

COUNTING SUPERCUSPIDAL REPRESENTATIONS OF p -ADIC GROUPS

A thesis submitted to the School of Mathematics of the University of East
Anglia in partial fulfilment of the requirements for the degree of Doctor of
Philosophy

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December 2012

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Acknowledgements

I would like to thank my supervisor Prof. Shaun Stevens for his enormous support, help and patience during the period of my study in UEA. This thesis would not have been possible to accomplish without him. It is an absolute honor and privilege to be one of his students and I must say I would not find a better supervisor.

I would like, also, to thank my friends in Norwich (in particular Badr Alharbi and Youssef Lazar), Great Britain and Saudi Arabia for their encouragement.

I would like to thank the high ministry of education in Saudi Arabia for granting me a scholarship to complete my degree and achieve my goal.

I dedicate this thesis to my parents Aljoahra and Nasser, who have been giving me support and love throughout my life, my brothers Nife and Khalifah, my sisters Najla, Monirah, Riaa, Assma and my young sister Shoq and my lovely wife Bashyer. I am so lucky to be part of this family.

Abstract

Let F be a non-archimedean local field with residual characteristic $p \neq 2$. In this thesis we will deduce a formula for the number of irreducible supercuspidal representations of $GL_N(F)$, $N \geq 1$, with $\omega_\pi(\varpi_F) = 1$ and level less than or equal to k . Following Blasco, we construct all irreducible supercuspidal representations of the unitary groups $U(1,1)(F/F_0)$ and $U(2,1)(F/F_0)$ by looking at their characteristic polynomials and then compute the number of all these representations according to their level.

Introduction

Let G be a p -adic group defined over a non-archimedean local field F with a finite residue field k_F of order $q = p^m, m \in \mathbb{N}$. In this thesis, we will be concerned with the smooth complex representations of the group G which are supercuspidal i.e. they are not subquotient of any parabolic induced representation. These representations can be constructed by inducing from certain compact open subgroups. Howe constructed the irreducible supercuspidal representations of $GL_N(F)$ for $p \nmid N$ in [14]. Moy in his paper [24] proved that all irreducible supercuspidal representations of $GL_N(F)$ are obtained this way. For any prime l , a construction of irreducible supercuspidal representations of $GL_l(F)$ was given by Carayol in [8] and Bushnell and Kutzko, in [6], generalized this construction to include the irreducible supercuspidal representations of $GL_N(F)$, for any N and p .

We are interested in counting supercuspidal representations, more precisely, in counting by the level up to unramified twist. Carayol in his paper [8] computed, via construction, the number of irreducible supercuspidal representations of $GL_l(F)$ of a fixed level and central character ω_π fixed on the uniformizer ϖ_F . We first look at $GL_N(F)$, generalizing Carayol but the approach is different. Rather than counting through the construction of supercuspidal representations, we generalize a result of Bushnell and Henniart in [5] to include non-integer level. The formula, in fact, is for the number of discrete series representations π of $GL_N(F)$ with fixed level $\ell(\pi) = k/N$ and $\omega_\pi(\varpi_F) = 1$, which is a multiple sum

$$|A_N(F, k)| = \sum_{m|N} \frac{N}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q^d - 1) q^{d\lceil \frac{k}{m} \rceil}.$$

This is proved using the same method which is passing via the Jacquet-Langlands correspondence to the representations of the division algebra D^\times . These representations are counted by Koch [18]. Theorem 9.3 in [38] says there is a bijection map between the union, over the divisors d of N , of the sets of irreducible supercuspidal representations of $GL_N(F)$ with fixed level and $\omega_\pi(\varpi_F)^d = 1$ and the set of discrete series $A_N(F, k)$. We used this bijection to get a formula for the number of supercuspidal representations of $GL_N(F)$.

In the case $N = l$, l is a prime, we recover Carayol's number. We can also simplify the number when N is a power of l . For $\ell(\pi) = 0$, the number of supercuspidal representations of $GL_N(F)$ become easier to count for any N , so, we will give a formula for the number of supercuspidal representations of $GL_N(F)$ of level zero and $\omega_\pi(\varpi_F) = 1$ for any N .

We denote the number of irreducible supercuspidal representations of $GL_l(F)$ of minimal level k and $\omega_\pi(\varpi_F) = 1$, up to equivalence by $\mathfrak{S}(l, lk)^m$. We will study some properties of zeta function of the form $\zeta_q(s) = \sum_{n \geq 0} \frac{\mathfrak{S}(l, n)^m}{q^{ns}}$.

Then we look at unramified unitary groups. Let F_0 be a non-archimedean local field with residual characteristic $p \neq 2$. Let F be an unramified quadratic extension of F_0 and let $\bar{}$ denotes the fixed points in F . The involution $\bar{}$ extend to the adjoint involution of a non-degenerate hermitian form on F^N . We define the unitary group G as follows

$$G = \{g \in GL_N(F) : g\bar{g} = 1\}.$$

We will study the irreducible supercuspidal smooth complex representations of G , for any N . For certain types of representations, which we call “totally split”, we have a formula for the number of irreducible supercuspidal representations of G of level k (the level must be integer).

Then we will study in more detail the irreducible supercuspidal representations of G , where G is either $U(1,1)(F/F_0)$ or $U(2,1)(F/F_0)$. For level zero, we will count the number of these representations for both groups. For positive level, let π be an irreducible supercuspidal representation of G of level $k > 0$, then π contains a fundamental skew stratum $[\Lambda, n, n-1, b]$, where $k = n/e$ and $e = e(\Lambda)$, by [25, Theorem 5.2]. We classify the representations π by looking at the characteristic polynomial $\varphi_b(X)$, which depends only on π . We will consider all possible $\varphi_b(X)$ and then compute the number of irreducible supercuspidal representations for each case. Finally we will give a formula for the number of irreducible supercuspidal representations of G of level less than k .

Chapter 1

PRELIMINARIES

1.1 Notations

In the beginning we start by fixing some notations that we are going to use regularly throughout this thesis. We fix a non-archimedean local field F with absolute value $|\cdot|$.

Let \mathcal{O}_F be the ring of integers of F ;

$$\mathcal{O}_F = \{x \in F : |x| \leq 1\}.$$

We denote the unique maximal ideal of the ring of integers \mathcal{O}_F by \mathcal{P}_F :

$$\mathcal{P}_F = \{x \in F : |x| < 1\}.$$

The maximal ideal, in fact, is principal i.e. \mathcal{P}_F is generated by one element, called a *uniformizer* of F . We fix a uniformizer ϖ_F . We define the group of units to be

$$\mathcal{O}_F^\times = \{x \in F : |x| = 1\}.$$

Any non-trivial ideal of \mathcal{O}_F has the form \mathcal{P}_F^m , for some integer m . The residue field of F is $k_F = \mathcal{O}_F / \mathcal{P}_F$ and it has characteristic p and size $|k_F| = q_F$.

Definition 1.1.1. We define a character of a field L to be a continuous map

$$\chi : L \longrightarrow \mathbb{C}^\times$$

which is homomorphism. The set of characters of the field L is denoted by \widehat{L} .

Definition 1.1.2. Let ψ be a non-trivial character of a local field F . We define the level of ψ to be the least integer m such that $\mathcal{P}_F^m \subset \text{Ker}(\psi)$. We denote the level of ψ by $\ell(\psi)$.

Lemma 1.1.3. Let ψ be a non-trivial character of F . Then there exists an integer $k \geq 0$ such that $\ell(\psi) = k$.

Proof. Let U be an open neighborhood of the identity in \mathbb{C}^\times contained in

$$\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}.$$

Now the subset $\psi^{-1}(U)$ of F is a neighborhood of the identity in F . Since the fractional ideals \mathcal{P}_F^m , $m \in \mathbb{Z}$, form a fundamental system of neighborhoods of the identity in F , then $\psi^{-1}(U)$ must contain \mathcal{P}_F^m for some integer m . Then $\psi(\mathcal{P}_F^m)$ is a group contained in U so $\psi(\mathcal{P}_F^m) = \{1\}$ i.e, $\operatorname{Ker}(\psi) \supset \mathcal{P}_F^m$. \square

Let $x = \varpi^n u$, $u \in \mathcal{O}_F^\times$, and let $\nu_F : F \rightarrow \mathbb{Z} \cup \{\infty\}$ be an additive valuation on F defined by $\nu_F(x) = n$ and $\nu(0) = \infty$.

Lemma 1.1.4. *Suppose ψ is a non-trivial character of F with level k . Then the map*

$$\begin{aligned} F &\longrightarrow \widehat{F} \\ \alpha &\longmapsto \alpha\psi \end{aligned}$$

is an isomorphism, where $\alpha\psi$ is the character of level $k - \nu_F(\alpha)$ defined by $\alpha\psi(x) = \psi(\alpha x)$.

Proof. The map is clearly injective. For any $\alpha, \beta \in F$, $(\alpha + \beta)\psi = \alpha\psi + \beta\psi$, so the map is a homomorphism.

We only need to prove it is surjective, so let $\phi \in \widehat{F}$ of level r . For $\alpha = \varpi_F^{k-r} u$, where $u \in \mathcal{O}_F^\times$, the character $\alpha\psi$ has level r so $\phi|_{\mathcal{P}_F^r} = \alpha\psi|_{\mathcal{P}_F^r}$. If $\alpha' = \varpi_F^{k-r} u'$, where $u' \in \mathcal{O}_F^\times$, then

$$\alpha\psi|_{\mathcal{P}_F^{r-1}} = \alpha'\psi|_{\mathcal{P}_F^{r-1}} \iff \alpha \equiv \alpha' \pmod{\mathcal{P}_F^{k-r+1}}.$$

We have $q_F - 1$ distinct non-trivial characters $\alpha\psi|_{\mathcal{P}_F^{r-1}}$ which are trivial on \mathcal{P}_F^r so one of them must be $\phi|_{\mathcal{P}_F^{r-1}}$, say $\alpha_1\psi$ where $\alpha_1 = \varpi_F^{r-k} u_1$. Keep doing this we obtained a Cauchy sequence $\{\alpha_n\}$, where $\alpha_n = u_n \varpi_F^{k-r}$, such that $\phi|_{\mathcal{P}_F^{r-n}} = \alpha_n\psi|_{\mathcal{P}_F^{r-n}}$ and

$$\alpha_{n+1}\psi|_{\mathcal{P}_F^{r-n}} = \alpha_n\psi|_{\mathcal{P}_F^{r-n}} \iff \alpha_{n+1} \equiv \alpha_n \pmod{\mathcal{P}_F^{n+k-r}}.$$

The Cauchy sequence $\{\alpha_n\}$ has a limit say α . Therefore, $\phi = \alpha\psi$. \square

The multiplicative group F^\times has a (unique) maximal compact subgroup denoted by $U_F = \mathcal{O}_F^\times$ which has a filtration

$$U_F^n = 1 + \mathcal{P}_F^n, \quad n \geq 1,$$

with $U_F^0 = U_F$.

Definition 1.1.5. *Let χ be a non-trivial character of F^\times . The level of χ is the least integer $n \geq 0$ such that χ is trivial on U_F^{n+1} . Again the level of χ will be denoted by $\ell(\chi)$.*

Lemma 1.1.6. *1. For any non-trivial character χ of F^\times , there exists $k \geq 0$ such that $\ell(\chi) = k$.*

2. For any $\zeta \in \mathbb{C}^\times$ and $k \geq 0$, the number of characters χ of F^\times which have level k and such that $\chi(\varpi_F) = \zeta$ is $(q_F - 1)q_F^k$.

Proof. The proof of (1) is similar to Lemma 1.1.3. For part (2), the number of the characters χ is equal to the index $[F^\times : \langle \varpi_F \rangle U_F^{k+1}]$ which is equal to $(q_F - 1)q_F^k$. \square

For any two integers n, m satisfying $1 \leq n < m \leq 2n$, the map $x \mapsto 1 + x$ induces an isomorphism [4, 1.8],

$$\mathcal{P}_F^n / \mathcal{P}_F^m \xrightarrow{\sim} U_F^n / U_F^m$$

Lemma 1.1.7. *Let ψ be a character of F of level one and n, m integers with $0 \leq m < n \leq 2m$. The map*

$$\begin{aligned} \mathcal{P}_F^{1-n} / \mathcal{P}_F^{1-m} &\longrightarrow (U_F^m / U_F^n)^\wedge \\ \alpha &\longmapsto \psi_{F,\alpha} \end{aligned}$$

is an isomorphism, where $\psi_{F,\alpha}(x) = \psi_F(\alpha(x-1))$.

Proof. See [4, Proposition 1.8]. \square

1.2 Modulus Character

In this section we start by introducing the notion of locally profinite groups.

Definition 1.2.1. *A group G is called a topological group if it is a topological space such that the multiplication map $G \times G \rightarrow G$ and the inversion map $G \rightarrow G$ are both continuous.*

Definition 1.2.2. *A topological group G is called locally profinite if any open neighborhood of the identity in G contains a compact open subgroup of G . Let $\Omega(G)$ denote the set of all compact open subgroups of G .*

Example 1.2.3. 1. *The additive group $(F, +)$ and the multiplicative group $(F^\times, *)$ are both locally profinite groups.*

2. *The general linear group $GL_n(F)$ is also a locally profinite group.*

Definition 1.2.4. *Let $K \in \Omega(G)$. The modulus character δ_G of G is defined by*

$$\delta_G(g) = [gKg^{-1} : K] \quad g \in G.$$

We say the group G is unimodular if $\delta_G(g) = 1$ for all $g \in G$.

Lemma 1.2.5. 1. *The modulus character δ_G is independent of choices $K \in \Omega(G)$.*

2. *The modulus character δ_G is trivial on any compact subgroup $K \in \Omega(G)$.*

Proof. See [35, I, 2.7] \square

Example 1.2.6. *The following groups are unimodular: compact subgroups- abelian groups- general linear group $GL_N(F)$.*

1.3 Smooth Representations

In this section we will give some of the basic definitions and results for the representations of locally profinite groups. We will focus only on *smooth* representations.

Definition 1.3.1. *Let G be a locally profinite group and V a complex vector space. A pair (π, V) is called a representation of G if*

$$\pi : G \longrightarrow \text{Aut}_{\mathbb{C}}(V)$$

is a homomorphism.

For $K \subseteq G$, we denote by V^K the set of all vectors v in V such that $\pi(k)v = v$, for all $k \in K$.

Definition 1.3.2. *Let (π, V) be a representation of a locally profinite group G . We say that the representation (π, V) is smooth if one of the following equivalent conditions holds:*

1. *for every $v \in V$, there is a compact open subgroup K of G such that*

$$\pi(k)v = v,$$

for all $k \in K$;

2. *the space V can be written as*

$$V = \bigcup_{K \in \Omega(G)} V^K;$$

3. *the stabilizer of v in G , $\text{Stab}_G(v) = \{g \in G : \pi(g)v = v\}$, is open for all $v \in V$.*

Let U be a subspace of V . Then U is called *G -invariant* if, for any $g \in G$ and $u \in U$, we have

$$\pi(g)u \in U.$$

Let (π_1, V_1) and (π_2, V_2) be smooth representations of G . The set $\text{Hom}_G(\pi_1, \pi_2)$ (or some people call it $\text{Hom}_G(V_1, V_2)$) is the space of linear maps $f : V_1 \rightarrow V_2$ such that

$$f \circ \pi_1(g) = \pi_2(g) \circ f, \quad g \in G \tag{1.1}$$

The representations (π_1, V_1) and (π_2, V_2) are *equivalent* if there exists a \mathbb{C} -isomorphism $f \in \text{Hom}_G(\pi_1, \pi_2)$.

Proposition 1.3.3. *Let (π_1, V_1) and (π_2, V_2) be smooth representations of G and $f \in \text{Hom}_G(\pi_1, \pi_2)$. Then we have*

1. *$\text{Ker}(f)$ is a G -invariant subspace of V_1 ;*
2. *$\text{Im}(f)$ is a G -invariant subspace of V_2 .*

Definition 1.3.4. *Let (π, V) be a smooth representation of G . Then:*

1. We define a subrepresentation of (π, V) to be a pair (ρ, U) where U is a G -invariant subspace of V and

$$\rho(g)u = \pi(g)u$$

for all $g \in G$ and $u \in U$.

2. A quotient of the representation (π, V) is the natural representation σ of G on the quotient space V/W , for W a G -invariant subspace of V ,

$$\sigma(g)(v + w) = \sigma(g)v + W.$$

3. A subquotient of the representation (π, V) is a quotient of a subrepresentation of (π, V) .

Definition 1.3.5. The representation (π, V) is admissible if, for any $K \in \Omega(G)$, the space V^K is finite-dimensional.

Definition 1.3.6. Let (π, V) be a smooth representation of G . Then (π, V) is called:

1. irreducible if the only subrepresentations of π are $\{0\}$ and V itself; otherwise we say the representation (π, V) is reducible;
2. semisimple if the representation π is a direct sum of irreducible subrepresentations;
3. finite type if there exists a finite subset $\{v_1, \dots, v_k\}$ of V such that V is spanned by

$$\{\pi(g)v_i : g \in G, 1 \leq i \leq k\}.$$

Lemma 1.3.7. Let (π, V) be a representation of G . The following are equivalent:

1. the representation (π, V) is semisimple;
2. for any subrepresentation W of π , there is a complementary subrepresentation W^\perp of π such that

$$V = W \oplus W^\perp.$$

Proof. See [4, Proposition 2.2]. □

Lemma 1.3.8. Let (π, V) be a smooth representation of G . Suppose that (π, V) is of finite type, then π has an irreducible quotient.

Proof. Consider the set

$$\Sigma = \{W : W \text{ is a subrepresentation of } \pi \text{ and, } W \neq V\}.$$

The representation π is of finite type, which implies that the set Σ is closed under union of chains, i.e for $W_1 \subset W_2 \subset \dots$, where $W_i \in \Sigma$, then $\cup_i W_i \in \Sigma$. By Zorn's lemma, there is a maximal element $W \in \Sigma$, so V/W is irreducible. □

Corollary 1.3.9. Let (π, V) be a smooth representation of G of finite type. Then π has an irreducible subquotient.

From now on suppose that G/K is countable, for any compact open subgroup K of G . Then we have the following lemma:

Lemma 1.3.10. *Suppose (π, V) is an irreducible smooth representation of G . Then the space V has countable dimension.*

Proof. Suppose $v \in V$ with $v \neq 0$. The smoothness of π implies that there exists a compact open subgroup K of G such that $v \in V^K$. The irreducibility of π implies that V is spanned by the set

$$\{\pi(g)v : g \in G\}$$

and since $v \in V^K$, then the set is

$$\{\pi(g)v : gK \in G/K\}.$$

This set is countable and, therefore, V has countable dimension. □

Proposition 1.3.11. *(Schur's lemma) Suppose (π, V) is an irreducible smooth representation of G . If $f \in \text{Hom}_G(\pi, \pi)$, then f is a scalar i.e. $\text{End}_G(V) = \mathbb{C}$.*

Proof. Let ϕ be a non-zero element in $\text{End}_G(V)$. From Proposition 1.3.3, we have that $\text{Ker}(\phi)$ and $\text{Im}(\phi)$ are both G -invariant subspaces of V . The representation π is irreducible, therefore, $\text{Ker}(\phi) = \{0\}$ and $\text{Im}(\phi) = V$. Thus ϕ is bijective and invertible so $\text{End}_G(V)$ is a division algebra over the complex numbers.

Now let v be a non-zero vector in V . Since the representation π is irreducible, V is spanned by the set

$$\{\pi(g)v : g \in G\}$$

so any element ϕ , in fact, is determined by $\phi(v)$. Since V has countable dimension, by Lemma 1.3.10, so does $\text{End}_G(V)$.

Now if $\phi \in \text{End}_G(V)$ but $\phi \notin \mathbb{C}$, then ϕ is transcendental over \mathbb{C} and the set $\mathbb{C}(\phi)$ is a field. Now consider the subset of $\mathbb{C}(\phi)$

$$\{(\phi - \alpha)^{-1} : \alpha \in \mathbb{C}\}$$

noting that $\phi - \alpha \neq 0$, for all $\alpha \in \mathbb{C}$. This is linearly independent and uncountable. However, this contradicts $\text{End}_G(V)$ having countable dimension. The only possible way is that $\text{End}_G(V) = \mathbb{C}$. □

Corollary 1.3.12. *Let (π, V) be an irreducible smooth representation of G and Z be the center of the group G . Then there exists a character $\omega_\pi : Z \rightarrow \mathbb{C}^\times$ such that:*

$$\pi(z)v = \omega_\pi(z)v,$$

for $z \in Z$ and $v \in V$.

Proof. Let $z \in Z$, then we have

$$\pi(g)\pi(z)v = \pi(z)\pi(g)v,$$

for all $g \in G$. Therefore, $\pi(z) \in \text{End}_G(\pi)$. By Schur's Lemma, we have $\text{End}_G(\pi) = \mathbb{C}$ so there exists $\omega_\pi : Z \rightarrow \mathbb{C}^\times$ such that $\pi(z)v = \omega_\pi(z)v$ for all $z \in Z$ and $v \in V$. Now it is easy to check that $\omega_\pi(z_1 z_2) = \omega_\pi(z_1)\omega_\pi(z_2)$, for any

$z_1, z_2 \in Z$. Let v be a non-zero vector in V . Since the representation (π, V) is smooth, then there exists a compact open subgroup K of G for which $v \in V^K$. Thus,

$$v = \pi(z)v = \omega_\pi(z)v,$$

and ω_π is trivial on $K \cap Z$ so ω_π is a character of Z . □

The character ω_π of Z is called the *central character* of the representation π .

Corollary 1.3.13. *If G is abelian then any irreducible smooth representation of G is one-dimensional.*

1.4 Invariants and Coinvariants

The space V^K , for some closed subgroup K of G , is the largest subspace of V on which the subgroup K acts trivially.

Proposition 1.4.1. *Let (π_i, V_i) be a smooth representation of G , where $i = 1, 2, 3$. Let $\sigma: V_1 \rightarrow V_2$ and $\rho: V_2 \rightarrow V_3$ be G -homomorphisms. Then*

$$V_1 \xrightarrow{\sigma} V_2 \xrightarrow{\rho} V_3$$

is an exact sequence if and only if

$$V_1^K \xrightarrow{\sigma} V_2^K \xrightarrow{\rho} V_3^K$$

is exact, for all compact open subgroups K in $\Omega(G)$.

Proof. Suppose $V_1 \xrightarrow{\sigma} V_2 \xrightarrow{\rho} V_3$ is exact. Obviously, we have $\sigma|_{V_1^K}$ is injective and

$$\text{Im}(\sigma|_{V_1^K}) \subset \text{Ker}(\rho|_{V_2^K}).$$

Let $v_2 \in \text{Ker}(\rho|_{V_2^K})$, then there exists $v_1 \in V_1$ such that $\sigma(v_1) = v_2$. Now, for $k \in K$,

$$\sigma \circ \pi_1(k)v_1 = \pi_2(k) \circ \sigma(v_1) = \pi_2(k)v_2 = v_2 = \sigma(v_1).$$

Since σ is injective, then $v_1 \in V_1^K$ and, therefore, the sequence $V_1^K \xrightarrow{\sigma} V_2^K \xrightarrow{\rho} V_3^K$ is exact.

Conversely, suppose $V_1^K \xrightarrow{\sigma} V_2^K \xrightarrow{\rho} V_3^K$ is exact for all $K \in \Omega(G)$. By the smoothness of π_1 , for any $v_1 \in V_1$, there exists $K \in \Omega(G)$ such that $v_1 \in V_1^K$ and by the exactness we have $\rho \circ \sigma(v_1) = 0$. Now suppose $v_2 \in V_2$ with $\rho(v_2) = 0$. By the smoothness of π_2 , there exists $K \in \Omega(G)$ such that $v_2 \in V_2^K$. By the exactness, $v_2 = \sigma(v_1)$ for some $v_1 \in V_1^K$. □

Proposition 1.4.2. *Let (π_i, V_i) be a smooth representation of G , where $i = 1, 2, 3$, and*

$$0 \rightarrow V_1 \xrightarrow{\varphi} V_2 \xrightarrow{\rho} V_3 \rightarrow 0$$

be an exact sequence. Then the following are equivalent:

1. *the representation π_2 is admissible;*

2. the representations π_1 and π_3 are both admissible.

Proof. By Lemma 1.4.1, the sequence

$$0 \rightarrow V_1^K \xrightarrow{\varphi} V_2^K \xrightarrow{\rho} V_3^K \rightarrow 0 \quad (1.2)$$

is exact for all $K \in \Omega(G)$. Suppose (1) is holds. The properties of exactness of (1.2) implies that the space V_1^K is isomorphic to a subspace of V_2^K so $\dim(V_1^K) \leq \dim(V_2^K)$. Thus, π_1 is admissible. Also by the exactness of (1.2), V_3^K is isomorphic to V_2^K/V_1^K so $\dim(V_3^K) \leq \dim(V_2^K)$, therefore, π_3 is admissible.

Now suppose (2) is holds. By the exactness of (1.2), $\dim(V_2^K) = \dim(V_1^K) + \dim(V_3^K)$. Therefore, π_2 is admissible. \square

Now put

$$V(K) = \text{Span}\{\pi(k)v - v : v \in V, k \in K\}.$$

The quotient $V_K = V/V(K)$ is the largest quotient on which K acts trivially.

Proposition 1.4.3. *Let (π_i, V_i) be a smooth representation of G where $i = 1, 2, 3$. Let $\sigma : V_1 \rightarrow V_2$ and $\rho : V_2 \rightarrow V_3$ be a G -homomorphism. Then*

$$V_1 \xrightarrow{\sigma} V_2 \xrightarrow{\rho} V_3$$

is an exact sequence if and only if

$$(V_1)_K \xrightarrow{\sigma} (V_2)_K \xrightarrow{\rho} (V_3)_K$$

is exact, for all compact open subgroups K in $\Omega(G)$.

Proof. See [35, Chapter I, Proposition 4.9]. \square

Corollary 1.4.4. *For any $K \in \Omega(G)$, we have:*

1. $V = V^K \oplus V(K)$;
2. $V_K \cong V^K$.

Definition 1.4.5. *Let N be a normal subgroup of G and let (σ, V) be a representation of the quotient group G/N and denote by*

$$\varphi : G \rightarrow G/N$$

the canonical homomorphism. We can define a representation of G as follow:

$$\begin{aligned} \pi : G &\longrightarrow \text{Aut}_F(V), \\ g &\longmapsto \sigma(\varphi(g)). \end{aligned}$$

The representation π is called the inflation of σ to G .

Remark 1.4.6. *The relationship between the inflation and (co)-invariants is*

$$\text{Hom}_G(\pi, \text{infl } \sigma) = \text{Hom}_{G/N}(\pi_N, \sigma),$$

where σ is a representation of G/N and $\text{infl } \sigma$ is the inflation of σ to G .

1.5 Contragredients

Let (π, V) be a smooth representation of the locally profinite group G . Denote by V^* the dual space $\text{Hom}_G(V, \mathbb{C})$ and

$$\langle, \rangle : V^* \times V \longrightarrow \mathbb{C}$$

the natural non-degenerate pairing. The space V^* is equipped with a representation π^* of G given by

$$\langle \pi^*(g)v^*, v \rangle = \langle v^*, \pi(g^{-1})v \rangle$$

where $g \in G$, $v^* \in V^*$ and $v \in V$. Note that the representation π being smooth, does not necessarily imply that π^* is also smooth. In fact, V^* is only smooth if V is finite-dimensional.

Definition 1.5.1. Let (π, V) be a smooth representation of G . Set

$$\tilde{V} = \bigcup_{K \in \Omega(G)} (V^*)^K.$$

Then \tilde{V} is a G -invariant subspace of V^* which gives a smooth representation $(\tilde{\pi}, \tilde{V})$ called the contragredient of (π, V) .

Lemma 1.5.2. Let (π, V) be a smooth representation of G and K in $\Omega(G)$ then $\tilde{V}^K \cong (V^K)^*$.

Proof. See [4, Proposition 2.8]. □

Remark 1.5.3. Let (π, V) is smooth representation of G and $(\tilde{\tilde{\pi}}, \tilde{\tilde{V}})$ be the contragredient of $(\tilde{\pi}, \tilde{V})$. It is not true, in general, that $\tilde{\tilde{\pi}} \cong \pi$. However, there is a condition that the representation (π, V) must satisfy in order to have $\tilde{\tilde{\pi}} \cong \pi$.

Proposition 1.5.4. Let (π, V) be a smooth representation of G . The following are equivalent:

- (π, V) is admissible.
- $(\tilde{\pi}, \tilde{V})$ is admissible.
- $\pi \cong \tilde{\tilde{\pi}}$.

Proof. [4, Proposition 2.9]. □

Proposition 1.5.5. Suppose that (π, V) is an admissible representation of G . Then the following are equivalent:

1. (π, V) is irreducible;
2. $(\tilde{\pi}, \tilde{V})$ is irreducible.

Proof. See [4, Proposition 2.10]. □

Let $(\pi_1, V_1), (\pi_2, V_2)$ be smooth representations of G and let $\sigma : V_1 \rightarrow V_2$ be a G -homomorphism. We define $\tilde{\sigma} : \tilde{V}_2 \rightarrow \tilde{V}_1$ as follows

$$\langle \tilde{\sigma}(\tilde{v}_2), v_1 \rangle = \langle \tilde{v}_2, \sigma(v_1) \rangle \quad v_1 \in V_1, v_2 \in V_2.$$

Lemma 1.5.6. *Let (π_i, V_i) be a smooth representation of G , where $i = 1, 2, 3$. Let $\sigma : V_1 \rightarrow V_2$ and $\rho : V_2 \rightarrow V_3$ be G -homomorphisms. If*

$$V_1 \xrightarrow{\sigma} V_2 \xrightarrow{\rho} V_3$$

is exact, then

$$\tilde{V}_3 \xrightarrow{\tilde{\rho}} \tilde{V}_2 \xrightarrow{\tilde{\sigma}} \tilde{V}_1$$

is also exact.

Proof. See [4, Lemma 2.10]. □

1.6 Structure of G

Here we will introduce some special subgroups of the group G which we will use regularly in the rest of this thesis.

A Borel subgroup of G is a maximal connected solvable subgroup.

Lemma 1.6.1. *Let B_1 and B_2 be Borel subgroups of G . Then there exists $g \in G$ such that ${}^g B_1 = B_2$ (where ${}^g B_1 = g^{-1} B_1 g$).*

Proof. See [15, Theorem 21.3] □

For $G = GL_N(F)$, the *standard Borel* subgroup B of G is the set of matrices in G of the form

$$B = \left\{ \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} : a_i \in F^\times \right\}$$

where $*$ represents entries in F .

Definition 1.6.2. *We define a parabolic subgroup of G to be a subgroup containing a Borel subgroup.*

A *standard parabolic* subgroup P of $GL_N(F)$ is a subgroup has the form

$$P = \begin{pmatrix} GL_{N_1}(F) & & * \\ & \ddots & \\ 0 & & GL_{N_r}(F) \end{pmatrix}$$

where (N_1, N_2, \dots, N_r) is a composition of N and $*$ represents entries in F .

Lemma 1.6.3. *The Bruhat Decomposition of $G = GL_N(F)$ is*

$$G = \bigcup_{w \in W} BwB$$

where W is the group of permutation matrices.

Proof. See [29, Theorem 3.2]. □

The *unipotent radical* N of a parabolic subgroup is its maximal connected normal unipotent subgroup. In the case of $G = GL_N(F)$, the unipotent radical of the standard parabolic subgroup P is a subgroup of the form

$$N = \begin{pmatrix} \mathbb{1}_{N_1} & & * \\ & \ddots & \\ 0 & & \mathbb{1}_{N_r} \end{pmatrix}$$

where $\mathbb{1}_{n_i}$ is the identity in $GL_{n_i}(F)$ and $*$ represents entries in F . N is the maximal connected normal unipotent subgroup.

A *Levi* subgroup of a standard parabolic subgroup P is a subgroup L such that $P = L \ltimes N$, in which case, $L \simeq P/N$. The Levi subgroup L of the standard parabolic subgroup in $GL_N(F)$ is called *standard* if it has the form

$$L = \begin{pmatrix} GL_{N_1}(F) & & 0 \\ & \ddots & \\ 0 & & GL_{N_r}(F) \end{pmatrix}$$

Remark 1.6.4. In the case $G = GL_N(F)$, we have that

$$L \cong \prod_{i=1}^r GL_{N_i}(F) \tag{1.3}$$

so L is a direct product of $GL_{N_i}(F)$. However, for any reductive p -adic group, it is not true, in general, that (1.3) is satisfied.

Lemma 1.6.5. The Iwasawa decomposition of G is

$$G = BK_0 = K_0B$$

where $K_0 = GL_N(\mathcal{O}_F)$.

Proof. See [27, Lemma p.38]. □

1.7 Induced Representations

Here in this section we will introduce the notion of induced representations. Suppose (π, V) is a smooth representation of G and H is a closed subgroup of G . To get a representation of H , we simply restrict the representation π to the subgroup H , such a restriction is denoted by $\text{Res}_H^G \pi$. The idea of induced representation is that of forming a smooth representation of G from a smooth representation of the subgroup H .

Definition 1.7.1. Let H be a closed subgroup of G and (σ, W) a smooth representation of H . The induced representation of G , denoted by $\text{Ind}_H^G \sigma$, is the space of all functions $f : G \rightarrow W$ such that

1. $f(hg) = \sigma(h)f(g)$ for all $h \in H$ and $g \in G$;

2. there is a compact open subgroup K of G such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$. This condition is to ensure that the representation is smooth.

The group G acts on the induced representation $\text{Ind}_H^G \sigma$ by

$$(g.f)(x) = f(xg)$$

for $f \in \text{Ind}_H^G \sigma$ and $g \in G$.

Induced representations have a fundamental property called Frobenius Reciprocity.

Proposition 1.7.2 (Frobenius Reciprocity). *Let (π, V) and (σ, W) be representation of G and H respectively, where H is a closed subgroup of G . Then the map*

$$\begin{aligned} \text{Hom}_G(\pi, \text{Ind}_H^G \sigma) &\longrightarrow \text{Hom}_H(\pi, \sigma) \\ \varphi &\longmapsto \psi_\sigma \circ \varphi \end{aligned}$$

is an isomorphism, where ψ_σ is the canonical H -homomorphism

$$\begin{aligned} \psi_\sigma : \text{Ind}_H^G \sigma &\longrightarrow W \\ f &\longmapsto f(1). \end{aligned}$$

Proof. See [4, 2.4]. □

Definition 1.7.3. A compactly induced representation, denoted by $\text{c-Ind}_H^G \sigma$, is the subspace of the induced representation $\text{Ind}_H^G \sigma$ which consist all functions in $\text{Ind}_H^G \sigma$ with compact support modulo H ; in another words, $f \in \text{c-Ind}_H^G \sigma$ if and only if there is a compact subset E of G such that

$$\text{Supp}(f) = HE.$$

Compact induction has its own form of Frobenius Reciprocity. Before we state the property we need the following canonical H -homomorphism, where H is open in G ,

$$\begin{aligned} \alpha_\sigma^c : W &\longrightarrow \text{c-Ind}_H^G \sigma \\ w &\longmapsto f_w, \end{aligned}$$

where $f_w(h) = \sigma(h)w$, $h \in H$.

Proposition 1.7.4. [4, Proposition 2.5] *Let H be an open subgroup of G , let (σ, W) be a smooth representation of H and (π, V) a smooth representation of G . The canonical map*

$$\begin{aligned} \text{Hom}_G(\text{c-Ind}_H^G \sigma, \pi) &\longrightarrow \text{Hom}_H(\sigma, \pi) \\ f &\longmapsto f \circ \alpha_\sigma^c, \end{aligned}$$

is an isomorphism.

Proposition 1.7.5. *Let (σ, W) be a smooth representation of a closed subgroup H . If $H \backslash G$ is compact and the representation σ is admissible then $\text{Ind}_H^G \sigma$ and $\text{c-Ind}_H^G \sigma$ are also admissible and, in fact, they are equal.*

Proof. See [35, 5.6]. □

Lemma 1.7.6. *Suppose (ρ_1, W_1) , (ρ_2, W_2) and (ρ_3, W_3) are smooth representations of a closed subgroup H . Suppose $\sigma : W_1 \rightarrow W_2$ and $\tau : W_2 \rightarrow W_3$ are H -homomorphisms. If*

$$W_1 \xrightarrow{\sigma} W_2 \xrightarrow{\tau} W_3$$

is exact then

$$\text{Ind}_H^G W_1 \xrightarrow{\text{Ind}(\sigma)} \text{Ind}_H^G W_2 \xrightarrow{\text{Ind}(\tau)} \text{Ind}_H^G W_3$$

and

$$\text{c-Ind}_H^G W_1 \xrightarrow{\text{c-Ind}(\sigma)} \text{c-Ind}_H^G W_2 \xrightarrow{\text{c-Ind}(\tau)} \text{c-Ind}_H^G W_3$$

are exact.

Proof. See [4, Propositions 2.4, 2.5]. □

Proposition 1.7.7. *Suppose H and K are subgroup of G and $H \subset K$ and (ρ, W) is a smooth representation of H . Then*

$$\text{Ind}_H^G \rho \cong \text{Ind}_K^G \text{Ind}_H^K \rho$$

and

$$\text{c-Ind}_H^G \rho \cong \text{c-Ind}_K^G \text{c-Ind}_H^K \rho.$$

Proof. See [35, 5.3]. □

1.8 Matrix Coefficients

Let G be a locally profinite group and Z be the center of G . Let (π, V) be a smooth representation of G .

Definition 1.8.1. *For $v \in V$ and $\tilde{v} \in \tilde{V}$, the map*

$$\begin{aligned} \varphi_{\tilde{v} \otimes v} : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \langle \tilde{v}, \pi(g)v \rangle \end{aligned}$$

is called the matrix coefficient of π associated to \tilde{v} and v .

Denote by $\mathcal{C}(\pi)$ the space spanned by the matrix coefficients $\varphi_{\tilde{v} \otimes v}$ of π .

Definition 1.8.2. *The representation π is called Z -compact if for all $v \in V$ and $\tilde{v} \in \tilde{V}$, the support of $\varphi_{\tilde{v} \otimes v}$ is compact modulo Z , i.e.*

$$\text{Supp}(\varphi_{\tilde{v} \otimes v}) = EZ$$

where E is compact.

The space $\tilde{V} \otimes V$ carries a smooth representation of the group $G \times G$. The map

$$\begin{aligned} \tilde{V} \otimes V &\longrightarrow \mathcal{C}(\pi) \\ \tilde{v} \otimes v &\longmapsto \varphi_{\tilde{v} \otimes v} \end{aligned}$$

is a $G \times G$ -homomorphism and is surjective.

Proposition 1.8.3. *1. If (π, V) is an irreducible Z -compact representation of G , then π is admissible.*

2. Let (π, V) be an irreducible admissible representation of G and suppose there exists $v \in V, \tilde{v} \in \tilde{V}$ such that $\varphi_{\tilde{v} \otimes v}$ is compact supported modulo Z . Then π is Z -compact.

Proof. See [4, Proposition 10.1] □

1.9 Supercuspidal Representations

In section (1.6), we introduced a Borel subgroup B , parabolic subgroups P , Levi subgroup L and the unipotent radical N .

We are going to build a representation π , starting with a representation σ of L . We inflate this representation to get a representation of P , $\text{infl}_L^P \sigma$. Finally we induce from P to G to get a representation of G . This procedure is called *parabolic induction* and is denoted by i_L^G . The *parabolic restriction* is the opposite of parabolic induction. We start with a representation π of G and then restrict π to P to get a representation $\text{Res}_P^G \pi$ of P . Finally a representation of L can be obtained by the N -coinvariants of $\text{Res}_P^G \pi$. We denote the parabolic restriction by r_L^G .

Lemma 1.9.1. *We have the following:*

1. i_L^G and r_L^G are exact.
2. i_L^G and r_L^G are transitive.
3. i_L^G respects admissibility.
4. r_L^G respect finite type.

Proof. See [35, Chapter II, 2.1] □

For an irreducible smooth representation (σ, W) of L , the Frobenius Reciprocity is

$$\text{Hom}_G(\pi, i_L^G \sigma) = \text{Hom}_L(r_L^G \pi, \sigma)$$

Definition 1.9.2. *Let (π, V) be a smooth representation of G . We say (π, V) is quasi-cuspidal if, for any smooth representation σ of a proper Levi subgroup L , we have*

$$\text{Hom}_G(\pi, i_L^G \sigma) = 0.$$

Let (π, V) be a smooth representation of G . We say (π, V) is *cuspidal* if it is:

1. quasi-cuspidal;
2. admissible.

Definition 1.9.3. *The representation π is called supercuspidal if it is not a subquotient of a parabolically induced representation.*

Theorem 1.9.4. *Let (π, V) be an irreducible smooth representation of G . Then the following are equivalent:*

1. (π, V) is supercuspidal;
2. (π, V) is Z -compact.

Proof. See [27, Corollary p.56]. □

Theorem 1.9.5. *(Jacquet) Let (π, V) be a smooth irreducible representation of G . Then there exists a parabolic subgroup $P = LN$ and an irreducible supercuspidal representation (σ, W) of L such that*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_L^G \sigma) \neq 0.$$

Proof. See [27, Theorem p.54]. □

Remark 1.9.6. *The Theorem 1.9.5, implies that the notion of irreducible supercuspidal and cuspidal representations of $GL_N(F)$ coincide.*

Theorem 1.9.7. *Let (π, V) be an irreducible smooth representation of $GL_N(F)$. Then (π, V) is admissible.*

Proof. See [27, Theorem p.55]. □

Corollary 1.9.8. *If (π, V) is an irreducible cuspidal representation of G then the contagredient representation $(\tilde{\pi}, \tilde{V})$ is also cuspidal.*

Theorem 1.9.9. *(Harish-Chandra) Let (π, V) be a smooth representation of G . Then the following are equivalent:*

1. The representation (π, V) is supercuspidal;
2. $r_L^G \pi = 0$.

Proof. See [27, Theorem p.52]. □

A Haar measure on G is a non-zero measure μ which is invariant under left-translation by G . Now we define a new concept of irreducible supercuspidal representations of G , that is the formal degree.

Definition 1.9.10. *The formal degree $d(\pi)$ of $\pi = \mathrm{c}\text{-Ind}_K^G \sigma$, where $K \in \Omega(G)$, is given by*

$$d(\pi) = \frac{\dim(\sigma)}{\mu(K)}.$$

which depends on the choice of the Haar measure.

1.10 Intertwining

In this section let G be any p -adic reductive group.

Definition 1.10.1. Suppose (ρ_i, W_i) is an irreducible smooth representation of a compact open subgroup K_i , where $i = 1, 2$. We say an element $g \in G$ intertwines ρ_1 with ρ_2 if

$$\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0,$$

where $K_1^g = g^{-1}K_1g$ and $\rho_1^g(x) = \rho(gxg^{-1})$, for $x \in K_1^g$. We say g intertwines a representation ρ of G if g intertwines ρ with itself.

Remark 1.10.2. If both ρ_1 and ρ_2 are characters then the definition is equivalent to:

$$\rho_1^g|_{K_1^g \cap K_2} = \rho_2|_{K_1^g \cap K_2}.$$

Definition 1.10.3. Let (π, V) be a smooth representation of G and (σ, W) an irreducible representation of a compact open subgroup K of G . We say π contains σ if

$$\mathrm{Hom}_K(\sigma, \pi) \neq 0.$$

Proposition 1.10.4. For $i = 1, 2$ let K_i be a compact open subgroup of G and ρ_i be an irreducible smooth representation of K_i . Let (π, V) be an irreducible smooth representation of G . Then if (π, V) contains both ρ_1 and ρ_2 then there exists $g \in G$ which intertwines ρ_1 with ρ_2 .

Proof. See [4, Proposition 11.1] □

Lemma 1.10.5. Let $g \in G$. Then g intertwines ρ_1 with ρ_2 if and only if g^{-1} intertwines ρ_2 with ρ_1 .

Proof. See [4, §11]. □

The following Theorem is a very significant result proved by Caryl in his paper [8, Proposition 1.5]. The importance of this theorem comes from the fact that it gives us a way to construct supercuspidal representations.

Theorem 1.10.6. Suppose K is an open subgroup of G such that K/Z is compact, where Z is the center of G . Let (ρ, W) be an irreducible smooth representation of K and suppose that $g \in G$ intertwines ρ if and only if $g \in K$. Then the compactly induced representation $\mathrm{c}\text{-Ind}_K^G \rho$ is irreducible and supercuspidal.

Proof. See [8, Proposition 1.5]. □

1.11 Lattice Sequences and Chains

In this section, let

$$\begin{aligned} V &= \text{an } N\text{-dimensional vector space over } F; \\ A &= \mathrm{End}_F(V); \\ G &= \mathrm{Aut}_F(V) \cong \mathrm{GL}_N(F). \end{aligned}$$

Definition 1.11.1. We say L is an \mathcal{O}_F -lattice in V if we can write it as

$$L = \left\{ \sum_{i=1}^N a_i v_i : a_i \in \mathcal{O}_F \right\}$$

where $\{v_1, \dots, v_N\}$ is a basis for V .

Definition 1.11.2. An \mathcal{O}_F -lattice sequence in V is a function Λ from \mathbb{Z} to the set of \mathcal{O}_F -lattices in V such that:

1. for $n \geq m$, integers, $\Lambda(n) \subseteq \Lambda(m)$;
2. there exists a positive integer $e = e(\Lambda)$, called the period of Λ , such that:

$$\Lambda(n + e) = \varpi_F \Lambda(n), \quad n \in \mathbb{Z}.$$

If the lattice sequence Λ is injective, we call it an \mathcal{O}_F -lattice chain.

Proposition 1.11.3. Suppose \mathcal{L} is an \mathcal{O}_F -lattice chain in V . The value of the period is $1 \leq e(\mathcal{L}) \leq N$.

Proof. Since \mathcal{L} is a chain, then we can write it as

$$\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$$

where L_i is an \mathcal{O}_F -lattice in V . Set $e = e(\mathcal{L})$ and consider the quotient

$$L_0/L_e = L_0/\varpi_F L_0.$$

This is a vector space over k_F of dimension N . The quotients L_i/L_e , for $0 \leq i \leq e$, form a flag of subspaces of L_0/L_e ,

$$\{0\} \subsetneq L_{e-1}/L_e \subsetneq \dots \subsetneq L_1/L_e \subsetneq L_0/L_e.$$

Therefore, the value which e can take is between 1 and N . □

Suppose Λ is an \mathcal{O}_F -lattice sequence in V . We define an \mathcal{O}_F -lattice $\mathfrak{A}(\Lambda)$ in A by:

$$\begin{aligned} \mathfrak{A}(\Lambda) &= \bigcap_{0 \leq i \leq e-1} \text{End}_{\mathfrak{o}_F}(\Lambda(i)) \\ &= \{x \in A : x\Lambda(i) \subseteq \Lambda(i), i \in \mathbb{Z}\}. \end{aligned}$$

The \mathcal{O}_F -lattice $\mathfrak{A}(\Lambda)$ in A forms a ring under multiplication and it is called an \mathcal{O}_F -order.

An \mathcal{O}_F -lattice sequence Λ in V gives a filtration of A by

$$\mathfrak{P}^n(\Lambda) = \{x \in A : x\Lambda(m) \subseteq \Lambda(m+n), \forall m \in \mathbb{Z}\},$$

for $n \in \mathbb{Z}$. This induces a valuation on A given by:

$$\begin{aligned} \nu_\Lambda : A &\longrightarrow \mathbb{Z} \cup \{\infty\} \\ x &\longmapsto \max\{n \in \mathbb{Z} : x \in \mathfrak{P}^n(\Lambda)\}. \end{aligned}$$

where we understand $\nu_\Lambda(0) = \infty$. We say $\mathfrak{P}^0(\Lambda)$ is the \mathcal{O}_F -order $\mathfrak{A}(\Lambda)$ determined by the chain associated to Λ . When $n = 1$, we say $\mathfrak{P}(\Lambda) = \mathfrak{P}^1(\Lambda)$ is the Jacobson radical.

Definition 1.11.4. An element $g \in G$ is called Λ -invertible if $\nu_\Lambda(g^{-1}) = -\nu_\Lambda(g)$. Equivalently,

$$g\Lambda(i) = \Lambda(i + \nu_\Lambda(g)) \quad \forall i \in \mathbb{Z}.$$

Let Λ be an \mathcal{O}_F -lattice sequence in V . We define a compact open subgroup $U(\Lambda)$ of G by

$$U(\Lambda) = U^0(\Lambda) = \mathfrak{A}(\Lambda)^\times.$$

This has a filtration given by

$$U^n(\Lambda) = 1 + \mathfrak{P}^n(\Lambda),$$

for any $n \geq 1$.

Remark 1.11.5. The unit subgroups $U^n(\Lambda)$, $n \geq 1$, are compact open subgroups of G and, moreover, they are normal in $U(\Lambda)$.

With the notations above, we have:

Lemma 1.11.6. Let $n, m \in \mathbb{Z}$ with $0 \leq m < n \leq 2m$. The map

$$\begin{aligned} \mathfrak{P}^m(\Lambda)/\mathfrak{P}^n(\Lambda) &\longrightarrow U^m(\Lambda)/U^n(\Lambda) \\ x + \mathfrak{P}^n(\Lambda) &\longmapsto (1+x)U^n(\Lambda) \end{aligned}$$

is an isomorphism.

Proof. See [7, Section 3]. □

Let \mathcal{L} be an \mathcal{O}_F -lattice chain in V . We define the G -normalizer of $\mathfrak{A}(\mathcal{L})$ to be the set

$$\begin{aligned} \mathcal{K}(\mathfrak{A}(\mathcal{L})) &= \{g \in G : g\mathfrak{A}(\mathcal{L})g^{-1} = \mathfrak{A}(\mathcal{L})\} \\ &= \{g \in G : gL \in \mathcal{L} = \mathcal{L}\}. \end{aligned}$$

Remark 1.11.7. The subgroup $F^\times U(\mathcal{L})$ is contained in $\mathcal{K}(\mathfrak{A}(\mathcal{L}))$. Moreover, $\mathcal{K}(\mathfrak{A}(\mathcal{L}))$ normalizes both subgroups $U(\mathcal{L})$ and $U^n(\mathcal{L})$.

1.12 Characters

Denote by \widehat{A} the set of characters of A . For any non-trivial character ψ of F , we can define a character ψ_A of A by

$$\psi_A = \psi \circ \text{tr}_A$$

where tr_A is the trace map.

Lemma 1.12.1. The map

$$\begin{aligned} A &\longrightarrow \widehat{A}, \\ \alpha &\longmapsto \alpha\psi_A, \end{aligned}$$

is an isomorphism where $\alpha\psi_A(x) = \psi_A(\alpha x)$.

Proof. See [37, Corollary p.41]. \square

Recall that a non trivial character ψ of F has level one if $\mathcal{P}_F \subseteq \text{Ker}(\psi)$ but $\mathcal{O}_F \not\subseteq \text{Ker}(\psi)$. From now we fix ψ to be a non-trivial character of F of level one. Suppose L is an \mathcal{O}_F -lattice in A and define the following set:

$$\begin{aligned} L^* &= \{x \in A : \psi_A(xy) = 1, \forall y \in L\} \\ &= \{x \in A : \text{tr}_A(xy) \in \mathcal{P}_F, \forall y \in L\} \end{aligned}$$

Remark 1.12.2. *The set L^* is also an \mathcal{O}_F -lattice with the following properties: let L_i be an \mathcal{O}_F -lattice in A for $i = 1, 2$ then,*

1. $L^{**} = L$.
2. $(L_1 + L_2)^* = L_1^* \cap L_2^*$.
3. $(L_1 \cap L_2)^* = L_1^* + L_2^*$.

Lemma 1.12.3. *Let Λ be an \mathcal{O}_F -lattice sequence in V . Then $(\mathfrak{P}^n(\Lambda))^* = \mathfrak{P}^{1-n}(\Lambda)$ and hence*

$$(\mathfrak{P}^m(\Lambda)/\mathfrak{P}^n(\Lambda))^\wedge \cong \mathfrak{P}^{1-n}(\Lambda)/\mathfrak{P}^{1-m}(\Lambda)$$

for any integers n, m with $m \leq n$.

Proof. See [7, 4.3]. \square

Corollary 1.12.4. *Let $n, m \in \mathbb{Z}$ with $0 \leq m < n \leq 2m + 1$ then we have the following isomorphism:*

$$\begin{aligned} \mathfrak{P}^{-n}/\mathfrak{P}^{-m} &\longrightarrow (U^{m+1}/U^{n+1})^\wedge, \\ b + \mathfrak{P}^{-m} &\longmapsto \psi_{A,b}|_{U^{m+1}}, \end{aligned}$$

where $\psi_{A,b}(x) = \psi(\text{tr}_A b(x-1))$. For brevity, we denote $\psi_{A,b}$ by ψ_b .

Proof. See [7, 4.3]. \square

Definition 1.12.5. *Suppose (π, V) is an irreducible smooth representation of G . Let $S(\pi)$ be the set of all pairs (Λ, n) , where Λ is an \mathcal{O}_F -lattice chain in V and n is a non-negative integer, such that the representation π contains the trivial character of $U^{n+1}(\Lambda)$. Define the normalized level $\ell(\pi)$ of π as:*

$$\ell(\pi) = \min\{n/e(\Lambda) : (\Lambda, n) \in S(\pi)\}.$$

We say $\ell(\pi)$ is minimal (or π is minimal) if $\ell(\pi) \leq \ell(\ell(\pi \otimes \chi))$, for all characters χ of F . We say π is minimal if $\ell(\pi)$ is minimal.

Theorem 1.12.6 (Existence). *Let (π, V) be an irreducible smooth representation of G , then the normalized level $\ell(\pi)$ exists.*

Proof. We prove this by showing that the set $S(\pi)$ is not empty. Let $v \in V$, then by the smoothness of π , there is a compact open subgroup K of G such that

$$\pi(k)v = v, \quad \forall k \in K.$$

The subgroup K contains $U^m(\Lambda)$ for some positive integer m . The representation π , therefore, contains the trivial character of $U^m(\Lambda)$ so the set $S(\pi)$ is not empty. Moreover, $\{n/e(\Lambda) : (\Lambda, n) \in S(\pi)\}$ is contained in the discrete series set $\{n/e : n, e \in \mathbb{N}, e \leq N\}$ so it has a minimal element. \square

1.13 Strata

Definition 1.13.1. A stratum in A is a quadruple $[\Lambda, n, r, b]$ consisting of an \mathcal{O}_F -lattice sequence Λ , integers $n > r \geq 0$ and an element $b \in \mathfrak{P}^{-n}(\Lambda)$.

Let $[\Lambda, n, r, b_1]$ and $[\Lambda, n, r, b_2]$ be two strata in A . We say they are equivalent if

$$b_1 \equiv b_2 \pmod{\mathfrak{P}^{-r}(\Lambda)}.$$

Remark 1.13.2. We say the stratum $[\Lambda, n, r, b]$ is trivial if $b \in \mathfrak{P}^{-r}(\Lambda)$. In this case the stratum is equivalent to $[\Lambda, n, r, 0]$.

If $0 \leq r < n \leq 2r + 1$, we can associate the character ψ_b of $U^r(\Lambda)/U^{n+1}(\Lambda)$ to the stratum $[\Lambda, n, r, b]$ and it depends only on the equivalence class of the stratum.

Definition 1.13.3. Let (π, V) be an irreducible smooth representation of G and let $[\Lambda, n, r, b]$ be a stratum with $n \leq 2r + 1$. We say π contains $[\Lambda, n, r, b]$ if it contains the character ψ_b .

Lemma 1.13.4. Let $[\Lambda, n, r, b]$ be a stratum in A . Suppose $b' \in b + \mathfrak{P}^{1-n}(\Lambda)$ is Λ -invertible and $\nu_\Lambda(b') = -n$. Then $\mathfrak{P}^{1-n}(\Lambda) \not\subseteq \mathfrak{P}^{-n}(\Lambda)$.

Proof. See [7, Proposition 3.5]. \square

Lemma 1.13.5. For $i = 1, 2$, let $[\Lambda_i, n_i, r_i, b_i]$ be strata in A . If $n_i \leq 2r_i + 1$, then the following are equivalent:

1. the element $g \in G$ intertwines the character ψ_{b_1} with the character ψ_{b_2} ;
2. the intersection

$$g^{-1}(b_1 + \mathfrak{P}^{1-r_1}(\Lambda_1))g \cap b_2 + \mathfrak{P}^{1-r_2}(\Lambda_2) \quad (1.4)$$

is non-empty.

Proof. (1) \Rightarrow (2). Suppose $g \in G$ intertwines ψ_{b_1} with ψ_{b_2} so

$$\psi_{b_1}^g(x) = \psi_{b_2}(x), \quad \text{for all } x \in U^{n_1}(\Lambda_1)^g \cap U^{n_2}(\Lambda_2)$$

where

$$\begin{aligned} \psi_{b_1}^g(x) = \psi_{b_1}(gxg^{-1}) &= \psi(\text{tr}_A b_1(gxg^{-1} - 1)) \\ &= \psi(\text{tr}_A g^{-1}b_1g(x - 1)) = \psi_{g^{-1}b_1g}(x), \end{aligned}$$

and $U^{n_1}(\Lambda_1)^g = 1 + g^{-1}\mathfrak{P}^{n_1}(\Lambda_1)g$. This implies

$$\psi(\text{tr}_A((g^{-1}b_1g - b_2)y)) = 1$$

for all $y = x - 1 \in \mathfrak{P}^{n_1}(\Lambda_1)^g \cap \mathfrak{P}^{n_2}(\Lambda_2)$. Therefore,

$$g^{-1}b_1g - b_2 \in (g^{-1}\mathfrak{P}^{n_1}(\Lambda_1)g \cap \mathfrak{P}^{n_2}(\Lambda_2))^*$$

By remark (1.12.2),

$$(\mathfrak{P}^{n_1}(\Lambda_1)^g \cap \mathfrak{P}^{n_2}(\Lambda_2))^* = (\mathfrak{P}^{n_1}(\Lambda_1)^g)^* + (\mathfrak{P}^{n_2}(\Lambda_2))^* = \mathfrak{P}^{1-n_1}(\Lambda_1)^g + \mathfrak{P}^{1-n_2}(\Lambda_2)$$

Therefore, there are $\beta_1 \in \mathfrak{P}^{1-n_1}(\Lambda_1)$ and $\beta_2 \in \mathfrak{P}^{1-n_2}(\Lambda_2)$ such that:

$$g^{-1}b_1g - b_2 = g^{-1}\beta_1g + \beta_2$$

Therefore,

$$g^{-1}(b_1 - \beta_1)g = b_2 + \beta_2 \in g^{-1}(b_1 + \mathfrak{P}^{1-n_1}(\Lambda_1))g \cap b_2 + \mathfrak{P}^{1-n_2}(\Lambda_2).$$

so the intersection is not empty.

(2) \Rightarrow (1). Now suppose $b \in g^{-1}(b_1 + \mathfrak{P}^{1-n_1}(\Lambda_1))g \cap b_2 + \mathfrak{P}^{1-n_2}(\Lambda_2)$ is not empty so there exist $\beta_1 \in \mathfrak{P}^{1-n_1}(\Lambda_1)$ and $\beta_2 \in \mathfrak{P}^{1-n_2}(\Lambda_2)$ such that:

$$b = g^{-1}(b_1 + \beta_1)g = b_2 + \beta_2$$

Now we have $\psi_b(x) = \psi_{g^{-1}b_1g}(x) = \psi_{b_1}^g(x)$ for all $x \in U^{n_1}(\Lambda_1)^g$. Also we have, $\psi_b(x) = \psi_{b_2}(x)$ for all $x \in U^{n_2}(\Lambda_2)$. Therefore, $\psi_b = \psi_{b_1}^g = \psi_{b_2}$ on $U^{n_1}(\Lambda_1)^g \cap U^{n_2}(\Lambda_2)$. \square

Remark 1.13.6. For arbitrary strata, we say that $g \in G$ intertwines $[\Lambda_1, n_1, r_1, b_1]$ with $[\Lambda_2, n_2, r_2, b_2]$ if g satisfies (1.4). We denote the set of elements in G which intertwines these strata by $\mathcal{I}_G([\Lambda_1, n_1, r_1, b_1], [\Lambda_2, n_2, r_2, b_2])$ and the set of elements in G which intertwine $[\Lambda_1, n_1, r_1, b_1]$ with itself by $\mathcal{I}_G([\Lambda_1, n_1, r_1, b_1])$.

Definition 1.13.7. A stratum $[\Lambda, n, r, b]$ in A is called fundamental if $r = n - 1$ and the coset $b + \mathfrak{P}^{1-n}(\Lambda)$ contains no nilpotent element.

Proposition 1.13.8. Let $[\Lambda, n, n - 1, b]$ be a stratum in A . Then the stratum $[\Lambda, n, n - 1, b]$ is not fundamental if and only if there exists a stratum $[\Lambda', n', n' - 1, b']$ such that:

1. $b + \mathfrak{P}^{1-n}(\Lambda) \subset \mathfrak{P}^{-n'}(\Lambda')$;
2. $n'/e(\Lambda') < n/e(\Lambda)$.

Proof. See [7, Theorem 5.3] \square

Lemma 1.13.9. Let $[\Lambda, n, n - 1, b]$ be a fundamental stratum which is intertwined with another stratum $[\Lambda', n', n' - 1, b']$ in A . Then $n/e(\Lambda) \leq n'/e(\Lambda')$ with equality if and only if $[\Lambda', n', n' - 1, b']$ is fundamental.

Proof. See [7, Corollary 5.4] \square

Theorem 1.13.10. *Let π be an irreducible smooth representation of G . Then one of the followings holds:*

1. *the representation π contains a fundamental stratum (positive level);*
2. *the representation π contains the trivial character of $U^1(\Lambda)$, for some \mathcal{O}_F -lattice chain Λ (level zero).*

Proof. For (1), suppose $\ell(\pi) > 0$. Then the representation π contains a character of $U^n(\Lambda)$ trivial on $U^{n+1}(\Lambda)$. This character has the form ψ_b , by Lemma 1.12.4, which corresponds to the stratum $[\Lambda, n, n-1, b]$. If this stratum is not fundamental, then by Proposition 1.13.8, there exists a stratum $[\Lambda', n', n'-1, b']$ such that

$$b + \mathfrak{P}^{1-n}(\Lambda) \subset \mathfrak{P}^{-n'}(\Lambda').$$

which implies $U^{n'+1}(\Lambda') \subset U^n(\Lambda)$, by duality. Proposition 1.13.8 also implies $n'/e(\Lambda') < n/e(\Lambda)$. Now the representation π contains the trivial character of $U^{n'+1}(\Lambda')$, contradicts with the minimality of $n/e(\Lambda)$. For (2), See [7, Theorem 5.5(a)]. \square

Lemma 1.13.11. *Let π be an irreducible smooth representation of $GL_N(F)$ and let χ be a character of F^\times . Suppose $\ell(\pi) \neq \ell(\chi)$, then:*

$$\ell(\pi \otimes \chi \circ \det) = \max\{\ell(\pi), \ell(\chi)\}.$$

Proof. We have two cases:

1. First case when $\ell(\pi) > \ell(\chi)$: the character $\chi \circ \det$ is trivial on $U_\Lambda^{e\ell(\pi)}$, where $e = e(\Lambda)$, and $\det(U^{e\ell(\pi)}(\Lambda)) = U_F^{\ell(\pi)}$. Therefore, the representation $\pi \otimes \chi \circ \det$ contains the trivial character of $U^{e\ell(\pi)+1}(\Lambda)$. Hence,

$$\ell(\pi \otimes \chi \circ \det) = \ell(\pi).$$

2. Second case when $\ell(\pi) < \ell(\chi)$: the representation $\pi \otimes \chi \circ \det$ contains the character $\chi \circ \det$ of $U^{e\ell(\chi)}(\Lambda)$ which associated to a fundamental scalar stratum of the form $[\Lambda, e\ell(\chi), e\ell(\chi) - 1, b]$ where $b \notin \mathfrak{P}^{1-e\ell(\chi)}$. Therefore,

$$\ell(\pi \otimes \chi \circ \det) = \ell(\chi).$$

\square

Corollary 1.13.12. *If $\ell(\pi) = n$ and $\ell(\pi \otimes \chi) < n$ then $\ell(\pi) = \ell(\chi)$.*

Definition 1.13.13. *Let $[\Lambda, n, n-1, b]$ be a stratum in A and $e = e(\Lambda)$. Let $y_b = \varpi_F^{n/g} b^{e/g} \in \mathfrak{A}(\Lambda)$ where $g = \gcd(n, e)$. We define the characteristic polynomial $\varphi_b(X)$ of the stratum $[\Lambda, n, n-1, b]$ to be the reduction of $\Phi(X)$ modulo \mathcal{P}_F , where $\Phi(X)$ is the characteristic polynomial of y_b .*

Remark 1.13.14. *The stratum $[\Lambda, n, n-1, b]$ is fundamental if and only if $\varphi_b(X) \neq X^N$.*

Definition 1.13.15. *A stratum $[\Lambda, n, n-1, b]$ in A is called split if the characteristic polynomial $\varphi_b(X)$ has at least two distinct prime factors.*

Definition 1.13.16. Let $b \in A$ and $E := F[b]$. We say the element b is minimal over F , where $e = e(E/F)$ is the ramification index, if E is a field and it satisfies:

1. $\gcd(\nu_F(b), e) = 1$;
2. the element $\varpi_F^{-\nu_F(b)} b^e + \mathcal{P}_E$ generates the field k_E/k_F .

Definition 1.13.17. Let $[\Lambda, n, n-1, b]$ be a stratum of period e . We say the stratum is simple if:

1. b is minimal;
2. the field $E^\times = F[b]^\times$ normalizes $\mathfrak{P}^0(\Lambda)$;
3. $\nu_\Lambda(b) = -n$.

Lemma 1.13.18. Let $[\Lambda, n, n-1, b]$ be a non-split fundamental stratum in A . Then there exists a simple stratum $[\Lambda', n', n'-1, b']$ such that

$$b + \mathfrak{P}^{1-n}(\Lambda') \subset b' + \mathfrak{P}^{1-n'}(\Lambda').$$

Proof. See [6, (2.3.4)]. □

Theorem 1.13.19. Let π be an irreducible supercuspidal representation of G . Then one of the following holds:

1. the representation π has level zero;
2. the representation contains a simple stratum.

Proof. See [7, Corollary 7.13]. □

Chapter 2

COUNTING SUPERCUSPIDAL REPRESENTATIONS OF $GL_N(F)$

In this section F is a non-archimedean local field with the ring of integers \mathcal{O}_F , the maximal ideal \mathcal{P}_F and the residue field k_F of size q .

2.1 Discrete Series Representations of $GL_N(F)$

In this section, we will study the discrete series representations of $GL_N(F)$, in particular, supercuspidals and the Steinberg representations. Let $G = GL_N(F)$.

Definition 2.1.1. *Let (π, V) be a smooth representation of G and χ a character of F^\times . The twist of π by χ is the smooth representation $(\pi\chi, V)$ of G defined by*

$$\pi\chi(g) = \pi(g)\chi(\det(g)), \quad g \in G$$

Lemma 2.1.2. *Let (π, V) be a smooth representation of G and χ a character of F^\times . Then if (π, V) is irreducible then so is $(\pi\chi, V)$. Moreover, the central character of $\pi\chi$ is*

$$\omega_{\pi\chi} = \omega_\pi \chi^N.$$

Proof. In $(\pi\chi, V)$, if U is a G -invariant subspace of V then U is also a G -invariant subspace of V in (π, V) , which is irreducible, so we must have U is either $\{0\}$ or V . Finally, for $z \in Z = F^\times$, $\det(z) = z^N$ so

$$\pi\chi(z) = \omega_\pi(z)\chi(\det(z)) = \omega_\pi \chi^N(z).$$

□

Definition 2.1.3. *Let (π, V) be an irreducible smooth representation of G . We say that the representation π is square-integrable modulo Z if the followings are hold:*

1. The central character ω_π of π is unitary i.e.

$$|\omega_\pi(z)| = 1, \quad \forall z \in Z.$$

2. The integral

$$\int_{G/Z} |\varphi_{\bar{v} \otimes v}(g)|^2 dg \quad (2.1)$$

is finite, where dg is a Haar measure on G/Z and $\varphi_{\bar{v} \otimes v}(g)$ is any matrix coefficient.

Definition 2.1.4. A smooth representation (π, V) of the group G is called discrete series if we can write the representation π as

$$\pi = \pi' \otimes \chi \circ \det$$

where π' is square-integrable mod Z and χ is a (quasi)-character of the multiplicative field F^\times .

Lemma 2.1.5. Let (π, V) be an irreducible supercuspidal representation of G . Then π is a discrete series representation.

Proof. The representation π is Z -compact so the support of the integral (2.1) is compact so it converges and the lemma then follows. \square

For a non-negative integer j , denote by $A_N(F, j)$ the set of equivalence classes of irreducible smooth representations π of G which have the following properties:

1. π is square-integrable mod center;
2. $\omega_\pi(\varpi_F) = 1$;
3. $\ell(\pi) \leq j/N$.

and put

$$A_N(F) = \bigcup_{j \geq 1} A_N(F, j).$$

2.1.1 Steinberg Representations

The Steinberg representation of $GL_N(F)$ is an example of a discrete series representation. We will define the Steinberg representations of $GL_N(F)$ and we will give some very important results.

Definition 2.1.6. Let $d = (d_i)$ and $n = \sum_{i=1}^r d_i$. Let

$$\pi = \pi_1 \otimes \cdots \otimes \pi_r$$

be an irreducible representation of a standard Levi subgroup L of G with π_i an irreducible representation of $GL_{d_i}(F)$. We define the normalized induction

$$\pi_1 \times \cdots \times \pi_r := \text{Ind}_{L, P}^G \pi$$

where P is the standard parabolic subgroup containing L .

Let $\nu = \nu_N$ be the character of G , defined by

$$\nu(g) = |\det(g)|, \quad g \in G,$$

where $|\cdot|$ is the absolute value on F .

Let d be a divisor of N , with $N = md$ and let π be an irreducible smooth supercuspidal representation of $GL_m(F)$. Let

$$\sigma(\pi, d) = \nu_m^{-(d-1)/2} \pi \times \nu_m^{1-(d-1)/2} \pi \times \cdots \times \nu_m^{(d-1)/2} \pi.$$

Proposition 2.1.7. *There is a unique irreducible quotient of $\sigma(\pi, d)$, denoted by $St(\pi, d)$ and called the Steinberg representation.*

Proof. See [35, Chapter III, Proposition 3.13]. □

There is a close relation between the discrete series and irreducible supercuspidal representations. This relation will depend on the divisors of N and is shown by the following theorem.

Theorem 2.1.8. *Let π be a discrete series representation of $GL_N(F)$. Then there are a unique divisor d of N and a unique irreducible supercuspidal representation π_0 of $GL_d(F)$ such that:*

$$\pi \cong St(\pi_0, N/d).$$

Proof. [38, Theorem 9.3]. □

Lemma 2.1.9. *Let $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ be an irreducible representation of L , a standard Levi subgroup of G . Let*

$$\Pi = \text{Ind}_{L,P}^G \pi.$$

Then any subquotient of Π has central character

$$\omega_\Pi = \omega_{\pi_1} \cdots \omega_{\pi_r}.$$

Proof. The center Z of G acts on Π as follows: let $z \in Z, g \in G$ and $f \in \Pi$, then

$$\begin{aligned} (z.f)(g) &= f(gz) = f(zg) \\ &= \delta_P(z)^{1/2}(z)\pi(z).f(g) \quad \text{as } z \in L. \end{aligned}$$

Since $\delta_P(z)^{1/2}(z) = 1$, we get

$$\begin{aligned} (z.f)(g) &= \pi_1 \otimes \cdots \otimes \pi_r(z)f(g) \\ &= \omega_{\pi_1} \cdots \omega_{\pi_r}(z)f(g) \\ &= \omega_\Pi(z)f(g). \end{aligned}$$

□

Proposition 2.1.10. *Suppose π is a discrete series representation of $GL_N(F)$ which is isomorphic to $St(\pi_0, N/d)$ as above. Then:*

1. $\ell(\pi) = \ell(\pi_0)$
2. $\omega_\pi(\varpi_F) = 1$ if and only if $\omega_{\pi_0}(\varpi_F)^{\frac{N}{d}} = 1$.

Proof. For part (1) Vigneras in [35, II 5.12] proves that the parabolic induction preserves the level i.e. every subquotient of $\text{Ind}_{L,P}^G \pi_0$ has the same level $\ell(\pi_0)$, (for π_0 irreducible). Therefore, $\ell(\pi) = \ell(\pi_0)$.

Part (2) follows from Lemma 2.1.9 . □

2.2 Representations of Division Algebras

In this section we will discuss the structure of central division algebras D . We will concentrate on the irreducible smooth representations of the locally profinite group D^\times .

Definition 2.2.1. *A central division algebra D over the field F is a finite dimensional F -algebra which is a division algebra with center F .*

Fix a central division algebra D over F of dimension N^2 . Let E/F be a separable extension of degree N and $M_N(E)$ be the set of $N \times N$ matrices with entries in E .

Proposition 2.2.2. *There is an embedding $E \hookrightarrow D$, unique up to D^\times -conjugacy. Moreover, we have the following isomorphism*

$$E \otimes_F D \cong M_N(E).$$

Proof. [9, §7] □

Lemma 2.2.3. *Let E be an extension of F of degree N centered in D . Then*

$$[E : F]^2 = [D : F] = [D : E]^2.$$

Proof. See [9, §7, Corollary 1.11]. □

Definition 2.2.4. *The reduce trace map $\text{Tr}_D : D \rightarrow F$ is given by*

$$\text{Tr}_D(d) = \text{tr}_{M_N(E)}(1 \otimes d), \quad d \in D.$$

The the reduced norm map $\text{Nr}_D : D^\times \rightarrow F^\times$ is defined by

$$\text{Nr}_D(d) = \det_{M_N(E)}(1 \otimes d), \quad d \in D.$$

Proposition 2.2.5. *For $x, y \in D, a \in F$, we have:*

1. $\text{Tr}_D(x + y) = \text{Tr}_D(x) + \text{Tr}_D(y)$;
2. $\text{Tr}_D(ax) = a \text{Tr}_D(x)$ and $\text{Tr}_D(1) = N$;
3. $\text{Tr}_D(xy) = \text{Tr}_D(yx)$;
4. $\text{Nr}_D(xy) = \text{Nr}_D(x) \text{Nr}_D(y)$;
5. $\text{Nr}_D(ax) = a^N \text{Nr}_D(x)$ and $\text{Nr}_D(1) = 1$.

Proof. See [9, §,7.3]. □

Now define

$$\begin{aligned}\nu_D : D &\longrightarrow \mathbb{Z} \cup \{\infty\} \\ x &\longmapsto \nu_F(N_D(x)),\end{aligned}$$

where ν_F as defined in Section 1.1. Then the map ν_D is a valuation on D so, for $x, y \in D$, put

$$|x|_D = q^{-\nu_D(x)}.$$

Then, for $x, y \in D$, we have:

$$|xy|_D = |x|_D |y|_D$$

and

$$|x + y|_D \leq \max\{|x|_D, |y|_D\}.$$

We define the ring of integers \mathcal{O}_D in D as follows:

$$\mathcal{O}_D = \{x \in D : |x|_D \leq 1\}.$$

The unique maximal ideal \mathcal{P}_D of \mathcal{O}_D is then:

$$\mathcal{P}_D = \{x \in D : |x|_D < 1\}.$$

The ideal \mathcal{P}_D is principal so it is generated by one element called a *uniformizer* of D and denoted by ϖ_D .

The ideal \mathcal{P}_D defines a filtration

$$\dots \subset \mathcal{P}_D^2 \subset \mathcal{P}_D^1 \subset \mathcal{P}_D^0 = \mathcal{O}_D \subset \mathcal{P}_D^{-1} \subset \mathcal{P}_D^{-2} \subset \dots$$

where \mathcal{P}_D^m , for $m \in \mathbb{Z}$, is defined as follows

$$\mathcal{P}_D^m = \varpi_D^m \mathcal{O}_D.$$

We define the residue field of D to be the quotient $k_D = \mathcal{O}_D / \mathcal{P}_D$. The size of the residue field k_D is q^N and the size of the quotient $\mathcal{O}_D / \mathcal{P}_D^m$ is then q^{mN} , where $m > 0$.

We define the unit group of D as

$$U_D = U_D^0 = \mathcal{O}_D^\times.$$

It has a filtration

$$U_D^m = 1 + \mathcal{P}_D^m, \quad m \geq 1.$$

Remark 2.2.6. *The subgroups U_D^m , $m \geq 0$, are compact open subgroups and they are normal in D^\times .*

Definition 2.2.7. *We define the level of a character θ of D to be the least integer m such that $\mathcal{P}_D^m \subset \text{Ker}(\theta)$. χ is trivial on U_D^{k+1} .*

We will denote the set of characters of D by \widehat{D} .

Lemma 2.2.8. *Let $\theta \neq 1$ be a character of D , then the map*

$$\begin{aligned} D &\longrightarrow \widehat{D} \\ a &\longmapsto \theta(ax) \end{aligned}$$

is an isomorphism.

Proof. Similar to Lemma 1.1.4. □

For each character ψ of F , we can define a character of D by setting

$$\psi_D = \psi \circ \text{Nr}_D.$$

Remark 2.2.9. *If the level of the character ψ is one then the character ψ_D is also of level one.*

Proposition 2.2.10. *For $0 \leq k \leq m < 2k$, the map*

$$\begin{aligned} \mathcal{P}_D^{-k} / \mathcal{P}_D^{-m} &\longrightarrow (U_D^{m+1} / U_D^{k+1})^\wedge \\ a &\longmapsto \psi_{D,a}(x) \end{aligned}$$

is an isomorphism, where $\psi_{D,a}(x) = \psi_D(a(x-1))$.

Proof. We only need to check the map is homomorphism which follows from the condition $0 \leq k \leq m < 2k$. □

Proposition 2.2.11. *The map*

$$\begin{aligned} \widehat{F^\times} &\longrightarrow \widehat{D^\times} \\ \chi &\longmapsto \chi \circ \text{Nr}_D. \end{aligned}$$

is an isomorphism.

Proof. See [37, Proposition 6, p.195]. □

Now consider an irreducible smooth representation π of the locally profinite group D^\times .

Remark 2.2.12. *As the quotient D^\times / F^\times is compact, all smooth representations of D^\times are discrete series and, in fact, cuspidal.*

Definition 2.2.13. *We define the level $\ell(\pi)$ of π to be the least integer k such that the representation π is trivial on U_D^{k+1} .*

Lemma 2.2.14 (Existence). *Let (π, V) be an irreducible smooth representation of D^\times . Then there exists a non-negative integer m such that $\ell(\pi) = m$*

Proof. Similar to the proof in Lemma 1.12.6. □

Definition 2.2.15. *Let (π, V) be a smooth representation of D^\times . We say the representation π is minimal if for any character χ of D^\times , we have*

$$\ell(\pi) \leq \ell(\pi\chi).$$

Now denote by $A(D, j)$, the set of equivalence classes of irreducible smooth representations π of D^\times which satisfy the following:

1. $\omega_\pi(\varpi_F) = 1$;
2. The restriction of π to the open compact subgroup U_D^{j+1} is trivial.

Koch has proved in his paper [18] the following result:

Lemma 2.2.16. *The set $A(D, j)$ is finite and the order of $A(D, j)$ is given by:*

$$|A(D, j)| = \sum_{m|n} \frac{n}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q^d - 1) q^{d[\frac{j}{m}]}$$

2.3 L -Functions and Local Constant for $GL_N(F)$

Here, let π be an irreducible smooth representation of G which we will assume either $GL_N(F)$ or D^\times . We will attach to this representation π two invariants: let s be a complex variable. The first invariant is the L -function $L(\pi, s)$ and the second one is called the local constant $\varepsilon(\pi, s, \psi)$, where ψ is a non-trivial character of F . The results of this section are originally in [11].

We are going to focus more on the L -function and the local constant of supercuspidal representations for $GL_N(F)$ and we will see later on that there is a connection between these invariants for irreducible supercuspidal representations of $GL_N(F)$ and others defined for irreducible smooth representations of D^\times .

We will fix the following notations. When $G = GL_N(F)$, then $A = M_N(F)$. The trace map on A is defined by $T_A = \text{tr}_{A/F}$ and the norm on A is $N_A = \det_A$. When G is the central division algebra D^\times , then A will be D . The trace map T_A on A will be the reduce trace map Tr_A and norm map N_A on A is the reduce norm map Nr_D .

Denote by $\mathcal{C}_0^\infty(A)$ the space of all functions $f : A \rightarrow \mathbb{C}$ which have compact support and are locally constant. Recall that: a function $f : A \rightarrow \mathbb{C}$ is called *locally constant* if for any element a in A , there exists a neighborhood U of a such that $f(u) = f(a)$, $\forall u \in U$.

Now fix a non-trivial character ψ of F . For $\Phi \in \mathcal{C}_0^\infty(A)$, we define the Fourier transform $\hat{\Phi}$ of Φ by

$$\hat{\Phi} = \int_G \Phi(x) \psi \circ T_A(xy) d\mu(y)$$

where μ is a Haar measure on A .

Now consider the integral

$$\zeta(\Phi, f, s) = \int_G \Phi(x) f(x) \|N_A(x)\|^s d\mu^*(x)$$

where μ^* is a Haar measure on G , $\Phi \in \mathcal{C}_0^\infty(A)$ and $f \in \mathcal{C}(\pi)$.

Set

$$\mathcal{Z}(\pi) = \left\{ \zeta \left(\Phi, f, s + \frac{N-1}{2} \right) : \Phi \in \mathcal{C}_0^\infty(A), f \in \mathcal{C}(\pi) \right\}.$$

Theorem 2.3.1. *There is a unique polynomial $P_\pi(X) \in \mathbb{C}[X]$ such that*

1. $P_\pi(0) = 1$;
2. $\mathcal{Z}(\pi)P_\pi(q^{-s}) \in \mathbb{C}[q^s, q^{-s}]$.

Proof. See [19, Theorem 2.3 and Corollary 2.4]. □

Definition 2.3.2. *The L-function $L(\pi, s)$ of π is defined by*

$$L(\pi, s) = \frac{1}{P_\pi(q^{-s})}$$

where s is a complex variable. Note that $L(\pi, s)$ is independent of the choice of Haar measure μ^* .

For $f \in \mathcal{C}(\pi)$ we define $\check{f} \in \mathcal{C}(\check{\pi})$ by $\check{f}(g) = f(g^{-1})$.

Theorem 2.3.3. *Let (π, V) be an irreducible smooth representation of G . For each $\Phi \in \mathcal{C}_0^\infty(A)$ and $f \in \mathcal{C}(\pi)$, there exists a unique function $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$ such that*

$$\zeta(\hat{\Phi}, \check{f}, \frac{N+1}{2} - s) = \gamma(\pi, s, \psi) \zeta(\Phi, f, \frac{N-1}{2} + s).$$

Proof. See [19, Theorem 2.3 (2)]. □

Definition 2.3.4. *We define the local constant of the representation π as follows*

$$\varepsilon(\pi, s, \psi) = \gamma(\pi, s, \psi) \frac{L(\pi, s)}{L(\check{\pi}, 1-s)}.$$

Corollary 2.3.5. *The function $\varepsilon(\pi, s, \psi)$ has the form $aq^{f(\pi, \psi)s}$ for some $a \in \mathbb{C}^\times$ and $f(\pi, \psi) \in \mathbb{Z}$. Moreover, the local constant $\varepsilon(\pi, s, \psi)$ satisfies the functional equation:*

$$\varepsilon(\pi, s, \psi) \varepsilon(\check{\pi}, 1-s, \psi) = \omega_\pi(-1).$$

Remark 2.3.6. *The integer $f(\pi, \psi)$ in Corollary 2.3.5 is called the conductor of π . The conductor is called minimal if π is minimal. Let π be irreducible supercuspidal representation of G of minimal conductor c , then [8, § 4] provides*

$$f(\pi, \psi) = r(m-1) + N.$$

Remark 2.3.7. *The local constant $\varepsilon(\pi, s, \psi)$ can be written as*

$$\varepsilon(\pi, s, \psi) = q^{f(\pi, \psi)(\frac{N-1}{2}-s)} \varepsilon(\pi, \frac{N-1}{2}, \psi)$$

for some $f(\pi, \psi) \in \mathbb{Z}$.

Proposition 2.3.8. *Let (π, V) be an irreducible smooth representation of G . Let ψ be a non-trivial character of F . For any $z \in Z(G)$, we have*

$$\varepsilon(\pi, s, z\psi)|z|^{N(1-s)/2}\varepsilon(\pi, s, \psi)^{-1} = \omega_\pi(z).$$

Proof. See [17, 1.3.9]. □

Proposition 2.3.9. *Let (π, V) be an irreducible supercuspidal representation of G . Then*

1. $Z(\pi) = \mathbb{C}[X, X^{-1}]$.
2. $L(\pi, s) = 1$;

Proof. See [17, 1.3.5]. □

Let ψ be a non-trivial character of F of level one. For the following see [3, (3.1.5), (3.2.11)], noting that it is independent of choice of ψ .

Remarks 2.3.10. *Let π_D be an irreducible representation of D^\times .*

1. *The L -function $L(\pi_D, \psi)$ is not trivial if and only if the dimension of π_D is one, in which case, the representation π_D is of level zero and $f(\pi_D, \psi) \leq 0$.*
2. *If the L -function $L(\pi_D, \psi)$ is trivial, then $\ell(\pi_D) = f(\pi_D, \psi) = m$, for some positive integer m .*

Recall that the set $A_N(F)$ is union of $A_N(F, j)$ over $j \geq 1$ as defined in section (2.1).

Lemma 2.3.11 ([5, Lemma 5]). *For $\pi \in A_N(F)$ and ψ a character of F of level one. Then one of the following occur:*

1. $f(\pi, \psi) = -1$ and $\ell(\pi) = 0$, or
2. $f(\pi, \psi) = N\ell(\pi)$.

2.4 Jacquet-Langlands Correspondence

In this section G denotes either $GL_N(F)$ or D^\times .

Let $g \in D$. The *reduce* characteristic polynomial of g is the characteristic polynomial of $1 \otimes g$ in $M_N(E)$ as in Proposition 2.2.2.

Definition 2.4.1. *Let $g \in G$, then we say g is:*

1. *regular semisimple if the element g has (reduce) characteristic polynomial with distinct roots over \bar{F} (algebraic closure of F).*
2. *elliptic if g is regular semisimple and the (reduce) characteristic polynomial of g is irreducible over F .*

We denote the set of regular semisimple elements in G by G^s .

Definition 2.4.2. *Let $g \in GL_N(F)$ and $g' \in D^\times$. We say g corresponds to g' , and we write $g \leftrightarrow g'$, if they have the same characteristic polynomial.*

Recall that $\mathcal{H}(G)$ is the set of all functions $f : G \rightarrow \mathbb{C}$, which are locally constant with compact support. Let (π, V) be an admissible representation of G and $f \in \mathcal{H}(G)$. We can define

$$\pi(f) = \int_G f(g)\pi(g)dg.$$

This makes sense as an operator $V \rightarrow V$ because the function $g \mapsto \pi(g)v$ is locally constant so the integral reduces to a finite sum. Moreover (see [13]), the trace $\text{tr}(\pi(f))$ is well-defined and there is a function χ_π on G^s such that

$$\text{tr}(\pi(f)) = \int_{G^s} \chi_\pi(g)f(g)dg$$

for all $f \in \mathcal{H}(G)$ with support in G^s .

Let $A_N^\diamond(F)$ be the set of equivalence classes of discrete series representations of the group $GL_N(F)$.

Let $A^\diamond(D)$ be the set of irreducible smooth representations of the group D^\times .

Theorem 2.4.3 (Jacquet-Langlands Correspondence). *There is a unique bijection*

$$\begin{array}{ccc} A_N^\diamond(F) & \longrightarrow & A^\diamond(D) \\ \pi & \longmapsto & \pi_D \end{array}$$

such that, for all $\pi \in A_N$,

$$\chi_\pi(g) = (-1)^{N-1}\chi_{\pi_D}(g')$$

for all $g \leftrightarrow g'$.

Proof. For $N = 2$, the Theorem was proved by Jacquet-Langlands and a sketch of the proof can be found in [4, 56]. For characteristic zero, it was proved by Rogawski in [30]. For the positive characteristic case, the Jacquet-Langlands correspondence was proved by Badulescu in [3]. \square

The Jacquet-Langlands correspondence has very important consequences which is relating the the L -functions and local constants in $A_N^\diamond(F)$ to the L -functions and local constants in $A^\diamond(D)$. For the following proposition see [1].

Proposition 2.4.4. *Suppose $\pi \leftrightarrow \pi_D$ in the Jacquet-Langlands correspondence.*

1. $L(\chi\pi, s) = L(\chi\pi_D, s)$ for all characters χ of F^\times .
2. For all $\pi \in A_N^\diamond(F)$ and all characters χ of F^\times ,

$$\varepsilon(\chi\pi, s, \psi) = (-1)^{N-1}\varepsilon(\chi\pi_D, s, \psi)$$

holds for all character ψ of F with $\psi \neq 1$.

Lemma 2.4.5. *If $\pi \leftrightarrow \pi_D$ in the Jacquet-Langlands correspondence then*

$$\omega_\pi = \omega_{\pi_D}$$

Proof. By Proposition 2.4.4(2), we have

$$\varepsilon(\pi, s, z, \psi) = (-1)^{N-1} \varepsilon(\pi_D, s, z, \psi)$$

for $z \in F^\times$ and $z.\psi(x) = \psi(zx)$. Using this fact, we have the equality

$$(-1)^{N-1} \varepsilon(\pi, s, z, \psi) |z|^{N(1-s)/2} \varepsilon(\pi, s, \psi)^{-1} = (-1)^{N-1} \varepsilon(\pi_D, s, z, \psi) |z|^{N(1-s)/2} \varepsilon(\pi_D, s, \psi)^{-1}.$$

Proposition 2.3.8 implies that

$$\omega_\pi(z) = \omega_{\pi_D}(z)$$

as required. \square

Now let

$$A(D) = \bigcup_{j \geq 1} A(D, j).$$

The Jacquet-Langlands correspondence implies that there is a unique bijection

$$\begin{aligned} A_N(F) &\longrightarrow A(D), \\ \pi &\longmapsto \pi_D. \end{aligned}$$

Let $\pi \in A_N(F)$ and $\pi_D \in A(D)$ be such that $\pi \mapsto \pi_D$. By Section (2.3), there are integers $f(\pi, \psi)$ and $f(\pi_D, \psi)$ such that:

$$\varepsilon(\pi, s, \psi) = q^{f(\pi, \psi)(\frac{N-1}{2}-s)} \varepsilon\left(\pi, \frac{N-1}{2}, \psi\right)$$

and

$$\varepsilon(\pi_D, s, \psi) = q^{f(\pi_D, \psi)(\frac{N-1}{2}-s)} \varepsilon\left(\pi_D, \frac{N-1}{2}, \psi\right).$$

Since the Jacquet-Langlands correspondence implies $\varepsilon(\pi, s, \psi) = \varepsilon(\pi_D, s, \psi)$, for all $s \in \mathbb{C}$, then we deduce that

$$f(\pi, \psi) = f(\pi_D, \psi). \quad (2.2)$$

Now we deduce a very useful proposition

Proposition 2.4.6. *The map*

$$\begin{aligned} A_N(F, j) &\longrightarrow A(D, j) \\ \pi &\longmapsto \pi_D \end{aligned}$$

is well-defined and a bijection.

Proof. Let ψ be an addition character of level one. We prove this proposition by showing the following are equivalent:

1. $\pi \in A_N(F, j)$;
2. $\pi_D \in A(D, j)$;

$$3. f(\pi, \psi) = f(\pi_D, \psi) \leq j.$$

This follows from Remarks (2.3.10), Lemma (2.3.11) and the equation (2.2). \square

Theorem 2.4.7. *The set $A_N(F, j)$ is finite and the order of $A_N(F, j)$ is given by*

$$|A_N(F, j)| = \sum_{m|N} \frac{N}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q^d - 1) q^{d[\frac{j}{m}]}.$$

Proof. Proposition 2.4.6 shows there is a bijection between the set $A(D, j)$ and the set $A_N(F, j)$ so they have the same size:

$$|A_N(F, j)| = |A(D, j)|.$$

The size of the set $A(D, j)$ is given in Lemma 2.2.16 and that completes the proof. \square

Remark 2.4.8. *This generalizes Theorem 1 of [5], where the result is proved for $j/N \in \mathbb{Z}$. The proof given here is essentially the same as that in [5].*

2.5 Counting Supercuspidal representations of $GL_N(F)$

In this section, we are going to compute the number of irreducible supercuspidal representations of $GL_N(F)$ under some conditions. For any positive integer N , we will deduce a formula for the number of these representations which will depend on the divisors of the integer N . However, when N is a prime, the formula becomes simpler as the only divisors of N are 1 and N .

A special case, is when these representations have level zero. We will give a general formula for the number of these representations for any positive integer N .

For a divisor d of N , let $S_d^N(F, j)$ be the set of equivalence classes of irreducible smooth supercuspidal representations of $GL_d(F)$ with the following properties:

1. $\ell(\pi) \leq j/d$;
2. $\omega_\pi(\varpi_F)^{N/d} = 1$.

Then $S_d^d(F, j)$ is the set of irreducible smooth supercuspidal representations of $GL_d(F)$ with $\ell(\pi) \leq j/d$ and $\omega_\pi(\varpi_F) = 1$.

Denote by Ψ , the set of characters of F^\times of order dividing N . Then we have the following lemma,

Lemma 2.5.1. *The map*

$$\begin{aligned} S_d^d(F, j) \times \Psi &\longrightarrow S_d^N(F, j) \\ (\pi, \chi) &\longmapsto \pi \otimes \chi \circ \det \end{aligned}$$

is surjective, with fibers

$$\{(\pi \otimes \psi \circ \det, \chi\psi^{-1}) : \psi \text{ is an unramified character of order dividing } d\}.$$

Proof. To prove the map is surjective, suppose $\pi \in S_d^N(F, j)$ so

$$\omega_\pi(\varpi_F)^{N/d} = 1.$$

Now pick $\zeta \in \mathbb{C}^\times$ such that $\zeta^d = \omega_\pi(\varpi_F)$. Then

$$\zeta^N = \omega_\pi(\varpi_F)^{N/d} = 1.$$

Define a character χ of F^\times by

$$\chi(\varpi_F^r u) = \zeta^{-r}$$

for $r \in \mathbb{Z}$ and $u \in \mathcal{O}_F^\times$. Then by Lemma 2.1.2, we get

$$\omega_{\pi \otimes \chi \circ \det}(\varpi_F) = \omega_\pi(\varpi_F) \omega_\chi(\varpi_F)^d = 1.$$

Therefore, $\pi \otimes \chi \circ \det \in S_d^d(F, j)$ and

$$(\pi \otimes \chi \circ \det, \chi^{-1} \circ \det) \longmapsto \pi.$$

Now let $\pi \in S_d^d(F, j)$ and $\chi \in \Psi$ such that $\pi \otimes \chi \circ \det \in S_d^N(F, j)$ and consider the set

$$\text{Fib}(\pi \otimes \chi) = \{(\pi', \chi') : \pi' \in S_d^d(F, j), \chi' \in \Psi \text{ and } \pi' \otimes \chi' \cong \pi \otimes \chi\}.$$

The set $\text{Fib}(\pi \otimes \chi)$ is not-empty since it contains (π, χ) . Now let $(\pi', \chi') \in \text{Fib}(\pi \otimes \chi)$ so $\pi' \cong \pi \otimes \chi \chi'^{-1}$. Put $\psi = \chi \chi'^{-1}$, then

$$\omega_{\pi'}(\varpi_F) = \omega_{\pi \otimes \psi}(\varpi_F) = \omega_\pi(\varpi_F) \psi(\varpi)^d = 1,$$

therefore, ψ is a character of F^\times of order divides d . □

Corollary 2.5.2. *The size of each fiber is d and*

$$|S_d^N(F, j)| = \frac{N}{d} |S_d^d(F, j)|.$$

For $r \in \mathbb{R}, r \geq 0$ and $n \in \mathbb{N}$, define $A_n(F, r)$ to be the set of equivalence classes of irreducible smooth representations of $GL_n(F)$ which have the following properties:

1. π is square-integrable mod center;
2. $\omega_\pi(\varpi_F) = 1$;

3. $\ell(\pi) \leq r/n$.

Remarks 2.5.3. 1. $A_n(F, r) = A_n(F, [r])$ because, for all discrete series of $GL_n(F)$ of level $\ell(\pi) \in \frac{1}{n}\mathbb{Z}$, we have

$$\ell(\pi) \leq \frac{r}{n} \Leftrightarrow \ell(\pi) \leq \frac{[r]}{n}.$$

2. The formula of Theorem 2.4.7 is also valid for real number r :

$$|A_n(F, r)| = \sum_{m|n} \frac{n}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q^d - 1) q^{d[\frac{r}{m}]}.$$

Since $[r/m] = [[r]/m]$ for any integer $m \geq 1$, then the formula for $|A_n(F, r)|$ and $|A_n(F, [r])|$ are equal term by term.

Now we deduce the the following Theorem which will provide us with a formula for the size of the set $S_N^N(F, j)$

Theorem 2.5.4. For any non-negative integer j , the set $S_N^N(F, j)$ is finite and its size is given by:

$$|S_N^N(F, j)| = \sum_{d|N} \mu\left(\frac{N}{d}\right) \cdot \frac{N}{d} \cdot |A_d(F, jd/N)|.$$

Proof. By Theorem 2.1.8, the following map

$$\begin{aligned} \bigcup_{d|N} S_d^N(F, jd/N) &\longrightarrow A_N(F, j) \\ \pi_0 &\longmapsto St(\pi_0, N/d) \end{aligned}$$

is a bijection.

Now for any two distinct divisors d_1, d_2 of N , we have

$$S_{d_1}^N(F, jd_1/N) \cap S_{d_2}^N(F, jd_2/N) = \emptyset,$$

Therefore,

$$|A_N(F, j)| = \sum_{d|N} |S_d^N(F, jd/N)|$$

By Corollary 2.5.2, we get

$$|A_N(F, j)| = \sum_{d|N} \frac{N}{d} |S_d^d(F, jd/N)|$$

Finally, using Möbius inversion, we conclude:

$$|S_N^N(F, j)| = \sum_{d|N} \mu\left(\frac{N}{d}\right) \frac{N}{d} |A_d(F, jd/N)|.$$

□

The previous Theorem works for any positive integer N . Let $N = p_1^{r_1} \dots p_s^{r_s}$, where p_i is a prime and $r_i \geq 1$ is an integer for $i \in \{1, \dots, s\}$. Put

$$J = \{p_1, \dots, p_s\}.$$

For a subset I of J , define

$$N_I = \prod_{p \in I} p$$

By Theorem 2.5.4, the size of the set $S_N^N(F, k)$ is

$$\begin{aligned} |S_N^N(F, k)| &= \sum_{I \subset J} \mu(N_I) \cdot N_I \cdot |A_{N/N_I}(F, k/N_I)| \\ &= \sum_{I \subset J} (-1)^{\#I} N_I |A_{N/N_I}(F, k/N_I)|. \end{aligned}$$

As we can see, this involve a deep calculations when N is large. For this reason, we are going to specialize the integer N in order to deduce a simpler formula.

We will denote the number of equivalence classes of irreducible supercuspidal representations of $GL_N(F)$ with $\omega_\pi(\varpi_F) = 1$ and $\ell(\pi) = k/N$ by $\mathfrak{S}(N, k)$.

Remark 2.5.5. Denote by $\mathfrak{S}(N, k)^m$, the number of equivalence classes of irreducible supercuspidal representations of $GL_N(F)$ with $\omega_\pi(\varpi_F) = 1$ and minimal level $\ell(\pi) = k/N$.

2.5.1 Level Zero Supercuspidal Representations of $GL_N(F)$

Here we will compute the number of irreducible supercuspidal representations of $GL_N(F)$ with $\omega_\pi(\varpi_F) = 1$ and $\ell(\pi) = 0$, for any positive integer N . The reason for considering the level zero supercuspidal representations is because the number of discrete series $|A_N(F, 0)|$ of $GL_N(F)$ becomes easy to calculate

$$|A_N(F, 0)| = \sum_{m|N} \frac{N}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q^d - 1)$$

and as a result of that $\mathfrak{S}(N, 0)$ becomes also easy to compute.

Remark 2.5.6. We have $\mathfrak{S}(N, 0) = \mathfrak{S}(N, 0)^m$.

Theorem 2.5.7. The number of the irreducible smooth supercuspidal representations of $GL_N(F)$, upto equivalence, for any integer $N \geq 1$, with $\ell(\pi) = 0$ and $\omega_\pi(\varpi_F) = 1$ is given by the following:

$$\mathfrak{S}(N, 0) = \frac{1}{N} \sum_{d|N} \mu\left(\frac{N}{d}\right) (q^d - 1)$$

Proof. For abbreviation put

$$f(m) = \sum_{d|m} \mu\left(\frac{m}{d}\right) (q^d - 1)$$

so we have:

$$|A_N(F, 0)| = \sum_{m|N} \frac{N}{m^2} f(m)$$

Now the number of equivalence classes of irreducible supercuspidal representations of $GL_N(F)$ of level zero and $\omega_\pi(\varpi_F) = 1$ is

$$\begin{aligned} \mathfrak{S}(N, 0) &= \sum_{d|N} \mu\left(\frac{N}{d}\right) \frac{N}{d} |A_d(F, 0)| \\ &= N \sum_{d|N} \mu\left(\frac{N}{d}\right) \left(\sum_{m|d} \frac{1}{m^2} f(m) \right) \\ &= N \sum_{m|N} \frac{1}{m^2} f(m) \sum_{\substack{d|N \\ m|d}} \mu\left(\frac{N}{d}\right) \\ &= N \sum_{m|N} \frac{1}{m^2} f(m) \sum_{\substack{d \\ \frac{d}{m}|N \\ m|d}} \mu\left(\frac{N/m}{d/m}\right) \end{aligned}$$

Now we have

$$\sum_{\substack{d \\ \frac{d}{m}|N \\ m|d}} \mu\left(\frac{N/m}{d/m}\right) = \begin{cases} 1 & \text{if } \frac{N}{m} = 1; \\ 0 & \text{otherwise} \end{cases}$$

Therefore, we deduce

$$\mathfrak{S}(N, 0) = \frac{1}{N} f(N).$$

as required. □

Corollary 2.5.8. *When the integer $N \geq 2$, then:*

$$\mathfrak{S}(N, 0) = \frac{1}{N} \sum_{d|N} \mu\left(\frac{N}{d}\right) q^d.$$

Remark 2.5.9. *The number in Corollary 2.5.8 is, in fact, the number of irreducible polynomials of degree N over the residue field k_F .*

Green's work in [12] shows that irreducible cuspidal representations of $GL_N(k_F)$ are parametrized by the regular orbits of characters of $k_{F_N}^\times$ (where k_{F_N}/k_F is the unique degree N extension), which are parametrized by irreducible polynomials of degree N over k_F .

Finally there is a bijection [7, §6]

$$\left\{ \begin{array}{l} \text{irreducible cuspidal} \\ \text{representations of} \\ GL_N(k_F) \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{irreducible supercuspidal} \\ \text{representations of } GL_N(F) \text{ with} \\ \ell(\pi) = 0 \text{ and } \omega_\pi(\varpi_F) = 1 \end{array} \right\}$$

given by

$$\sigma \mapsto \text{Ind}_{F \times GL_N(\mathcal{O}_F)}^{GL_N(F)} \tilde{\sigma},$$

where $\tilde{\sigma}(\varpi_F^k g) = \sigma(\bar{g})$, for $k \in \mathbb{Z}$ and $g \in GL_N(F)$ and \bar{g} is the reduction mod \mathcal{P}_F of g .

2.5.2 Positive non-Integral Level Supercuspidal Representations of $GL_N(F)$

In this section, we will compute $\mathfrak{S}(N, k)^m$ for any positive integer N with $(N, k) = 1$ for some integer $k \geq 1$. The reason for imposing the condition $(N, k) = 1$ is to make the counting simpler because we have:

$$\left[\frac{k}{m} \right] = \left[\frac{k-1}{m} \right] \quad (2.3)$$

for any divisor $m > 1$ of N . When $(N, k) \neq 1$, then (2.3) is not true anymore and, therefore, the computations become more complicated. Before we count the supercuspidals we need the following lemma.

Lemma 2.5.10. *Any representation in $S_N^N(F, k)$ with a level $\ell(\pi) = k/N$ is minimal.*

Proof. Follows from Lemma 1.13.11. □

Corollary 2.5.11. *We have $\mathfrak{S}(N, k) = \mathfrak{S}(N, k)^m$.*

Lemma 2.5.12. *The number of irreducible supercuspidal representations of $GL_N(F)$ of minimal level $\ell(\pi) = k/N$ and $\omega_\pi(\varpi_F) = 1$ is*

$$\mathfrak{S}(N, k) = N(q_F - 1)^2 q_F^{N\ell(\pi)-1}.$$

Proof. By Lemma 2.5.10 and Theorem 2.5.4, we get

$$\begin{aligned} \mathfrak{S}(l, k)^m &= |S_l^l(F, k)| - |S_l^l(F, k-1)| \\ &= \sum_{d|N} \mu\left(\frac{N}{d}\right) \frac{N}{d} \left(|A_d(F, kdN^{-1})| - |A_d(F, (k-1)dN^{-1})| \right). \end{aligned}$$

By (2.3) and Theorem 2.4.7, we obtain

$$\begin{aligned} \mathfrak{S}(l, k)^m &= |A_N(F, k)| - |A_N(F, k-1)| \\ &= \sum_{m|N} \frac{N}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q_F^d - 1) \left(q_F^{d \left[\frac{k}{m} \right]} - q_F^{d \left[\frac{k-1}{m} \right]} \right) \\ &= N(q_F - 1)^2 q_F^{N\ell(\pi)-1}. \end{aligned}$$

□

2.5.3 Positive Non-Integral Level Supercuspidal Representations of $GL_{l^r}(F)$

In this section, we deal with a special case when $N = l^r$ for some integer $r > 0$ and l a prime. We will compute $\mathfrak{S}(l^r, k)$ for some integer $k \geq 1$. When $(l^r, k) = 1$, this case was investigated in previous section and the number is

$$\mathfrak{S}(l^r, k) = l^r (q_F - 1)^2 q_F^{l^r \ell(\pi) - 1}.$$

Suppose $(l^r, k) = l^t$, where $t > 0$. For any divisor $m = l^i$ of l^r , if $i > t$ then

$$\left[\frac{k}{m} \right] = \left[\frac{k-1}{m} \right]$$

otherwise,

$$\left[\frac{k-1}{m} \right] = \left[\frac{k}{m} \right] - 1.$$

If $i \leq t$, we can write

$$\left[\frac{k}{l^i} \right] = l^{r-i} \ell(\pi)$$

Any irreducible supercuspidal representation in $S_{l^r}^{l^r}(F, k)$ with level k/l^r is minimal by Lemma 2.5.10. This implies that $\mathfrak{S}(l^r, k) = \mathfrak{S}(l^r, k)^m$ and

$$\mathfrak{S}(l^r, k) = |S_{l^r}^{l^r}(F, k)| - |S_{l^r}^{l^r}(F, k-1)|.$$

Now using Theorem 2.5.4, we obtained:

$$\begin{aligned} \mathfrak{S}(l^r, k) &= \underbrace{|A_{l^r}(F, k)| - |A_{l^r}(F, k-1)|}_X \\ &\quad - \underbrace{l(|A_{l^{r-1}}(F, kl^{-1})| - |A_{l^{r-1}}(F, (k-1)l^{-1})|)}_Y. \end{aligned}$$

First we compute X .

$$\begin{aligned} X &= \sum_{m|l^t} \frac{l^r}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q_F^d - 1) \left(q_F^{d \left[\frac{k}{m} \right]} - q_F^{d \left[\frac{k-1}{m} \right]} \right) \\ &= l^r (q_F - 1)^2 q_F^{l^r \ell(\pi) - 1} \\ &\quad + l^{r-2(1)} \left((q_F^l - 1)^2 q_F^{l^r \ell(\pi) - l} - (q_F - 1)^2 q_F^{l^{r-1} \ell(\pi) - 1} \right) \\ &\quad \vdots \\ &\quad + l^{r-2(t)} \left((q_F^{l^t} - 1)^2 q_F^{l^r \ell(\pi) - l^t} - (q_F^{l^{t-1}} - 1)^2 q_F^{l^{r-t} \ell(\pi) - l^{t-1}} \right). \end{aligned}$$

Thus

$$\begin{aligned} X &= \sum_{i=0}^t l^{r-2i} (q_F^{l^i} - 1)^2 q_F^{l^r \ell(\pi) - l^i} \\ &\quad - \sum_{i=1}^t l^{r-2i} (q_F^{l^{i-1}} - 1)^2 q_F^{l^{r-1} \ell(\pi) - l^{i-1}}. \end{aligned}$$

Now we compute Y . We will use $[[kl^{-1}]/m] = [k/ml]$.

$$\begin{aligned}
Y &= \sum_{m|l^{t-1}} \frac{l^r}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q_F^d - 1) \left(q_F^{d[\frac{k}{ml}]} - q_F^{d[\frac{k-1}{ml}]} \right) \\
&= l^r (q_F - 1)^2 q_F^{l^{r-1}\ell(\pi)-1} \\
&+ l^{r-2(1)} \left((q_F^l - 1)^2 q_F^{l^{r-1}\ell(\pi)-l} - (q_F - 1)^2 q_F^{l^{r-2}\ell(\pi)-1} \right) \\
&\quad \vdots \\
&+ l^{r-2(t)} \left((q_F^{l^{t-1}} - 1)^2 q_F^{l^{r-1}\ell(\pi)-l^{t-1}} - (q_F^{l^{t-2}} - 1)^2 q_F^{l^{r-2}\ell(\pi)-l^{t-2}} \right).
\end{aligned}$$

so

$$\begin{aligned}
Y &= \sum_{i=0}^{t-1} l^{r-2i} (q_F^{l^i} - 1)^2 q_F^{l^{r-1}\ell(\pi)-l^i} \\
&\quad - \sum_{i=1}^{t-1} l^{r-2i} (q_F^{l^{i-1}} - 1)^2 q_F^{l^{r-2}\ell(\pi)-l^{i-1}}.
\end{aligned}$$

Finally we subtract Y from X to get $\mathfrak{S}(l^r, k)$.

$$\begin{aligned}
\mathfrak{S}(l^r, k) &= \sum_{i=0}^t l^{r-2i} (q_F^{l^i} - 1)^2 (q_F^{l^r\ell(\pi)-l^i} - q_F^{l^{r-1}\ell(\pi)-l^i}) \\
&\quad - \sum_{i=1}^t l^{r-2i} (q_F^{l^{i-1}} - 1)^2 (q_F^{l^{r-1}\ell(\pi)-l^{i-1}} - q_F^{l^{r-2}\ell(\pi)-l^{i-1}}) \\
&\quad + l^{r-2t} \left[(q_F^{l^t} - 1)^2 q_F^{l^{r-1}\ell(\pi)-l^t} - (q_F^{l^{t-1}} - 1)^2 q_F^{l^{r-2}\ell(\pi)-l^{t-1}} \right].
\end{aligned}$$

2.5.4 Positive Integral Level Supercuspidal Representations of $GL_l(F)$

Again we are considering N to be a prime l .

Lemma 2.5.13. *There exists a representation π in $S_l^1(F, lk)$ with level $\ell(\pi) = k$ which is not minimal.*

Proof. See [7, §9.2]. □

Corollary 2.5.14. *We have $\mathfrak{S}(l, lk) > \mathfrak{S}(l, lk)^m$.*

Lemma 2.5.15. *The number of equivalence classes of irreducible supercuspidal representations of $GL_l(F)$ of level $\ell(\pi) = k$ and $\omega_\pi(\varpi_F) = 1$ is*

$$\mathfrak{S}(l, lk) = \frac{1}{l} (q_F^l - 1)^2 q_F^{l(\ell(\pi)-1)} - \left(\frac{1}{l} + l\right) (q_F - 1)^2 q_F^{\ell(\pi)-1} + l (q_F - 1)^2 q_F^{l\ell(\pi)-1}.$$

Proof. This is straight computations:

$$\begin{aligned}
\mathfrak{S}(l, lk) &= |S_l^l(F, lk)| - |S_l^l(F, lk-1)| \\
&= |A_l(F, lk)| - |A_l(F, lk-1)| \\
&\quad - l(|A_1(F, k)| - |A_1(F, k-1)|) \\
&= \sum_{m|l} \frac{l}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q_F^d - 1) \left(q_F^{d\lceil \frac{lk}{m} \rceil} - q_F^{d\lceil \frac{lk-1}{m} \rceil} \right) \\
&\quad - l(q_F - 1)^2 q_F^{k-1}.
\end{aligned}$$

□

Lemma 2.5.16. *The map*

$$S_N^N(F, kN-1) \times \Psi_{\leq k} \longrightarrow S_N^N(F, kN)$$

has fibers

$$\{(\pi\psi, \chi\psi^{-1}) : \psi \text{ is a character of } F^\times \text{ of level } \leq k-1\}.$$

The size of each fiber is $(q-1)q^{k-1}$ and the size of the image is $q|S_N^N(kN-1)|$.

Proof. The map is not surjective, since all minimal representations in $S_N^N(F, kN)$ of level k are not in the image as we cannot reduce the level. Now let $\pi \in S_N^N(F, kN-1)$ and $\chi \in \Psi_{\leq k}$ such that $\pi \otimes \chi \in S_N^N(F, kN)$. Let $\text{Fib}(\pi \otimes \chi)$ denote the fiber of $\pi \otimes \chi$, then $\text{Fib}(\pi \otimes \chi)$ is not-empty as it contains the pair (π, χ) . Now let $(\pi', \chi') \in \text{Fib}(\pi \otimes \chi)$, then $\pi' \otimes \chi' \cong \pi \otimes \chi$ so $\pi' \cong \pi \otimes \chi\chi'^{-1}$. Put $\psi = \chi\chi'^{-1}$, then by Lemma 1.13.11, we must have $\ell(\psi) < k$, so $\ell(\psi) \leq k-1$. Therefore,

$$\text{Fib}(\pi \otimes \chi) = \{(\pi\psi, \chi\psi^{-1}) : \psi \text{ is a character of } F^\times \text{ of level } \leq k-1\}.$$

By Lemma 1.1.6, the size of each fiber is $(q_F - 1)q_F^{k-1}$ and $|\Psi_{\leq k}| = (q_F - 1)q_F^k$. Therefore, the size of the image of this map is

$$\frac{(q_F - 1)q_F^k}{(q_F - 1)q_F^{k-1}} |S_N^N(F, kN-1)|.$$

□

Corollary 2.5.17. *The number of equivalence classes of irreducible minimal supercuspidal representations of $GL_l(F)$ with $\omega_\pi(\varpi_F) = 1$ and $\ell(\pi) = k \in \mathbb{Z}$ is*

$$\mathfrak{S}(l, lk)^m = \frac{1}{l} (q_F^l - q_F) (q_F^l - 1) q_F^{l(\ell(\pi)-1)}.$$

Proof. By the previous lemma, the cardinality $\mathfrak{S}(l, lk)^m$ is obtained by

$$\mathfrak{S}(l, lk)^m = |S_l^l(F, lk)| - q_F |S_l^l(F, lk-1)|.$$

Now using Theorem 2.5.4, we get

$$\begin{aligned}
\mathfrak{S}(l, lk)^m &= |A_l(F, lk)| - q_F |A_l(F, lk-1)| \\
&= \sum_{m|l} \frac{l}{m^2} \sum_{d|m} \mu\left(\frac{m}{d}\right) (q_F^d - 1) \left(q_F^{d\lceil \frac{lk}{m} \rceil} - q_F^{d\lceil \frac{lk-1}{m} \rceil + 1} \right).
\end{aligned}$$

The result then follows. □

2.5.5 Carayol's Number

In Carayol's paper [8], he studied irreducible smooth supercuspidal representations of $GL_N(F)$, $N = rs$. In particular when N is prime, he gave a formula for the number of irreducible very cuspidal representations of $\mathcal{K}(\mathfrak{A}_s)$ under certain conditions. We will go through his formula and we will see how to get the number irreducible supercuspidals of $GL_N(F)$ from this formula. Eventually, we will compare this number with the number we got in Lemma (2.5.12) and Lemma (2.5.17).

Carayol defines the notion of a "very cuspidal" representation of $\mathcal{K}(\mathfrak{A}_s)$ of level $\ell(\pi) = (m-1)/s$ and then he proves the following Theorem:

Theorem 2.5.18. [8, Theorem 8.1]

1. If N is a prime, then any irreducible supercuspidal representation of $GL_N(F)$ of conductor mN , minimal, is induced from a unique very cuspidal representation of $\mathcal{K}(\mathfrak{A}_1)$ of level $\ell(\pi) = m-1$.
2. Let $m \geq 2$ such that $m-1$ is prime to N . Then any irreducible supercuspidal representation of G of conductor $n+m-1$ is induced from a unique very cuspidal representation of $\mathcal{K}(\mathfrak{A}_N)$ of level $\ell(\pi) = (m-1)/N$.

Now denote the number of very cuspidal representations of $\mathcal{K}(\mathfrak{A}_s)$ of level $\ell(\pi) = (m-1)/s$ with $\omega_\pi(\varpi_F) = 1$ by $\mathfrak{n}_s(N, m-1)$. Carayol in [8, 8.5] gave a formula for $\mathfrak{n}_s(N, m-1)$ which is:

$$\mathfrak{n}_s(N, m-1) = sa_r(q_F^r - 1)q_F^{r(s\ell(\pi)-1)}, \quad m \geq 2$$

where

$$a_r = \frac{1}{r} |k_{(r)}^{reg}|$$

for $k_{(r)}$ the extension of k of degree r , and

$$k_{(r)}^{reg} = \{\alpha \in k_{(r)}^\times : k(\alpha) = k_{(r)}\}.$$

Now we will compare Carayol's numbers with the number we deduced for the representations of $GL_l(F)$. Set $N = l$,

1. If $\ell(\pi) \in \mathbb{Z}$ then $s = 1$, $r = l$. The number of very cuspidal representations of $GL_l(F)$ with $\omega_F(\varpi_F) = 1$ and level $\ell(\pi)$ is

$$\mathfrak{n}_1(l, m-1) = \frac{1}{l} (q_F^l - q_F)(q_F^l - 1)q_F^{l(\ell(\pi)-1)}.$$

Comparing this number and the number in Lemma (2.5.17), we deduce,

$$\mathfrak{n}_1(l, \ell(\pi)) = \mathfrak{S}(l, l\ell(\pi))^m,$$

where the letter m denotes the minimality.

2. If $\ell(\pi) \notin \mathbb{Z}$, then $s = l$, $r = 1$. The number of irreducible supercuspidal representations of $GL_l(F)$ with $\omega_F(\varpi_F) = 1$ and level $\ell(\pi)$ is

$$\mathbf{n}_l(l, \ell(\pi)) = l(q_F - 1)^2 q_F^{\ell(\pi) - 1}$$

This number matches the number in Lemma (2.5.12),

$$\mathbf{n}_l(l, \ell(\pi)) = \mathfrak{S}(l, \ell(\pi)).$$

2.6 Zeta Function

Here in this section, we will study the zeta function of the number of irreducible supercuspidal representations of $G = GL_l(F)$ with $\omega_\pi(\varpi_F) = 1$ and minimal level n/l . Put

$$\zeta_q(s) = \sum_n \frac{\mathfrak{S}(l, n)^m}{q^{ns}}.$$

Recall that

$$\mathfrak{S}(l, n)^m = \begin{cases} \frac{1}{l}(q^l - q) & \text{if } n = 0 \\ l(q-1)^2 q^{n-1} & \text{if } l \nmid n \\ \frac{1}{l}(q^l - q)(q^{l-1} - 1)q^{n-l} & \text{if } l|n \end{cases}$$

First we will put the function $\zeta_q(s)$ in the form $(1 - q^{-s})^{-1}$, so

$$\begin{aligned} \zeta_q(s) &= \mathfrak{S}(G, 0) + \sum_{l \nmid n} \frac{\mathfrak{S}(l, n)^m}{q^{ns}} + \sum_{l|n} \frac{\mathfrak{S}(l, n)^m}{q^{(1-s)n}} \\ &= \frac{1}{l}(q^l - q) + l(q-1)^2 q^{-1} \sum_{l \nmid n} q^{(1-s)n} + \frac{1}{l}(q^l - q)(q^l - 1)q^{-l} \sum_{l|n} q^{(1-s)n}. \end{aligned}$$

but

$$\begin{aligned} \sum_{l \nmid n} q^{(1-s)n} &= \sum_n q^{(1-s)n} - \sum_{l|n} q^{(1-s)n} \\ &= \sum_n q^{(1-s)n} - \sum_m (q^{(1-s)l})^m \\ &= \frac{1}{1 - q^{1-s}} - \frac{1}{1 - q^{(1-s)l}}. \end{aligned}$$

Therefore, we get

$$\zeta_q(s) = \frac{1}{l}(q^l - q) + l(q-1)^2 q^{-1} \frac{1}{1 - q^{1-s}} + \left[\frac{1}{l}(q^l - q)(q^l - 1)q^{-l} - l(q-1)^2 q^{-1} \right] \frac{1}{1 - q^{(1-s)l}}.$$

We can see that $\zeta_q(s)$ is rational function in q^{1-s} converges on the half-plane $\operatorname{Re}(s) > 0$ and diverges when $\operatorname{Re}(s) \leq 0$. The function $\zeta_q(s)$ has simple poles at $s = 1 + \left(\frac{2\pi i}{l \log q} \right)$, for $i \in \mathbb{Z}$.

Lemma 2.6.1.

$$\lim_{q \rightarrow \infty} \left(\frac{1}{1 - q^{1-s}} \right) = \begin{cases} 0 & \text{if } \operatorname{Re}(s) < 1 \\ 1 & \text{if } \operatorname{Re}(s) > 1 \end{cases}$$

Proof. This follows from

$$\lim_{q \rightarrow \infty} |q|^z = \begin{cases} 0 & \text{if } \operatorname{Re}(z) < 1 \\ \infty & \text{if } \operatorname{Re}(z) > 1 \end{cases}$$

□

Lemma 2.6.2.

$$\lim_{q \rightarrow \infty} \left(\frac{l\zeta_q(s)}{q^l} \right) = \begin{cases} 1 & \text{if } \operatorname{Re}(s) < 1 \\ 2 & \text{if } \operatorname{Re}(s) > 1 \end{cases}$$

Proof. We have

$$\frac{l\zeta_q(s)}{q^l} = \left(1 - \frac{1}{q^{l-1}}\right) + \frac{(q-1)^2}{q^{l+1}} \cdot \frac{1}{1 - q^{1-s}} + \left[\left(1 - \frac{1}{q^{l-1}}\right) \left(1 - \frac{1}{q^l}\right) - \frac{(q-1)^2}{q^{l+1}} \right] \cdot \frac{1}{1 - q^{(1-s)l}}.$$

Now we apply Lemma 2.6.1, so $\frac{l\zeta_q(s)}{q^l}$ has limit 1 if $\operatorname{Re}(s) < 1$ and 2 if $\operatorname{Re}(s) > 1$. □

Chapter 3

GENERAL UNRAMIFIED UNITARY GROUPS

3.1 Notations

In this chapter we fix F_0 a non-archimedean local field with residual characteristic $p \neq 2$ and $\mathcal{O}_0, \mathcal{P}_0, k_0, \varpi_0$ as defined in chapter one.

Let F be an unramified quadratic extension of F_0 with Galois involution $x \mapsto \bar{x}$. Fix an additive character ψ_0 of F_0 of level one. We can define a character ψ_F of F by setting

$$\psi_F = \psi_0 \circ \text{tr}_{F/F_0}$$

where tr_{F/F_0} is the trace map. Then ψ_F has level one. Let

$$\begin{aligned} V &= \text{an } F\text{-vector space of dimension } N \\ A &= \text{End}_F(V) \cong \mathbb{M}_N(F) \\ \tilde{G} &= \text{Aut}_F(V) \cong GL_N(F) \end{aligned}$$

Definition 3.1.1. *Let $\varepsilon = \pm$. An ε -hermitian form on V*

$$h : V \times V \longrightarrow F$$

is a non-degenerate sesquilinear form such that:

$$h(v, w) = \varepsilon \overline{h(w, v)}$$

The ε -hermitian h induces an adjoint involution on A , which we denoted by bar $\bar{}$ given by: for any $a \in A$

$$h(av, w) = h(v, \bar{a}w),$$

for all $v, w \in V$. For the field F embedded diagonally in A , the two involutions on F coincide.

Properties: For any $a, b \in A$:

1. $\overline{\bar{a}} = a$;

2. $\overline{ab} = \overline{b\bar{a}}$;
3. $\overline{a+b} = \bar{a} + \bar{b}$.

Define

$$F^1 = \{x \in F : N_{F/F_0}(x) = 1\},$$

where N_{F/F_0} is the norm map, $N_{F/F_0}(x) = x\bar{x}$.

Definition 3.1.2. We define the unitary group G by:

$$\begin{aligned} G &= \{g \in \tilde{G} : h(gv, gw) = h(v, w), \forall v, w \in V\} \\ &= \{g \in \tilde{G} : g\bar{g} = 1\}. \end{aligned}$$

An element a in A is called *skew* if it satisfies $a + \bar{a} = 0$. We put

$$A_- = \{a \in A : a + \bar{a} = 0\}.$$

We say the element a in A is *symmetric* if $\bar{a} = a$ and we put

$$A_+ = \{a \in A : \bar{a} = a\}.$$

3.2 Lattice Dual

In section (1.11), we defined lattice sequences in V . Now we need to define the notion of self dual lattice.

Definition 3.2.1. Let Λ be an \mathcal{O}_F -lattice sequence in V . We define the dual lattice sequence Λ^\sharp of Λ by

$$\Lambda^\sharp(k) = \{x \in V : h(x, \Lambda(1-k)) \subseteq \mathcal{P}_F\}.$$

Definition 3.2.2. An \mathcal{O}_F -lattice sequence Λ is called self-dual if $\Lambda^\sharp = \Lambda$.

Lemma 3.2.3. [31, Lemma 1.2.1] Let Λ be a self dual lattice sequence in V and \mathfrak{P}^n be the associated filtration. Then

$$\overline{\mathfrak{P}^n} = \mathfrak{P}^n,$$

for each $n \in \mathbb{Z}$.

We can write $\mathfrak{P}^n = \mathfrak{P}^n(\Lambda)$ as a direct sum

$$\mathfrak{P}^n = \mathfrak{P}_-^n \oplus \mathfrak{P}_+^n$$

where

$$\mathfrak{P}_-^n = \mathfrak{P}^n \cap A_- \quad \text{and} \quad \mathfrak{P}_+^n = \mathfrak{P}^n \cap A_+.$$

3.3 Parahoric Subgroups

Let Λ be a self-dual \mathcal{O}_F -lattice sequence in V . We defined compact open subgroups $U(\Lambda)$ and $U^n(\Lambda)$ of \tilde{G} in section (1.11),

Now we define a parahoric subgroup of G by:

$$P = P(\Lambda) = U(\Lambda) \cap \tilde{G}$$

It has a filtration given by:

$$P^n = P^n(\Lambda) = U^n(\Lambda) \cap G \quad n \geq 1.$$

Lemma 3.3.1. [20, 2.14(a)] *Let Λ be a self dual \mathcal{O}_F -lattice sequence in V . The quotient P/P^1 is isomorphic to*

$$P^* = \{x \in \mathfrak{A}(\Lambda)/\mathfrak{P} : x\bar{x} = 1\}.$$

Proof. By [20, 2.11], the map:

$$\varphi : P \longrightarrow P^*$$

is surjective. Now the kernel of the map φ is

$$\text{Ker}(\varphi) = \{x \in P : x \in 1 + \mathfrak{P}\} = P^1$$

By the first isomorphism theorem we have

$$P/P^1 \cong P^*.$$

□

Definition 3.3.2. *Let x be a skew element in A such that $\det(1 - \frac{x}{2}) \neq 0$. We define the Cayley map transform C of x by*

$$C(x) = \left(1 + \frac{x}{2}\right) \left(1 - \frac{x}{2}\right)^{-1}.$$

Proposition 3.3.3. [21, 2.13(c)] *For each $n \geq 1$, the Cayley map provides a bijection*

$$\begin{aligned} \varphi : \mathfrak{P}_-^n &\longrightarrow P^n \\ x &\longmapsto C(x). \end{aligned}$$

Corollary 3.3.4. [21, 2.13(d)] *For $2m \geq n \geq m \geq 1$, there is an isomorphism of abelian groups*

$$\varphi : \mathfrak{P}_-^m / \mathfrak{P}_-^n \longrightarrow P^m / P^n$$

induced by

$$x \longmapsto 1 + x.$$

Proof. We only need to show that the map is a homomorphism and it easily follows from $2m \geq n \geq m \geq 1$. □

We define the trace map on A/F_0 by

$$\mathrm{tr}_0 = \mathrm{tr}_{F/F_0} \circ \mathrm{tr}_{A/F} : A \rightarrow F_0$$

Let S be an \mathcal{O}_F -lattice in A . Suppose S is stable under the involution i.e. $\overline{S} = S$, then

$$(S_-)^* := \{a \in A_- : \mathrm{tr}_0(aS_-) \subset \mathcal{P}_0\} = (S^*)_-.$$

where $S_- = S \cap A_-$. In particular

$$(\mathfrak{P}_-^n)^* = ((\mathfrak{P}^n)^*)_- = \mathfrak{P}_-^{1-n}.$$

by Lemma (1.12.3).

Proposition 3.3.5. [37, II.5] *The following map*

$$\begin{aligned} A_- &\longrightarrow (A_-)^\wedge \\ x &\longmapsto (y \mapsto \psi_0(\mathrm{tr}_0(xy))) \end{aligned}$$

is an isomorphism of abelian groups.

Lemma 3.3.6. [22, 4.19] *For $2m \geq n \geq m \geq 1$, there exists a P -equivariant isomorphism of abelian groups*

$$\begin{aligned} \mathfrak{P}_-^{1-n}/\mathfrak{P}_-^{1-m} &\longrightarrow (P^m/P^n)^\wedge \\ b + \mathfrak{P}_-^{1-m} &\longmapsto \psi_b \end{aligned}$$

where $\psi_b(x) = \psi_0(\mathrm{tr}_0(b(x-1)))$ for $x \in \mathfrak{p}_0^m$.

Proof. This follows from Corollary 3.3.4. □

In the following let $b_1, b_2 \in A_-$ and $\widetilde{\psi}_{b_1}, \widetilde{\psi}_{b_2}$ be characters of $U^n(\Lambda)$ trivial on $U^{n+1}(\Lambda)$, for some lattice sequence Λ and let ψ_{b_1} and ψ_{b_2} be the characters $\widetilde{\psi}_{b_1}, \widetilde{\psi}_{b_2}$ restricted to $P^n(\Lambda)$.

Lemma 3.3.7. *Let $g \in G$. Then the following are equivalent:*

1. g intertwines ψ_{b_1} with ψ_{b_2} ;
2. g intertwines $\widetilde{\psi}_{b_1}$ with $\widetilde{\psi}_{b_2}$.

Proof. The implication (1) \Rightarrow (2) is clear. Now let g intertwines $\widetilde{\psi}_{b_1}$ with $\widetilde{\psi}_{b_2}$, then

$$g(b_1 + \mathfrak{P}^{1-n})g^{-1} \cap (b_2 + \mathfrak{P}^{1-n}) \neq \emptyset.$$

Thus there exists $\beta_i \in \mathfrak{P}^{1-n}$, $i = 1, 2$, such that:

$$g(b_1 + \beta_1)g^{-1} = b_2 + \beta_2.$$

Since $\mathfrak{P}^{1-n} = \mathfrak{P}_-^{1-n} \oplus \mathfrak{P}_+^{1-n}$, there exists $x_i \in \mathfrak{P}_-^{1-n}$ and $y_i \in \mathfrak{P}_+^{1-n}$ such that $\beta_i = x_i + y_i$, $i = 1, 2$. Then

$$g(b_1 + x_1)g^{-1} + gy_1g^{-1} = (b_2 + x_2) + y_2.$$

Since $g(b_1 + x_1)g^{-1} \in A_-$ and $gy_1g^{-1} \in A_+$ then, using $A = A_- \perp A_+$, we get

$$g(b_1 + x_1)g^{-1} = b_2 + x_2 \quad \text{and} \quad gy_1g^{-1} = y_2.$$

Therefore,

$$g(b_1 + \mathfrak{P}_-^{1-n})g^{-1} \cap (b_2 + \mathfrak{P}_-^{1-n}) \neq \emptyset$$

and g intertwines ψ_{b_1} with ψ_{b_2} . □

3.4 Skew Strata

Definition 3.4.1. A skew stratum in A is a quadruple $[\Lambda, n, r, b]$ consisting of a self dual \mathcal{O}_F -lattice sequence, integers $n > r \geq 0$ and an element $b \in \mathfrak{P}_F^{-n}(\Lambda)$.

Two skew strata $[\Lambda, n, r, b_1]$ and $[\Lambda, n, r, b_2]$ are equivalent if:

$$b_1 \equiv b_2 \pmod{\mathfrak{P}_F^{-r}(\Lambda)}.$$

Let π be an irreducible smooth representation of G and $[\Lambda, n, r, b]$ a skew stratum with $n \leq 2r + 1$. We say the representation π contains $[\Lambda, n, r, b]$ if it contains the character ψ_b of $P^{r+1}(\Lambda)$.

Definition 3.4.2. Let π be an irreducible smooth representation of G and $S(\pi)$ be the set of pairs (Λ, n) , where Λ is a self dual \mathcal{O}_F -lattice sequence in V and $n \geq 0$, such that the representation π contains the trivial character of $P^{n+1}(\Lambda)$. Then we define the normalized level of π by:

$$\ell(\pi) = \min\{n/e(\Lambda) : (\Lambda, n) \in S(\pi)\}.$$

Theorem 3.4.3. Let π be an irreducible smooth representation of G . Then, either

1. π contains a skew fundamental stratum $[\Lambda, n, n-1, b]$ (positive level); or
2. There exists a self dual lattice sequence Λ such that the restriction of π to the group $P(\Lambda)$ contains an irreducible representation σ trivial on $P^1(\Lambda)$ (level zero).

Proof. See [25, Theorem 5.2]. □

Let $[\Lambda, n, n-1, b]$ be a skew fundamental stratum in A and $e = e(\Lambda)$. If π is irreducible smooth representation of G contains $[\Lambda, n, n-1, b]$, then $\ell(\pi) = n/e$, by [28, §3 and §6] and [25, Theorem 5.2]. The characteristic polynomial $\varphi_b \in k_F[X]$ is the characteristic polynomial of $y = \varpi_F^{n/g} b^{e/g} \in \mathfrak{A}(\Lambda)$ modulo \mathcal{P}_F , where $g = \gcd(n, e)$.

Remark 3.4.4. If y is skew then $\overline{\varphi_b(X)} = \varphi_b(-X)$ and if y is symmetric then $\overline{\varphi_b(X)} = \varphi_b(X)$. Note that, for b skew, y is skew if e/g is odd and symmetric if e/g is even.

Now if the integers n and r satisfy:

$$n \leq 2r + 1$$

then the equivalence class of the stratum $[\Lambda, n, r, b]$ corresponds to the character ψ_b of $P^{r+1}(\Lambda)$ which is trivial on $P^{n+1}(\Lambda)$ by Lemma 3.3.6.

Lemma 3.4.5. *Let π be an irreducible smooth representation of G and let χ be a character of F^1 . Suppose $\ell(\pi) \neq \ell(\chi)$, then:*

$$\ell(\pi \otimes \chi \circ \det) = \max\{\ell(\pi), \ell(\chi)\}.$$

Proof. The proof is essentially the same as Lemma 1.13.11. □

Corollary 3.4.6. *Let π be an irreducible smooth representation of G and let χ be a character of F^1 . If $\ell(\pi \otimes \chi \circ \det) < \ell(\pi)$, then $\ell(\pi) = \ell(\chi)$.*

Proposition 3.4.7. [2, Proposition 4.1] *Suppose N is a prime. Let π be an irreducible smooth representation of G which contains a non-split fundamental skew stratum. Then π either:*

1. *contains a skew fundamental scalar stratum i.e. there exists a character χ of F^1 such that*

$$\ell(\pi \otimes \chi \circ \det) < \ell(\pi); \quad \text{or,}$$

2. *is induced from a compact open subgroup K of G .*

Let $[\Lambda, n, r, b]$ be a skew stratum in A . We define the formal intertwining of the stratum in G by:

$$\mathcal{I}_G[\Lambda, n, r, b] = \{g \in G : g^{-1}(b + \mathfrak{P}_-^r)g \cap (b + \mathfrak{P}_-^r) \neq \emptyset\}$$

If $n \leq 2r + 1$, then this is the same as the intertwining of the character ψ_b of P^{r+1} , by [32, §4].

Lemma 3.4.8. *Let $[\Lambda, n, n-1, b_1]$ and $[\Lambda, n, n-1, b_2]$ be fundamental skew strata in A which are intertwined. Then they have the same characteristic polynomials.*

Proof. Suppose that $g \in G$ intertwines ψ_{b_1} with ψ_{b_2} , then

$$g^{-1}(b_1 + \mathfrak{P}_-^{1-n})g \cap (b_2 + \mathfrak{P}_-^{1-n})$$

is not empty. If b lies in the intersection, then $g^{-1}bg \in b_1 + \mathfrak{P}_-^{1-n}$ and $b \in b_2 + \mathfrak{P}_-^{1-n}$. The characteristic polynomial of $y_1 = g^{-1}\varpi_F^{n/g}b^{e/g}g$ modulo \mathcal{P}_F is $\varphi_{b_1}(X)$ and the characteristic polynomial of $y_2 = \varpi_F^{n/g}b^{e/g}$ modulo \mathcal{P}_F is $\varphi_{b_2}(X)$. Since any two conjugate elements have the same characteristic polynomial, then we deduce $\varphi_{b_1}(X) = \varphi_{b_2}(X)$. □

Definition 3.4.9. *A semi-minimal skew stratum is a stratum $[\Lambda, n, n-1, b]$ with a decomposition $V = V^1 \perp \dots \perp V^l$ into orthogonal subspaces, such that*

1. $\Lambda(k) = \bigoplus_{i=1}^l (\Lambda(k) \cap V^i)$, for each $k \in \mathbb{Z}$; we write Λ^i for the (self-dual) lattice sequence given by $\Lambda^i(k) = \Lambda(k) \cap V^i$;
2. $b = \sum_{i=1}^l b_i$, where $b_i \in \text{End}_F(V^i)$;
3. each $[\Lambda^i, n, n-1, b_i]$ is either minimal or $b_i = 0$;
4. for $i \neq j$, each $[\Lambda^i \perp \Lambda^j, n, n-1, b_i + b_j]$ is not null, i.e. $b_1 + b_2 \neq 0$, or equivalent to a minimal stratum.

Theorem 3.4.10. *Let $[\Lambda, n, r, b]$ be a semi-minimal skew stratum in A . Then the intertwining of the stratum $[\Lambda, n, r, b]$ in G is*

$$\mathcal{I}_G[\Lambda, n, r, b] = P^{n-r}(B^\times \cap G)P^{n-r}$$

where B is the centralizer of b in A .

Proof. See [32, Theorem 4.15] □

Definition 3.4.11. *Let $b \in \mathbb{M}_N(k)$, for a finite field k . Then we say b is:*

- regular if all eigenvalue are distinct and skew.
- semi-simple if it is diagonalizable (over algebraic closure extension of k) i.e. there exists $g \in GL_N(\bar{k})$ such that gbg^{-1} is diagonal.

Let

$$G(k_F) = \{g \in \text{Aut}_{k_F}(\tilde{V}) : \tilde{h}(g\underline{v}, g\underline{w}) = \tilde{h}(\underline{v}, \underline{w}), \underline{v}, \underline{w} \in \tilde{V}\}$$

where \tilde{V} is a k_F -vector space and \tilde{h} is an ε -hermitian form on \tilde{V} , where $\varepsilon = \pm$. Now we have the following lemma

Lemma 3.4.12. *Suppose b_1 and b_2 are regular semi-simple elements in $\mathbb{M}_N(k_F)$. Then they are conjugate in $G(k_F)$ if and only if they have the same characteristic polynomials.*

Proof. There exists a splitting $b_1 = \sum_{i=1}^l \alpha_i$ and $b_2 = \sum_{i=1}^l \beta_i$ such that

$$C_{GL_N}(b_1) = k_F[b_1]^\times = \bigoplus_{i=1}^l k_F[\alpha_i] \quad \text{and} \quad C_{GL_N}(b_2) = k_F[b_2]^\times = \bigoplus_{i=1}^l k_F[\beta_i].$$

Note that when $l = 1$ then b_1 and b_2 called simple. If the elements b_1 and b_2 are conjugate by $g \in G(k_F)$; then we can take $b_1 = g^{-1}b_2g$ so clearly they have the same characteristic polynomial. Now suppose they have the same characteristic polynomial so $\varphi_{b_1}(X) = \varphi_{b_2}(X)$. Now, by elementary algebra, there exists an element $g \in GL_N(k_F)$ such that

$$g^{-1}b_1g = b_2 \tag{3.1}$$

Now, for $i = 1, 2$, we have $b_i = -\bar{b}_i$ if b_i is skew or $b_i = \bar{b}_i$ if b_i is symmetric. We apply the involution to both sides of (3.1), and get

$$(\bar{g})^{-1}\bar{b}_1\bar{g} = \bar{b}_2$$

thus

$$(\bar{g})^{-1}b_1\bar{g} = b_2 \tag{3.2}$$

By (3.1) and (3.2) we have:

$$(\bar{g}g)b_1(\bar{g}g)^{-1} = b_1$$

so $\bar{g}g$ lies in the centralizer of b_1 in $GL_N(k_F)$. Now put $z = \bar{g}g$, so we can write $z = \sum_{i=1}^l z_i$, where $z_i \in k_{F_i}$ and $F_i = F[\alpha_i]$.

Since $z = \bar{z}$, then we also have $z_i = \bar{z}_i$ for all i . The map $N_{k_{F_i}/k_{F_0}} : k_{F_i}^\times \rightarrow k_{F_0}^\times$ is surjective so there exists $h_i \in k_{F_i}^\times$ such that

$$\bar{h}_i h_i = N_{k_{F_i}/k_{F_0}}(h_i) = \bar{z}_i z_i$$

Put $h = \sum_{i=1}^l h_i$, then $\bar{h}h = \bar{g}g$. Hence, $gh^{-1} = \bar{h}^{-1}\bar{g} \in G(k_F)$ and

$$(gh^{-1})b_1(gh^{-1})^{-1} = g(h^{-1}b_1h)g^{-1} = gb_1g^{-1} = b_2$$

Therefore the elements b_1 and b_2 are conjugate by $gh^{-1} \in G(k_F)$. \square

3.5 Maximal Simple Skew Strata

A semi-minimal skew stratum $[\Lambda, n, n-1, b]$ is *maximal simple* if $l = 1$ in Definition 3.4.9 and $[F[b] : F] = N$. Since $F[b]/F$ is of degree N , there is a unique lattice chain \mathcal{L} normalized by $F[b]^\times$, and it is self dual. Moreover, the only lattice sequences normalized by $F[b]^\times$ are multiple of \mathcal{L} so we may, and we will, assume Λ is a chain.

Let $P = P(\Lambda)$ and $\mathfrak{P} = \mathfrak{P}(\Lambda)$. The following is an analogue of [4, Lemma 16.1]

Lemma 3.5.1. *Let $[\Lambda, n, n-1, b_i]$, $i = 1, 2$, be maximal simple strata in A with $n \geq 1$. Suppose $g \in G$ intertwines ψ_{b_1} with ψ_{b_2} on P^n . Then:*

1. $g \in P$;
2. the characters $\psi_{b_1}|_{P^n}$ and $\psi_{b_2}|_{P^n}$ are conjugate by the element g ; equivalently, the cosets $b_1 + \mathfrak{P}_-^{1-n}$ and $b_2 + \mathfrak{P}_-^{1-n}$ are conjugate by g .

Proof. (1) The element g intertwines the character ψ_{b_1} with ψ_{b_2} on P^n , so:

$$g^{-1}(b_1 + \mathfrak{P}_-^{1-n})g \cap (b_2 + \mathfrak{P}_-^{1-n}) \neq \emptyset$$

Suppose b lies in the intersection so $b \in b_2 + \mathfrak{P}_-^{1-n}$. Since the stratum is simple, the element b_2 is minimal over F and $[F[b_2] : F] = N$, then by [6, proposition 2.2.2], b is minimal. We have $F[b]^\times$ normalizes Λ . If \mathcal{L} is the chain of \mathfrak{A}_Λ -lattices in V then \mathcal{L} is the chain of all $\mathfrak{o}_{F[b]}$ -lattices in V .

Also, $b \in g^{-1}(b_1 + \mathfrak{P}_-^{1-n})g$ so $gbg^{-1} \in b_1 + \mathfrak{P}_-^{1-n}$. Now \mathcal{L} is the chain of all $\mathcal{O}_{F[gbg^{-1}]}$ -lattices in V so $g^{-1}\mathcal{L}$ is the chain of all $\mathcal{O}_{F[b]}$ -lattices. By uniqueness, $g^{-1}\mathcal{L} = \mathcal{L}$ so $g \in \mathcal{K}(\mathcal{L}) \cap G = P$.

(2) The intersection between the cosets $g^{-1}(b_1 + \mathfrak{P}_-^{1-n})g = g^{-1}b_1g + \mathfrak{P}_-^{1-n}$ and $b_2 + \mathfrak{P}_-^{1-n}$ is not empty so they are equal. \square

The following is analogous to [4, Lemma 15.2]

Theorem 3.5.2. *For $i = 1, 2$, let $[\Lambda, n, n-1, b_i]$ be a maximal simple skew stratum with $n \geq 1$. The characters ψ_{b_i} of the group $P^{[\frac{n}{2}]+1}$ are intertwined in G if and only if they are conjugate by an element of P .*

Proof. Suppose $g \in G$ intertwines the characters ψ_{b_i} of $P^{\lfloor \frac{n}{2} \rfloor + 1}$ then:

$$g^{-1} \left(b_1 + \mathfrak{P}_-^{\lfloor \frac{n}{2} \rfloor} \right) g \cap \left(b_1 + \mathfrak{P}_-^{\lfloor \frac{n}{2} \rfloor} \right) \neq \emptyset$$

Since $\mathfrak{P}_-^{\lfloor \frac{n}{2} \rfloor} \subset \mathfrak{P}_-^{1-n}$. Then

$$g^{-1} (b_1 + \mathfrak{P}_-^{1-n}) g \cap (b_1 + \mathfrak{P}_-^{1-n}) \neq \emptyset$$

By Lemma 3.5.1(1), $g \in P$. The cosets $g^{-1} (b_1 + \mathfrak{P}_-^{\lfloor n/2 \rfloor}) = g^{-1} b_1 g + \mathfrak{P}_-^{\lfloor n/2 \rfloor}$ and $(b_2 + \mathfrak{P}_-^{\lfloor n/2 \rfloor})$ intersect so they are equal. \square

Lemma 3.5.3. *Let $[\Lambda, n, n-1, b]$ be a maximal simple skew stratum in A . Then the number of the characters $\psi_b|_{P^{\lfloor \frac{n}{2} \rfloor + 1}}$ which extend the character $\psi_b|_{P^n}$ is*

$$\left[(1 + \mathcal{P}_E)^1 : \left(1 + \mathcal{P}_E^{\lfloor \frac{n+1}{2} \rfloor} \right)^1 \right]$$

up to G -intertwining.

Proof. Now Theorem 3.5.2 implies that G -intertwining is the same as $P(\Lambda)$ -conjugacy. Any $g \in P$ conjugating two extensions must normalize the character $\psi_b|_{P^n}$, so $g \in E^1 P^1$ by Theorem 3.4.10.

Let Γ denote the set of characters $\psi|_{P^{\lfloor \frac{n}{2} \rfloor + 1}}$ which extend $\psi_b|_{P^n}$. The group $E^1 P^1$ acts on the set Γ and the stabilizer of $\psi|_{P^{\lfloor \frac{n}{2} \rfloor + 1}}$ is $E^1 P^{\lfloor \frac{n+1}{2} \rfloor}$ by Theorem 3.4.10, so, by the orbit-stabilizer theorem, the orbit of $\psi|_{P^{\lfloor \frac{n}{2} \rfloor + 1}}$ has size

$$\left[E^1 P^1 : E^1 P^{\lfloor \frac{n+1}{2} \rfloor} \right]$$

The lemma follows since

$$|\Gamma| = \left[P^{\lfloor \frac{n}{2} \rfloor + 1} : P^n \right]$$

\square

For a maximal simple skew stratum $[\Lambda, n, n-1, b]$ with radical \mathfrak{P} , define the following compact subgroup:

$$J = E^1 P^{\lfloor \frac{n+1}{2} \rfloor}.$$

Denote by $\mathcal{R}(\Lambda, \psi_b)$ the set of equivalence classes of irreducible representations η of J such that the restriction to $P^{\lfloor \frac{n}{2} \rfloor + 1}$ contains ψ_b (or, equivalently by Theorem 3.4.10, is a multiple of ψ_b).

Theorem 3.5.4. *The representation $\text{Ind}_J^G \eta$ is irreducible and supercuspidal, for $\eta \in \mathcal{R}(\Lambda, \psi_b)$.*

Proof. Since η is an irreducible smooth representation of J and the intertwining of η in G is J by Theorem 3.4.10, then Theorem 1.10.6 implies that the representation $\text{Ind}_J^G \eta$ is irreducible and supercuspidal. \square

Theorem 3.5.5 (Uniqueness). *For $i = 1, 2$, let $[\Lambda, n_i, n_i - 1, b_i]$ be a maximal simple stratum in A . Let $\eta_i \in \mathcal{R}(\Lambda, \psi_{b_i})$. Suppose that the representations*

$$\pi_i = \text{Ind}_{J_i}^G \eta_i, \quad i = 1, 2,$$

are equivalent. Then there exists $g \in G$ such that:

$$J_2 = g^{-1} J_1 g \quad \text{and} \quad \eta_2 \cong \eta_1^g.$$

Proof. We may assume $\pi_1 = \pi_2 = \pi$. The representation π contains the stratum $[\Lambda, n_i, n_i - 1, b_i]$, $i = 1, 2$, and since the two strata fundamental then they have the same level as they are contained in the same representation π . Thus, $e_1 = e_2$ and $n_1 = n_2$. Now π contains the characters ψ_{b_1} and ψ_{b_2} on $P^{[\frac{n}{2}]+1}(\Lambda)$ so they intertwine in G by Proposition 1.10.4. By Theorem 3.5.1, they are conjugate by an element in $P(\Lambda)$:

$$\psi_{b_2} = \psi_{b_1}^g, \quad g \in P(\Lambda).$$

Now J_i is the normalizer of the character $\psi_{b_i}|_{P^{[\frac{n}{2}]+1}(\Lambda)}$. Therefore, J_2 and J_1 are conjugate by g .

Finally, consider the representation $\eta_3 = \eta_1^g$ of J_2 . Now $\eta_3|_{P^{[\frac{n}{2}]+1}(\Lambda)}$ is a multiple of ψ_{b_2} and η_3 is intertwined with η_2 by an element $h \in G$. The element h also intertwines ψ_{b_2} on $P^{[\frac{n}{2}]+1}(\Lambda)$. By Theorem 3.4.10, $h \in J_2$. Hence h fixes η_2 and we conclude that $\eta_3 \cong \eta_2$. \square

3.6 Essentially Scalar

Definition 3.6.1. *Let $[\Lambda, ne, ne - 1, b]$ be a fundamental skew stratum in A where $e = e(\Lambda)$. We say the stratum $[\Lambda, ne, ne - 1, b]$ is essentially scalar if the characteristic polynomial of $y = \varpi_F^n b$ has the form*

$$\varphi_b(X) = (X - \alpha)^N$$

for some $\alpha \in k_F$ and $\alpha = -\bar{\alpha}$. An irreducible supercuspidal representation π of G is essentially scalar if it contains an essentially scalar skew simple stratum.

Lemma 3.4.8, implies that if two strata intertwine in G , then they have the same characteristic polynomials. Let $[\Lambda, n, n - 1, b_i]$ be an essentially scalar skew simple stratum in A , $i = 1, 2$. Suppose that

$$\varphi_{b_1}(X) = (X - \alpha_1)^N = (X - \alpha_2)^N = \varphi_{b_2}(X)$$

for some $\alpha_i \in k_F$. Then $\alpha_1 = \alpha_2$ and by Lemma 3.4.12 b_1, b_2 are conjugate so they intertwine.

Now we deduce the following lemma

Lemma 3.6.2. *The number of fundamental essentially scalar skew simple strata in A of level n up to G -intertwining is $q - 1$.*

Lemma 3.6.3. *[2, §4] Let π be an irreducible supercuspidal representation of G which contains an essentially scalar skew stratum $[\Lambda, ne, ne - 1, b]$. There exists a character χ of G such that the representation $\pi \otimes \chi$ contains a skew stratum of level strictly less than the level of the stratum $[\Lambda, ne, ne - 1, b]$.*

Let $\text{Rep}(G, n)^{es}$ be the set of equivalence classes of essentially scalar irreducible representations π of G of level n and $\text{Rep}(G, < n)$ be the set of equivalence classes of irreducible supercuspidal representations of G of level less than n . Let Ψ denote the set of characters of G of level less or equal n , then we have the proposition:

Proposition 3.6.4. *The map*

$$\begin{aligned} \text{Rep}(G, < n) \times \Psi &\longrightarrow \text{Rep}(G, n)^{es} \\ (\pi, \chi) &\longmapsto \pi \otimes \chi \end{aligned}$$

is surjective. Moreover, the fibers of this map are

$$\{(\pi\psi^{-1}, \chi\psi) : \psi \text{ is a character of } F^1 \text{ of level } k \text{ less than } n\}$$

Proof. Let $\pi \in \text{Rep}(G, n)^{es}$. By Lemma 3.6.3, there exists a character χ such that

$$\ell(\pi \otimes \chi) < \ell(\pi) = n.$$

Moreover, we must have $\ell(\chi) = \ell(\pi) = n$, by Corollary 3.4.6.

The representation $\pi \otimes \chi$, therefore, lies in $\text{Rep}(G, < n)$ and since $\ell(\chi) = \ell(\chi^{-1}) = n$, then $\chi^{-1} \in \Psi$ and $(\pi \otimes \chi, \chi^{-1})$ maps to π .

Suppose $\pi \in \text{Rep}(G, < n)$ and $\chi \in \Psi$ such that $\pi \otimes \chi \in \text{Rep}(G, n)^{es}$. Consider the set

$$\text{Fib}(\pi \otimes \chi) = \{(\pi', \chi') : \pi' \in \text{Rep}(G, < n), \chi' \in \Psi \text{ and } \pi' \otimes \chi' \cong \pi \otimes \chi\}.$$

The set $\text{Fib}(\pi \otimes \chi)$ contains (π, χ) so it is not-empty. Let $\pi' \in \text{Rep}(G, < n)$ and $\chi' \in \Psi$ such that $\pi' \otimes \chi' = \pi \otimes \chi$, thus, $\pi' \cong \pi \otimes \chi\chi'^{-1}$. But $\ell(\pi') < n = \ell(\pi)$ so, by Lemma 3.4.5, we must have $\ell(\chi\chi'^{-1}) < n$. Put $\chi\chi'^{-1} = \psi^{-1}$, for some character ψ of F^1 of level less than n . Then $\pi \cong \pi \otimes \chi\chi'^{-1} = \pi\psi^{-1}$ and

$$(\pi', \chi') = (\pi\psi^{-1}, \chi\psi)$$

for all characters ψ of F^1 of level less than n . □

Corollary 3.6.5. *The size of each fiber in Proposition 3.6.4 is $(q+1)q^{n-2}$.*

Corollary 3.6.6. *We have*

$$|\text{Rep}(G, n)^{es}| = (q-1)|\text{Rep}(G, < n)|.$$

Proof. We have $|\Psi| = (q+1)(q-1)q^{n-2}$ and the result follows. □

3.7 Construction of Supercuspidal Representations

In this section, we are going to generalize section (3.5) to construct irreducible supercuspidal representations of the unitary group G that contain a semi-minimal

stratum. These representations are not the only supercuspidals of G . There are others coming from a more general construction, as we will see in chapter 5.

Let π be an irreducible supercuspidal representation of G which contains a semi-minimal skew stratum $[\Lambda, n, n-1, b]$ so π has level n/e , where $e = e(\Lambda)$. Moreover, we assume the notation of Definition 3.4.9, that each $[\Lambda^i, n, n-1, b_i]$ is a maximal simple stratum, i.e.

$$[F[b_i] : F] = \dim_F(V^i).$$

Lemma 3.7.1. *Any two semi-minimal skew strata which intertwine are conjugate.*

Proof. If two semi-minimal skew strata intertwine, then they have the same characteristic polynomials by Lemma 3.4.8. Now the proof follows from Lemma 3.4.12. \square

Let $P = P(\Lambda)$ and $\mathfrak{P} = \mathfrak{P}(\Lambda)$. We have a decomposition

$$V = V^1 \perp \dots \perp V^l,$$

where $i \in \{1, \dots, l\}$ and Λ^i is a self-dual lattice sequence in V^i . Put

$$E = \bigoplus_{i=1}^l E_i$$

where $E_i = F[b_i]$ is a field with involution (the restriction of the adjoint involution on A) and fixed field $E_{i,0}$. Define

$$\mathcal{P}_E = \bigoplus_{i=1}^l \mathcal{P}_{E_i}$$

and

$$E^1 = \prod_{i=1}^l E_i^1.$$

Now define the following subgroups

$$\begin{aligned} {}^1H = {}^1H(\Lambda) &= P^{[\frac{n}{2}]+1} \cdot (1 + \mathcal{P}_E)^1 \\ {}^1J = {}^1J(\Lambda) &= P^{[\frac{n+1}{2}]} \cdot (1 + \mathcal{P}_E)^1 \\ J = J(\Lambda) &= P^{[\frac{n}{2}]+1} \cdot E^1. \end{aligned}$$

If n is odd, then ${}^1J = {}^1H$.

Now we start with the construction. The representation π contains the character $\psi_b|P^n$. By Lemma 3.3.6, the character $\psi_b|P^n$ extends to a character of $P^{[\frac{n}{2}]+1}$ and the total number of these characters is

$$\left[\mathfrak{P}_-^{1-n} : \mathfrak{P}_-^{-[\frac{n}{2}]} \right].$$

We will denote the set of G -intertwining classes of characters $\psi|P^{[\frac{n}{2}]+1}$ which extend $\psi_b|P^n$ by $\text{Ext}(\psi_b, P^n)$.

- Remark 3.7.2.** 1. *The intertwining is not usually an equivalence relation, however, Lemma 3.7.1 implies that the G -intertwining is the same as G -conjugacy, so here the intertwining is, indeed an equivalence relation.*
2. *Any character which extend $\psi_b|P^n$ has the form $\psi_{b'}$ with $[\Lambda', n, n-1, b']$ still maximal semi-minimal and*

$$b' \equiv b \pmod{\mathfrak{P}^{1-n}(\Lambda)},$$

so we may as well called it ψ_b .

Now consider the quotient

$${}^1H/P^{[\frac{n}{2}]+1} \cong \prod_{i=1}^l (1 + \mathcal{P}_{E_i})^1 / \left(1 + \mathcal{P}_{E_i}^{[\frac{n}{2e_i}]+1}\right)^1$$

where $e_i = e(\Lambda^i)$. The quotient is a product of cyclic groups and the group 1H normalizes the character ψ_b , therefore, the character $\psi_b|P^{[\frac{n}{2}]+1}(\Lambda)$ extends to a character θ_b of 1H . We denote the set of characters θ which extend $\psi_b|P^{[\frac{n}{2}]+1}(\Lambda)$ by $\Gamma(\Lambda, \psi_b)$

Lemma 3.7.3. *Let $\theta_1, \theta_2 \in \Gamma(\Lambda, \psi_b)$. They intertwine if and only if $\theta_1 = \theta_2$.*

Proof. Let $g \in G$ intertwines θ_1 with θ_2 . Then g must intertwines the character ψ_b and by Theorem 3.4.10, $g \in J = E^1 P^{[\frac{n+1}{2}]}$, but J normalizes both characters θ_1 and θ_2 . Therefore, $\theta_1 = \theta_2$. \square

We deduce

$$|\Gamma(\Lambda, \psi_b)| = [{}^1H : P^{[\frac{n}{2}]+1}].$$

Now consider the case ${}^1J \neq {}^1H$.

Lemma 3.7.4 (Heisenberg). *Given $\theta \in \Gamma(\Lambda, \psi_b)$ There is a unique irreducible representation η_θ of 1J such that the restriction of η_θ to 1H is a multiple of θ . Moreover, $\dim(\eta_\theta) = \sqrt{[{}^1J : {}^1H]}$.*

Proof. The sequence

$$0 \rightarrow {}^1H \rightarrow {}^1J \rightarrow {}^1J/{}^1H \rightarrow 0 \tag{3.3}$$

is exact. The paring

$$\begin{aligned} {}^1J/{}^1H \times {}^1J/{}^1H &\longrightarrow \mathbb{C}^\times \\ (x, y) &\longmapsto \theta(x^{-1}y^{-1}xy) \end{aligned}$$

is non-degenerate as if $\theta(x^{-1}y^{-1}xy) = 1$, for all $y \in {}^1J$ then x must be in 1H , by [33, Proposition 4.1]. Clearly the paring is alternating.

By Theorem [3, Proposition 8.3.3] there is a unique irreducible representation η_θ of 1J such that the restriction of η_θ to 1H is a multiple of θ and $\dim(\eta_\theta) = \sqrt{[{}^1J : {}^1H]}$. \square

Carayol's Result:

Suppose that \bar{H} is a group which is a central extension of an abelian group K by another abelian group A i.e. the short sequence

$$1 \rightarrow A \rightarrow H \rightarrow K \rightarrow 1$$

is exact and A is in the central of K . Let χ be a character of A and

$$\begin{aligned} [,] : H \times H &\longrightarrow A \\ (h_1, h_2) &\longmapsto \chi([h_1, h_2]) \end{aligned}$$

a non-degenerate bilinear form, where $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$. There exists a unique irreducible representation r_χ of dimension $d = |K|^{1/2}$ such that the restriction of r_χ to A contains χ .

Suppose we another short exact sequence

$$1 \rightarrow H \rightarrow H_1 \rightarrow B \rightarrow 1$$

with B is abelian group and A is in the center of H_1 . Then H_1 normalizes the representation r_χ .

Carayol in his paper [8, §4.2] show the following result

Lemma 3.7.5. *Let A, H, K, H_1 as above and suppose*

$$B \cong \mathbb{Z}/\alpha_1\mathbb{Z} \times \dots \times \mathbb{Z}/\alpha_m\mathbb{Z}.$$

If $\gcd(\alpha_1, \dots, \alpha_m, d) = 1$, then the representation r_χ extends to a representation \tilde{r}_χ of H such that the restriction of r to A contains χ .

We apply Carayol's Lemma with $A = {}^1H/\text{Ker}(\theta_b)$, $H = {}^1J/\text{Ker}(\theta_b)$, $K = {}^1J/{}^1H$, $H_1 = J$ and

$$B \cong k_{E_1} \times \dots \times k_{E_l}.$$

Clearly $\gcd(q_{E_1}, \dots, q_{E_l}, d) = 1$ and J normalizes the representation η_b , therefore, the representation η_b of 1J extends to an irreducible representation η of J .

Denote by $\mathcal{R}(\Lambda, \theta)$, the set of equivalence classes of irreducible representations η of J such that the restriction to 1J is a multiple of θ .

Lemma 3.7.6. *If $\eta_i \in \mathcal{R}(\Lambda, \theta_i)$ and $\theta_i \in \Gamma(\Lambda, \psi_b)$, $i = 1, 2$, and η_1, η_2 intertwine, then*

$$\eta_1 \cong \eta_2 \quad \text{and} \quad \theta_1 = \theta_2.$$

Proof. Let $g \in G$ intertwine η_1 with η_2 then it must intertwine $\psi_b|P^{[\frac{n+1}{2}]}$ with itself. By Theorem 3.4.10, $g \in J$ so g normalizes both η_1 and η_2 . Hence, $\eta_1 \cong \eta_2$ and restricting to 1H , we get $\theta_1 = \theta_2$. \square

Finally, we induce the representation η of J to the unitary group G to get an irreducible supercuspidal representation.

Lemma 3.7.7. *The representation $\text{c-Ind}_J^G \eta$ is irreducible and supercuspidal and equivalent to π .*

Proof. The intertwining in G of η is contained in

$$\mathcal{I}_G(\psi_b|P^{\lfloor \frac{n+1}{2} \rfloor}) = J$$

by Theorem 3.4.10. By Theorem (1.10.6), the induced representation $\text{c-Ind}_J^G \eta$ is irreducible and supercuspidal. \square

A summary of the construction is shown by the following diagram:

$$\begin{array}{ccc}
 \text{c-Ind}_{J(\Lambda)}^G \eta & & G \\
 | & & | \\
 \eta & & J(\Lambda) \\
 | & & | \\
 \eta_\theta & & {}^1J(\Lambda) \\
 | & & | \\
 \theta & & {}^1H(\Lambda) \\
 | & & | \\
 \psi_b & & P^{\lfloor \frac{n}{2} \rfloor + 1}(\Lambda) \\
 | & & | \\
 \psi_b & & P^n(\Lambda)
 \end{array}$$

Denote the number of irreducible supercuspidal representations of G which contain a semi-minimal skew stratum $[\Lambda, n, n-1, b]$, up to G -intertwining (conjugacy) by $\mathfrak{s}(G, n/e)^{sm}$, where $e = e(\Lambda)$. It is given by

$$\boxed{\mathfrak{s}(G, \Lambda, n/e)^{sm} = \sum_{(b, \Lambda)} \text{Str}(\Lambda, n/e)^{ms} |\text{Ext}(\psi_b, P^n)| \cdot |\Gamma(\Lambda, \psi_b)| \cdot |\mathcal{R}(\Lambda, \theta)|} \quad (3.4)$$

where $\text{Str}(\Lambda, n/e)$ is the number of semi-minimal skew strata $[\Lambda, n, n-1, b]$ in A of level n/e up to G -intertwining.

3.8 Level Zero Supercuspidals of G

Here in this section, we will consider irreducible smooth representations of G of level zero. A full classification of these representations is given by Morris in his paper [23].

Let π be an irreducible smooth representation of G of level $\ell(\pi) = 0$. By Theorem 3.4.3, there exists a self dual lattice Λ such that the restriction of π to $P = P(\Lambda)$ contains an irreducible representation σ which is trivial on P^1 . The short sequence

$$0 \rightarrow P^1 \rightarrow P \rightarrow P^* \rightarrow 0$$

is exact, where $P^* = P/P^1$.

Let $\tilde{\sigma}$ be the inflation of σ to P and π be the compact induction of $\tilde{\sigma}$ from P to G . The following Theorem will be useful to decide whether the representation π is supercuspidal or not.

Theorem 3.8.1. *The representation π is supercuspidal if and only if $P(\Lambda)$ is maximal (i.e $e(\Lambda)$ is 1 or 2) and σ is cuspidal.*

Proof. See [23, Proposition 4.1 and Theorem 4.8]. □

Lemma 3.8.2. *Suppose P, P' are maximal and σ, σ' cuspidals. Let $\pi = \text{Ind}_P^G \sigma$ and $\pi' = \text{Ind}_{P'}^G \sigma'$. Then $\pi \cong \pi'$ if and only if σ and σ' conjugate.*

Proof. See [26, Proposition 6.1 and 6.2]. □

To count the number of irreducible supercuspidal representations of G of level zero, we only need to compute the cuspidal representation σ of P^* . The reason is we can inflate the representation σ to P then we induce it to G which is by previous Theorem equivalent to π . Also by the previous Theorem, the representation π is supercuspidal if and only if P is maximal.

Denote by $s(G, 0)$, the number of irreducible cuspidal representations π of G of level zero. Fix the measure μ on G such that $\mu(K) = 1$, where K is the maximal parahoric subgroup of period one.

3.8.1 Level Zero Supercuspidals of $U(1, 1)$

Here we have two cases and in both cases we have

$$P^* = U(1, 1)(k_F/k_{F_0}).$$

The number of irreducible cuspidal representations of $P^* = K/K_1$ can be found in Ennola's paper [10]. We will use his notation for the trace characters of the cuspidal representations, without recalling their definition since we are only interested in counting them and their dimensions. The notation $\chi_{f(q)}^{\underline{t}}$ denotes that the representation has dimension $f(q)$ and the \underline{t} is some parameters.

The irreducible cuspidals of P^* is given by the character $\chi_{q-1}^{(t,u)}$ where $t, u = 1, \dots, q+1$, $t < u$.

The number of irreducible cuspidals of $U(1, 1)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ is:

$$\frac{1}{2}q(q+1)$$

Theorem 3.8.3. *We have*

$$s(U(1, 1)(F/F_0), 0) = q(q+1).$$

Proof. The number $s(U(1, 1)(F/F_0), 0)$ is twice of the number of irreducible cuspidals of $U(1, 1)(\mathbb{F}_{q^2}/\mathbb{F}_q)$. □

Proposition 3.8.4. *The formal degree of $\pi = \text{Ind}_P^G \sigma$, where P is K or L , is $q-1$.*

Proof. We have $d(\pi) = \dim(\tilde{\sigma})$ which is $q-1$, by [10, §6]. □

3.8.2 Level Zero Supercuspidals of $U(2, 1)$

Here we have

$$P^* = \begin{cases} U(2, 1)(k_F/k_{F_0}) & \text{if } e(\Lambda) = 1 \\ U(1, 1)(k_F/k_{F_0}) \times U(1)(k_F/k_{F_0}) & \text{if } e(\Lambda) = 2 \end{cases}$$

The number of irreducible cuspidal representations of P^* for both periods $e(\Lambda) = 1$ and 2 can be found in Ennola's paper [10] in section 7.

When Λ has period one, then the representation σ is cuspidal if it is one of the following characters:

1. $\chi_{(q^2-q)}^{(t)}$ for $1 \leq t \leq q+1$. The number of these characters is $q+1$.
2. $\chi_{(q+1)(q^2-1)}^{(t)}$ for $1 \leq t \leq q^3$, $t \not\equiv 0 \pmod{q^2-q+1}$ and if $t_1 = q^2t, t_2 = q^4t \pmod{q^3+1}$, then

$$\chi^{(t)} = \chi^{(t_1)} = \chi^{(t_2)}.$$

The number of characters $\chi_{(q+1)(q^2-1)}^{(t)}$ is $q(q^2-1)/3$.

3. $\chi_{(q-1)(q^2-q+1)}^{(t,u,v)}$, where $t, u, v = 1, \dots, q+1$ and $t < u < v$. The number of these characters is $q(q^2-1)/6$.

Therefore, the number of irreducible cuspidal representations of P^* is

$$(q+1) + \left\lceil \frac{1}{3}q(q^2-1) \right\rceil + \left\lceil \frac{1}{6}q(q^2-1) \right\rceil = (q+1) + \frac{1}{2}q(q^2-1).$$

Proposition 3.8.5. *The formal degree of $\pi = \text{c-Ind}_K^G \tilde{\sigma}$ is*

$$d(\pi) = \begin{cases} q(q-1) & \text{if } \sigma = \chi_{(q^2-q)}^{(t)} \\ (q+1)(q^2-1) & \text{if } \sigma = \chi_{(q+1)(q^2-1)}^{(t)} \\ (q-1)(q^2-q+1) & \text{if } \sigma = \chi_{(q-1)(q^2-q+1)}^{(t,u,v)} \end{cases}$$

Proof. Since $\mu(K) = 1$, then $d(\pi) = \dim(\sigma)$ and the dimensions are given in [10, §7]. \square

When Λ has period two (P conjugate to L , we may assume $P = L$), then the number of irreducible supercuspidals of G of level zero is equal to the number of irreducible cuspidal representations of $U(1, 1)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ and $U(1)(\mathbb{F}_{q^2}/\mathbb{F}_q)$. The irreducible cuspidals of $U(1, 1)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ are given by the characters $\chi_{q-1}^{t,u}$ where $1 \leq t < u \leq q+1$ and the number of irreducible cuspidals of $U(1, 1)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ is $q(q+1)/2$. The number of irreducible cuspidal representations of $U(1)(\mathbb{F}_{q^2}/\mathbb{F}_q)$ is $q+1$. Therefore,

$$\frac{1}{2}q(q+1)^2.$$

Proposition 3.8.6. *The formal degree of $\pi = \text{c-Ind}_L^G \tilde{\sigma}$ is $(q-1)(q^2-q+1)$.*

Proof. We have $d(\pi) = \dim(\sigma)$ which is given in [10, §7]. \square

We, then, deduce the following Theorem,

Theorem 3.8.7. *We have*

$$s(U(2, 1)(F/F_0), 0) = (q + 1)(q^2 + 1).$$

3.9 Counting Totally Split Supercuspidals

In this section let $U(\lceil \frac{N}{2} \rceil, \lfloor \frac{N}{2} \rfloor)(F/F_0)$ and $N \geq 2$. Let $K = U(N)(\mathcal{O}_F)$ be the maximal compact parahoric subgroup of G of period one equipped with a filtration

$$K_i = \{x \in K : x \equiv I \pmod{\varpi_F^i}\}.$$

where I is Iwahori subgroup. The quotient K/K_1 is isomorphic to $U(N)(k_F/k_{F_0})$. Let τ_N be the size of the group K/K_1 , then (see [36, p.33])

$$\tau_N = q^{\frac{1}{2}N(N-1)} \prod_{i=1}^N (q^i - (-1)^i).$$

The other maximal parahoric subgroup is $L = \mathfrak{A}_L$ which has period two, where

$$\mathfrak{A}_L = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F & \cdots & \mathcal{O}_F & \mathcal{P}_F^{-1} \\ \mathcal{P}_F & \mathcal{O}_F & & & \mathcal{O}_F \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathcal{P}_F & \cdots & \cdots & \mathcal{P}_F & \mathcal{O}_F \end{pmatrix}$$

It is equipped with a filtration

$$L_{2m+1} = (1 + \varpi_F^m \mathfrak{P}_L^1) \cap G, \quad L_{2m} = (1 + \varpi_F^m \mathfrak{A}_L) \cap G.$$

where

$$\mathfrak{P}_L^1 = \begin{pmatrix} \mathcal{P}_F & \mathcal{O}_F & \cdots & \mathcal{O}_F & \mathcal{P}_F^{-1} \\ \vdots & \mathcal{P}_F & \ddots & & \mathcal{O}_F \\ \vdots & & \ddots & \ddots & \vdots \\ \mathcal{P}_F & & & \ddots & \mathcal{O}_F \\ \mathcal{P}_F^2 & \mathcal{P}_F & \cdots & \cdots & \mathcal{P}_F \end{pmatrix}$$

The quotient L/L_1 is isomorphic to $U(2r)(k_F/k_{F_0}) \times U(N - 2r)(k_F/k_{F_0})$ for some $1 \leq r < N/2$. Let $\tau_N(L)$ be the size of L/L_1 , then

$$\tau_N^L = \tau_{2r} \cdot \tau_{N-2r}.$$

We also have

$$[L_1 : L_{3m-i}] = \begin{cases} q^{mN^2} & \text{if } i = 0 \\ q^{mN^2 - r^2(N-r)^2} & \text{if } i = 1 \end{cases}$$

Definition 3.9.1. *A semi-minimal skew stratum $[\Lambda, n, n-1, b]$ is called totally split if the characteristic polynomial of $y = \varpi_F^{n/g} b^{e/g}$, where $e = e(\Lambda)$ and $g = (n, e)$, has the form*

$$\varphi_b(X) = (X - a_1) \cdots (X - a_N)$$

where $a_i = -\bar{a}_i$ for all $i \in \{1, \dots, N\}$ and $a_i \neq a_j$ for $i \neq j$.

Remark 3.9.2. *This is the special case of section 3.7 where $b_i \in F$, for each i , $i = 1, \dots, l$. In this case e divides n .*

Here we will compute the number these representations up to equivalence. Let π be an irreducible supercuspidal representations of G that contains a totally split semi-minimal skew stratum $[\Lambda, n, n-1, b]$, so π has level $k = n/e \in \mathbb{Z}$. We have two cases for the period of Λ that are period one and period two.

Lemma 3.9.3. *The number of totally split skew strata in A of level n/e up to G -conjugacy is*

$$\text{Str}(k)^{ts} = \frac{1}{N!} q(q-1) \cdots (q-N).$$

Proof. Let $[\Lambda, n, n-1, b]$ be a totally split skew stratum in A with characteristic polynomial

$$\varphi_b(X) = (X - a_1) \cdots (X - a_N)$$

where $a_i = -\bar{a}_i$ for all $i \in \{1, \dots, N\}$ and $a_i \neq a_j$ for $i \neq j$. By Lemma 3.1.1, it is sufficient to compute the number of possibilities of $\varphi_b(X)$. The element a_1 has the form $x_1 \sqrt{\varepsilon}$ for some $x_1 \in k_{F_0}$ so we have q possibilities including $a_1 = 0$. The element $a_2 \in$ has $q-1$ possibilities as $a_1 \neq a_2$. We do this procedure with all elements a_i , we deduce

$$\text{Str}(n/e)^{ts} = \frac{1}{N!} q(q-1) \cdots (q-(N-1))$$

as required. □

Now consider the character ψ_b of

$$P^n = P^n(\Lambda) = \begin{pmatrix} P^n(\Lambda^1) & & * \\ & \ddots & \\ * & & P^n(\Lambda^n) \end{pmatrix}$$

As we saw in previous section $\psi_b|P^n(\Lambda)$ extends to a character of $P^{\lfloor \frac{n}{2} \rfloor + 1}(\Lambda)$.

Proposition 3.9.4. *The number of characters $\psi_b|P^{\lfloor n/2 \rfloor + 1}(\Lambda)$ which extend $\psi_b|P^n(\Lambda)$ up to G -intertwining is*

$$|\text{Ext}(\psi_b, P^n)| = q^{N \lfloor \frac{k-1}{2} \rfloor}$$

Proof. By Theorem 3.4.10, we have

$$\mathcal{I}_G(\psi_b|P^n) = (F^1)^N P^1.$$

Let Γ be the set of characters $\psi|P^{\lfloor \frac{n}{2} \rfloor + 1}$ which extend $\psi_b|P^n(\Lambda)$. Suppose ψ_{b_1} and ψ_{b_2} are characters in Γ which intertwine by $g \in G$, then $g \in P$ and

$$g(b_1 + \mathfrak{P}^{1-n}(\Lambda))g^{-1} = b_2 + \mathfrak{P}^{1-n}(\Lambda)$$

thus, the characters ψ_{b_1} and ψ_{b_2} are conjugated by g in P . Now the group $(F^1)^N P^{\lfloor \frac{n+1}{2} \rfloor}$ acts on Γ by G -intertwining. The stabilizer of $\psi_b|P^{\lfloor \frac{n}{2} \rfloor + 1}$ is $(F^1)^N P^{\lfloor \frac{n+1}{2} \rfloor}$. By the orbit stabilizer theorem, the orbit of $\psi_b|P^{\lfloor \frac{n}{2} \rfloor + 1}$ has size

$$\left[(F^1)^N P^1 : (F^1)^N P^{\lfloor \frac{n+1}{2} \rfloor} \right].$$

Since $|\Gamma| = [P^{\lfloor \frac{n}{2} \rfloor + 1} : P^n]$, we then deduce

$$|\text{Ext}(\psi_b, P^n)| = \frac{[P^{\lfloor \frac{n}{2} \rfloor + 1} : P^n]}{\left[(F^1)^N P^1 : (F^1)^N P^{\lfloor \frac{n+1}{2} \rfloor} \right]}$$

Since $[P^1 : P^{\lfloor (n+1)/2 \rfloor}] = [P^{\lfloor n/2 \rfloor + 1} : P^n]$ and $P^{i+1} \cap (F^1)^N = ((1 + \mathcal{P}_F^{\lfloor i/e \rfloor + 1})^1)^N$, then we get

$$|\text{Ext}(\psi_b, P^n)| = \left[(1 + \mathcal{P}_F)^1 : \left(1 + \mathcal{P}_F^{\lfloor \frac{k-1}{2} \rfloor + 1} \right)^1 \right]^N.$$

□

Now as the stratum is completely split, then

$${}^1H = P^{\lfloor n/2 \rfloor + 1} \cdot ((1 + \mathcal{P}_F)^1)^N$$

where $((1 + \mathcal{P}_F)^1)^N$ represents the group

$$\begin{pmatrix} (1 + \mathcal{P}_F)^1 & & 0 \\ & \ddots & \\ 0 & & (1 + \mathcal{P}_F)^1 \end{pmatrix}$$

Proposition 3.9.5. *We have*

$$\Gamma(\Lambda, \psi_b) = q^{N \lfloor \frac{k}{2} \rfloor}.$$

Proof. By construction, ${}^1H/P^{\lfloor \frac{n}{2} \rfloor + 1}$ is a product of cyclic groups and

$$\begin{aligned} \Gamma(\Lambda, \psi_b) &= [{}^1H : P^{\lfloor n/2 \rfloor + 1}] \\ &= \left[((1 + \mathcal{P}_F)^1)^N : ((1 + \mathcal{P}_F)^1)^N \cap P^{\lfloor \frac{n}{2} \rfloor + 1} \right]. \end{aligned}$$

where

$$((1 + \mathcal{P}_F)^1)^N \cap P^{\lfloor \frac{n}{2} \rfloor + 1} = \left(\left(1 + \mathcal{P}_F^{\lfloor \frac{k}{2} \rfloor + 1} \right)^1 \right)^N.$$

□

The character θ of 1H extends uniquely to an irreducible representation η_θ of 1J where

$${}^1J = P^{\lfloor \frac{n+1}{2} \rfloor}(\Lambda) ((1 + \mathcal{P}_F)^1)^N.$$

The group J is

$$J = P^{\lfloor \frac{n+1}{2} \rfloor} (F^1)^N.$$

The quotient $J/{}^1J$, therefore, is

$$(F^1 / (1 + \mathcal{P}_F)^1)^N.$$

Proposition 3.9.6. *We have*

$$|\mathcal{R}(\Lambda, \theta)| = (q+1)^N.$$

Proof. The size of $\mathcal{R}(\Lambda, \theta)$ is the index $[F^1 : (1 + \mathcal{P}_F)^1]^N$. □

Theorem 3.9.7. *The number of irreducible supercuspidal representations of G of level n which contain a totally split stratum $[\Lambda, n, n-1, b]$ of level k for a fixed Λ is*

$$s^{ts}(G, \Lambda, k) = \frac{1}{N!} q \cdots (q - (N-1)) (q+1)^N q^{N(k-1)}.$$

Remark 3.9.8. *Let $s^{ts}(G, k)$ be the number of irreducible supercuspidal representations of G of level k containing a totally split skew stratum. If \mathfrak{n} is the number of lattice sequences Λ , up to conjugacy, such that $P(\Lambda)$ is a maximal, then*

$$s^{ts}(G, k) = \mathfrak{n} \cdot s^{ts}(G, \Lambda, k).$$

Suppose π is irreducible supercuspidal representation of G containing two totally split skew strata $[\Lambda, n, n-1, b]$ and $[\Lambda', n', n'-1, b']$ with same level k and same characteristic polynomial.

Question: *Is there $g \in G$ such that g conjugate the two strata?*

We know that the two strata intertwine but we do not know whether they conjugate or not. However, if they are conjugate, then the number of conjugacy classes of $P(\Lambda)$ is 2^{N-1} so in this case $s^{ts}(G, k) = 2^{N-1} s^{ts}(G, \Lambda, k)$. Therefore, $s(G, k)$ is bounded in which case, so

$$s^{ts}(G, \Lambda, k) \leq s^{ts}(G, k) \leq 2^{N-1} s^{ts}(G, \Lambda, k).$$

Proposition 3.9.9. *Let $\pi = \text{Ind}_J^G \eta$ be an irreducible supercuspidal representation of G which contains a completely split skew stratum. Then the formal degree of π is*

$$d(\pi) = \begin{cases} \tau_N (q+1)^{-N} q^{\frac{1}{2}(N^2-N)(n-1)} & \text{if } e = 1 \\ q^{\frac{1}{2}N(N-1)-2r(N-2r)} \cdot \frac{(q^N+1)\tau_N^L \cdot q^\delta}{(q+1)(q+1)^N q^{\lfloor (n-1)/4 \rfloor}} & \text{if } e = 2 \end{cases}$$

where $\delta = [L_1 : L_{n/2}]$.

Proof. First consider $e = 1$. When n is odd then the dimension of η_b is one and when n is even the the dimension of η_b is $[{}^1J : {}^1H]^{1/2}$ so

$$\begin{aligned} [{}^1J : {}^1H] &= \left[K_{[\frac{n+1}{2}]} \left((1 + \mathcal{P}_F)^1 \right)^N : K_{[\frac{n}{2}]+1} \left((1 + \mathcal{P}_F)^1 \right)^N \right] \\ &= \left[K_{[\frac{n+1}{2}]} : K_{[\frac{n}{2}]+1} \right] \left[\left(\left(1 + \mathcal{P}_F^{[\frac{n+1}{2}]} \right)^1 \right)^N : \left(\left(1 + \mathcal{P}_F^{[\frac{n}{2}]+1} \right)^1 \right)^N \right]^{-1} \\ &= q^{N^2-N}. \end{aligned}$$

By the definition of formal degree, we have $d(\pi) = \dim(\eta)\mu(J)^{-1}$, where $J = K_{[\frac{n+1}{2}]}(F^1)^N$, so

$$\begin{aligned}
\mu(J)^{-1} &= [K : J] = \left[K : K_{\lfloor \frac{n+1}{2} \rfloor} (F^1)^N \right] \\
&= \frac{[K : K_{\lfloor \frac{n+1}{2} \rfloor}]}{\left[K_{\lfloor \frac{n+1}{2} \rfloor} (F^1)^N : K_{\lfloor \frac{n+1}{2} \rfloor} \right]} \\
&= \frac{[K : K_1] \cdot [K_1 : K_{\lfloor \frac{n+1}{2} \rfloor}]}{\left[F^1 : (1 + \mathcal{P}_F)^1 \right]^N \left[(1 + \mathcal{P}_F)^1 : (1 + \mathcal{P}_F^{\lfloor \frac{n+1}{2} \rfloor})^1 \right]^N} \\
&= \frac{[K : K_1] q^{N^2(\lfloor \frac{n+1}{2} \rfloor - 1)}}{(q+1)^N q^{N(\lfloor \frac{n+1}{2} \rfloor - 1)}} \\
&= \tau_N (q+1)^{-N} q^{(N^2 - N)(\lfloor \frac{n+1}{2} \rfloor - 1)}.
\end{aligned}$$

Thus, the formal degree of π when $e = 1$ is

$$d(\pi) = \tau_N (q+1)^{-N} q^{\frac{1}{2}(N^2 - N)(n-1)}.$$

Now consider $e = 2$. Since the level of π must be integer then n must be even so the dimension of the representation η is d where

$$\begin{aligned}
d^2 &= [L_{\lfloor \frac{n+1}{2} \rfloor} ((1 + \mathcal{P}_F)^1)^N : L_{\lfloor \frac{n}{2} \rfloor + 1} ((1 + \mathcal{P}_F)^1)^N] \\
&= \frac{[L_{\lfloor \frac{n+1}{2} \rfloor} : L_{\lfloor \frac{n}{2} \rfloor + 1}]}{\left[(1 + \mathcal{P}_F^{\lfloor \frac{n+1}{2} \rfloor})^1 : (1 + \mathcal{P}_F^{\lfloor \frac{n}{2} \rfloor + 1})^1 \right]^N} \\
&= q^{N^2 - 2r(N-2r)} q^{-N}.
\end{aligned}$$

so $d = q^{N(N-1)/2 - 2r(N-2r)}$. Now we compute $\mu(J)^{-1}$.

$$\begin{aligned}
\mu(J)^{-1} &= \left[K : L_{\lfloor \frac{n+1}{2} \rfloor} (F^1)^N \right] \\
&= \frac{[K : L] \cdot [L : L_1] \cdot [L_1 : L_{\lfloor (n+1)/2 \rfloor}]}{\left[F^1 : (1 + \mathcal{P}_F^{\lfloor (n-1)/4 \rfloor})^1 \right]^N}.
\end{aligned}$$

Thus,

$$d(\pi) = q^{N(N-1)/2 - 2r(N-2r)} \cdot (q^N + 1)(q+1)^{-1} (q+1)^{-N} \tau_N^L \cdot q^{\delta - \lfloor (n-1)/4 \rfloor}$$

□

Chapter 4

SUPERCUSPIDAL REPRESENTATIONS OF UNRAMIFIED $U(1, 1)(F/F_0)$

4.1 Parahoric Subgroups and Filtrations

Now we start with defining the group G . Let F be an unramified quadratic extension of F_0 so we can write $F = F_0[\sqrt{\epsilon}]$ for some unit element ϵ .

$$\begin{aligned} V &= \text{an } F\text{-vector space of dimension 2} \\ A &= \text{End}_F(V) \cong \mathbb{M}_2(F). \end{aligned}$$

Let h be an ϵ -hermitian form on V given by $h(\underline{v}, \underline{w}) = \underline{v}J\overline{\underline{w}}$, for $\underline{v}, \underline{w} \in V$, where

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The unitary group G is

$$G = \{g \in GL_2(F) : g^T J \bar{g} = J\}.$$

In $GL_2(F)$, there is the special linear group $SL_2(F)$, which is the set of elements in $GL_2(F)$ of determinant one. Similarly we define the special unitary group, denoted by $SU(1, 1)(F/F_0)$, to be

$$SU(1, 1)(F/F_0) = \{g \in U(1, 1)(F/F_0) : \det(g) = 1\}.$$

Now the sequence

$$0 \rightarrow SU(1, 1)(F/F_0) \rightarrow U(1, 1)(F/F_0) \xrightarrow{\det} F^1 \rightarrow 0$$

is exact.

Remark 4.1.1. *The subgroup $SU(1,1)(F/F_0)$ is, in fact, isomorphic to $SL_2(F)$ since any element in $SU(1,1)(F/F_0)$ is a conjugate of an element in $SL_2(F_0)$ by*

$$\begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}.$$

The group G has three conjugacy classes of parahoric subgroup in which two of them are maximal. The maximal parahoric subgroups are

$$K = U(1,1)(\mathcal{O}_F) = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{O}_F & \mathcal{O}_F \end{pmatrix} \cap G.$$

and

$$L = \{g \in G : g \in \mathfrak{A}_L\}$$

where

$$\mathfrak{A}_L = \begin{pmatrix} \mathcal{O}_F & \mathcal{P}_F^{-1} \\ \mathcal{P}_F & \mathcal{O}_F \end{pmatrix}$$

The non-maximal parahoric subgroup is the Iwahori subgroup, denoted by I , which is

$$I = \{g \in G : g \in \mathfrak{A}_I\}$$

where

$$\mathfrak{A}_I = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{O}_F \end{pmatrix}.$$

For each of these parahoric subgroups, we have a standard filtration. Let $K_0 = K$ and

$$\mathfrak{A}_K = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{O}_F & \mathcal{O}_F \end{pmatrix}.$$

The standard filtration on K is

$$K_m = (1 + \mathfrak{P}_K^m) \cap G, \quad m \geq 1.$$

where $\mathfrak{P}_K^m = \varpi_F^m \mathfrak{A}_K$ and $\mathfrak{P}_K = \text{rad } \mathfrak{A}_K$. We have

$$K/K_1 \cong U(1,1)(k_F).$$

Proposition 4.1.2. *The group $U(1,1)(k_F/k_{F_0})$ has order $(q^2 - q)(q + 1)^2$ and has $(q + 1)^2$ conjugacy classes.*

Proof. See [10, §6]. □

For the parahoric subgroup L , the standard filtration is

$$L_m = (1 + \mathfrak{P}_L^m) \cap G \quad m \geq 1.$$

where $\mathfrak{P}_L^m = \varpi_F^m \mathfrak{A}_L$ and $\mathfrak{P}_L = \text{rad } \mathfrak{A}_L$. The quotient L/L_0 is isomorphic to $U(1,1)(k_F)$ which has size $q(q - 1)(q + 1)^2$, by Proposition 4.1.2.

Finally, the standard filtration for the Iwahori subgroup is:

$$I_m = (1 + \Pi^m \mathfrak{P}_I) \cap G \quad m \geq 1.$$

where

$$\Pi = \begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{P}_I = \text{rad } \mathfrak{A}_I = \Pi \mathfrak{A}_I.$$

The quotient I/I_1 is isomorphic to

$$T(I) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} : a \in k_F^\times \right\}.$$

and has size $q^2 - 1$.

Let P be a parahoric subgroup. Then by Lemma 3.3.4, for $1 \leq i < j \leq 2i$, the quotient P^i/P^j is abelian and isomorphic to $\mathfrak{P}_P^i/\mathfrak{P}_P^j$. Lemma 3.3.6, implies that there is an isomorphism,

$$\mathfrak{P}_P^{1-j}/\mathfrak{P}_P^{1-i} \rightarrow (P^i/P^j)^\wedge$$

4.2 Level Zero Supercuspidals

Let π be an irreducible supercuspidal representations of G of level zero. By Theorem 3.4.3 there exists a self-dual lattice sequence Λ such that the restriction of π to $P = P(\Lambda)$ contains an irreducible representation σ which is trivial on P^1 . Note that the parahoric subgroup P must be maximal and σ cuspidal by Theorem 3.8.1. Here we have two maximal parahoric subgroups of period one and in this case

$$P^* = P/P^1 = U(1,1)(k_F/k_{F_0}).$$

We have seen in Section 3.8.1 that all irreducible cuspidal representations of P^* , given by the characters $\chi_{q-1}^{(t,u)}$, where $t, u = 1, \dots, q+1$, $t < u$ and the number of these representations is $q(q+1)/2$.

The representation π is equivalent to $\text{c-Ind}_P^G \tilde{\sigma}$, where $\tilde{\sigma}$ is the inflation of σ to P . The formal degree of π is $q-1$, by Proposition 3.8.4, for a fixed Haar measure μ on G such that $\mu(K) = 1$.

Denote by $s(G, 0)$, the number of irreducible supercuspidal representations of G of level zero. By Theorem 3.8.3, we have

$$s(G, 0) = q(q+1).$$

4.3 Positive Level Supercuspidals

Any irreducible supercuspidal representation π of G of positive level k must contain a skew fundamental stratum $[\Lambda, n, n-1, b]$, by Theorem 3.4.3, where $k = n/e$ and $e = e(\Lambda)$. Let $s(G, k)$ be the number of irreducible supercuspidal representations of G of level k and $S(G, k)$ be the number of irreducible supercuspidal representations of G of level strictly less than k . We classify π by looking at the characteristic polynomial $\varphi_b(X)$, which depends only on π . Here we have three different cases as follows:

- (a) The first case b is split (in fact totally split) but not G -split. Here the characteristic polynomial $\varphi_b(X) = (X-\alpha)(X-\beta)$, where $\alpha, \beta \in k_F$, $\alpha = -\bar{\alpha}$, $\beta = -\bar{\beta}$ and $\alpha \neq \beta$.

(b) The second case b is simple i.e. it one of the following:

(a) $\varphi_b(X)$ is irreducible and $k \in \mathbb{Z}$;

(b) This is the case when $y = \bar{y}$ so $\varphi_b(X) = (X - \alpha)^2$, $\alpha \in k_F$, $\alpha = \bar{\alpha}$ and $k \notin \mathbb{Z}$.

(c) The last case is b essentially scalar i.e. $\varphi_b(X) = (X - \alpha)^2$, with $\alpha \in k_F$, $\alpha = -\bar{\alpha}$ and $k \in \mathbb{Z}$.

Now we will study each case in more details.

4.3.1 Totally Split

Let $[\Lambda, n, n-1, b]$ be totally split i.e. the characteristic polynomial $\varphi_b(X)$ has the form $(X - \alpha)(X - \beta)$ where $\alpha = -\bar{\alpha}$, $\beta = -\bar{\beta}$ and $\alpha \neq \beta$. In this case, the period e divides n . Let $\text{Str}(k)^{ts}$ be the number of totally split strata of level $k = n/e$, then $\text{Str}(k)^{ts} = q(q-1)/2$, by Lemma 3.9.3.

The representation π is constructed as follows: the character $\psi_b|P^n$ extends to a character $\psi_b|P^{[n/2]+1}$, by Lemma 3.3.6, and the number of these extensions up to G -intertwining is $q^{2([\frac{n+1}{2}]-1)}$, by Proposition 3.9.4. Let $\Gamma(\Lambda, \psi_b)$ be the set of characters θ_b of ${}^1H = P^{[n/2]+1} \left((1 + \mathcal{P}_F)^1 \right)^2$ which extends $\psi_b|P^{[n/2]+1}$. By Lemma 3.7.3, any two characters in $\Gamma(\Lambda, \psi_b)$ intertwine if and only if they are equal. The size of the set $\Gamma(\Lambda, \psi_b)$ is $q^{2[n/2]}$, by Proposition 3.9.5. By Heisenberg Lemma, there is a unique irreducible representation η_b of ${}^1J = P^{[(n+1)/2]} \left((1 + \mathcal{P}_F)^1 \right)^2$ such that the restriction of η_b to 1H contains the character θ_b . Now let $\mathcal{R}(\Lambda, \theta_b)$ be the set of representations η of $J = P^{[(n+1)/2]} (F^1)^2$ which extend η_b . By Lemma 3.7.5, $\mathcal{R}(\Lambda, \theta_b)$ is non-empty and by Proposition 3.9.6, the size of $\mathcal{R}(\Lambda, \theta_b)$ is $(q+1)^2$. The representation $\text{c-Ind}_J^G \eta$ is irreducible and supercuspidal and equivalent to π , by Theorem 3.7.7. The formal degree of π , by Proposition 3.9.9, is

$$d(\pi) = (q-1)q^n.$$

Finally, let $s_a(G, \Lambda, k)$ be the number of irreducible totally split supercuspidal representations of G of level k which contain $[\Lambda, n, n-1, b]$. By Theorem 3.9.7, we have

$$s_a(G, \Lambda, k) = \begin{cases} (q-1)f_a(q)q^{2k} & \text{if } k \in \mathbb{Z}; \\ 0 & \text{if } k \notin \mathbb{Z}. \end{cases}$$

where $f_a(q) = q^{-1}(q+1)^2/2$.

Let $s_a(G, k)$ be the number of irreducible totally split supercuspidal representations of G of level k . By Remark 3.9.8, the number $s_a(G, k)$ is bounded, so

$$s_a(G, \Lambda, k) \leq s_a(G, k) \leq 2s_a(G, \Lambda, k).$$

For purpose of computing $s(G, k)$ and $S(G, k)$, we will assume that $s_a(G, k) = s_a(G, \Lambda, k)$.

4.3.2 Simple

Let $[\Lambda, n, n-1, b]$ be skew simple stratum. In this section, we will classify all irreducible supercuspidal representations of G which contain $[\Lambda, n, n-1, b]$, we will call these representations *simple*. Denote by $\text{Str}(k)^s$, $k = n/e$, the number of simple strata $[\Lambda, n, n-1, b]$ up to G -intertwining. We will compute $\text{Str}(k)^s$ and then we will count the number of these representations of fixed level k which we will denote the number by $s_b(G, k)$.

Integral Level

Suppose $\varphi_b(X)$ is irreducible of degree 2, so $[\Lambda, n, n-1, b]$ has integral level and Λ is a multiple of a period one lattice chain.

Lemma 4.3.1. *There is no skew simple stratum $[\Lambda, n, n-1, b]$ such that $\varphi_b(X)$ is irreducible of degree 2.*

Proof. Let $y = \varpi_{\mathbb{F}}^n b$ and

$$\varphi_b(X) = X^2 + a_1 X + a_0 \in k_F[X]$$

be the characteristic polynomial of y . The element b is skew so $b = -\bar{b}$ and $y = -\bar{y}$.

If $\varphi_b(X)$ is irreducible, then $[k_F[y] : k_F] = 2$ and $k_F[y]$ has a unique involution extending $\bar{}$ with $y = -\bar{y}$. But then the fixed field of $\bar{}$ has index 2 in $k_F[y]$ so (by uniqueness of finite fields of given order) the fixed field is k_F which is absurd since $\bar{}$ is not trivial on k_F . \square

The previous Lemma shows that there is no simple skew stratum of integral level with $\varphi_b(X)$ irreducible so we have the following corollaries:

Corollary 4.3.2.

$$\text{Str}(k)^s = 0 \quad \forall k > 0.$$

Corollary 4.3.3. *There is no irreducible simple supercuspidal representation of G of integral level i.e*

$$s_b(G, n) = 0$$

for all $n \in \mathbb{Z}$.

Non-Integral Level

Suppose the stratum $[\Lambda, n, n-1, b]$ is simple of level in $\frac{1}{2}\mathbb{Z}$ but not in \mathbb{Z} i.e the period of Λ is two and $(n, e(\Lambda)) = 1$ so n must be odd, say, $n = 2m + 1$. Let $P = P(\Lambda)$, then P is conjugate to the Iwahori group I so we may assume $P = I$. We start by computing the number of maximal simple skew strata of half-integer level.

Lemma 4.3.4. *The number of maximal simple skew strata in A of level $k = (2m + 1)/2$, up to G -intertwining, is*

$$\text{Str}(k)^s = q - 1.$$

Proof. Here the skew simple element b may be assumed to lie in

$$\left(\begin{array}{cc} 0 & \mathcal{P}_F^{-m-1} \setminus \mathcal{P}_F^{-m} \\ \mathcal{P}_F^{-m} \setminus \mathcal{P}_F^{1-m} & 0 \end{array} \right) \cap A_-.$$

The total number of simple skew strata of level $(2m+1)/2$ is $(q-1)^2$. We can write

$$b = \begin{pmatrix} 0 & \varpi_F^{-m-1}x \\ \varpi_F^{-m}y & 0 \end{pmatrix},$$

where $x + \bar{x} = 0$, $y + \bar{y} = 0$ and $x, y \in \mathcal{O}_F^\times$. By Theorem 3.5.1, intertwining in G is the same as L -conjugacy. Now conjugating b by

$$g = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} \in P/P^1,$$

where $a \in \mathcal{O}_F^\times$, we have

$$gbg^{-1} = \begin{pmatrix} 0 & \varpi_F^{-m-1}(a\bar{a})x \\ \varpi_F^{-m}(a\bar{a})^{-1}y & 0 \end{pmatrix}.$$

Fix some $\delta \in \mathcal{O}_F^\times$ such that $\delta + \bar{\delta} = 0$. Then $\overline{\delta x^{-1}} = \delta x^{-1}$ so, since F/F_0 is unramified, there is an $a \in \mathcal{O}_F^\times$ such that

$$a\bar{a} = \delta x^{-1}.$$

Therefore, a conjugate of b has the form

$$\begin{pmatrix} 0 & \varpi_F^{-m-1}\delta \\ \varpi_F^{-m}z\delta^{-1} & 0 \end{pmatrix}$$

where $z \in \mathcal{O}_F \setminus \mathcal{P}_F$ and $z = \bar{z}$. For different z , the characteristic polynomial $\varphi_b(X)$ are distinct so the strata cannot intertwine. We have $(q-1)$ choices for z so

$$\text{Str}(\Lambda, 2m+1, b)^s = q-1.$$

□

The character $\psi_b|P^{2m+1}$ extends to $\psi_b|P^{m+1}$, by Lemma 3.3.6. We denote the set of characters $\psi_b|P^{m+1}$ which extend $\psi_b|P^{2m+1}$ up to G -intertwining (P -conjugacy by Theorem 3.5.2) by $\text{Ext}(\psi_b, P^{2m+1})$. By Lemma 3.5.3

$$|\text{Ext}(\psi_b, P^{2m+1})| = [P^1 E^1 : P^{m+1} E^1].$$

Since $e(\Lambda|_{\mathcal{O}_E}) = e(\Lambda_{\mathcal{O}_F})/e(E/F) = 1$, then

$$|\text{Ext}(\psi_b, P^{2m+1})| = [(1 + \mathcal{P}_E)^1 : (1 + \mathcal{P}_E^{m+1})^1] = q_{E_0}^m = q^m.$$

The character $\psi_b|P^{m+1}$ extends to a character θ_b of ${}^1H = P^{m+1}(1 + \mathcal{P}_E)^1$ and the set of such extensions is denoted by $\Gamma(\Lambda, \psi_b)$. Any two extensions in $\Gamma(\Lambda, \psi_b)$ intertwine if and only if they are equal, by Lemma 3.7.3. Thus, the number of

characters θ_b of ${}^1H = P^{m+1}(1+\mathcal{P}_E)^1$ which extend $\psi_b|_{P^{m+1}}$ up to G -intertwining is

$$\begin{aligned} |\Gamma(\Lambda, \psi_b)| &= [{}^1H : P^{m+1}] \\ &= [(1+\mathcal{P}_E)^1 \cap P^1 : (1+\mathcal{P}_E)^1 \cap P^{m+1}] \\ &= q_{E_0}^m = q^m. \end{aligned}$$

Finally, for any given character θ_b of ${}^1H = {}^1J$, the character θ_b extends to a representation η of $J = P^{m+1}F^1$ (in fact a character), by Lemma 3.7.5, and we denote the set of these extensions by $\mathcal{R}(\Lambda, \theta_b)$. Lemma 3.7.6, implies that any two extensions in $\mathcal{R}(\Lambda, \theta_b)$ intertwine if and only they are equal. Thus,

$$\begin{aligned} |\mathcal{R}(\Lambda, \theta_b)| &= [J : {}^1J] \\ &= [E^1 : (1+\mathcal{P}_E)^1] = q_{E_0} + 1 = q + 1. \end{aligned}$$

Theorem 4.3.5. *The number of irreducible simple supercuspidal representations π of G of level $k = (2m+1)/2$ is*

$$s_b(G, k) = (q-1)f_b(q)q^{2k}.$$

up to equivalence classes, where $f_b(q) = q^{-1}(q+1)$.

Proof. By equation (3.4), the number of irreducible supercuspidal representations π of G up to equivalence is

$$s_b(G, k) = \text{Str}(k)^s \cdot |\text{Ext}(\psi_b, P^{2m+1})| \cdot |\Gamma(\Lambda, \psi_b)| \cdot |\mathcal{R}(\Lambda, \theta_b)|.$$

which leads to $s_b(G, k) = (q-1)f_b(q)q^{2k-1}$. □

Proposition 4.3.6. *Let Λ be of period two and $\pi = \text{Ind}_J^G \eta$ be an irreducible simple supercuspidal representation of G of level $(2m+1)/2$. Then the formal degree of π is*

$$d(\pi) = (q^2 - 1)q^m$$

Proof. Since $n = 2m+1$ is odd, then ${}^1J = {}^1H$ so η is, in fact, character so $\dim(\eta) = 1$. Now using $[K : P] = q+1$ and $e(\Lambda|_{\mathcal{P}_E}) = 1$, we get

$$\begin{aligned} d(\pi) &= \mu(P^{m+1}E^1)^{-1} = [K : P^{m+1}E^1] \\ &= \frac{[K : P^{m+1}]}{[P^{m+1}E^1 : P^{m+1}]} \\ &= \frac{[K : P] \cdot [P : P^1] \cdot [P^1 : P^{m+1}]}{[E^1 : (1+\mathcal{P}_E)^1] [(1+\mathcal{P}_E)^1 : (1+\mathcal{P}_E^{m+1})^1]} \\ &= \frac{(q+1)(q^2-1)q^{2m}}{(q+1)q^m} = (q^2-1)q^m. \end{aligned}$$

□

4.3.3 Essentially Scalar

Let $[\Lambda, n, n-1, b]$ be essentially scalar skew stratum i.e. $k = n/e \in \mathbb{Z}$ and the characteristic polynomial $\varphi_b(X)$ has the form $(X - \alpha)^2$, where $\alpha \in k_F^\times$ and $\alpha = -\bar{\alpha}$. Let π be an irreducible essentially scalar supercuspidal representation of G which contains $[\Lambda, n, n-1, b]$. If $\text{Str}(k)^{es}$ is the number of essentially scalar skew strata of level k up to G -intertwining, then $\text{Str}(k)^{es} = q - 1$, by Lemma 3.6.2.

By Lemma 3.6.3, there exists a character χ of G such that the representation $\pi \otimes \chi$ contains a skew stratum of level strictly less than k .

In Proposition 3.6.4, we have shown that the map

$$\begin{aligned} \text{Rep}(G, < k) \times \Psi &\longrightarrow \text{Rep}(G, k)^{es} \\ (\pi, \chi) &\longmapsto \pi \otimes \chi \end{aligned}$$

is surjective where Ψ is the set of characters of F^1 of level k , and the size of each fiber of this map is $(q+1)q^{k-2}$, by Corollary 3.6.5. Let $s_c(G, k)$ is the number of essentially scalar supercuspidal representations of G of level k . Then by Corollary 3.6.6, we have:

$$s_c(G, k) = \begin{cases} (q-1)S(G, k) & \text{if } k \in \mathbb{Z}; \\ 0 & \text{if } k \notin \mathbb{Z}. \end{cases}$$

4.4 Computing $s(G, k)$ and $S(G, k)$

We have classified all irreducible supercuspidal representations of G for any positive level. We have seen that they are one of the following: totally split, simple and essentially scalar. In each case we counted the number of irreducible supercuspidal representations of G of level k . In this section, we will give general formulas for $s(G, k)$ and $S(G, k)$.

The number $s(G, k)$ is obtained by

$$s(G, k) = s_a(G, k) + s_b(G, k) + s_c(G, k).$$

Thus,

$$s(G, k) = \begin{cases} s_a(G, k) + s_c(G, k) & \text{if } k \in \mathbb{Z}; \\ s_b(G, k) & \text{if } k \notin \mathbb{Z}. \end{cases}$$

For $k \in \mathbb{Z}$, we have

$$S(G, k+1) = s(G, k + \frac{1}{2}) + s(G, k) + S(G, k) \quad (4.1)$$

$$= (s_b(G, k + \frac{1}{2}) + s_a(G, k)) + qS(G, k) \quad (4.2)$$

Now $s_a(G, k) = (q-1)f_a(q)q^{2k}$ and $s_b(G, k) = (q-1)f_b(q)q^{2k}$ for some functions $f_a(q), f_b(q)$ independent of the choice of k . Now we replace $s_a(G, k)$ and

$s_b(G, k + \frac{1}{2})$ in (4.2) by $(q-1)f_a(q)q^{2k}$ and $(q-1)qf_b(q)q^{2k}$, we get

$$\begin{aligned} S(G, k+1) &= (q-1)(f_a(q) + qf_b(q))q^{2k} + qS(G, k) \\ &= (q-1)(f_a(q) + qf_b(q)) [q^{2k} + q^{2k-1}] + q^2S(G, k-1) \\ &\quad \vdots \\ &= (q-1)(f_a(q) + qf_b(q)) [q^{2k} + q^{2k-1} + \dots + q^{k+1}] + q^k s(G, 0). \end{aligned}$$

Therefore, we deduce

$$S(G, k+1) = (f_a(q) + qf_b(q)) q^{k+1} (q^k - 1) + q^k s(G, 0) \quad (4.3)$$

In section (4.2), we have $s(G, 0) = q(q+1)$ so

$$\boxed{S(G, k+1) = q^{k+1} [f(q)(q^k - 1) + q(q+1)]} \quad (4.4)$$

where $f(q) = f_a(q) + qf_b(q)$.

When $k \notin \mathbb{Z}$, then $k + 1/2 \in \mathbb{Z}$ so

$$\boxed{S(G, k) = S(G, k + \frac{1}{2}) - s_b(G, k)}.$$

Now using the formula 4.4, we get

$$\boxed{S(G, k) = q^{k+\frac{1}{2}} \left[f(q) \left(q^{k-\frac{1}{2}} - 1 \right) + f_b(q)(q-1)q^{k-\frac{1}{2}} + q(q+1) \right]}$$

Finally we summarize the number of irreducible supercuspidal representations of G by the following table:

	$k \in \mathbb{Z}$	$k \notin \mathbb{Z}$
$s(G, k), k = 0$	$q(q+1)$	-
$s_a(G, k)$	$\frac{1}{2}q(q^2 - 1)(q+1)q^{2(k-1)}$	0
$s_b(G, k)$	0	$(q^2 - 1)q^{2k-1}$
$s_c(G, k)$	$(q-1)q^k(f(q)(q^{k-1} - 1) + q(q+1))$	0
$S(G, k)$	$q^{k+1} [f(q)(q^k - 1) + q(q+1)]$	$f(q)(q^{2k} - q^{k+\frac{1}{2}}) + f_b(q)(q-1)q^{2k} + q(q+1)q^{k+\frac{1}{2}}$

where

$$\begin{aligned} f_a(q) &= \frac{1}{2}q^{-1}(q+1)^2 \\ f_b(q) &= q^{-1}(q+1) \\ f(q) &= f_a(q) + qf_b(q) = \frac{1}{2}(q+1)(q^{-1} + 3). \end{aligned}$$

Chapter 5

SUPERCUSPIDAL REPRESENTATIONS OF $U(2, 1)(F/F_0)$

In this chapter, Let $F = F_0[\sqrt{\epsilon}]$, $\epsilon \in \mathcal{O}_F^\times$, be an unramified quadratic extension of F_0 . Let

$$\begin{aligned} V &= \text{an } F\text{-vector space of dimension 3,} \\ A &= \text{End}_F(V) \cong \mathbb{M}_3(F), \\ \tilde{G} &= \text{Aut}_F(V) \cong GL_3(F). \end{aligned}$$

Let $\{\underline{e}_1, \underline{e}_0, \underline{e}_{-1}\}$ be the standard basis for V . Let h be the hermitian form on V given by $h(\underline{v}, \underline{w}) = \underline{v}^T J \overline{\underline{w}}$, for $\underline{v}, \underline{w} \in V$, where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We define the unitary group $G = U(2, 1)(F/F_0)$ by

$$G = \{g \in GL_3(F) : g^T J \bar{g} = J\}.$$

5.1 Parahoric Subgroups and Filtrations

Let Λ be a self-dual \mathcal{O}_F -lattice sequence. We can form the parahoric subgroup $P(\Lambda)$. The group G has three conjugacy classes of parahoric subgroups equipped with standard filtrations. Two of these classes are maximal (the level zero irreducible smooth representations of G are constructed from these subgroups as we studied in section (3.8)). The non-maximal is the Iwahori subgroup. The subgroup $P(\Lambda)$ is maximal if and only if the period of Λ is either one or two.

◆**Period One**

This case happens when Λ has period one and $P(\Lambda)$ is maximal and conjugate to

$$K = U(2, 1)(\mathcal{O}_F) = GL_3(\mathcal{O}_F) \cap G.$$

Let $K_0 = K$. The standard filtration of K is, then, given by

$$K_m = (1 + \mathfrak{P}_K^m) \cap G, \quad m \geq 1.$$

where $\mathfrak{P}_K^m = \varpi_F^m \mathfrak{A}_K$ and

$$\mathfrak{A}_K = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{O}_F & \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{O}_F & \mathcal{O}_F & \mathcal{O}_F \end{pmatrix}.$$

The quotient of K_0 modulo K_1 is

$$K_0/K_1 \cong U(2, 1)(k_F/k_{F_0}).$$

Lemma 5.1.1. *The size of the group $U(2, 1)(k_F/k_{F_0})$ is $q^3(q+1)^2(q^2-1)(q^2-q+1)$.*

Proof. See [16, Lemma 3.16]. □

The size of the index $[K_i : K_{i+1}]$ is q^9 , for $i \geq 1$. We will fix once and for all, a Haar measure μ on G for which $\mu(K) = 1$.

◆**Period Two**

Here Λ has period two and this is the second of the maximal parahoric subgroups. Let

$$\mathfrak{P}_L^0 = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F & \mathcal{P}_F^{-1} \\ \mathcal{P}_F & \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{P}_F & \mathcal{O}_F \end{pmatrix}, \quad \mathfrak{P}_L^1 = \begin{pmatrix} \mathcal{P}_F & \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{P}_F & \mathcal{O}_F \\ \mathcal{P}_F^2 & \mathcal{P}_F & \mathcal{P}_F \end{pmatrix}.$$

The group $P(\Lambda)$ is conjugate to

$$L_0 = L = \mathfrak{P}_L^0 \cap G$$

The standard filtration of L is

$$L_{2m+1} = (1 + \varpi_F^m \mathfrak{P}_L^1) \cap G, \quad L_{2m} = (1 + \varpi_F^m \mathfrak{P}_L^0) \cap G.$$

We have

$$L_0/L_1 \cong U(1, 1)(k_F/k_{F_0}) \times U(1)(k_F/k_{F_0}).$$

Thus,

$$[L : L_1] = q(q-1)(q+1)^3.$$

We also have, for $i \geq 1$,

$$[L_{2i} : L_{2i+1}] = q^5 \quad \text{and} \quad [L_{2i-1} : L_{2i}] = q^4.$$

◆ **Period Three**

This is the (non-maximal) Iwahori parahoric subgroup. The parahoric subgroup $P(\Lambda)$ is conjugate to the Iwahori subgroup

$$I = \mathfrak{A}_I \cap G$$

where

$$\mathfrak{A}_I = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{P}_F & \mathcal{O}_F \end{pmatrix}.$$

The standard filtration on I is

$$I_m = (1 + \mathfrak{P}_I^m) \cap G.$$

where \mathfrak{P}_I is the radical of \mathfrak{A}_I and $\mathfrak{P}_I^m = \Pi^m \mathfrak{A}_I$ where

$$\Pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varpi_F & 0 & 0 \end{pmatrix}.$$

The quotient of I_0 modulo I_1 is isomorphic to

$$GL_1(k_F) \times U(1, 1)(k_F/k_{F_0}).$$

◆ **Period Four**

Let $\{\underline{e}_1, \underline{e}_0, \underline{e}_{-1}\}$ be the standard basis of V and put $V^1 = \langle \underline{e}_1, \underline{e}_{-1} \rangle$ and $V^2 = \langle \underline{e}_0 \rangle$. Let $\mathcal{L}^1 = \{L_i^1 : i \in \mathbb{Z}\}$ be the period 2 self-dual lattice chain on V^1 given by $L_0^1 = \mathcal{O}_E \underline{e}_1 \oplus \mathcal{O}_E \underline{e}_{-1}$, $L_1^1 = \mathcal{P}_E \underline{e}_1 \oplus \mathcal{O}_E \underline{e}_{-1}$ and $\mathcal{L}^2 = \{L_i^2 : i \in \mathbb{Z}\}$ be the period 1 self-dual lattice chain on V^2 given by $L_0^2 = \mathcal{O}_E \underline{e}_0$. In order to take the “direct sum” of these chains to obtain a self-dual lattice sequence in V , we must first scale them to period 4; then we get Λ given by

$$\begin{aligned} \Lambda(-1) &= L_0^1 \oplus L_0^2 \\ \Lambda(0) &= L_1^1 \oplus L_0^2 \\ \Lambda(1) &= L_1^1 \oplus L_1^2 \\ \Lambda(2) &= L_1^1 \oplus L_1^2 \end{aligned}$$

This is the sequence for which $P^m(\Lambda)$ gives the non-standard filtration of the Iwahori subgroup: for any integers i, j with $0 \leq j < 4$, and $4i + j > 0$ the non-standard filtration of I is

$$I_{4i+j}^* = (1 + \varpi_F^i \mathfrak{P}_{I^*}^j) \cap G$$

where

$$\begin{aligned} \mathfrak{P}_{I^*}^0 &= \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{P}_F & \mathcal{O}_F \end{pmatrix}, \mathfrak{P}_{I^*}^1 = \begin{pmatrix} \mathcal{P}_F & \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{P}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{P}_F & \mathcal{P}_F \end{pmatrix}, \\ \mathfrak{P}_{I^*}^2 &= \begin{pmatrix} \mathcal{P}_F & \mathcal{P}_F & \mathcal{O}_F \\ \mathcal{P}_F & \mathcal{P}_F & \mathcal{P}_F \\ \mathcal{P}_F & \mathcal{P}_F & \mathcal{P}_F \end{pmatrix}, \mathfrak{P}_{I^*}^3 = \begin{pmatrix} \mathcal{P}_F & \mathcal{P}_F & \mathcal{P}_F \\ \mathcal{P}_F & \mathcal{P}_F & \mathcal{P}_F \\ \mathcal{P}_F^2 & \mathcal{P}_F & \mathcal{P}_F \end{pmatrix}. \end{aligned}$$

Fix a Haar measure μ on G such that $\mu(K) = 1$. Then

Proposition 5.1.2. *Let $P = P(\Lambda)$ be a parahoric subgroup and $e = e(\Lambda)$. For $m \geq 0$, we have*

$$\mu(P^{m+1}) \begin{cases} q^3(q+1)(q^2-1)(q^3+1)q^{9m} & \text{if } P \text{ conjugate to } K \\ (q+1)(q^2-1)(q^3+1)q^{4m+[m/2]} & \text{if } P \text{ conjugate to } L \\ (q+1)(q^2-1)(q^3+1)q^{3m} & \text{if } P \text{ conjugate to } I \text{ and } e = 3 \\ q(q+1)(q^2-1)(q^3+1)q^{2m} & \text{if } P \text{ conjugate to } I \text{ and } e = 4 \end{cases}$$

5.2 Level Zero Supercuspidals

Let π be an irreducible smooth representation of G of level zero. By Theorem 3.4.3, there exists a self-dual \mathcal{O}_F -lattice sequence such that the restriction of π to $P = P(\Lambda)$ contains an irreducible representation σ trivial on P^1 . The group $P^* = P/P^1$ is either $U(2, 1)(k_F/k_{F_0})$, if $e = 1$, or $U(1, 1)(k_F/k_{F_0}) \times U(1)(k_F/k_{F_0})$, if $e = 2$. We are not considering the period 3 or 4, since the supercuspidals do not arise from non-maximal P , by Theorem 3.8.1. If $\tilde{\sigma}$ is the inflation of σ to P , then $\text{c-Ind}_P^G \tilde{\sigma}$ is irreducible and supercuspidal and equivalent to π , by Theorem 3.8.1. The formal degree of π is given in Propositions 3.8.5 and 3.8.6. Finally, by Theorem 3.8.7, the number of irreducible supercuspidal representations of G of level zero is

$$s(G, 0) = (q+1)(q^2+1).$$

5.3 Positive Level Supercuspidals

Let π be an irreducible smooth representation of G of positive level k . Then by Theorem 3.4.3, π contains a fundamental skew stratum $[\Lambda, n, n-1, b]$, where $k = n/e$, $e = e(\Lambda)$. We classify the representation π by looking at the characteristic polynomial $\varphi_b(X)$ which depends only on the representation π not the stratum $[\Lambda, n, n-1, b]$.

Definition 5.3.1. *A skew fundamental stratum $[\Lambda, n, n-1, b]$ is called G -split if the characteristic polynomial $\varphi_b(X)$ has the form*

$$\varphi_b(X) = (X - \alpha)(X - (-1)^{e/g}\bar{\alpha})\varphi_{b_2}(X)$$

where $\alpha \neq (-1)^{e/g}\bar{\alpha}$.

Theorem 5.3.2. *Let π be an irreducible smooth representation of G which contains a G -split stratum. Then π is not supercuspidal.*

Proof. See [34, Theorem 4.9]. □

For this reason, we will not consider G -split strata as we only interested in supercuspidals

Here we have six possible cases for $\varphi_b(X)$, in which two of them are non-integral level. The possibilities for $\varphi_b(X)$ are:

1. $\varphi_b(X) = (X - \alpha)^3$ with $\alpha \in k_F^\times$, $\alpha = -\bar{\alpha}$ and $k \notin \mathbb{Z}$. This is the maximal simple case where the period of Λ is 3 and $k = n/3$ with $(3, n) = 1$.

2. $\varphi_b(X) = X(X-\alpha)^2$ with $\alpha = \bar{\alpha}$ and $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. This is the second possibility for the non-integral level case.
3. $\varphi_b(X) = (X-\alpha)^3$ with $\alpha \in k_F^\times$, $\alpha = -\bar{\alpha}$ and $k \in \mathbb{Z}$. This is the essentially scalar case.
4. $\varphi_b(X) = (X-\alpha)(X-\beta)(X-\gamma)$ with $\alpha = -\bar{\alpha}$, $\beta = -\bar{\beta}$, $\gamma = -\bar{\gamma}$ and α, β, γ are distinct. This is the totally split case.
5. $\varphi_b(X)$ is irreducible. This is the maximal simple case with $k \in \mathbb{Z}$.
6. Finally, there is the split, but not totally split, case i.e $\varphi_b(X) = \varphi_{b_1}(X)\varphi_{b_2}(X)$ where $\varphi_{b_2}(X)$ is polynomial in k_F of degree 1 and $\varphi_{b_1}(X)$ is polynomial of degree 2 coprime to $\varphi_{b_2}(X)$ but not irreducible, by Lemma 4.3.1. So $\varphi_{b_1}(X) = (X-\alpha_1)^2$ and $\varphi_{b_2}(X) = (X-\alpha_2)$, where $\alpha_1 = -\bar{\alpha}_1$, $\alpha_2 = -\bar{\alpha}_2$ and $\alpha_1 \neq \alpha_2$.

We will study each case and then will count the number of supercuspidals of G . Then we will give formulas for the number $s(G, k)$ of irreducible supercuspidal representations of G of level k and the number $S(G, k)$ of irreducible supercuspidal representations of G of level strictly less than k .

5.3.1 Maximal Simple and $k \in \frac{1}{3}\mathbb{Z}$

Let $[\Lambda, n, n-1, b]$ be maximal simple of period 3 i.e $E = F[b]$ is a ramified extension of F of degree 3. The characteristic polynomial $\varphi_b(X)$ has the form $\varphi_b(X) = (X-\alpha)^3$ with $\alpha \in k_F^\times$, $\alpha = -\bar{\alpha}$.

We can assume that the lattice sequence Λ is a chain as in section (3.5) and the maximal parahoric subgroup $P = P(\Lambda)$ is conjugate to the Iwahori subgroup I , so we may assume they are equal. Now $e(\Lambda|_{\mathcal{O}_E}) = e(\Lambda|_{\mathcal{O}_F})/e(E/F) = 1$ so

$$P^m(\Lambda) \cap E = 1 + \mathcal{P}_E^m, \quad \forall m \geq 1.$$

Let $\text{Str}(k)^s$ be the number of maximal simple strata in A of level k up to G -intertwining. Then

Lemma 5.3.3. *We have*

$$\text{Str}(k)^s = q - 1.$$

Proof. Since $(3, n) = 1$, then $n = 3m + 1$ or $n = 3m + 2$ for $m \geq 0$. We will prove the lemma for $n = 3m + 2$ and the proof is exactly the same for $n = 3m + 1$. We have

$$b \in \mathfrak{P}_-^{-3m-2}(\Lambda)/\mathfrak{P}_-^{-3m-1}(\Lambda).$$

We can choose b to lie in

$$\left(\begin{array}{ccc} 0 & \mathfrak{p}_F^{-m-1} \setminus \mathfrak{p}_F^{-m} & 0 \\ 0 & 0 & \mathfrak{p}_F^{-m-1} \setminus \mathfrak{p}_F^{-m} \\ \mathfrak{p}_F^{-m} \setminus \mathfrak{p}_F^{1-m} & 0 & 0 \end{array} \right) \cap A_.$$

The number of the simple skew strata in A of level $(3m+2)/3$ up to equivalence is

$$(q_F - 1)(q - 1).$$

Put

$$b = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & -\bar{x} \\ y & 0 & 0 \end{pmatrix}$$

where $x \in \mathfrak{p}_F^{-m-1} \setminus \mathfrak{p}_F^{-m}$ and $y \in \mathfrak{p}_F^{-m} \setminus \mathfrak{p}_F^{1-m}$ with $y + \bar{y} = 0$. By Lemma 3.5.1, the group $P(\Lambda)$ acts on the set of simple skew strata of level $(3m+2)/3$ by conjugation. Now if $g \in P^1(\Lambda)$, then g^{-1} is also in $P^1(\Lambda)$ so we can write $g = 1 + X$ and $g^{-1} = 1 + Y$ where $X, Y \in \mathfrak{P}$. We have

$$gbg^{-1} \equiv b \pmod{\mathfrak{P}^{1-n}}$$

so we need only consider conjugating by g in

$$H = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & (\bar{a})^{-1} \end{pmatrix} : a \in k_F^\times, d \in k_F^1 \right\},$$

where we identify elements of k_F^\times with their Teichmüller lifts in \mathcal{O}_F^\times , since $P(\Lambda) = HP^1(\Lambda)$.

Let $a \in k_F^\times$ be the unique element such that $x \equiv \varpi_F^{-m-1}a \pmod{\mathcal{P}_F^{-m}}$. Now we conjugate the element b by

$$g = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\bar{a})^{-1} \end{pmatrix}$$

We get

$$g^{-1}bg = \begin{pmatrix} 0 & \varpi_F^{-m-1} & 0 \\ 0 & 0 & \varpi_F^{-m-1} \\ \varpi_F^{-m}y' & 0 & 0 \end{pmatrix},$$

for some skew $y' \in \mathcal{O}_F^\times$. The number of choices for y' is $q-1$. The size of the stabilizer of the stratum $[\Lambda, 3m+2, 3m+1, b]$ is

$$|\text{Stab}_H(b)| = q+1.$$

Every orbit has size $|H|/|\text{Stab}_H(b)|$ elements, which is equal to $q^2-1 = q_F-1$. By Lemma 3.5.1, the G -intertwining of ψ_b is the same as $P(\Lambda)$ -conjugacy. Now the number of $P(\Lambda)$ -conjugacy classes of simple skew strata of period three which is:

$$\text{Str}(k)^s = \frac{(q_F-1)(q-1)}{(q_F-1)} = q-1.$$

□

The representation π is constructed as follows: The character $\psi_b|P^n$ extends to a character of $P^{[n/2]+1}$, which we also call ψ_b , by Lemma 3.3.6. Recall that $\text{Ext}(\psi_b, P^n)$ denotes the set of characters $P^{[n/2]+1}$ which extend $\psi_b|P^n$

up to G -intertwining. Then the size of $\text{Ext}(\psi_b, P^n)$ is $q^{\lfloor (n+1)/2 \rfloor - 1}$, by Lemma 3.5.3. Let ${}^1H = P^{\lfloor n/2 \rfloor + 1}(1 + \mathcal{P}_E)^1$; then $\Gamma(\psi_b, {}^1H)$ is the set of characters θ_b of 1H such that the restriction of θ_b to $P^{\lfloor n/2 \rfloor + 1}$ is ψ_b . By Lemma 3.7.3, any $\theta, \theta' \in \Gamma(\psi_b, {}^1H)$ intertwine if and only if they are equal. Moreover, the size of $\Gamma(\psi_b, {}^1H)$ is $q^{\lfloor n/2 \rfloor}$ which is independent of the extension ψ to $P^{\lfloor n/2 \rfloor + 1}$. Let ${}^1J = P^{\lfloor (n+1)/2 \rfloor}(1 + \mathcal{P}_E)^1$, then by Heisenberg Lemma, there is a unique irreducible representation η_b of 1J such that the restriction of η_b to 1H contains the character θ_b . Now Let $\mathcal{R}(\Lambda, \theta_b)$ denote the set of representations of $J = P^{\lfloor (n+1)/2 \rfloor}E^1$ which extend η_b . If $\eta_i \in \mathcal{R}(\Lambda, \theta_i)$ and $\theta_i \in \Gamma(\Lambda, \psi_b)$, $i = 1, 2$, and η_1, η_2 intertwine, then $\eta_1 \cong \eta_2$ and $\theta_1 = \theta_2$, by Lemma 3.7.6. The set $\mathcal{R}(\Lambda, \theta_b)$ is non-empty, by Lemma 3.7.5, and it has size $q + 1$, independent of θ_b . Finally, the representation $\text{c-Ind}_J^G \eta$ is irreducible and supercuspidal and equivalent to π , by Theorem 3.5.4.

If $s_3(G, k)$ denotes the number of irreducible simple supercuspidal representations of G of level k up to equivalence, then

Theorem 5.3.4. *We have*

$$s_3(G, k) = (q^2 - 1)q^{3k-1}$$

for $k = n/3 \notin \mathbb{Z}$.

Lemma 5.3.5. *Let π be an irreducible simple supercuspidal representation of G of level $k = n/3 \notin \mathbb{Z}$. Then the formal degree of π is:*

$$d(\pi) = (q^2 - 1)(q + 1)^3 q^{3k-1}$$

Proof. We have $\pi \cong \text{c-Ind}_J^G \eta$ where η is an irreducible representation of J which extends η_b . The dimension of η is equal to the dimension of η_b which is $[{}^1J : {}^1H]^{1/2}$, which is 1 if n is odd and q when n is even. Now we compute $\mu(J)^{-1}$. Since $P = I$, then

$$\mu(J)^{-1} = \mu \left(I_{\left[\frac{n+1}{2} \right]} \right)^{-1} \cdot \left[E^1 : \left(1 + \mathcal{P}_E^{\left[\frac{n+1}{2} \right]} \right)^1 \right]^{-1}.$$

The value of $\mu(I_{\lfloor (n+1)/2 \rfloor})^{-1}$ is $(q + 1)(q^2 - 1)(q^3 + 1)q^{3(\lfloor (n+1)/2 \rfloor - 1)}$ by Lemma 5.1.2 and the size of the index $[E^1 : (1 + \mathcal{P}_E^{\lfloor (n+1)/2 \rfloor})^1]$ is $(q + 1)q^{\lfloor (n+1)/2 \rfloor - 1}$. Finally,

$$d(\pi) = \dim(\eta)\mu(J)^{-1} = (q^2 - 1)(q^3 + 1)q^{3k-1}.$$

□

5.3.2 Level in $\frac{1}{2}\mathbb{Z}$

Let $[\Lambda, n, n - 1, b]$ be split, but not G -split, of level $k = n/e \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. The characteristic polynomial $\varphi_b(X)$ has the form $X(X - \alpha)^2$ with $\alpha = \bar{\alpha}$. There exist finite dimensional vector spaces V^i over F , $i = 1, 2$, in which $h|_{V^i \times V^i}$ is non-degenerate and $V = V^1 \oplus V^2$ is orthogonal, and

1. $\Lambda = \Lambda^1 \oplus \Lambda^2$, where $\Lambda^i(k) = \Lambda(k) \cap V^i$, $k \in \mathbb{Z}$ and Λ^i is a self dual lattice sequence in V^i ;
2. $b = b_1 \oplus b_2$ for $b_i \in A^i = \text{End}_F(V^i)$ and $\nu_{\Lambda^i}(b_i) = -n$.

Then we can write

$$[\Lambda, n, n-1, b] = [\Lambda^1, n, n-1, b_1] \oplus [\Lambda^2, n, n-1, b_2],$$

with E_1/F is ramified quadratic and $E_2 = F$ where $E_i = F[b_i]$ for $i = 1, 2$. We may assume $b_2 = 0$ because $n/e \notin \mathbb{Z}$ but, for $b_2 \neq 0$, $v_F(b_2) \in \mathbb{Z}$ so $b_2 \equiv 0 \pmod{\mathfrak{P}^{1-n}(\Lambda)}$.

Remark 5.3.6. *Since the stratum $[\Lambda^i, n, n-1, b_i]$, $i = 1, 2$, is a maximal simple skew stratum, the lattice sequence Λ^i , $i = 1, 2$, is a multiple of some chain \mathcal{L}^i so*

$$e_i = e(\Lambda^i) = m_i e(\mathcal{L}^i)$$

for some positive integer m_i . In fact, $e(\mathcal{L}^1) = 2$ as E_1/F is ramified and $e(\mathcal{L}^2) = 1$ as V^2 is 1-dimensional. So $m_2 = e(\Lambda_{\mathcal{O}_F}) = 2m_1$ and $n = tm_1$, for some $t \in \mathbb{Z} \setminus 2\mathbb{Z}$.

Remark 5.3.7. *We have two cases for the group $U(V^1)$:*

- (a) *Let $\{\underline{e}_1, \underline{e}_0, \underline{e}_{-1}\}$ be the standard basis of V with $h(\underline{e}_0, \underline{e}_0) = 1$. Put $V^1 = \langle \underline{e}_1, \underline{e}_{-1} \rangle$ and $V^2 = \langle \underline{e}_0 \rangle$, then $U(V^1) = U(1, 1)(F/F_0)$ and $U(V^2) = U(1)(F/F_0)$. Here the period of Λ is 4 and $P^m(\Lambda)$ gives the non-standard filtration of the Iwahori subgroup, as we explained in Section 5.1.*
- (b) *Let $\{e_1, e_0, e_{-1}\}$ be a basis for V such that $h(e_0, e_0) = 1$ and $h(e_1, e_1) = h(e_{-1}, e_{-1}) = \varpi_F$. Put $V^1 = \langle e_1, e_{-1} \rangle$ and $V^2 = \langle e_0 \rangle$, then $U(V^1) = U(2)(F/F_0)$ and $U(V^2) = U(1)(F/F_0)$. Now let $\Lambda^1 = \{L_i^1 : i \in \mathbb{Z}\}$ be the multiple of period 2 self-dual lattice on V^1 given by $L_0^1 = \mathcal{O}_E e_1 \oplus \mathcal{O}_E e_{-1}$, $L_1^1 = \mathcal{P}_E e_1 \oplus \mathcal{O}_E e_{-1}$ and $\Lambda^2 = \{L_i^2 : i \in \mathbb{Z}\}$ be a self-dual lattice sequence on V^2 of a period multiple of 1 given by $L_0^2 = \mathcal{O}_E e_0$. Then the direct sum of these self-dual lattices is a self-dual lattice sequence Λ given by*

$$\begin{aligned} \Lambda(-1) &= L_0^1 \oplus L_0^2 \\ \Lambda(0) &= L_1^1 \oplus L_0^2. \end{aligned}$$

The period of Λ is 2 and $P^m(\Lambda)$ gives filtration on L .

Let $\text{Str}(k)^s$ be the number of split, but not G -split, skew strata in A of level k upto G -intertwining, then since $b_2 = 0$, we deduce:

Lemma 5.3.8. *We have $\text{Str}(t)^m = q - 1$.*

Proof. When $U(V^1) = U(1, 1)(F/F_0)$, then the proof given by Lemma 4.3.4. Now let $U(V^1) = U(2)(F/F_0)$, where V^1 has the basis $\{e_1, e_{-1}\}$ with $h(e_1, e_1) = h(e_{-1}, e_{-1}) = \varpi_F$. Now

$$\text{if } b = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, \text{ then } \bar{b} = \begin{pmatrix} \bar{x} & \bar{z}\varpi_F^{-1} \\ \bar{y}\varpi_F & \bar{w} \end{pmatrix}.$$

So $b = -\bar{b}$ if and only if $x = -\bar{x}$, $w = -\bar{w}$ and $z = \varpi_F \bar{y}$. The element $b \in \mathfrak{P}^{-2m-1}/\mathfrak{P}^{-2m}$ so it has the form

$$b = \begin{pmatrix} 0 & y \\ \varpi_F \bar{y} & 0 \end{pmatrix}$$

where $y \in k_F^\times$. We conjugate b by element g in P/P^1 , so let

$$g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Then

$$g^{-1}bg = \begin{pmatrix} 0 & a\bar{b}y \\ \varpi_F \bar{a}b\bar{y} & 0 \end{pmatrix}$$

Finally, the number of orbits is $\text{Str}(t)^m = |k_F^\times|/|k_F^1| = q - 1$. \square

Define the following subgroups

$$\begin{aligned} {}^1H = {}^1H(\Lambda) &= (P^1 \cap B).P\left[\frac{n}{2}\right]^+ \\ {}^1J = {}^1J(\Lambda) &= (P^1 \cap B).P\left[\frac{n+1}{2}\right] \\ J = J(\Lambda) &= (P \cap B).P\left[\frac{n+1}{2}\right]. \end{aligned}$$

where B is the centralizer of b in G and $P = P(\Lambda)$.

Suppose π is an irreducible supercuspidal representation of G which contains the stratum $[\Lambda, n, n-1, b]$, then $\ell(\pi) = k$ and the character $\psi_{b_1 \oplus 0}$ of P^n is contained in π . By Lemma 3.3.6, the character $\psi_{b_1 \oplus 0}|P^n$ extends to a character $\psi_{b_1 \oplus b_2}|P^{[n/2]^+}$. We denote the set of characters $\psi_{b_1 \oplus b_2}|P^{[n/2]^+}$ which extend $\psi_{b_1 \oplus 0}|P^n$ up to G -intertwining by $\text{Ext}(\psi_{b_1 \oplus 0}, P^n)$.

Lemma 5.3.9. *We have*

$$|\text{Ext}(\psi_{b_1 \oplus 0}, P^n)| = q^{\left[\frac{t-1}{2}\right]} q^{\left[\frac{t-1}{4}\right]}.$$

Proof. By [32, Theorem 4.9], we have

$$\mathcal{I}_G\left(\psi_{b_1 \oplus b_2}|P\left[\frac{n}{2}\right]^+\right) \subseteq J\mathcal{I}_L\left(\psi_{b_1 \oplus b_2}|P\left[\frac{n}{2}\right]^+\right)J$$

where $L = U(1, 1)(F/F_0) \times U(1)(F/F_0)$. Since J normalizes the character $\psi_{b_1 \oplus 0}$, then

$$|\text{Ext}(\psi_{b_1 \oplus 0}, P^n)| = |\text{Ext}(\psi_{b_1}, P^n(\Lambda^1))| \cdot |\text{Ext}(\psi_0, P^n(\Lambda^2))|.$$

By Lemma 3.5.3, we get

$$\begin{aligned} |\text{Ext}(\psi_{b_1 \oplus 0}, P^n)| &= \left[(1 + \mathcal{P}_{E_1})^1 : \left(1 + \mathcal{P}_{E_1}^{\left[\frac{t-1}{2}\right]^+}\right)^1 \right] \cdot \left[(1 + \mathcal{P}_F)^1 : \left(1 + \mathcal{P}_F^{\left[\frac{t-1}{4}\right]^+}\right)^1 \right] \\ &= q^{\left[\frac{t-1}{2}\right]} q^{\left[\frac{t-1}{4}\right]}. \end{aligned}$$

\square

Consider the quotient ${}^1H/P^{[n/2]+1}$. It is isomorphic to

$$(P^1(\Lambda) \cap B) / \left(P^{\lfloor \frac{n}{2} \rfloor + 1}(\Lambda) \cap B \right) \quad (5.1)$$

Since $P^{i+1}(\Lambda) \cap B = P_{E_1}^{\lfloor i/e_1 \rfloor + 1}(\Lambda^1) \times P_F^{\lfloor i/e_2 \rfloor + 1}(\Lambda^2)$, for $i \geq 0$, then (5.1) is isomorphic to

$$\left(P_{E_1}^1(\Lambda^1) / P_{E_1}^{\lfloor \frac{t}{2} \rfloor + 1}(\Lambda^1) \right) \times \left(P_F^1(\Lambda^1) / P_F^{\lfloor \frac{t}{4} \rfloor + 1}(\Lambda^2) \right).$$

This is a product of cyclic groups and since the group 1H normalizes the character $\psi_{b_1 \oplus b_2} |P^{[n/2]+1}$ then the character $\psi_{b_1 \oplus b_2}$ extends to a character θ_b of 1H and the set of such extensions is denoted by $\Gamma(\Lambda, \psi_{b_1 \oplus b_2})$. Any two characters $\theta_1, \theta_2 \in \Gamma(\Lambda, \psi_{b_1 \oplus b_2})$ intertwine in G if and only if they are equal, by Lemma 3.7.3. We, then, have the following proposition,

Proposition 5.3.10. *The size of the set $\Gamma(\Lambda, \psi_{b_1 \oplus b_2})$ is $q^{\lfloor t/2 \rfloor} q^{\lfloor t/4 \rfloor}$, independent of the choice of $\psi_{b_1 \oplus b_2}$.*

Proof. The size of $\Gamma(\Lambda, \psi_{b_1 \oplus b_2})$ is

$$|\Gamma(\Lambda, \psi_{b_1 \oplus b_2})| = \left[P_{E_1}^1(\Lambda^1) : P_{E_1}^{\lfloor \frac{t}{2} \rfloor + 1}(\Lambda^1) \right] \cdot \left[P_F^1(\Lambda^1) : P_F^{\lfloor \frac{t}{4} \rfloor + 1}(\Lambda^2) \right].$$

□

By Heisenberg Lemma 3.7.4, there is a unique irreducible representation η_b of 1J such that the restriction of η_b to 1H contains the character θ_b .

Lemma 5.3.11. *The representation η_b extends to a representation κ of J .*

Proof. This follows from Lemma 3.7.5 since

$$J/{}^1J \cong k_{E_1}^1 \times k_F^1.$$

□

Recall that $\mathcal{R}(\Lambda, \theta_b)$ is the set of representations κ of J which extend η_b .

Lemma 5.3.12. *Let $\kappa_1, \kappa_2 \in \mathcal{R}(\Lambda, \theta_b)$. If they intertwine, then $\kappa_1 \cong \kappa_2$.*

Proof. If $g \in G$ intertwines κ_1 with κ_2 then it must intertwine the character θ_b (so g also intertwines $\psi_{b_1 \oplus b_2}$). Therefore $g \in J$, but then g normalizes both κ_1 and κ_2 . Hence $\kappa_1 \cong \kappa_2$. □

Now we can deduce

$$|\mathcal{R}(\Lambda, \theta_b)| = |k_{E_1}^1| \cdot |k_F^1| = (q+1)^2.$$

which is independent of the choice of θ_b . Now the representation $\text{c-Ind}_J^G \kappa$ is irreducible and supercuspidal, by Theorem 1.10.6, since the intertwining of κ in G is J .

Let $s_2(G, k)$ be the number of irreducible supercuspidal representations of G of level k up to equivalence, and Let $s_2(G, e, k)$ be the number of irreducible supercuspidal representations of G of level k and Λ has period e up to equivalence, then we have the following Theorem

Theorem 5.3.13. *We have*

$$s_2(G, e, k) = \begin{cases} (q-1)(q+1)^2 q^{3k-\frac{3}{2}} & \text{if } 2k \equiv 1 \pmod{4}; \\ (q-1)(q+1)^2 q^{3k-\frac{5}{2}} & \text{if } 2k \equiv 3 \pmod{4}. \end{cases}$$

and $s_2(G, k) = 2s_2(G, e, k)$.

Now we compute the formal degree for $\text{c-Ind}_J^G \kappa$.

Lemma 5.3.14. *The formal degree of π is*

$$d(\pi) = \begin{cases} (q-1)(q^3+1)q^{2k+\lceil \frac{k-1}{2} \rceil - \gamma_t - 1} & \text{if } e = 2; \\ (q-1)(q^3+1)q^{4k-\gamma_t+1} & \text{if } e = 4. \end{cases}$$

where $\gamma_t = \lceil (t-1)/2 \rceil + \lceil (t-1)/4 \rceil$

Proof. When $e(\Lambda) = 2$, then n must be odd so the dimension of η is one. When $e(\Lambda) = 4$, then n must have the form $2t$, where t is positive odd integer, so the dimension of η is $[{}^1J : {}^1H]^{1/2}$ which is equal to q . Now we compute $\mu(J)^{-1}$.

$$\mu(J)^{-1} = \mu \left(P \left[\frac{n+1}{2} \right] \right)^{-1} \left[P \cap B : P \left[\frac{n+1}{2} \right] \cap B \right]^{-1}.$$

By Proposition 5.1.2, $\mu(P^{[(n+1)/2]})^{-1}$ is $(q+1)(q^2-1)(q^3+1)q^{n-1+\lceil (n-1)/4 \rceil}$ when $e = 2$ (n is odd and $n = 2k$) and $(q+1)(q^2-1)(q^3+1)q^n$ (n is even) when $e = 4$, by Proposition 5.1.2. The size of the index $[P \cap B : P^{[(n+1)/2]} \cap B]$ is $(q+1)^2 q^{\lceil (t-2)/2 \rceil} q^{\lceil (t-1)/4 \rceil}$. \square

5.3.3 Essentially Scalar

Let π be essentially scalar i.e $k \in \mathbb{Z}$ so $e|n$ and the characteristic polynomial $\varphi_b(X)$ has the form $(X-\alpha)^3$, where $\alpha \in k_F^\times$ and $\alpha = -\bar{\alpha}$. If $\text{Str}(k)^{es}$ denotes the number of essentially scalar skew strata in A of level k up to G -intertwining, then by Lemma 3.6.2, $\text{Str}(k)^{es} = q - 1$.

By Lemma 3.6.3, there exists a character χ of G such that the twist of π by χ contains a skew stratum of level strictly less than k . By Proposition 3.6.4, the map

$$\begin{aligned} \text{Rep}(G, < k) \times \Psi &\longrightarrow \text{Rep}(G, k)^{es} \\ (\pi, \chi) &\longmapsto \pi \otimes \chi \end{aligned}$$

is surjective and the size of each fiber of this map is $(q+1)q^{k-2}$ by Corollary 3.6.5. Let $s_1^{es}(G, k)$ denotes the number of irreducible essentially scalar supercuspidal representations of G of level k , then by Corollary 3.6.6, we have

$$s^{es}(G, k) = (q-1)S(G, k).$$

5.3.4 Maximal Simple and $k \in \mathbb{Z}$

Here in this section we will consider $[\Lambda, n, n-1, b]$ to be maximal simple of integral level k i.e $e = e(\Lambda) = 1$ (so $k = n$) and $E = F[b]$ is an unramified extension of F of degree 3. Moreover, the characteristic polynomial $\varphi_b(X)$ is irreducible. As in section (3.5) we can assume that Λ is a chain. The parahoric subgroup $P = P(\Lambda)$ is conjugate to K , so we may assume that $P = K$. Thus, $e(\Lambda|_{\mathcal{O}_E}) = 1$ so

$$P^m \cap E^1 = (1 + \mathcal{P}_E^m)^1, \quad \forall m \geq 1.$$

Theorem 5.3.15. *The number of maximal simple skew unramified strata of level $k \in \mathbb{Z}$ up to G -intertwining is*

$$\text{Str}(k)^s = \frac{1}{3}q(q^2 - 1).$$

Proof. The number of maximal simple skew strata of level k is equal to the number of characteristic polynomials of y , by Lemma 3.4.8. Every such polynomial is the minimal polynomial of some skew element in k_F , where $[k_F : k_{F_0}] = 3$. The number of skew elements is q^3 , all but q of which are not in k_{F_0} . If y is one of these elements, then the minimal characteristic polynomial of y is of the required form, but has 3 roots (each of which has the same minimal characteristic polynomial) so the number of irreducible polynomials is $(q^3 - q)/3$. \square

We constructed π in the general case in section (3.7) and, in specific case, the irreducible simple supercuspidal representation of G of level $n/3 \notin \mathbb{Z}$ in section (5.3.1) so we will not repeat the construction but we will give some facts we need.

The number of characters of $P^{[n/2]+1}$ which extends $\psi_b|P^n$ up to G -intertwining is $|\text{Ext}(\psi_b, P^n)| = q_{E_{1,0}}^{[(n+1)/2]-1} = q^{3([(n+1)/2]-1)}$, by Lemma 3.5.3. The number of characters θ_b of ${}^1H = P^{[n/2]+1}(1 + \mathcal{P}_E)^1$ which extend $\psi_b|P^{[n/2]+1}$ is $|\Gamma(\Lambda, \psi_b)| = q_{E_{1,0}}^n = q^{3[n/2]}$, by Section (3.7). The number of representations η of $J = P^{[(n+1)/2]}E^1$ which extend $\eta_b|{}^1J$, where ${}^1J = P^{[(n+1)/2]}(1 + \mathcal{P}_E)^1$, is $|\mathcal{R}(\Lambda, \theta_b)| = (q^3 + 1)$. Finally, if $s_1(G, k)$ denotes the number of irreducible simple supercuspidal representations of G of level $k \in \mathbb{Z}$, up to equivalence then we deduce

Theorem 5.3.16. *We have*

$$s_1(G, k) = \frac{1}{3}q(q^2 - 1)(q^3 + 1)q^{3(k-1)}.$$

Now we compute the formal degree for π

Lemma 5.3.17. *Let $\pi \cong \text{c-Ind}_J^G \eta$, for $\eta \in \mathcal{R}(\Lambda, \psi_b)$, be an irreducible simple supercuspidal representation of G of level $k \in \mathbb{Z}$. Then the formal degree of π is:*

$$d(\pi) = (q^2 - 1)(q + 1)^3 q^{3k-1}$$

Proof. The dimension of η is $[{}^1J : {}^1H]^{1/2}$ which is 1 if n is odd and q^3 if n is even. Now we compute $\mu(J)^{-1}$, Note that $P = K$ so

$$\mu(J)^{-1} = [K : J] = \mu \left(P \left[\frac{n+1}{2} \right] \right)^{-1} \cdot \left[F^1 : \left(1 + \mathcal{P}_E \left[\frac{n+1}{2} \right] \right)^1 \right]^{-1}$$

By Proposition 5.1.2, $\mu(P^{[(n+1)/2]})^{-1} = q^3(q+1)(q^2-1)(q^3+1)q^{9([(n+1)/2]-1)}$ and since $k = n$, then

$$d(\pi) = (q^2 - 1)(q + 1)q^{3k}.$$

□

5.3.5 Totally Split

Let $[\Lambda, n, n-1, b]$ be totally split i.e $k = n/e \in \mathbb{Z}$ (so $e|n$) and the characteristic polynomial $\varphi_b(X)$ has the form

$$(X - \alpha_1)(X - \alpha_2)(X - \alpha_3)$$

where $\alpha \in k_F^\times$ with $\alpha_i = -\overline{\alpha_i}$, $i = 1, 2, 3$, and $\alpha_j \neq \alpha_i$ for $j \neq i$. If $\text{Str}(k)^{ts}$ is the number of totally split skew strata in A of level k up to G -intertwining, then

$$\text{Str}(k)^{ts} = \frac{1}{6}q(q-1)(q-2),$$

by Lemma 3.9.3.

Now π is constructed as follows: first we extend $\psi_b|P^n$ to $\psi_b|P^{[n/2]+1}$ and the number of these extensions up to G -intertwining is $|\text{Ext}(\psi_b, P^n)| = q^{3([k/2]-1)}$, by Lemma 3.9.4. The character $\psi_b|P^{[n/2]+1}$ extends to a character θ_b of ${}^1H = P^{[n/2]+1}((1 + \mathcal{P}_F)^1)^3$ in which any two extensions do not intertwine by Lemma 3.7.3. The number of these extensions is $|\Gamma(\Lambda, \psi_b)| = q^{3[k/2]}$, by Proposition 3.9.5. By Heisenberg Lemma, there is a unique irreducible representation η_b of ${}^1J = P^{[(n+1)/2]}((1 + \mathcal{P}_F)^1)^3$ such that the restriction of η_b to 1H contains θ_b . Now the representation η_b extends to a representation η of $J = P^{[(n+1)/2]}(F^1)^3$, by Lemma 3.7.5, and any two extensions intertwine if and only if they are equivalent, by Lemma 3.7.6. The set of the representations η of J which extend η_b is denoted by $\mathcal{R}(\Lambda, \theta_b)$ and the size of this set is $(q+1)^3$, by Proposition 3.9.6. Now the representation $\text{c-Ind}_J^G \eta$ is irreducible and supercuspidal and equivalent to π , Theorem 3.7.7.

Denote by $s^{ts}(G, \Lambda, k)$, the number of irreducible totally split supercuspidal representations of G of level $k = n/e$ up to equivalence, then

Theorem 5.3.18. *We have*

$$s^{ts}(G, \Lambda, k) = \frac{1}{6}q(q-1)(q-2)(q+1)^3q^{3(k-1)}.$$

Recall that $s^{ts}(G, k)$ is the number of irreducible supercuspidal representations of G of level k which contain a skew totally split stratum. By Remark 3.9.8, we have

$$s^{ts}(G, \Lambda, k) \leq s^{ts}(G, k) \leq 4s^{ts}(G, \Lambda, k).$$

We will assume $s^{ts}(G, k) = s^{ts}(G, \Lambda, k)$.

We conclude this section by computing the formal degree of π .

Lemma 5.3.19. *Let $\pi \cong \text{c-Ind}_J^G \eta$, for $\eta \in \mathcal{R}(\Lambda, \psi_b)$, be an irreducible totally split supercuspidal representation of G of level $k = n/e$. Then the formal degree of π is:*

$$d(\pi) = \begin{cases} (q-1)(q^2-q+1)q^{3k} & \text{if } e = 1 \\ (q-1)(q^2-q+1)q^{3k-2} & \text{if } e = 2 \text{ and } n \equiv 0 \pmod{4} \\ (q-1)(q^2-q+1)q^{3k-1} & \text{if } e = 2 \text{ and } n \equiv 2 \pmod{4} \end{cases}$$

Proof. When $e = 1$, the dimension of η is 1 if n is odd and q^3 if n is even. By Proposition 5.1.2, we have $\mu(P^{[(n+1)/2]})^{-1} = q^3(q+1)(q^2-1)(q^3+1)q^{9([(n+1)/2]-1)}$. When $e = 2$, then n must be even and the dimension of η is q^2 and by Proposition 5.1.2, we have $\mu(P^{[(n+1)/2]})^{-1} = (q+1)(q^2-1)(q^3+1)q^{4([(n+1)/2]-1)+[(n-1)/4]}$. Finally,

$$\mu(J)^{-1} = \mu\left(P^{\lfloor \frac{n+1}{2} \rfloor}\right)^{-1} \cdot \left[F^1 : \left(1 + \mathcal{P}_F^{\lfloor \frac{n-1}{2e} \rfloor + 1} \right)^1 \right]^{-3}$$

and the size of the index $[F^1 : (1 + \mathcal{P}_F^{[(n-1)/2e]+1})^1]$ is $(q+1)q^{[(n-1)/2e]+1}$. \square

5.3.6 Split

Let $[\Lambda, n, n-1, b]$ be split, but not totally split. The characteristic polynomial $\varphi_b(X)$ has the form

$$(X - \alpha)^2(X - \beta), \quad \alpha, \beta \in k_F$$

with $\alpha = -\bar{\alpha}$, $\beta = -\bar{\beta}$ and $\alpha \neq \beta$.

There exist finite-dimensional vector spaces V^i over F , $i = 1, 2$, in which $h|_{V^i \times V^i}$ is non-degenerate and $V = V^1 \oplus V^2$ is orthogonal, and

1. $\Lambda = \Lambda^1 \oplus \Lambda^2$, where $\Lambda^i(k) = \Lambda(k) \cap V^i$, $k \in \mathbb{Z}$ and Λ^i is a self dual lattice sequence in V^i .
2. $b = b_1 \oplus b_2$ for $b_i \in A^i = \text{End}_F(V^i)$ and $\nu_{\Lambda^1}(b_2) = -n$ and b_2 is Λ^2 -invertible;

Remark 5.3.20. *For i is 1 or 2, the space V^i has dimension one and b_i is scalar. Twisting π by a character of G , we may assume that V^1 has dimension two with b_1 is Λ^1 -invertible and V^2 has dimension one. The group $G^2 = U(V^2)$, then, is $U(1)(F/F_0)$ and $G^1 = U(V^1)$ is either $U(2)(F/F_0)$ or $U(1,1)(F/F_0)$ depending on the period e , by Remark 5.3.7.*

There are $q(q-1)$ choices for $\varphi_b(X)$ but, by twisting, we can assume $\alpha = 0$, $\beta \neq 0$ which leaves us with $q-1$ choices for $\varphi_b(X)$.

Remark 5.3.21. *The level k of π is in \mathbb{Z} i.e. e divides n .*

We can write the stratum $[\Lambda, n, n-1, b]$ as follows:

$$[\Lambda^1, n, n-1, b_1] \oplus [\Lambda^2, n, n-1, b_2].$$

where $b_2 \in F$ is a \mathcal{L}^2 -invertible with valuation $-k$ where Λ^2 is a multiple of some chain \mathcal{L}^2 of period 1 so $P^{m+1}(\Lambda^2) = P^{\lceil m/e \rceil + 1}(\mathcal{L}^2)$.

If $\{\underline{e}_{-1}, \underline{e}_1\}$ (or $\{e_1, e_{-1}\}$) and $\{\underline{e}_0\}$ (or $\{e_0\}$) are Witt bases for V^1 and V^2 respectively, then the union of these is a bases for V and the form h on V has the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varpi_F & 0 \\ 0 & 0 & \varpi_F \end{pmatrix}.$$

We can represents b as a matrix

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

where $b_i \in \text{End}(V^i)$, $\overline{b_i} = -b_i$, $i = 1, 2$. We will write all matrices with respect of the basis $\{\underline{e}_1, \underline{e}_{-1}, \underline{e}_0\}$ (or $\{e_1, e_{-1}, e_0\}$) in this section.

In the following we define some \mathcal{O}_F -lattices in A which are stable under the involution.

$$\begin{aligned} {}^r \mathfrak{h}_1 &= \begin{pmatrix} \mathfrak{P}^r(\Lambda^1) & \mathfrak{P}_{12}^{\lceil \frac{n}{2} \rceil + 1}(\Lambda) \\ \mathfrak{P}_{21}^{\lceil \frac{n}{2} \rceil + 1}(\Lambda) & \mathfrak{P}^n(\Lambda^2) \end{pmatrix} \\ {}^r \mathfrak{h}_2 &= \begin{pmatrix} \mathfrak{P}^{r+1}(\Lambda^1) & \mathfrak{P}_{12}^{\lceil \frac{n}{2} \rceil + 1}(\Lambda) \\ \mathfrak{P}_{21}^{\lceil \frac{n}{2} \rceil + 1}(\Lambda) & \mathfrak{P}^{n+1}(\Lambda^2) \end{pmatrix} \\ {}^r \mathfrak{J}_1 &= \begin{pmatrix} \mathfrak{P}^r(\Lambda^1) & \mathfrak{P}_{12}^{\lceil \frac{n+1}{2} \rceil}(\Lambda) \\ \mathfrak{P}_{21}^{\lceil \frac{n+1}{2} \rceil}(\Lambda) & \mathfrak{P}^n(\Lambda^2) \end{pmatrix}. \end{aligned}$$

Now put

$${}^r H_i = (1 + {}^r \mathfrak{h}_i) \cap G \quad \text{and} \quad {}^r J_1 = (1 + {}^r \mathfrak{J}_1) \cap G$$

where $i = 1, 2$. When $r = n$, put $H_i = {}^n H_i$, $i = 1, 2$, and $J_1 = {}^n J_1$ (${}^n \mathfrak{h}_i = \mathfrak{h}_i$ and ${}^n \mathfrak{j}_1 = \mathfrak{j}_1$). If n is odd, then ${}^n H_1 = {}^n J_1$.

Proposition 5.3.22. *1. The sets H_i , $i = 1, 2$, and J_1 are compact open subgroups of G .*

2. The map $x \mapsto 1 + x$ induces an isomorphism

$$\mathfrak{h}_1^- / \mathfrak{h}_2^- \longrightarrow H_1 / H_2.$$

3. We have

$$\mathfrak{h}_1^* = \begin{pmatrix} \mathfrak{P}^{1-n}(\Lambda^1) & \mathfrak{P}_{12}^{-\lceil \frac{n}{2} \rceil}(\Lambda) \\ \mathfrak{P}_{21}^{-\lceil \frac{n}{2} \rceil}(\Lambda) & \mathfrak{P}^{1-n}(\Lambda^2) \end{pmatrix} \quad \text{and} \quad \mathfrak{h}_2^* = \begin{pmatrix} \mathfrak{P}^{-n}(\Lambda^1) & \mathfrak{P}_{12}^{-\lceil \frac{n}{2} \rceil}(\Lambda) \\ \mathfrak{P}_{21}^{-\lceil \frac{n}{2} \rceil}(\Lambda) & \mathfrak{P}^{-n}(\Lambda^2) \end{pmatrix}.$$

Proof. See [20, §2]. □

We define the character ψ_b of H_1 as

$$\psi_b(1 + x) = \psi_A(bx), \quad x \in \mathfrak{h}_1.$$

Theorem 5.3.23. *We have*

$$\mathcal{I}_G(\psi_b|_{H_1}) = J_1 (\mathcal{I}_{G^1}(\psi_{b_1}) \times \mathcal{I}_{G^2}(\psi_{b_2})) J_1$$

where ψ_{b_i} is the character $P^n(\Lambda^i)$, $i = 1, 2$.

Proof. See [3, 5.2.2]. □

The sequence

$$1 \rightarrow H_1/H_2 \rightarrow J_1/H_2 \rightarrow J_1/H_1 \rightarrow 1$$

is exact and, in fact, H_1/H_2 lies in the center of J_1/H_2 . Therefore, J_1/H_2 is a central extension of J_1/H_1 by H_1/H_2 .

Lemma 5.3.24. *The pairing*

$$(x, y) \mapsto \psi_b([x, y]), \quad \text{for } x, y \in J_1,$$

is a non-degenerate alternating bilinear form on J_1/H_1 .

Proof. See [33, Proposition 4.1]. □

Lemma 5.3.25. *There exists a unique irreducible representation η_b of J_1 such that the restriction of η_b to H_1 contains the character ψ_b (in fact, it is a multiple of ψ_b). Moreover,*

$$\dim(\eta_b) = \sqrt{[J_1 : H_1]}.$$

Proof. See Lemma 3.7.4. □

The following Lemma is important for our classification of π .

Lemma 5.3.26. *There exists a self dual lattice sequence Λ' and an integer $r \in \{0, \dots, ke\}$ such that π contains either:*

1. ($r > 0$) the character $\psi_{b'_1 \oplus b_2}$ of ${}^r H_1(\Lambda')$ such that the stratum $[\Lambda'^1, r, r - 1, b'_1]$ is fundamental.
2. ($r = 0$) the irreducible representation of

$${}^0 H_1(\Lambda') / {}^1 H_1(\Lambda') \cong P(\Lambda'^1) / P^1(\Lambda'^1) \times P^{ke}(\Lambda'^2) / P^{ke+1}(\Lambda'^2).$$

and such representation has the form $\rho \otimes \psi_{b_2}$ with ρ is cuspidal.

Proof. See [2, Lemma 5.6.1]. □

Here we have four different cases for the representation π depending on the type of the stratum $[\Lambda'^1, r, r - 1, b'_1]$. These cases are:

- (a) essentially scalar;
- (b) split, but not G split;
- (c) minimal non scalar;

(d) $r = 0$.

Remark 5.3.27. *The reason we do not consider G -split is because the representation π is supercuspidal (see [34, Theorem 4.9]).*

Before we investigate each case, we need the following Lemma: let ψ_b be a character of ${}^r H_1(\Lambda) = (1 + {}^r \mathfrak{h}_1(\Lambda)) \cap G$ and $\psi = \psi_{b'}$ be characters of ${}^{r'} H(\Lambda') = (1 + {}^{r'} \mathfrak{h}_1(\Lambda')) \cap G$ and $L = U(1, 1)(F/F_0) \times U(1)(F/F_0)$. Let $L = \tilde{L} \cap G$ and let $\tilde{\psi}_b$ and $\tilde{\psi}_{b'}$ be characters of ${}^r \tilde{H}_1(\Lambda) = 1 + {}^r \mathfrak{h}_1(\Lambda)$ and ${}^{r'} \tilde{H}_1(\Lambda') = 1 + {}^{r'} \mathfrak{h}_1(\Lambda')$. The characters $\tilde{\psi}_b$ and $\tilde{\psi}_{b'}$ are defined the same as ψ_b and $\psi_{b'}$.

Lemma 5.3.28. $\mathcal{I}_G(\psi_b, \psi_{b'}) = {}^n J_1(\Lambda) \mathcal{I}_L(\psi_b|_{{}^r H_1(\Lambda) \cap L}, \psi_{b'}|_{{}^{r'} H_1(\Lambda') \cap L}) {}^n J_1(\Lambda')$.

Proof. Surely we have

$$\mathcal{I}_G(\tilde{\psi}_b, \tilde{\psi}_{b'}) \subseteq \mathcal{I}_G(\tilde{\psi}_b|_{{}^n \tilde{H}_1(\Lambda)}, \tilde{\psi}_{b'}|_{{}^n \tilde{H}_1(\Lambda')}).$$

By restricting $\tilde{\psi}_b, \tilde{\psi}_{b'}$ to ${}^n \tilde{H}_1(\Lambda), {}^n \tilde{H}_1(\Lambda')$, we get $b = 0 \oplus b_2 = b'$. Now let $g \in \mathcal{I}_G(\tilde{\psi}_b|_{{}^n \tilde{H}_1(\Lambda)}, \tilde{\psi}_{b'}|_{{}^n \tilde{H}_1(\Lambda')})$, then there exists $x \in {}^n \mathfrak{h}_1^*(\Lambda)$ and $x' \in {}^n \mathfrak{h}_1^*(\Lambda')$ such that

$$g^{-1}(b+x)g = b+x'.$$

where

$${}^n \mathfrak{h}_1^*(\Lambda) = \begin{pmatrix} \mathfrak{P}^{1-n}(\Lambda^1) & \mathfrak{P}^{-[\frac{n}{2}]}(\Lambda) \\ \mathfrak{P}^{-[\frac{n}{2}]}(\Lambda) & \mathfrak{P}^{1-n}(\Lambda^2) \end{pmatrix}, \quad {}^n \mathfrak{h}_1^*(\Lambda') = \begin{pmatrix} \mathfrak{P}^{1-n}(\Lambda'^1) & \mathfrak{P}^{-[\frac{n}{2}]}(\Lambda') \\ \mathfrak{P}^{-[\frac{n}{2}]}(\Lambda') & \mathfrak{P}^{1-n}(\Lambda'^2) \end{pmatrix}$$

By [2, Lemma 5.10(2)], there exists $j \in {}^n \tilde{J}_1(\Lambda) = 1 + {}^n \mathfrak{j}_1(\Lambda)$ and $j' \in {}^n \tilde{J}_1(\Lambda') = 1 + {}^n \mathfrak{j}_1(\Lambda')$ such that

$$j(b+x)j^{-1} = b+y \in (b + {}^n \mathfrak{h}_1^*(\Lambda)) \cap \begin{pmatrix} A^{11} & 0 \\ 0 & A^{22} \end{pmatrix}$$

and

$$j'(b+x)j'^{-1} = b+y' \in (b + {}^n \mathfrak{h}_1^*(\Lambda')) \cap \begin{pmatrix} A^{11} & 0 \\ 0 & A^{22} \end{pmatrix}$$

where $y \in {}^n \mathfrak{h}_1^*(\Lambda)$ and $y' \in {}^n \mathfrak{h}_1^*(\Lambda')$. Now put $h = jjj'^{-1}$, then we have

$$h(b+y)h^{-1} = b+y' \Rightarrow h_{12} = h_{21} = 0 \Rightarrow h \in \tilde{L}.$$

This implies that

$$\mathcal{I}_{GL_3}(\tilde{\psi}_b, \tilde{\psi}_{b'}) \subseteq {}^n \tilde{J}_1(\Lambda) \mathcal{I}_{\tilde{L}}(\tilde{\psi}_b|_{{}^n \tilde{H}_1(\Lambda) \cap \tilde{L}}, \tilde{\psi}_{b'}|_{{}^n \tilde{H}_1(\Lambda) \cap \tilde{L}}) {}^n \tilde{J}_1(\Lambda').$$

Now by [33, Corollary 2.5], we get

$$\mathcal{I}_G(\psi_b, \psi_{b'}) = \mathcal{I}_{GL_3}(\tilde{\psi}_b, \tilde{\psi}_{b'}) \cap G$$

and by the same proof as Theorem 2.3 in [32], we deduce

$$\mathcal{I}_G(\psi_b, \psi_{b'}) \subseteq {}^n \tilde{J}_1(\Lambda) \tilde{L} {}^n \tilde{J}_1(\Lambda') \cap G = {}^n J_1(\Lambda) \mathcal{I}_L(\psi_b|_{{}^r H_1(\Lambda) \cap L}, \psi_{b'}|_{{}^{r'} H_1(\Lambda') \cap L}) {}^n J_1(\Lambda'),$$

since the groups ${}^n J_1(\Lambda), {}^n J_1(\Lambda')$ normalize the characters $\psi_b, \psi_{b'}$. \square

In particular, this Lemma implies that the four cases (a), (b), (c) and (d) are disjoint i.e we cannot get the same supercuspidal representation from different cases. The reason is that if π contains two characters $\psi_{b'_1 \oplus b_2}$ and $\psi_{b''_1 \oplus b_2}$ of ${}^r H(\Lambda')$, by Lemma 5.3.28, the strata $[\Lambda'^1, r, r-1, b'_1]$ and $[\Lambda'^1, r, r-1, b''_1]$ intertwine. By Lemma 3.4.8, they have the same characteristic polynomial so they must be the same type.

(a) $[\Lambda'^1, r, r-1, b'_1]$ is essentially scalar

Here we will consider the stratum $[\Lambda'^1, r, r-1, b'_1]$ to be essentially scalar so $t = r/e_1 \in \mathbb{Z}$, where $e_1 = e(\Lambda'^1)$ and $\varphi_{b'_1}(X) = (X - \alpha)^2$, where $\alpha \in k_F^\times$ and $\alpha = -\bar{\alpha}$.

Let $A_{<t}$ be the set of irreducible supercuspidal representations of G of level k which contain a character $\psi_{b''_1 \oplus b_2}$ of ${}^{r''} H(\Lambda'')$ with $r''/e(\Lambda'') < t$, where b_2 is fixed. Let A_t^{es} be the set of irreducible supercuspidal representations of G of level k which contain a character $\psi_{b'_1 \oplus b_2}$ of ${}^r H(\Lambda')$ with b'_1 is essentially scalar with $t = r/e(\Lambda')$ and b_2 fixed. Let Ψ_t denote the set of characters of F^1 of level t . Then we have the following Lemma

Proposition 5.3.29. *For $\pi \in A_{<t}$ and a character χ of F^1 of level t , we have $\pi \otimes \chi \circ \det \in A_t^{es}$.*

Proof. By Lemma 3.4.5 we have

$$\ell(\pi \otimes \chi \circ \det) = \max\{\ell(\pi), \ell(\chi)\} = \ell(\pi).$$

Suppose π contains $\psi_{b''_1 \oplus b_2}$ of ${}^{r''} H(\Lambda'')$, with $r''/e(\Lambda'') < t$ and put $r = te(\Lambda'')$. The restriction of the character χ to the group ${}^r H(\Lambda'')$ is a character of the form ψ_c , where $c = \varpi_F^{-r} c_0 \in (\mathcal{P}_F^{-r})_-$, defined by

$$\psi_c(1+x) = \psi_F \circ \det(cx), \quad 1+x \in {}^r H(\Lambda').$$

Now the restriction of the representation $\pi \otimes \chi \circ \det$ to the group ${}^r H(\Lambda'')$ contains the character $\psi_{c+b''_1 \oplus c+b_2}$. If $v_{\Lambda''_1}(b''_1) = r'' < r$ then $\psi_{c+b''_1 \oplus c+b_2} = \psi_{c \oplus c+b_2}$ and $[\Lambda''^1, r, r-1, c]$ is scalar. Thus $\pi \otimes \chi \circ \det \in A_t^{es}$. \square

Theorem 5.3.30. *Let $\pi, \pi' \in A_{<t}$ and χ, χ' be characters of F^1 of level t . If $\pi \otimes \chi \circ \det \cong \pi' \otimes \chi' \circ \det$ then $\ell(\chi' \chi^{-1}) < t$.*

Proof. If $\ell(\chi' \chi^{-1}) = t$, then by Proposition 5.3.29, $\pi' \otimes \chi' \chi^{-1} \circ \det \in A_t^{es}$. On the other hand, the representation $\pi' \otimes \chi' \chi^{-1} \circ \det$ is isomorphic to π which lies in $A_{<t}$. This is impossible as π contains some character $\psi_{b''_1 \oplus b_2}$ of ${}^{r''} H_1(\Lambda'')$ within $r''/e(\Lambda'') < t$ and $\pi' \otimes \chi' \chi^{-1} \circ \det$ contains some character $\psi_{b'_1 \oplus b_2}$ of ${}^r H_1(\Lambda'^1)$ and the two characters intertwine. By Lemma 5.3.28, $[\Lambda''^1, r'', r''-1, b''_1]$ intertwines with $[\Lambda'^1, r, r-1, b'_1]$ and since they are both fundamental, then they have the same level, contradiction. \square

Lemma 5.3.31. *The map*

$$\begin{aligned} A_{<t} \times \Psi_t &\longrightarrow A_t^{es} \\ (\pi, \chi) &\longmapsto \pi \otimes \chi. \end{aligned}$$

is surjective.

Proof. Let $\pi \in A_t^{es}$, then π contains a character $\psi_{b'_1 \oplus b_2}$ of ${}^r H_1(\Lambda')$ with b'_1 essentially scalar with $t = r/e$ and b_2 fixed. There exists a character χ of F^1 such that $\chi \circ \det |P^r(\Lambda') = \psi_\lambda$, for some $\lambda \in \varpi_F^{-r} \mathcal{O}_F^\times$. Now the character χ has level t . The representation $\pi \otimes \chi^{-1} \circ \det$ contains the character $\psi_{b'_1 - \lambda \oplus b_2 - \lambda}$ of ${}^r H_1(\Lambda')$, but $b'_1 \equiv \lambda \pmod{\mathfrak{P}^{1-n}(\Lambda^1)}$, since $r \leq 1 - n$, and hence $\pi \otimes \chi^{-1} \circ \det \in A_{<t}$. \square

Lemma 5.3.32. *The fibers of the map in Lemma 5.3.31 is*

$$X = \{(\pi\psi, \chi\psi^{-1}) : \psi \text{ is a character of } F^1 \text{ of level } < t\}.$$

and the size of each fiber is $(q+1)q^{t-2}$.

Proof. Let $\pi \in A_{<t}$ and $\chi \in \Psi$ such that $\pi \otimes \chi \in A_t^{es}$ and consider the set

$$\text{Fib}(\pi \otimes \chi) = \{(\pi', \psi) : \pi' \in A_{<t}, \text{ for some } t, \chi' \in \Psi_t \text{ and } \pi' \otimes \chi' \cong \pi \otimes \chi\}.$$

The set $\text{Fib}(\pi \otimes \chi)$ is not-empty since it contains the pair (π, χ) . Now let $\pi' \in A_{<t}$ and $\chi' \in \Psi_t$ such that $\pi' \otimes \chi' \cong \pi \otimes \chi$. So we have $\pi' \cong \pi \otimes \chi \chi'^{-1}$, within $\ell(\chi \chi'^{-1}) < t$, by Theorem 5.3.30. Put $\psi^{-1} = \chi \chi'^{-1}$, a character ψ of F^1 of level $< t$. Now $\pi' \cong \pi \otimes \chi \chi'^{-1} \cong \pi \psi^{-1}$ and

$$(\pi', \chi') = (\pi \psi^{-1}, \chi \psi).$$

\square

Let $s_6^{es}(t)$ denote the number of irreducible supercuspidal representations of G of level k which contain a character $\psi_{b'_1 \oplus b_2}$ of ${}^r H(\Lambda')$ where b'_1 is essentially scalar and b_2 is fixed. Let $S_6^*(t)$ denote the number of irreducible supercuspidal representations of G of level strictly less than t which contain a character $\psi_{b'_1 \oplus b_2}$ of ${}^r H(\Lambda')$ where b'_1 is minimal non-scalar or split and b_2 is fixed. The size of Ψ_t is $(q+1)(q-1)q^{t-2}$, by Corollary 3.6.6, so we have the following

Corollary 5.3.33. *We have*

$$s_6^{es}(t) = (q-1)S_6^*(k)$$

(b) $[\Lambda^1, r, r-1, b'_1]$ is split, but not G -split

Let $[\Lambda^1, r, r-1, b'_1]$ be a split, but not G -split, stratum. Then $t = r/e \in \mathbb{Z}$ and

$$\varphi_{b'_1}(X) = (X - \alpha_1)(X - \alpha_2)$$

where $\alpha_i = -\bar{\alpha}_i$, $i = 1, 2$, and $\alpha_1 \neq \alpha_2$. If $\text{Str}_b(t)^{sp}$ denotes the number of split strata of level t , up to G -intertwining then $\text{Str}_b(t)^{sp} = q(q-1)/2$, by Lemma 3.9.3. In fact, the number $\text{Str}_b(t)^{sp}$ is for the case $U(V^1) = U(1,1)(F/F_0)$, however, in the case $U(V^1) = U(2)(F/F_0)$, the number $\text{Str}_b(t)^{sp}$ is the same and, also, the same proof.

Remark 5.3.34. Note that the group $U(V^1)$ is either $U(1,1)(F/F_0)$ or $U(2)(F/F_0)$. Although they are different groups, but the construction of supercuspidals is exactly the same.

We construct the representation π as follows: π contains the character $\psi_{b'_1 \oplus b_2}$ of ${}^r H_1(\Lambda')$ which extends to $\psi_{b'_1 \oplus b_2}|^{[r/2]+1} H_1(\Lambda')$. The set of these extensions up to G -intertwining is denoted by $\text{Ext}(\psi_{b'_1 \oplus b_2}, {}^r H_1(\Lambda'))$ and the size of this set is

$$\begin{aligned} |\text{Ext}(\psi_{b'_1 \oplus b_2}, {}^r H_1(\Lambda'))| &= \left[(1 + \mathcal{P}_F)^1 : \left(1 + \mathcal{P}_F^{\lfloor \frac{t-1}{2} \rfloor + 1} \right)^1 \right]^2 \\ &= q^{2 \lfloor \frac{t-1}{2} \rfloor}. \end{aligned}$$

Now the character $\psi_{b'_1 \oplus b_2}|^{[r/2]+1} H_1(\Lambda')$ extends a character $\theta_{b'_1 \oplus b_2}$ of ${}^1 H = [r/2]+1 H_1(\Lambda') \cdot (B \cap G) \cap P^1(\Lambda')$, where B is the centralizer of $b'_1 \oplus b_2$ in A . We denote the set of these extensions by $\Gamma(\Lambda', \psi_{b'_1 \oplus b_2})$. Let $\theta_1, \theta_2 \in \Gamma(\Lambda', \psi_{b'_1 \oplus b_2})$. Let $g \in G$ intertwines both characters, then $g \in J = \lfloor (r+1)/2 \rfloor J_1 \cdot (B \cap G) \cap P^1(\Lambda')$. The group J normalizes both characters θ_1, θ_2 , so $\theta_1 = \theta_2$. Now the quotient ${}^1 H / [r/2]+1 H_1(\Lambda')$ is isomorphic to

$$(1 + \mathcal{P}_F)^1 / \left(1 + \mathcal{P}_F^{\lfloor t/2 \rfloor + 1} \right)^1 \times (1 + \mathcal{P}_F)^1 / \left(1 + \mathcal{P}_F^{\lfloor t/2 \rfloor + 1} \right)^1 \times (1 + \mathcal{P}_F)^1 / (1 + \mathcal{P}_F^k)^1.$$

Thus, $|\Gamma(\Lambda', \psi_{b'_1 \oplus b_2})| = q^{2 \lfloor t/2 \rfloor} q^{k-1}$. By Heisenberg Lemma, there is a unique irreducible representation $\eta_{b'_1 \oplus b_2}$ of ${}^1 J = \lfloor (r+1)/2 \rfloor J_1 \cdot (B \cap G) \cap P^1(\Lambda')$ such that the restriction of $\eta_{b'_1 \oplus b_2}$ to ${}^1 H$ contains $\theta_{b'_1 \oplus b_2}$. By Lemma 3.7.5, the representation $\eta_{b'_1 \oplus b_2}$ of ${}^1 J$ extends to a representation η of J and the number of these extensions is $|\mathcal{R}(\Lambda', \theta_{b'_1 \oplus b_2})| = (q+1)^3$. Now the representation $\text{c-Ind}_J^G \eta$ is irreducible and supercuspidal and equivalent to π .

The number of irreducible supercuspidal representations of G of level k which contain a character $\psi_{b'_1 \oplus b_2}$ of ${}^r H(\Lambda')$, where b'_1 is split and both Λ'^1, b_2 fixed denoted by $s_b(\Lambda'^1, t)$. The number of irreducible supercuspidal representations of G of level k which contain a character $\psi_{b'_1 \oplus b_2}$ of ${}^r H(\Lambda')$, where b'_1 is split and b_1 fixed denoted by $s_b(t)$. so

$$s_b(\Lambda', t) = \frac{1}{2}(q-1)(q+1)^3 q^{2(t-1)} q^k$$

Remark 5.3.35. We have asked a question in Remark 3.9.8 about whether there exists an element $g \in U(1,1)$ such that g conjugate two fundamental skew strata $[\Lambda'^1, r, r-1, b]$ and $[\Lambda''^1, r, r-1, b]$ or not? In either case the number $s_b(t)$ is bounded

$$s_b(\Lambda'^1, t) \leq s_b(t) \leq 4s_b(\Lambda'^1, t).$$

We will assume that $s_b(t) = s_b(\Lambda'^1, t)$.

(c) $[\Lambda^1, r, r-1, b'_1]$ is minimal non-scalar

Now suppose π contains a character $\psi_{b'_1 \oplus b_2}$ of ${}^r H_1(\Lambda')$ where b'_1 is minimal non-scalar, where $0 < r < n$.

Remark 5.3.36. *Lemma 4.3.1 implies there is no skew maximal simple stratum of period one in the 2-dimensional space V^1 , therefore, Λ^1 must be a multiple of some chain \mathcal{L}^1 of period two and E_1/F is ramified, $E_1 = F[b'_1]$, so*

$$e(\Lambda^1|_{\mathcal{O}_F}) = 2m, \quad m \geq 1.$$

We also have

$$e(\Lambda^1|_{\mathcal{O}_{E_1}}) = \frac{e(\Lambda^1|_{\mathcal{O}_F})}{e(E/F)} = m,$$

where $E_1 = F[b'_1]$. Then also r is an odd multiple of m , say, $r = tm$, where $t = 2t' + 1$. It is odd multiple because the stratum $[\Lambda^1, r, r-1, b'_1]$ has level in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ as E_1/F is ramified.

Let $\text{Str}(t)^m$ is the number of minimal non-scalar skew strata in A^1 of level $t = r/e$, up to P -conjugacy.

Lemma 5.3.37. *We have $\text{Str}(t)^m = q - 1$.*

Proof. When $U(V^1) = U(1, 1)(F/F_0)$, then the proof by Lemma 4.3.4 and when $U(V^1) = U(2)(F/F_0)$, the proof given by Lemma 5.3.8. \square

Remark 5.3.38. *The construction of π in both cases $U(V^1) = U(1, 1)(F/F_0)$ or $U(2)(F/F_0)$ are exactly the same.*

Denote by $\text{Ext}(\psi_{b'_1 \oplus b_2}, {}^r H_1)$, the set of G -intertwining classes of characters $\psi_{b'_1 \oplus b_2}|^{[\frac{r}{2}]^+} H_1$ which extend $\psi_{b'_1 \oplus b_2}|^r H_1$. Now define the following subgroups

$$\begin{aligned} {}^1 H &= H_1 \left(P^{[\frac{r}{2}]^+}(\Lambda^1) (1 + \mathcal{P}_{E_1})^1 \times (1 + \mathcal{P}_F)^1 \right) \\ {}^1 J &= J_1 \left(P^{[\frac{r+1}{2}]}(\Lambda^1) (1 + \mathcal{P}_{E_1})^1 \times (1 + \mathcal{P}_F)^1 \right) \\ J &= J_1 \left(P^{[\frac{r+1}{2}]}(\Lambda^1) E_1^{-1} \times F^1 \right). \end{aligned}$$

Lemma 5.3.39. *1. The character $\psi_{b'_1 \oplus b_2}|^r H_1$ extends to a character of $^{[\frac{r}{2}]^+} H_1$ and any such extension has the form $\psi_{b''_1 \oplus b_2}$ where*

$$b''_1 \equiv b'_1 \pmod{\mathfrak{P}_-^{1-r}(\Lambda^1)}.$$

2. The number of these extension characters up to G -intertwining is

$$|\text{Ext}(\psi_{b'_1 \oplus b_2}, {}^r H_1)| = q^{t'}.$$

Proof. For part (1), consider the quotient $^{[\frac{r}{2}]^+} H_1(\Lambda')/{}^r H_1(\Lambda')$. It is abelian and isomorphic to

$$\begin{pmatrix} P^{[\frac{r}{2}]^+}(\Lambda^1)/P^r(\Lambda^1) & 0 \\ 0 & 1 \end{pmatrix}.$$

We can see the change only in the block $P^{[\frac{r}{2}]+1}(\Lambda'^1)/P^r(\Lambda'^1)$. By Lemma 3.3.6, the character $\psi_{b'_1}|P^r(\Lambda'^1)$ extends to a character $\psi_{b''_1}$ of $P^{[\frac{r}{2}]+1}(\Lambda'^1)$, where

$$b''_1 \equiv b'_1 \pmod{\mathfrak{P}_-^{1-r}(\Lambda'^1)}.$$

For part (2), since J normalizes any extension, the number of characters of $[\frac{r}{2}]+1H_1$ which extend $\psi_{b'_1 \oplus b_2}|^r H_1$ up to G -intertwining is equal to number of characters $\psi_{b'_1}|P^{[\frac{r}{2}]+1}(\Lambda'^1)$ which extend $\psi_{b'_1}|P^r(\Lambda'^1)$ up to G^1 -intertwining, by Lemma 5.3.28. The number is given by Lemma 3.5.3 so

$$\begin{aligned} |\text{Ext}(\psi_{b'_1 \oplus b_2}, [\frac{r}{2}]+1H_1)| &= |\text{Ext}(\psi_{b'_1}, P^{[\frac{r}{2}]+1}(\Lambda'^1))| \\ &= [E_1^1 \cap P^1(\Lambda'^1) : E_1^1 \cap P^{[\frac{r+1}{2}]}(\Lambda'^1)] \\ &= [(1 + \mathcal{P}_{E_1})^1 : (1 + \mathcal{P}_{E_1}^{t'+1})^1]. \end{aligned}$$

□

Lemma 5.3.40. *The character $\psi_{b'_1 \oplus b_2}|^{[\frac{r}{2}]+1}H_1(\Lambda')$ extends to a character $\theta_{b'_1 \oplus b_2}$ of 1H .*

Proof. Consider the quotient ${}^1H/[\frac{r}{2}]+1H_1$. It is isomorphic to

$$\left((1 + \mathcal{P}_{E_1})^1 / (1 + \mathcal{P}_{E_1}^{t'+1})^1 \right) \times \left((1 + \mathcal{P}_F)^1 / (1 + \mathcal{P}_F^k)^1 \right).$$

It is a product of cyclic groups and since 1H normalizes the character $\psi_{b'_1}$, then, the character $\psi_{b'_1 \oplus b_2}|^{[\frac{r}{2}]+1}H_1(\Lambda')$ does extend to $\theta_{b'_1 \oplus b_2}|{}^1H$. □

We denote the set of characters $\theta_{b'_1 \oplus b_2}$ of 1H which extend the characters $\psi_{b'_1 \oplus b_2}|^{[\frac{r}{2}]+1}H_1(\Lambda')$ by $\Gamma(\Lambda', \psi_{b'_1 \oplus b_2})$. Any two distinct extensions in $\Gamma(\Lambda', \psi_{b'_1 \oplus b_2})$, do not intertwine by Lemma 3.7.3 and Lemma 5.3.28.

Corollary 5.3.41. *We have*

$$|\Gamma(\Lambda', \psi_{b'_1 \oplus b_2})| = q^{t'} q^{k-1}.$$

Proof. Since none of the characters in $\Gamma(\Lambda', \psi_{b'_1 \oplus b_2})$ intertwine, then, the size of the set $\Gamma(\Lambda', \psi_{b'_1 \oplus b_2})$ is, in fact, the index

$$[{}^1H : [\frac{r}{2}]+1H_1] = \left[(1 + \mathcal{P}_{E_1})^1 / (1 + \mathcal{P}_{E_1}^{t'+1})^1 \right] \cdot \left[(1 + \mathcal{P}_F)^1 / (1 + \mathcal{P}_F^k)^1 \right].$$

□

Proposition 5.3.42 (Heisenberg). *There is a unique irreducible representation $\eta_{b'_1 \oplus b_2}$ of 1J such that the restriction of $\eta_{b'_1 \oplus b_2}$ to 1H contains the character $\psi_{b'_1 \oplus b_2}$. Moreover,*

$$\dim(\eta_{b'_1 \oplus b_2}) = \sqrt{[{}^1J : {}^1H]}.$$

Proof. See [2, Lemma 5.4(i)]. □

Lemma 5.3.43. *There exists a representation η of J which extends $\eta_{b'_1 \oplus b_2}$.*

Proof. See [2, 5.7.1]. □

Denote by $\mathcal{R}(\Lambda', \theta_{b'_1 \oplus b_2})$, the set of irreducible representations η of J which extend $\eta_{b'_1 \oplus b_2}$ of 1J and the restriction of η to 1H contain $\theta_{b'_1 \oplus b_2}$.

Lemma 5.3.44. *Let $\eta_1, \eta_2 \in \mathcal{R}(\Lambda', \theta_{b'_1 \oplus b_2})$. Then they intertwine in G if and only if $\eta_1 \cong \eta_2$.*

Proof. If $g \in G$ intertwines η_1 with η_2 , then it must intertwine the character $\psi_{b'_1 \oplus b_2}$ so $g \in J$. Therefore, $\eta_1 \cong \eta_2$ since J normalizes both η_1 and η_2 . □

Corollary 5.3.45. *We have*

$$|\mathcal{R}(\Lambda', \theta_{b'_1 \oplus b_2})| = (q+1)^2.$$

Proof. The quotient $J/{}^1J$ is isomorphic to

$$E_1^1/(1 + \mathcal{P}_{E_1})^1 \times F^1/(1 + \mathcal{P}_F)^1.$$

Thus,

$$\begin{aligned} |\mathcal{R}(\Lambda', \theta_{b'_1 \oplus b_2})| &= [E_1^1 : (1 + \mathcal{P}_{E_1})^1][F^1 : (1 + \mathcal{P}_F)^1] \\ &= (q_{E_{1,0}} + 1)(q+1) = (q+1)^2. \end{aligned}$$

as $q_{E_{0,1}} = q$. □

Theorem 5.3.46. *If π contains $\eta \in \mathcal{R}(\Lambda', \theta_{b'_1 \oplus b_2})$ then the representation $\text{c-Ind}_J^G \eta$ is irreducible and supercuspidal and equivalent to π .*

Proof. See [2, Proposition 5.7]. □

The following diagram summarize the construction of π :

$$\begin{array}{ccc} \text{c-Ind}_J^G \eta & & G \\ \downarrow & & \downarrow \\ \eta & & J \\ \downarrow & & \downarrow \\ \eta_{b'_1 \oplus b_2} & & {}^1J \\ \downarrow & & \downarrow \\ \theta_{b'_1 \oplus b_2} & & {}^1H \\ \downarrow & & \downarrow \\ \psi_{b'_1 \oplus b_2} & & [\frac{E}{2}]^{+1} H_1 \\ \downarrow & & \downarrow \\ \psi_{b'_1 \oplus b_2} & & {}^r H_1 \end{array}$$

Let $s_6^c(t, k)$ denote the number of irreducible supercuspidal representations of G of level $k = n/e$ which contain a character $\psi_{b_1' \oplus b_2}$ of ${}^r H_1(\Lambda')$ with b_1 is minimal non-scalar, b_2 fixed and $t = (2t' + 1)/2$. Let $s_6^c(U(V^1), t, k)$ denote the number of irreducible supercuspidal representations of G (with $G^1 = U(V^1)$) of level $k = n/e$ which contain a character $\psi_{b_1' \oplus b_2}$ of ${}^r H_1(\Lambda')$ with b_1 is minimal non-scalar, b_2 fixed and $t = (2t' + 1)/2$. Then

Theorem 5.3.47. *We have*

$$s_6^c(t, k) = 2q(q-1)(q+1)^2 q^{2t-1} q^{k-1}.$$

Proof. We have

$$s_6^c(U(1, 1)(F/F_0), t, k) = s_6^c(U(2)(F/F_0), t, k) = q(q-1)(q+1)^2 q^{2t-1} q^{k-1}.$$

The result follow since $s_6^c(t, k) = s_6^c(U(1, 1)(F/F_0), t, k) + s_6^c(U(2)(F/F_0), t, k)$. \square

Lemma 5.3.48. *The formal degree of $\pi = \text{c-Ind}_J^G \eta$ is*

$$d(\pi) = \begin{cases} (q-1)(q^3+1)q^{4k-[(t-1)/2]-1} & \text{if } e(\Lambda) = 2 \\ (q-1)(q^3+1)q^{4k-\gamma+2} & \text{if } e(\Lambda) = 4 \end{cases}$$

where $\gamma = [(t-1)/2] + [(k-1)/2]$ and $k = n/e$.

Proof. The period e of Λ' is either 2, in this case $r = 2t' + 1$, or 4, in this case $r = 2t$ with t is odd. In both cases n must be even as $k = n/e \in \mathbb{Z}$. The dimension of η is $[{}^1 J : {}^1 H]^{1/2}$, where ${}^1 J / {}^1 H$ is isomorphic to

$$\begin{pmatrix} 1 & \mathfrak{P}_{12}^{[\frac{n+1}{2}]}(\Lambda') / \mathfrak{P}_{12}^{[\frac{n}{2}]+1}(\Lambda') \\ \mathfrak{P}_{21}^{[\frac{n+1}{2}]}(\Lambda') / \mathfrak{P}_{21}^{[\frac{n}{2}]+1}(\Lambda') & 1 \end{pmatrix}$$

if $e = 2$, or

$$\begin{pmatrix} P^{[\frac{r+1}{2}]}(\Lambda') / P^{[\frac{r}{2}]+1}(\Lambda') & \mathfrak{P}_{12}^{[\frac{n+1}{2}]}(\Lambda') / \mathfrak{P}_{12}^{[\frac{n}{2}]+1}(\Lambda') \\ \mathfrak{P}_{21}^{[\frac{n+1}{2}]}(\Lambda') / \mathfrak{P}_{21}^{[\frac{n}{2}]+1}(\Lambda') & 1 \end{pmatrix}$$

if $e = 4$. Therefore,

$$\dim(\eta) = \begin{cases} q & \text{if } e = 2 \\ q^2 & \text{if } e = 4 \end{cases}$$

Now we compute $\mu(J)^{-1}$,

$$\mu(J)^{-1} = \mu\left(P^{[\frac{n+1}{2}]}(\Lambda)\right)^{-1} \cdot [J : {}^1 J]^{-1} \cdot [{}^1 J : P^{[\frac{n+1}{2}]}(\Lambda)]^{-1}.$$

By Remark 5.3.7, $P(\Lambda)$ is either L or I with the non-standard filtration, depending of the period of Λ . By Proposition 5.1.2, if $e(\Lambda) = 2$, then $\mu(P^{[(n+1)/2]})^{-1} =$

$(q+1)(q^2-1)(q^3+1)q^{4(\lfloor (n+1)/2 \rfloor - 1) + \lfloor (n-1)/4 \rfloor}$ and if $e(\Lambda) = 4$, then $\mu(P^{\lfloor (n+1)/2 \rfloor})^{-1} = q(q+1)(q^2-1)(q^3+1)q^{2(\lfloor (n+1)/2 \rfloor - 1)}$. The size of the index $[J : {}^1J]$ is $(q+1)^2$ and the quotient ${}^1J/P^{\lfloor \frac{n+1}{2} \rfloor}$ is isomorphic to

$$\left((1 + \mathcal{P}_{E_1})^1 / \left(1 + \mathcal{P}_{E_1}^{\lfloor \frac{t-1}{2} \rfloor + 1} \right)^1 \right) \times \left((1 + \mathcal{P}_F)^1 / \left(1 + \mathcal{P}_F^{\lfloor \frac{k-1}{2} \rfloor + 1} \right)^1 \right).$$

where $n = tm$. Thus,

$$\mu(J)^{-1} = \mu\left(P^{\lfloor \frac{n+1}{2} \rfloor}(\Lambda)\right)^{-1} \cdot (q+1)^{-2} q^{-\gamma}.$$

□

(d) $r = 0$:

Let π be an irreducible smooth representation of G which contains the character $\psi_{0 \oplus b_2}$ of

$${}^1H_1 = {}^1H_1(\Lambda') = H_1(P^1(\Lambda') \times P^{ke}(\Lambda'^2)).$$

Remark 5.3.49. *The element b_2 is non-zero and lies in $\mathcal{P}_F^{-n} \setminus \mathcal{P}_F^{1-n}$ so we have $q-1$ different choices for b_2 . Moreover, none of the characters $\psi_{0 \oplus b_2}$ intertwine since for each b_2 , we have different characteristic polynomial.*

Define the following subgroups:

$$\begin{aligned} {}^1H &= {}^1H_1\left(P^1(\Lambda'^1) \times (1 + \mathcal{P}_F)^1\right) \\ {}^1J &= J_1\left(P^1(\Lambda'^1) \times (1 + \mathcal{P}_F)^1\right) \\ J &= J_1\left(P(\Lambda'^1) \times F^1\right). \end{aligned}$$

Lemma 5.3.50. *The character $\psi_{0 \oplus b_2}$ of 1H_1 extends to a character $\theta_{0 \oplus b_2}$ of 1H .*

Proof. Consider the quotient

$${}^1H / {}^1H_1 \cong 1 \times (1 + \mathcal{P}_F)^1 / (1 + \mathcal{P}_F^k)^1.$$

It is abelian. The group 1H_1 normalizes the character $\psi_{0 \oplus b_2}$, therefore, we can extend $\psi_{0 \oplus b_2}|_{{}^1H_1}$ to a character $\theta_{0 \oplus b_2}$ of 1H . □

Let $\Gamma(\Lambda', \psi_{0 \oplus b_2})$ be the set of characters $\theta_{0 \oplus b_2}$ of 1H which extend $\psi_{0 \oplus b_2}|_{{}^1H_1}$. Any two extensions in $\Gamma(\Lambda', \psi_{0 \oplus b_2})$ intertwine if and only if they are equal, by Lemma 3.7.3.

Corollary 5.3.51. *We have*

$$|\Gamma(\Lambda', \psi_{0 \oplus b_2})| = q^{k-1}.$$

Proof. None of the characters in $\Gamma(\Lambda', \psi_{0\oplus b_2})$ intertwine so

$$|\Gamma(\Lambda', \psi_{0\oplus b_2})| = [(1 + \mathcal{P}_F)^1 : (1 + \mathcal{P}_F^k)^1].$$

□

Proposition 5.3.52 (Heisenberg). *There is a unique irreducible representation $\eta_{0\oplus b_2}$ of 1J such that the restriction of $\eta_{0\oplus b_2}$ to 1H contains the character $\theta_{0\oplus b_2}$. Moreover $\dim(\eta_{0\oplus b_2}) = \sqrt{[{}^1J : {}^1H]}$*

Proof. See [2, §5.8].

□

Lemma 5.3.53. *The intertwining of $\eta_{0\oplus b_2}$ in G is*

$$\mathcal{I}_G(\eta_{0\oplus b_2}) = {}^1J(G^1 \times G^2){}^1J.$$

Moreover,

$$\dim(\mathcal{I}_g(\eta_{0\oplus b_2})) = \begin{cases} 1 & \text{if } g \in {}^1J(G^1 \times G^2){}^1J; \\ 0 & \text{Otherwise} \end{cases}$$

Proof. See [2, Lemma 5.8(i)].

□

Definition 5.3.54. *A b -extension of the representation $\eta_{0\oplus b_2}$ of 1J is a representation κ of J such that:*

1. $\kappa|{}^1J = \eta$ and
2. the representation κ is intertwined by the whole of $G^1 \times G^2$.

Lemma 5.3.55. [2, Lemma 5.8(ii)] *There exists a b -extension κ of η .*

Theorem 5.3.56. *Let κ be a b -extension of η and ξ, ξ' be the inflation to J of some irreducible representations of $J/{}^1J$. Then*

1. For any $h \in G^1 \times G^2$, the intertwining spaces $I_h(\kappa \otimes \xi, \kappa \otimes \xi')$ and $I_h(\xi, \xi')$ have the same dimension.
2. $I_G(\kappa \otimes \xi, \kappa \otimes \xi') = JI_{G^1 \times G^2}(\xi, \xi')J$.

Proof. Let g intertwine $\kappa \otimes \xi$ with $\kappa \otimes \xi'$ so it must intertwine $\kappa \otimes \xi|_{J_1}$ with $\kappa \otimes \xi'|_{J_1}$ which are both multiple of η . Therefore, g intertwines η . By Lemma 5.3.53, we have $g \in {}^1J(G^1 \times G^2){}^1J$. Thus we may assume $g \in G^1 \times G^2$. Let V_1 be the space of κ and V_2 the space of ξ and V_2' the space of ξ' . Let $\phi \in I_g(\kappa \otimes \xi, \kappa \otimes \xi')$ so

$$(\kappa \otimes \xi')(h) \circ \phi = \phi \circ (\kappa \otimes \xi)^g(h) \tag{5.2}$$

for all $h \in J \cap {}^gJ$. We may write

$$\phi = \sum_j S_j \otimes T_j$$

where $S_j \in \text{End}_{\mathbb{C}}(V_1)$ and $T_j \in \text{Hom}_{\mathbb{C}}(V_2, V_2')$ and $\{T_i\}$ is linearly independent. Suppose $h \in {}^1J \cap ({}^1J)^g$ so

$$(\kappa \otimes \xi')(h) \circ \sum_j S_j \otimes T_j = \sum_j S_j \otimes T_j \circ (\kappa \otimes \xi)^g(h) \quad (5.3)$$

Since ${}^1J \subset \text{Ker}(\xi) \cap \text{Ker}(\xi')$, then we have

$$\sum_j (\kappa(h) \circ S_j - S_j \circ \kappa^g(h)) \otimes T_j = 0$$

The set $\{T_j\}$ is linearly independent so

$$\kappa(h) \circ S_j = S_j \circ \kappa^g(h), \quad \text{for each } j.$$

Thus $S_j \in \mathcal{I}_g(\kappa|_{{}^1J}) = \mathcal{I}_g(\eta)$ for all j . Now we have

$$\mathcal{I}_G(\kappa) = J(G^1 \times G^2)J = {}^1J(G^1 \times G^2){}^1J = \mathcal{I}_G(\eta)$$

and by Lemma 5.3.53, we have $\dim_{\mathbb{C}}(\mathcal{I}_g(\eta)) = 1$. Let $S \in \mathcal{I}_g(\kappa)$ then $S_j = \lambda_j S$ for some $\lambda_j \in \mathbb{C}$, so

$$\phi = \sum_j \lambda_j S \otimes T_j = \sum_j S \otimes (\lambda_j T_j) = S \otimes \sum_j \lambda_j T_j = S \otimes T,$$

where $S \in \mathcal{I}_g(\kappa)$ and $T = \sum_j \lambda_j T_j \in \text{End}_{\mathbb{C}}(V_2)$.

Now let $h \in J \cap J^g$. We rewrite the equation (5.2) as follows

$$(\kappa(h) \circ S - S \circ \kappa^g(h)) \otimes (\xi'(h) \circ T - T \circ \xi^g(h)) = 0.$$

Since $S \in \mathcal{I}_g(\kappa)$, then we get

$$\xi'(h) \circ T = T \circ \xi^g(h)$$

so $T \in \mathcal{I}_g(\xi)$.

Now fix $S \in \mathcal{I}_g(\kappa)$ the linear map

$$\begin{aligned} \mathcal{I}_g(\xi) &\longrightarrow \mathcal{I}_g(\kappa \otimes \xi) \\ T &\longmapsto S \otimes T \end{aligned}$$

is well defined and a bijection. \square

Theorem 5.3.57. *The representation $\text{Ind}_J^G \kappa \otimes \eta$ is irreducible and supercuspidal and equivalent to π .*

Proof. See [2, Proposition 5.7]. \square

Corollary 5.3.58. *The following map:*

$$\left\{ \begin{array}{l} \text{irreducible cuspidal} \\ \text{representations of } J/{}^1J \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{irreducible supercuspidal} \\ \text{representations containing } \psi_{0 \oplus b_2} \end{array} \right\}$$

given by

$$\xi \mapsto \text{Ind}_J^G \kappa \otimes \xi$$

is a bijection.

The following diagram summarizes the construction of π :

$$\begin{array}{ccc}
\text{c-Ind}_J^G \kappa \otimes \xi & & G \\
\downarrow & & \downarrow \\
\eta & & J \\
\downarrow & & \downarrow \\
\eta\theta_{0\oplus b_2} & & {}^1J \\
\downarrow & & \downarrow \\
\theta_{0\oplus b_2} & & {}^1H \\
\downarrow & & \downarrow \\
\psi_{0\oplus b_2} & & {}^1H_1
\end{array}$$

Corollary 5.3.58 implies that the number of irreducible supercuspidal representations of G which contain the character $\psi_{0\oplus b_2}$ of 1H_1 is equal to the number of irreducible cuspidal representations of $J/{}^1J$. Now consider the quotient

$$J/{}^1J \cong U(1, 1)(k_F/k_{F_0}) \times U(1)(k_F/k_0). \quad (5.4)$$

The number of irreducible cuspidal representations of $J/{}^1J$ can be found in [10, §5.6]. The number is as follows: the number of irreducible cuspidal representations of $U(1, 1)(k_F/k_{F_0})$ is $q(q+1)/2$ and the irreducible cuspids for $U(1)(k_F/k_0)$ is $(q+1)$. Multiplying these numbers we get the number of irreducible cuspids of $J/{}^1J$ which is

$$|\mathcal{R}(\Lambda', \theta_{0\oplus b_2})| = \frac{1}{2}q(q+1)^2.$$

Let $s_6^d(0, k)$ denote the number of irreducible supercuspidal representations of G of level k which contains the character ${}^0H_1(\Lambda')$. Then

Theorem 5.3.59. *We have*

$$s_6^d(0, k) = \frac{1}{2}q(q+1)(q^2-1)q^{k-1}.$$

Lemma 5.3.60. *The formal degree of π is*

$$d(\pi) = \begin{cases} (q-1)(q^2-q+1)q^{\frac{9}{2}k-\gamma-\frac{5}{2}} & \text{if } e(\Lambda) = 1 \text{ and } n \text{ odd} \\ (q-1)(q^2-q+1)q^{\frac{9}{2}k-\gamma-5} & \text{if } e(\Lambda) = 1 \text{ and } n \text{ even} \\ (q-1)(q^2-q+1)q^{2k-\lceil \frac{t-1}{2} \rceil -2} & \text{if } e(\Lambda) = 2 \end{cases}$$

where $\gamma = \lceil (t-1)/2 \rceil + \lceil (k-1)/2 \rceil$.

Proof. We have $d(\pi) = \dim(\kappa \otimes \eta)/\mu(J)$ where $\dim(\kappa \otimes \eta) = \dim(\kappa) \cdot \dim(\eta)$. The dimension of ξ is $q-1$, by [10, §6]. The dimension of κ is $[{}^1J : {}^1H]^{1/2}$, where ${}^1J/{}^1H$ is isomorphic to

$$\begin{pmatrix} 1 & \mathfrak{P}_{12}^{[\frac{n+1}{2}]}(\Lambda')/\mathfrak{P}_{12}^{[\frac{n}{2}]+1}(\Lambda') \\ \mathfrak{P}_{21}^{[\frac{n+1}{2}]}(\Lambda')/\mathfrak{P}_{21}^{[\frac{n}{2}]+1}(\Lambda') & 1 \end{pmatrix}$$

Thus,

$$\dim(\kappa) = \begin{cases} 1 & \text{if } e = 1 \text{ and } n \text{ is odd} \\ q^2 & \text{if } e = 1 \text{ and } n \text{ is even} \\ q & \text{if } e = 2 \end{cases}$$

Now we compute $\mu(J)^{-1}$. When $U(V^1) = U(2)(F/F_0)$, then $e(\Lambda') = 2$ and when $U(V^1) = U(1,1)(F/F_0)$ then $e(\Lambda')$ is either 1 or 2.

$$\mu(J)^{-1} = \mu\left(P^{[\frac{n+1}{2}]}(\Lambda')\right)^{-1} \cdot [J : {}^1J]^{-1} \cdot \left[{}^1J : P^{[\frac{n+1}{2}]}(\Lambda')\right]^{-1}$$

By (5.4), the quotient $J/{}^1J$ has size $q(q+1)^2(q^2-1)$. The quotient ${}^1J/P^{[\frac{n+1}{2}]}(\Lambda')$ is isomorphic to

$$(1 + \mathcal{P}_{E_1})^1 / \left(1 + \mathcal{P}_{E_1}^{[\frac{t-1}{2}]+1}\right)^1 \times (1 + \mathcal{P}_F)^1 / \left(1 + \mathcal{P}_F^{[\frac{k-1}{2}]}(\Lambda')\right)^1,$$

so it has size q^γ . Finally, the value of $\mu(P^{[(n+1)/2]}(\Lambda'))^{-1}$ is given by Proposition 5.1.2 which is $q^3(q+1)(q^2-1)(q^3+1)q^{9([\frac{(n+1)/2]-1})}$, if $e(\Lambda') = 1$ and $(q+1)(q^2-1)(q^3+1)q^{4([\frac{(n+1)/2]-1}]+[\frac{(n-1)/4}]}$, if $e(\Lambda') = 2$. \square

Computing $S^{sp}(G, k)$, $s^*(t)$ and $S^*(t)$:

Let $S^{sp}(G, k)$ be the number of the irreducible supercuspidal representations of G of level $k = n/e \in \mathbb{Z}$ which contains a split skew stratum $[\Lambda, n, n-1, b]$, but not totally split. Denote by $s^*(t)$, the number of irreducible supercuspidal representations of G containing the character $\psi_{b'_1 \oplus b_2}$ of ${}^rH_1(\Lambda')$ with $t = r/e$. Let $S^*(t)$ denotes the number of irreducible supercuspidal representations of G containing the character $\psi_{b'_1 \oplus b_2}$ of ${}^rH_1(\Lambda')$ with $r/e < t$.

First we summarize the number of the cases (a),(b),(c) and (d) by the following table:

	$t \in \mathbb{Z}$	$t \notin \mathbb{Z}$
$s_d(0)$	$\frac{1}{2}(q+1)(q^2-1)q^k$	-
$s_b(t)$	$(q-1)f_b(q)q^{2t}q^k$	0
$s_c(t)$	0	$(q-1)f_c(t)q^{2t}q^k$
$s_a^{es}(t)$	$(q-1)S^*(t)$	0

where $f_b(q) = q^{-2}(q+1)^3$ and $f_c(q) = q^{-1}(q+1)^2$.

The number $s^*(t)$ is

$$s^*(t) = \begin{cases} (q-1)f_c(q)q^{2t}q^k + (q-1)S^*(t) & \text{if } t \in \mathbb{Z} \\ (q-1)f_b(q)q^{2t}q^k & \text{if } t \notin \mathbb{Z} \\ \frac{1}{2}(q+1)(q^2-1)q^k & \text{if } t = 0 \end{cases}$$

Suppose $t \in \mathbb{Z}$, then

$$\begin{aligned} S^*(t+1) &= s^*(t + \frac{1}{2}) + s^*(t) + S^*(t) \\ &= (q-1)f_b(q)q^k q^{2t+1} + (q-1)f_c(q)q^k q^{2t} + qS^*(t) \\ &= q^k [(q-1)(qf_b(q) + f_c(q))q^{2t}] + qS^*(t). \end{aligned}$$

Now

$$\begin{aligned} S^*(t+1) &= q^k [(q-1)(qf_b(q) + f_c(q))q^{2t}] + qS^*(t) \\ &= q^k [(q-1)(qf_b(q) + f_c(q))(q^{2t} + q^{2t-1})] + q^2 S^*(t-1) \\ &\quad \vdots \\ &= q^k [(q-1)(qf_b(q) + f_c(q))(q^{2t} + q^{2t-1} + \dots + q^{t+1})] + q^{t+1} S^*(0) \end{aligned}$$

Thus

$$S^*(t+1) = q^k q^{t+1} [(qf_b(q) + f_c(q))(q^t - 1) + \frac{1}{2}(q+1)(q^2 - 1)]$$

We have computed all irreducible split supercuspidal representations of G of level k and the number is, in fact, the sum of $s^*(t)$ over every possible t which is between 0 and k multiply by the number of the twists which is q , so $s^{sp}(G, k) = qS^*(k)$ and therefore,

$$s^{sp}(G, k) = q^{3k}(qf_b(q) + f_c(q)) + q^{2k+1} \left(\frac{1}{2}(q+1)(q^2 - 1) - (qf_b(q) + f_c(q)) \right)$$

5.4 Computing $s(G, k)$ and $S(G, k)$

We start this section by summarizing the number of irreducible supercuspidal representations of G of level k that we obtained in previous sections:

	$k \in \mathbb{Z}$	$k \notin \mathbb{Z}$
$s_a(G, k), k = 0$	$(q+1)(q^2+1)$	-
$s_1(G, k)$	$\frac{1}{3}q(q^2-1)(q+1)^3q^{3(k-1)}$	0
$s_2(G, k)$	0	$(q^2-1)f_2(q)q^{3k}$
$s_3(G, k)$	0	$(q^2-1)q^{3k-1}$
$s^{ts}(G, k)$	$\frac{1}{6}q(q-1)(q-2)(q+1)^3q^{3(k-1)}$	0
$s^{sp}(G, k)$	$f_1^{sp}(q)q^{3k} + f_2^{sp}(q)q^{2k}$	0
$s^{es}(G, k)$	$(q-1)S(G, k)$	0

where $f_1^{sp}(q) = qf_b(q) + f_c(q)$, $f_2^{sp}(q) = \left(\frac{1}{2}(q+1)(q^2-1) - (qf_b(q) + f_c(q)) \right)$ and

$$f_2(q) = \begin{cases} 2(q+1)q^{-\frac{3}{2}} & \text{if } k \equiv 1 \pmod{4} \\ 2(q+1)q^{-\frac{5}{2}} & \text{if } k \equiv 3 \pmod{4} \end{cases}$$

Put

$$\begin{aligned} f_1(q) &= \frac{1}{3}q^{-2}(q+1)^3 \\ f_3(q) &= q^{-1} \\ f^{ts}(q) &= \frac{1}{6}q^{-2}(q-2)(q+1)^2 \end{aligned}$$

Now we have

$$s(G, k) = \begin{cases} (q+1)(q^2+1) & \text{if } k = 0 \\ s_1(G, k) + s^{ts}(G, k) + s^{es}(G, k) + s^{sp}(G, k) & \text{if } k \in \mathbb{Z} \\ s_2(G, k) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \\ s_3(G, k) & \text{if } k \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z} \end{cases}$$

If $k \in \mathbb{Z}$, then

$$\begin{aligned} S(G, k+1) &= s_3(G, k + \frac{2}{3}) + s_2(G, k + \frac{1}{2}) + s_3(G, k + \frac{1}{3}) + s(G, k) + S(G, k) \\ &= s_3(G, k + \frac{2}{3}) + s_2(G, k + \frac{1}{2}) + s_3(G, k + \frac{1}{3}) + s_1(G, k) + s^{sp}(G, k) \\ &+ s^{es}(G, k) + s^{ts}(G, k) + S(G, k) \\ &= (q^2 - 1)((q+1) + f_2(q)q^{\frac{3}{2}} + f_1(q) + f^{ts}(q))q^{3k} + s^{sp}(G, k) + qS(G, k), \end{aligned}$$

Now put $f(q) = (q+1) + f_2(q)q^{\frac{3}{2}} + f_1(q) + f^{ts}(q)$, then

$$\begin{aligned} S(G, k+1) &= (q^2 - 1)f(q)q^{3k} + qS^*(k) + qS(G, k) \\ &= (q^2 - 1)f(q)(q^{3k} + q^{3(k-1)+1}) + (s^{sp}(G, k) + s^{sp}(G, k-1)) + q^2S(G, k-1) \\ &\quad \vdots \\ &= (q^2 - 1)f(q)(q^{3k} + q^{3(k-1)+1} + \dots + q^{k+2}) + \sum_{i=1}^k s^{sp}(G, i) + q^{k+1}S(G, 0), \end{aligned}$$

Thus we deduce the following

$$\boxed{S(G, k+1) = f(q)q^{k+2}(q^{k+1} - 1) + f_1^{sp}(q)q^3 \frac{q^{3(k+1)} - 1}{q^3 - 1} + f_2^{sp}(q)q^2 \frac{q^{2(k+1)} - 1}{q^2 - 1} + q^{k+1}(q+1)(q^2+1)}$$

Bibliography

- [1] Alexandru Ioan Badulescu. Orthogonalité des caractères pour GL_n sur un corps local de caractéristique non nulle. *Manuscripta Math.*, 101(1):49–70, 2000.
- [2] Laure Blasco. Description du dual admissible de $U(2,1)(F)$ par la théorie des types de C. Bushnell et P. Kutzko. *Manuscripta Math.*, 107(2):151–186, 2002.
- [3] Colin J. Bushnell and Albrecht Fröhlich. *Gauss sums and p -adic division algebras*, volume 987 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.
- [4] Colin J. Bushnell and Guy Henniart. *The local Langlands conjecture for $GL(2)$* , volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
- [5] Colin J. Bushnell and Guy Henniart. Counting the discrete series for $GL(n)$. *Bull. Lond. Math. Soc.*, 39(1):133–137, 2007.
- [6] Colin J. Bushnell and Philip C. Kutzko. *The admissible dual of $GL(N)$ via compact open subgroups*, volume 129 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.
- [7] Colin J. Bushnell and Philip C. Kutzko. Semisimple types in GL_n . *Compositio Math.*, 119(1):53–97, 1999.
- [8] H. Carayol. Représentations cuspidales du groupe linéaire. *Ann. Sci. École Norm. Sup. (4)*, 17(2):191–225, 1984.
- [9] P. M. Cohn. *Algebra. Vol. 3*. John Wiley & Sons Ltd., Chichester, second edition, 1991.
- [10] Veikko Ennola. On the characters of the finite unitary groups. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 323:35, 1963.
- [11] Roger Godement and Hervé Jacquet. *Zeta functions of simple algebras*. Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin, 1972.
- [12] J. A. Green. The characters of the finite general linear groups. *Trans. Amer. Math. Soc.*, 80:402–447, 1955.

- [13] Harish-Chandra. A submersion principle and its applications. In *Geometry and analysis*, pages 95–102. Indian Acad. Sci., Bangalore, 1980.
- [14] Roger E. Howe. Tamely ramified supercuspidal representations of GL_n . *Pacific J. Math.*, 73(2):437–460, 1977.
- [15] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.
- [16] David Carl Jabon. *The supercuspidal representations of $U(2,1)$ and $GSP(4)$ via Hecke algebra isomorphisms*. ProQuest LLC, Ann Arbor, MI, 1989. Thesis (Ph.D.)—The University of Chicago.
- [17] Hervé Jacquet. Principal L -functions of the linear group. In *Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 63–86. Amer. Math. Soc., Providence, R.I., 1979.
- [18] Helmut Koch. Eisensteinsche Polynomfolgen und Arithmetik in Divisionalgebren über lokalen Körpern. *Math. Nachr.*, 104:229–251, 1981.
- [19] Alberto Minguez. Fonctions zeta l -modulaires. *Nagoya Math. J.*, 2000.
- [20] Lawrence Morris. Some tamely ramified supercuspidal representations of symplectic groups. *Proc. London Math. Soc. (3)*, 63(3):519–551, 1991.
- [21] Lawrence Morris. Tamely ramified supercuspidal representations of classical groups. I. Filtrations. *Ann. Sci. École Norm. Sup. (4)*, 24(6):705–738, 1991.
- [22] Lawrence Morris. Tamely ramified supercuspidal representations of classical groups. II. Representation theory. *Ann. Sci. École Norm. Sup. (4)*, 25(3):233–274, 1992.
- [23] Lawrence Morris. Level zero \mathbf{G} -types. *Compositio Math.*, 118(2):135–157, 1999.
- [24] Allen Moy. Local constants and the tame Langlands correspondence. *Amer. J. Math.*, 108(4):863–930, 1986.
- [25] Allen Moy and Gopal Prasad. Unrefined minimal K -types for p -adic groups. *Invent. Math.*, 116(1-3):393–408, 1994.
- [26] Allen Moy and Gopal Prasad. Jacquet functors and unrefined minimal k -types. *Commentarii Mathematici Helvetici*, 71:98–121, 1996.
- [27] Fiona Murnaghan. *Representations of reductive p -adic groups*. <http://www.math.toronto.edu/murnaghan/course.html>, 2009.
- [28] Yu J.-K. Pan, S.-Y. Unrefined minimal k -types for p -adic classical groups. *Prepublication*.
- [29] Dipendra Prasad and A. Raghuram. *Representation theory of $GL(n)$ over non-Archimedean local fields*. www.math.okstate.edu/araghur/ictp.pdf.

- [30] Jonathan D. Rogawski. Representations of $GL(n)$ and division algebras over a p -adic field. *Duke Math. J.*, 50(1):161–196, 1983.
- [31] Shaun Stevens. Types and representations of p -adic symplectic groups. *Thesis*, 1998.
- [32] Shaun Stevens. Double coset decompositions and intertwining. *Manuscripta Math.*, 106(3):349–364, 2001.
- [33] Shaun Stevens. Intertwining and supercuspidal types for p -adic classical groups. *Proc. London Math. Soc. (3)*, 83(1):120–140, 2001.
- [34] Shaun Stevens. Semisimple characters for p -adic classical groups. *Duke Math. J.*, 127(1):123–173, 2005.
- [35] Marie-France Vignéras. *Représentations l -modulaires d'un groupe réductif p -adique avec $l \neq p$* , volume 137 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1996.
- [36] G. E. Wall. On the conjugacy classes in the unitary, symplectic and orthogonal groups. *J. Austral. Math. Soc.*, 3:1–62, 1963.
- [37] André Weil. *Basic number theory*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the second (1973) edition.
- [38] A. V. Zelevinsky. Induced representations of reductive p -adic groups. II. On irreducible representations of $GL(n)$. *Ann. Sci. École Norm. Sup. (4)*, 13(2):165–210, 1980.