0 < |L_s(x)|_s < \varepsilon$ with $Q_s(x) \in \overline{c_s}$ for all $s \in S$.

Proving density for pairs using this way appears to be discouraging from the time it does not give satisfaction for a single quadratic in the previous remark. However it may be possible to arrive to density using the strategy of ([B94], §8) and the proof of Theorem 5.1.1.

Open problem

We conclude by mentioning a conjecture of Gorodnik (see [Gor04], Conjecture 15) which concerns the assumption (2) of Theorem 3.1.3 in the real case. It is conjectured that the condition (2), that is, $Q|_{L=0}$ is isotropic can be replaced by the weaker assumption that $\alpha Q + \beta L^2$ is isotropic for any real numbers α, β such that $(\alpha, \beta) \neq (0, 0)$. With this assumption we cannot reduce to dimension 4 and we are led to classify an unmanageable number of intermediate subgroups. It is noteworthy that for the moment it does not exist any complete proof of the Oppenheim conjecture or neither any of all its variants which does not make use of Ratner's theorems.

Chapter 6

Final remarks

6.1 Gorodnik's conjecture on density for pairs

The problem of finding optimal conditions which ensure that $B(\mathbb{Z}^n)$ is dense in \mathbb{R}^2 for $n \geq 4$ for a given pair $B = (Q, L)$ is still an open problem. The best known answer is given by Theorem 3.1.3. It is conjectured by A.Gorodnik (see [Gor04], Conjecture 15) that the assumptions of the theorem could be weakened.

Conjecture 6.1.1 (Gorodnik). *Let $B = (Q, L)$ be a pair consisting in one nondegenerate quadratic and one nonzero linear form in dimension $n \geq 4$. Suppose that*

1. *For every $\beta \in \mathbb{R}$, $Q + \beta L^2$ is indefinite*
2. *For every $(\alpha, \beta) \neq (0, 0)$, with $\alpha, \beta \in \mathbb{R}$, $\alpha Q + \beta L^2$ is not rational*

Then $B(\mathbb{Z}^n)$ is dense in \mathbb{R}^2 .

The first condition is necessary for density to hold. The main issue is that this condition contrarily to the condition that $Q|_{L=0}$ is indefinite does not allow us to reduce to the four dimensional case. Hence all the strategy of the proof of Theorem 3.1.3 become needless regarding the impossibility to classify all the intermediate subgroups in high dimension.

6.2 Lower Bounds for pairs (Q, L) in Dimension $n \geq 4$

A possible method could be to apply the strategy of Dani-Margulis as we did in Theorem 3.3.1 for dimension three, in order to give a asymptotic lower bound of the values of pairs at integral points. The situation of the dimension $n \geq 4$ is more intricate, the method of Gorodnik used to show density of $\{(Q(x), L(x)) : x \in \mathbb{Z}^n\}$ does not consist to show that $x = g\Gamma$ is generic for some unipotent action. This classical argument used in the original deduction of the Oppenheim conjecture fails because the stabiliser of such pairs (Q, L)

is not maximal. This phenomenon is due to the existence of non-generic points, the set $\mathcal{S}(H)$ such points is called the set of singular points with respect to the action of the stabiliser H for which the H -orbits are not dense in G/Γ .

Let $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, we denote by μ the G -invariant probability of the space of unimodular lattices G/Γ .

Let (Q, L) a pair satisfying the conditions of Theorem 3.1.3 ([Gor04], Theorem1),

1. $Q|_{L=0}$ is *indefinite*
2. No linear combinaison over \mathbb{R} of Q and L^2 is *rational*.

There exists some $g \in G$ such that $(Q, L) = (Q_0^g, L_0^g)$ where (Q_0, L_0) is given for (Q, L) of type (I) (resp. type (II)) by

$$(Q_0(x), L_0(x)) = (x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2, x_n) \text{ for } p = 1, \dots, n \text{ or}$$

$$(Q_0(x), L_0(x)) = (x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{n-2}^2 + x_{n-1}x_n, x_n) \text{ for } p = 1, \dots, \left[\frac{n-2}{2}\right].$$

Let $U = \{u_t | t \in \mathbb{R}\}$ defined as in §3.3.2, it is a unipotent one-parameter subgroup of H . One can apply Ratner theorem's to obtain a connected Lie subgroup F of G containing $g^{-1}Hg$ such that $\overline{\{u_t x | t \in \mathbb{R}\}} = Fx$ where $x = g\Gamma$ and such that the orbit Fx is supported by a F -invariant measure μ_x . Instead of inequality (3.5) we have

$$\frac{1}{(l-1)T} \int_T^{lT} \int_{B_j} \tilde{\chi}(u_t m g \Gamma) d\sigma(m) dt \geq \omega^2 \sigma(B_j) \int_{Fx} \varphi d\mu_x$$

where $\varphi \in \mathcal{C}_c(G/\Gamma)$ and ω are taken such that

$$0 \leq \varphi \leq \tilde{\chi} \text{ and } \int_{G/\Gamma} \varphi d\mu_x \geq \omega \int_{G/\Gamma} \tilde{\chi} d\mu_x.$$

Using the same computations of the previous section in the proof of Theorem 4.1.1 we get the following inequality

$$\#(S_j(T) \cap g\mathbb{Z}^n) \geq l^{-2} \omega^5 \lambda(S_j(T)) \int_{Fx} \tilde{\chi} \frac{d\mu_x}{\lambda(\Delta)}.$$

Finally following the end of the proof of the Theorem 4.1.1, the adding term propagates and we get,

$$\#(\mathcal{D}(\mathbb{Z}) \cap r\Omega) \geq (1-\theta) \lambda(\mathcal{D}(\mathbb{R}) \cap r\Omega) \int_{Fx} \tilde{\chi} \frac{d\mu_x}{\lambda(\Delta)}. \quad (6.1)$$

Let us define the following quantity

$$\Theta_\tau(x) := \int_{Fx} \tilde{\chi}_\Delta \frac{d\mu_x}{\lambda(\Delta)}.$$

The definition of the function Θ_τ depends on the choice of $\tau > 0$, we fix it such that it satisfies the condition of lemma 4.2.2. The function Θ_τ is well defined on G/Γ and its image lies in $[0, 1]$. This quantity can be seen as the defect of equidistribution of the orbit $\{u_t x : t \in \mathbb{R}\}$ of x in G/Γ .

It is not easy to find a lower bound for $\Theta_\tau(x)$ which is independent from τ and x only using the definition. Moreover it suffices to understand Θ_τ on its restriction to the set of singular points $\mathcal{S}(U)$, since it is identically equal to 1 on the set of generic points $\mathcal{G}(U) = G/\Gamma - \mathcal{S}(U)$ (see §1.4.1) by Siegel's integral formula. The main issue is that $\mathcal{S}(U)$ is a subset of null measure with respect to the Haar measure μ on G/Γ .

6.2.1 Minoration of $\Theta_\tau(x)$

Finding a lower bound for $\#(\mathcal{D}(\mathbb{Z}) \cap r\Omega)$ for large r reduces to find a minoration of $\Theta_\tau(x)$ after inequality (6.1),

$$\#(\mathcal{D}(\mathbb{Z}) \cap r\Omega) \geq (1 - \theta)\lambda(\mathcal{D}(\mathbb{R}) \cap r\Omega) \Theta_\tau(x) \text{ for any } \theta > 0.$$

The left hand-side is known to be > 0 by density in Theorem 3.1.3. The proof of the Theorem 3.1.3 can be reduced to the case of the dimension 4, therefore F can be considered as a subgroup of $\text{SL}(4, \mathbb{R})$ which contains $g^{-1}Hg$ where H unipotently generated subgroup of G which stabilises the pair (Q_0, L_0) . Even in low dimensional cases, classifying all the intermediate subgroups F seems very complicated. One can also think to work out this issue using harmonic analysis in $L^2(G/\Gamma)$ since the function $\widetilde{\chi}_\Delta$ is clearly in $L^2(G/\Gamma)$. One can try to use Siegel-Eisenstein series associated to χ_Δ instead of Siegel transform, given by:

$$E_\Delta(s, \Lambda) = \sum_{v \in \Lambda - \{0\}} \chi_{\Delta, s}(v) \text{ for any } \Lambda \in G/\Gamma.$$

This sum converges for s complex such that $\text{Re}(s) > n$ and $\chi_{\Delta, s}(v)$ is taken to be the Mellin transform of χ_Δ . Using this series one obtain an other expression for $\widetilde{\chi}_\Delta$ using Mellin inversion:

$$\widetilde{\chi}_\Delta(y) = \int_{\text{Re}(s) > n} E_\Delta(s, y) \frac{ds}{2i\pi} \text{ for any } y \in G/\Gamma.$$

Hence we obtain the following expression for Θ_τ

$$\Theta_\tau(x) = \frac{1}{\lambda(\Delta)} \int_{Fx} \int_{\text{Re}(s) > n} E_\Delta(s, y) \frac{ds}{2i\pi} d\mu_x(y)$$

where $x = g\Gamma$ given by the condition on the pair $(Q, L) = (Q_0^g, L_0^g)$.

By Fubini's theorem, we arrive to

$$\Theta_\tau(x) = \frac{1}{\lambda(\Delta)} \int_{\operatorname{Re}(s) > n} \frac{ds}{2i\pi} \int_{Fx} E_\Delta(s, y) d\mu_x(y).$$

Using the analogy with toroidal integral and Hecke's formula, it would be interesting to get a simpler expression in order to find lower estimate for Θ_τ

$$\int_{Fx} E_\Delta(s, y) d\mu_x(y)$$

but unfortunately such a formula does not exist for unipotent orbits.

It would be interesting to try to find lower bounds for $\#(\mathcal{D}(\mathbb{Z}) \cap r\Omega)$ under the conditions of the conjecture instead of trying to arrive by straightforward methods to density. A minoration of the Θ as defined previously by a positive quantity would be a crucial step towards the conjecture.

Bibliography

- [B63] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, *Topology* **2** (1963), 111-122.
- [B91] A. Borel, *Linear algebraic groups*, GTM **126** Springer-Verlag, (second enlarged edition), (1991), 288 pp.
- [B94] A. Borel, *Values of indefinite quadratic forms at integral points and flows on spaces of lattices*, *Bull. of the Amer. Math. Soc.*, Volume **32**, Number **2**, (1995), 184-204.
- [BH62] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, *Ann. of Math.* **75** (1962), 485-535.
- [BP88] A. Borel and G. Prasad, *Valeurs de formes quadratiques aux points entiers*, *C.R. Acad. Sci. Paris Ser. I Math.* **307** (1988) 217-220.
- [BP92] A. Borel and G. Prasad, *Values of isotropic quadratic forms at S -integral points*, *Compositio Math.* **83** (1992) 347-372.
- [BT73] A. Borel and J. Tits, *Homomorphismes "abstrait" de groupes algebriques simples* *Ann. of Math.* **97** (1973) 499-571
- [Ch34] S. Chowla, *A theorem on irrational indefinite quadratic forms*, *J. London Math. Soc.* **9** (1934), 162-163.
- [Dan81] S.G. Dani, *Invariant measures and minimal sets of horospherical flows*, *Inventiones Math.* **64** (1981), 357-385.
- [Dan82] S.G. Dani, *On uniformly distributed orbits of certain horocycle flows*, *Ergodic and Dynamical systems*, **2** (1982), 139-158.
- [Dan84] S.G. Dani, *On orbits of unipotent flows on homogeneous spaces*, *Ergodic and Dynamical systems*, **4** (1984), 139-158.
- [Dan86a] S.G. Dani, *Orbits of horospherical flows*, *Duke Math. J.* **53** (1986), 177-188.
- [Dan86b] S.G. Dani, *On orbits of unipotent flows on homogeneous spaces II*, *Ergodic and Dynamical systems*, **4** (1984), 139-158.

- [DanSm] S.G. Dani and J.Smilie *Uniform distribution of horocycle flows for fuschian groups*, Duke Math. J. **71** (1993), 143-180.
- [Dan00] S.G. Dani, *On values of linear and quadratic forms at integral points*, Number theory, **107-119**, Trends Math., Birkhauser, Basel, 2000.
- [Dan08] S.G. Dani, *Simultaneous diophantine approximation with quadratic and linear forms*, J.Modern Dynamics **2** (2008), 129-139 (with the special issue dedicated to G.A. Margulis)
- [DavH] H. Davenport and H. Heilbronn, *On indefinite quadratic forms in five variables*, J. London Math. Soc. **21** (1946) 185-193.
- [DavR] H. Davenport and H. Ridout, *Indefinite quadratic forms in five variables*, Proc. London Math. Soc. **9** (1959) 544-555.
- [DM90] S.G. Dani and G.A Margulis, *Orbit closures of generic unipotent flows on homogeneous space of $SL(3, \mathbb{R})$* , Math. Ann. **286** (1990) 101-128
- [DM93] S. G. Dani and G. A. Margulis, *Limit distributions of orbits of unipotent flows and values of quadratic forms*, Adv. in Soviet Math. **16** (1993) 91-137.
- [DWMo] D. Witt Morris, *Ratner's theorem on unipotent flows*, University of Chicago Press,
- [EW] M. Einsiedler and T. Ward, *Ergodic theory with a view towards number theory*, Springer-Verlag, (2010).
- [EKL06] M. Einsiedler, A. Katok and E. Lindenstrauss *Invariant measures and the set of exception to Littlewood's conjecture*. Annals of Math. **164** (2006) 513-560
- [Esk10] A. Eskin, *Unipotent flows and applications*, Homogeneous spaces, moduli spaces and arithmetic, Clay Math. Proc., **10**, Amer. Math. Soc., Providence, RI, (2010) 71-129.
- [EMM98] A. Eskin, G. A. Margulis and S. Mozes, *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*, Ann. of Math. **147** (1998), 93-141.
- [EMM05] A. Eskin, G. A. Margulis and S. Mozes, *Quadratic forms of signature (2,2) and eigenvalue spacings on flat 2-tori*, Ann. of Math. **161** (2005), 679-725.
- [EV08] J. Ellenberg and A. Venkatesh, *Local-global principles for representation of quadratic forms*. Invent. Math. **171** no.2 (2008) 257-279
- [Furst72] H. Furstenberg, *The unique ergodicity of horocycle flow*, Recent Advances in Topological dynamics, Lecture Notes in Math. **318**, Springer New-York, (1972), 95-115.

- [Furst76] H. Furstenberg, *A note on Borel's density theorem*, Proc. Amer. Soc. vol. **55** (1976), 209-212.
- [Hedl] G. Hedlund, *Fuchsian groups and transitive horocycles*, Duke Math Journal **2** (1936), 530-542.
- [Gor04] A. Gorodnik *Oppenheim conjecture for pairs consisting of a quadratic form and a linear form*, Trans. Amer. Math. Soc. **356** (2004), no. 11, 4447–4463
- [Ger] L.J. Gerstein *Basic quadratic forms*, Graduate Studies in Math. vol. **90**, Amer. Math. Soc. (2008) **356** (2004), no. 11, 4447–4463
- [KM] D. Kleinbock and G. A. Margulis, *Bounded orbits of quasiunipotent flows on homogeneous spaces*, Transl. Amer. Math. Soc. **171** (1996), 141-172.
- [Kn69] M. Knesner, *Lectures on Galois cohomology of classical groups*, India TIFR Bombay 1969, 158 p.
- [L] S. Lang *Algebraic number theory, second ed.* Graduate texts in Mathematics Vol. **110**, Springer-Verlag, New-York 1994.
- [Mar86] G.A Margulis, *Lie groups and ergodic theory, Algebra some current trends*(L.L. Avramov, ed.) Proc. Varna 1986, Lecture Notes in Math. vol. **1352**, Springer, New-York, 130-146.
- [Mar89a] G.A Margulis, *Indefinite quadratic forms and unipotent flows on homogeneous spaces*, Banach Center Publish. , vol. **23**, Polish Scientific Publishers, Warsaw, 1989.
- [Mar89b] G.A Margulis, *Discrete subgroups and Ergodic theory, Number theory, Trace formulas and Discrete subgroups* (symposium in honour of A. Selberg) Academic Press, San Diego, CA, (1989) 377-398.
- [Mar97] G.A Margulis, *Oppenheim Conjecture* (Fields Medalists' lectures), World scientific series in 20th century Mathematics, vol. **5** (1997), ed. M.Atiyah, D. Iagolnitzer, Singapore University Press, 272-328.
- [MT92] G.A Margulis and G. Tomanov, *Measure rigidity for algebraic groups over local fields*, C.R. Acad Sci. Paris Ser. I Math. **315** (1992) 1221-1226.
- [MT94] G.A Margulis and G. Tomanov, *Invariant measures for actions of unipotent flows over local fields on homogeneous spaces*, Invent. Math. **116**(1-3) (1994) 347-392.
- [Mau10] F. Maucourant, *An nonhomogeneous orbit of a diagonal subgroup*, Ann. of Math. (2) **171**, (2010) 557-570.

- [Mo66] C.C. Moore, *Ergodicity of flows on homogeneous spaces*, Amer. J. Math. **88** (1966) 154-178.
- [Op1] A. Oppenheim, *The minima of indefinite quaternary quadratic forms*, Proc. Nat. Acad. Sci. USA, **15** (1929) 724-727.
- [Op2] A. Oppenheim, *The minima of indefinite quaternary quadratic forms*, Ann. of Math., **32** (1931) 271-298.
- [Op3] A. Oppenheim, *The minima of quaternary quadratic forms of signature zero*, Proc. London Math. Soc. **37** (1934) 63-81.
- [Op4] A. Oppenheim, *Values of quadratic forms I*, Quat. J. of Math., Oxford ser. (2) **4** (1953) 54-59.
- [Op5] A. Oppenheim, *Values of quadratic forms II*, Quat. J. of Math., Oxford ser. (2) **4** (1953) 60-66.
- [Op6] A. Oppenheim, *Values of quadratic forms III*, Monatsh. Math. **57** (1953) 97-101.
- [OV91] A.L. Onishchik and E.B. Vinberg (Eds.), *Lie groups and Lie algebras III*, Encyclopaedia of mathematical sciences, Vol. 41, Springer-Verlag, (1991), 253 pp.
- [P69] V.P. Platonov, *The problem of strong approximation and the Kneser-Tits conjecture*, math USSR **Izv** **3** (1969) 1139-1147; Addendum, *ibid.* **4** (1970) 784-786.
- [Pra] G. Prasad, *Elementary proof of a theorem of Bruhat-Tits-Rousseau and a theorem of Tits*, Bull. Soc. Math. Fr. , **110**, (1982) 197-202.
- [PR] V.P. Platonov and A.S.Rapinchuk, *Algebraic groups and number theory*, Volume **139**, Academic Press (1994) 614p.
- [PV] A.D. Pollington and S. Velani, *A problem on simultaneous diophantine approximation: Littlewood's conjecture*, Acta Math. **185** (2000), no. 2, 287-306.
- [PVb] A.D. Pollington and S. Velani, *On simultaneously badly approximate numbers*, J. London Math. Soc. **21** (2002), no.1, 29-40.
- [Ratn90] M. Ratner, *On measure rigidity of unipotent subgroups of semisimple groups*, Acta Math. **165**, (1990), 229-309.
- [Ratn91a] M. Ratner, *On Raghunathan's measure conjecture*, Ann. of Math. **134**, (1991), 545-607.
- [Ratn91b] M. Ratner, *Raghunathan's topological conjecture and distribution of unipotent flows*, Duke Math. J. **63**, (1991) 235-280.

- [Ratn92] M. Ratner, *Raghunathan's conjecture for $SL(2, \mathbb{R})$* , Isr. J. of Math. **80** (1992), 1-31.
- [R93] M. Ratner, *Raghunathan's conjectures for p -adic Lie groups*, International Mathematics Research Notices, **No.5** (1993), 141-146.
- [Ser73] J.P. Serre, *A course in arithmetic*, GTM, vol. **7**, Springer New-York (second ed.), (1973).
- [Ser92] J.P. Serre, *Lie algebras and Lie groups*, Lecture Notes in Math., vol. **1500**, Springer New-York (second ed.), (1992).
- [Sh91] N. Shah, *Uniformly distributed orbits of certain flows on homogeneous spaces*, Math. Ann. **289** (1991) 315-334.
- [Sh96] N. Shah, *Limit distribution of expanding translates of certain orbits on homogeneous spaces*, Proc. Indian Acad. Sci. Math. Sci. **106** no 2. (1996), 105–125.
- [Shap11] U.Shapira, *A solution of a problem of Cassels and diophantine properties of cubic numbers*, Ann. of Math. (1) **173**, (2011) 543-557.
- [Sieg] C. L. Siegel, *A mean value theorem in geometry of numbers*, Ann. of Math. **46** (1945), 340-347.
- [Zim84] R.J. Zimmer, *Ergodic Theory and Semisimple Groups*, Monographs in Mathematics Vol. **81**, Birkhäuser, (1984)