Games **2010**, *1*, 159-167; doi:10.3390/g1020159

OPEN ACCESS
games
ISSN 2073-4336
www.mdpi.com/journal/games

Article

Balanced Weights and Three-Sided Coalition Formation

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Received: 5 May 2010; in revised form: 10 June 2010 / Accepted: 15 June 2010 / Published: 25 June 2010

Abstract: We consider three-sided coalition formation problems when each agent is concerned about his local status as measured by his relative rank position within the group of his own type and about his global status as measured by the weighted sum of the average rankings of the other types of groups. We show that a core stable coalition structure always exists, provided that the corresponding weights are balanced and each agent perceives the two types of status as being substitutable.

Keywords: core; hedonic games; three-sided matching

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1. Introduction

The dependence of an agent's utility on the identity of the members of his or her coalition was recognized in the seminal paper of [1], and formally introduced as a hedonic coalition formation game by [2,3]. The model consists of two components, namely, a finite set of players and a preference relation for each player defined over the coalitions that a player may belong to. The outcome of such a game is a partition of the player set into coalitions called a coalition structure.

In this paper we extend the hedonic coalition formation model by distinguishing between three different types of players and allowing for agents' preferences to be type-dependent. Our model can also be seen as an extension of the three-sided matching model in which agents' preferences are defined over subsets of players from all possible players' types. Within this framework, we study the existence of coalition structures that are immune against coalitional deviations.

As it is well known (cf. [4–9]), one needs strong restrictions on agents' preferences even for the standard three-sided matching model to assure that a stable three-sided matching exists. Therefore, it is not surprising that in the more general game we consider, suitable preference restrictions are needed to show the existence of a core stable coalition structure. In particular, we assume players' preferences to be based on the substitutability between local and global status that an agent attains in the corresponding coalitions (cf. [10,11]). In this framework, we show that if global status is weighted in a balanced way, then a core stable coalition structure always exists.

2. Notation and Definitions

Let N^a, N^b, and N^c be three disjoint and finite sets of agents of type a, b, and c, respectively. For each player $i \in N := N^a \cup N^b \cup N^c$ we denote by $\mathcal{N}_i = \{X \subseteq N \mid i \in X\}$ the collection of all coalitions containing i. A partition π of N is called a coalition structure. For each coalition structure π and each player $i \in N$, we denote by $\pi(i)$ the coalition in π containing player i, *i.e.*, $\pi(i) \in \pi$ and $i \in \pi(i)$. Further, we assume that each player $i \in N$ is endowed with a preference $\succeq_i$ over $\mathcal{N}_i$, *i.e.*, a binary relation over $\mathcal{N}_i$ which is reflexive, complete, and transitive. Denote by $\succ_i$ and $\sim_i$ the strict and indifference relation associated with $\succeq_i$ and by $\succeq := (\succeq_1, \succeq_2, \ldots, \succeq_n)$ a profile of preferences $\succeq_i$ for all $i \in N$. A player's preference relation over coalitions canonically induces a preference relation over coalition structures in the following way: For any two coalition structures π and π', player i weakly prefers π to π' if and only if he weakly prefers "his" coalition in π to the one in π', *i.e.*, $\pi \succeq_i' \pi$ if and only if $\pi(i) \succeq_i' \pi(i)$. Hence, we assume that players' preferences over coalition structures are purely hedonic. That means, they are completely characterized by their preferences over coalitions. Finally, a *hedonic game* $(N, \succeq)$ is a pair consisting of the set of players and a preference profile. Given a hedonic game $(N, \succeq)$, a coalition structure π of N is *core stable* if there does not exist a nonempty coalition X such that $X \succ_i \pi(i)$ holds for each $i \in X$.

3. Balanced Weights and Status-Based Preferences

Let $|N^a| = n_a$, $|N^b| = n_b$, and $|N^c| = n_c$ with $n_a \leq n_b \leq n_c$. We assume that every agent $i^d \in N^a \cup N^b \cup N^c$ is assigned a rank r_{i^d} which induces a unique ordering of agents within each type such that any two consecutive players' ranks differ by one unit. The highest ranked agents of each type all have rank n_c, and the lowest ranked agents in the sets N^a, N^b, and N^c have ranks $n_c - n_a + 1$, $n_c - n_b + 1$, and 1, respectively. We take the rank of the empty set to be equal to zero. Underlying the ranking of agents may be a distribution of abilities or material endowment. By letting the highest ranked agents of all types have equal ranks we rule out scale effects. The assumption that the lowest ranked a-agent may have a higher rank than the lowest ranked b- and c-agent reflects the scarcity of a-type individuals relative to those of b- and c-types.

In addition to his preference $\succeq_{i^a}$, each agent $i^a \in N^a$ is characterized by a non-negative weight vector $\left(w_{i^a}^b, w_{i^a}^c\right)$. The corresponding vectors for $i^b \in N^b$ and $i^c \in N^c$ are $\left(w_{i^b}^a, w_{i^b}^c\right)$ and $\left(w_{i^c}^a, w_{i^c}^b\right)$, respectively. We call these weights *balanced* if the following two conditions hold:

(1) for each $i^a \in N^a$, $i^b \in N^b$, and $i^c \in N^c$: $w_{i^a}^b + w_{i^a}^c = w_{i^b}^a + w_{i^b}^c = w_{i^c}^a + w_{i^c}^b = 1$.

(2) for each $d \in \{a, b, c\}$, and all $i^{d'} \in N^{d'}$ and $i^{d''} \in N^{d''}$ with $d', d'' \in \{a, b, c\} \setminus \{d\}$, $d' \neq d''$: $w^d_{i^{d'}} + w^d_{i^{d''}} = 1$.

One possible interpretation of such a restriction on the weights is as follows. Each agent has exactly one unit of communication time that is efficiently allocated between the other two types of groups, as reflected in (1). Similarly, condition (2) imposes that the communication time of agents of two different types with agents of the third type sums up to one. It follows from the balancedness condition that assigning a value to some $w^d_{i^{d'}}$, $d \neq d'$, determines also the rest of the weights. For instance, if we fix $w^b_{i^a} \in [0, 1]$, then (1) and (2) imply $w^b_{i^a} = w^c_{i^b} = w^a_{i^c}$ and $1 - w^b_{i^a} = w^c_{i^a} = w^b_{i^c} = w^a_{i^b}$.[1] Thus, this condition inherits and strengthens the spirit of cyclicity from the standard three-sided matching model (cf. [5]).

By using the above setup, let us now define agents' preferences in a hedonic game with status-based preferences and balanced weights. For $d \in \{a, b, c\}$, each $i^d \in N^d$ and each $X, Y \in \mathcal{N}_{i^d}$, $X \succeq_{i^d} Y$ if and only if

$$r_{i^d} - \frac{\sum_{i \in X \cap N^d} r_i}{|X \cap N^d|} + \sum_{d' \in \{a,b,c\} \setminus \{d\}} w^{d'}_{i^d} \cdot \frac{\sum_{i \in X \cap N^{d'}} r_i}{|X \cap N^{d'}|} \geq r_{i^d} - \frac{\sum_{i \in Y \cap N^d} r_i}{|Y \cap N^d|} + \sum_{d' \in \{a,b,c\} \setminus \{d\}} w^{d'}_{i^d} \cdot \frac{\sum_{i \in Y \cap N^{d'}} r_i}{|Y \cap N^{d'}|}$$

Thus, in such a hedonic game, each agent is concerned about his local status as measured by his relative rank position within the group of his own type and about his global status as measured by the weighted sum of the average rankings of the other types of groups (cf. [11]). Moreover, each agent perceives the two types of status as being substitutable.

Theorem 1 *Let $(N, \succeq)$ be a hedonic game with status-based preferences and balanced weights. Then the set of core stable coalition structures is non-empty.*

Consider the coalition structure $\pi = \{N\}$. We will show that π is core stable.

Let $Y \subseteq N$ and $d \in \{a, b, c\}$. In what follows we will denote by $r^d(Y)$ ($r_d(Y)$) the rank of the highest (lowest) ranked d-agent in Y.

First, for all $d \in \{a, b, c\}$ we have by construction that

$$\frac{\sum_{i^d \in N^d} r_{i^d}}{|N^d|} = \frac{r_d(N^d) + r^d\left(N^d\right)}{2}$$

In addition, since the highest ranked agents in all types have the same rank and the weights are balanced, we have for the status of agent $i^d \in N^d$ in the coalition structure $\pi = \{N\}$ that

$$\begin{aligned}
& r_{i^d} - \frac{\sum_{i \in N^d} r_i}{|N^d|} + \sum_{d' \in \{a,b,c\} \setminus \{d\}} w^{d'}_{i^d} \cdot \frac{\sum_{i \in N^{d'}} r_i}{|N^{d'}|} \\
= \; & r_{i^d} - \frac{r_d(N^d) + r^d\left(N^d\right)}{2} + \sum_{d' \in \{a,b,c\} \setminus \{d\}} w^{d'}_{i^d} \cdot \frac{r_{d'}(N^{d'}) + r^{d'}\left(N^{d'}\right)}{2} \\
= \; & r_{i^d} - \frac{r_d(N^d)}{2} - \frac{r^d\left(N^d\right)}{2} + \sum_{d' \in \{a,b,c\} \setminus \{d\}} w^{d'}_{i^d} \cdot \frac{r_{d'}(N^{d'})}{2} + \sum_{d' \in \{a,b,c\} \setminus \{d\}} w^{d'}_{i^d} \cdot \frac{r^{d'}\left(N^{d'}\right)}{2} \\
= \; & r_{i^d} - \frac{r_d(N^d)}{2} + \sum_{d' \in \{a,b,c\} \setminus \{d\}} w^{d'}_{i^d} \cdot \frac{r_{d'}(N^{d'})}{2}
\end{aligned}$$

[1] We thank one anonymous referee for pointing this out.

Next, we will show that there is no coalition $X \subseteq N$ that blocks the constructed coalition structure π. Suppose, on the contrary, that such a coalition exists.

First, suppose that $X \subseteq N^d$ for some $d \in \{a, b, c\}$. Let j be the lowest ranked member of X. This agent's status in X is given by

$$r_j - \frac{\sum_{k \in X} r_k}{|X|} \leq 0$$

The same agent's status in π, however, is given by

$$r_j - \frac{r_d(N^d)}{2} + \sum_{d' \in \{a,b,c\} \setminus \{d\}} w_{i^d}^{d'} \cdot \frac{r_{d'}(N^{d'})}{2} > 0$$

This establishes a contradiction to X blocking π.

Suppose next that agents of all the types are contained in X and let $X \cap N^a = A$, $X \cap N^b = B$, and $X \cap N^c = C$. Then, for $i^a \in A$, $i^b \in B$, and $i^c \in C$ one should have

$$\begin{aligned}
r_{i^a} - \frac{\sum_{i \in A} r_i}{|A|} + w_{i^a}^b \cdot \frac{\sum_{j \in B} r_j}{|B|} + w_{i^a}^c \cdot \frac{\sum_{k \in C} r_k}{|C|} &> r_{i^a} - \frac{r_a(N^a)}{2} + w_{i^a}^b \cdot \frac{r_b(N^b)}{2} + w_{i^a}^c \cdot \frac{r_c(N^c)}{2}, \\
r_{i^b} - \frac{\sum_{j \in B} r_j}{|B|} + w_{i^b}^a \cdot \frac{\sum_{i \in A} r_i}{|A|} + w_{i^b}^c \cdot \frac{\sum_{k \in C} r_k}{|C|} &> r_{i^b} - \frac{r_b(N^b)}{2} + w_{i^b}^a \cdot \frac{r_a(N^a)}{2} + w_{i^b}^c \cdot \frac{r_c(N^c)}{2}, \\
r_{i^c} - \frac{\sum_{k \in C} r_k}{|C|} + w_{i^c}^a \cdot \frac{\sum_{i \in A} r_i}{|A|} + w_{i^c}^b \cdot \frac{\sum_{j \in B} r_j}{|B|} &> r_{i^c} - \frac{r_c(N^c)}{2} + w_{i^c}^a \cdot \frac{r_a(N^a)}{2} + w_{i^c}^b \cdot \frac{r_b(N^b)}{2}
\end{aligned}$$

Summing up, we get

$$\begin{aligned}
\frac{\sum_{i \in A} r_i}{|A|} + \frac{\sum_{j \in B} r_j}{|B|} + \frac{\sum_{k \in C} r_k}{|C|} \quad < \quad & (1 - w_{i^b}^a - w_{i^c}^a) \cdot \frac{r_a(N^a)}{2} + \left(1 - w_{i^a}^b - w_{i^c}^b\right) \cdot \frac{r_b(N^b)}{2} \\
& + (1 - w_{i^a}^c - w_{i^b}^c) \cdot \frac{r_c(N^c)}{2} \\
& + (w_{i^b}^a + w_{i^c}^a) \cdot \frac{\sum_{i \in A} r_i}{|A|} + \left(w_{i^a}^b + w_{i^c}^b\right) \cdot \frac{\sum_{j \in B} r_j}{|B|} \\
& + (w_{i^a}^c + w_{i^b}^c) \cdot \frac{\sum_{k \in C} r_k}{|C|}
\end{aligned}$$

Since the weights are balanced, we have

$$\frac{\sum_{i \in A} r_i}{|A|} + \frac{\sum_{j \in B} r_j}{|B|} + \frac{\sum_{k \in C} r_k}{|C|} < \frac{\sum_{i \in A} r_i}{|A|} + \frac{\sum_{j \in B} r_j}{|B|} + \frac{\sum_{k \in C} r_k}{|C|}$$

which is a contradiction.

Take finally the case where X contains two agents' types, say a and b, and let $X \cap N^a = A$ and $X \cap N^b = B$. As X is a blocking coalition, one should have for $i^a \in A$ and $i^b \in B$ that

$$\begin{aligned}
r_{i^a} - \frac{\sum_{i \in A} r_i}{|A|} + w_{i^a}^b \cdot \frac{\sum_{j \in B} r_j}{|B|} &> r_{i^a} - \frac{r_a(N^a)}{2} + w_{i^a}^b \cdot \frac{r_b(N^b)}{2} + w_{i^a}^c \cdot \frac{r_c(N^c)}{2}, \\
r_{i^b} - \frac{\sum_{j \in B} r_j}{|B|} + w_{i^b}^a \cdot \frac{\sum_{i \in A} r_i}{|A|} &> r_{i^b} - \frac{r_b(N^b)}{2} + w_{i^b}^a \cdot \frac{r_a(N^a)}{2} + w_{i^b}^c \cdot \frac{r_c(N^c)}{2}
\end{aligned}$$

Summing up, and applying the balancedness of the weights, we get

$$(1 - w_{i^b}^a) \cdot \frac{\sum_{i \in A} r_i}{|A|} + \left(1 - w_{i^a}^b\right) \cdot \frac{\sum_{j \in B} r_j}{|B|} < (1 - w_{i^b}^a) \cdot \frac{r_a(N^a)}{2} + \left(1 - w_{i^a}^b\right) \cdot \frac{r_b(N^b)}{2} - \frac{r_c(N^c)}{2}$$

Since $w^a_{i^b}, w^b_{i^a} \in [0,1]$, $\frac{\sum_{i\in A} r_i}{|A|} \geq r_a(N^a)$, and $\frac{\sum_{j\in B} r_j}{|B|} \geq r_b(N^b)$, we have again a contradiction. We conclude that X cannot block $\pi = \{N\}$.

Next we study in more detail the core stable outcomes when there is an equal number of a-, b-, and c-type of agents. For this, we need to introduce some new notation. Let $d \in \{a,b,c\}$. We denote by $r_{i^d}^{\pi\left(i^d\right)}$ the status that agent $i^d \in N^d$ attains in a coalition structure π, *i.e.*,

$$r_{i^d}^{\pi\left(i^d\right)} := r_{i^d} - \frac{\sum_{i\in\pi(i^d)\cap N^d} r_i}{|\pi(i^d)\cap N^d|} + \sum_{d'\in\{a,b,c\}\setminus\{d\}} w^{d'}_{i^d} \cdot \frac{\sum_{i\in\pi(i^d)\cap N^{d'}} r_i}{|\pi(i^d)\cap N^{d'}|}$$

Furthermore, we define the set $\Pi := \left\{\pi : r_i^{\pi(i)} \geq r_i \text{ for all } i \in N\right\}$ and note that $\Pi \neq \emptyset$ as, for instance, the coalition structure $\left\{\left\{i^a, i^b, i^c\right\}_{r_{i^a}=r_{i^b}=r_{i^c}}\right\}$ belongs to Π when $|N^a| = |N^b| = |N^c|$.

Theorem 2 *Let* $(N, \succeq)$ *be a hedonic game with status-based preferences and balanced weights, and* $|N^a| = |N^b| = |N^c|$. *If* $\pi \in \Pi$, *then* π *is core stable.*

Proof. Take $\pi \in \Pi$ and suppose that there is a coalition X that blocks π.

Consider first the case where X contains only one type of agents. Then, one should have

$$r_i - \frac{\sum_{j\in X} r_j}{|X|} > r_i^{\pi(i)} \geq r_i$$

which is a contradiction. Thus, X cannot be blocking π.

Suppose next that X contains two agents' types, say a and b (considering other two agents' types does not alter the proof argument), and let $X \cap N^a = A$ and $X \cap N^b = B$. For $i^a \in A$ and $i^b \in B$ one should then have

$$r_{i^a} - \frac{\sum_{i\in A} r_i}{|A|} + w^b_{i^a} \cdot \frac{\sum_{j\in B} r_j}{|B|} > r_{i^a}^{\pi(i^a)} \geq r_{i^a}, r_{i^b} - \frac{\sum_{j\in B} r_j}{|B|} + w^a_{i^b} \cdot \frac{\sum_{i\in A} r_i}{|A|} > r_{i^b}^{\pi\left(i^b\right)} \geq r_{i^b}$$

Summing up, we get

$$w^a_{i^b} \cdot \frac{\sum_{i\in A} r_i}{|A|} + w^b_{i^a} \cdot \frac{\sum_{j\in B} r_j}{|B|} > \frac{\sum_{i\in A} r_i}{|A|} + \frac{\sum_{j\in B} r_j}{|B|}$$

which is a contradiction since $w^a_{i^b}, w^b_{i^a} \in [0,1]$.

Take finally the case where agents of all the types are contained in X and let $X \cap N^a = A$, $X \cap N^b = B$, and $X \cap N^c = C$. Then, one should have

$$\begin{aligned} r_{i^a} - \tfrac{\sum_{i\in A} r_i}{|A|} + w^b_{i^a} \cdot \tfrac{\sum_{j\in B} r_j}{|B|} + w^c_{i^a} \cdot \tfrac{\sum_{k\in C} r_k}{|C|} &> r_{i^a}^{\pi(i^a)} \geq r_{i^a}, \\ r_{i^b} - \tfrac{\sum_{j\in B} r_j}{|B|} + w^a_{i^b} \cdot \tfrac{\sum_{i\in A} r_i}{|A|} + w^c_{i^b} \cdot \tfrac{\sum_{k\in C} r_k}{|C|} &> r_{i^b}^{\pi\left(i^b\right)} \geq r_{i^b}, \\ r_{i^c} - \tfrac{\sum_{k\in C} r_k}{|C|} + w^a_{i^c} \cdot \tfrac{\sum_{i\in A} r_i}{|A|} + w^b_{i^c} \cdot \tfrac{\sum_{j\in B} r_j}{|B|} &> r_{i^c}^{\pi(i^c)} \geq r_{i^c} \end{aligned}$$

Summing up, we get

$$\begin{aligned} &(w^a_{i^b} + w^a_{i^c}) \cdot \frac{\sum_{i\in A} r_i}{|A|} + \left(w^b_{i^a} + w^b_{i^c}\right) \cdot \frac{\sum_{j\in B} r_j}{|B|} + (w^c_{i^a} + w^c_{i^b}) \cdot \frac{\sum_{k\in C} r_k}{|C|} \\ > \; & \frac{\sum_{i\in A} r_i}{|A|} + \frac{\sum_{j\in B} r_j}{|B|} + \frac{\sum_{k\in C} r_k}{|C|} \end{aligned}$$

which is again a contradiction since the weights are balanced. We conclude that π is core stable.

Notice that the reverse implication of Theorem 2 does not hold as the following example shows.

Example 1 Consider $N^d = \{1^d, 2^d, 3^d\}$ for each $d \in \{a, b, c\}$. The agents are ranked as follows: $r_{1^d} = 1$, $r_{2^d} = 2$, and $r_{3^d} = 3$ for each $d \in \{a, b, c\}$. Let the weight vector be such that $w_{i^d}^{d'} = 0.5$ for all $i^d \in N^d$ with $d, d' \in \{a, b, c\}$, $d' \neq d$. Consider the coalition structure $\pi = \{\{1^a\}, \{2^a, 1^b, 1^c\}, \{3^a, 2^b, 3^b, 2^c, 3^c\}\}$. It is easy to compute the status of each agent in π: $r_{1^a}^{\pi(1^a)} = 0$, $r_{2^a}^{\pi(2^a)} = 1$, $r_{3^a}^{\pi(3^a)} = 2.5$, $r_{1^b}^{\pi(1^b)} = r_{1^c}^{\pi(1^c)} = 1.5$, $r_{2^b}^{\pi(2^b)} = r_{2^c}^{\pi(2^c)} = 2.25$, and $r_{3^b}^{\pi(3^b)} = r_{3^c}^{\pi(3^c)} = 3.25$. We will show that π is core stable.

As every agent's status in π is at least as high as when being alone, no agent can block π by himself. Moreover, it is clear from the proof of Theorem 1 that there cannot be a blocking coalition $X \subseteq N^d$ with $|X| \geq 2$ for any $d \in \{a, b, c\}$ (the status of the lowest ranked member of X is non-positive).

Furthermore, notice that $r_{i^d}^{\pi(i^d)} > r_{i^d}$ for all $i^d \in N^d$ with $d \in \{b, c\}$. Suppose, by contradiction, that there is a blocking coalition $X \subseteq N^b \cup N^c$ with $X \cap N^b = B \neq \emptyset$ and $X \cap N^c = C \neq \emptyset$. Therefore, there must be agents $i^b, i^c \in X$ for whom

$$\begin{aligned} r_{i^b} - \frac{\sum_{j \in B} r_j}{|B|} + 0.5 \cdot \frac{\sum_{k \in C} r_k}{|C|} &> r_{i^b}^{\pi(i^b)} > r_{i^b} \\ r_{i^c} - \frac{\sum_{k \in C} r_k}{|C|} + 0.5 \cdot \frac{\sum_{j \in B} r_j}{|B|} &> r_{i^c}^{\pi(i^c)} > r_{i^c} \end{aligned}$$

Summing up, we get $-0.5 \cdot \left(\frac{\sum_{j \in B} r_j}{|B|} + \frac{\sum_{k \in C} r_k}{|C|} \right) > 0$. This leads to a contradiction as a group's average rank is strictly positive.

Next, suppose that $X \subseteq N^a \cup N^b$ with $X \cap N^a = A \neq \emptyset$ and $X \cap N^b = B \neq \emptyset$. One can show that if π is blocked by X, then for an agent $i^b \in B$, it must hold that $r_{i^b} - \frac{\sum_{j \in B} r_j}{|B|} + 0.5 \cdot \frac{\sum_{i \in A} r_i}{|A|} > r_{i^b}^{\pi(i^b)} > r_{i^b}$, which implies that $0.5 \cdot \frac{\sum_{i \in A} r_i}{|A|} > \frac{\sum_{j \in B} r_j}{|B|}$. This is only satisfied in the case when $\frac{\sum_{j \in B} r_j}{|B|} = 1$ and $\frac{\sum_{i \in A} r_i}{|A|} \in \{2.5, 3\}$, *i.e.*, we have $A \in \{\{2^a, 3^a\}, \{3^a\}\}$. Notice that if $A = \{2^a, 3^a\}$, then agent 2^a has no incentive to participate in X as his ranking in this case would be $2 - 2.5 + 0.5 \cdot 1 = 1$ which equals his ranking in the partition π. Analogously, if $A = \{3^a\}$, then the ranking of 3^a in X would be $3 - 3 + 0.5 \cdot 1 = 0.5 < 2.5 = r_{3^a}^{\pi(3^a)}$, *i.e.*, agent 3^a has no incentive to participate in X either. Similarly, one can show that there is no coalition $X \subseteq N^a \cup N^c$ with $X \cap N^a \neq \emptyset$ and $X \cap N^c \neq \emptyset$ which blocks π.

Last, suppose that there is a blocking coalition X with $X \cap N^a = A \neq \emptyset$, $X \cap N^b = B \neq \emptyset$ and $X \cap N^b = C \neq \emptyset$. Consider an agent $i^b \in B$ and an agent $i^c \in C$. If π is blocked by X, this implies that

$$\begin{aligned} r_{i^b} - \frac{\sum_{j \in B} r_j}{|B|} + 0.5 \cdot \frac{\sum_{i \in A} r_i}{|A|} + 0.5 \cdot \frac{\sum_{k \in C} r_k}{|C|} &> r_{i^b}^{\pi(i^b)} \\ r_{i^c} - \frac{\sum_{k \in C} r_k}{|C|} + 0.5 \cdot \frac{\sum_{i \in A} r_i}{|A|} + 0.5 \cdot \frac{\sum_{j \in B} r_j}{|B|} &> r_{i^c}^{\pi(i^c)} \end{aligned}$$

Summing up, we get $\frac{\sum_{i\in A} r_i}{|A|} > 0.5\cdot\frac{\sum_{j\in B} r_j}{|B|} + 0.5\cdot\frac{\sum_{k\in C} r_k}{|C|} + (r_{i^b}^{\pi(i^b)} - r_{i^b}) + (r_{i^c}^{\pi(i^c)} - r_{i^c})$. As $\frac{\sum_{j\in B} r_j}{|B|} \geq 1$, $\frac{\sum_{k\in C} r_k}{|C|} \geq 1$, $r_{i^b}^{\pi(i^b)} - r_{i^b} \geq 0.25$, and $r_{i^c}^{\pi(i^c)} - r_{i^c} \geq 0.25$, this implies that $\frac{\sum_{i\in A} r_i}{|A|} > 1.5$.

Notice finally that, for X to block π, one should have for each $i^a \in A$ that

$$r_{i^a} - \frac{\sum_{i\in A} r_i}{|A|} + 0.5\cdot\frac{\sum_{j\in B} r_j}{|B|} + 0.5\cdot\frac{\sum_{k\in C} r_k}{|C|} > r_{i^a}^{\pi(i^a)}$$

Using $\frac{\sum_{i\in A} r_i}{|A|} > 0.5\cdot\frac{\sum_{j\in B} r_j}{|B|} + 0.5\cdot\frac{\sum_{k\in C} r_k}{|C|} + 0.5$, we get $r_{i^a} - 0.5 > r_{i^a}^{\pi(i^a)}$ for each $i^a \in A$. This implies that $3^a \notin A$. In addition, recall that $\frac{\sum_{i\in A} r_i}{|A|} > 1.5$. Since $3^a \notin A$, this implies that $1^a \notin A$, hence, $A = \{2^a\}$. Therefore, $1^b \notin B$ and $1^c \notin C$, which in turn implies that $B = \emptyset$ and $C = \emptyset$ as no agent among $2^b, 3^b, 2^c$ and 3^c can obtain a higher status.

Let us finally consider a hedonic game with status-based preferences and individual weights that are defined on coalitions and are, therefore, not balanced. As our last example shows, a core stable coalition structure may fail to exist in this case.

Example 2 Let $N^d = \{1^d, 2^d\}$ for $d \in \{a, b, c\}$. Let $r_{1^d} = 1$ and $r_{2^d} = 2$ for all $d \in \{a, b, c\}$. Consider the following non-balanced weight vector:

$$\begin{aligned}
1^a: &\ w_{1^a}^{\{1^c\}} = 1/2,\ w_{1^a}^{\{2^b\}} = 2/3,\ w_{1^a}^{\{1^c2^c\}} = w_{1^a}^{\{2^c\}} = 1,\ w_{1^a}^{\{1^b\}} = 3/2,\ w_{1^a}^{\{1^b2^b\}} = 4;\\
2^a: &\ w_{2^a}^{\{1^b\}} = 1/2,\ w_{2^a}^{\{2^b\}} = w_{2^a}^{\{1^c\}} = w_{2^a}^{\{1^c2^c\}} = 1,\ w_{2^a}^{\{2^c\}} = w_{2^a}^{\{1^b2^b\}} = 2;\\
1^b: &\ w_{1^b}^{\{2^a\}} = 1/2,\ w_{1^b}^{\{2^c\}} = 9/10,\ w_{1^b}^{\{1^a\}} = w_{1^b}^{\{1^c2^c\}} = 1,\ w_{1^b}^{\{1^c\}} = 3/2,\ w_{1^b}^{\{1^a2^a\}} = 3;\\
2^b: &\ w_{2^b}^{\{1^c\}} = w_{2^b}^{\{2^a\}} = 1/2,\ w_{2^b}^{\{1^a2^a\}} = w_{2^b}^{\{1^c2^c\}} = 1,\ w_{2^b}^{\{1^a\}} = w_{2^b}^{\{2^c\}} = 2;\\
1^c: &\ w_{1^c}^{\{1^a\}} = w_{1^c}^{\{2^b\}} = 0,\ w_{1^c}^{\{1^b\}} = w_{1^c}^{\{2^a\}} = 1/2,\ w_{1^c}^{\{1^a2^a\}} = w_{1^c}^{\{1^b2^b\}} = 1;\\
2^c: &\ w_{2^c}^{\{2^a\}} = w_{2^c}^{\{2^b\}} = w_{2^c}^{\{1^a2^a\}} = w_{2^c}^{\{1^b2^b\}} = 1,\ w_{2^c}^{\{1^b\}} = 3,\ w_{2^c}^{\{1^a\}} = 10/3
\end{aligned}$$

We will show that there is no core stable coalition structure.

First, consider coalition structure $\pi_1 = \{N^a \cup N^b \cup N^c\}$. The status of each player in the given coalition structure can be computed easily: $r_{1^a}^{\pi_1(1^a)} = 7$, $r_{2^a}^{\pi_1(2^a)} = 5$, $r_{1^b}^{\pi_1(1^b)} = 11/2$, $r_{2^b}^{\pi_1(2^b)} = 7/2$, $r_{1^c}^{\pi_1(1^c)} = 5/2$, and $r_{2^c}^{\pi_1(2^c)} = 7/2$. This coalition structure can be blocked by coalition $\{2^a, 2^b, 2^c\}$ in which the blocking players $2^a, 2^b, 2^c$ can obtain status of 6, 5, and 4, respectively, by forming a coalition.

Similarly, one can show that coalition structures $\{\{1^a, 2^a, 1^b, 1^c\}, \{2^b, 2^c\}\}$, $\{\{1^a, 2^a, 1^c\}, \{1^b, 2^b, 2^c\}\}$, $\{\{1^a, 2^a, 2^c\}, \{1^b, 2^b, 1^c\}\}$, $\{\{1^a, 1^b, 2^b\}, \{2^a, 1^c, 2^c\}\}$, $\{1^a, 1^b, 2^b, 1^c\}, \{2^a, 2^c\}\}$ and $\{\{2^a, 1^b, 2^b, 1^c\}, \{1^a, 2^c\}\}$ can be blocked by $\{2^a, 2^b, 2^c\}$.

Next, consider coalition structure $\pi_2 = \{\{1^a, 1^b, 1^c\}, \{2^a, 2^b, 2^c\}\}$. The status of players 1^a, 1^b, and 1^c in this coalition structure is 2, 5/2, and 1/2, respectively. The status of players 2^a, 2^b and 2^c is given above. Coalition structure π_2 is blocked by $1^a, 2^b$and 2^c who can form a coalition and obtain status of 10/3, 6, and 16/3.

Similarly, one can show that coalition structures $\{\{1^a, 2^a, 1^b, 1^c, 2^c\}, \{2^b\}\}$, $\{\{1^a, 2^b, 1^c\}, \{2^a, 1^b, 2^c\}$, $\{\{2^a, 1^b, 2^b, 1^c, 2^c\}, \{1^a\}\}$, $\{\{1^a, 1^b\}, \{2^a, 2^b, 1^c, 2^c\}\}$ and $\{\{1^a, 2^a, 1^b, 2^c\}, \{2^b, 1^c\}\}$ can be blocked by coalition $\{1^a, 2^b, 2^c\}$.

Next, consider coalition structure $\pi_3 = \{\{1^a, 2^b, 2^c\}, \{2^a, 1^b, 1^c\}\}$. The status of players 2^a, 1^b, and 1^c in this coalition structure is $3/2$, $5/2$, and $3/2$, respectively, and the status of the remainder of the players is given above. This coalition structure is blocked by coalition $\{1^a, 1^b, 2^c\}$ where the members obtain a higher status of $7/2$, $28/10$, and $19/3$.

Similarly, one can show that coalition structures $\{\{1^a, 2^a, 2^b, 1^c, 2^c\}, \{1^b\}\}$, $\{\{2^a, 1^b, 2^b, 2^c\}, \{1^a, 1^c\}\}$, $\{\{1^a, 2^b\}, \{2^a, 1^b, 1^c, 2^c\}\}$ and $\{\{1^a, 2^a, 2^b, 2^c\}, \{1^b, 1^c\}\}$ are blocked by $\{1^a, 1^b, 2^c\}$.

Next, consider coalition structure $\pi_4 = \{\{1^a, 1^b, 2^c\}, \{2^a, 2^b, 1^c\}\}$. Players' status is $r_{1^a}^{\pi_4(1^a)} = 7/2$, $r_{2^a}^{\pi_4(2^a)} = 3$, $r_{1^b}^{\pi_4(1^b)} = 28/10$, $r_{2^b}^{\pi_4(2^b)} = 3/2$, $r_{1^c}^{\pi_4(1^c)} = 1$, and $r_{2^c}^{\pi_4(2^c)} = 19/3$. This coalition structure is blocked by players $1^a, 2^a, 1^b, 2^b$ who can obtain status $11/2$, $7/2$, 4, and 2, respectively, by forming a coalition.

Similarly, one can show that coalition structures $\{\{1^a, 1^b, 1^c, 2^c\}, \{2^a, 2^b\}\}$ and $\{\{1^a, 2^a, 2^b\}, \{1^b, 1^c, 2^c\}\}$ are blocked by $\{1^a, 2^a, 1^b, 2^b\}$.

Next, consider coalition structure $\pi_5 = \{\{1^a, 2^a, 1^b, 2^b\}, \{1^c\}, \{2^c\}\}$. Note that $r_{1^c}^{\pi_5(1^c)} = r_{2^c}^{\pi_5(2^c)} = 0$. This coalition structure is blocked by the grand coalition, π_1.

Similarly, one can easily show that coalition structures $\{\{1^b, 2^b, 1^c, 2^c\}, \{1^a\}, \{2^a\}\}$, $\{\{1^a, 2^a, 2^b, 1^c\}, \{1^b, 2^c\}\}$ and $\{\{1^a, 2^a, 1^c, 2^c\}, \{1^b\}, \{2^b\}\}$ are blocked by the grand coalition.

Consider coalition structures $\pi_6 = \{\{1^a, 1^b, 2^b, 1^c, 2^c\}, \{2^a\}\}$ where $r_{1^a}^{\pi_6(1^a)} = 15/2$, $r_{2^a}^{\pi_6(2^a)} = 0$, $r_{1^b}^{\pi_6(1^b)} = 2$, $r_{2^b}^{\pi_6(2^b)} = 4$, $r_{1^c}^{\pi_6(1^c)} = 1$, and $r_{2^c}^{\pi_6(2^c)} = 16/3$. This coalition structure is blocked by $\{2^a, 1^b, 1^c\}$ whose members can obtain status of $3/2$, $5/2$ and $3/2$, respectively.

Similarly, we can show that coalition structure $\{\{1^a, 1^b, 2^b, 2^c\}, \{2^a, 1^c\}\}$ can be blocked by coalition $\{2^a, 1^b, 2^c\}$.

In all of the remainder of the coalition structures a blocking coalition can be found easily. There is either at least one player who forms a blocking coalition by himself, e.g., consider coalition structure $\{\{1^a, 2^b, 1^c, 2^c\}, \{2^a, 1^b\}\}$ where the status of player 1^c is $-1/2$; or there are at lest two players of distinct types who have status of 0 and by forming a coalition obtain strictly higher status, e.g., consider coalition structure $\{\{2^a, 2^b, 2^c\}, \{1^a\}, \{1^b\}, \{1^c\}\}$ where players 1^a, 1^b and 1^c have 0 status and can obtain strictly positive status by forming coalition $\{1^a, 1^b, 1^c\}$.

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