On Homomorphisms Between Specht Modules For the Ariki-Koike Algebra

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Abstract

Specht modules occupy a position of central importance in the representation theory of both the symmetric and Iwahori-Hecke algebras, and there is hence considerable interest in achieving a greater understanding of their structure. To this end, over the past thirty years there has been much study undertaken of the homomorphism spaces between these modules, with a particular emphasis being placed upon the construction of explicit homomorphisms between Specht modules.

Being a generalization of the Iwahori-Hecke algebra of type A, Specht modules are of a similar importance to the Ariki-Koike algebra. In this thesis we provide and analogue of James's kernel intersection theorem, the latter having been a key tool in the study of homomorphisms between Specht modules in the setting of both the symmetric group and the Iwahori-Hecke algebra of type A. We also provide and outline of how this result may be used to construct homomorphisms between Specht modules for the Iwahori-Hecke algebra of type B. Additionally, as a byproduct of this work, we include a sufficient condition for certain kinds of commonly encountered tableaux to determine homomorphisms between analogues of Young's permutation modules and the Specht modules.

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This thesis is dedicated to the memories of all those we've lost over the years.

Contents

Chapter 1

Introduction

First described by Ariki and Koike [4] and, independently, by Broué and Malle [7], the Ariki-Koike algebra $\mathscr{H}_{r,n}$ can most easily be thought of as a particular deformation of the group algebra of the complex reflection group $\mathbb{Z}/r\mathbb{Z}\wr\mathfrak{S}_n$. In the case of Ariki and Koike's paper, their algebra arises as a natural generalization of the Iwahori-Hecke algebras of type A and type B, both of which can be considered special cases of the Ariki-Koike algebra. Broué and Malle, on the other hand, were concerned with constructing such deformations for a more general class of complex reflection groups, being motivated by possible, and presently still conjectured, applications to Lie theory (see, for instance, [6] for further details).

In addition to its obvious connections to the study of Iwahori-Hecke algebras of type A and type B, and those conjectured to exist with Lie theory, the Ariki-Koike algebra also appears in other related areas of mathematics. For instance, in knot theory the Ariki-Koike algebra has led in part to a generalization of the Birman-Murakami-Wenzl algebras [25] that play a role in the study of non-intersecting knots in a solid torus. There is also considerable current interest in the deep relationship between certain topics in quantum group theory and the representation theory Ariki-Koike algebra. Although specific examples of this relationship will be reviewed in due course of this introduction, we refer the reader to [3] and [36] for a broader, more comprehensive account of this topic.

The purpose of this chapter is to provide a brief overview of historical and more recent developments in the representation theory of the Ariki-Koike algebra. We focus in particular on those most relevant to this thesis, i.e. those pertaining to the study of spaces of homomorphisms between pairs of Specht modules. Before continuing, the reader should be aware that in this thesis we concentrate almost entirely upon the Ariki-Koike algebra as defined over a basis due to Dipper, James, Mathas, and Murphy [21]. As such, the introduction we provide is very much confined to this setting and rather limited in scope. Unfortunately, this means that a number of topics and a wealth of research conducted in other settings is not discussed. In compensation, the reader is directed to more complete surveys provided in [22], [40], and [34].

The Representation Theory of $\mathscr{H}_{r,n}$

Much is already known about the representation theory of the Ariki-Koike algebra. Not surprisingly given its origins, this has been found to have a number of features in common with that of the Iwahori-Hecke algebra of Type A and the symmetric group $\mathfrak{S}_n$; indeed, much of the development of this subject has been a process of attempting to generalize results from these two better known structures. In this section we provide an account of some of the most important, aspects of the established representation theory of $\mathscr{H}_{r,n}$ relevant to the material appearing in later chapters.

Cellular Algebras, Cell Modules, and Irreducible Modules

One of the most important feature shared with the Iwahori-Hecke algebra of type A is that the Ariki-Koike algebra is 'cellular'. Motivated by properties of the much studied Kazhdan-Lusztig canonical basis of the former algebra [34], the notion of a cellular algebra was introduced by Graham and Lehrer in [24]. One very useful property of such algebras is that a great deal of their representation theory can be determined from the cellular structure alone. For instance, once identified as being cellular, the irreducible modules of the algebra in question are immediately characterized as being the simple heads, relative to a particular bilinear form arising from the multiplicative properties of cellular algebras, of a class of so-called 'cell' modules.

As this suggests, these cell modules occupy an important position in the representation theory of cellular algebras, and determining their structure is consequently of much interest. To this end, a major contemporary research direction is focused on decomposition numbers, these being the multiplicities with which each simple module appears as a composition factor of a given cell module, computing these being equivalent to computing the multiplicities of cell modules appearing as composition factors of the principal indecomposable modules.

Before we consider the Ariki-Koike algebra, it's worthwhile mentioning that a given algebra is defined as being cellular if it has a cellular basis, this being a basis satisfying certain 'nice' multiplicative properties. However, such an algebra may have more than one cellular basis, and those modules that comprise the cell modules differs over different cellular basis. For instance, the cell modules for the Iwahori-Hecke algebra of type A defined by Kazhdan and Lusztig in [34], Dipper and James in [10], and Murphy [44] differ, although they are isomorphic (see [44] and [42]).

The Representation Theory of the Ariki-Koike Algebra

As has already been mentioned, the Ariki-Koike algebra is cellular. The cellular basis of the Ariki-Koike algebra that we work with in this thesis is due to Dipper, James, and Mathas [21], this being a generalization the Murphy basis of the Iwahori-Hecke algebra of type A constructed in [44] and [45]. This basis is indexed by pairs of standard Young tableaux, giving the representation theory of the Ariki-Koike algebra a combinatorial flavour very reminiscent of that of the Iwahori-Hecke algebras and the symmetric group. In particular, there are a number of striking similarities with the representation theory of the Iwahori-Hecke algebra of type A as set out in [39]. The cell modules with respect to this basis are the Specht modules, as defined in [21], themselves a generalization of the Specht modules that occur in the representation theory of the Iwahori-Hecke algebra of type A. Here, each Specht module S^{λ}, where λ is a multipartition, is defined as a quotient of a particular right ideal M^{λ} of $\mathscr{H}_{r,n}$, and a clear strategy for constructing homomorphisms $\hat{\Theta}: S^{\lambda} \to S^{\mu}$ between Specht modules is to analyse the structure of homomorphisms $\Theta: M^{\lambda} \to S^{\mu}$ and determine when we can factor these homomorphisms through S^{λ}. This topic will be discussed in much greater detail once we have covered some of the most important developments in the study of the Ariki-Koike algebra.

The simple $\mathscr{H}_{r,n}$-modules were fully classified in arbitrary characteristic by Ariki in [2] and Ariki and Mathas in [5]. These simple modules are indexed by combinatorial objects known as 'Kleshchev Multipartitions', which can be considered as a (highly non-trivial) generalization of the e-restricted partitions appearing in [10] indexing the irreducible modules for the Iwahori-Hecke algebra of type A. The proof of this classification draws heavily on the theory of crystal bases of quantum groups developed by Kashiwara [33] and is covered in depth in [26]. A development closely related to this classification of the simple modules, and one that again makes use of the connection between $\mathscr{H}_{r,n}$ and quantum groups, is that in characteristic zero the decomposition numbers of the Ariki-Koike algebra are known, this being due to results of Ariki [1], Uglov [47], and James and Mathas [32]. Despite this being a major development, it should be remarked that the decomposition numbers are known only 'in principle', in so much as they can be calculated by a recursive algorithm. As such, even in characteristic zero, there remains much interest in studying decompositions, particularly with a view to finding a closed formula. More generally, a major open problem is James' Conjecture [30], the hypothesis of which being that, in certain specific cases, the decomposition numbers of the Iwahori-Hecke algebra of type A are the same in both prime and zero characteristic. Although a vast subject and one well beyond the scope of this thesis, some recent developments in this area and background to the problem of solving James' Conjecture are given in [16], [17] and [23].

A useful tool in the study of the Ariki-Koike algebra is the cyclotomic q-Schur algebra introduced in [21], this being the endomorphism algebra of the direct sum of certain right ideals of $\mathscr{H}_{r,n}$. As in the case of the Iwahori-Hecke algebra of type A and the q-Schur algebra, and the symmetric group and the Schur algebra, a close relationship exists between the representation theory of these two structures. This is particularly the case for the decomposition numbers and block structure of the two algebras (see [31] and [39, Theorem 5.5] respectively). In fact, it is the q-Schur algebra that provides much of the motivation for the Dipper-James-Mathas-Murphy basis, since this can be 'lifted' to a basis of the q-Schur algebra in the sense that elements of the latter can be expressed as a linear combination of elements of the former.

Finally, an important and more recent development in the representation theory of the Ariki-Koike algebra, and one well beyond the scope of this thesis, involves the Khovanov-Lauda-Rouquier algebra. First introduced in [35] and, independently, in [46], these are graded algebras arising from ongoing efforts to categorify quantum groups. Their relevance here is that the Ariki-Koike algebra is in fact a special case of the Khovanov-Lauda-Rouquier algebra, an isomorphism between the Ariki-Koike algebra and a certain Khovanov-Lauda-Rouquier algebra being due to Brundan and Kleshchev [8]. This discovery establishes a non-trivial $\mathbb{Z}$-grading of the Ariki-Koike algebra, and both this grading and the more general connection with the Khovanov-Lauda-Rouquier algebras have been the subject of considerable interest since they were first established.

Homomorphisms Between Specht Modules

In the case of the symmetric group and the Iwahori-Hecke algebra of type A, the structure of the space of homomorphisms between Specht modules has attracted considerable attention. We now provide a summary of developments in this area that are most relevant to this thesis.

The Symmetric Group

Two of the first major results relating to the study of Specht modules and the homomorphism spaces between them are the kernel intersection theorem [28, Corollary 17.18], which expresses each Specht module as an intersection of certain homomorphisms, and the semistandard homomorphism theorem [28, Theorem 13.13]. As long as the characteristic of the ground field of the group algebra is not equal to 2, the latter result provides a basis for the homomorphism space from a given Specht module to one of Young's permutation modules. The theorem still provides a basis of such homomorphism spaces in the case where the characteristic of the ground field is equal to two, but only for a certain class of Specht modules. The significance of these results to this thesis is that they have provided much of the theoretical foundation and inspiration for many later developments, including our own.

In [19] Fayers and Lyle decompose the homomorphism space between Specht modules as a tensor product of homomorphism spaces between Specht modules of 'smaller' algebras, these being obtained by removing rows and columns from the partitions indexing the former Specht modules. A similar process was used by James [29] and Donkin [13] to study the decomposition numbers of the symmetric group algebra.

This was subsequently used by Fayers and Martin in [20], building upon and generalizing an earlier result of Carter and Payne [9], to determine when the homomorphism space between certain pairs of Specht modules are non-zero. It is significant that, whereas the result of Carter and Payne employs an argument based in geometry, the methods employed by Fayers and Martin are far more constructive and achieve a similar aim via combinatorial reasoning; in particular, the combinatorics of tableaux are used to construct explicit examples of homomorphisms, this being an approach that forms much of the basis of this thesis. As an explicit example of such results being used to investigate the structure of Specht modules, a recent paper [12] by Dodge and Fayers partly uses [20] to exhibit what are, in the authors' own words, the first new examples of decomposable Specht modules to be found in the last thirty years. The last published instance of such examples of decomposable Specht modules being due to Murphy in 1980 [43].

The Iwahori-Hecke Algebra of Type A

One of the first results pertaining to the study of the space of homomorphisms between Specht modules of the Iwahori-Hecke algebra of type A was the generalization of the kernel intersection theorem, this appearing in [10]. A generalization of the semistandard homomorphism theorem subsequently appeared in [11], and thus set the stage for a similar analysis of homomorphism spaces and programme of research as that applied previously to the representation theory of the Symmetric group.

Following the work of Fayers and Martin in [20] and working in the same combinatorial setting, Lyle produced a similar construction of homomorphisms between Specht modules for the Iwahori-Hecke algebra of type A [37]. This, combined with [14] led to a classification of the reducible Specht modules when $q \neq -1$; when $q = -1$ or $\mathbb{F}$ is of characteristic zero we have only a necessary condition for the reducibility of Specht modules [15].

This method of constructing homomorphisms between Specht modules applies only to the classical non-cellular setting of Dipper and James. In the more modern view of the Iwahori-Hecke algebra of type A as a cellular algebra, our definition of the Specht modules differs considerably and hence we cannot call upon the kernel intersection theorem of Dipper and James. Addressing this problem, a 'cellular analogue' of this theorem was recently developed by Lyle [38].

In general, the constructions of homomorphisms in either setting is a difficult and very technical matter, one that is wholly dependent on our ability to express homomorphisms in terms of the elements of a certain basis of the homomorphism space. This has meant that, until recently, such constructions have been limited to individual special cases. Fayers addresses this in [18] by providing an algorithm for computing the homomorphism space between arbitrary pairs of Specht modules, thus removing one of the biggest obstacles faced when studying such spaces.

The Ariki-Koike Algebra and this Thesis

To date there has been little research into the homomorphism spaces between Specht modules for the Ariki-Koike algebra. In this thesis we generalize [38], providing an analogue of the kernel intersection theorem appropriate to the Dipper-James-Mathas-Murphy basis of the Ariki-Koike algebra. In principle, this allows us to identify when a homomorphism $\Theta : M^\lambda \to S^\mu$ can be factored through S^λ to provide a homomorphism $\hat{\Theta} : S^\lambda \to S^\mu$, as well as a means of constructing homomorphisms between Specht modules using basis elements of $\mathrm{Hom}_{\mathscr{H}_{r,n}}(M^\lambda, S^\mu)$. Although such constructions are generally difficult in practice, we apply this result to construct homomorphisms between certain pairs of Specht modules, much in the same way that Lyle constructed 'one-node homomorphisms' between Specht modules for the Iwahori-Hecke algebra of type A in [37]. We also provide a sufficient condition for tableaux to determine homomorphisms between Young permutation modules of the Ariki-Koike algebra.

Unfortunately, our results are more limited than those of [20], [37], or [38]. This is because we cannot currently say when those homomorphisms in $\mathrm{Hom}_{\mathscr{H}_{r,n}}(S^\lambda, S^\mu)$ that can be extended to elements of $\mathrm{Hom}_{\mathscr{H}_{r,n}}(M^\lambda, S^\mu)$ comprise the entire homomorphism space $\mathrm{Hom}_{\mathscr{H}_{r,n}}(S^\lambda, S^\mu)$. An analogue, described later in Conjecture 1, of the semistandard homomorphism theorem mentioned earlier would resolve this question, and it is believed that such an analogue is possible. As such, a natural continuation of the research presented in this thesis would consist of:

- Deducing an analogue of the semistandard homomorphism theorem;
- Generalizing the results appearing in [18] to the Ariki-Koike algebra. Doing so would allow us to methodically extend our method of constructing homomorphisms considerably further than the special cases considered in this thesis. The author believes that this should be relatively straightforward; and
- Providing a necessary condition for tableaux to determine a homomorphism to complement the sufficient condition given in this thesis.

Main Results and an Outline of the Thesis Structure

Chapters 3, 4, and 5 contain a number of new results. Much of our main focus is developing the theoretical machinery necessary to construct homomorphisms between Specht modules of the Ariki-Koike algebra by factoring homomorphisms $\Theta : M^\lambda \to S^\mu$ through S^λ. Our approach to this problem utilizes the basis for $\mathrm{Hom}_{\mathscr{H}_{r,n}}(M^\lambda, S^\mu)$ provided by those homomorphisms $\Theta_S : M^\lambda \to S^\mu$ indexed by an important variety of combinatorial objects called the semistandard μ-tableaux of type λ.

During the course of the author's research it became apparent that it was not at all clear if and when an element of the more general set of μ-tableaux of type λ determines a homomorphism and when it does not. We provide a partial solution to this dilemma in the form of a simple necessary condition for a μ-tableau of type λ to determine a homomorphism $\Theta_{\mathsf{S}} : M^\lambda \to S^\mu$. This condition depends purely on the combinatorics of the tableau in question.

Theorem (Theorem 3.4.2)**.** *Let λ and μ be multipartitions. A row-semistandard μ tableaux S of type λ determines a homomorphism $\Theta_{\mathsf{S}} : M^\lambda \to S^\mu$ whenever the entries in S adhere to a certain ordering.*

In addition to being of interest in its own right, this result is an important technical necessity when it comes to the business of actually constructing homomorphisms.

We also show that a necessary and sufficient condition for a homomorphism $\Theta : M^\lambda \to S^\mu$ to factor through S^λ is that a certain two-sided ideal $M^\lambda \cap \check{\mathscr{H}}^\lambda$ is contained in the kernel of Θ. The central result of our thesis is that, for every multicomposition λ, this ideal is generated (as a right ideal) by a finite family of elements $m_\lambda \mathfrak{d}^{(s)}_{d,t}$ and $m_\lambda \mathfrak{l}^{(s)}$ of $\mathscr{H}_{r,n}$.

Theorem (Theorem 3.3.16)**.** *Let $\mathfrak{J}$ be the right ideal of $\mathscr{H}_{r,n}$ generated by the sets*

$$\mathbf{D}(\lambda) = \left\{ m_\lambda \mathfrak{d}^{(s)}_{d,t} : (d,t,s) \in \operatorname{def}(\lambda, \mathfrak{d}) \right\}$$

and

$$\mathbf{L}(\lambda) = \left\{ m_\lambda \mathfrak{l}^{(s)} : s \in \operatorname{def}(\lambda, \mathfrak{l}) \right\}.$$

Then $\mathfrak{J} = M^\lambda \cap \check{\mathscr{H}}^\lambda$.

As a corollary, this theorem then provides us with another necessary and sufficient condition for homomorphisms $\Theta : M^\lambda \to S^\mu$. More to the point, with this result in place, we can consider the image of Θ of only a finite number of elements of $\mathscr{H}_{r,n}$ when trying to construct a homomorphism between Specht modules S^λ and S^μ from Θ.

Theorem (Corollary 3.3.17)**.** *Let $\Theta : M^\lambda \to S^\mu$ be a homomorphism. Then the following statements are equivalent:*

- $\Theta(m_\lambda h) = 0$ *for every* $h \in \mathscr{H}_{r,n}$ *with* $m_\lambda h \in \check{\mathscr{H}}$;
- $\Theta\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right) = 0$ *for every* $(d,t,s) \in \operatorname{def}(\lambda, \mathfrak{d})$ *and* $\Theta\left(m_\lambda \mathfrak{k}^{(s)}\right) = 0$ *for every* $s \in \operatorname{def}(\lambda, \mathfrak{l})$*; and*
- Θ *factors through* S^λ.

Finally, we apply the previous result to a specific class of pairs of Specht modules. In this setting we derive an explicit condition for when there exists a non-zero homomorphism between Specht modules.

This condition depends upon the residues of the Young diagrams associated with the multipartitions indexing the Specht modules we're working with. Informally speaking, the residue of a particular node appears as a scalar expressed in terms of the parameters $q, Q_1, \ldots, Q_r$, and so in effect the theorem places a restriction upon which Ariki-Koike algebras exhibit non-zero homomorphisms between the Specht modules being considered.

Theorem (Theorem 4.7.3). *Let λ and μ be multipartitions such that μ is constructed from λ by the deletion of a removable node $\mathbf{r}$ in a single component x and the adjoining of an addable node $\mathbf{a}$ to component $x-1$. Then there exists a non-zero homomorphism $\hat{\Theta} : S^\lambda \to S^\mu$ whenever $\mathrm{res}_\mu(\mathbf{a}) = \mathrm{res}_\lambda(\mathbf{r})$*

So, for instance, if the residue of $\mathbf{a}$ is q^2Q_1 and the residue of $\mathbf{r}$ is $q^{-1}Q_2$ we have that a non-zero homomorphism exists between S^λ and S^μ whenever $Q_1 = q^{-3}Q_2$.

The definition of all terms and objects used in this summary can be found in Chapter 2 and Section 3.2.

The structure of the thesis is as follows:

Chapter 2 Here we collect together those fundamental definitions and results from the representation theory of both the Ariki-Koike algebra and the Iwahori-Hecke algebra of type A that we will need in the course of this thesis. We conclude with a discussion of some important topics such as the classification of the irreducible modules of the Ariki-Koike algebra, although this is not directly relevant from a technical perspective to the rest of the thesis.

Chapter 3 In this chapter we prove the main result of this thesis, that being an analogue of the kernel intersection theorem of James in the setting of the Murphy basis of the Ariki-Koike algebra, and provide an outline of how this can be used to construct homomorphisms between Specht modules. In doing so we set the stage for the remainder of the thesis, providing many of the definitions and technical results that will be needed in chapter four.

Chapter 4 Here we provide an application of the results from the previous chapter by deriving a condition that describes when it is possible to construct homomorphisms between certain related Specht modules for the Iwahori-Hecke algebra of type B, this being a special case of the Ariki-Koike algebra.

Chapter 5 Concluding remarks are provided, including a brief discussion of current research and possible future development of the material contained in this thesis. We also provide a conjecture regarding expressing tableaux explicitly in terms of the generators of $\mathscr{H}_{r,n}$

Chapter 2

Background

The aim of this chapter is to provide a summary of the standard definitions and results later chapters require. In most cases we will adopt the notation and terminology found in [40]. We also describe the classification of the irreducible modules of the Ariki-Koike algebra in section 2.8, although this topic will not feature directly in the remainder of the thesis.

2.1 The Ariki-Koike Algebra

Let $\mathbb{F}$ be a field and let $q, Q_1, Q_2, \ldots, Q_r$ be non-zero elements of $\mathbb{F}$ with $q \neq 1$.

Definition 2.1.1. For each pair of positive integers n and r *the Ariki-Koike algebra* is the unital associative algebra with generators $T_0, T_1, \ldots, T_{n-1}$, subject to the following relations:

$$\begin{aligned} (T_0 - Q_1)(T_0 - Q_2)\cdots(T_0 - Q_r) &= 0 \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0 \\ (T_i - q)(T_i + 1) &= 0 && \text{for } 1 \leq i \leq n-1, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && \text{for } 1 \leq i < n-1, \\ T_iT_j &= T_jT_i && \text{for } 0 \leq i < j-1 < n-1. \end{aligned}$$

As in the introduction, we let $\mathscr{H}_{r,n}$ or, more simply, $\mathscr{H}$ denote this algebra.

Remark. The condition that $q \neq 1$ is necessary since the corresponding theory for $q = 1$ requires a 'degenerate' version of the Ariki-Koike algebra, which we do not go into here.

For each $1 \leq i \leq n-1$, let s_i be the simple transposition $s_i = (i, i+1)$. If $w \in \mathfrak{S}_n$ and $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced expression for w, we set $T_w = T_{i_1}T_{i_2}\cdots T_{i_k}$. Due to Matsumoto's Theorem on reduced expressions [41], T_w is independent of the choice of reduced expression for w, and so is well defined. We shall write $T_{i_1,i_2,\ldots,i_k}$ for $T_{i_1}T_{i_2}\cdots T_{i_k}$ whenever the latter would be too cumbersome.

If $\ell(w)$ is the length of the permutation $w \in \mathfrak{S}_n$, a consequence of the relations defining $\mathscr{H}$ is the following multiplication formula:
Proposition 2.1.1. *Suppose that $w \in \mathfrak{S}_n$, then*

$$T_w T_{s_i} = \begin{cases} T_{ws_i} & \text{if } \ell(ws_i) > \ell(w), \\ qT_{ws_i} + (q-1)T_w & \text{otherwise.} \end{cases}$$

Note that the subalgebra generated by $T_1, T_2, \ldots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra of type A, which we define properly in section 2.6. Much of the work in this thesis uses this fact in order to apply results from the representation theory of this algebra to the study of the Ariki-Koike algebra.
Definition 2.1.2. For each $1 \le k \le n$, define the element $L_k \in \mathscr{H}_{r,n}$ by

$$L_k := q^{1-k} T_{k-1} T_{k-2} \cdots T_1 T_0 T_1 \cdots T_{k-2} T_{k-1}.$$

These elements are an analogue of the *Jucys-Murphy elements* that occur in the representation theory of $\mathscr{H}_n$ and $\mathfrak{S}_n$, occupying a similar role by virtue of generating a large abelian subalgebra of $\mathscr{H}_{r,n}$. These elements also play an crucial part in this thesis, much of the technical details of which being concerned with studying how they interact with one another and the generators of $\mathscr{H}_{r,n}$.
Proposition 2.1.2 ([21, (2.1)])**.**

1. *Let $1 \le l \le r$. If $j \ne k$, then T_j and $\prod_{i=1}^{k}(L_i - Q_l)$ commute.*
2. *$L_i T_j = T_j L_i$ whenever $j \ne i, i-1$.*

2.2 Multipartitions, Diagrams and Tableaux

Recall that:

- A *composition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers such that $|\lambda| = \sum_i \lambda_i = n$. It is convention to write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, where l is the largest positive integer such that λ_l is non-zero. We refer to a given term λ_i in this sequence is the i-th *row* of λ. ;
- A *partition* λ is a composition satisfying the additional condition that $\lambda_i \ge \lambda_{i+1}$ for all $i \ge 1$.
- A *multicomposition* of n in r parts is an r-tuple $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ of compositions such that $|\lambda| = \sum_k |\lambda^{(k)}| = n$. For $1 \le k \le r$, the k-th *component* or *part* of λ refers to the composition $\lambda^{(k)}$, and $\lambda_i^{(k)}$ denotes the i-th row of the k-th component.
- A multicomposition in which each component is a partition is a *multipartition*.

Clearly, compositions and partitions can be regarded as multicompositions and multipartitions in 1-part respectively.

The set of multicompositions of n in r parts is partially ordered under the *dominance relation*: μ is said to *dominate* λ, in which case we write $\lambda \trianglelefteq \mu$, if and only if

$$\sum_{k=1}^{l-1} |\lambda^{(k)}| + \sum_{i=1}^{s} \lambda_i^{(l)} \leq \sum_{k=1}^{l-1} |\mu^{(k)}| + \sum_{i=1}^{s} \mu_i^{(l)} \tag{2.1}$$

for all positive integers l and s. We write $\lambda \triangleleft \mu$ if $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$.

The *diagram* $[\lambda]$ of a multicomposition λ is a sequence of Young diagrams, each corresponding to the composition forming the corresponding component of λ. More formally,

Definition 2.2.1. The diagram $[\lambda]$ of a multicomposition λ is the set

$$[\lambda] = \left\{(i,j,k) \in \mathbb{N}^2 \times \{1,2,\ldots,r\} : 1 \leq j \leq \lambda_i^{(k)}\right\}.$$

The elements of $[\lambda]$ are referred to as the *nodes* of the diagram.

Represented graphically as an array of boxes, we take the coordinates of the triple (i,j,k) to refer to the row, column, and component in which that node appears. For this reason, we will refer to each term of a given component as a *row* of that component.

If λ is a multipartition, two important classes of nodes of $[\lambda]$ are the removable and addable nodes:

- $\mathbf{i} \in [\lambda]$ is *removable* whenever the diagram $[\lambda] \setminus \{\mathbf{i}\}$ is that of a multipartition; and
- a triple $\mathbf{i} \in \mathbb{N}^2 \times \{1,2,\ldots,r\}$ and such that $\mathbf{i} \notin [\lambda]$ is an *addable node* whenever the diagram $[\lambda] \cup \{\mathbf{i}\}$ is that of a multipartition.

Definition 2.2.2. Given a multicomposition λ, a *λ-tableau* is a bijection $\mathfrak{t} : [\lambda] \to \{1,2,\ldots,n\}$, and may be represented visually as filling the nodes of $[\lambda]$ with entries taken from $\{1,2,\ldots,n\}$.

As we shall see, certain tableaux index the basis elements of $\mathscr{H}_{r,n}$, the study of which is often a matter of investigating the combinatorial properties of tableaux. A λ-tableau $\mathfrak{t}$ is *row standard* if its entries are increasing along the rows of each component, and *standard* if λ is a multipartition and the entries increase both along the rows and down the columns of each component. We let $\mathrm{RStd}(\lambda)$ denote the set of row standard λ-tableaux for each multicomposition λ and, if λ is also a multipartition, $\mathrm{Std}(\lambda)$ the set of standard λ-tableaux.

The *initial* λ-tableau $\mathfrak{t}^\lambda$ is the standard λ-tableau in which each node $(i,j,k) \in [\lambda]$ contains the entry

$$\sum_{k=1}^{c-1} |\lambda^{(k)}| + \sum_{i=1}^{a-1} \lambda_i^{(c)} + b.$$

We say that a node $(i,j,k) \in [\lambda]$ is *higher than* a node $(x,y,z) \in [\lambda]$ if $\mathfrak{t}^\lambda(i,j,k) < \mathfrak{t}^\lambda(x,y,z)$, otherwise we say that (i,j,k) is *lower than* (x,y,z).

The elements of the symmetric group $\mathfrak{S}_n$ most naturally act on the set of tableaux by permuting the entries.

Definition 2.2.3. If λ is a multicomposition, define $\mathscr{D}_\lambda$ to be the set

$$\mathscr{D}_\lambda = \left\{ w \in \mathfrak{S}_n : \mathfrak{t}^\lambda \cdot w \text{ is row-standard} \right\}$$

and $\mathfrak{S}_\lambda$ to be the subgroup of $\mathfrak{S}_n$ consisting of all permutations that stabilize the rows of $\mathfrak{t}^\lambda$.

It is not difficult to see that [39, Proposition 3.3] implies that $\mathscr{D}_\lambda$ is a complete set of right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. Moreover, we have the following:

Proposition 2.2.1 ([39, Proposition 3.3])**.** *If $w \in \mathfrak{S}_\lambda$ and $d \in \mathscr{D}_\lambda$, then $T_w T_d = T_{wd}$ and $\ell(wd) = \ell(w) + \ell(d)$.*

If $\mathfrak{t}$ is a λ–tableau, let $d(\mathfrak{t})$ denote the unique permutation such that $\mathfrak{t} = \mathfrak{t}^\lambda \cdot d(\mathfrak{t})$.

Definition 2.2.4. For each $1 \le x \le n$ the *residue* of x in $\mathfrak{t}$ is defined as

$$\operatorname{res}_\mathfrak{t}(x) = q^{j-i} Q_k$$

where $\mathfrak{t}(i,j,k) = x$. We write $\operatorname{res}_\lambda(x)$ for $\operatorname{res}_{\mathfrak{t}^\lambda}(x)$.

Example 1. *Suppose that $\lambda = ((2,2,1),(3,1))$, then*

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\, , \begin{array}{|c|c|c|}\hline 6 & 7 & 8 \\ \hline 9 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \right).$$

If $\mathfrak{t}$ is the row standard λ-tableau given by

$$\mathfrak{t} = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 9 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\, , \begin{array}{|c|c|c|}\hline 4 & 7 & 8 \\ \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \right),$$

then $d(\mathfrak{t}) = (2,3)(4,5,9,6)$. The residue of 9 in each tableau is given by $\operatorname{res}_\lambda(9) = q^{-1}Q_2$ and $\operatorname{res}_\mathfrak{t}(9) = q^{-2}Q_1$.

Using the dominance order on multipartitions, we can impose a partial order on the set of standard λ–tableau thus: Let k be a positive integer with $1 \le k \le n$. For each standard λ–tableau $\mathfrak{t}$, let $\mathfrak{t} \downarrow k$ be the sub–tableau consisting of the nodes of $[\lambda]$ in which only the integers 1 to k appear as entries. Also, let $\operatorname{Shape}(\mathfrak{t} \downarrow k)$ denote the multipartition associated with $\mathfrak{t} \downarrow k$. Then, for standard λ-tableaux $\mathfrak{s}$ and $\mathfrak{t}$, define the relation $\mathfrak{s} \trianglelefteq \mathfrak{t}$ by

$$\mathfrak{s} \trianglelefteq \mathfrak{t} \quad \text{if and only if} \quad \operatorname{Shape}(\mathfrak{s} \downarrow k) \trianglelefteq \operatorname{Shape}(\mathfrak{t} \downarrow k) \quad \text{for every } k = 1,2,\ldots,n. \tag{2.2}$$

We write $\mathfrak{s} \triangleleft \mathfrak{t}$ if $\mathfrak{s} \trianglelefteq \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$.

2.3 A Cellular Basis of $\mathscr{H}_{r,n}$ and Specht Modules

The following basis of $\mathscr{H}_{r,n}$ is due to Dipper, James, and Mathas [21]. For each multicomposition λ let $\mathfrak{S}_\lambda$ be the row stabilizer of $\mathfrak{t}^\lambda$ and define elements x_λ and u_λ^+ of $\mathscr{H}_{r,n}$

by

$$x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w \quad \text{and} \quad u_\lambda^+ = \prod_{s=2}^{r} \prod_{i=1}^{|\lambda^{(1)}|+\cdots+|\lambda^{(s-1)}|} (L_i - Q_s).$$

Applying the relations in Proposition 2.1.2 shows that $x_\lambda u_\lambda^+ = u_\lambda^+ x_\lambda$. Set $m_\lambda = x_\lambda u_\lambda^+$ and let M^λ denote the right $\mathscr{H}_{r,n}$-module $m_\lambda \mathscr{H}_{r,n}$.

Define $* : \mathscr{H}_{r,n} \to \mathscr{H}_{r,n}$ to the anti-isomorphism given by $T_i^* = T_i$ for all $0 \le i < n$ and set $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})}^* m_\lambda T_{d(\mathfrak{t})}$ for each multicomposition λ and pair of λ-tableaux $\mathfrak{s}$ and $\mathfrak{t}$.

Theorem 2.3.1 ([21, Theorem 3.26])**.** *A cellular basis of $\mathscr{H}_{r,n}$ over $\mathbb{F}$ is given by the set*

$$\{m_{\mathfrak{s}\mathfrak{t}} : \lambda \textit{ is a multipartition}, \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda)\}$$

Note that $T_w^* = T_{w^{-1}}$ for each $w \in \mathfrak{S}_n$ and $L_k^* = L_k$ for $1 \le k \le n$.

For each multipartition λ, let $\check{\mathscr{H}}_{r,n}^\lambda$ be the $\mathbb{F}$–module with basis

$$\left\{m_{\mathfrak{s}\mathfrak{t}} : \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\mu), \mu \text{ is a multipartition such that } \lambda \lhd \mu\right\}.$$

Via the properties of cellular algebras, $\check{\mathscr{H}}_{r,n}^\lambda$ is a two–sided ideal of $\mathscr{H}_{r,n}$.

Definition 2.3.1. If λ is a multipartition, we define the *Specht module* S^λ to be the right $\mathscr{H}_{r,n}$-module generated by $\check{\mathscr{H}}_{r,n}^\lambda + m_\lambda$.

Note that $\check{\mathscr{H}}_{r,n}^\lambda$ is not contained in $m_\lambda \mathscr{H}_{r,n}$, so the Specht module is not a quotient module in the accepted sense.

The set

$$\left\{\check{\mathscr{H}}_{r,n}^\lambda + m_{\mathfrak{t}^\lambda \mathfrak{t}} : \mathfrak{t} \in \mathrm{Std}(\lambda)\right\}$$

forms a basis of S^λ as a module over $\mathbb{F}$, the elements of which we denote as $m_\mathfrak{t} = \check{\mathscr{H}}_{r,n}^\lambda + m_{\mathfrak{t}^\lambda \mathfrak{t}}$ for $\mathfrak{t} \in \mathrm{Std}(\lambda)$.

Proposition 2.3.2 ([31, Proposition 3.7])**.** *Suppose that $1 \le k \le n$ and let $\mathfrak{s}$ and $\mathfrak{t}$ be standard λ–tableaux. Then, there exist scalars $r_\mathfrak{v} \in \mathbb{F}$ such that*

$$m_{\mathfrak{s}\mathfrak{t}} L_k = \mathrm{res}_\mathfrak{t}(k) m_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{v}\in\mathrm{Std}(\lambda)\\ \mathfrak{v}\rhd\mathfrak{t}}} r_\mathfrak{v} m_{\mathfrak{s}\mathfrak{v}} \mod \check{\mathscr{H}}_{r,n}^\lambda.$$

An extremely useful corollary of this proposition is that $m_\lambda L_k = \mathrm{res}_\lambda(k) m_\lambda$ modulo $\check{\mathscr{H}}_{r,n}^\lambda$ for any given multipartition λ; this follows immediately from the fact that $m_\lambda = m_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda}$ and that no λ-tableau dominates $\mathfrak{t}^\lambda$.

2.4 Semistandard Tableaux and Homomorphisms

Definition 2.4.1. Let λ and μ be multicompositions. A *μ-tableau of type λ* is a mapping

$$\mathtt{T} : [\mu] \to \mathbb{N} \times \{1, \ldots, r\}$$

such that the number of nodes $\mathbf{n} \in [\mu]$ with $\mathtt{T}(\mathbf{n}) = (i,k)$ for a given i and k is $\lambda_i^{(k)}$.

We call the codomain of $\mathtt{T}$ the *entries of* $\mathtt{T}$ and impose a total ordering $\preceq$ on this set thus:

$$(i,k) \preceq (j,l) \text{ if and only if } k < j, \text{ or } k = j \text{ and } i \le l.$$

A μ-tableau of type λ is *row-semistandard* whenever the entries are non-decreasing along the rows of each component, and the set of all row-semistandard μ-tableaux of type λ is denoted $\mathcal{T}_r(\mu,\lambda)$.

Definition 2.4.2. Suppose now that λ is a multicomposition, μ is a multipartition, and $\lambda \trianglelefteq \mu$. Then we say that a μ-tableau $\mathtt{T}$ of type λ is *semistandard* whenever:

1. $\mathtt{T}$ is row-semistandard;
2. the entries are strictly increasing down each column of every component; and
3. if $(i,j,k) \in [\mu]$ and $\mathtt{T}(i,j,k) = (a,c)$, then $k \le c$.

Let $\mathcal{T}_0(\mu,\lambda)$ denote the set of all semistandard μ–tableaux of type λ. If $\mathfrak{t}$ is a μ-tableau, let $\lambda(\mathfrak{t})$ be the μ-tableau of type λ in which the node $(i,j,k) \in [\mu]$ is occupied by (a,c) if the entry $\mathfrak{t}(i,j,k)$ appears in row a of component c of $\mathfrak{t}^\lambda$. We use $\mathtt{T}^\lambda$ to denote the tableau $\lambda(\mathfrak{t}^\lambda)$.

When it comes to actually writing down examples of such tableaux, we will let i_k represent the entry (i,k), such a format being better suited to displaying as the entry in a diagram.

Example 2. *If* $\lambda = ((3,2,1),(2,1))$, $\mu = ((4,3,1),(1))$, *and* $\mathfrak{t}$ *is given by*

$$\mathfrak{t} = \left(\begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 9 \\ \hline 2 & 5 & 7 & \\ \cline{1-3} 4 & & & \\ \cline{1-1} \end{array} , \begin{array}{|c|} \hline 8 \\ \hline \end{array} \right),$$

then

$$\lambda(\mathfrak{t}) = \left(\begin{array}{|c|c|c|c|} \hline 1_1 & 1_1 & 3_1 & 2_2 \\ \hline 1_1 & 2_1 & 1_2 & \\ \cline{1-3} 2_1 & & & \\ \cline{1-1} \end{array} , \begin{array}{|c|} \hline 1_2 \\ \hline \end{array} \right) \qquad \mathtt{T}^\lambda = \left(\begin{array}{|c|c|c|} \hline 1_1 & 1_1 & 1_1 \\ \hline 2_1 & 2_1 & \\ \cline{1-2} 3_1 & & \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 1_2 & 1_2 \\ \hline 2_2 & \\ \cline{1-1} \end{array} \right).$$

Let λ be a multicomposition and μ a multipartition. For $\mathtt{S} \in \mathcal{T}_0(\mu,\lambda)$ and $\mathfrak{t} \in \operatorname{Std}(\mu)$ we define

$$m_{\mathtt{S}\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \operatorname{Std}(\mu) \\ \lambda(\mathfrak{s}) = \mathtt{S}}} T^*_{d(\mathfrak{s})} m_\mu T_{d(\mathfrak{t})}.$$

The importance of these elements is given by the following theorem, due to Dipper, James, and Mathas.

Theorem 2.4.1 ([21, Theorem 4.14])**.** *Suppose that* λ *is a multicomposition of* n*. Then* M^λ *is free as an* $\mathbb{F}$*-module with basis*

$$\left\{ m_{\mathtt{S}\mathfrak{t}} : \mathtt{S} \in \mathcal{T}_0(\mu,\lambda), \mathfrak{t} \in \operatorname{Std}(\mu) \textit{ for each multipartition } \mu \right\}.$$

The right ideals M^λ of $\mathscr{H}_{r,n}$ are, in a sense, an analogue of Young's permutation modules for the group algebra of $\mathfrak{S}_n$. Referring to these modules as *permutation modules*, each Specht module arises as the quotient of the permutation module corresponding to the same indexing multipartition.

One of the reasons that semistandard tableaux are important to us is that they determine a set of linearly independent, non-zero homomorphisms from permutation modules to Specht modules: Let λ be a multicomposition and μ a multipartition with $\lambda \trianglelefteq \mu$ and let $\mathtt{T}$ be a semistandard μ tableau of type λ. By [21, Proposition 4.9] we may define a non-zero homomorphism from M^λ to S^μ by

$$\Theta_{\mathtt{T}}(m_\lambda h) = \check{\mathscr{H}}^\mu + \left(m_\mu \sum_{\substack{\mathfrak{t} \in \mathrm{Std}(\mu) \\ \lambda(\mathfrak{t}) = \mathtt{T}}} T_{d(\mathfrak{t})} \right) h \tag{2.3}$$

for $h \in \mathscr{H}$. We call such homomorphisms *semistandard*.

The fact that semistandard homomorphisms are non-zero follows from the definition of $\check{\mathscr{H}}^\mu$, although not obviously so.

Lemma 2.4.2. *$\Theta_{\mathtt{T}} : M^\lambda \to S^\mu$ is non-zero for every $\mathtt{T} \in \mathscr{T}_0(\mu, \lambda)$.*

Proof. By definition every element of $\check{\mathscr{H}}^\mu$ is a linear combination of the set

$$\left\{ T^*_{d(\mathfrak{s})} m_\nu T_{d(\mathfrak{t})} : \mu \lhd \nu \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\nu) \right\}$$

If $\Theta_{\mathtt{T}}$ were not non-zero, then we would have

$$\sum_{\substack{\mathfrak{t} \in \mathrm{Std}(\mu) \\ \lambda(\mathfrak{t}) = \mathtt{T}}} m_\mu T_{d(\mathfrak{t})} \in \check{\mathscr{H}}^\mu.$$

But this is impossible since both the elements $m_\mu T_{d(\mathfrak{t})}$ in the above sum and those in the basis of $\check{\mathscr{H}}^\mu$ are elements of the standard basis of $\mathscr{H}$, and hence linearly independent. □

Example 3. *Suppose that $\lambda = ((3,2,1),(2,2))$, $\mu = ((4,2,1),(2,1))$, and that $\mathtt{T} \in \mathscr{T}_0(\mu, \lambda)$ is given by*

$$\mathtt{T} = \left(\begin{array}{|c|c|c|c|}\hline 1_1 & 1_1 & 1_1 & 2_1 \\ \hline 2_1 & 3_1 \\ \cline{1-2} 1_2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1_2 & 2_2 \\ \hline 2_2 \\ \cline{1-1} \end{array} \right).$$

Then

$$\begin{aligned} \Theta_{\mathtt{T}}(m_\lambda h) = \check{\mathscr{H}}^\mu &+ (1+T_1)(1+T_2+T_{2,1})(1+T_3+T_{3,2}+T_{3,2,1}) \\ &\times (1+T_5)(1+T_8) \\ &\times (L_1 - Q_2)(L_2 - Q_2) \cdots (L_7 - Q_2) \\ &\times (1+T_4)(1+T_7)(1+T_9)h \end{aligned}$$

for each $h \in \mathscr{H}$. *Since* $(1+T_1)(1+T_2+T_{2,1})(1+T_3+T_{3,2}+T_{3,2,1})(1+T_5)(1+T_8) = x_\mu$ *and* $(L_1-Q_2)(L_2-Q_2)\cdots(L_7-Q_2) = u_\mu^+$ *we have that*

$$\Theta_{\mathrm{T}}(m_\lambda h) = \check{\mathscr{H}}^\mu + m_\mu(1+T_4)(1+T_7)(1+T_9)h.$$

Of considerable importance is the fact that semistandard homomorphisms are linearly independent, a property that follows from [21, Theorem 6.6]. In fact, the semistandard homomorphisms provide a basis for the space of all homomorphisms from M^λ to S^μ which factor through M^μ. The following conjecture appears in the 'folklore'.

Conjecture 1. *If* $q \neq -1$ *and* $Q_i \neq Q_j$ *whenever* $i \neq j$*, then the homomorphisms defined in* (2.3) *provide a basis for the entire homomorphism space* $\mathrm{Hom}_{\mathscr{H}}\left(M^\lambda, S^\mu\right)$.

2.5 Quantum Integers and the Quantum Characteristic

Suppose that $\alpha \geq 0$ is an integer and define $[\alpha] \in \mathbb{F}$ by

$$\begin{aligned} &[0] = 0 \\ &[\alpha] = 1 + q + q^2 + \cdots + q^{\alpha-1} && \text{for } \alpha > 0. \end{aligned}$$

These are known as the *quantum integers* and play a role in the combinatorics of the rest of the paper. We may also define *quantum factorials* thus

Set $[0]! = 1$ and $[\alpha]! = [0][1]\cdots[\alpha]$ for $\alpha > 0$ and, for $\alpha \geq \beta \geq 0$ set

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{[\alpha]!}{[\beta]![\alpha-\beta]!}.$$

Definition 2.5.1. The *quantum characteristic* of $\mathbb{F}$ is the positive integer e that is minimal such that $[e] = 0$. If no such integer exists, we set $e = \infty$.

Note that $e = \operatorname{char}\mathbb{F}$ whenever $q = 1$, otherwise q is a primitive e-th root of unity in $\mathbb{F}$. Throughout this thesis, we will concentrate only on the case where $q \neq -1$.

2.6 The Iwahori-Hecke Algebra of Type A

A number of our results rely on the ability to 'reduce' certain problems to the setting of the Iwahori-Hecke algebra of type A. This subsection introduces the standard definitions and results from this setting that we will need.

A further reason for including this subsection is that due to the similarities between the notation commonly used in the setting of the Iwahori-Hecke algebra of type A and relating to the Ariki-Koike algebra, we are forced to use our own notation for the latter in order to distinguish between the two cases and avoid confusion.

Definition 2.6.1. The *Iwahori-Hecke algebra of type A* is the associative unital algebra with generators $t_1, t_2, \ldots, t_{n-1}$ such that

$$\begin{aligned} t_i^2 &= (q-1)t_i + q && (1 \le i \le n-1) \\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1} && (1 \le i < n-1) \\ t_i t_j &= t_j t_i && (1 \le i < j-1 \le n-1). \end{aligned}$$

As we remarked in section 2.1, the Iwahori-Hecke algebra of type A is isomorphic to the subalgebra of the Ariki-Koike algebra generated by the elements $T_1, \ldots, T_{n-1}$ of $\mathscr{H}_{r,n}$. We will denote this latter algebra by $\mathscr{H}_n$.

Let λ be a composition, then the *diagram* of λ is the set

$$[\lambda] = \left\{(i,j) \in \mathbb{N}^2 : 1 \le j \le \lambda_i\right\},$$

and a λ-tableau is a bijection $\mathfrak{t} : [\lambda] \to \{1, 2, \ldots, n\}$. As with the Ariki-Koike algebra, we say that such a tableau is *row-standard* whenever its entries increase along the rows and *standard* when, in addition to being row-standard, its entries also increase down the columns. Moreover, compositions and tableaux in this setting may be partially ordered in a manner identical to that (2.1) and (2.2)

Definition 2.6.2. If λ is a multicomposition, set $y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$ and define N^λ to be the right $\mathscr{H}_n$-module $y_\lambda \mathscr{H}_n$. If λ is a partition, we define $\check{\mathscr{H}}_n^\lambda$ to be the two-sided ideal of $\mathscr{H}_n$ with an $\mathbb{F}$-basis given by

$$\left\{T^*_{d(\mathfrak{s})} y_\nu T_{d(\mathfrak{t})} : \nu \text{ a partition of } n \text{ with } \lambda \lhd \nu, \text{ and } \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\nu)\right\}.$$

It's worth remarking that these modules are an analogue of both the $\mathscr{H}_{r,n}$-modules M^λ and the Young permutation modules of the symmetric group.

2.7 Applying the Representation Theory of $\mathscr{H}_n$ to $\mathscr{H}_{r,n}$

A key technique used in this thesis consists of reducing problems in $\mathscr{H}_{r,n}$ to problems in an Iwahori-Hecke algebra and applying results from this latter setting. One way in which we can do this is by 'stacking' the components on a multicomposition. Another, equally important but more restrictive in when it can be applied, method concerns studying the individual components of a multicomposition.

2.7.1 Stacking Components of Multicompositions

Let λ be a multipartition of n in r parts and let $\alpha(\lambda)$ be the composition of n determined by 'stacking' the components of $[\lambda]$ on top of one another such that $[\lambda^{(i+1)}]$ appears immediately below $[\lambda^{(i)}]$ for all $1 \le i \le r-1$. More formally, we have

Definition 2.7.1. If λ is a multicomposition of n given by

$$\lambda = \left(\left(\lambda_1^{(1)},\dots,\lambda_{\rho_1(\lambda)}^{(1)}\right),\left(\lambda_1^{(2)},\dots,\lambda_{\rho_2(\lambda)}^{(2)}\right),\dots,\left(\lambda_1^{(r)},\dots,\lambda_{\rho_r(\lambda)}^{(r)}\right)\right),$$

then $\alpha(\lambda)$ is the composition of n for which each row given by

$$\alpha(\lambda)_i = \lambda_k^{(j)}$$

where j and k are such that $i = \sum_{l=1}^{j-1}\rho_l(\lambda)+k$.

With this definition in place, we immediately have that $x_\lambda = y_{\alpha(\lambda)}$ and that the diagram $[\alpha(\lambda)]$ of $\alpha(\lambda)$ is the set

$$[\alpha(\lambda)] = \left\{(i,j)\in\mathbb{N}^2 : 1\le j\le \sum_{k=1}^{r}\rho_k(\lambda)\right\}$$

We can associate an $\alpha(\lambda)$-tableau in $\mathscr{H}_n$ to a λ-tableau in $\mathscr{H}_{r,n}$ in what amounts to a similar fashion. If $\mathfrak{t}$ is a λ-tableau, let $\alpha(\mathfrak{t}) : [\alpha(\lambda)] \to \{1,2,\dots,n\}$ be the $\alpha(\lambda)$-tableau given by

$$(i,j)\mapsto \mathfrak{t}(x,j,z)$$

where x and z are such that $i = \sum_{l=1}^{z-1}\rho_l(\lambda)+x$.

Example 4. *Let $\lambda = ((2,1),(3,3))$, and*

$$\mathfrak{t} = \left(\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 4 & 6 & 9 \\ \hline 2 & 7 & 8 \\ \hline\end{array}\right).$$

Then $\alpha(\lambda) = (2,1,3,3)$ and

$$\alpha(\mathfrak{t}) = \begin{array}{|c|c|c|}\cline{1-2} 1 & 5 \\ \cline{1-2} 3 \\ \hline 4 & 6 & 9 \\ \hline 2 & 7 & 8 \\ \hline\end{array}$$

Lemma 2.7.1. *If $\mathfrak{t}$ is a row standard λ-tableau, then*

1. *$\alpha(\mathfrak{t})$ is a row standard $\alpha(\lambda)$-tableau.*

2. *$T_{d(\mathfrak{t})} = T_{d(\alpha(\mathfrak{t}))}$*

Proof. Immediate. □

Writing $m_\lambda T_{d(\mathfrak{t})} = u_\lambda^+ y_{\alpha(\lambda)} T_{d(\alpha(\mathfrak{t}))}$ we may therefore apply results from the representation theory of $\mathscr{H}_n$ to $y_{\alpha(\lambda)} T_{d(\alpha(\mathfrak{t}))}$.

Given multicompositions μ and λ, we can perform the same kind of procedure on μ-tableaux of type λ for multicompositions. If $\mathtt{S} \in \mathscr{T}(\mu,\lambda)$, we define $\alpha(\mathtt{S})$ to be the $\alpha(\mu)$-tableau of type $\alpha(\lambda)$ formed by stacking the components of $\mathtt{S}$ in the same manner as was done for λ-tableaux. To complete this process, relabel every entry of the form u_v appearing in $\mathtt{S}$ using the rule

$$(u,v)\mapsto u+\rho_1(\lambda)+\dots+\rho_{v-1}(\lambda).$$

Lemma 2.7.2. *If* S *is a row-semistandard μ-tableau of type λ, then $\alpha(\mathtt{S})$ is a row-semistandard $\alpha(\mu)$-tableau of type $\alpha(\lambda)$.*

Proof. Suppose that $j < l$ are positive integers and that $\mathtt{S}(x,j,z) = (u,v)$ and $\mathtt{S}(x,l,z) = (s,t)$. If S is row-semistandard, then $(u,v) \preceq (s,t)$. Therefore, either $v = t$ and $u \leq s$ or $v < t$, and hence

$$u + \rho_1(\lambda) + \cdots + \rho_{v-1}(\lambda) \leq s + \rho_1(\lambda) + \cdots + \rho_{t-1}(\lambda).$$

Our proof now follows from the construction of $\alpha(\mu)$. □

Example 5. *If $\mu = ((3,2),(2,1,1))$, then $\alpha(\mu) = (3,2,2,1,1)$. Additionally, if $\lambda = ((2,1),(3,2,1))$ and $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ is given by*

$$\mathtt{S} = \left(\begin{array}{|c|c|c|} \hline 1_1 & 1_1 & 2_2 \\ \hline 2_1 & 1_2 \\ \cline{1-2} \end{array}\, , \begin{array}{|c|c|} \hline 1_2 & 1_2 \\ \hline 2_2 \\ \cline{1-1} 3_2 \\ \cline{1-1} \end{array} \right)$$

then

$$\alpha(\mathtt{S}) = \begin{array}{|c|c|c|} \hline 1 & 1 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

2.7.2 Individual Components of a Multicomposition

Under certain circumstances, we may also use the fact that each component of a multicomposition is a composition in order to work with an Iwahori-Hecke algebra of type A or something isomorphic to such an algebra.

Definition 2.7.2. For positive integers m and s, let σ_s^m be the isomorphism from $\mathfrak{S}_s$ to the symmetric group on $\{m+1, \ldots, m+s\}$ given by

$$\sigma_s^m((i,i+1)) = ((i+m, i+1+m)) \quad \text{for each } 1 \leq i \leq s-1$$

and set $\mathfrak{S}_{m,s} = \sigma_s^m \mathfrak{S}_s$.

For each multicomposition, let $\overline{\lambda}^{(i)} = \sum_{j=1}^{i-1} |\lambda^{(j)}|$ for each $1 \leq i \leq r$. An immediate consequence of this definition is that, for each multicomposition λ, then $\mathfrak{S}_{\lambda^{(i)}} = \mathfrak{S}_{\overline{\lambda}^{(i)}, |\lambda^{(i)}|}$ for every $1 \leq i \leq r$; that is to say, $\mathfrak{S}_{\lambda^{(i)}}$ is the group of permutations in $\mathfrak{S}_n$ that permute only the entire appearing in the i-th component of $\mathtt{t}^{\lambda}$.

Definition 2.7.3. For each $1 \leq i \leq r$, define $\mathscr{H}(\lambda^{(i)})$ be the subalgebra of $\mathscr{H}_{r,n}$ generated by the set

$$\left\{ T_{\overline{\lambda}^{(i)}+1}, T_{\overline{\lambda}^{(i)}+2}, \ldots, T_{\overline{\lambda}^{(i+1)}-1} \right\}.$$

We may then also define an isomorphism $\zeta_i^{\lambda} : \mathscr{H}_{|\lambda^{(i)}|} \to \mathscr{H}\left(\lambda^{(i)}\right)$ via

$$\zeta_i^{\lambda} : T_j \mapsto T_{\overline{\lambda}^{(i)}+j}$$

for each $1 \le j \le |\lambda^{(i)}|-1$.

The following lemma collects together some properties of the objects just defined that will help us transition between the Ariki-Koike algebra $\mathscr{H}_{r,n}$ and certain Iwahori-Hecke algebras of type A.

Lemma 2.7.3. *Let λ be a multicomposition. Then*

1. $\zeta_i^\lambda(y_{\lambda^{(i)}})\zeta_j^\lambda(y_{\lambda^{(j)}}) = \zeta_j^\lambda(y_{\lambda^{(j)}})\zeta_i^\lambda(y_{\lambda^{(i)}})$ *whenever $i \neq j$;*

2. $x_\lambda = \prod_{i=1}^r \zeta_i^\lambda(y_{\lambda^{(i)}})$*; and*

3. *if $w \in \mathfrak{S}_{\lambda^{(i)}}$ for any $1 \le i \le r$ and $v \in \mathfrak{S}_{|\lambda^{(i)}|}$ is such that $\sigma^{\overline{\lambda}^{(i)}}_{|\lambda^{(i)}|}(v) = w$ then*

$$x_\lambda T_w = \left(\zeta_1^\lambda\left(y_{\lambda^{(1)}}\right)\cdots\zeta_{i-1}^\lambda\left(y_{\lambda^{(i-1)}}\right)\right)\zeta_i^\lambda\left(y_{\lambda^{(i)}}T_v\right)\left(\zeta_{i+1}^\lambda\left(y_{\lambda^{(i+1)}}\right)\cdots\zeta_r^\lambda\left(y_{\lambda^{(r)}}\right)\right).$$

Proof. Writing $x_\lambda = \prod_{i=1}^r x_{\lambda^{(i)}}$, where $x_{\lambda^{(i)}} = \sum_{w\in\mathfrak{S}_{\lambda^{(i)}}} T_w$, the first part of the lemma follows immediately. As for the second part, if $w \in \mathfrak{S}_{\lambda^{(i)}}$, then T_w commutes with all $x_{\lambda^{(j)}}$ with $i \neq j$; hence, $x_\lambda T_w = \left(x_{\lambda^{(1)}}\cdots x_{\lambda^{(i-1)}}\right)x_{\lambda^{(i)}}T_w\left(x_{\lambda^{(i+1)}}\cdots x_{\lambda^{(r)}}\right)$. □

Let λ be a multicomposition. The importance of the lemma above and preceding definition is that we may express certain λ-tableaux in terms of tableaux for its individual components. Our requirements are such that we will address this matter on a case by case basis whenever we have cause to do so, rather than provide a general description of this situation.

2.8 Further Topics from the Representation Theory of the Ariki-Koike Algebra

Although not of direct relevance to this thesis, it would be remiss to write a thesis on the representation theory of an algebra and not discuss the irreducible modules for that algebra. This subsection therefore presented as a summary of such topics.

Irreducible $\mathscr{H}$-Modules

One important feature of the theory of cellular algebras is that it provides a partial classification of irreducible modules of such algebras. Phrased purely in terms of the Ariki-Koike algebra, this classification proceeds thus: Let λ be a multipartition. There is a unique symmetric and associative bilinear map $S^\lambda \times S^\lambda \to \mathbb{F}$ given by

$$\langle m_\mathfrak{s}, m_\mathfrak{t}\rangle m_{\mathfrak{u}\mathfrak{v}} = m_{\mathfrak{u}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{v}} \mod \check{\mathscr{H}}^\lambda$$

where $\mathfrak{u},\mathfrak{v} \in \mathrm{Std}(\lambda)$. Defining $\mathrm{rad}\, S^\lambda$ to be the radical of this bilinear form and setting $D^\lambda = S^\lambda/\mathrm{rad} S^\lambda$, we can now state this partial classification, due to Graham and Lehrer [24], of the irreducible modules of $\mathscr{H}$.

Theorem 2.8.1. *The set* $\{D^\lambda : \lambda$ *is such that* $D^\lambda \neq 0\}$ *is a complete set of pairwise inequivalent irreducible* $\mathscr{H}$*-modules.*

Following Ariki [2], we now complete this classification by identifying those multipartitions λ such that $D^\lambda \neq 0$. Let λ be a multipartition and say that a node $\mathbf{x} = (a,b,c) \in [\lambda]$ is an *i-node* if $i = \mathrm{res}_\lambda(\mathbf{x})$. If $\mathbf{x}$ is an i-node, we say that it is

1. *normal* if

   (a) whenever $\mathbf{y}$ is a removable i-node lower than $\mathbf{x}$, there are more removable i-nodes between $\mathbf{x}$ and $\mathbf{y}$ than there are addable i-nodes, and

   (b) there are at least as many removable i-nodes as there are addable i-nodes lower than $\mathbf{x}$;

2. *good* if there are no normal i-nodes higher than $\mathbf{x}$.

Note that (a) doesn't necessarily imply (b). We may have an instance where the lowest removable i-node occurs above a number of addable i-nodes

A multipartition μ is *Kleshchev* if μ is the empty multipartition or $[\mu] = [\lambda] \cup \{\mathbf{x}\}$ for a Kleshchev multipartition λ and a good node $\mathbf{x}$. In the most general case, Kleshchev multipartitions correspond precisely to the irreducible modules of $\mathscr{H}$, as this next theorem, due to Ariki, demonstrates.

Theorem 2.8.2 ([2]). *Suppose that* $q \neq 1$ *and* $Q_s \neq 0$ *for* $1 \leq s \leq r$*, and that* λ *is a multipartition of n, Then* $D^\lambda \neq 0$ *if and only if* λ *is Kleshchev.*

The appearance of Kleshchev multipartitions in the representation theory of $\mathscr{H}$ seems slightly mysterious and rather technical at first, especially compared with their analogues in the representation theory of $\mathscr{H}_n$ and $\mathfrak{S}_n$. That they do occupy such a role is due to the intimate connection between the Ariki-Koike algebra and quantum algebra, demonstrating how non-trivial the process of constructing such analogues can be when dealing with the Ariki-Koike algebra. We refer the reader to [3] for further details and references on this connection.

Chapter 3

A Cellular Analogue of James's Kernel Intersection Theorem

3.1 Introduction

Let λ be a partition and, for positive integers d and $0 \le t < \lambda_{d+1}$, let $\nu^{d,t}$ be the composition given by

$$\nu_i^{d,t} = \begin{cases} \lambda_i + \lambda_{i+1} - t & \text{if } i = d, \\ t & \text{if } i = d+1, \\ \lambda_i & \text{otherwise.} \end{cases}$$

In [10], each Specht module S^λ for $\mathscr{H}_n$ is defined as a particular submodule of M^λ, with the following theorem characterizes these as the intersection of the kernels of a family of homomorphisms $\psi_{d,t} : M^\lambda \to M^{\nu^{d,t}}$. Note that these are not the same as the Specht modules previously defined and which we consider throughout the remainder of this thesis.

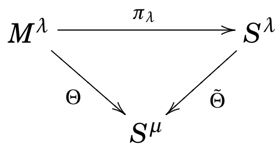
Theorem 3.1.1 ([10, Theorem 7.5])**.** *If λ is a partition of n, then*

$$\bigcap_{d \ge 1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi_{d,t} = S^\lambda.$$

This is the *kernel intersection theorem*, a corollary of which is that the image of a homomorphism $\Theta : S^\mu \to M^\lambda$ lies in S^λ, if and only if $\psi_{d,t} \circ \Theta = 0$ for every d and t. We may use this to identify and, with some more work, construct explicit homomorphisms between Specht modules, this being the strategy adopted in [20] and [37] for the symmetric group and Iwahori-Hecke algebra of type A respectively.

With respect to the cellular basis of Murphy, the definition of the Specht modules for $\mathscr{H}_n$ differs substantially from that of [10]. In particular, in that other setting a Specht module S^λ appears as a submodule of M^λ, which isn't true in the setting of the cellular basis we are working with. As a result, Theorem 3.1.1 is in our case false and we therefore require an alternative means of constructing homomorphisms in this setting. Just such a means appearing in [38], in which we are provided with an analogue of the kernel intersection theorem for the Murphy basis of $\mathscr{H}_n$ and the corresponding Specht modules in this setting. In this chapter we generalize this result further, extending it to the Ariki-Koike algebra with respect to the Dipper-James-Murphy basis defined in subsection 2.3.

The underlying philosophy of this chapter is that, for a given multipartition λ, a homomorphism $\Theta : M^\lambda \to S^\mu$ factors through S^λ if and only if $\Theta(m_\lambda h) = 0$ for all $h \in \mathscr{H}_{r,n}$ with $m_\lambda h \in \check{\mathscr{H}}^\lambda_{r,n}$. If this is the case, then Θ determines a homomorphism $\tilde{\Theta} : S^\lambda \to S^\mu$ such that the following diagram commutes:



where $\pi_\lambda : M^\lambda \to S^\lambda$ is the projection given by $\pi_\lambda(m_\lambda h) = \check{\mathscr{H}}^\lambda_{r,n} + m_\lambda h$ for all $h \in \mathscr{H}_{r,n}$. In particular, we prove that this condition is equivalent to $\Theta(m_\lambda h) = 0$ for only a finite family of elements $h \in \mathscr{H}_{r,n}$. Note that $\Theta(m_\lambda h) = 0$ for every $h \in \mathscr{H}_{r,n}$ with $m_\lambda h \in \check{\mathscr{H}}^\lambda_{r,n}$ if and only if $\Theta(x) = 0$ for every $x \in M^\lambda \cap \check{\mathscr{H}}^\lambda_{r,n}$. Therefore, we can restate our objective as being the construction of a finite set of generators of the right ideal $M^\lambda \cap \check{\mathscr{H}}^\lambda_{r,n}$.

This is the same reasoning as that employed in [38], and the set of generators we construct here is in fact a generalization of those appearing in that paper. Indeed, the relationship between the two is sufficiently close that there will be times at which, in light of section 2.7, we can use the results of [38] in proving our own.

3.2 Setting the Stage

Here we specify some of the notation and objects upon which the results of this chapter depend and that occupy a central role throughout this thesis. Let λ be a multicomposition of n:

- for every $1 \le i \le r$, let $\rho_i(\lambda) = \max\{x : \lambda^{(i)}_x \ne 0\}$; and
- for every pair of integers i and j with $1 \le i \le r$ and $0 \le j \le \rho_i(\lambda)$,

$$\overline{\lambda}^{(i)}_j = \sum_{k=1}^{i-1} |\lambda^{(k)}| + \sum_{l=1}^{j} \lambda^{(i)}_l \quad \text{and} \quad \overline{\lambda}^{(i)} = \sum_{k=1}^{i-1} |\lambda^{(i)}|.$$

Now let $\eta = (\eta_1, \eta_2, \dots, \eta_l)$ be a composition of n and, for each positive integer m, let σ^m_n be the isomorphism from $\mathfrak{S}_n$ to the symmetric group on $\{m+1, \dots, m+n\}$ given by $\sigma^m_n : (i, i+1) \mapsto (m+i, m+i+1)$ for every $1 \le i \le n-1$. We set $\mathscr{D}_{m,\eta} = \sigma^m_n \mathscr{D}_\eta$, where $\mathscr{D}_\eta$ is the the set of minimal length coset representatives of $\mathfrak{S}_\eta$ in $\mathfrak{S}_n$.

Definition 3.2.1.
$$\mathrm{C}(m:\eta) = \mathrm{C}(m:(\eta_1,\eta_2,\ldots,\eta_l)) = \sum_{w\in\mathscr{D}_{m,\eta}} T_w.$$

For a positive integer m, let $f_m : \{1,2,\ldots,n\} \to \{m+1,m+2,\ldots,m+n\}$ be the function given by $i \mapsto i+m$. Using the definition of $\mathscr{D}_\eta$, $\mathrm{C}(m:\eta)$ can be thought of as the sum of terms T_w such that $(f_m \circ \mathfrak{t}^\lambda)\cdot w$ is row standard[1].

Example 6. *Suppose that* $m = 4$ *and* $\eta = (2,2)$*, then*

$$f_4 \circ \mathfrak{t}^\eta = \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 & 8 \\ \hline\end{array}$$

and $\mathrm{C}(4:(2,2)) = 1 + T_6 + T_{6,5} + T_{6,7} + T_{6,7,5} + T_{6,7,5,6}$.

We now define two families of elements of $\mathscr{H}_{r,n}$ associated with each multipartition of n, calling these $\mathfrak{d}$- and $\mathfrak{l}$-elements. As we will see, these elements determine our chosen generators of $M^\lambda \cap \check{\mathscr{H}}^\lambda$, and may be thought of as performing a similar role in $\mathscr{H}_{r,n}$ as the homomorphisms $\psi_{d,t}$ do in the representation theory of the symmetric group or in the Dipper-James version of the representation theory of the Iwahori-Hecke algebra of type A.

$\mathfrak{d}$-elements of $\mathscr{H}$: For a multipartition λ of n, let $\mathrm{def}(\lambda,\mathfrak{d})$ be the set

$$\mathrm{def}(\lambda,\mathfrak{d}) = \left\{(d,t,s)\in\mathbb{N}^3 : 1\le s\le r, 1\le d<\rho_s(\lambda), 1\le t\le \lambda^{(s)}_{d+1}\right\}$$

and define
$$\mathfrak{d}^{(s)}_{d,t} = \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1};\left(\lambda^{(s)}_d,t\right)\right)$$
for each $(d,t,s)\in\mathrm{def}(\lambda,\mathfrak{d})$.

These are a direct generalization of the $h_{d,t}$ elements of $\mathscr{H}_n$ appearing in [38]. In fact, one way in which $\mathfrak{d}^{(s)}_{d,t}$ arises is to consider the partition $\lambda^{(s)}$ and the Iwahori-Hecke algebra $\mathscr{H}_{|\lambda^{(s)}|}$: we can take the element $h_{d,t}$ associated with $\lambda^{(s)}$ and set $\mathfrak{d}^{(s)}_{d,t} = \zeta^\lambda_{|\lambda^{(s)}|}(h_{d,t})$, where $\zeta^\lambda_{|\lambda^{(s)}|} : \mathscr{H}_{|\lambda^{(s)}|} \to \mathscr{H}(\lambda^{(s)})$ is the isomorphism defined in section 2.7.

$\mathfrak{l}$-elements of $\mathscr{H}$: For a multipartition λ of n, let $\mathrm{def}(\lambda,\mathfrak{l})$ be

$$\mathrm{def}(\lambda,\mathfrak{l}) = \left\{s\in\mathbb{Z} : 1\le s<r, \text{ and } \lambda^{(s+1)}\neq\emptyset\right\}$$

and define
$$\mathfrak{l}^{(s)} = \left(L_{\overline{\lambda}^{(s+1)}+1} - Q_{s+1}\right).$$

[1]This is something of an abuse of terminology, since, strictly speaking, $f_m\circ\mathfrak{t}^\lambda$ is not a tableau in the way we have defined the term. However, it has the advantage of making some kind of intuitive sense, and permitting it avoids having to have an overly complicated definition of tableaux that we use only once.

for each $s \in \operatorname{def}(\lambda, \mathfrak{l})$.

Should we wish to express the fact that a given $\mathfrak{d}$- or $\mathfrak{l}$-element is that which is associated with a particular multipartition λ, we will write $\mathfrak{d}^{(s)}_{d,t}(\lambda)$ or $\mathfrak{l}^{(s)}(\lambda)$.

Example 7. *Let* $\lambda = ((3,1),(2,2),(2,1,1))$. *Then*

$$\mathfrak{d}^{(1)}_{1,1} = 1 + T_3 + T_3T_2 + T_3T_2T_1 \qquad \mathfrak{d}^{(2)}_{1,1} = 1 + T_6 + T_6T_5$$

$$\mathfrak{d}^{(2)}_{1,2} = 1 + T_6 + T_6T_5 + T_6T_7 + T_6T_7T_5 + T_6T_7T_5T_6$$

$$\mathfrak{d}^{(3)}_{1,1} = 1 + T_{10} + T_{10}T_9 \qquad \mathfrak{d}^{(3)}_{2,1} = 1 + T_{11}$$

$$\mathfrak{l}^{(1)} = L_5 - Q_2 \qquad \mathfrak{l}^{(2)} = L_9 - Q_3.$$

For a multipartition λ of n, our set of generators of $M^\lambda \cap \check{\mathscr{H}}^\lambda$ will be the union of the sets

$$\mathbf{D}(\lambda) = \left\{ m_\lambda \mathfrak{d}^{(s)}_{d,t} : (d,t,s) \in \operatorname{def}(\lambda, \mathfrak{d}) \right\} \quad \text{and} \quad \mathbf{L}(\lambda) = \left\{ m_\lambda \mathfrak{l}^{(s)} : s \in \operatorname{def}(\lambda, \mathfrak{l}) \right\}.$$

More formally, let $\mathfrak{I}_\lambda$ be the right ideal of $\mathscr{H}$ generated by $\mathbf{D}(\lambda) \cup \mathbf{L}(\lambda)$. Then,

Claim 3.2.1. For each multipartition λ of n, $\mathfrak{I}_\lambda = M^\lambda \cap \check{\mathscr{H}}^\lambda$.

Much of the remainder of this chapter is dedicated to proving this claim, from which our main theorem then follows.

Theorem 3.2.1. *Let* $\Theta : M^\lambda \to S^\mu$ *be a homomorphism. Then the following are equivalent:*

1. $\Theta\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right) = 0$ *for every* $(d,t,s) \in \operatorname{def}(\lambda, \mathfrak{d})$ *and* $\Theta\left(m_\lambda \mathfrak{k}^{(s)}\right) = 0$ *for every* $s \in \operatorname{def}(\lambda, \mathfrak{l})$*;*
2. $\Theta(m_\lambda h) = 0$ *for every* $h \in \mathscr{H}_{r,n}$ *with* $m_\lambda h \in \check{\mathscr{H}}^\lambda$*; and*
3. Θ *factors through* S^λ.

Proof. The equivalence of 2. and 3. is straightforward and was dealt with in the introduction to this chapter, whereas the equivalence of 1. and 2. follows (almost) immediately from claim 3.2.1.

1. ⇒ 2. If $m_\lambda h \in \check{\mathscr{H}}^\lambda$, then $m_\lambda h \in M^\lambda \cap \check{\mathscr{H}}^\lambda$. Thus, $\Theta(m_\lambda h)$ is zero, since $\mathfrak{I}_\lambda = M^\lambda \cap \check{\mathscr{H}}^\lambda$ is generated as a right ideal by $\mathbf{D}(\lambda) \cap \mathbf{L}(\lambda)$.

2. ⇒ 1. Immediate. □

3.2.1 Applications to the Construction of Homomorphisms

Here we provide some motivation for Theorem 3.2.1 with a discussion of how we can use it to construct homomorphisms between Specht modules. We also take this as an opportunity to introduce further objects and notation that will feature in the rest of this thesis.

Each $\mathfrak{d}$- and $\mathfrak{l}$-element can be associated with a multicomposition corresponding to a diagram formed by rearranging certain nodes of λ.

Definition 3.2.2. For any $(s,d,t) \in \mathrm{def}(\lambda,\mathfrak{d})$ define the multicomposition $\lambda \cdot \mathfrak{d}^{(s)}_{d,t}$ by

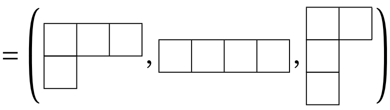
$$\left(\lambda \cdot \mathfrak{d}^{(s)}_{d,t}\right)^{(l)}_i = \begin{cases} \lambda^{(s)}_d + t & \text{if } l = s \text{ and } i = d, \\ \lambda^{(s)}_{d+1} - t & \text{if } l = s \text{ and } i = d+1, \\ \lambda^{(l)}_i & \text{otherwise.} \end{cases}$$

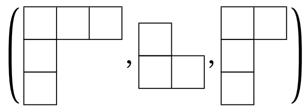
For any $s \in \mathrm{def}(\lambda,\mathfrak{l})$ define the multicomposition $\lambda \cdot \mathfrak{l}^{(s)}$ by

$$\left(\lambda \cdot \mathfrak{l}^{(s)}\right)^{(k)}_j = \begin{cases} 1 & \text{if } k = s \text{ and } j = \rho_s(\lambda)+1, \\ \lambda^{(s+1)}_1 - 1 & \text{if } k = s+1 \text{ and } j = 1, \\ \lambda^{(k)}_j & \text{otherwise.} \end{cases}$$

In other words, $\lambda \cdot \mathfrak{d}^{(s)}_{d,t}$ is the multicomposition whose diagram is formed by raising the last t nodes of the $(d+1)$-th row of the s-th component of $[\lambda]$ to the end of the d-th row of the same component. Similarly $\lambda \cdot \mathfrak{l}^{(s)}$ is the multicomposition whose diagram is obtained by removing the last node from the first row of component $s+1$ of $[\lambda]$ and inserting a new row consisting of a single node at the bottom of component s.

Example 8. *Let* $\lambda = ((3,1),(2,2),(2,1,1))$. *Then*

$\left[\lambda \cdot \mathfrak{d}^{(2)}_{1,2}\right] =$  *,and*

$\left[\lambda \cdot \mathfrak{l}^{(1)}\right] =$  .

Now suppose that $\Theta : M^\lambda \to S^\mu$ is a homomorphism of the form

$$\Theta = \sum_{\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)} a_{\mathtt{S}} \Theta_{\mathtt{S}}$$

where $a_{\mathtt{S}} \in \mathbb{F}$ and $\Theta_{\mathtt{S}} : M^\lambda \to S^\mu$ is the homomorphism determined by $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$. We shall see in Proposition 4.1.2 and Section 4.5 that, for each $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ and $(s,d,t) \in \mathrm{def}(\lambda,\mathfrak{d})$, we can express $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right)$ as an $\mathbb{F}$-linear combination of elements of the form

$$\Theta_{\mathtt{X}}\left(m_{\lambda \cdot \mathfrak{d}^{(s)}_{d,t}}\right),$$

where

$$\Theta_{\mathtt{X}} : M^{\lambda \cdot \mathfrak{d}^{(s)}_{d,t}} \to S^\mu$$

is a homomorphism for each $\mathtt{X} \in \mathscr{T}_0\left(\mu, \lambda \cdot \mathfrak{d}^{(s)}_{d,t}\right)$. Propositions 4.4.6 and 4.4.8 taken together with Section 4.5 perform a similar role for $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{l}^{(s)}\right)$ for each $s \in \mathrm{def}(\lambda,\mathfrak{l})$. Applying Theorem 3.2.1 is then a matter of collecting like terms and, using the linear independence of semistandard homomorphisms, identifying when these are zero. A further reason for introducing the multicompositions given in Definition 3.2.2 and the homomorphisms determined by them is that, since every such $\mathtt{S}$ is semistandard, these homomorphisms are linearly independent by [21, Corollary 6.14]; therefore, we can be sure that such linear combinations are non-zero.

Constructing Homomorphisms Between Specht Modules: an Example

Let $\lambda = ((2,2),(2,1))$ and $\mu = ((5),(2))$. Then $\mathbf{D}(\lambda) = \{m_\lambda \mathfrak{d}^{(1)}_{1.1}, m_\lambda \mathfrak{d}^{(1)}_{2,1}, m_\lambda \mathfrak{d}^{(2)}_{11}\}$ and $\mathbf{L}(\lambda) = \{m_\lambda \mathfrak{l}^{(1)}\}$, where

$$\mathfrak{d}^{(1)}_{1,1} = 1 + T_2 + T_{2,1} \qquad \mathfrak{d}^{(1)}_{2,2} = 1 + T_2 + T_{2,1} + T_{2,3} + T_{2,3,1} + T_{2,3,1,2}$$
$$\mathfrak{d}^{(2)}_{1.1} = 1 + T_6 + T_{6,5} \qquad \mathfrak{l}^{(1)} = L_5 - Q_2,$$

and $\mathscr{T}_0(\mu,\lambda) = \{\mathtt{S}_1, \mathtt{S}_2\}$, where

$$\mathtt{S}_1 = \left(\boxed{1_1|1_1|2_1|2_1|1_2}, \boxed{1_2|2_2}\right) \qquad \mathtt{S}_2 = \left(\boxed{1_1|1_1|2_1|2_1|2_2}, \boxed{1_2|1_2}\right),$$

and the homomorphisms determined by the tableaux $\mathtt{S}_1$ and $\mathtt{S}_2$ are given by

$$\Theta_{\mathtt{S}_1}(m_\lambda h) = \check{\mathscr{H}}^\mu + m_\mu(1+T_5)h, \qquad \Theta_{\mathtt{S}_2}(m_\lambda h) = \check{\mathscr{H}}^\mu + m_\mu T_{5,6} h$$

for every $h \in \mathscr{H}$.

Suppose now that $\Theta \in \operatorname{Hom}_{\mathscr{H}}(M^\lambda, S^\mu)$ is of the form

$$\Theta = \alpha_1 \Theta_{\mathtt{S}_1} + \alpha_2 \Theta_{\mathtt{S}_2}$$

for some $\alpha_1, \alpha_2 \in \mathbb{F}$. If we define $\mathtt{X}_1, \mathtt{X}_2 \in \mathscr{T}_0(\mu, \lambda\cdot\mathfrak{d}^{(1)}_{1,1})$, $\mathtt{X}_3, \mathtt{X}_4 \in \mathscr{T}_0(\mu, \mathfrak{d}^{(1)}_{2.1}\cdot\lambda)$, $\mathtt{X}_5 \in \mathscr{T}_0(\mu, \mathfrak{d}^{(2)}_{1,1}\cdot\lambda)$, and $\mathtt{B} \in \mathscr{T}_0(\mu, \mathfrak{l}^{(1)}\cdot\lambda)$ by

$$\mathtt{X}_1 = \left(\boxed{1_1|1_1|1_1|2_1|1_2}, \boxed{1_2|2_2}\right) \qquad \mathtt{X}_2 = \left(\boxed{1_1|1_1|1_1|2_1|2_2}, \boxed{1_2|1_2}\right)$$
$$\mathtt{X}_3 = \left(\boxed{1_1|1_1|1_1|1_1|1_2}, \boxed{1_2|2_2}\right) \qquad \mathtt{X}_4 = \left(\boxed{1_1|1_1|1_1|1_1|2_2}, \boxed{1_2|1_2}\right)$$
$$\mathtt{X}_5 = \left(\boxed{1_1|1_1|2_1|2_1|1_2}, \boxed{1_2|1_2}\right) \qquad \mathtt{B} = \left(\boxed{1_1|1_1|2_1|2_1|3_1}, \boxed{1_2|2_2}\right)$$

we have

$$\begin{aligned}
\Theta_{\mathtt{S}_1}\left(m_\lambda \mathfrak{d}^{(1)}_{1,1}\right) &= \Theta_{\mathtt{S}_1}(m_\lambda)\mathfrak{d}^{(1)}_{1,1} = \check{\mathscr{H}}^\mu + m_\mu\left(1 + T_2 + T_{2,1}\right)(1+T_5)\\
&= \check{\mathscr{H}}^\mu + \left(1+q+q^2\right) m_\mu (1+T_5)\\
&= \left(1+q+q^2\right)\psi_{\mathtt{X}_1}\left(m_{\mathfrak{d}^{(1)}_{1,1}\cdot\lambda}\right)
\end{aligned}$$

and so, performing the same calculation for each element of $\mathbf{D}(\lambda)$,

$$\begin{aligned}
\Theta\left(m_\lambda \mathfrak{d}^{(1)}_{1,1}\right) &= \alpha_1\left(1+q+q^2\right)\Theta_{\mathtt{X}_1}\left(m_{\mathfrak{d}^{(1)}_{1,1}\cdot\lambda}\right) + \alpha_2\left(1+q+q^2\right)\Theta_{\mathtt{X}_2}\left(m_{\mathfrak{d}^{(1)}_{1,1}\cdot\lambda}\right)\\
\Theta\left(m_\lambda \mathfrak{d}^{(1)}_{2,1}\right) &= \alpha_1(1+q)\left(1+q+q^2\right)\Theta_{\mathtt{X}_3}\left(m_{\mathfrak{d}^{(1)}_{1,1}\cdot\lambda}\right)\\
&\quad + \alpha_2(1+q)\left(1+q+q^2\right)\Theta_{\mathtt{X}_4}\left(m_{\mathfrak{d}^{(1)}_{1,1}\cdot\lambda}\right)\\
\Theta\left(m_\lambda \mathfrak{d}^{(2)}_{1,1}\right) &= \left(\alpha_1(1+q) + \alpha_2 q^2\right)\Theta_{\mathtt{X}_5}\left(m_{\mathfrak{d}^{(1)}_{1,1}\cdot\lambda}\right).
\end{aligned}$$

Setting each of these equations to zero, we see that $\alpha_1 = \alpha_2 = 0$, and hence $\theta \equiv 0$, whenever $e \neq 3$, and so let us assume that $e = 3$. In this case $\Theta\left(m_\lambda \mathfrak{d}^{(1)}_{1.1}\right)$ and $\Theta\left(m_\lambda \mathfrak{d}^{(1)}_{2,1}\right)$ are zero and

$$\Theta\left(m_\lambda \mathfrak{d}^{(2)}_{1,1}\right) = 0 \Leftrightarrow \alpha_2 = -q^{-2}(1+q)\alpha_1. \tag{3.1}$$

This leaves us with only $\Theta\left(m_\lambda \mathfrak{l}^{(1)}\right)$ to contend with. Let $i \le n$. Recall that the residue of i in $\mathfrak{t}^\mu$ is given by

$$\mathrm{res}_{\mathfrak{t}^\mu}(i) = q^{y-x} Q_z,$$

where $(x,y,z) \in [\mu]$ is such that $\mathfrak{t}^\mu(x,y,z) = i$. Using the fact, due to proposition 2.3.2, that $m_\mu L_i = \mathrm{res}_\mu(i) m_\mu$ and

$$T_i L_i = L_{i+1} T_i - (q-1) L_{i+1}$$

we have, upon a direct calculation,

$$\begin{aligned}
\Theta\left(m_\lambda \mathfrak{l}^{(1)}\right) &= \alpha_1 \Theta_{\mathrm{S}_1}(m_\mu)(L_5 - Q_2) + \alpha_2 \Theta_{\mathrm{S}_2}(m_\mu)(L_5 - Q_2) \\
&= \check{\mathscr{H}}^\mu + \alpha_1 m_\mu (1+T_5)(L_5 - Q_2) + \alpha_2 m_\mu T_{5,6}(L_5 - Q_2) \\
&= \check{\mathscr{H}}^\mu + \alpha_1 m_\mu (L_5 + L_6 T_5 - (q-1)L_6 - Q_2 - Q_2 T_5) \\
&+ \alpha_2 m_\mu (L_6 T_{5,6} - (q-1) L_6 T_6 - Q_2 T_{5,6}) \\
&= \check{\mathscr{H}}^\mu + \alpha_1 m_\mu (q^4 Q_1 + Q_2 T_5 - (q-1)Q_2 - Q_2 - Q_2 T_5) \\
&+ \alpha_2 m_\mu (Q_2 T_{5,6} - q(q-1)Q_2 - Q_2 T_{5,6}) \\
&= \left(\alpha_1\left(q^4 Q_1 - q Q_2\right) - \alpha_2 q(q-1) Q_2\right) \Theta_{\mathrm{B}}\left(m_{\lambda \cdot \mathfrak{l}^{(1)}}\right).
\end{aligned} \tag{3.2}$$

Note that $q(q-1)Q_2$ appears in the penultimate line due to the fact that $T_6 \in \mathfrak{S}_\mu$, and so $m_\mu T_6 = q m_\mu$.

Now, substituting the conclusion of (3.1) into (3.2) yields

$$\Theta\left(m_\lambda \mathfrak{l}^{(1)}\right) = \left(q^4 Q_1 - q^{-1} Q_2\right) \Theta_{\mathrm{B}}\left(m_{\lambda \cdot \mathfrak{l}^{(1)}}\right);$$

therefore, when $e = 3$, the homomorphism $\Theta : M^\lambda \to S^\mu$ given by $\Theta = \alpha \Theta_{\mathrm{S}_1} + \alpha_2 \Theta_{\mathrm{S}_2}$ factors through S^λ if and only if

$$-q^{-2}(1+q)\alpha_1 = \alpha_2 \qquad \text{and} \qquad \alpha_1\left(q^4 Q_1 - q^{-1} Q_2\right) = 0.$$

Moreover, this then easily provides us with a homomorphism $\hat{\Theta} \in \mathrm{Hom}_{\mathscr{H}}(S^\lambda, S^\mu)$. For instance, setting $\alpha_1 = 1$ we have that

$$\hat{\Theta}(m_\lambda h) = \check{\mathscr{H}}^\mu + \Theta_{\mathrm{S}_1}(m_\lambda h) - q^{-2}(1+q)\Theta_{\mathrm{S}_2}(m_\lambda h)$$

is just such a homomorphism, from which we may also conclude that $\mathrm{Hom}_{\mathscr{H}}(S^\lambda, S^\mu)$ is non-empty if $(q^4 Q_1 - q^{-1} Q_2) = 0$.

3.3 Proof of Claim 3.2.1

Our starting point is to provide a basis of $M^\lambda \cap \check{\mathscr{H}}^\lambda$ when regarded as an $\mathbb{F}$-module. The remainder of the proof then consisting of showing that each element of this basis can be expressed as a linear combination of elements from $\mathbf{D}(\lambda)$ and $\mathbf{L}(\lambda)$.

Lemma 3.3.1. *The right ideal $M^\lambda \cap \check{\mathscr{H}}^\lambda$ is free as an $\mathbb{F}$-module with basis*

$$\{m_{\mathtt{St}} : \mathtt{S} \in \mathscr{T}_0(\nu,\lambda), \mathfrak{t} \in \mathrm{Std}(\nu) \text{ for } \nu \text{ a multipartition with } \lambda \lhd \nu\}.$$

Proof. By Theorem 2.4.1, every element of $M^\lambda \cap \check{\mathscr{H}}^\lambda$ can be expressed as a linear combination of elements $m_{\mathtt{St}}$ with $\mathtt{S} \in \mathscr{T}_0(\nu,\lambda)$ and $\mathfrak{t} \in \mathrm{Std}(\nu)$, with ν ranging over the multipartitions of n. Moreover, these elements are linearly independent over $\mathbb{F}$. The lemma then follows from the basis for $\check{\mathscr{H}}^\lambda$ given in Section 2.3. □

Since $m_{\mathtt{St}} = m_{\mathtt{St}^\nu} T_{d(\mathfrak{t})}$ for every $\mathfrak{t} \in \mathrm{Std}(\nu)$, we will have $M^\lambda \cap \check{\mathscr{H}}^\lambda \subseteq \mathfrak{J}_\lambda$ if we have that

$$m_{\mathtt{St}^\nu} = \sum_{(d,t,s)\in \mathrm{def}(\lambda,\mathfrak{d})} \gamma^{(s)}_{d,t} m_\lambda \mathfrak{d}^{(s)}_{d,t} h^{(s)}_{d,t} + \sum_{u \in \mathrm{def}(\lambda,\mathfrak{l})} \gamma^{(u)} m_\lambda \mathfrak{l}^{(u)} h^{(u)} \in \mathfrak{J}_\lambda \tag{3.3}$$

where $\gamma^{(s)}_{d,t}, \gamma^{(u)} \in \mathbb{F}$ and $h^{(s)}_{d,t}, h^{(u)} \in \mathscr{H}$, for every $\mathtt{S} \in \mathscr{T}_0(\nu,\lambda)$ with ν a multipartition of n dominating μ. In fact, we will prove a slightly more general result with the aim of gaining a sufficient condition for when row-semistandard tableaux determine homomorphisms between permutation modules. As a first step towards this goal we provide the following easy but very useful lemma.

Lemma 3.3.2. *Let $w \in \mathfrak{S}_n$ be such that, for every i, $w(i)$ appears in the same component of $\mathfrak{t}^\nu$ as does i. Then $T_w u^+_\nu = u^+_\nu T_w$.*

Proof. Since w only permutes entries within each component of $\mathfrak{t}^\nu$, we can write w as a product $w = w_1 w_2 \cdots w_r$ where $w_i \in \mathfrak{S}_{\overline{\nu}^{(i)},\nu^{(i)}}$. Each w_i fixes all entries not in the set $\{\overline{\nu}^{(i)}+1, \overline{\nu}^{(i)}+2, \ldots, \overline{\nu}^{(i)}+|\nu^{(i)}|\}$, and so w does not involve $s_{\overline{\nu}^{(i)}}$ for any $1 \le i \le r$. Therefore, T_w may be expressed as $T_{w_1} T_{w_2} \cdots T_{w_r}$, each term commuting with u^+_ν by Proposition 2.1.2 and the definition of u^+_ν. □

The following notation mimics that appearing in [38].

Definition 3.3.1. Let ν and λ be multicompositions and let $\mathtt{S}$ be a ν-tableau of type λ:

- For each pair of integers (i,j) with $1 \le j \le r$ and $1 \le i \le \rho_j(\lambda)$ and (k,l) with $1 \le l \le r$ and $1 \le k \le \rho_l(\nu)$, define $\mathtt{S}^{(i,j)}_{(k,l)}$ to be the number of entries appearing in the k-th row of the l-th component of $\mathtt{S}$ that are equal to (i,j).
- For $1 \le j \le r$ and $1 \le i \le \rho_j(\lambda)$, let $\Gamma^{\mathtt{S}}_{(x,y)}$ be the sequence

$$\mathtt{S}^{(1,1)}_{(i,j)}, \mathtt{S}^{(2,1)}_{(i,j)}, \ldots, \mathtt{S}^{(\rho_1(\lambda),1)}_{(i,j)}, \mathtt{S}^{(1,2)}_{(i,j)}, \mathtt{S}^{(2,2)}_{(i,j)}, \ldots, \mathtt{S}^{(\rho_2(\lambda),2)}_{(i,j)}, \ldots, \mathtt{S}^{(1,r)}_{(i,j)}, \mathtt{S}^{(2,r)}_{(i,j)}, \ldots, \mathtt{S}^{(\rho_r(\lambda),r)}_{(i,j)}.$$

- Let S be a ν-tableau of type λ and let $\mathrm{t_S}$ be the row-standard λ-tableau in which i occupies a node in row u of component v if the place occupied by i in t^{ν} is occupied by (u,v) in S. If $\mathrm{t_S} = \mathrm{t}^{\lambda} \cdot w$, we set $T_{\mathrm{S}} = T_w$. This is a generalization of the 1_{S} notation introduced in [10].

Our next lemma makes use of the definitions and general reasoning outlined in section 2.7.

Lemma 3.3.3. *Suppose that λ and ν are multipartitions of n and that $\mathrm{S} \in \mathscr{T}_r(\nu,\lambda)$. Then*

$$m_{\mathrm{St}^{\nu}} = x_{\lambda} T_{\mathrm{S}} u_{\nu}^{+} \prod_{i,j\geq 1} \mathrm{C}\left(\overline{\nu}_{i-1}^{(j)} : \Gamma_{(i,j)}^{\mathrm{S}}\right).$$

Proof. Recall that $x_{\nu} = y_{\alpha(\nu)}$ and consider the row-semistandard $\alpha(\nu)$-tableau $\alpha(\mathrm{S})$ of type $\alpha(\lambda)$. This allows us to write

$$m_{\mathrm{St}^{\nu}} = \sum_{\substack{x\in\mathscr{D}_{\nu} \\ \lambda(\mathrm{t}^{\nu}x)=\mathrm{S}}} T_x^{*} x_{\nu} u_{\nu}^{+} = \left(\sum_{\substack{z\in\mathscr{D}_{\gamma} \\ \iota(\mathrm{t}^{\gamma}z)=\mathrm{T}}} T_z^{*} y_{\gamma} \right) u_{\nu}^{+} = y_{\mathrm{Tt}^{\gamma}} u_{\nu}^{+},$$

where

- the first equality is a consequence of the definition of $m_{\mathrm{St}^{\nu}}$ given in Subsection 2.4 and the fact that S is semistandard forces the tableaux $\mathrm{t}^{\nu} \cdot x$ to be standard; and
- $\gamma = \alpha(\nu)$, $\iota = \alpha(\lambda)$, and $\mathrm{T} = \alpha(\mathrm{S})$.

Note that, since the expressions involving γ and ι are merely the 'stacked' versions of

$$\sum_{\substack{x\in\mathscr{D}_{\nu} \\ \lambda(\mathrm{t}^{\nu}x)=\mathrm{S}}} T_x^{*} x_{\nu},$$

the x's and z's being considered are in fact the same permutations

Applying [38, Corollary 3.7] to $y_{\alpha(\mathrm{S})\alpha(\mathrm{t})^{\alpha(\nu)}}$ yields

$$\begin{aligned} y_{\mathrm{Tt}^{\gamma}} &= y_{\alpha(\lambda)} T_{\alpha(\mathrm{S})} \prod_{i\geq 1} \mathrm{C}\left(\alpha(\nu)_{i-1} : \alpha(\mathrm{S})_i^1, \ldots, \alpha(\mathrm{S})_i^{\rho(\alpha)}\right) \\ &= x_{\lambda} T_{\mathrm{S}} \prod_{i,j\geq 1} \mathrm{C}\left(\overline{\nu}_{i-1}^{(j)} : \Gamma_{(i,j)}^{\mathrm{S}}\right). \end{aligned}$$

By definition, for each i and j the term $\mathrm{C}\left(\overline{\nu}_{i-1}^{(j)} : \Gamma_{(i,j)}^{\mathrm{S}}\right)$ consists only of elements T_w of $\mathfrak{S}_{\nu^{(i)}}$; therefore, $\mathrm{C}\left(\overline{\nu}_{i-1}^{(j)} : \Gamma_{(i,j)}^{\mathrm{S}}\right) u_{\nu}^{+} = u_{\nu}^{+}\mathrm{C}\left(\overline{\nu}_{i-1}^{(j)} : \Gamma_{(i,j)}^{\mathrm{S}}\right)$, by Lemma 3.3.2. This then completes the proof. □

Example 9. *Let $\lambda = ((3,2,2),(3,1,1))$ and $\nu = ((4,3,2,1),(2))$ and let $\mathrm{S} \in \mathscr{T}_0(\nu,\lambda)$ be given by*

$$\mathrm{S} = \left(\begin{array}{|c|c|c|c|} \hline 1_1 & 1_1 & 1_1 & 1_2 \\ \hline 2_1 & 2_1 & 3_1 \\ \cline{1-3} 3_1 & 3_2 \\ \cline{1-2} 2_2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1_2 & 1_2 \\ \hline \end{array} \right)$$

Then

$$\begin{aligned} m_{\mathtt{St}^\nu} &= (1+T_8+T_8T_9)(1+T_6)T_{7,6,5,4}T_{11,10,9}T_{11,10}m_\nu \\ &= x_\lambda T_{7,6,5,4}T_{11,10,9}T_{11,10}u_\nu^+(1+T_3+T_{3,2}+T_{3,2,1})(1+T_6+T_{6,5})(1+T_8) \\ &= x_\lambda T_{\mathtt{S}} u_\nu^+ \prod_{i,j\ge 1} \mathrm{C}\left(\overline{\nu}_{i-1}^{(j)} : \Gamma_{(i,j)}^{\mathtt{S}}\right). \end{aligned}$$

As a consequence of this lemma and we need only focus on how $T_{\mathtt{S}}$ interacts with u_ν^+ for each multipartition ν of n dominating λ and every $\mathtt{S} \in \mathcal{T}_0(\lambda,\nu)$ in order to show that $m_{\mathtt{St}^\nu}$ is of the form given in 3.3. This breaks down into two cases:

- when all of the components of λ are the same size as their counterparts in ν; and
- when at least one component of ν is larger than the corresponding component of λ.

Of these two cases, the former is considerably easier and so we consider this first. To help distinguish between these two cases, we will informally view ν as being constructed from λ by a process moving nodes around the diagram $[\lambda]$. In the first case, $[\nu]$ can be formed by moving nodes within the components of $[\lambda]$, which we will describe as ν being a *component-wise shift* of λ. The second case, where ν is in part formed by moving nodes between components, is described as ν as being a *cross-component shift* of λ.

Before we study these two cases, recall that one of the conditions for a tableau $\mathtt{S} \in \mathcal{T}(\nu,\lambda)$ to be semistandard, as given in Definition 2.4.2, is that

$$\text{if } (a,b,c) \in [\nu] \text{ and } \mathtt{S}(a,b,c) = (i,k), \text{ then } c \le k \tag{3.4}$$

Subsequent results throughout this thesis depend heavily on our non-semistandard tableaux satisfying this condition, and so we define $\mathcal{T}_{r,0}(\nu,\lambda)$ as the set of row-semistandard ν-tableaux of type λ satisfying (3.4).

Definition 3.3.2. We say that $\mathtt{S}$ is *quasi-semistandard* whenever $\mathtt{S} \in \mathcal{T}_{r,0}(\nu,\lambda)$.

3.3.1 Component-wise shifts

Let λ and ν be multicompositions such that $\lambda \lhd \nu$ and $|\lambda^{(s)}| = |\nu^{(s)}|$ for every integer s with $1 \le s \le r$. Then:

- the s-th components of each multipartition are partitions of the same size, which we denote by n_s;
- since $\overline{\nu}^{(s)} = \overline{\lambda}^{(s)}$ and $\lambda \unlhd \nu$ we have that $\lambda^{(s)} \unlhd \nu^{(s)}$ for every value of s; and
- the set of entries appearing in a given component of $\mathtt{t}^\lambda$,

  $$\left\{\overline{\lambda}_{(s)}+1, \overline{\lambda}_{(s)}+2, \dots, \overline{\lambda}_{(s+1)}\right\},$$

  is the same the set of entries appearing in the corresponding component of $\mathtt{t}^\nu$.

Lemma 3.3.4. *Let* $\mathrm{S} \in \mathscr{T}_{r,0}(\nu,\lambda)$ *and let* $w_{\mathrm{S}} \in \mathfrak{S}_n$ *be such that* $T_{\mathrm{S}} = T_w$. *Then, for all* i, $1 \le i \le n$ *and* $w_{\mathrm{S}}(i)$ *appear in the same component of* t^λ.

Proof. We proceed by induction on the component index s. Let $s = 1$ and suppose that i appears in the first component of t^λ. If $w_{\mathrm{S}}(i)$ does not appear in the first component of t^λ, then at least one node $(x,y,1) \in [\nu]$ such that $\mathrm{S}(x,y,1) = (u,v)$ for some $v \ge 2$. Since $|\lambda^{(1)}| = |\nu^{(1)}|$ we must there for have some entry of the form $(a,1)$ appearing in a component of S lower than the first, contradiction our assumption that S is quasi-semistandard.

Suppose now that the lemma holds for all values of s less than some arbitrary k and that i appears in component k of t^λ. Suppose also that $w_{\mathrm{S}}(i)$ does not appear in component k of t^λ. Since S is quasi-semistandard and $|\lambda^{(k)}| = |\nu^{(k)}|$, we have that $w_{\mathrm{S}}(i)$ must appear in a component lower than k. But then component k of S contains a term of the form (u,v) with $v \ge k+1$, meaning that there is an entry of the form (a,k) appearing in some component l of S other than k. By our inductive hypothesis, $l \ge k$, but this contradicts our assumption that S is quasi-semistandard. Thus, i and $w_{\mathrm{S}}(i)$ both appear in component k of t^λ and, by induction, the lemma is proved. □

For each $1 \le j \le r$, let $w_j \in \mathfrak{S}_{\lambda^{(j)}}$ be such that the s-th component of $\mathrm{t}^\lambda \cdot w_{\mathrm{S}}$ is the same as that of $\mathrm{t}^\lambda \cdot w_j$. Setting $T_{\mathrm{S}^{(j)}} = T_{w_j}$, we have a decomposition of T_{S} in terms of the elements $T_{\mathrm{S}^{(j)}}$.

Lemma 3.3.5. *Let* λ *and* ν *be multipartitions such that* $\lambda \trianglelefteq \nu$ *and* $|\lambda^{(i)}| = |\nu^{(i)}|$ *for all* $1 \le i \le r$. *Then, whenever* $\mathrm{S} \in \mathscr{T}_{r,0}(\nu,\lambda)$, *the elements* $T_{\mathrm{S}^{(1)}}, \dots, T_{\mathrm{S}^{(r)}}$ *commute pairwise and*

$$T_{\mathrm{S}} = \prod_{j=1}^{r} T_{\mathrm{S}^{(j)}}.$$

Moreover,

$$T_{\mathrm{S}} \prod_{x,y \ge 1} \mathrm{C}\left(\overline{\nu}^{(y)}_{(x-1)} : \Gamma^{\mathrm{S}}_{(x,y)}\right) = \prod_{y=1}^{r} \left(T_{\mathrm{S}^{(y)}} \prod_{x \ge 1} \mathrm{C}\left(\overline{\nu}^{(y)}_{(x-1)} : \Gamma^{\mathrm{S}}_{(x,y)}\right)\right).$$

Proof. Since $w_j \in \mathfrak{S}_{\lambda^{(j)}}$ for each $1 \le j \le r$, we may write $w = w_1 w_2 \cdots w_r$. Therefore, $T_{w_j} T_{w_k} = T_{w_k} T_{w_j}$ whenever $j \ne k$, due to the defining relations of $\mathscr{H}$, and so

$$T_{\mathrm{S}} = \prod_{j=1}^{r} T_{\mathrm{S}^{(j)}}.$$

By similar reasoning, $T_{\mathrm{S}^{(j)}}$ commutes with $\mathrm{C}\left(\overline{\nu}^{(y)}_{x-1} : \Gamma^{\mathrm{S}}_{(x,y)}\right)$ whenever $j \ne y$, since each term of $\mathrm{C}\left(\overline{\nu}^{(y)}_{x-1} : \Gamma^{\mathrm{S}}_{(x,y)}\right)$ is an element of $\mathfrak{S}_{\lambda^{(i)}}$. □

We are now almost in a position to present the main result of this subsection, that being that $m_{\mathrm{S}\mathrm{t}^\nu} \in \mathfrak{I}_\lambda$ for every $\mathrm{S} \in \mathscr{T}_0(\nu,\lambda)$ whenever ν is a component-wise shift of λ. Its proof requires a fairly technical factorization of x_λ and relies on being able to set certain problems in the context of Iwahori-Hecke algebras of type A as detailed in Section 2.7. In particular, we will make use of the following result, due to Lyle.

Theorem 3.3.6 ([38, Theorem 2.3]). *Let $\mathscr{H}_n$ be the Iwahori-Hecke algebra of type A and let λ be a partition of n. Then $N^\lambda \cap \check{\mathscr{H}}_n^\lambda = \mathfrak{I}_\lambda$.*

Here N^λ and $\check{\mathscr{H}}_n^\lambda$ refer to the 'type A' analogues, described in Definition 2.6.2, of M^μ and $\check{\mathscr{H}}^\mu$ (where λ is a partition of n and μ a multipartition). In this case $\mathfrak{I}_\lambda$ refers to the ideal of $\mathscr{H}_n$ generated by elements $y_\lambda \mathfrak{d}^{(1)}_{d,t}(\lambda)$.

We will first prove that a result analogous to Theorem 3.3.6 holds for component-wise shifts. Doing so will require us to make use of the the isomorphism $\zeta_i^\lambda : \mathscr{H}_{|\lambda^{(i)}|} \to \mathscr{H}\left(\lambda^{(i)}\right)$ given in Definition 2.7.3 and our setting

$$x_{\lambda\setminus\lambda^{(k)}} = \prod_{i=1}^{k-1} \zeta_i^\lambda\left(y_{\lambda^{(i)}}\right) \prod_{j=k+1}^{r} \zeta_j^\lambda\left(y_{\lambda^{(j)}}\right)$$

for each $1 \le k \le r$. Note that, as a consequence of Lemma 2.7.3, we have $x_\lambda = x_{\lambda\setminus\lambda^{(k)}}\zeta_k^\lambda\left(y_{\lambda^{(k)}}\right)$. In what follows, for $\mathtt{S}$ a quasi-semistandard ν-tableau of type λ, we write

$$m_{\mathtt{S}\mathfrak{t}^\nu} = \sum_{\substack{\mathfrak{s}\in \mathrm{RStd}(\nu)\\ \lambda(\mathfrak{s})=\mathtt{S}}} T^*_{d(\mathfrak{s})} m_\nu \tag{3.5}$$

Proposition 3.3.7. *Let λ and ν be multipartitions with $\lambda \lhd \nu$ and $|\lambda^{(i)}| = |\nu^{(i)}|$ for all $1 \le i \le r$. If $\mathtt{S} \in \mathscr{T}_{r,0}(\nu,\lambda)$, then $m_{\mathtt{S}\mathfrak{t}^\nu} \in M^\lambda$. Moreover, if $\mathtt{S}$ is semistandard, then $m_{\mathtt{S}\mathfrak{t}^\nu} \in \mathfrak{I}_\lambda$.*

Proof. If $|\lambda^{(i)}| = |\nu^{(i)}|$ for all $1 \le i \le r$, then $u_\lambda^+ = u_\nu^+$. Also, by Lemma 3.3.3, we have

$$m_{\mathtt{S}\mathfrak{t}^\nu} = \sum_{\substack{\mathfrak{s}\in \mathrm{RStd}(\nu)\\ \lambda(\mathfrak{s})=\mathtt{S}}} m_{\mathfrak{s}\mathfrak{t}^\nu} = \left(x_\lambda T_{\mathtt{S}} u_\lambda^+\right) \prod_{x,y\ge 1} \mathrm{C}\left(\overline{\nu}_{x-1}^{(y)} : \Gamma^{\mathtt{S}}_{(x,y)}\right).$$

By Lemma 3.3.2, $T_{\mathtt{S}} u_\lambda^+ = u_\lambda^+ T_{\mathtt{S}}$, and so we can write

$$\begin{aligned} m_{\mathtt{S}\mathfrak{t}^\nu} &= x_\lambda u_\lambda^+ T_{\mathtt{S}} \prod_{x,y\ge 1} \mathrm{C}\left(\overline{\nu}_{x-1}^{(y)} : \Gamma^{\mathtt{S}}_{(x,y)}\right) \\ &= m_\lambda T_{\mathtt{S}} \prod_{x,y\ge 1} \mathrm{C}\left(\overline{\nu}_{x-1}^{(y)} : \Gamma^{\mathtt{S}}_{(x,y)}\right). \end{aligned}$$

Thus, our first statement is true, this obviously being an element of M^λ.

To show that $m_{\mathtt{S}}\mathfrak{t}^\mu \in \mathfrak{I}_\lambda$ whenever $\mathtt{S}$ is semistandard, suppose that k is the least value for which $\lambda^{(k)} \lhd \nu^{(k)}$. Lemma 3.3.5 and the fact that $T_{\mathtt{S}^{(i)}} = 1$ for all $1 \le i < k$ allows us to write

$$m_{\mathtt{S}\mathfrak{t}^\nu} = u_\lambda^+ x_\lambda \prod_{j=k}^{r} \left(T_{\mathtt{S}^{(j)}} \prod_{i=1}^{\rho_j(\nu)} \mathrm{C}\left(\overline{\nu}_{i-1}^{(s)} : \Gamma^{\mathtt{S}}_{(i,j)}\right)\right). \tag{3.6}$$

Writing $x_\lambda = x_{\lambda\setminus\lambda^{(k)}}\zeta_k^\lambda\left(y_{\lambda^{(k)}}\right)$ and substituting into (3.6) yields

$$m_{\mathtt{S}\mathfrak{t}^\nu} = u_\lambda^+ x_{\lambda\setminus\lambda^{(k)}} \left(\zeta_k^\lambda\left(y_{\lambda^{(k)}}\right) T_{\mathtt{S}^{(k)}} \prod_{i=1}^{\rho_k(\nu)} \mathrm{C}\left(\overline{\nu}_{i-1}^{(k)} : \Gamma^{\mathtt{S}}_{(i,k)}\right)\right) h, \tag{3.7}$$

where h consists of all remaining terms of the product that appears in (3.6). Our proof proceeds by first showing that

$$\zeta_k^\lambda\left(y_{\lambda^{(k)}}\right)T_{\mathrm{S}^{(j)}}\prod_{i=1}^{\rho_k(\nu)}\mathrm{C}\left(\overline{\nu}_{i-1}^{(s)}:\Gamma_{(i,k)}^{\mathrm{S}}\right)\in\zeta_k^\lambda\left(N^{\lambda^{(k)}}\cap\check{\mathscr{H}}_{|\lambda^{(k)}|}^{\lambda^{(k)}}\right),$$

By setting $\mathtt{T}\in\mathscr{T}(\nu^{(k)},\lambda^{(k)})$ to be the tableau obtained by relabeling the entries of $\mathrm{S}^{(k)}$ according to the rule

$$\mathrm{S}^{(k)}(i,j,k)=(l,k)\Rightarrow\mathtt{T}(i,j)=l$$

we now have that

$$\zeta_k^\lambda\left(y_{\lambda^{(k)}}\right)T_{\mathrm{S}^{(k)}}\prod_{i=1}^{\rho_k(\nu)}\mathrm{C}\left(\overline{\nu}_{i-1}^{(k)}:\Gamma_{(i-1,k)}^{\mathrm{S}}\right)=\zeta_k^\lambda\left(y_{\mathtt{T}\mathfrak{t}^{\nu^{(k)}}}\right).$$

Note that this rule means that $\mathtt{T}$ is a semistandard $\nu^{(k)}$-tableaux of type $\lambda^{(k)}$ whenever S is semistandard. This follows from the fact that the condition $|\lambda^{(k)}|=|\nu^{(k)}|$ for every $1\le k\le r$ means that all the entries appearing in the k-th component of S are all of the form $(\square,k)$, which, in turn, means that l is non-decreasing along the rows of this component and strictly increasing down its columns. Hence,

$$y_{\mathtt{T}\mathfrak{t}^{\nu^{(k)}}}\in N^{\lambda^{(k)}}\cap\check{\mathscr{H}}_{|\lambda^{(k)}|}^{\lambda^{(k)}}.$$

By Theorem 3.3.6 this implies that

$$\zeta_k^\lambda\left(y_{\mathtt{T}\mathfrak{t}^{\nu^{(k)}}}\right)=\zeta_k^\lambda\left(y_{\lambda^{(k)}}\sum_{d,t}\mathfrak{d}_{d,t}^{(1)}(\lambda^{(k)})h'\right)=\left(\zeta_k^\lambda\left(y_{\lambda^{(k)}}\right)\sum_{d,t}\mathfrak{d}_{d,t}^{(k)}(\lambda)\right)\zeta_k^\lambda\left(h'\right)$$

where $1\le d$, $1\le t\le\lambda_{k+1}^{(k)}$, and $h'\in\mathscr{H}_{|\lambda^{(n)}|}$, and so our proof is complete since substituting this into (3.7) yields

$$m_{\mathrm{S}\mathfrak{t}^\nu}=\left(m_\lambda\sum_{d,t}\mathfrak{d}_{d,t}^{(k)}(\lambda)\right)\zeta_k^\lambda\left(h'\right)h\in\mathfrak{I}_\lambda.\qquad\square$$

3.3.2 Cross component shifts

As in the case for component-wise shifts, we use Lemma 3.3.3 to write

$$m_{\mathrm{S}\mathfrak{t}^\nu}=\left(x_\lambda T_{\mathrm{S}}\prod_{x,y\ge1}\mathrm{C}\left(\overline{\nu}_{(x-1,y)}:\Gamma_{(x,y)}^{\mathrm{S}}\right)\right)u_\nu^+$$

with the aim of expressing the right hand side as an element of $\mathfrak{I}_\lambda$ whenever $\lambda\lhd\nu$ and $\mathrm{S}\in\mathscr{T}_0(\nu,\lambda)$. Here the situation we're interested in is where $|\lambda^{(i)}|<|\nu^{(i)}|$ for at least one $1\le i\le r-1$. As before, we have

$$\prod_{x,y\ge1}\mathrm{C}\left(\overline{\nu}_{(x-1,y)}:\Gamma_{(x,y)}^{\mathrm{S}}\right)u_\nu^+=u_\nu^+\prod_{x,y\ge1}\mathrm{C}\left(\overline{\nu}_{(x-1,y)}:\Gamma_{(x,y)}^{\mathrm{S}}\right)$$

and so we focus our attention upon $x_\lambda T_{\mathrm{S}} u_\nu^+$.

In this case, it is not in general true that $T_{\mathrm{S}} u_\nu^+ = u_\nu^+ T_{\mathrm{S}}$. Our intent, therefore, is to determine how T_{S} interacts with u_ν^+, with the aim of writing $x_\lambda T_{\mathrm{S}} u_\nu^+$ in the form given in (3.3).

In order to proceed, we must introduce a particular family of λ-tableaux. As before, let λ and ν be multipartitions with $\lambda \lhd \nu$, and, recalling Definition 3.3.2, let $\mathrm{S} \in \mathcal{T}_{r,0}(\nu, \lambda)$. The tableaux in question will be defined recursively as follows:

1. Recalling Definition 3.3.1, set $\mathfrak{t}_{\mathrm{S}(0)} = \mathfrak{t}_{\mathrm{S}}$;
2. For $1 \le i < r-1$ define $\mathfrak{t}_{\mathrm{S}(i)}$ to be the λ-tableau obtained from $\mathfrak{t}_{\mathrm{S}(i-1)}$ by setting the entries in the first i components to be the same as in $\mathfrak{t}^\lambda$. Note that the first i components of $\mathfrak{t}_{\mathrm{S}}$ contain only entries from the set $\{1, 2, \ldots, \overline{\nu}_{(i+1)}\}$ and so there are then $\tau_i = \sum_{j=1}^{i}(|\nu^{(j)}| - |\lambda^{(j)}|)$ entries from the first i components of $\mathfrak{t}^\nu$ appearing in the remaining $r-i$ components of our tableau. Labelling these $\{x_1, x_2, \ldots, x_{\tau_i}\}$, such that $x_i < x_j$ whenever $i < j$, we replace these elements using the rule

$$x_k \longmapsto \overline{\lambda}_{(i)} + k.$$

Example 10. *If* $\lambda = ((2,1),(2,2),(2,2,1))$ *and* $\nu = ((3,2,1),(3,1),(2))$, *and* $\mathrm{S} \in \mathcal{T}_0(\nu, \lambda)$ *is given by*

$$\mathrm{S} = \left(\begin{array}{|c|c|c|} \hline 1_1 & 1_1 & 2_3 \\ \hline 2_1 & 2_2 \\ \cline{1-2} 1_3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline 1_2 & 1_2 & 3_3 \\ \hline 2_2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1_3 & 2_3 \\ \hline \end{array} \right)$$

Then

$$\mathfrak{t}_{\mathrm{S}(0)} = \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 5 & 10 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 6 & 11 \\ \hline 3 & 12 \\ \hline 9 \\ \cline{1-1} \end{array} \right) \qquad \mathfrak{t}_{\mathrm{S}(1)} = \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 5 & 10 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 6 & 11 \\ \hline 4 & 12 \\ \hline 9 \\ \cline{1-1} \end{array} \right)$$

$$\mathfrak{t}_{\mathrm{S}(2)} = \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 6 & 7 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 9 & 11 \\ \hline 8 & 12 \\ \hline 10 \\ \cline{1-1} \end{array} \right).$$

Remark. It is in the second part of this construction that the fact that S is quasi-semistandard is necessary. Otherwise, τ_i may not represent the number of entries from the first i components of $\mathfrak{t}^\nu$ appearing in the last $r-i$ components of $\mathfrak{t}_{\mathrm{S}(i)}$.

With this definition in mind, if $w \in \mathfrak{S}_n$ is the permutation such that $\mathfrak{t}_{\mathrm{S}} = \mathfrak{t}^\lambda \cdot w$, set $T_{\mathrm{S}} = T_w$. In order to better describe how T_{S} interacts with u_μ^+, we introduce the following notation.

- For any given multicomposition α of n, let $\overline{u}^+_{\alpha^{(i)}}$ refer to the factor of u_α^+ corresponding to the first i components of α:

$$\overline{u}^+_{\alpha^{(i)}} = \prod_{j=2}^{i+1} \prod_{k=1}^{|\alpha^{(1)}|+\cdots+|\alpha^{(j-1)}|} (L_k - Q_j)$$

- Similarly, we specify $\underline{u}^+_{\alpha^{(i)}}$ as the factors of u^+_α corresponding to the last $(r-1)-i$ components:

$$\underline{u}^+_{\alpha^{(i)}} = \prod_{j=i+1}^{r} \prod_{k=1}^{|\alpha^{(1)}|+\cdots+|\alpha^{(j-1)}|} (L_k - Q_j).$$

- Finally, if γ is another multicomposition of n, with $\alpha \trianglelefteq \gamma$, let $u^+_{\gamma^{(i)}\setminus\alpha^{(i)}}$ signify the 'difference' between u^+_γ and u^+_α at the i-th component:

$$u^+_{\gamma^{(i)}\setminus\alpha^{(i)}} = \prod_{j=\overline{\alpha}^{(i)}+1}^{\overline{\gamma}^{(i)}} (L_j - Q_{i+1}).$$

Example 11. *Suppose that* $\alpha = ((2,1),(2,2),(2,2,1))$. *Then*

$$\begin{aligned}\overline{u}^+_{\alpha^{(2)}} &= (L_1-Q_2)(L_2-Q_2)(L_3-Q_2)\\ &\quad\times(L_1-Q_3)(L_2-Q_3)(L_3-Q_3)(L_4-Q_3)(L_5-Q_3)(L_6-Q_3)(L_7-Q_3)\\ \underline{u}^+_{\alpha^{(2)}} &= (L_1-Q_3)(L_2-Q_3)(L_3-Q_3)(L_4-Q_3)(L_5-Q_3)(L_6-Q_3)(L_7-Q_3),\end{aligned}$$

and, if $\gamma = ((3,2,1),(3,1),(2))$, *then*

$$u^+_{\gamma^{(2)}\setminus\alpha^{(2)}} = (L_8-Q_3)(L_9-Q_3)(L_{10}-Q_3).$$

The following lemma expresses how the terms of T_{S} in a sense filter out the extraneous terms of u^+_ν, the result being something of the form $u^+_\lambda h$ for some $h \in \mathscr{H}$. Subsequent results then establish that whatever is to the right of u^+_λ after this filtering procedure is completed consists of a linear combination of the elements of $\mathbf{D}(\lambda)$ and $\mathbf{L}(\lambda)$ (multiplied on the right by elements of $\mathscr{H}$).

Lemma 3.3.8. *Let λ and ν be multipartitions of n in r parts with $\lambda \lhd \nu$. If* S *is a quasi-semistandard ν-tableau of type λ, then, for every* $0 \le i \le r-1$

$$T_{\mathrm{S}} u^+_\nu = \left(T_{\mathrm{S}(i)}\overline{u}^+_{\lambda^{(i)}}\underline{u}^+_{\nu^{(i+1)}}\right) u^+_{\nu^{(i)}\setminus\lambda^{(i)}} h_i$$

for some $h_i \in \mathscr{H}$.

Proof. Let w_{S} be the permutation associated with T_{S} and for each $0 \le i \le r-1$ let $w_{\mathrm{S}(i)}$ be the unique permutation such that $\mathfrak{t}_{\mathrm{S}(i)} = \mathfrak{t}^\lambda \cdot w_{\mathrm{S}(i)}$. Furthermore, let w_i be the permutation such that $\mathfrak{t}_{\mathrm{S}(i-1)} = \mathfrak{t}_{\mathrm{S}(i)} \cdot w_i$. Our strategy is to compare the lengths of w_{S} and $w_{\mathrm{S}(i)}w_i$ in order to show that the latter is reduced for all i, and that T_{w_i} commutes 'enough' with u^+_μ. Recall that, for $w \in \mathfrak{S}_n$, Dyer's reflection cocycle is the set

$$N(w) = \{(j,k) \in \mathfrak{S}_n : 1 \le j < k \le n \text{ and } jw > kw\}$$

and that the length $\ell(w)$ of w is the same as the cardinality of $N(w)$. Hence the length of w_{S} is equal to the number of pairs (j,k) where $j<k$ and where k occupies a node of $\mathfrak{t}_{\mathrm{S}}$ higher than that j occupies.

It is trivial to see that the lemma is true when $i = 0$; in this case $T_{S(i)} = T_S$, $\overline{u}^+_{\lambda^{(i)}} = u^+_{\nu^{(i)}\setminus\lambda^{(i)}} = 1$, and $\underline{u}^+_{\nu^{(i+1)}} = u^+_\nu$. Assume now that the lemma is true for an arbitrary value $i = k$, so that

$$T_S u^+_\nu = \left(T_{S(k)}\overline{u}^+_{\lambda^{(k)}}\underline{u}^+_{\nu^{(k+1)}}\right) u^+_{\nu^{(k)}\setminus\lambda^{(k)}} h_k.$$

For any node $x \in [\lambda]$, let $n_{S(i)}(x)$ be the number of entries in $t_{S(i)}$ less than $t_{S(i)}(x)$ situated in lower nodes. By its definition $w_{S(k+1)}$ fixes the first $k+1$ components of t^λ, so

$$\begin{aligned}\ell(w_{S(k)}) &= \sum_{x\in[\lambda^{(k+1)}]} n_{S(k)}(x) + \sum_{x\notin[\lambda^{(k+1)}]} n_{S(k)}(x)\\ &= \sum_{x\in[\lambda^{(k+1)}]} n_{S(k)}(x) + \ell(w_{S(k+1)}),\end{aligned}$$

where the equality $\ell(w_{S(k+1)}) = \sum_{x\notin[\lambda^{(k+1)}]} n_{S(k)}(x)$ follows immediately from the construction of the tableau $t_{S(k+1)}$.

We will now show that for every element of the set

$$\left\{\left(t_{S(k)}(x), t_{S(k)}(y)\right) : x \in \left[\lambda^{(k+1)}\right], t_{S(k)}(x) > t_{S(k)}(y), \text{ and } x \text{ lower than } y\right\}$$

there is an element of $N(w_{k+1})$ and vice versa, and so

$$\sum_{x\in[\lambda^{(k+1)}]} n_{S(k)}(x) = N(w_{k+1}) = \ell(w_{k+1}).$$

By our construction of $t_{S(k+1)}$ from $t_{S(k)}$, w_{k+1} permutes only the elements of $\Omega = \{\overline{\lambda}_{(k)} + 1, \dots, \overline{\nu}_{(k+1)}\}$. Additionally, if x and y are nodes of the final $r-(k+1)$ components of $[\lambda]$ such that $t_{S(k+1)}(x)$ and $t_{S(k+1)}(y)$ take entries from the set $\{\overline{\lambda}_{(k+1)} + 1, \dots, \overline{\nu}_{(k+1)}\}$, then

$$t_{S(k+1)}(x)w_{k+1} < t_{S(k+1)}(y)w_{k+1} \Leftrightarrow t_{S(k)}(x) < t_{S(k)}(y). \tag{3.8}$$

Suppose that a and b are such that $a < b$ and $aw_{k+1} > bw_{k+1}$. If $x, y \in [\lambda]$ are such that $t_{S(k+1)}(x) = a$ and $t_{S(k+1)}(y) = b$, then $x \in [\lambda^{(k+1)}]$; otherwise, (3.8) results in a contradiction. Since $a < b$ and the first $k+1$ components of $t_{S(k+1)}$ are the same as those of t^λ, we therefore conclude that y is lower than x. Conversely, suppose that x and y are such that $x \in [\lambda^{(k+1)}]$, y is lower than x, and $t_{S(k)}(x) > t_{S(k)}(y)$. Since $x \in [\lambda^{(k+1)}]$ and y is lower than x, we have that $t_{S(k+1)}(x) < t_{S(k+1)}(y)$.

We now complete our proof of the lemma: $\ell(w_{S(k)}) = \ell(w_{S(k+1)}) + \ell(w_{k+1})$, and so $w_{S(k)} = w_{S(k+1)} \cdot w_{k+1}$ is reduced; this in turn implies that $T_{S(k)} = T_{S(k+1)}T_{w_{k+1}}$. Finally, since w_{k+1} fixes all entries outside of Ω, we see that $T_{w_{k+1}}$ commutes with $\overline{u}^+_{\lambda^{(k)}}\underline{u}^+_{\nu^{(k+1)}}$. Combining these two facts then yields

$$T_S u^+_\nu = \left(T_{S(k+1)}\overline{u}^+_{\lambda^{(k+1)}}\underline{u}^+_{\nu^{(k+2)}}\right) u^+_{\nu^{(k+1)}\setminus\lambda^{(k+1)}} T_{w_{k+1}} u^+_{\nu^{(k)}\setminus\lambda^{(k)}} h,$$

as required. □

Example 12. *Let* $\lambda = ((2,1,1),(3,1),(2,1))$ *and* $\nu = ((4,2),(3,1),(1))$*, and let* $\mathtt{S} \in \mathscr{T}_0(\nu,\lambda)$ *be given by*

$$\mathtt{S} = \left(\begin{array}{|c|c|c|c|} \hline 1_1 & 1_1 & 3_1 & 1_3 \\ \hline 2_1 & 2_2 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|} \hline 1_2 & 1_2 & 1_2 \\ \hline 2_3 \\ \cline{1-1} \end{array}, \begin{array}{|c|} \hline 1_3 \\ \hline \end{array} \right)$$

Then

$$\begin{aligned} T_{\mathtt{S}} u_\nu^+ &= T_{8,7,6,5} T_{8,7,6} T_{10} T_{3,4} \\ &\quad \times ((L_1 - Q_2)\cdots(L_6 - Q_2))((L_1 - Q_3)\cdots(L_{10} - Q_3)) \\ &= \underbrace{T_{8,7,6,5} T_{8,7,6} T_{10}}_{T_{\mathtt{S}(1)}} \underbrace{((L_1 - Q_2)\cdots(L_4 - Q_2))}_{\overline{u}^+_{\lambda(1)}} \underbrace{((L_1 - Q_3)\cdots(L_{10} - Q_3))}_{\underline{u}^+_{\nu(2)}} \\ &\quad \times \underbrace{(L_5 - Q_2)(L_6 - Q_2)}_{u^+_{\nu(1)\setminus\lambda(1)}} T_{3,4} \\ &= \underbrace{T_{10}}_{T_{\mathtt{S}(2)}} \underbrace{((L_1 - Q_2)\cdots(L_4 - Q_2))((L_1 - Q_3)\cdots(L_8 - Q_2))}_{\overline{u}^+_{\lambda(2)} = u^+_\lambda} \\ &\quad \times \underbrace{(L_9 - Q_3)(L_{10} - Q_3)}_{u^+_{\nu(2)\setminus\lambda(2)}} T_{8,7,6,5} T_{8,7,6} (L_5 - Q_2)(L_6 - Q_2) T_{3,4} \end{aligned}$$

It's worth remarking that in the above example, $T_{\mathtt{S}(r-1)} = T_{\mathtt{S}(2)} = T_{10}$ commutes past $u_\lambda^+(L_9 - Q_3)$ and so, since $\mathfrak{l}^{(2)} = L_9 - Q_3$ we can write $m_{\mathtt{S}\mathfrak{t}^\nu} = m_\lambda \mathfrak{l}^{(2)} h$ for some $h \in \mathscr{H}$.

We might well ask what happens when $|\lambda^{(r)}| = |\nu^{(r)}|$, since in this case u_ν^+ does not contain a factor of $\mathfrak{l}^{(r-1)} = \left(L_{\overline{\lambda}^{(r)}} - Q_r\right)$. To answer this question, suppose that k is a positive integer and maximal such that $|\lambda^{(k)}| \neq |\nu^{(k)}|$. If $\mathtt{S}$ is a quasi-semistandard ν-tableau of type λ, then each of the final $r-k$ components of $\mathtt{S}$ are a permutation of the entries appearing in the corresponding component of the tableau $\lambda(\mathfrak{t}^\nu)$. Hence we can define another semistandard ν-tableau of type λ, which we denote by $\mathtt{R}$, constructed from $\mathtt{S}$ by replacing all the last $r-k$ components with those of $\lambda(\mathfrak{t}^\nu)$.

Corollary 3.3.9. *Suppose that* λ *and* ν *are both* r*-multicompositions of* n *such that* $\lambda \lhd \nu$ *and suppose that* k *is the maximal positive integer such that* $|\lambda^{(k)}| \neq |\nu^{(k)}|$*. If* $\mathtt{S} \in \mathscr{T}_{r,0}(\nu,\lambda)$*, then*

$$x_\lambda T_{\mathtt{S}} u_\nu^+ = x_\lambda T_{\mathtt{R}(k-1)} u_\lambda^+ u^+_{\nu(k-1)\setminus\lambda(k-1)} h$$

for some $h \in \mathscr{H}$*. Moreover,* $u^+_{\nu(k-1)\setminus\lambda(k-1)} \neq 1$*.*

Proof. If $|\lambda^{(l)}| = |\nu^{(l)}|$ for all $l \geq k$, then the permutation determining $T_{\mathtt{S}}$ fixes the final $r-l$ components of $\mathfrak{t}^\nu$. We may then write this permutation as a product uv where $u \in \mathfrak{S}_X$ and $v \in \mathfrak{S}_Y$ for

$$X = \left\{1, 2, \ldots, \overline{\nu}_{(k+1)}\right\} \quad \text{and} \quad Y = \left\{\overline{\nu}_{(k+1)} + 1, \overline{\nu}_{(k)} + 2, \ldots, n\right\}.$$

Hence $T_{\mathtt{S}} = T_u T_v$, and, by definition, $T_u = T_{\mathtt{R}}$. Given that v only permutes the entries of Y within the components of $\mathfrak{t}^\nu$ they occupy, T_v commutes with u_ν^+ and

$$x_\lambda T_{\mathtt{S}} u_\nu^+ = x_\lambda T_{\mathtt{R}} u_\nu^+ T_v.$$

Applying Lemma 3.3.8 then yields

$$x_\lambda T_{\mathtt{S}} u_\nu^+ = x_\lambda T_{\mathtt{R}(k-1)} \overline{u}^+_{\lambda(k-1)} \underline{u}^+_{\nu(k)} u^+_{\lambda(k-1)\setminus\nu(k-1)} h$$

for $h \in \mathscr{H}$. However, by the definition of k, $\overline{u}^+_{\nu(k)} = \overline{u}^+_{\lambda(k)}$ and so

$$x_\lambda T_{\mathtt{S}} u_\nu^+ = \left(x_\lambda T_{\mathtt{R}(k-1)} u_\lambda^+\right) u^+_{\nu(k-1)\setminus\lambda(k-1)} h. \qquad \square$$

Using the fact that $u^+_{\nu(k-1)\setminus\lambda(k-1)} = \mathfrak{l}^{(l-1)} h$ for some $h \in \mathscr{H}$, we may rewrite the conclusion of Corollary 3.3.9 in the following, more suggestive form:

$$x_\lambda T_{\mathtt{S}} u_\nu^+ = x_\lambda T_{\mathtt{R}(k-1)} u_\lambda^+ \mathfrak{l}^{(k-1)} h$$

for some $h \in \mathscr{H}$.

We next attempt to better describe $T_{\mathtt{R}(k-1)}$ in a form more suited to our purposes. Throughout we will fix k as the maximal integer x such that $|\nu^{(x)}| \neq |\lambda^{(x)}|$.

For $1 \le s \le t$, define

$$\pi(t,s) = (s, s+1, \ldots, t)$$
$$\mathrm{D}(t,s) = T_{\pi(t,s)} = T_{t-1} T_{t-2} \cdots T_s$$

Lemma 3.3.10 ([38, Lemma 3.8]). *Suppose that $\mathtt{A}$ is an α-tableau of type β for compositions α and β, and let $\mathfrak{a}(0) = \mathfrak{t}^\beta$. If we define $\mathfrak{a}(i)$ recursively by*

$$\mathfrak{a}(i) = \mathfrak{a}(i-1)\pi(i^*, i),$$

where i^* occupies the same node in $\mathfrak{a}(i-1)$ as does i in $\mathfrak{t}_{\mathtt{A}}$, then

$$T_{\mathtt{A}} = \prod_{j=1}^{n-1} \mathrm{D}(j^*, j).$$

It's worth remarking that Lemma 3.3.10 was originally stated for partitions and row-semistandard tableaux, but can be modified as we have done since the lemma depends on neither of these restrictions. The statement and its notation can then be easily adapted to multicompositions and the setting of the Ariki-Koike algebra: if λ and μ are multipartitions of n and $\mathtt{A} \in \mathscr{T}_0(\mu, \lambda)$ we consider the compositions $\alpha(\lambda)$ and $\alpha(\mu)$ and the $\alpha(\mu)$-tableau $\alpha(\mathtt{A})$-of type $\alpha(\lambda)$, and use the fact that $T_{\mathtt{A}} = T_{\alpha(\mathtt{A})}$.

Lemma 3.3.11. *Let λ and ν be multipartitions with $\lambda \lhd \nu$ such that there is at least one $1 \le i \le r$ with $|\lambda^{(i)}| < |\nu^{(i)}|$ and let $\mathtt{S}$ be a quasi-semistandard ν-tableau of type λ. Then*

$$x_\lambda T_{\mathtt{S}} u_\nu^+ = x_\lambda u_\lambda^+ \mathrm{D}((\overline{\lambda}_{(k)} + 1)^*, \overline{\lambda}_{(k)} + 1) \mathfrak{l}^{(k-1)} h$$

for an element $h \in \mathscr{H}$.

Proof. By Corollary 3.3.9 and the fact that S is quasi-semistandard, we can write

$$x_\lambda T_{\mathrm{S}} u_\nu^+ = x_\lambda T_{\mathrm{R}(k-1)} u_\lambda^+ u_{\nu(k-1)\setminus\lambda(k-1)}^+ h'$$

for some $h' \in \mathscr{H}$. By definition, $u_{\nu(k-1)\setminus\lambda(k-1)}^+ = \mathfrak{l}^{(k-1)} h''$ for some $h'' \in \mathscr{H}$ and so we can rewrite this as

$$x_\lambda T_{\mathrm{S}} u_\nu^+ = x_\lambda T_{\mathrm{R}(k-1)} u_\lambda^+ \mathfrak{l}^{(k-1)} h'' h'.$$

Applying Lemma 3.3.10 to $T_{\mathrm{R}(k-1)}$, and recalling that , we have

$$T_{\mathrm{R}(k-1)} = \prod_{i=\overline{\lambda}_{(k)}+1}^{\rho_r(\lambda)} \mathrm{D}(i^*, i) = \mathrm{D}\left(\left(\overline{\lambda}_{(k)}+1\right)^*, \overline{\lambda}_{(k)}+1\right) \prod_{j=\overline{\lambda}_{(k)}+2}^{\rho_r(\lambda)} \mathrm{D}(j^*, j),$$

where the first equality results from the fact that the first $k-1$ components of $\mathfrak{t}_{\mathrm{S}(k-1)}$ are identical to those of $\mathfrak{t}^\lambda$ and hence $\mathrm{D}(i^*, i) = 1$ for every $i < \overline{\lambda}_{(k)} + 1$.

The product

$$\prod_{j=\overline{\lambda}_{(k)}+2}^{\rho_r(\lambda)} \mathrm{D}(j^*, j)$$

commutes with $u_\lambda^+ \mathfrak{l}^{(k-1)}$ in its entirety since $j > \overline{\lambda}_{(k)} + 1$ and so the result follows due to the fact that

$$\mathrm{D}\left(\left(\overline{\lambda}_{(k)}+1\right)^*, \overline{\lambda}_{(k)}+1\right) = T_{\left(\overline{\lambda}_{(k)}+1\right)^*-1} \cdots T_{\overline{\lambda}_{(k)}+2} T_{\overline{\lambda}_{(k)}+1}$$

commutes with u_λ^+, but not necessarily with $u_\lambda^+ \mathfrak{l}^{(k-1)}$. □

Note that $\mathrm{D}((\overline{\lambda}_{(k)}+1)^*, \overline{\lambda}_{(k)}+1) = 1$ when $(\overline{\lambda}_{(k)}+1)^* = \overline{\lambda}_{(k)}+1$, so that $x_\lambda T_{\mathrm{S}} u_\nu^+ = m_\lambda \mathfrak{l}^{(k-1)} h$ for some element $h \in \mathscr{H}$. Our proof that $m_{\mathrm{S}\mathfrak{t}^\nu} \in \mathfrak{I}_\lambda$ is then complete in the cross-component case if $\mathrm{D}(x^*, x)$ can be expressed as a linear combination of elements of the form $\mathfrak{d}_{d,t}^{(s)} h$ and $\mathfrak{l}^{(u)} h$ for $h \in \mathscr{H}$.

For this, we will need the following characterization of $\mathfrak{d}_{d,t}^{(s)}$ due to Lyle [38, Lemma 3.9]: for $m, a, b \geq 0$, define

$$\langle m, a, b \rangle := \{\mathbf{i} = (i_1, i_2, \ldots, i_b) : m+1 \leq i_1 < i_2 < \cdots < i_b \leq m+a+b\}.$$

If (a, b) is a composition, then

$$\mathrm{C}(m; (a,b)) = \sum_{\mathbf{i} \in \langle m,a,b \rangle} \prod_{l=1}^{b} \mathrm{D}(m+a+l, i_l).$$

Therefore,

$$\mathfrak{d}_{d,t}^{(s)} = \sum_{\mathbf{i} \in \langle \overline{\lambda}_{(d-1,s)}, \lambda_d^{(s)}, t \rangle} \prod_{l=1}^{t} \mathrm{D}\left(\overline{\lambda}_{(d,s)} + l, i_l\right), \tag{3.9}$$

for every $(d, t, s) \in \operatorname{def}(\lambda, \mathfrak{d})$.

Lemma 3.3.12. *For $1 \le l \le \rho_k(\lambda)$, we have*

$$\begin{aligned}\mathrm{D}\left(\overline{\lambda}_{(l,k)}+1,\overline{\lambda}_{(k)}+1\right) = {} & \mathfrak{d}^{(k)}_{l,1}\mathfrak{d}^{(k)}_{l-1,1}\cdots\mathfrak{d}^{(k)}_{1,1} \\ & -\mathfrak{d}^{(k)}_{l-1,1}\mathfrak{d}^{(k)}_{l-2,1}\cdots\mathfrak{d}^{(k)}_{1,1}h_{l-1} \\ & \vdots \\ & -\mathfrak{d}^{(k)}_{2,1}\mathfrak{d}^{(k)}_{1,1}h_2 \\ & -\mathfrak{d}^{(k)}_{1,1}h_1 \\ & -h_0,\end{aligned}$$

where $h_0, h_1, \ldots, h_{l-1} \in \mathscr{H}$ and h_0 commutes with $u^+_\lambda\restriction^{(k-1)}$ and h_i commutes with u^+_λ for every $1 \le i \le l-1$.

Proof. If $l = 1$, then using (3.9) with $s = k$ and $t = 1$ gives us

$$\mathrm{D}\left(\overline{\lambda}_{(1,k)}+1,\overline{\lambda}^{(k)}+1\right) = \mathfrak{d}^{(r)}_{1,1} - \sum_{i=\overline{\lambda}_{(k)}+2}^{\overline{\lambda}_{(1,k)}+1} \mathrm{D}\left(\overline{\lambda}_{(1,k)}+1,i\right)$$

and we can take the sum on the right hand side as our h_0. Suppose then that the lemma is true for some arbitrary value of l. Since

$$\mathrm{D}\left(\overline{\lambda}_{(l+1,k)}+1,\overline{\lambda}_{(k)}+1\right) = \mathrm{D}(\overline{\lambda}_{(k+1,k)}+1,\overline{\lambda}_{(l,k)}+1)\mathrm{D}(\overline{\lambda}_{(l,k)}+1,\overline{\lambda}_{(k)}+1)$$

and

$$\mathrm{D}(\overline{\lambda}_{(l+1,k)}+1,\overline{\lambda}_{(l,k)}+1) = \mathfrak{d}^{(r)}_{l+1,1} - \sum_{j=\overline{\lambda}_{(l,k)}+2}^{\overline{\lambda}_{(l+1,k)}+1} \mathrm{D}(\overline{\lambda}_{(l+1,k)}+1,j),$$

we have

$$\begin{aligned}\mathrm{D}(\overline{\lambda}_{(l+1,k)}+1,\overline{\lambda}_{(k)}+1) = {} & \mathfrak{d}^{(k)}_{l+1,1}\mathfrak{d}^{(k)}_{l,1}\cdots\mathfrak{d}^{(k)}_{1,1} \\ & -\mathfrak{d}^{(k)}_{l,1}\mathfrak{d}^{(k)}_{l-1,1}\cdots\mathfrak{d}^{(k)}_{1,1}\left(\sum_{j=\overline{\lambda}_{(l,k)}+2}^{\overline{\lambda}_{(l+1,k)}+1}\mathrm{D}(\overline{\lambda}_{(l+1,k)}+1,j)\right) \\ & \vdots \\ & -\mathfrak{d}^{(k)}_{2,1}\mathfrak{d}^{(k)}_{1,1}\left(\mathfrak{d}^{(k)}_{l+1,1} - \sum_{j=\overline{\lambda}_{(l,k)}+2}^{\overline{\lambda}_{(l+1,k)}+1}\mathrm{D}(\overline{\lambda}_{(l+1,k)}+1,j)\right)h_2 \\ & -\mathfrak{d}^{(k)}_{1,1}\left(\mathfrak{d}^{(k)}_{l+1,1} - \sum_{j=\overline{\lambda}_{(l,k)}+2}^{\overline{\lambda}_{(l+1,k)}+1}\mathrm{D}(\overline{\lambda}_{(l+1,k)}+1,j)\right)h_1 \\ & -\left(\mathfrak{d}^{(k)}_{l+1,1} - \sum_{j=\overline{\lambda}_{(l,k)}+2}^{\overline{\lambda}_{(l+1,k)}+1}\mathrm{D}(\overline{\lambda}_{(l+1,k)}+1,j)\right)h_0.\end{aligned}$$

It is routine to check that $\mathfrak{d}^{(k)}_{l+1,1}$ and $\sum \mathrm{D}(\overline{\lambda}_{(l+1,k)}+1,j)$ both commute with $\mathfrak{d}^{(k)}_{x,1}\mathfrak{d}^{(k)}_{x-1,1}\cdots\mathfrak{d}^{(k)}_{1,1}$ for every $x \le k-1$ and $x \le k$ respectively.

That

$$\mathfrak{d}^{(k)}_{l+1,1} - \sum_{j=\overline{\lambda}_{(l,k)}+2}^{\overline{\lambda}_{(l+1,k)}+1} \mathrm{D}(\overline{\lambda}_{(l+1,k)}+1,j)$$

commutes with $u^+_\lambda \mathfrak{l}^{(k-1)}$ follows from both terms being elements of the symmetric group acting on the set $\{\overline{\lambda}_{(l,k)}+1,\ldots,\overline{\lambda}_{(l+1,k)}\}$. The given expression can then be written in terms of generators T_i of $\mathscr{H}$ with $i \ne \overline{\lambda}_{(k)}+1$. □

Recall that when S is a quasi-semistandard ν-tableau of type λ, we have

$$T_{\mathrm{S}} u^+_\nu = u^+_\lambda \mathrm{D}((\overline{\lambda}_{(k)}+1)^*, \overline{\lambda}_{(k)}+1)\mathfrak{l}^{(k-1)} h, \tag{3.10}$$

for some $h \in \mathscr{H}$. We now show that we can rewrite the right hand side of (3.10) in the form we need in order to show that $m_{\mathrm{St}^\nu} \in \mathfrak{I}_\lambda$ in the cross-component case.
Lemma 3.3.13. *Let* S *be a quasi-semistandard* ν*-tableau of type* λ *and let*

$$u^+_\lambda \mathrm{D}((\overline{\lambda}_{(k)}+1)^*, \overline{\lambda}_{(k)}+1)\mathfrak{l}^{(k-1)}$$

be as in (3.10)*. Then there is some* l *with* $1 \le l \le \rho_k(\lambda)$ *such that*

$$u^+_\lambda \mathrm{D}((\overline{\lambda}_{(k)}+1)^*, \overline{\lambda}_{(k)}+1)\mathfrak{l}^{(k-1)} = u^+_\lambda \left(\sum_{i=1}^{l-1} r_i \mathfrak{d}^{(k)}_{i,1} h_i - \mathfrak{l}^{(k-1)} h'_0 \right)$$

for $h'_0, h'_1, \ldots, h'_{l-1} \in \mathscr{H}$ *and* $r_i \in \mathbb{F}$*.*

Proof. Since S is quasi-semistandard, the λ-tableau $\mathfrak{t}_{\mathrm{R}(k-1)}$ defined previously is row-standard. We also have, by definition that the first $k-1$ components of $\mathfrak{t}_{\mathrm{R}(k-1)}$ are identical to those of $\mathfrak{t}^\lambda$. Hence we have that the entry $(\overline{\lambda}_{(k)}+1)^*$ appears at the beginning of a particular row of component k; hence, $\mathrm{D}(x^*,x)$ is of the form

$$\mathrm{D}\left(\overline{\lambda}_{(l-1,k)}+1, \overline{\lambda}_{(k)}+1\right)$$

for some l with $1 \le l \le \rho_k(\lambda)$; in fact, l is the number of the row containing $(\overline{\lambda}_{(k)}+1)$.

The statement now follows from Lemma 3.3.12. Indeed, by definition $\mathfrak{d}^{(k)}_{i,1}$ commutes with u^+_λ for all values of i, and so, using the notation in Lemma 3.3.12, we can take

$$h'_i = \mathfrak{d}^{(k)}_{i-1,1}\mathfrak{d}^{(k)}_{i-2,1}\cdots\mathfrak{d}^{(k)}_{1,1} h_i$$

for each $1 \le i \le l-1$, and $h'_0 = h_0$. □

Proposition 3.3.14. *Let* λ *and* μ *be multipartitions of* n *with* $\lambda \lhd \nu$ *and such that there is at least one* $1 \le i \le r$ *with* $|\lambda^{(i)}| < |\nu^{(i)}|$*. If* $\mathrm{S} \in \mathscr{T}_{r,0}(\nu,\lambda)$*, then* $m_{\mathrm{St}^\nu} \in \mathfrak{I}_\lambda$*.*

Proof. If S is a quasi-semistandard, then Lemma 3.3.3 and Lemma 3.3.11 shows that we can write m_{St^ν} as

$$x_\lambda u_\lambda^+ \mathrm{D}((\overline{\lambda}_{(k)}+1)^*, \overline{\lambda}_{(k)}+1)\mathfrak{l}^{(k-1)} h$$

for some $h \in \mathscr{H}$. Applying Lemma 3.3.13 then completes the proof. □

3.3.3 Generating $M^\lambda \cap \check{\mathscr{H}}^\lambda$

Recall that we take $\mathfrak{I}_\lambda$ to be generated as a right ideal of $\mathscr{H}$ by the sets

$$\mathbf{D}(\lambda) = \left\{ m_\lambda \mathfrak{d}_{d,t}^{(s)} : (d,t,s) \in \mathrm{def}(\lambda, \mathfrak{d}) \right\} \quad \text{and} \quad \mathbf{L}(\lambda) = \left\{ m_\lambda \mathfrak{l}^{(s)} : s \in \mathrm{def}(\lambda, \mathfrak{l}) \right\}.$$

Let λ and ν be compositions of n and let $\mathrm{S} \in \mathscr{T}_r(\mu, \lambda)$. A useful result from the representation theory of the Iwahori-Hecke Algebra of type A [39, 4.6] is that

$$\sum_{\substack{x \in \mathscr{D}_\gamma \\ \nu(\mathrm{t}^\lambda x) = \mathrm{T}}} y_\lambda T_x = h y_\nu \tag{3.11}$$

for some $h \in \mathscr{H}_n$. Using this ability to regard certain constructions in $\mathscr{H}$ as objects in $\mathscr{H}_n$ and properties of the latter algebra, showing that $\mathfrak{I}_\lambda$ is contained in $M^\lambda \cap \check{\mathscr{H}}^\lambda$ is relatively straight forward.

Lemma 3.3.15. *For every r-multipartition λ of n*

$$\mathfrak{I}_\lambda \subseteq M^\lambda \cap \check{\mathscr{H}}^\lambda.$$

Proof. Fixing $(d,t,s) \in \mathrm{def}(\lambda, \mathfrak{d})$, let $\nu = \lambda \cdot \mathfrak{d}_{d,t}^{(s)}$ and consider $m_\lambda \mathfrak{d}_{d,t}^{(s)}$. The terms $\mathfrak{d}_{d,t}^{(s)}$ and u_λ^+ commute with one another, since $u_\nu^+ = u_\lambda^+$ and $\mathfrak{d}_{d,t}^{(s)} \in \mathscr{H}(\mathfrak{S}_\nu)$; hence,

$$m_\lambda \mathfrak{d}_{d,t}^{(s)} = \left(x_\lambda \mathfrak{d}_{d,t}^{(s)} \right) u_\lambda^+.$$

Let S be the row-semistandard λ-tableau of type ν derived from T^λ by changing the first t entries in row $d+1$ of component s from $(d+1, s)$ to (d, s). We have

$$x_\lambda \mathfrak{d}_{d,t}^{(s)} = x_\lambda \sum_{\substack{x \in \mathscr{D}_\lambda \\ \nu(\mathrm{t}^\lambda x) = \mathrm{S}}} T_x = y_{\alpha(\lambda)} \sum_{\substack{x \in \mathscr{D}_{\alpha(\lambda)} \\ \alpha(\nu)(\mathrm{t}^{\alpha(\lambda)} x) = \alpha(\mathrm{S})}} T_x,$$

and so, by (3.11),

$$m_\lambda \mathfrak{d}_{d,t}^{(s)} = h y_{\alpha(\nu)} u_\nu^+ = h x_\nu u_\nu^+ = h m_\nu$$

for some $h \in \mathscr{H}_n$; thus, $m_\lambda \mathfrak{d}_{d,t}^{(s)} \in M^\lambda \cap \check{\mathscr{H}}^\lambda$ since ν dominates λ.

Now fix $s \in \operatorname{def}(\lambda, \mathfrak{l})$ and let $\nu = \lambda \cdot \mathfrak{l}^{(s)}$. If S is the row-semistandard λ-tableau of type ν derived from T^λ by changing the entry in the first node of the first row of component $s+1$ from $(1, s+1)$ to $(\rho_s(\lambda)+1, s)$, then

$$\sum_{\substack{x \in \mathscr{D}_\lambda \\ \nu(\mathrm{t}^\lambda x) = \mathrm{S}}} T_x = 1,$$

and so, again by (3.11), we have

$$x_\lambda = h x_\nu$$

for an element $h \in \mathscr{H}$. Since $u_\lambda^+ \mathfrak{l}^{(s)} = u_\nu^+$ and $\lambda \lhd \nu$ we have that $m_\lambda \mathfrak{l}^{(s)} = x_\lambda u_\nu^+ = h x_\nu u_\nu^+ = h m_\nu \in M^\lambda \cap \check{\mathscr{H}}^\lambda$, as required. □

We may now prove the main result of this chapter by collecting together the most important results so far encountered in this chapter.

Theorem 3.3.16. *Let $\mathfrak{I}_\lambda$ be the right ideal of $\mathscr{H}_{r,n}$ generated by the sets*

$$\mathbf{D}(\lambda) = \left\{ m_\lambda \mathfrak{d}_{d,t}^{(s)} : (d,t,s) \in \operatorname{def}(\lambda, \mathfrak{d}) \right\}$$

and

$$\mathbf{L}(\lambda) = \left\{ m_\lambda \mathfrak{l}^{(s)} : s \in \operatorname{def}(\lambda, \mathfrak{l}) \right\}.$$

Then $\mathfrak{I}_\lambda = M^\lambda \cap \check{\mathscr{H}}^\lambda$.

Proof. By Lemma 3.3.1, the right ideal $M^\lambda \cap \check{\mathscr{H}}^\lambda$ is free as an $\mathbb{F}$-module with basis

$$\{m_{\mathrm{St}} : \mathrm{S} \in \mathscr{T}_0(\nu, \lambda), \mathrm{t} \in \operatorname{Std}(\nu) \text{ for } \nu \text{ a multipartition with } \lambda \lhd \nu\}.$$

Proposition 3.3.14 shows us that $m_{\mathrm{St}^\nu} \in \mathfrak{I}_\lambda$ whenever there exists some $1 \le i \le r$ such that $|\lambda^{(i)}| \ne |\nu^{(i)}|$, whilst Lemma 3.3.7 does the same for the case where $|\lambda^{(i)}| = |\nu^{(i)}|$ for every $1 \le i \le r$; therefore, $M^\lambda \cap \check{\mathscr{H}}^\lambda \subseteq \mathfrak{I}_\lambda$. Lemma 3.3.15 then completes the proof, since its conclusion is that $\mathfrak{I}_\lambda \subseteq M^\lambda \cap \check{\mathscr{H}}^\lambda$ for every multipartition λ. □

Via our discussion in the introduction to this chapter, we immediately have the (important) corollary of this result.

Corollary 3.3.17. *Let $\Theta : M^\lambda \to S^\mu$ be a homomorphism. Then the following are equivalent:*

- $\Theta(m_\lambda h) = 0$ *for every* $h \in \mathscr{H}_{r,n}$ *with* $m_\lambda h \in \check{\mathscr{H}}$*;*
- $\Theta\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) = 0$ *for every* $(d,t,s) \in \operatorname{def}(\lambda, \mathfrak{d})$ *and* $\Theta\left(m_\lambda \mathfrak{l}^{(s)}\right) = 0$ *for every* $s \in \operatorname{def}(\lambda, \mathfrak{l})$*; and*
- Θ *factors through* S^λ.

3.4 Homomorphisms and Quasi-Semistandard Tableaux

Before we continue on to constructing homomorphisms in the next chapter, we provide a useful consequence of the proof of Theorem 3.3.16. As we saw in Section 2.4, every semistandard tableaux $\mathrm{S} \in \mathcal{T}_0(\mu,\lambda)$ determines a homomorphism $\Theta_{\mathrm{S}} : M^\lambda \to S^\lambda$; in this section we prove that the same is true of quasi-semistandard tableaux. Recall that $M^{\lambda *}$ is the image of M^λ under the anti-isomorphism $* : \mathscr{H} \to \mathscr{H}$; that is, $M^{\lambda *} = \mathscr{H} m_\lambda$.

Lemma 3.4.1 ([21, Corollary 5.17])**.** *Suppose that λ and μ are multicompositions of n. Then* $\mathrm{Hom}_{\mathscr{H}}(M^\lambda, M^\mu) \cong M^{\lambda *} \cap M^\mu$ *via the map $\varphi \mapsto \varphi(m_\lambda)$.*

The following proposition and its corollary establishes the property of being quasi-semistandard as a sufficient condition for a row-semistandard tableau in $\mathcal{T}(\mu,\lambda)$ to determine an element of $\mathrm{Hom}_{\mathscr{H}}(M^\lambda, M^\mu)$ and $\mathrm{Hom}_{\mathscr{H}}(M^\lambda, S^\mu)$ respectively.

Proposition 3.4.2. *Let λ and μ be multipartitions of n with $\lambda \lhd \mu$ and let* S *be a row-semistandard μ-tableau of type λ. Setting*

$$\Theta_{\mathrm{S}}(m_\lambda h) = \left(\check{\mathscr{H}}^\mu + m_\mu \sum_{\substack{\mathfrak{s} \in \mathrm{RStd}(\mu) \\ \lambda(\mathfrak{s}) = \mathrm{S}}} T_{d(\mathfrak{s})} \right) h$$

for each $h \in \mathscr{H}$ defines a homomorphism from M^λ to S^μ whenever S *satisfies* (3.4)*, i.e. when* S *is quasi-semistandard..*

Proof. By Propositions 3.3.14 and 3.3.7, we have that $m_{\mathrm{S}\mathfrak{t}^\mu} \in M^\lambda$; hence, $m_{\mathrm{S}\mathfrak{t}^\mu} \in M^\lambda \cap \mathscr{H} m_\mu$. Applying the anti-isomorphism $* : \mathscr{H} \to \mathscr{H}$ to $m_{\mathrm{S}\mathfrak{t}^\mu}$ yields $m_{\mathfrak{t}^\mu \mathrm{S}} \in \mathscr{H} m_\lambda \cap M^\mu$. The remainder follows from Lemma 3.4.1. □

Chapter 4

One Node Homomorphisms

4.1 Chapter Introduction

Before introducing this chapter, which for the most part concerns a very specific class of multipartitions, we provide a couple of more generally applicable results. Let λ and μ be multipartitions such that $\lambda \lhd \mu$, and recall that each quasi-semistandard tableau $\mathtt{S} \in \mathscr{T}_{0,r}(\mu,\lambda)$ determines a homomorphism $\Theta_{\mathtt{S}} : M^{\lambda} \to S^{\mu}$ via

$$\Theta_{\mathtt{S}}(m_{\lambda}h) = \check{\mathscr{H}}^{\mu} + m_{\mu} \sum_{\substack{\mathfrak{s}\in \mathrm{RStd}(\mu) \\ \lambda(\mathfrak{s})=\mathtt{S}}} T_{d(\mathfrak{s})}h.$$

Definition 4.1.1. Let $\Omega^{\mathtt{S}}_{(x,y)}$ be the sequence

$$\begin{aligned}
&\mathtt{S}^{(x,y)}_{(1,1)}, \mathtt{S}^{(x,y)}_{(2,1)}, \dots, \mathtt{S}^{(x,y)}_{(\rho_1(\lambda),1)}, \\
&\mathtt{S}^{(x,y)}_{(1,2)}, \mathtt{S}^{(x,y)}_{(2,2)}, \dots, \mathtt{S}^{(x,y)}_{(\rho_2(\lambda),2)}, \\
&\vdots \\
&\mathtt{S}^{(x,y)}_{(1,r)}, \mathtt{S}^{(x,y)}_{(2,r)}, \dots, \mathtt{S}^{(x,y)}_{(\rho_r(\lambda),r)}.
\end{aligned}$$

The first result of this chapter serves a similar purpose to that of Lemma 3.3.3 in the previous chapter, in so much as it expresses an object in which we're interested in a form more suited to our aims.

If λ and μ are multicompositions with $\lambda \unlhd \mu$, and $\mathtt{S}$ is a μ-tableau of type λ, we set first($\mathtt{S}$) to be the 'largest' μ-tableau such that $\lambda(\text{first}(\mathtt{S})) = \mathtt{S}$ in the dominance ordering on μ-tableaux. More intuitively, first($\mathtt{S}$) is the μ-tableau such that

- $\lambda(\text{first}(\mathtt{S})) = \mathtt{S}$, and

- if i and j both appear in the same row of $\mathfrak{t}^\lambda$ and $i<j$, then i appears in first($\mathtt{S}$) occupying a node of $[\mu]$ higher than that occupied by j.

Lemma 4.1.1. *Let μ and λ be multicompositions of n with $\lambda \lhd \mu$, and let $\mathtt{S}$ be a μ-tableau of type λ. Then*

$$m_\mu \sum_{\substack{\mathfrak{s}\in \mathrm{RStd}(\mu)\\ \lambda(\mathfrak{s})=\mathtt{S}}} T_{d(\mathfrak{s})} = m_\mu T_{d(\mathrm{first}(\mathtt{S}))} \prod_{\mathbf{i}=(i_1,i_2)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right).$$

Proof. Immediate from the definitions. □

Our second result calculates $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right)$ for each $(d,t,s) \in \mathrm{def}(\lambda,\mathfrak{d})$ in terms of a linear combination of elements $\Theta_{\mathtt{X}}(m_\nu)$, where $\nu = \lambda\cdot\mathfrak{d}^{(s)}_{d,t}$ and $\mathtt{X} \in \mathscr{T}_{0,r}(\mu,\nu)$. Its statement requires us to first expand some of the notation established in the previous chapter.

Definition 4.1.2. If λ and μ are multicomposition and $\mathtt{S} \in \mathscr{T}(\mu,\lambda)$, let $\mathtt{S}^{(x,y)}_{>\mathbf{i}}$ be the number of entries of the form (x,y) occupying a node of $\mathtt{S}$ lower than $\mathbf{i}$.

Proposition 4.1.2 ([38, Proposition 2.7])**.** *Let λ and μ be multipartitions of n with $\lambda \lhd \mu$ and fix an element (d,t,s) of* $\mathrm{def}(\lambda,\mathfrak{d})$*. Suppose that $\mathtt{S}$ is a semistandard μ-tableau of type λ and let $\mathscr{A}$ be the set of row–semistandard μ-tableau of type $\nu = \lambda\cdot\mathfrak{d}^{(s)}_{d,t}$ obtained from $\mathtt{S}$ by replacing t entries of the form $(d+1,s)$ with (d,s). Then each $\mathtt{X}\in\mathscr{A}$ is quasi-semistandard and hence $\Theta_{\mathtt{X}}$ defined in* (3.3.10) *is a homomorphism $M^\nu \to S^\mu$. Moreover,*

$$\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right) = \sum_{\mathtt{X}\in\mathscr{A}} \left(\prod_{\mathbf{i}} q^{\mathtt{S}^{(d,s)}_{>\mathbf{i}}\left(\mathtt{X}^{(d,s)}_{\mathbf{i}} - \mathtt{S}^{(d,s)}_{\mathbf{i}}\right)} \begin{bmatrix} \mathtt{X}^{(d,s)}_{\mathbf{i}} \\ \mathtt{S}^{(d,s)}_{\mathbf{i}} \end{bmatrix}\right) \Theta_{\mathtt{X}}(m_\nu),$$

where $\mathbf{i}$ *runs over the rows in* $\mathtt{S}$.

Proof. By the discussion above, we may write

$$x_\mu \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\mu)\\ \lambda(\mathfrak{s})=\mathtt{S}}} T_{d(\mathfrak{s})} = y_{\alpha(\mu)} \sum_{\substack{\alpha(\mathfrak{s})\in\mathrm{RStd}(\alpha(\mu))\\ \alpha(\lambda)(\alpha(\mathfrak{s}))=\alpha(\mathtt{S})}} T_{d(\alpha(\mathfrak{s}))}.$$

This defines a homomorphism $N^{\alpha(\lambda)} \to N^{\alpha(\mu)}$, where these are the 'type A' permutation modules described in Definition 2.6.2, by [39, Equation 4.6]. Working in type A, we can adapt the proof of [38, Proposition 2.7] almost immediately; namely, by applying the anti-isomorphism $*$ to [37, Proposition 2.14]. □

Example 13. *Let $\lambda = ((3,2,2,1),(2,1))$ and $\mu = ((4,3,2),(2))$, and let $\mathtt{S}$ be given by*

$$\mathtt{S} = \left(\begin{array}{|c|c|c|c|} \hline 1_1 & 1_1 & 1_1 & 2_2 \\ \hline 2_1 & 2_1 & 3_1 \\ \cline{1-3} 3_1 & 4_1 \\ \cline{1-2} \end{array}\;,\; \begin{array}{|c|c|} \hline 1_2 & 1_2 \\ \hline \end{array} \right).$$

Then

$$\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(1)}_{2,1}\right) = (1+q+q^2)\Theta_{\mathtt{X}_1}(m_\nu) + \Theta_{\mathtt{X}_2}(m_\nu),$$

where

$$\mathtt{X}_1 = \left(\begin{array}{llll} 1_1 & 1_1 & 1_1 & 2_2 \\ 2_1 & 2_1 & 2_1 & \\ 3_1 & 4_1 & & \end{array}, \begin{array}{ll} 1_2 & 1_2 \end{array} \right) \quad \textit{and} \quad \mathtt{X}_2 = \left(\begin{array}{llll} 1_1 & 1_1 & 1_1 & 2_2 \\ 2_1 & 2_1 & 3_1 & \\ 2_1 & 4_1 & & \end{array}, \begin{array}{ll} 1_2 & 1_2 \end{array} \right).$$

Notice that the tableaux $\mathtt{X} \in \mathcal{T}_{0,r}(\mu, \nu)$ referred to in Proposition 4.1.2 are not necessarily semistandard, and so the various $\Theta_{\mathtt{X}}$ involved in each linear combination may not be linearly independent. This leads to the possibility that such a combination is in fact zero. In order to address this complication, much of this chapter is dedicated to expressing each $\Theta_{\mathtt{X}}$ in terms of semistandard (and hence linearly independent) homomorphisms.

In [18], Fayers provided an algorithm for expressing homomorphisms $\Theta_{\mathtt{S}} : S^\lambda \to M^\mu$ in type A that are indexed by non-semistandard tableaux as a linear combination of semistandard homomorphisms. Unfortunately, we cannot presently make use of this algorithm even in type A due to the differences between the Specht modules appearing in that paper, which are the Specht modules in the sense of [10] that we discuss as the beginning of Chapter 3, and the Specht modules that arise as a result of the cellular basis we work with here. However, the author is currently working with Lyle in the hope that we will be able to fully generalize [18] to the setting of the Ariki-Koike algebra.

For the remainder of this chapter, we apply the results from the previous one to construct homomorphisms between certain pairs of Specht modules for the Ariki-Koike algebra with $r = 2$ (otherwise known as the Iwahori-Hecke algebra of type B. Suppose that λ and μ are bipartitions such that $\lambda \trianglelefteq \mu$ and

$$\mu = \left(\left(\lambda_1^{(1)}, \ldots, \lambda_i^{(1)} + 1, \ldots, \lambda_{\rho_1(\lambda)}^{(1)} \right), \left(\lambda_1^{(2)}, \ldots, \lambda_j^{(2)} - 1, \ldots, \lambda_{\rho_2(\lambda)}^{(2)} \right) \right) \tag{4.1}$$

for a given pair of integers i and j. In other words, μ and λ are such that $[\mu]$ can be formed from $[\lambda]$ by removing a single node from the second component and adding it to the first. When μ is related to λ in this way we will call a homomorphism $\Theta : M^\lambda \to S^\mu$ a *one-node homomorphism*, these forming the focus of this chapter. We will actually concentrate on the case where $i = 1$ and $j = \rho_2(\lambda)$, from which the more general case can be inferred.

4.2 Characterizing Tableaux

With λ and μ multipartitions and as specified at the end of section 4.1, here we collect a number of properties of semistandard μ-tableaux of type λ. This will culminate in a characterisation of such tableaux in terms of the entries occupying a certain class of nodes of $[\mu]$.

Lemma 4.2.1. *If* $\mathtt{S} \in \mathcal{T}_0(\mu, \lambda)$*, then there is exactly one node* $\mathbf{n} \in \left[\mu^{(1)}\right]$ *such that* $\mathtt{S}(\mathbf{n}) = (i, 2)$*, for some* $1 \le i \le \rho_2(\lambda)$*.*

Proof. There are exactly $|\lambda^{(1)}| = |\mu^{(1)}| - 1$ entries of the form $(x, 1)$ appearing in $\mathtt{S}$. All of these must occur in the first component if $\mathtt{S}$ is to be semistandard, leaving precisely one unoccupied node $\mathbf{n} \in \left[\mu^{(1)}\right]$. The proof of the lemma is then complete, since there are $|\lambda^{(1)}| = |\mu^{(1)}| + 1$ of entries of the form $(i,2)$ and the same number of nodes of $[\mu]$ yet to be assigned an entry by $\mathtt{S}$. □

The following lemma demonstrates that every semistandard μ-tableau of type λ is completely determined by which entries occupy nodes in the set

$$\mathscr{E}_\mu = \left\{(i,j,k) \in [\mu] : j = \mu_i^{(k)}\right\}.$$

Since the elements of this set depend only on the values taken by $\mathbf{a} = (i,k)$, we write $\left(i,\mu_i^{(k)},k\right)$ as $\mathrm{e}_{\mathbf{a}}$.

Lemma 4.2.2. *Let* $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$*. Then* $\mathtt{S}(i,j,k) = (i,k)$ *for every* $(i,j,k) \in [\mu]$ *with* $j \neq \mu_i^{(k)}$.

Proof. The first row of $\mathtt{S}$ must contain all of those entries of the form $(1,1)$. Were this not the case, our assumption that $\mathtt{S}$ is semistandard would be contradicted as there would be at least one such entry in either:

- a lower row of the first component, violating condition 2 of Definition 2.4.2; or
- a lower component than the first, violating condition 3.

Since $\mu_1^{(1)} = \lambda_1^{(1)} + 1$ and $\mathtt{S}$ is row-semistandard, there is precisely one node occupied by an entry other than $(1,1)$ and this node must be situated at the end of the row; therefore, the lemma holds when $(i,k) = (1,1)$.

Now let (i,k) be such that $(1,1) \prec (i,k)$. If the lemma holds for every row x and every component z such that $(x,z) \prec (i,k)$, then, by either condition 2 or condition 3 of Definition 2.4.2, each entry of the form (x,z) occupies a node higher than row i of component k. There are

$$\left|\lambda^{(k-1)}\right| + \sum_{l=1}^{i-1} \lambda_l^{(k)}$$

such entries and

$$\left|\lambda^{(k-1)}\right| + \sum_{l=1}^{(i-1)} \lambda_l^{(k)} + 1$$

possible nodes they can occupy, and so there is at most one entry of the form (i,k) occupying one of these available nodes. Employing a similar argument, the $\lambda_i^{(k)} = \mu_i^{(k)}$ nodes of row i in component k contain at least $\lambda_i^{(k)} - 1$ instances of this entry. The lemma now follows since the remaining entry in this row is equal to or greater than (i,k) and $\mathtt{S}$ is row-semistandard. □

We shall prove that $\mathtt{S}$ is in fact completely determined by the set of nodes $\mathrm{e}_{\mathbf{a}}$ where $\mathbf{a} = (i,k)$ is such that either:

- $i < \rho_k(\mu)$ and $(i+1,k) \prec \mathtt{S}(\mathrm{e}_{\mathbf{a}})$; or
- $i = \rho_k(\mu)$ and $(1,k+1) \prec \mathtt{S}(\mathrm{e}_{\mathbf{a}})$.

We denote this set $\mathscr{E}_{\mathrm{S}}$.

Example 14. *Let* $\lambda = ((4,3,2,1),(3,2,1,1))$ *and* $\mu = ((5,3,2,1),(3,2,1))$, *and let* $\mathrm{S} \in \mathscr{T}_0(\mu,\lambda)$ *be given by*

$$\mathrm{S} = \left(\begin{array}{lllll} 1_1 & 1_1 & 1_1 & 1_1 & 3_1 \\ 2_1 & 2_1 & 2_1 & & \\ 3_1 & 4_1 & & & \\ 2_2 & & & & \end{array} , \begin{array}{lll} 1_2 & 1_2 & 1_2 \\ 2_2 & 4_2 & \\ 3_2 & & \end{array} \right).$$

In this case, $\mathscr{E}_{\mathrm{S}} = \{(1,5,1),(4,1,1),(2,2,2)\}$.

If instead S *is given by*

$$\mathrm{S} = \left(\begin{array}{lllll} 1_1 & 1_1 & 1_1 & 1_1 & 2_1 \\ 2_1 & 2_1 & 3_1 & & \\ 3_1 & 4_1 & & & \\ 1_2 & & & & \end{array} , \begin{array}{lll} 1_2 & 1_2 & 2_2 \\ 2_2 & 3_2 & \\ 4_2 & & \end{array} \right),$$

then $\mathscr{E}_{\mathrm{S}} = \emptyset$.

Lemma 4.2.3. *Let* $\mathrm{S} \in \mathscr{T}_0(\mu,\lambda)$ *and let* $\mathbf{j},\mathbf{k} \in \mathbb{N}^2$ *be such that* $\mathrm{e}_{\mathbf{j}}, \mathrm{e}_{\mathbf{k}} \in \mathscr{E}_{\mathrm{S}}$*; then* $\mathrm{S}(\mathrm{e}_{\mathbf{j}}) \prec \mathrm{S}(\mathrm{e}_{\mathbf{k}})$ *if and only if* $\mathbf{j} \prec \mathbf{k}$.

Proof. Suppose that $\mathrm{S}(\mathrm{e}_{\mathbf{j}}) \prec \mathrm{S}(\mathrm{e}_{\mathbf{k}})$ and $\mathbf{k} \preceq \mathbf{j}$. If $\mathbf{j} = (j_1, j_2)$, then there are exactly

$$\left|\mu^{(j_2-1)}\right| + \sum_{i=1}^{j_2} \mu_i^{(j_2)} - 1$$

nodes of $[\mu]$ higher than $\mathrm{e}_{\mathbf{j}}$. Moreover, since $\mu_1^{(1)} = \lambda_1^{(1)} + 1$ and $\mu_x^{(z)} = \lambda_x^{(z)}$ for all $(x,z) \neq (1,1)$ and $(x,z) \neq (\rho_2(\lambda),2)$, this is also the number of entries in S that are less than or equal to (j_1,j_2). That S is semistandard means that all such nodes must be occupied by these entries and, since $\mathrm{e}_{\mathbf{k}}$ is higher than $\mathrm{e}_{\mathbf{k}}$, this contradicts our assumption that $\mathrm{S}(\mathrm{e}_{\mathbf{j}}) \prec \mathrm{S}(\mathrm{e}_{\mathbf{k}})$.

Conversely, suppose that $\mathbf{j} \prec \mathbf{k}$ and that $\mathbf{k} = (k_1,k_2)$. Then, by similar reasoning, all nodes higher than $\mathrm{e}_{\mathbf{k}}$ are occupied by entries less than or equal to (k_1,k_2); therefore, $\mathrm{S}(\mathrm{e}_{\mathbf{j}}) \preceq (k_1,k_2) \prec \mathrm{S}(\mathrm{e}_{\mathbf{k}})$. □

Proving that S is completely determined by the entries it maps to each node of the set $\mathscr{E}_{\mathrm{S}}$ depends on describing how S assigns entries to the remaining nodes of $[\mu]$. Lemma 4.2.2 already accomplishes this for those nodes not appearing at the ends of the rows of $[\mu]$, and so we now focus on those nodes that do.

Proposition 4.2.4. *Let* S *be a semistandard* μ*-tableau of type* λ *and let* $\mathbf{k} = (k_1,k_2)$ *for* $1 \leq k_2 \leq 2$ *and* $1 \leq k_1 \leq \rho_2(\mu)$*:*

1. *we have*

$$\mathrm{S}(\mathrm{e}_{\mathbf{k}}) = \begin{cases} (k_1+1,k_2) & \text{if } k_1 < \rho_{k_2}(\lambda) \\ (1,k_2+1) & \text{if } k_1 = \rho_{k_2}(\lambda) \text{ and } k_2 = 1 \end{cases}$$

whenever

(a) $\mathbf{k} \prec \min\{\mathbf{j} : \mathrm{e}_{\mathbf{j}} \in \mathscr{E}_{\mathrm{S}}\}$,

(b) $\mathrm{S}(\mathrm{e}_\mathbf{j}) \preceq \mathbf{k} \prec \min\{\mathbf{l} : \mathrm{e}_\mathbf{l} \in \mathscr{E}_\mathrm{S}$ *and* $\mathrm{S}(\mathrm{e}_\mathbf{j}) \prec \mathrm{e}_\mathbf{l}\}$ *for* $\mathbf{j}$ *such that* $\mathrm{e}_\mathbf{j} \in \mathscr{E}_\mathrm{S}$, *or*

(c) $\mathrm{S}(\mathrm{e}_\mathbf{j}) \preceq \mathbf{k} \prec (\rho_2(\lambda), 2)$ *for* $\mathbf{j}$ *such that* $\mathrm{S}(\mathrm{e}_\mathbf{j}) \neq (\rho_2(\lambda), 2)$ *and* $\mathbf{j} = \max\{\mathbf{l} : \mathrm{e}_\mathbf{l} \in \mathscr{E}_\mathrm{S}\}$ *(should such a value* $\mathbf{j}$ *exist);*

2. $\mathrm{S}(\mathrm{e}_\mathbf{k}) = (k_1, k_2)$ *whenever* $\mathbf{j} \prec \mathbf{k} \prec \mathrm{S}(\mathrm{e}_\mathbf{j})$ *for any* $\mathbf{j}$ *such that* $\mathrm{e}_\mathbf{j} \in \mathscr{E}_\mathrm{S}$*; and*

3. *If* $\mathbf{k} = (\rho_2(\lambda), 2)$*, then* $\mathrm{S}(\mathrm{e}_\mathbf{k}) = (\rho_2(\lambda), 2)$.

Proof. For the first part of the lemma we prove only (a) in the case where $k_1 < \rho_{k_2}(\lambda)$, the remainder of 1. and the remaining case being sufficiently similar to this to warrant omission.

1. (a) Suppose that $\mathbf{k} \prec \min\{\mathbf{j} : \mathrm{e}_\mathbf{j} \in \mathscr{E}_\mathrm{S}\}$; therefore, $\mathrm{S}(\mathrm{e}_\mathbf{k})$ is equal to either (k_1, k_2) or $(k_1 + 1, k_2)$. The first row of the first component of $[\mu]$ consists of $\mu_1^{(1)} = \lambda_1^{(1)} + 1$ nodes, $\lambda_1^{(1)}$ of which being assigned by entries of the form $(1,1)$ by S. Since S is semistandard, $\mathrm{S}(\mathrm{e}_\mathbf{k}) = (2,1)$ if $\mathbf{k} = (1,1)$. If $\mathbf{k} \neq (1,1)$ and the statement is true for all $\mathbf{l} \prec \mathbf{k}$, then there are

$$\left|\mu^{(k_2-1)}\right| + \sum_{i=1}^{k_1} \mu_i^{(k_2)} - 1$$

nodes higher than $\mathrm{e}_\mathbf{k}$ and these must all be assigned entries less than or equal to $\mathbf{k}$. In particular, these include all possible instances of $\mathbf{k}$: $\mathrm{S}(\mathrm{e}_{(k_1-1,k_2)}) = \mathbf{k}$ and there are $\mu_{k_1}^{(k_2)} - 1$ further instances of $\mathbf{k}$ occupying the nodes of row k_1, component k_2, giving a total of $\mu_{k_1}^{(k_2)} = \lambda_{k_1}^{(k_2)}$ entries. Therefore $\mathrm{S}(\mathrm{e}_\mathbf{k}) \neq \mathbf{k}$ and hence $\mathrm{S}(\mathrm{e}_\mathbf{k}) = (k_1 + 1, k_2)$.

2. Our proof is essentially the same as that of 1. (a). Suppose that $\mathbf{j} = (j_1, j_2)$ is such that $\mathrm{e}_\mathbf{j} \in \mathscr{E}_\mathrm{S}$ and that $\mathbf{j} \prec \mathbf{k} \prec \mathrm{S}(\mathrm{e}_\mathbf{j})$. There are

$$\left|\mu^{(j_2-1)}\right| + \sum_{i=1}^{j_1} \mu_i^{(j_2)} - 1$$

nodes higher than $\mathbf{j}$ and the same number of entries less than or equal to $\mathbf{j}$. Since $\mathrm{S}(\mathrm{e}_\mathbf{j}) \neq \mathbf{j}$ and S, these nodes must occupy the nodes in question. There are now two possibilities to consider: either $j_1 = \rho_{j_2}(\mu)$ or $j_1 < \rho_{j_2}(\mu)$. Since the proof proceeds in an identical fashion for both possibilities, we will omit the latter case. If $\mathbf{k} = (j_1 + 1, j_2)$, then $\mathrm{S}(\mathrm{e}_\mathbf{j}) \neq \mathbf{k}$ and, since S is semistandard, row $j_1 + 1$ of component j_2 contains all entries of the form $\mathbf{k}$ that appear in S. There are $\lambda_{j_1}^{(j_2)}$ such entries and $\mu_{j_1}^{(j_2)} = \lambda_{j_2}^{(j_2)}$; therefore $\mathrm{S}(\mathrm{e}_\mathbf{k}) = \mathbf{k}$. Our inductive step mimics that of 1. (a), completing the proof. □

We conclude this section by remarking that each of its results apply, subject to changing some of the entries, to the more general case when μ is given by (4.1): Fix positive integers $k \leq \rho_1(\lambda)$ and $j \leq \rho_2(\lambda)$. If $\mu_i^{(1)} = \lambda_i^{(1)}$ for all $1 \leq i \leq k$, then $\mathrm{S}(i,1) = (i,1)$. Similarly, if

$\mu_i^{(2)} = \lambda_i^{(2)}$ for all $j \le i \le \rho_2(\lambda)$, then $\mathtt{S}(i,2) = (i,2)$. In this way, the set $\mathscr{E}_{\mathtt{S}}$ still completely determines $\mathtt{S}$.

4.3 Describing Homomorphisms

We now use the results of the previous subsection to refine our description of semistandard homomorphisms $\Theta_{\mathtt{S}} : M^\lambda \to S^\mu$. A great deal of the most important aspect of this description depends solely upon the images of the elements of $\mathscr{E}_{\mathtt{S}}$ under $\mathtt{S}$.

Definition 4.3.1. Let $\mathbf{j} = (j_1, j_2)$ be such that $\mathrm{e}_{\mathbf{j}} \in \mathscr{E}_{\mathtt{S}}$. If $\mathtt{S}(\mathrm{e}_{\mathbf{j}}) = (x,y)$, define $\alpha_{\mathbf{j}} = \mathfrak{t}^\mu(\mathrm{e}_{\mathbf{j}})$ and $\omega_{\mathbf{j}} = \overline{\lambda}^{(y)}_{x-1} + 1$.

Additionally, for integers $1 \le a, b \le n-1$, set $\overrightarrow{T}(a,b) = T_a T_{a+1} \cdots T_{b-1}$, adopting the convention that $\overrightarrow{T}(a,b) = 1$ whenever $a = b$.

Proposition 4.3.1. *Let* $\mathtt{S}$ *be a row-semistandard* μ*-tableau of type* λ*. Then*

$$\Theta_{\mathtt{S}}(m_\lambda h) = \left(\check{\mathscr{H}}^\mu + m_\mu \prod_{\mathbf{j} \in \mathscr{E}_{\mathtt{S}}} \overrightarrow{T}\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}\right) \prod_{\mathbf{i}=(i_1,i_2)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right) \right) h,$$

where $1 \le i_2 \le 2$ *and* $1 \le i_1 \le \rho_{i_2}(\lambda)$*, for all* $h \in \mathscr{H}$*.*

Proof. By Lemma 4.1.1

$$\Theta_{\mathtt{S}}(m_\lambda h) = \left(\check{\mathscr{H}}^\mu + m_\mu T_{d(\mathrm{first}(\mathtt{S}))} \prod_{\mathbf{i}=(i_1,i_2)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right) \right) h,$$

and so we must show that

$$\prod_{\mathbf{j} \in \mathscr{E}_{\mathtt{S}}} \overrightarrow{T}\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}\right) = T_{d(\mathrm{first}(\mathtt{S}))}.$$

Suppose that $\mathbf{j} = (j_1, j_2)$ is any node with $\mathrm{e}_{\mathbf{j}} \in \mathscr{E}_{\mathtt{S}}$. By definition, $d(\mathrm{first}(\mathtt{S}))$ sends $\alpha_{\mathbf{j}}$ to the smallest integer in row x_1 of component x_2 of the tableau $\mathfrak{t}^\lambda$ with $(x_1, x_2) = \mathtt{S}(\mathrm{e}_{\mathbf{j}})$. By Lemma 4.2.3 and the definition of $\mathrm{first}(\mathtt{S})$

$$d(\mathrm{first}(\mathtt{S})) : \alpha_{\mathbf{j}} \mapsto \omega_{\mathbf{j}} \mapsto \omega_{\mathbf{j}} - 1 \mapsto \cdots \mapsto \alpha_{\mathbf{j}} + 1 \mapsto \alpha_{\mathbf{j}}.$$

If $\mathbf{k}$ is minimal such that $\mathrm{e}_{\mathbf{k}} \in \mathscr{E}_{\mathtt{S}}$ and $\mathbf{j} < \mathbf{k}$, then, again by Lemma 4.2.3,

$$d(\mathrm{first}(\mathtt{S})) : i \mapsto i$$

for all integers i. Hence $d(\mathrm{first}(\mathtt{S}))$ consists of disjoint cycles

$$\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}, \omega_{\mathbf{j}} - 1, \omega_{\mathbf{j}} - 2, \ldots, \alpha_{\mathbf{j}} + 1\right),$$

for every $\mathbf{j} \in \mathscr{E}_{\mathtt{S}}$. ☐

Example 15. *Let* $\lambda = ((3,2,1),(2,2,1))$ *and* $\mu = ((4,2,1),(2,2))$, *and let* $\mathrm{S} \in \mathscr{T}_0(\mu,\lambda)$ *be given by*

$$\mathrm{S} = \left(\begin{array}{|c|c|c|c|} \hline 1_1 & 1_1 & 1_1 & 2_2 \\ \hline 2_1 & 2_1 \\ \cline{1-2} 3_1 \\ \cline{1-1} \end{array} \ , \ \begin{array}{|c|c|} \hline 1_2 & 1_2 \\ \hline 2_2 & 3_2 \\ \hline \end{array} \right).$$

We have $\mathscr{E}_{\mathrm{S}} = \{(1,1)\}$, $\alpha_{(2,1)} = 4$, *and* $\omega_{(2,1)} = 9$, *and so* $\vec{T}(\alpha_{(2,1)},\omega_{(2,1)}) = T_{4,5,6,7,8}$ *and* $\Theta_{\mathrm{S}}(m_\lambda) = \check{\mathscr{H}}^\mu + m_\mu T_{4,5,6,7,8}(1+T_9)$.

The subsequent lemma lays out some useful properties of the description of $\Theta_{\mathrm{S}}(m_\lambda h)$ given in Proposition 4.3.1.

Lemma 4.3.2. *Let* S *be a semistandard* μ*-tableau of type* λ *and let* $\mathbf{j} = (j_1,j_2)$ *and* $\mathbf{k} = (k_1,k_2)$ *be elements of* $\mathscr{E}_{\mathrm{S}}$. *If* $\mathbf{j} \neq \mathbf{k}$, *then:*

1. $\vec{T}(\alpha_{\mathbf{j}},\omega_{\mathbf{j}})$ *and* $\vec{T}(\alpha_{\mathbf{k}},\omega_{\mathbf{k}})$ *commute with one another;*
2. $\mathrm{C}(\overline{\lambda}^{(j_2)}_{j_1} : \Omega^{\mathrm{S}}_{\mathbf{j}})$ *and* $\mathrm{C}(\overline{\lambda}^{(k_2)}_{k_1} : \Omega^{\mathrm{S}}_{\mathbf{k}})$ *commute with one another; and*
3. $\vec{T}(\alpha_{\mathbf{j}},\omega_{\mathbf{j}})$ *and* $\mathrm{C}\left(\overline{\lambda}^{(k_2)}_{k_1} : \Omega^{\mathrm{S}}_{\mathbf{k}}\right)$ *commute with one another.*

Furthermore, if $\mathbf{j}$ *is not the unique node of* $[\mu]$ *for which* S *is occupied by an entry of the form* $(\square,2)$ *and* $\mathbf{k} \neq (1,2)$, *then the element* $\mathfrak{l}^{(1)} = (L_{|\lambda^{(1)}|+1} - Q_2)$ *commutes with both*

$$\vec{T}(\alpha_{\mathbf{j}},\omega_{\mathbf{j}}) \quad \text{and} \quad \mathrm{C}(\overline{\lambda}^{(k_2)}_{k_1} : \Omega^{\mathrm{S}}_{\mathbf{k}}).$$

Proof. 1. That $\vec{T}(\alpha_{\mathbf{j}},\omega_{\mathbf{j}})$ and $\vec{T}(\alpha_{\mathbf{k}},\omega_{\mathbf{k}})$ commute follows immediately from the fact that their indexing permutations are disjoint; see the proof of Proposition 4.3.1.

2. The permutations indexing the terms of $\mathrm{C}(\overline{\lambda}^{(j_2)}_{j_1-1} : \Omega^{\mathrm{S}}_{\mathbf{j}})$ are all elements of the symmetric group on $\{\overline{\lambda}^{(j_2)}_{j_1-1}+1,\dots,\overline{\lambda}^{(j_2)}_{j_1}\}$ whilst those indexing the terms of $\mathrm{C}(\overline{\lambda}^{(k_2)}_{k_1-1} : \Omega^{\mathrm{S}}_{\mathbf{k}})$ are elements of that on the set $\{\overline{\lambda}^{(k_2)}_{k_1-1}+1,\dots,\overline{\lambda}^{(k_2)}_{k_1}\}$.

3. Our proof proceeds in a similar fashion to the previous two cases: we need only the additional observation that
$$\mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathrm{S}}_{\mathbf{i}}\right) = 1$$
for all $\mathbf{i} = (i_1,i_2)$ with $\mathbf{j} \prec \mathbf{i} \prec \mathrm{S}(\mathrm{e}_{\mathbf{j}})$. This follows from the fact that all entries of the form (i_1,i_2) are confined to a single row of S, by Proposition 4.2.4. □

A consequence of Lemma 4.3.2 is that we can rewrite our semistandard homomorphisms in a form which is much more convenient for computing the action of $\Theta_{\mathrm{S}}\left(m_\lambda \mathfrak{l}^{(1)}\right)$.

Corollary 4.3.3. *Let* S *be a semistandard* μ*-tableau of type* λ, *Then*

$$\Theta_{\mathrm{S}}\left(m_\lambda \mathfrak{l}^{(1)}\right) = \check{\mathscr{H}}^\mu + m_\mu \vec{T}(\alpha_{\mathbf{n}},\omega_{\mathbf{n}})\mathrm{C}\left(|\lambda^{(1)}| : \Omega^{\mathrm{S}}_{(1,2)}\right)\mathfrak{l}^{(1)} h_{\mathrm{S}},$$

where

$$h_{\mathrm{S}} = \prod_{\substack{\mathbf{j}\in\mathscr{E}_{\mathrm{S}}\\ \mathbf{j}\neq\mathbf{n}}} \vec{T}(\alpha_{\mathbf{j}},\omega_{\mathbf{j}}) \prod_{\mathbf{i}=(i_1,i_2)\neq(1,2)} \mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathrm{S}}\right).$$

Proof. An immediate consequence of Lemma 4.3.2. □

4.4 The Image of $m_\lambda\mathfrak{l}^{(1)}$ under a Semistandard Homomorphism

Our intention here is to provide an analogue of Proposition 4.1.2 for the elements $m_\lambda\mathfrak{l}^{(s)}$. Even in the case we consider, where μ and λ are identical bar the position of a single node, this is so much more lengthy and technical than the proof of Proposition 4.1.2.

We begin with the following lemma specifying how the Jucys-Murphy element L_i acts on the generators T_i and T_{i+1}.

Lemma 4.4.1. *Recall that* $L_i = q^{1-i}T_{i-1,i-2,\dots,1,0,1,\dots,i-2,i-1}$. *For all* $1 \le i \le n$*:*

1. $T_iL_{i+1} = (q-1)L_{i+1} + L_iT_i$*;*
2. $T_iL_i = L_{i+1}T_i - (q-1)L_{i+1}$*;*
3. $T_jL_i = L_iT_j$ *whenever* $j \neq i$ *or* $i-1$*, and*
4. *let* a *and* b *be positive integers with* $a < b$*, then is equal to*

$$\vec{T}(a,b)L_i = \begin{cases} L_a\,\vec{T}(a,b) + (q-1)\sum_{j=a+1}^{b} L_j\,\vec{T}(a,j-1)\,\vec{T}(j,b) & \text{if } b = i,\\ L_{i+1}\,\vec{T}(a,b) - (q-1)L_{i+1}\,\vec{T}(a,i)\,\vec{T}(i+1,b) & \text{if } b > i \text{ and } a \le i,\\ L_i\,\vec{T}(a,b) & \text{if } b < i \text{ or } a > i \end{cases}$$

Proof. 1., 2., and 3. follow immediately from direct calculation, and the latter two statements of the fourth part are an immediate consequence of these. That

$$\vec{T}(a,i-1)L_i = L_a\,\vec{T}(a,i-1) + (q-1)\sum_{j=a}^{i-1} L_j\,\vec{T}(a,j-1)\,\vec{T}(j,i-1)$$

follows from repeated application of 1. □

Fix $\mathrm{S} \in \mathscr{T}_0(\mu,\lambda)$ and, as in the previous section, let $\mathbf{n} = (n_1,n_2,n_3)$ be the unique node in the first component of $[\mu]$ for which $\mathrm{S}(n_1,n_2,n_3) = (x,2)$, where x is a positive integer with $1 \le x \le \rho_2(\lambda)$. The form taken by $\Theta_{\mathrm{S}}(m_\lambda\mathfrak{l}^{(1)})$ depends on whether or not $x = 1$, and so we consider each of these two cases in turn.

The case where $x = 1$: For $1 \leq z \leq \rho_1(\lambda) - n_1$ and $\nu = \lambda \cdot \mathfrak{l}^{(1)}$, define tableaux $W_0, W_z \in \mathscr{T}(\mu, \nu)$ by

$$\mathtt{W}_0(i,j,k) = \begin{cases} (\rho_1(\lambda)+1,1) & \text{if } (i,j,k) = \mathbf{n}, \\ \mathtt{S}(i,j,k) & \text{otherwise,} \end{cases} \tag{4.2}$$

and

$$W_z(i,j,k) = \begin{cases} (\rho_1(\lambda)+1,1) & \text{if } (i,j,k) = \left(n_1+z, \mu^{(1)}_{n_1+z}, 1\right), \\ \mathtt{S}\left(n_1+z, \mu^{(1)}_{n_1+z}, 1\right) & \text{if } (i,j,k) = \mathbf{n}, \\ \mathtt{S}(i,j,k) & \text{otherwise} \end{cases} \tag{4.3}$$

for all $(i,j,k) \in [\mu]$. $\mathtt{W}_0$ and, for all specified values of z, $\mathtt{W}_z$ satisfy the hypothesis of Proposition 3.4.2, and so define homomorphisms $\Theta_{\mathtt{W}_0}, \Theta_{\mathtt{W}_z} : M^\nu \to S^\mu$. We explicitly describe this homomorphism below.

Lemma 4.4.2.

$$\Theta_{\mathtt{W}_0}(m_\nu h) = \left(\check{\mathscr{H}}^\mu + m_\mu \prod_{\mathbf{j} \in \mathscr{E}_\mathtt{S}} \vec{T}(\alpha_\mathbf{j}, \omega_\mathbf{j}) \prod_{\mathbf{i} \neq (1,2)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^\mathtt{S}_\mathbf{i}\right)\right) h$$

and

$$\Theta_{\mathtt{W}_z}(m_\nu h) = \left(\check{\mathscr{H}}^\mu + m_\mu \prod_{\mathbf{j} \in \mathscr{E}_\mathtt{S} \setminus \{\mathbf{n}\}} \vec{T}(\alpha_\mathbf{j}, \omega_\mathbf{j}) \, \vec{T}\left(\alpha_\mathbf{n}, \overline{\lambda}^{(1)}_{n_1+z-1} + 1\right) \vec{T}\left(\overline{\mu}^{(1)}_{n_1+z}, \omega_\mathbf{n}\right) \right.$$
$$\left. \times \prod_{\mathbf{i} \neq (1,2)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^\mathtt{S}_\mathbf{i}\right) \mathrm{C}\left(\overline{\lambda}^{(1)}_{n_1+z-1} : 1, \lambda^{(1)}_{n_1+z} - 1\right)\right) h$$

for every $h \in \mathscr{H}$.

Proof. The tableau $\mathtt{W}_0$ is constructed from $\mathtt{S}$ by relabelling the single entry of the form $(1,2)$ appearing in the first component as $(\rho_1(\lambda)+1,1)$. Hence

$$T_{d(\mathrm{first}(\mathtt{W}_0))} = T_{d(\mathrm{first}(\mathtt{S}))} = \prod_{\mathbf{j} \in \mathscr{E}_\mathtt{S}} \vec{T}(\alpha_\mathbf{j}, \omega_\mathbf{j}).$$

By definition, if $\mathbf{i} \in [\mu]$ is such that $\mathtt{S}(\mathbf{i}) \neq (1,2)$, then $\mathtt{S}(\mathbf{i}) = \mathtt{W}_0(\mathbf{i})$; therefore,

$$\mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^\mathtt{S}_\mathbf{i}\right) = \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathtt{W}_0}_\mathbf{i}\right)$$

for all $\mathbf{i} \neq (1,2)$. Also, since the entries of the form 1_2 are all confined to the first row of the second component of $\mathtt{W}_0$, we have

$$\mathrm{C}\left(|\lambda^{(1)}| : \Omega^{\mathtt{W}_0}_{(1,2)}\right) = 1.$$

This proves the first part of the proposition.

For the second part, $\mathrm{first}(\mathtt{S})(\mathbf{i}) = \mathrm{first}(\mathtt{W}_z)(\mathbf{i})$ for every $\mathbf{i} \neq \mathbf{n}$ and $\mathbf{i} \neq \mathrm{e}_{(n_1+z,1)}$. Applying a similar argument as in the proof of Proposition 4.3.1 then yields

$$T_{d(\mathrm{first}(\mathtt{W}_z))} = \prod_{\mathbf{j} \in \mathscr{E}_\mathtt{S} \setminus \{\mathbf{n}\}} \vec{T}(\alpha_\mathbf{j}, \omega_\mathbf{j}) \, \vec{T}\left(\alpha_\mathbf{n}, \overline{\lambda}^{(1)}_{n_1+z-1} + 1\right) \vec{T}\left(\overline{\mu}^{(1)}_{n_1+z}, \omega_\mathbf{n}\right).$$

Additionally, by the construction of $\mathtt{W}_z$,

$$\mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right) = \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathtt{W}_z}_{\mathbf{i}}\right)$$

whenever $\mathbf{i} \neq (1,2)$ or $\mathbf{i} \neq (n_1+z,1)$. Since

$$\mathrm{C}\left(|\lambda^{(1)}| : \Omega^{\mathtt{W}_z}_{(1,2)}\right) = 1,$$

we are left with only

$$\mathrm{C}\left(\overline{\lambda}^{(1)}_{n_1+z-1} : \Omega^{\mathtt{S}}_{(n_1+z,1)}\right) \quad \text{and} \quad \mathrm{C}\left(\overline{\lambda}^{(1)}_{n_1+z-1} : \Omega^{\mathtt{W}_z}_{(n_1+z,1)}\right)$$

to consider. By Lemma 4.2.4, all entries of this form $(n_1+z,1)$ are contained in the same row of $\mathtt{S}$, and so

$$\mathrm{C}\left(\overline{\lambda}^{(1)}_{n_1+z-1} : \Omega^{\mathtt{S}}_{(n_1+z,1)}\right) = 1.$$

On the other hand, in $\mathtt{W}_z$ one entry of the form $(n_1+z,1)$ appears as $\mathtt{W}_z(\mathrm{e}_{\mathbf{n}})$, with the remaining $\lambda^{(1)}_{n_1+z}-1$ entries all appearing in row n_1+z; hence,

$$\mathrm{C}\left(\overline{\lambda}^{(1)}_{n_1+z-1} : \Omega^{\mathtt{W}_z}_{(n_1+z,1)}\right) = \mathrm{C}\left(\overline{\lambda}^{(1)}_{n_1+z-1} : 1, \lambda^{(1)}_{n_1+z}-1\right).$$

Combining these observations yields the stated expression for $\Theta_{\mathtt{W}_z}(m_\nu)$. □

We can write

$$\begin{aligned}\Theta_{\mathtt{W}_0}(m_\nu) &= \check{\mathscr{H}}^\mu + m_\mu \vec{T}(\alpha_{\mathbf{n}},\omega_{\mathbf{n}}) \prod_{\substack{\mathbf{j}\in\mathscr{E}_{\mathtt{S}}\\ \mathbf{j}\neq\mathbf{n}}} \vec{T}(\alpha_{\mathbf{j}},\omega_{\mathbf{j}}) \prod_{\mathbf{i}\neq(1,2)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right)\\ &= \check{\mathscr{H}}^\mu + m_\mu \vec{T}(\alpha_{\mathbf{n}},\omega_{\mathbf{n}}) h_{\mathtt{S}} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\Theta_{\mathtt{W}_z}(m_\nu) = \check{\mathscr{H}}^\mu &+ m_\mu \vec{T}\left(\alpha_{\mathbf{n}}, \overline{\lambda}^{(1)}_{n_1+z-1}+1\right) \vec{T}\left(\overline{\mu}^{(1)}_{n_1+z},\omega_{\mathbf{n}}\right)\\ &\times h_{\mathtt{S}} \mathrm{C}\left(\overline{\lambda}^{(1)}_{n_1+z-1} : 1, \lambda^{(1)}_{n_1+z}-1\right). \end{aligned} \tag{4.5}$$

We let $h_{\mathtt{W}_z}$ denote $h_{\mathtt{S}}\mathrm{C}\left(\overline{\lambda}^{(1)}_{n_1+z-1} : 1, \lambda^{(1)}_{n_1+z}-1\right) = h_{\mathtt{S}}\mathrm{C}\left(\overline{\lambda}^{(1)}_{n_1+z-1} : \Omega^{\mathtt{W}_z}_{(n_1+z,1)}\right)$.

We will show that $\Theta_{\mathtt{S}}(m_\lambda \mathfrak{l}^{(1)})$ can be expressed as a linear combination of $\Theta_{\mathtt{W}_0}(m_\nu)$ and, for each z with $1 \leq z \leq \rho_1(\lambda) - n_1$, $\Theta_{\mathtt{W}_z}(m_\nu)$. In order to simplify the proof, we need the following three results.

Lemma 4.4.3.

$$m_\mu \vec{T}(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1)\left(L_{|\lambda^{(1)}|+2} - Q_2\right) \in \check{\mathscr{H}}^\mu$$

Proof. $\vec{T}(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1)$ and $\left(L_{|\lambda^{(1)}|+2} - Q_2\right)$ commute with one another, and, by the remark following Proposition 2.3.2, we have $m_\mu L_{|\lambda^{(1)}|+2} = m_\mu L_{|\mu^{(1)}|+1} \equiv Q_2 m_\mu \mod \check{\mathscr{H}}^\mu$. □

Lemma 4.4.4.

$$(q-1)Q_2 m_\mu \overrightarrow{T}(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1) L_{|\lambda^{(1)}|+2} \sum_{i=1}^{\lambda_1^{(2)}-1} \overrightarrow{T}\left(|\lambda^{(1)}|+2, |\lambda^{(1)}|+1+i\right) \tag{4.6}$$

is equal to

$$\left(q^{\lambda_1^{(2)}-1}-1\right) Q_2 m_\mu \overrightarrow{T}(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1)$$

modulo $\check{\mathscr{H}}^\mu$.

Proof. The term $\overrightarrow{T}(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1)$ commutes past everything that appears to its right in (4.6), and $m_\mu L_{|\lambda^{(2)}|+2} = Q_2 m_\mu$; hence, equation (4.6) is equal to

$$(q-1)Q_2 m_\mu \sum_{i=1}^{\lambda_1^{(2)}-1} \overrightarrow{T}\left(|\lambda^{(1)}|+2, |\lambda^{(1)}|+1+i\right) \overrightarrow{T}(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1). \tag{4.7}$$

By [39, Corollary 3.4] $m_\mu \overrightarrow{T}\left(|\lambda^{(1)}|+2, |\lambda^{(1)}|+1+i\right) = q^{i+1-2} m_\mu$ for every $1 \le i \le \lambda_1^{(2)} - 1$. Therefore, (4.7) is in turn equal to

$$(q-1)Q_2 \left(\sum_{y=0}^{\lambda_1^{(2)}-2} q^y\right) m_\mu \overrightarrow{T}(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1). \qquad \square$$

Lemma 4.4.5. *For every* $n_1+1 \le y \le \rho_1(\mu)$*,*

$$m_\mu \sum_{k=\overline{\mu}_{y-1}^{(1)}+1}^{\overline{\mu}_y^{(1)}} \overrightarrow{T}(\alpha_{\mathbf{n}}, k-1) \overrightarrow{T}\left(k, |\lambda^{(1)}|+1\right) L_k \tag{4.8}$$

is equal to

$$q^{\mu_y^{(1)}-y} Q_1 m_\mu \overrightarrow{T}\left(\alpha_{\mathbf{n}}, \overline{\lambda}_{y-1}^{(1)}+1\right) \overrightarrow{T}\left(\overline{\mu}_y^{(1)}, |\lambda^{(1)}|+1\right) \sum_{k=\overline{\lambda}_{y-1}^{(1)}+2}^{\overline{\lambda}_y^{(1)}+1} \overrightarrow{T}\left(\overline{\lambda}_{y-1}^{(1)}+1, k-1\right)$$

modulo $\check{\mathscr{H}}^\mu$.

Proof. Using Proposition 2.3.2 to evaluate $m_\mu L_k$ for each k, we have that (4.8) is equal to

$$m_\mu \sum_{k=\overline{\mu}_{y-1}^{(1)}+1}^{\overline{\mu}_x^{(1)}} q^{k-y} Q_1 \overrightarrow{T}(\alpha_{\mathbf{n}}, k-1) \overrightarrow{T}\left(k, |\lambda^{(1)}|+1\right). \tag{4.9}$$

Writing $\overrightarrow{T}\left(k, |\lambda^{(1)}|+1\right)$ as $\overrightarrow{T}\left(k, \overline{\mu}_y^{(1)}\right) \overrightarrow{T}\left(\overline{\mu}_y^{(1)}, |\lambda^{(1)}|+1\right)$ and observing that

$$\overrightarrow{T}(\alpha_{\mathbf{n}}, k-1) \overrightarrow{T}\left(k, \overline{\mu}_y^{(1)}\right) = \overrightarrow{T}\left(k, \overline{\mu}_y^{(1)}\right) \overrightarrow{T}(\alpha_{\mathbf{n}}, k-1),$$

yields

$$m_\mu \vec{T}(\alpha_{\mathbf{n}}, k-1)\vec{T}(k, \overline{\mu}_y^{(1)}) = q^{\mu_y^{(1)}-k} m_\mu \vec{T}(\alpha_{\mathbf{n}}, k-1).$$

Substituting this into (4.9) then allows us to write (4.8) as

$$q^{\mu_y^{(1)}-y} Q_1 m_\mu \sum_{k=\overline{\mu}_{y-1}^{(1)}+1}^{\overline{\mu}_y^{(1)}} \vec{T}(\alpha_{\mathbf{n}}, k-1)\vec{T}\left(\overline{\mu}_y^{(1)}, |\lambda^{(1)}|+1\right), \tag{4.10}$$

and, since $\overline{\mu}_{y-1}^{(1)} = \overline{\lambda}_{y-1}^{(1)} + 1$ we have that

$$\begin{aligned}\vec{T}(\alpha_{\mathbf{n}}, k-1) =& \vec{T}\left(\alpha_{\mathbf{n}}, \overline{\mu}_{y-1}^{(1)}\right)\vec{T}\left(\overline{\mu}_{y-1}^{(1)}, k-1\right) \\ =& \vec{T}\left(\alpha_{\mathbf{n}}, \overline{\lambda}_{y-1}^{(1)}+1\right)\vec{T}\left(\overline{\lambda}_{y-1}^{(1)}+1, k-1\right)\end{aligned}$$

Moreover, $\vec{T}\left(\overline{\lambda}_{y-1}^{(1)}+1, k-1\right)$ and $\vec{T}\left(\overline{\mu}_y^{(1)}, |\lambda^{(1)}|+1\right)$ commute for all stated values of k. Applying this fact to (4.10) then yields

$$\begin{aligned}(4.8) = q^{\mu_y^{(1)}-y} Q_1 m_\mu \vec{T}\left(\alpha_{\mathbf{n}}, \overline{\lambda}_{y-1}^{(1)}+1\right)\vec{T}\left(\overline{\mu}_y^{(1)}, |\lambda^{(1)}|+1\right) \\ \times \sum_{k=\overline{\lambda}_{y-1}^{(1)}+2}^{\overline{\lambda}_y^{(1)}+1} \vec{T}\left(\overline{\lambda}_{y-1}^{(1)}+1, k-1\right),\end{aligned}$$

as required. □

With the previous three lemmas in place, we're now ready to precisely describe the image of $m_\lambda \mathfrak{l}^{(1)}$ under $\Theta_{\mathtt{S}}$ for a given $\mathtt{S} \in \mathscr{T}_0(\mu, \lambda)$.

Proposition 4.4.6. *Let* $\mathtt{S} \in \mathscr{T}_0(\mu, \lambda)$ *and let* $\nu = \lambda \cdot \mathfrak{l}^{(1)}$. *Then*

$$\begin{aligned}\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{l}^{(1)}\right) =& \left(\mathrm{res}_\mu(\mathbf{n}) - \mathrm{res}_\lambda\left(1, \lambda_1^{(2)}, 2\right)\right)\Theta_{\mathtt{W}_0}(m_\nu) \\ &+ (q-1)\sum_{z=1}^{\rho_1(\mu)-n_1} \mathrm{res}_\mu\left(n_1+z, \mu_{n_1+z}^{(1)}, 1\right)\Theta_{\mathtt{W}_z}(m_\nu).\end{aligned}$$

Proof. By Corollary 4.3.3, we have

$$\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{l}^{(1)}\right) = \check{\mathscr{H}}^\mu + m_\mu \vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}}) \mathrm{C}\left(|\lambda^{(1)}| : \Omega_{(1,2)}^{\mathtt{S}}\right)\mathfrak{l}^{(1)} h_{\mathtt{S}}$$

and so we begin by considering

$$m_\mu \vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}}) \mathrm{C}\left(|\lambda^{(1)}| : \Omega_{(1,2)}^{\mathtt{S}}\right)\mathfrak{l}^{(1)}.$$

By the definition of $\mathtt{S}$ and the fact that $\mathtt{S}(\mathbf{n}) = (1,2)$,

$$\begin{aligned}\mathrm{C}\left(|\lambda^{(1)}| : \Omega_{(1,2)}^{\mathtt{S}}\right) =& \mathrm{C}\left(|\lambda^{(1)}| : 1, \lambda_1^{(2)}-1\right) \\ =& 1 + \sum_{i=1}^{\lambda_1^{(2)}-1} \vec{T}\left(|\lambda^{(1)}|+1, |\lambda^{(1)}|+1+i\right).\end{aligned}$$

Multiplying on the right by $\mathfrak{l}^{(1)}$ and applying the fourth part of Lemma 4.4.1 gives us that

$$\mathrm{C}\left(|\lambda^{(1)}| : \Omega^{\mathrm{S}}_{(1,2)}\right)\mathfrak{l}^{(1)}$$

is equal to

$$\left(L_{|\lambda^{(1)}|+1} - Q_2\right) + \left(\left(L_{|\lambda^{(1)}|+2} - Q_2\right)\sum_{i=1}^{\lambda_1^{(2)}-1} \vec{T}\left(|\lambda^{(1)}|+1, |\lambda^{(1)}|+1+i\right)\right)$$
$$- \left((q-1)L_{|\lambda^{(1)}|+2}\sum_{i=1}^{\lambda_1^{(2)}-1} \vec{T}\left(|\lambda^{(1)}|+2, |\lambda^{(1)}|+1+i\right)\right).$$

By Lemma 4.4.3 and Lemma 4.4.4 respectively, multiplying on the left by $m_\mu \vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}})$, and noting that $\omega_{\mathbf{n}} = |\lambda^{(1)}|+1$, has the result that the second term is killed off and the third term gives us

$$-\left(q^{\lambda_1^{(2)}-1} - 1\right) Q_2 m_\mu \sum_{i=1} \vec{T}(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1).$$

As for the first term, a consequence of Lemma 4.4.1 is that

$$m_\mu \vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}})\left(L_{|\lambda^{(1)}|+1} - Q_2\right) = m_\mu\left(L_{\alpha_{\mathbf{n}}} - Q_2\right)\vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}}) \tag{4.11}$$

$$+(q-1)m_\mu \sum_{j=1}^{|\lambda^{(1)}|-\alpha_{\mathbf{n}}+1} L_{\alpha_{\mathbf{n}}+j}\vec{T}(\alpha_{\mathbf{n}}, \alpha_{\mathbf{n}}+j-1)\vec{T}\left(\alpha_{\mathbf{n}}+j, |\lambda^{(1)}|+1\right) \tag{4.12}$$

Fix an integer z with $1 \le z \le \rho_1(\mu) - n_1$ and set $y = n_1 + z$ so that (4.12) is equal to

$$(q-1)m_\mu \sum_{x=n_1+1}^{\rho_1(\mu)} \sum_{k=\overline{\mu}_{y-1}^{(1)}+1}^{\overline{\mu}_y^{(1)}} L_k \vec{T}(\alpha_{\mathbf{n}}, k-1)\vec{T}\left(k, |\lambda^{(1)}|+1\right).$$

Lemma 4.4.5 then gives us that $\Theta_{\mathrm{S}}(m_\lambda \mathfrak{l}^{(s)})$ is equal to

$$m_\mu\left(L_{\alpha_{\mathbf{n}}} - q^{\lambda_1^{(2)}-1}Q_2\right)\vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}})h_{\mathrm{S}}$$
$$+(q-1)\left(\sum_{x=n_1+1}^{\rho+1(\mu)} q^{\mu_y^{(1)}-y} Q_1 m_\mu \vec{T}\left(\alpha_{\mathbf{n}}, \overline{\lambda}_{y-1}^{(1)}+1\right)\vec{T}\left(\overline{\mu}_y^{(1)}, |\lambda^{(1)}|+1\right)\right.$$
$$\left.\times \sum_{k=\overline{\lambda}_{y-1}^{(1)}+2}^{\overline{\lambda}_y^{(1)}+1} \vec{T}\left(\overline{\lambda}_{y-1}^{(1)}+1, k-1\right)h_{\mathrm{S}}\right).$$

Comparison with (4.4) and (4.5) then completes our proof since:

- $\sum_{k=\overline{\lambda}_{y-1}^{(1)}+2}^{\overline{\lambda}_y^{(1)}+1} \vec{T}\left(\overline{\lambda}_{y-1}^{(1)}+1, k-1\right)h_{\mathrm{S}} = \mathrm{C}\left(\overline{\lambda}_{y-1}^{(1)} : \Omega^{\mathtt{W}_z}_{(y,1)}\right)h_{\mathrm{S}} = h_{\mathtt{W}_z}$;
- $m_\mu L_{\alpha_{\mathbf{n}}} \equiv \mathrm{res}_\mu(\mathbf{n}) m_\mu \mod \check{\mathscr{H}}^\mu$;

- $q^{\lambda_1^{(2)}-1}Q_2 = \mathrm{res}_\lambda(1,\lambda_2^{(1)},2)$; and
- $q^{\mu_y^{(1)}-y}Q_1 = \mathrm{res}_\mu\left((n_1+z,\mu_{n_1+z}^{(1)},1)\right)$. □

With this case complete, we now turn our attention to the case where $\mathtt{S}(\mathrm{e}_{\mathbf{n}}) = (x,2)$ and $x \neq 1$. This case is simpler than the last, in terms of both statement its proof. One way of thinking about both is that $\mathfrak{l}^{(1)}$ 'acts' on the tableau S thus, as we shall see in Proposition 4.4.8

- if $(x,2) = (1,2)$, then $\mathfrak{l}^{(1)}$ gradually moves $(x,2)$ row by row down through its initial position in S; and
- if $(x,2) > (1,2)$, then $\mathfrak{l}^{(1)}$ swaps the positions of $(x,2)$ and an entry of the form $(1,2)$ such that the resulting tableau is row-semistandard.

The case where $1 < x$: Define the μ–tableau U of type $\nu = \lambda \cdot \mathfrak{l}^{(1)}$ by

$$\mathtt{U}(i,j,k) = \begin{cases} (\rho_1(\lambda)+1,1) & \text{if } (i,j,k) = \mathbf{n}, \\ (x,2) & \text{if } (i,j,k) = \left(1,\mu_1^{(2)},2\right), \\ \mathtt{S}(i,j,k) & \text{otherwise.} \end{cases} \tag{4.13}$$

Like $\mathtt{W}_0$ and $\mathtt{W}_z$, this tableau satisfies Proposition 3.4.2, and so $\Theta_{\mathtt{U}} : M^\nu \to S^\mu$ is a homomorphism.

Lemma 4.4.7. *Let $\nu = \lambda \cdot \mathfrak{l}^{(1)}$. Then*

$$\Theta_{\mathtt{U}}(m_\nu h) = \check{\mathscr{H}}^\mu + m_\mu \prod_{\mathbf{j} \in \mathscr{E}_{\mathtt{S}} \setminus \{\mathbf{n}\}} \vec{T}(\alpha_{\mathbf{j}},\omega_{\mathbf{j}})\,\vec{T}\left(\alpha_{\mathbf{n}},\overline{\lambda}^{(2)}+1\right)\vec{T}\left(\overline{\mu}_1^{(2)},\omega_{\mathbf{j}}\right) \times \prod_{\mathbf{i}\neq(1,2)} \mathrm{C}\left(\overline{\lambda}_{i_1-1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathtt{S}}\right) h$$

for all $h \in \mathscr{H}$.

Proof. This is very similar to the proof of Lemma 4.4.2, and so we only provide a sketch proof. Since all of the $\lambda_1^{(2)}-1$ entries of the form $(1,2)$ in U reside in the first row of the second component, we have

$$\mathrm{C}\left(|\lambda^{(1)}| : \Omega_{(1,2)}^{\mathtt{U}}\right) = 1.$$

Now consider entries of the form $(x,2)$. In S we see that

$$\mathtt{S}_{\mathbf{n}}^{(x,2)} = 1 = \mathtt{U}_{(1,2)}^{(x,2)} \quad \text{and} \quad \mathtt{S}_{(x,2)}^{(x,2)} = \lambda_x^{(2)} - 1 = \mathtt{U}_{(x,2)}^{(x,2)}.$$

Thus

$$\mathrm{C}\left(\overline{\lambda}_{x-1}^{(2)} : \Omega_{(x,2)}^{\mathtt{S}}\right) = \mathrm{C}\left(\overline{\lambda}_{x-1}^{(2)} : \Omega_{(x,2)}^{\mathtt{U}}\right).$$

Equality of the remaining terms follows immediately from the definition of U. □

As a result of this lemma, we can in this case write

$$\Theta_{\mathrm{U}}(m_\lambda) = \check{\mathscr{H}}^\mu + m_\mu \vec{T}\left(\alpha_{\mathbf{n}}, \overline{\lambda}^{(2)}+1\right)\vec{T}\left(\overline{\mu}_1^{(2)}, \omega_{\mathbf{j}}\right)h_{\mathrm{S}}. \tag{4.14}$$

Our next result concludes this subsection, expressing $\Theta_{\mathrm{S}}(m_\lambda \mathfrak{l}^{(1)})$ in terms of $\Theta_{\mathrm{U}}(m_\nu)$ when $\mathrm{S}(\mathrm{e}_{\mathbf{n}}) = (x,2)$ for $x > 1$.

Proposition 4.4.8. *Let* S *be a semistandard* μ*-tableau of type* λ*. Then*

$$\Theta_{\mathrm{S}}\left(m_\lambda \mathfrak{l}^{(1)}\right) = -(q-1)\mathrm{res}_\mu(\mathrm{e}_{(1,2)})\Theta_{\mathrm{U}}(m_\nu).$$

Proof. By Lemma 4.2.4, all entries of the form $(1,2)$ occupy the first row of the second component of S whenever $1 < x$, hence

$$\mathrm{C}\left(|\lambda^{(1)}| : \Omega^{\mathrm{U}}_{(1,2)}\right) = 1$$

and so Corollary 4.3.3 means that we need only consider $m_\mu \vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}})\mathfrak{l}^{(1)}$. By part 4 of Lemma 4.4.1 we have

$$\begin{aligned}\vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}})\mathfrak{l}^{(1)} &= \left(L_{|\lambda^{(1)}|+2} - Q_2\right)\vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}}) \\ &\quad - (q-1)L_{|\lambda^{(1)}|+2}\vec{T}\left(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1\right)\vec{T}\left(|\lambda^{(1)}|+2, \omega_{\mathbf{n}}\right),\end{aligned}$$

and evaluating $m_\mu L_{|\lambda|^{(1)}+2} = m_\mu L_{|\mu^{(1)}|+1} = Q_2 m_\mu$ gives us that

$$m_\mu \vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}})\mathfrak{l}^{(1)} = -(q-1)Q_2 m_\mu \vec{T}\left(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1\right)\vec{T}\left(|\lambda^{(1)}|+2, \omega_{\mathbf{n}}\right).$$

Taking $|\lambda^{(1)}|+2 = |\mu^{(1)}|+1$ we can rewrite $\vec{T}(|\lambda^{(1)}|+2, \omega_{\mathbf{n}})$ as

$$\vec{T}\left(|\mu^{(1)}|+1, \overline{\mu}_1^{(2)}\right)\vec{T}\left(\overline{\mu}_1^{(2)}, \omega_{\mathbf{n}}\right).$$

Using this, along with the fact that $\vec{T}\left(|\mu^{(1)}|+1, \overline{\mu}_1^{(2)}\right)$ commutes with $\vec{T}\left(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1\right)$ and

$$\vec{T}\left(|\mu^{(1)}|+1, \overline{\mu}_1^{(2)}\right) \in \left\{T_w \in \mathscr{H} : w \in \mathfrak{S}_\mu\right\}$$

we have

$$\begin{aligned}m_\mu \vec{T}(\alpha_{\mathbf{n}}, \omega_{\mathbf{n}})\mathfrak{l}^{(1)}h_{\mathrm{S}} &= -(q-1)q^{\mu_1^{(2)}-1}Q_2 m_\mu \vec{T}\left(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1\right)\vec{T}\left(\overline{\mu}_1^{(2)}, \omega_{\mathbf{n}}\right)h_{\mathrm{S}} \\ &= -(q-1)\mathrm{res}_\mu(\mathrm{e}_{(1,2)})m_\mu \vec{T}\left(\alpha_{\mathbf{n}}, |\lambda^{(1)}|+1\right)\vec{T}\left(\overline{\mu}_1^{(2)}, \omega_{\mathbf{n}}\right)h_{\mathrm{S}}.\end{aligned}$$

The statement then follows immediately from Lemma 4.4.7 and comparison with (4.14).
□

Example 16. *Let $\lambda = ((2,1),(1,1))$ and $\mu = ((3,1),(1))$, and let $\mathtt{S}_1, \mathtt{S}_2 \in \mathscr{T}(\mu,\lambda)$ be given by*

$$\mathtt{S}_1 = \left(\begin{array}{|c|c|c|}\hline 1_1 & 1_1 & 1_2 \\ \hline 2_1 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 2_2 \\ \hline\end{array}\right) \quad \textit{and} \quad \mathtt{S}_2 = \left(\begin{array}{|c|c|c|}\hline 1_1 & 1_1 & 2_2 \\ \hline 2_1 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1_2 \\ \hline\end{array}\right).$$

Then, if $\Theta = \Theta_{\mathtt{S}_1} + \Theta_{\mathtt{S}_2}$

$$\Theta\left(m_\lambda \mathfrak{l}^{(1)}\right) = \left(q^2 Q_1 - qQ_2\right)\Theta_{\mathtt{X}}(m_\nu) + (q-1)q^{-1}Q_1\Theta_{\mathtt{Y}}(m_\nu),$$

where

$$\mathtt{X} = \left(\begin{array}{|c|c|c|}\hline 1_1 & 1_1 & 3_1 \\ \hline 2_1 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 2_2 \\ \hline\end{array}\right) \quad \textit{and} \quad \mathtt{Y} = \left(\begin{array}{|c|c|c|}\hline 1_1 & 1_1 & 2_1 \\ \hline 3_1 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 2_2 \\ \hline\end{array}\right).$$

4.5 Manipulating Maps: Semi-standardization

To recap, we can now calculate $\Theta_{\mathtt{S}}\left(m_\lambda \eth^{(s)}_{d,t}\right)$ and $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{l}^{(s)}\right)$ for every $(d,t,s) \in \mathrm{def}(\lambda, \eth)$ and every $s \in \mathrm{def}(\lambda,\eth)$ for a given $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$, in the sense that we may write

$$\Theta_{\mathtt{S}}\left(m_\lambda \eth^{(s)}_{d,t}\right) = \sum_i \alpha_{\mathtt{X}_i}\Theta_{\mathtt{X}_i}(m_\nu) \tag{4.15}$$

and

$$\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{l}^{(1)}\right) = \sum_i \beta_{\mathtt{W}_i}\Theta_{\mathtt{W}_i}\left(m_\eta\right), \tag{4.16}$$

where

- $\alpha_{\mathtt{X}_i}, \beta_{\mathtt{W}_i} \in \mathbb{F}$,
- $\nu = \lambda \cdot \eth^{(s)}_{d,t}$, and $\eta = \lambda \cdot \mathfrak{l}^{(s)}$; and
- $\mathtt{X}_i \in \mathscr{T}_{r,0}(\mu,\nu)$ and $\mathtt{W}_i \in \mathscr{T}_{r,0}(\mu,\eta)$.

Although the various $\Theta_{\mathtt{X}_i}$ and $\Theta_{\mathtt{W}_i}$ are homomorphisms, we cannot say if they are linearly independent. As a result, it may be the case that (4.15) and (4.16) are zero. To address this, the results in this section provide us with a way of writing (4.15) and (4.16) in terms of the images of m_ν and m_η under semistandard, and hence linearly independent, homomorphisms. For want of a better term, we will call this process *semi-standardization*

However, this does not entirely avoid the problem that we may end up unwittingly constructing the zero homomorphism. It may be the case that we construct a homomorphism from the semistandard tableaux presented in this section only to find that the coefficients we choose are all zero, such as when coefficients contain $[e]$ as a factor, where e is the quantum characteristic defined in Definition 2.5.1. This problem will be discussed more fully in Section 4.7.

4.5.1 Semi-standardizing $\Theta\left(m_\lambda \partial^{(s)}_{d,t}\right)$

Motivated by convenience and the fact that $\mathtt{S}(x,y,z) = (x,z)$ for every $y \le \mu^{(z)}_x$, we take to writing

$$\mathtt{S} = \left(\mu : j^{(1)}_1, j^{(1)}_2, \ldots, j^{(1)}_{\rho_1(\mu)}, j^{(2)}_1, j^{(2)}_2, \ldots, j^{(2)}_{\rho_2(\mu)}\right),$$

where $j^{(z)}_x$ is to be taken as referring to the entry $\mathtt{S}(x, \mu^{(z)}_x, z)$. In order to avoid any ambiguity we shall write $\partial^{(s)}_{d,t}(\mu)$ to denote the element $\partial^{(s)}_{d,t}$ is an element of $\mathbf{D}(\mu)$, rather than the more usual $\mathbf{D}(\lambda)$.

Let s and t be fixed. For each particular value of d there are four possible configurations of $\mathtt{S}$ that we need to consider, these being given below. They can be, fairly imprecisely, summarized, as representing the situation when (d,s) appears in a row higher than d (Case I and Case II), and when (d,s) appears in row d.

<u>**Case I**</u>

$$\mathtt{S} = \left(\lambda : j^{(1)}_1, \ldots, j^{(s)}_{d-1}, (d+1,s), j^{(s)}_{d+1}, \ldots, j^{(2)}_{\rho_2(\mu)}\right) \tag{4.17}$$

where $j^{(s)}_{d+1} > (d+1,s)$ and either $d = s = 1$ or $j^{(l)}_k = (d,s)$ for some $(k,l) < (d,s)$.

<u>**Case II**</u>

$$\mathtt{S} = \left(\lambda : j^{(1)}_1, \ldots, j^{(s)}_{d-1}, j^{(s)}_d, (d+1,s), j^{(s)}_{d+2}, \ldots, j^{(s)}_{\rho_2(\mu)}\right) \tag{4.18}$$

where $j^{(s)}_d > (d+1,s)$ and either $d = s = 1$ or $j^{(l)}_k = (d,s)$ for some $(k,l) < (d,s)$.

<u>**Case III**</u>

$$\mathtt{S} = \left(\lambda : j^{(1)}_1, \ldots, j^{(s)}_{d-1}, (d,s), (d+1,s), j^{(s)}_{d+2}, \ldots, j^{(2)}_{\rho_2(\mu)}\right) \tag{4.19}$$

where $j^{(l)}_k > (d+1,s)$ for some $(k,l) < (d,s)$.

<u>**Case IV**</u>

$$\mathtt{S} = \left(\lambda : j^{(1)}_1, \ldots, j^{(s)}_{k-1}, (d+1,s), j^{(s)}_{k+1} \ldots, j^{(s)}_{d-1}, (d,s), j^{(s)}_{d+1}, \ldots, j^{(2)}_{\rho_2(\mu)}\right) \tag{4.20}$$

for some $1 \le k \le d-1$ and where $j^{(s)}_{d+1} > (d+1,s)$.
Example 17. *Let* $\mathtt{S}$ *be given by*

$$\left(\begin{array}{|c|c|c|c|}\hline 1_1 & 1_1 & 1_1 & 2_1 \\ \hline 2_1 & 2_1 & 2_2 \\ \cline{1-3} 3_1 & 3_1 \\ \cline{1-2} 4_1 & 4_1 \\ \cline{1-2}\end{array}\;,\;\begin{array}{|c|c|}\hline 1_2 & 1_2 \\ \hline 3_2 \\ \cline{1-1}\end{array}\right).$$

Here $\mathtt{A}$ *falls into case I for* $(d,s) = (1,1)$*, case II for* $(d,s) = (2,1)$*, case III for* $(d,s) = (3,1)$*, and case IV for* $(d,s) = (1,2)$*.*

Before we continue and consider each case (and its sub-cases) we introduce some technical Lemmas. These go some way towards letting us describe how the elements of $\mathbf{D}(\lambda)$ interact with certain tableaux.

Lemma 4.5.1 ([38, Lemma 3.13])**.** *Let $m \geq 0$ and let $\eta = (\eta_1, \ldots, \eta_l)$ be a composition. Suppose that $0 \leq x \leq l$. Then*

$$\mathrm{C}\left(m : \eta_1, \ldots, \eta_l\right) = \mathrm{C}(m : \eta_1, \ldots, \eta_x)\mathrm{C}(m + \overline{\eta}_x : \eta_{x+1}, \ldots, \eta_l)\mathrm{C}(m : \overline{\eta}_x, \overline{\eta}_l - \overline{\eta}_x).$$

The next two statements are introduced in order to help us better study the $\Theta_{\mathrm{X}_i}(m_\nu)$ terms described in the introduction to this subsection.

Lemma 4.5.2. *Let $\eta = (\eta_1, \eta_2)$ be a composition and let $\mathfrak{t}$ be a row-standard η-tableau with entries taken from $\{x+1, x+2, \ldots, x+|\eta|\}$ such that $\mathfrak{t}(1, \eta_1) = x + |\eta|$. Then*

$$T_{d(\mathfrak{t})} = T_{x+\eta_1} T_{x+\eta_1+1} \cdots T_{x+|\eta|-1} T_w$$

where $w \in \mathscr{D}_{x,(\eta_1-1,\eta_2)}$.

Proof. Let v be the permutation $(x+\eta_1, x+\eta_2, x+\eta_2-1, x+\eta_2-2, \ldots, x+\eta_1+1)$. Since $v \in \sigma^x \mathfrak{S}_\eta$, that we can write T_{vw} as $T_v T_w$ follows from [39, Proposition 3.13]. □

Proposition 4.5.3. *For all $x \geq 0$ and every $\eta_1, \eta_2 \geq 1$,*

$$\mathrm{C}\left(x : (\eta_1, \eta_2)\right) = \left(T_{x+\eta_1} T_{x+\eta_1+1} \cdots T_{x+|\eta|-1}\right) \mathrm{C}\left(x : (\eta_1 - 1, \eta_2)\right) + \mathrm{C}\left(x : (\eta_1, \eta_2 - 1)\right).$$

Proof. Let $\eta = (\eta_1, \eta_2)$ and consider the set of row-standard η-tableaux with entries taken from the set $\{x+1, x+2, \ldots, x+|\eta|\}$. This set can be partitioned into two subsets: that in which the entry $x+|\eta|$ occupies the node $(2, \eta_2)$, and that in which $x+|\eta|$ occupies the node $(1, \eta_1)$. Due to Lemma 4.5.2, the former provides us with the term

$$\left(T_{x+\eta_1} T_{x+\eta_1+1} \cdots T_{x+|\eta|-1}\right) \mathrm{C}\left(x : (\eta_1 - 1, \eta_2)\right),$$

with the latter providing the remaining term. □

From here on in, for every $1 \leq i \leq r$, let $x_{\mu^{(i)}}$ denote the image of $y_{\mu^{(i)}}$ under the isomorphism $\zeta_i^\mu : \mathscr{H}_{|\mu^{(i)}|} \to \mathscr{H}_{\overline{\mu}^{(i)}, \mu^{(i)}}$ defined in Section 2.7. The next result, in which this isomorphism plays a prominent role, is a limited generalization of the well known result in the setting of the Iwahori-Hecke algebra; that being if η is a partition and $\mathfrak{t}$ is a standard η-tableau in which i and $i+1$ appear in the same column, then $y_{\mathfrak{t}^\eta \mathfrak{t}} T_i = -y_{\mathfrak{t}^\eta \mathfrak{t}}$ modulo higher terms.

Proposition 4.5.4. *Suppose that $d, s \geq 1$ and that $\mu_{d-1}^{(s)} = \mu_d^{(s)}$. Then*

$$m_\mu \mathrm{C}\left(\overline{\mu}_{d-1}^{(s)} - 1 : 1, \mu_d^{(s)}\right) \in \check{\mathscr{H}}^\mu$$

Proof. Factorizing x_μ we can write

$$m_\mu \mathrm{C}\left(\overline{\mu}^{(s)}_{d-1}-1:1,\mu^{(s)}_d\right)=u^+_\mu x_{\mu\setminus\mu^{(s)}}x_{\mu^{(s)}}\mathrm{C}\left(\overline{\mu}^{(s)}_{d-1}-1:1,\mu^{(s)}_d\right).$$

Setting $\gamma=\mu^{(s)}$, we have that

$$x_{\mu^{(s)}}\mathrm{C}\left(\overline{\mu}^{(s)}_{d-1}-1:1,\mu^{(s)}_d\right)$$

corresponds under the isomorphism $\zeta^\mu_s:\mathscr{H}_{|\mu^{(s)}|}\to\mathscr{H}_{\mu^{(s)}}$ defined in Section 2.7 to the sum

$$\sum_{i=0}^{\mu^{(s)}} y_{\mathfrak{t}^\gamma\mathfrak{t}_i}$$

where, for every $0\le i\le\mu^{(s)}$, $\mathfrak{t}_i$ is the row-standard γ-tableau such that

$$\mathfrak{t}_i(x,y)=\begin{cases}\mathfrak{t}^\gamma(x,y)+i & \text{if }(x,y)=(d-1,\gamma_{d-1}),\\ \mathfrak{t}^\gamma(x,y)-1 & \text{if }y=d\text{ and }x\le i,\\ \mathfrak{t}^\gamma(x,y) & \text{otherwise.}\end{cases}$$

Apart from when $i=\mu^{(s)}$, each such γ-tableau is standard. When $i=\mu^{(s)}$ we have $y_{\mathfrak{t}^\gamma\mathfrak{t}_i}=y_{\mathfrak{t}^\gamma\mathfrak{t}_{i-1}}T_i$ and since $\mu^{(s)}-1$ and $\mu^{(s)}$ are in the same column of $\mathfrak{t}_i$, we then have

$$y_{\mathfrak{t}^\gamma\mathfrak{t}_{\mu^{(s)}}}=-y_{\mathfrak{t}^\gamma\mathfrak{t}_{\mu^{(s)}-1}}+\sum_{i=0}^{\mu^{(s)}-2}r_{\mathfrak{t}_i}y_{\mathfrak{t}^\gamma\mathfrak{t}_i}\mod\check{\mathscr{H}}^\gamma_{|\gamma|},$$

for some $r_{\mathfrak{t}_i}\in\mathbb{F}$, by [39, Corollary 3.21]. Moreover, applying [39, Corollary 3.19] yields $r_i=-1$ for each $0\le u\le\mu^{(s)}-2$; hence

$$\sum_{i=0}^{\mu^{(s)}} y_{\mathfrak{t}^\gamma\mathfrak{t}_i}\in\check{\mathscr{H}}^\gamma_{|\gamma|}.$$

This means that

$$x_{\mu^{(s)}}\mathrm{C}\left(\overline{\mu}^{(s)}_{d-1}-1:1,\mu^{(s)}_d\right)\tag{4.21}$$

is a linear combination of terms of the form $\zeta_\gamma(n_{\mathfrak{st}})$ where $\mathfrak{s}$ and $\mathfrak{t}$ are standard ξ-tableaux, with ξ ranging over the partitions of $|\gamma|$ dominating γ.

We also have that $\zeta_\gamma\left(T^*_{d(\mathfrak{s})}\right),\zeta_\gamma\left(T_{d(\mathfrak{t})}\right)\in\mathscr{H}(\mathfrak{S}_\gamma)$, and so these terms commute with u^+_μ; thus, (4.21) is a linear combination of terms of the form

$$\zeta_\gamma\left(T^*_{d(\mathfrak{s})}\right)u^+_\mu x_{\mu\setminus\mu^{(s)}}\zeta_\gamma(n_\xi)\zeta_\gamma\left(T_{d(\mathfrak{t})}\right)\in\check{\mathscr{H}}^\mu.\qquad\square$$

With these technical results in place, we are now ready to continue on to the main topic of this section: the semi-standardization of non-semi-standard homomorphisms. Much of this section has a slight similarity to the previous one, in so much as our dominant strategy is to define tableaux that suit our needs, along with the homomorphisms determined by them, and then deduce our results by various comparative methods.

Throughout this section, take ν to denote the multicomposition $\lambda\cdot\mathfrak{d}^{(s)}_{d,t}$. Since we consider mainly arbitrary values of d and s, there will be no danger of ambiguity in not specifying values for (d,t,s) in our notation.

Case I Suppose that $d > 1$ and that $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ is as in (4.17). For $t \geq 1$, let $\mathtt{X}_1, \mathtt{X}_2 \in \mathscr{T}(\mu,\lambda)$ be given by

$$\mathtt{X}_1(x,y,z) = \begin{cases} (d,s) & \text{if } (x,y,z) = (d,\mu_d^{(s)},s) \text{ or } (d+1,b,s) \text{ with } b \leq t-1 \\ \mathtt{S}(x,y,z) & \text{otherwise,} \end{cases}$$

and

$$\mathtt{X}_1(x,y,z) = \begin{cases} (d,s) & \text{if } (x,y,z) = (d+1,b,s) \text{ for } b \leq t, \\ \mathtt{S}(x,y,z) & \text{otherwise.} \end{cases}$$

That is to say, $\mathtt{X}_1$ and $\mathtt{X}_2$ are the two possible μ-tableaux of type $\lambda \cdot \mathfrak{d}_{d,t}^{(s)}$ formed from $\mathtt{S}$ by replacing t many entries $(d+1,s)$ with (d,s).

Lemma 4.5.5. *The homomorphisms $\Theta_{\mathtt{X}_1}, \Theta_{\mathtt{X}_2} : M^\nu \to S^\mu$ are given by*

$$\Theta_{\mathtt{X}_1}(m_\nu) = \check{\mathscr{H}}^\mu + m_\mu T_{d(\mathrm{first}(\mathtt{S}))} \prod_{\mathbf{i} \neq (d+1,s),(d,s)} \mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathtt{S}}\right) \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)}, t-1\right)$$

and

$$\Theta_{\mathtt{X}_2}(m_\nu) = \check{\mathscr{H}}^\mu + m_\mu T_{d(\mathrm{first}(\mathtt{S}))} \vec{T}\left(\overline{\mu}_d^{(s)}, \overline{\mu}_d^{(s)} + t\right) \prod_{\mathbf{i} \neq (d,s),(d+1,s)} \mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathtt{S}}\right)$$
$$\times \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)} - 1, t\right) \mathrm{C}\left(\overline{\lambda}_d^{(s)} : 1, \lambda_{d+1}^{(s)} - t - 1\right).$$

Proof. That $T_{d(\mathrm{first}(\mathtt{X}_1))} = T_{d(\mathrm{first}(\mathtt{S}))}$ is immediate and, by Lemma 4.1.1, provides us with the required equality for $\Theta_{\mathtt{X}_1}(m_\nu)$. In the case of $\Theta_{\mathtt{X}_2}(m_\nu)$, the proof then proceeds in a manner identical to that of Propositions 4.4.2 and 4.4.7. □

Corollary 4.5.6. *The homomorphisms $\Theta_{\mathtt{X}_1}, \Theta_{\mathtt{X}_2} : M^\nu \to S^\mu$ are given by*

$$\Theta_{\mathtt{X}_1}(m_\nu) = \check{\mathscr{H}}^\mu + m_\mu \mathrm{C}\left(\overline{\mu}_{d-1}^{(s)} : \lambda_d^{(s)}, t-1\right) \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)} + t - 1\right)$$
$$\times T_{d(\mathrm{first}(\mathtt{S}))} \prod_{\mathbf{i} \neq (d+1,s),(d,s)} \mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathtt{S}}\right)$$

and

$$\Theta_{\mathtt{X}_2}(m_\nu) = \check{\mathscr{H}}^\mu + m_\mu \vec{T}\left(\overline{\mu}_d^{(s)}, \overline{\mu}_d^{(s)} + t\right) \mathrm{C}\left(\overline{\mu}_{d-1}^{(s)} : \lambda_d^{(s)} - 1, t\right)$$
$$\times \mathrm{C}\left(\overline{\lambda}_d^{(s)} : 1, \lambda_{d+1}^{(s)} - t - 1\right) \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)} + t - 1\right)$$
$$\times T_{d(\mathrm{first}(\mathtt{S}))} \prod_{\mathbf{i} \neq (d,s),(d+1,s)} \mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathtt{S}}\right).$$

Proof. By [37, Lemma 3.13], we have

$$\mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)}, t-1\right) = \mathrm{C}\left(\overline{\mu}_{d-1}^{(s)} : \lambda_d^{(s)}, t-1\right) \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)} + t - 1\right)$$

and

$$\mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)} - 1, t\right) = \mathrm{C}\left(\overline{\mu}_{d-1}^{(s)} : \lambda_d^{(s)} - 1, t\right) \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)} + t - 1\right).$$

Now, recall that $T_{d(\text{first}(\mathtt{S}))} = \prod_{\mathbf{j}\in\mathscr{E}_{\mathtt{S}}} \vec{T}(\alpha_{\mathbf{j}},\omega_{\mathbf{j}})$. By the properties of $\mathtt{S}$ detailed in subsection 3.1, we have that $\mathbf{i} \prec \mathbf{j}$ if and only if $\mathtt{S}(\mathrm{e}_{\mathbf{i}}) \prec \mathtt{S}(\mathrm{e}_{\mathbf{j}})$ and so T_i is not involved in $T_{d(\text{first}(\mathtt{S}))}$ for any $i \in \{\overline{\mu}^{(s)}_{d-1}+1,\ldots,\overline{\lambda}^{(s)}_{d+1}\}$. From this we deduce that the necessary factors of $\Theta_{\mathtt{X}_1}(m_\nu)$ and $\Theta_{\mathtt{X}_2}(m_\nu)$ commute with $T_{d(\text{first}(\mathtt{S}))}$ to yield the expressions given in the corollary. □

With this Corollary in place, we're now in a position to refine and, if necessary, semi-standardize the expression for $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right)$ given in Proposition 4.1.2. Although so far we have concentrated on the case where $d > 1$, the following proposition also holds for $d = 1$ with only minor alterations of the proof being necessary.

Proposition 4.5.7. *If $t = 1$ and either $\mu^{(s)}_{d-1} \neq \mu^{(s)}_{d}$ or $j^{(s)}_{d-1} = (d,s)$ then*

$$\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right) = q^{\lambda^{(s)}_{d+1}-1}\left[\lambda^{(s)}_{d} - \lambda^{(s)}_{d+1} + 1\right]\Theta_{\mathtt{X}_1}(m_\nu).$$

Otherwise, $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right) \in \check{\mathscr{H}}^{\mu}$.

Proof. By Proposition 4.1.2, we have $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right) = a_1\Theta_{\mathtt{X}_1}(m_\nu) + a_2\Theta_{\mathtt{X}_2}(m_\nu)$ for constants $a_1, a_2 \in \mathbb{F}$. Suppose that $t > 1$. Then by, Corollary 4.5.6,

$$\begin{aligned}\Theta_{\mathtt{X}_1}(m_\nu) &= \check{\mathscr{H}}^{\mu} + m_\mu \mathrm{C}\left(\overline{\mu}^{(s)}_{d-1} : \lambda^{(s)}_{d}, t-1\right)h \\ &= \check{\mathscr{H}}^{\mu} + m_\mu \mathfrak{d}^{(s)}_{d,t-1}(\mu)h\end{aligned}$$

for a certain $h \in \mathscr{H}$, and so $\Theta_{\mathtt{X}_1}(m_\nu) \in \check{\mathscr{H}}^{\mu}$. Now, we have

$$\begin{aligned}\Theta_{\mathtt{X}_2(m_\nu)} &= \check{\mathscr{H}}^{\mu} + m_\mu\left(\mathrm{C}(\overline{\lambda}^{(s)}_{d-1} : \mu^{(s)}_{d}, t) - \mathrm{C}(\overline{\lambda}^{(s)}_{d-1} : \mu^{(s)}_{d}, t-1)\right)h \\ &= \check{\mathscr{H}}^{\mu} + m_\mu\left(\mathfrak{d}^{(s)}_{d,t}(\mu) - \mathfrak{d}^{(s)}_{d,t-1}(\mu)\right)h\end{aligned}$$

for a certain $h \in \mathscr{H}$; hence $\Theta_{\mathtt{X}_2}(m_\nu) \in \check{\mathscr{H}}^{\mu}$.

Suppose now that $t = 1$ and that either $\mu^{(s)}_{d-1} \neq \mu^{(s)}_{d}$ or $j^{(s)}_{d-1} \neq (d,s)$. Setting this value of t in Proposition 4.5.3 gives us

$$\Theta_{\mathtt{X}_1}(m_\nu) = \check{\mathscr{H}}^{\mu} + m_\mu \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : 1, \lambda^{(s)}_{d}\right) T_{d(\text{first}(\mathtt{S}))} \prod_{\mathbf{i}\neq(d+1,s),(d,s)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right)$$

and

$$\begin{aligned}\Theta_{\mathtt{X}_2}(m_\nu) = \check{\mathscr{H}}^{\mu} + m_\mu \vec{T}\left(\overline{\mu}^{(s)}_{d}, \overline{\mu}^{(s)}_{d}+1\right)\mathrm{C}\left(\overline{\mu}^{(s)}_{d-1} : \lambda^{(s)}_{d}-1, 1\right) \\ \times \mathrm{C}\left(\overline{\lambda}^{(s)}_{d} : 1, \lambda^{(s)}_{d+1}-2\right)\mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : 1, \lambda^{(s)}_{d}\right) \\ \times T_{d(\text{first}(\mathtt{S}))} \prod_{\mathbf{i}\neq(d,s),(d+1,s)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right).\end{aligned}$$

By Proposition 4.5.3 this is equal to

$$\check{\mathscr{H}}^{\mu} + m_\mu \left(\mathfrak{d}_{d,1}^{(s)}(\mu) - 1\right) \mathrm{C}\left(\overline{\lambda}_d^{(s)} : 1, \lambda_{d+1}^{(s)} - 2\right) \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)}\right) \times T_{d(\mathrm{first}(\mathtt{S}))} \prod_{\mathbf{i} \neq (d,s),(d+1,s)} \mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathtt{S}}\right),$$

and, since $m_\mu \mathfrak{d}_{d,1}^{(s)} \in \check{\mathscr{H}}^{\mu}$, this gives us

$$\begin{aligned}
\Theta_{\mathtt{X}_2}(m_\nu) &= \check{\mathscr{H}}^{\mu} - m_\mu \mathrm{C}\left(\overline{\lambda}_d^{(s)} : 1, \lambda_{d+1}^{(s)} - 2\right) \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)}\right) \\
&\quad \times T_{d(\mathrm{first}(\mathtt{S}))} \prod_{\mathbf{i} \neq (d,s),(d+1,s)} \mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathtt{S}}\right) \\
&= \check{\mathscr{H}}^{\mu} - \left[\lambda_{d+1}^{(s)} - 1\right] m_\mu \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)}\right) T_{d(\mathrm{first}(\mathtt{S}))} \prod_{\mathbf{i} \neq (d,s),(d+1,s)} \mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathtt{S}}\right) \\
&= -\left[\lambda_{d+1}^{(s)} - 1\right] \Theta_{\mathtt{X}_1}(m_\nu).
\end{aligned}$$

The proof is then complete by the direct calculation of $a_1 = [\lambda_d^{(s)}]$ and $a_2 = 1$ when $t = 1$.

Finally, suppose that $\mu_{d-1}^{(s)} = \mu_d^{(s)}$ and that $j_{d-1}^{(s)} = (d,s)$. Then

$$\Theta_{\mathtt{X}_1}(m_\nu) = \check{\mathscr{H}}^{\mu} + m_\mu \mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)} : 1, \lambda_d^{(s)}\right) T_{d(\mathrm{first}(\mathtt{S}))} \prod_{\mathbf{i} \neq (d,s),(d+1,s)} \mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)} : \Omega_{\mathbf{i}}^{\mathtt{S}}\right).$$

Proposition 4.5.4 then completes the proof. □

Finally, for Case I, we consider the case when $(d,s) = (1,1)$.

Proposition 4.5.8. *If* $t = 1$*, then*

$$\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}_{1,1}^{(1)}\right) = q^{\lambda_2^{(1)} - 1} \left[\lambda_1^{(1)} - \lambda_2^{(1)} + 2\right] \Theta_{\mathtt{X}_1}(m_\nu).$$

Otherwise, $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}_{1,t}^{(1)}\right) = \check{\mathscr{H}}^{\mu}$.

Proof. In both cases, the proof is virtually identical to that for when $(d,s) \neq (1,1)$. The only real difference is that $a_1 = [\lambda_1^{(1)} + 1]$, rather than $[\lambda_1^{(1)}]$. □

We now move on to cover the other three cases. Given the similarities between the proofs so far and those yet to come, we will suppress many of the details, in most cases simply stating what the relevant tableaux are and the form of the homomorphisms they determine as and when they are needed without labouring too much over justification.

Case II Suppose that $d>1$ and that $\mathtt{S}\in\mathcal{T}_0(\mu,\lambda)$ is as in (4.18). For $t\geq 1$, let $\mathtt{X}_3\in\mathcal{T}(\mu,\lambda)$ be given by

$$\mathtt{X}_3(x,y,z)=\begin{cases}(d,s) & \text{if } (x,y,z)=(d+1,b,s) \text{ with } b\leq t\\ \mathtt{S}(x,y,z) & \text{otherwise.}\end{cases}$$

That is, $\mathtt{X}_3$ is the μ-tableaux of type ν formed from $\mathtt{S}$ by replacing t many entries $(d+1,s)$ with (d,s).

Lemma 4.5.9.

$$\Theta_{\mathtt{X}_3}(m_\nu)=\check{\mathscr{H}}^\mu+m_\mu T_{d(\text{first}(\mathtt{S}))}\prod_{\mathbf{i}\neq(d,s)}\mathrm{C}\left(\overline{\lambda}_{i_2}^{(i_2)}:\Omega_{\mathbf{i}}^{\mathtt{S}}\right)\mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)}:1,\lambda_d^{(s)},t\right)$$

Proof. That $T_{d(\text{first}(\mathtt{S}))}=T_{d(\text{first}(\mathtt{X}_3))}$ is immediate by the construction of $\mathtt{X}_3$, and so the result is a consequence of Lemma 4.1.1. □

As in the previous case, we can almost immediately state a more useful expression for the homomorphism $\Theta_{\mathtt{X}_3}$:

Corollary 4.5.10.

$$\begin{aligned}\Theta_{\mathtt{X}_3}(m_\nu)=\check{\mathscr{H}}^\mu+m_\mu\,\vec{T}\left(\overline{\mu}_d^{(s)},\overline{\mu}_d^{(s)}+t\right)\mathrm{C}\left(\overline{\mu}_{d-1}^{(s)}:\lambda_d^{(s)},t\right)\vec{T}\left(\overline{\mu}_d^{(s)}+t,\right)\\ \times\prod\vec{T}\left(\alpha_{\mathbf{j}},\omega_{\mathbf{j}}\right)\prod_{\mathbf{i}\neq(d,s)}\mathrm{C}\left(\overline{\lambda}_{i_1}^{(i_2)}:\Omega_{\mathbf{i}}^{\mathtt{S}}\right)\mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)}:1,\lambda_d^{(s)}+t-1\right).\end{aligned}$$

Proof. By Lemma 4.5.1 we have

$$\mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)}:1,\lambda_d^{(s)},t\right)=\mathrm{C}\left(\overline{\mu}_{d-1}^{(s)}:\lambda_d^{(s)},t\right)\mathrm{C}\left(\overline{\lambda}_{d-1}^{(s)}:1,\lambda_d^{(s)}+t-1\right)$$

and

$$T_{d(\text{first}(\mathtt{S}))}=\vec{T}\left(\alpha_{\mathbf{k}},\omega_{\mathbf{k}}\right)\prod_{\substack{\mathbf{j}\in\mathscr{E}_{\mathtt{S}}\\ \mathbf{j}\neq\mathbf{k}}}\vec{T}\left(\alpha_{\mathbf{j}},\omega_{\mathbf{j}}\right)=\vec{T}\left(\overline{\mu}_d^{(s)},\omega_{\mathbf{k}}\right)\prod_{\substack{\mathbf{j}\in\mathscr{E}_{\mathtt{S}}\\ \mathbf{j}\neq\mathbf{k}}}\vec{T}\left(\alpha_{\mathbf{j}},\omega_{\mathbf{j}}\right).$$

Therefore, since we may write $\vec{T}\left(\overline{\mu}_d^{(s)},\omega_{\mathbf{k}}\right)=\vec{T}\left(\overline{\mu}_d^{(s)},\overline{\mu}_d^{(s)}+x\right)\vec{T}\left(\overline{\mu}_d^{(s)}+x,\omega_{\mathbf{k}}\right)$, the remainder of the proof is now a matter of multiplying $\mathrm{C}\left(\overline{\mu}_{d-1}^{(s)}:\lambda_d^{(s)},t\right)$ through to the left. □

Let $\mathtt{X}_4\in\mathcal{T}_0(\mu,\nu)$ be the tableau given by

$$\mathtt{X}_4(x,y,z)=\begin{cases}(d,s) & \text{if } (x,y,z)=(d,\mu_d^{(s)},s)\\ j_d^{(s)} & \text{if } (x,y,z)=(d+1,\mu_{d+1}^{(s)},s)\\ \mathtt{S}(x,y,z) & \text{otherwise.}\end{cases}$$

$\mathtt{X}_4$ is then the tableau obtained by swapping the position of the $(d+1,s)$ at the end of row $d+1$ with that of $j_d^{(s)}$ and then replacing the former with (d,s).

Proposition 4.5.11. *If $t>1$, then $\Theta_{\mathtt{S}}(m_\nu)\in\check{\mathscr{H}}^\mu$. Otherwise, $\Theta_{\mathtt{S}}(m_\nu)=-q^{\mu_{d+1}^{(s)}-1}\Theta_{\mathtt{X}_4}(m_\nu)$.*

Proof. Using Proposition 4.5.3 and Corollary 4.5.10 we have

$$\begin{aligned}
\Theta_{\mathtt{X}_3}(m_\nu) &= \check{\mathscr{H}}^\mu + m_\mu \left(\mathrm{C}\left(\overline{\mu}^{(s)}_{d-1} : \lambda^{(s)}_d + 1, t\right) - \mathrm{C}\left(\overline{\mu}^{(s)}_{d-1} : \lambda^{(s)}_d + 1, t-1\right)\right) \\
&\times \vec{T}\left(\overline{\mu}^{(s)}_d + t,\right) \prod \vec{T}\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}\right) \prod_{\mathbf{i} \neq (d,s)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1} : \Omega^{\mathrm{S}}_{\mathbf{i}}\right) \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : 1, \lambda^{(s)}_d + t - 1\right) \\
&= \check{\mathscr{H}}^\mu + m_\mu \left(\mathfrak{d}^{(s)}_{d,t}(\mu) - \mathfrak{d}^{(s)}_{d,t-1}(\mu)\right) \vec{T}\left(\overline{\mu}^{(s)}_d + t,\right) \prod \vec{T}\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}\right) \\
&\times \prod_{\mathbf{i} \neq (d,s)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1} : \Omega^{\mathrm{S}}_{\mathbf{i}}\right) \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : 1, \lambda^{(s)}_d + t - 1\right).
\end{aligned}$$

The case when $t > 1$ is now immediate.

If $t = 1$, then

$$\begin{aligned}
\Theta_{\mathtt{X}_3}(m_\nu) &= \check{\mathscr{H}}^\mu - m_\mu \vec{T}\left(\overline{\mu}^{(s)}_d + 1, \overline{\mu}^{(s)}_{d+1}\right) \vec{T}\left(\overline{\mu}^{(s)}_{d+1}, \omega_{\mathbf{k}}\right) \prod \vec{T}\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}\right) \\
&\times \prod_{\mathbf{i} \neq (d,s)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1} : \Omega^{\mathrm{S}}_{\mathbf{i}}\right) \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : 1, \mu^{(s)}_d\right) \\
&= \check{\mathscr{H}}^\mu - q^{\mu^{(s)}_{d+1} - 1} m_\mu \vec{T}\left(\overline{\mu}^{(s)}_{d+1}, \omega_{\mathbf{k}}\right) \prod \vec{T}\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}\right) \\
&\times \prod_{\mathbf{i} \neq (d,s)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1} : \Omega^{\mathrm{S}}_{\mathbf{i}}\right) \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : 1, \mu^{(s)}_d\right) \\
&= -q^{\mu^{(s)}_{d+1} - 1} \Theta_{\mathtt{X}_4}(m_\nu).
\end{aligned}$$

The second equality comes about due to the fact that $m_\mu T_i = q m_\mu$ whenever $s_i \in \mathfrak{S}_\mu$ (see, for instance, [39, Lemma 3.2]), which is the case when $\overline{\mu}^{(s)}_d + 1 \le 1 < \overline{\mu}^{(s)}_{d+1}$. □

We are not yet done with this case, as it might be that $\mu^{(s)}_{d+1} = \mu^{(s)}_{d+2}$ and $j^{(s)}_{d+2} < j^{(s)}_d$. Here $\mathtt{X}_4$ is still not semistandard as we would have $j^{(s)}_d$ appearing in a high node of the same column $j^{(s)}_{d+2}$ occupies. In fact, we may have that $\mu^{(s)}_{d+1} = \mu^{(s)}_{d+l} < \mu^{(s)}_{d+l+1}$, with $j^{(s)}_d$ being larger than a number or even all the entries appearing in the nodes below its position. Fortunately, in such a situation it is possible to swap the position of $j^{(s)}_d$ with that of the entry immediately below it, a process that introduces a factor of -1. By the properties of tableaux given in Proposition 4.2.4, we have that if $j^{(s)}_d > j^{(s)}_{d+2}$, then $j^{(s)}_{d+2} = (d+2, s)$.

Let $\mathtt{X} \in \mathscr{T}(\mu, \lambda)$ be the tableau formed from $\mathtt{X}_4$ by swapping the positions of $j^{(s)}_d$ and $(d+2, s)$. That is:

$$\mathtt{X}(i,j,k) = \begin{cases} (d+2, s) & \text{if } (i,j,k) = \mathrm{e}_{d+1,s}, \\ j^{(s)}_d & \text{if } (i,j,k) = \mathrm{e}_{d+2,s}, \\ \mathtt{X}_4(i,j,k) & \text{otherwise.} \end{cases}$$

Proposition 4.5.12. *With* $\mathtt{X}$ *If* $\mathtt{X}_4$ *are defined as above, then* $\Theta_{\mathtt{X}}(m_\nu) = -\Theta_{\mathtt{X}_4}(m_\nu)$*.*

Proof. We have that $\Theta_{\mathtt{X}_4}(m_\nu)$ is given by

$$\check{\mathscr{H}}^{\mu} + m_\mu \vec{T}\left(\overline{\mu}^{(s)}_{d+1}, \omega_{\mathbf{k}}\right) \prod \vec{T}\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}\right) \prod_{\mathbf{i} \neq (d,s)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right) \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : 1, \mu^{(s)}_{d}\right)$$

$$= \check{\mathscr{H}}^{\mu} + m_\mu \vec{T}\left(\overline{\mu}^{(s)}_{d+1}, \overline{\mu}^{(s)}_{d+2}\right) \vec{T}\left(\overline{\mu}^{(s)}_{d+2}, \omega_{\mathbf{k}}\right) \prod \vec{T}\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}\right) \prod_{\mathbf{i} \neq (d,s)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right) \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : 1, \mu^{(s)}_{d}\right)$$

Now we have

$$\begin{aligned} m_\mu \vec{T}\left(\overline{\mu}^{(s)}_{d+1}, \overline{\mu}^{(s)}_{d+2}\right) &= m_\mu \left(\mathrm{C}\left(\overline{\mu}^{(s)}_{d+1} - 1 : 1, \mu^{(s)}_{d+2}\right) - \mathrm{C}\left(\overline{\mu}^{(s)}_{d+1} - 1 : 1, \mu^{(s)}_{d+2} - 1\right)\right) \\ &= -m_\mu \mathrm{C}\left(\overline{\mu}^{(s)}_{d+1} - 1 : 1, \mu^{(s)}_{d+2} - 1\right) \\ &= -m_\mu \mathrm{C}\left(\overline{\lambda}^{(s)}_{d+1} : 1, \mu^{(s)}_{d+2} - 1\right) \end{aligned}$$

by Proposition 4.5.4. Hence

$$\begin{aligned} \Theta_{\mathtt{X}_4}(m_\nu) &= \check{\mathscr{H}}^{\mu} - m_\mu \mathrm{C}\left(\overline{\lambda}^{(s)}_{d+1} : 1, \mu^{(s)}_{d+2} - 1\right) \vec{T}\left(\overline{\mu}^{(s)}_{d+2}, \omega_{\mathbf{k}}\right) \\ &\quad \times \prod \vec{T}\left(\alpha_{\mathbf{j}}, \omega_{\mathbf{j}}\right) \prod_{\mathbf{i} \neq (d,s)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right) \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : 1, \mu^{(s)}_{d}\right) \\ &= -\Theta_{\mathtt{X}}(m_\nu). \end{aligned}$$

□

Notice that if we still don't have a semistandard homomorphism, we may repeat this procedure as many times as we can: gradually eliminating terms from $\vec{T}\left(\overline{\mu}^{(s)}_{d+1}, \overline{\mu}^{(s)}_{d+2}\right)$ as we did in the proof translates into shifting $j^{(s)}_{d}$ further down the column it occupies, one node at a time.

<u>Case III</u> Let $d > 1$ and let $\mathtt{S} \in \mathscr{T}_0(\mu, \nu)$ be as in (4.19). For $t \geq 1 \in \mathscr{T}(\mu, \nu)$, let $\mathtt{X}_5$ be given by

$$\mathtt{X}_5(x,y,z) = \begin{cases} (d,s) & \text{if } x = d+1, y \leq t, \text{ and } z = s, \\ \mathtt{S}(x,y,z) & \text{otherwise.} \end{cases}$$

Proposition 4.5.13. *For* $t \geq 1$, $\Theta_{\mathtt{S}}(m_\lambda \mathfrak{d}^{(s)}_{d,t}) = \check{\mathscr{H}}^{\mu}$.

Proof. For some element $a \in \mathbb{F}$ we have

$$\begin{aligned} \Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right) = a\Theta_{\mathtt{X}_5}(m_\nu) &= \check{\mathscr{H}}^{\mu} + a m_\mu T_{d(\text{first}(\mathtt{S}))} \prod_{\mathbf{i}=(x,z)} \mathrm{C}\left(\overline{\lambda}^{(z)}_{x-1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right) \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1} : \lambda^{(s)}_{d}, t\right) \\ &= \check{\mathscr{H}}^{\mu} + a m_\mu \mathrm{C}\left(\overline{\mu}^{(s)}_{d-1} : \mu^{(s)}_{d}, t\right) T_{d(\text{first}(\mathtt{S}))} \prod_{\mathbf{i}=(x,z)} \mathrm{C}\left(\overline{\lambda}^{(z)}_{x-1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right) \\ &= \check{\mathscr{H}}^{\mu} + a m_\mu \mathfrak{d}^{(s)}_{d,t}(\mu) T_{d(\text{first}(\mathtt{S}))} \prod_{\mathbf{i}=(x,z)} \mathrm{C}\left(\overline{\lambda}^{(z)}_{x-1} : \Omega^{\mathtt{S}}_{\mathbf{i}}\right) = \check{\mathscr{H}}^{\mu} \end{aligned}$$

□

Case IV Let $d > 1$ and let $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ be as in (4.20). Let $\mathtt{X}_6, \mathtt{X}_7 \in \mathscr{T}(\mu,\nu)$ be given by

$$\mathtt{X}_6(x,y,z) = \begin{cases} (d,s) & \text{if } x = d+1, y \le t, z = s, \\ \mathtt{S}(x,y,z) & \text{otherwise} \end{cases}$$

and

$$\mathtt{X}_7(x,y,z) = \begin{cases} (d,s) & \text{if } x = d+1, y \le t-1, z = s, \text{ and } (x,y,z) = (k,\mu_k^{(1)},1) \\ \mathtt{S}(x,y,z) & \text{otherwise.} \end{cases}$$

Proposition 4.5.14. *If* $t = 1$, *then* $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) = q^{\lambda_d^{(s)}} \Theta_{\mathtt{X}_7}(m_\nu)$. *Otherwise,* $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) = \check{\mathscr{H}}^\mu$

Proof. By Proposition 4.1.2 there are coefficients α_6 and α_7 such that

$$\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) = \alpha_1 \Theta_{\mathtt{X}_6}(m_\nu) + \alpha_2 \Theta_{\mathtt{X}_7}(m_\lambda).$$

We first show that $\Theta_{\mathtt{X}_7}(m_\nu) \in \check{\mathscr{H}}^\mu$ whenever $t > 1$. In this case we can write

$$\Theta_{\mathtt{X}_7}(m_\nu) = \check{\mathscr{H}}^\mu + m_\mu T_{d(\text{first}(\mathtt{X}_7))} \mathrm{C}\left(\overline{\lambda}_{d-1}^{(1)} : 1, \lambda_d^{(1)}, t-1\right) h$$

for some $h \in \mathscr{H}$. By Lemma 4.5.1 we can write

$$\begin{aligned} \mathrm{C}\left(\overline{\lambda}_{d-1}^{(1)} : 1, \lambda_d^{(1)}, t-1\right) &= \mathrm{C}\left(\overline{\lambda}_{d-1}^{(1)} + 1 : \lambda_d^{(1)}, t-1\right) \mathrm{C}\left(\overline{\lambda}_{d-1}^{(1)} : 1, \lambda_d^{(1)} + t - 1\right) \\ &= \mathrm{C}\left(\overline{\mu}_{d-1}^{(1)} + 1 : \mu_d^{(1)}, t-1\right) \mathrm{C}\left(\overline{\lambda}_{d-1}^{(1)} : 1, \lambda_d^{(1)} + t - 1\right) \end{aligned}$$

Observing that

$$\mathrm{C}\left(\overline{\mu}_{d-1}^{(1)} + 1 : \mu_d^{(1)}, t-1\right) = \mathfrak{d}_{d,t-1}^{(1)}(\mu)$$

commutes past $T_{d(\text{first}(\mathtt{X}_7)}$ then shows that $\Theta_{\mathtt{X}_7}(m_\nu) \in \check{\mathscr{H}}^\mu$ for $t > 1$. The proof that $\Theta_{\mathtt{X}_6}(m_\nu) \in \check{\mathscr{H}}^\mu$ follows a similar course. □

4.5.2 Semi-standardizing $\Theta\left(m_\lambda \mathfrak{l}^{(s)}\right)$

The case for $m_\lambda \mathfrak{l}^{(s)}$ is considerably easier than that for $m_\lambda \mathfrak{d}_{d,t}^{(s)}$. In this case we encounter only those tableaux of a form given in (4.2),(4.3), and (4.13). By their construction, the only way in which those tableaux can fail to be semistandard is if $\mu_d^{(s)} = \mu_{d+l}^{(s)}$ for integers d, s, and l and the tableau happens to deposit an entry directly above a smaller entry in the end nodes of one of these rows. Dealing with this case, the proof of Proposition 4.5.12 can easily be adapted to this situation.

Proposition 4.5.15. *Suppose that* $\mu_d^{(s)} = \mu_{d+l}^{(s)}$ *for some d, s, and l. Let* $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ *and let* $\mathtt{W}_0 \in \mathscr{T}_0\left(\mu, \lambda \cdot \mathfrak{l}^{(1)}\right)$ *be as it was defined in (4.2). If* $\mathtt{S}$ *is such that* $\mathtt{W}_0(\mathrm{e}_{m,s}) > \mathtt{W}_0(\mathrm{e}_{m+1,s})$ *for some* $d \le m < d+l$, *then* $\Theta_{\mathtt{W}_0}\left(m_{\lambda \cdot \mathfrak{l}^{(1)}}\right) = -\Theta_{\mathtt{X}}\left(m_{\lambda \cdot \mathfrak{l}^{(1)}}\right)$, *where* $\mathtt{X} \in \mathscr{T}\left(\mu, \lambda \cdot \mathfrak{l}^{(1)}\right)$ *is the tableau formed from* $\mathtt{W}_0$ *by swapping the positions of* $\mathtt{W}_0(\mathrm{e}_{m,s})$ *and* $\mathtt{W}_0(\mathrm{e}_{m+1,s})$.

An analogous statement holds for $\mathtt{W}_z$ *and* $\mathtt{U}$.

4.6 Killing off Homomorphisms.

Let $\Theta : M^\lambda \to S^\mu$ be a homomorphism such that $\Theta = \sum_{\mathrm{S}\in\mathscr{T}_0(\mu,\lambda)} f_{\mathrm{S}}\Theta_{\mathrm{S}}$ for constants $f_{\mathrm{S}} \in \mathbb{F}$. Our objective is to determine a set of conditions that determine when $\Theta\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) = 0$ for all $(d,t,s) \in \operatorname{def}(\lambda,\mathfrak{d})$ and when $\Theta\left(m_\lambda \mathfrak{l}^{(1)}\right) = 0$. This amounts to identifying, using the machinery developed in the the previous section, and collecting like terms resulting from the act of evaluating $\Theta\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right)$ and $\Theta\left(m_\lambda \mathfrak{l}^{(1)}\right)$.

Although fairly simple in terms of the reasoning underpinning this section, in practice this is a lengthy and technical procedure, especially with regards to $m_\lambda \mathfrak{l}^{(1)}$. We divide this section up into two parts in the hope that this will aid clarity.

4.6.1 When does $\Theta\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) = 0$?

Let μ be a multipartition and recall that $\mathrm{e}_{(j,k)} = \left(j, \mu_j^{(k)}, k\right) \in [\mu]$ for every (j,k) with $1 \le k \le r$ and $1 \le j \le \rho_k(\mu)$. For each $\mathrm{S} \in \mathscr{T}_0(\mu,\lambda)$, let $\mathrm{S}_{j,k} = \mathrm{S}\left(\mathrm{e}_{(j,k)}\right)$ and let $\mathbf{E}_{\mathrm{S}}$ be the sequence

$$\mathrm{S}_{1,1}, \dots, \mathrm{S}_{\rho_1(\mu),1}, \mathrm{S}_{1,2}, \dots, \mathrm{S}_{\rho_2(\mu),2}$$

Using the fact, proved in Lemma 4.2.2, that each S is completely determined by $\mathrm{S}(\mathrm{e}_{(j,k)})$, where $\mathrm{e}_{(j,k)} \in \mathscr{E}_\mu$, we write $\mathrm{S} = \left(\mu : \mathbf{E}_{\mathrm{S}}\right)$. This adapts the notation used in [37] to $\mathscr{H}_{2,n}$.

Suppose that $\mathrm{S} = \left(\mu : \mathbf{E}_{\mathrm{S}}\right) \in \mathscr{T}_0(\mu,\lambda)$ and that $\mathrm{T} \in \mathscr{T}_0(\mu,\lambda)$ is obtained from S by swapping the positions of the entries (u,v) and (x,y), but keeping all other entries the same. Defining the pair $(\hat{r}(x),\hat{c}(y))$ by $\mathrm{S}_{\hat{r}(x),\hat{c}(y)} = (x,y)$, we then write

$$f_{\mathrm{T}} = f\left(\mathrm{S} : \underset{(x,y)}{\overset{(\hat{r}(u),\hat{c}(v))}{\downarrow}}, \underset{(u,v)}{\overset{(\hat{r}(x),\hat{c}(y))}{\downarrow}}\right)$$

Where

$$\underset{(x,y)}{\overset{(\hat{r}(u),\hat{c}(v))}{\downarrow}}$$

is to be read as 'node $\left(\hat{r}(u), \mu_{\hat{r}(u)}^{\hat{c}(v)}, \hat{c}(v)\right)$ is occupied by the entry (x,y)'. This notation is readily extended to rearranging more than two entries of a tableau.

We are almost in a position to describe what conditions the various coefficients f_{S} must satisfy so that $\Theta\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) = 0$ for a given $(d,t,s) \in \operatorname{def}(\lambda,\mathfrak{d})$. Before that, we need to consider what it means to 'collect like terms' in the expression of $\Theta\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right)$.

Suppose that $(d,s) > (1,1)$, that $\mathrm{S} \in \mathscr{T}_0(\mu,\lambda)$ is such that $\mathrm{S}_{d,s} = (d,s)$, and that $\mu_d^{(s)} = \mu_{d+l}^{(s)}$ for some $1 \le l \le \rho_s(\mu)$. We can ignore the case where S is as in (4.19), since this would mean that $\Theta_{\mathrm{S}}\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) \in \check{\mathscr{H}}^\mu$, and so we conclude that S is as in (4.20); i.e.,

$$\mathrm{S} = \left(j_1^{(1)}, \dots, (d+1,s), \dots, j_{d-1}^{(s)}, (d,s), (d+2,s), (d+3,s), \dots, (d+l,s), j_{d+l}^{(s)}, \dots, j_{\rho_2(\mu)}^{(2)}\right).$$

We then have that $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) = \alpha_{\mathtt{X}}\Theta_{\mathtt{X}}(m_\nu)$ for an element $\alpha_{\mathtt{X}} \in \mathbb{F}$, $\nu = \lambda \cdot \mathfrak{d}_{d,t}^{(s)}$, and where $\mathtt{X} \in \mathscr{T}_{r,0}(\mu,\nu)$ is given by

$$\mathtt{X} = \left(j_1^{(1)}, \ldots, (d,s), \ldots, j_{d-1}^{(s)}, (d,s), (d+2,s), (d+3,s), \ldots, (d+l,s), j_{d+l}^{(s)}, \ldots, j_{\rho_2(\mu)}^{(2)}\right).$$

Therefore, we need to identify those tableaux $\mathtt{T} \in \mathscr{T}_0(\mu,\lambda)$ such that $\Theta_{\mathtt{X}}(m_\nu)$ features in the expression due to Proposition 4.1.2 (and taking section 4.5 into account) for $\Theta_{\mathtt{T}}\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right)$. These are precisely the tableaux obtained from $\mathtt{S}$ by swapping the positions of $(d+1,s)$ with that of $(d,s),(d+2,s),(d+3,s),\ldots,(d+l,s)$, or $j_{d+l}^{(s)}$ and rearranging the latter collection of entries so that the result is semistandard.

Moreover, these are the only configurations we need to consider when $(d,s) > (1,1)$ and $\mu_d^{(s)} = \mu_{d+l}^{(s)}$. Suppose otherwise and let $\mathtt{S}$ be such that $\mathtt{S}_{d,s} \neq (d,s)$, Then (d,s) must appear in a row higher than d, and so

$$\mathtt{S} = \left(j_1^{(1)}, \ldots, (d,s), \ldots, j_{d-1}^{(s)}, (d+1,s), (d+2,s), (d+3,s), \ldots, (d+l,s), j_{d+l}^{(s)}, \ldots, j_{\rho_2(\mu)}^{(2)}\right).$$

But then we end up with exactly the same collection of tableaux as we did when we took $\mathtt{S}$ to be such that $\mathtt{S}_{d,s} = (d,s)$.

Proposition 4.6.1. *Let $\Theta : M^\lambda \to S^\mu$ be the homomorphism given by*

$$\Theta = \sum_{\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)} f_{\mathtt{S}} \Theta_{\mathtt{S}}.$$

Then

$$\Theta\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right) = 0$$

for every $(d,t,s) \in \operatorname{def}(\lambda;\mathfrak{d})$ if and only if the following conditions hold:

- *If $(1,1) < (d,s)$ and $\mu_d^{(s)} = \mu_{d+l}^{(s)}$ for some $1 \le l \le \rho_s(\mu) - d$. Then, for each $\mathtt{S} \in \mathscr{T}_o(\mu,\lambda)$ of the form*

$$\mathtt{S} = \big(\mu : \mathtt{S}_{1,1}, \ldots, \mathtt{S}_{\hat{r}(d)-1,\hat{c}(s)}, (d+1,s), \mathtt{S}_{\hat{r}(d)+1,\hat{c}(s)}, \ldots, \\ \ldots, \mathtt{S}_{d-1,s}, (d,s), (d+2,s), \ldots, (d+l,s), \mathtt{S}_{d+l,s}, \ldots \mathtt{S}_{\rho_2(\mu),2}\big)$$

  we have

$$q^{\lambda_d^{(s)}} f_{\mathtt{S}} + q^{\lambda_{d+1}^{(s)}-1} f\left(\mathtt{S} : \begin{smallmatrix} (\hat{r}(d,s),\hat{c}(d,s)) \\ \downarrow \\ (d,s) \end{smallmatrix}, \begin{smallmatrix} (d,s) \\ \downarrow \\ (d+1,s) \end{smallmatrix}\right) = 0;$$

- *If $(1,1) < (d,s)$ and $\mu_d^{(s)} > \mu_{d+1}^{(s)} = \mu_{d+l}^{(s)} > \mu_{d+l+1}^{(s)}$, then for each $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ of the form*

$$\mathtt{S} = \big(\mu : \mathtt{S}_{1,1}, \ldots, \mathtt{S}_{\hat{r}(d,s)-1,\hat{c}(d,s)}(d+1,s), \mathtt{S}_{\hat{r}(d,s)+1,\hat{c}(d,s)}, \ldots, \\ \ldots, \mathtt{S}_{d-1,s}(d,s), (d+2,s), \ldots, (d+l,s), \mathtt{S}_{d+l,s}, \ldots, \mathtt{S}_{\rho_2(\mu),2}\big)$$

we have

$$
\begin{aligned}
q^{\lambda_d^{(s)}} f_{\mathtt{S}} &+ q^{\lambda_{d+1}^{(s)}-1}\left[\lambda_d^{(s)} - \lambda_{d+1}^{(s)} + 1\right] f\left(\mathtt{S} : \substack{(\hat{r}(d,s),\hat{c}(d,s)) \\ \downarrow \\ (d,s)}, \substack{(d,s) \\ \downarrow \\ (d+1,s)}\right) \\
&- q^{\lambda_{d+1}^{(s)}-1} f\left(\mathtt{S} : \substack{(\hat{r}(d,s),\hat{c}(d,s)) \\ \downarrow \\ (d,s)}, \substack{(d,s) \\ \downarrow \\ (d+2,s)}, \substack{(d+1,s) \\ \downarrow \\ (d+1,s)}\right) \\
&\vdots \\
&+ (-1)^{l-1} q^{\lambda_{d+1}^{(s)}-1} f\left(\mathtt{S} : \substack{(\hat{r}(d,s),\hat{c}(d,s)) \\ \downarrow \\ (d,s)}, \substack{(d,s) \\ \downarrow \\ (d+l,s)}, \substack{(d+1,s) \\ \downarrow \\ (d+1,s)}, \substack{(d+2,s) \\ \downarrow \\ (d+2,s)}, \ldots, \substack{(d+l-1,s) \\ \downarrow \\ (d+l-1,s)}\right) \\
&+ (-1)^{l} q^{\lambda_{d+1}^{(s)}-1} f\left(\mathtt{S} : \substack{(\hat{r}(d,s),\hat{c}(d,s)) \\ \downarrow \\ (d,s)}, \substack{(d,s) \\ \downarrow \\ \mathtt{S}_{d+l,s}}, \substack{(d+1,s) \\ \downarrow \\ (d+1,s)}, \substack{(d+2,s) \\ \downarrow \\ (d+2,s)}, \ldots, \substack{(d+l,s) \\ \downarrow \\ (d+l,s)}\right) = 0;
\end{aligned}
$$

and

- *If* $(d,s) = (1,1)$ *and* $\lambda_2^{(1)} = \lambda_{l+1}^{(1)} > \lambda_{l+2}^{(1)}$, *then for each* $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ *of the form*

$$\mathtt{S} = \left(\mu : 2, 3, \ldots, l+1, \mathtt{S}_{l+1,1}, \mathtt{S}_{l+2,1}, \ldots, \mathtt{S}_{\rho_2(\mu),2}\right)$$

then

$$
\begin{aligned}
\left[\lambda_1^{(1)} - \lambda_2^{(1)} + 2\right] f(\mathtt{S}) &- f\left(\mathtt{S} : \substack{(1,1) \\ \downarrow \\ (3,1)}, \substack{(2,1) \\ \downarrow \\ (2,1)}\right) \\
&+ f\left(\mathtt{S} : \substack{(1,1) \\ \downarrow \\ (4,1)}, \substack{(2,1) \\ \downarrow \\ (2,1)}, \substack{(3,1) \\ \downarrow \\ (3,1)}\right) \\
&\vdots \\
(-1)^{l-1} &f\left(\mathtt{S} : \substack{(1,1) \\ \downarrow \\ (l+1,1)}, \substack{(2,1) \\ \downarrow \\ (2,1)}, \substack{(3,1) \\ \downarrow \\ (3,1)}, \ldots, \substack{(l,1) \\ \downarrow \\ (l,1)}\right) \\
(-1)^{l} &f\left(\mathtt{S} : \substack{(1,1) \\ \downarrow \\ j_{l+1}^{(1)}}, \substack{(2,1) \\ \downarrow \\ (2,1)}, \substack{(3,1) \\ \downarrow \\ (3,1)}, \ldots, \substack{(l+1,1) \\ \downarrow \\ (l+1,l)}\right) = 0.
\end{aligned}
$$

Proof. In the second case, we have already verified that we have collected all 'like terms', the reader being invited to verify that this is also true of the remaining two cases. The coefficients with which they appear follow immediately from section 4.5. □

In the next section, we will specify coefficients $f_{\mathtt{S}}$ that satisfy these conditions. Along with the results in this subsection, our method of doing this is very similar to that appearing in [37], as should be expected given the close relationship between that paper and this thesis. What is entirely different is the role played by $\mathfrak{l}^{(1)}$ in determining when Θ gives rise to a homomorphism of Specht modules, this being the subject matter of the next subsection.

4.6.2 When Does $\Theta(m_\lambda \mathfrak{l}^{(1)}) = 0$?

Collecting like terms in this case is slightly more complicated due to the more involved expressions for the $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{l}^{(1)}\right)$ in terms of homomorphisms $\Theta_{\mathtt{W}} : M^{\lambda \cdot \mathfrak{l}^{(1)}} \to S^\mu$ evaluated at

$m_{\lambda\cdot\mathfrak{l}^{(1)}}$, and the reader will benefit greatly from keeping in mind Propositions 4.4.6 and 4.4.8 when going through many of the statements in this subsection.

Suppose that $\mathtt{S} \in \mathcal{T}_0(\mu,\lambda)$ is such that $\mathtt{S}_{m,1} = (1,2)$ for an integer $1 \le m \le \rho_1(\lambda)$ and recall that one of the terms involved in the expression for $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{l}^{(1)}\right)$ is $\Theta_{\mathtt{W}_0}\left(m_{\lambda\cdot\mathfrak{l}^{(1)}}\right)$ where $\mathtt{W}_0$ is constructed from $\mathtt{S}$ by replacing the entry $\mathtt{S}_{m,1} = (1,2)$ with $(\rho_1(\lambda)+1,1)$. Our strategy is to identify, for each such $\mathtt{S}$, those other tableaux $\mathtt{T}$ for which $\Theta_{\mathtt{W}_0}\left(m_{\lambda\cdot\mathfrak{l}^{(1)}}\right)$ arises when evaluating the homomorphisms they determine at $m_\lambda \mathfrak{l}^{(1)}$.

To do this, we will need some more notation. Recall the definitions of the tableaux $\mathtt{W}_0, \mathtt{W}_z$, and $\mathtt{U} \in \mathcal{T}_0\left(\mu, \lambda\cdot\mathfrak{l}^{(1)}\right)$ constructed from a given tableau $\mathtt{S}$, as given in (4.2), (4.3), and (4.13) respectively: let $\mathbf{n} = (n_1,n_2,n_3)$ be the unique node in the first component of $[\mu]$ for which $\mathtt{S}(n_1,n_2,n_3) = (x,2)$, where x is a positive integer with $1 \le x \le \rho_2(\lambda)$, and let z be an integer with $1 \le z \le \rho_1(\lambda) - n_1$. If $x = 1$, then

$$\mathtt{W}_0(i,j,k) = \begin{cases} (\rho_1(\lambda)+1,1) & \text{if } (i,j,k) = \mathbf{n}, \\ \mathtt{S}(i,j,k) & \text{otherwise,} \end{cases}$$

$$W_z(i,j,k) = \begin{cases} (\rho_1(\lambda)+1,1) & \text{if } (i,j,k) = \left(n_1+z, \mu^{(1)}_{n_1+z}, 1\right), \\ \mathtt{S}\left(n_1+z, \mu^{(1)}_{n_1+z}, 1\right) & \text{if } (i,j,k) = \mathbf{n}, \\ \mathtt{S}(i,j,k) & \text{otherwise} \end{cases}$$

for $(i,j,k) \in [\mu]$. If, on the other hand, $x > 1$, then

$$\mathtt{U}(i,j,k) = \begin{cases} (\rho_1(\lambda)+1,1) & \text{if } (i,j,k) = \mathbf{n}, \\ (x,2) & \text{if } (i,j,k) = \left(1, \mu^{(2)}_1, 2\right), \\ \mathtt{S}(i,j,k) & \text{otherwise.} \end{cases}$$

for $(i,j,k) \in [\mu]$.

When we wish to avoid any possibility of ambiguity and specify the tableaux they are constructed from, we will write $\mathtt{W}_0(\mathtt{S})$,$\mathtt{W}_z(\mathtt{S})$, and $\mathtt{U}(\mathtt{S})$.

We now specify all possible ways such like terms can appear. This depends greatly on the form taken by the multipartition μ, and so we proceed on a case by case basis.

Suppose that $\mathtt{S} \in \mathcal{T}_0(\mu,\lambda)$ is such that $\mathtt{S}_{m,1} = (1,2)$ for some $2 \le m \le \rho_1(\lambda)$ and suppose that $\mu^{(1)}_m < \mu^{(1)}_{m-1}$ and $\mu^{(2)}_1 > \mu^{(2)}_2$. Then there is an integer $1 \le l < m$ with $\mathrm{e}_{l,1} \in [\mu]$ removable and $\mathtt{S}_{l,1} = (m,1)$: were this node not removable, then part 2. of Proposition 4.2.4 implies that $(m,1)$ would appear directly above an entry $(x,1)$ with $x < m$, contradicting the fact that $\mathtt{S}$ is semistandard.

Next, define tableaux $\mathtt{T},\mathtt{R} \in \mathcal{T}_0(\mu,\lambda)$ by

$$\mathtt{T}_{x,y} = \begin{cases} (1,2) & \text{if } (x,y) = (1,2), \\ \mathtt{S}_{1,2} & \text{if } (x,y) = (m,1), \\ \mathtt{S}_{x,y} & \text{otherwise} \end{cases} \tag{4.22}$$

and

$$\mathtt{R}_{x,y} = \begin{cases} (1,2) & \text{if } (x,y) = (l,1) \\ (m,1) & \text{if } (x,y) = (m,1) \\ \mathtt{S}_{x,y} & \text{otherwise.} \end{cases} \tag{4.23}$$

Lemma 4.6.2. *Fix an integer m with $2 \le m \le \rho_1(\lambda)$ and suppose that $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ is such that $\mathtt{S}_{m,1} = (1,2)$, Then $\mathtt{T}$ and $\mathtt{R}$ are the only tableaux such that $\mathtt{W}_z(\mathtt{T}) = \mathtt{W}_0(\mathtt{S})$ and $\mathtt{U}(\mathtt{R}) = \mathtt{W}_0(\mathtt{S})$.*

Proof. Suppose that $\mathtt{X} \in \mathscr{T}_0(\mu,\lambda)$ is such that $\mathtt{X}_{k,1} = (1,2)$ for an integer $1 \le k \le \rho_1(\lambda)$ and $\mathtt{W}_0(\mathtt{X}) = \mathtt{W}_0(\mathtt{S})$. Since $\mathtt{W}_0(\mathtt{X})$ is formed from $\mathtt{X}$ merely by relabeling $(1,2)$ as $(\rho_1(\lambda)+1,1)$, we must have that $\mathtt{X} = \mathtt{S}$.

Suppose now that $\mathtt{X}$ is such that $\mathtt{W}_z(\mathtt{X}) = \mathtt{W}_0(\mathtt{S})$ for some $k \le z \le m$. Since $\mathtt{W}_z(\mathtt{X})$ is obtained from $\mathtt{X}$ by swapping the positions of $(1,2)$ and $(k+z,1)$ and then replacing $(1,2)$ with $(\rho_1(\lambda)+1,1)$, leaving the positions of all other entries unchanged, we have that $k = l$, $z = m-l$, and $\mathtt{X} = \mathtt{T}$.

Finally, suppose that $\mathtt{X}$ is such that $\mathtt{X}_{k,1} = (u,2)$ for some $u > 1$ and that $\mathtt{U}(\mathtt{X}) = \mathtt{W}_0(\mathtt{S})$. The tableau $\mathtt{U}(\mathtt{X})$ is constructed from $\mathtt{X}$ by swapping the positions of $(u,2)$ and $\mathtt{S}_{1,2} = (1,2)$, and then replacing the latter with $(\rho_1(\lambda)+1,1)$. Our task is then complete, since this means that $k = m$ and $(u,2) = \mathtt{S}_{1,2}$, so that $\mathtt{X} = \mathtt{R}$. □

Suppose now that $\mathtt{S}$ is as before, but $\mu_1^{(2)} = \mu_l^{(2)}$ for some $1 \le l \le \rho_2(\mu)$. Notice that in this situation, $\mathtt{S}_{i,2} = (i+1,2)$ for every $1 \le i \le l$. With $\mathtt{T}$ as in (4.22), define a family of tableaux $\mathtt{P}^i \in \mathscr{T}_0(\mu,\lambda)$ by

$$\mathtt{P}^i_{x,y} = \begin{cases} (x,y) & \text{if } y = 2 \text{ and } 1 \le x \le i, \\ \mathtt{S}_{i,2} & \text{if } (x,y) = (m,1), \\ \mathtt{S}_{x,y} & \text{otherwise.} \end{cases} \tag{4.24}$$

Lemma 4.6.3. *Fix an integer m with $2 \le m \le \rho_1(\lambda)$ and suppose that $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ is such that $\mathtt{S}_{m,1} = (1,2)$. If $\mu_m^{(1)} < \mu_{m-1}^{(1)}$ and $\mu_1^{(2)} = \mu_l^{(2)}$ for some $1 \le l \le \rho_2(\mu)$, then $\mathtt{T}$ and the $\mathtt{P}^i$ are the only tableaux such that $\mathtt{W}_z(\mathtt{T}) = \mathtt{W}_0(\mathtt{S})$ and $\mathtt{U}(\mathtt{P}^i) = \mathtt{W}_0(\mathtt{S})$.*

Proof. We dealt with $\mathtt{T}$ in Lemma 4.6.2, and, by construction, $\mathtt{U}(\mathtt{P}^i) = \mathtt{W}_0(\mathtt{S})$ for all i. To see that these are the only tableaux for which this is true, suppose that $\mathtt{X} \in \mathscr{T}_0(\mu,\lambda)$ is such that $\mathtt{U}(\mathtt{X}) = \mathtt{W}_0(\mathtt{S})$. By construction, $\mathtt{X}_{x,y} = \mathtt{S}_{x,y}$ for every $(x,y) \ne (m,1)$ such that $y \ne 2$ or $x > i$. For each of the remaining values of (x,y) we have that $\mathtt{X}_{x,y} \in \{(1,2),(2,2),\ldots,(i,2),\mathtt{S}_{i,2}\}$, but the only semistandard tableaux with these properties are the $\mathtt{P}^i$. □

Let $\mathtt{S}$ be as before and let μ be such that $\mu_m^{(1)} = \mu_l^{(1)}$ for some $1 \le l \le m$ and $\mu_1^{(2)} > \mu_2^{(2)}$. In this case there are integers k and w with $l \le k \le m$ and $1 \le w < l$, such that $\mathtt{S}_{w,1} = (k,1)$. With this in mind, define $\mathtt{Q} \in \mathscr{T}_0(\mu,\lambda)$ by

$$\mathtt{Q}_{x,y} = \begin{cases} (1,2) & \text{if } (x,y) = (w,1), \\ (x,y) & \text{if } y = 1 \text{ and } k \le x \le m, \\ \mathtt{S}_{x,y} & \text{otherwise,} \end{cases} \tag{4.25}$$

and let $\mathtt{R}$ be as defined in (4.23).

Lemma 4.6.4. *Fix an integer m, with $2 \le m \le \rho_1(\lambda)$, and suppose that $\mathtt{S} \in \mathcal{T}_0(\mu,\lambda)$ is such that $\mathtt{S}_{m,1} = (1,2)$. If $\mu^{(1)}_m = \mu^{(1)}_l$ for some $1 \le l \le m$ and $\mu^{(2)}_1 > \mu^{(2)}_2$, then $\mathtt{Q}$ and $\mathtt{R}$ are the only tableaux such that $\mathtt{W}_z(\mathtt{Q}) = \mathtt{W}_0(\mathtt{S})$ and $\mathtt{U}(\mathtt{R}) = \mathtt{W}_0(\mathtt{S})$.*

Proof. $\mathtt{R}$ has already been dealt with in Lemma 4.6.2, and, by construction $\mathtt{Q}$ is the unique tableaux satisfying the statement of the Lemma. □

Remark. By Proposition 4.2.4 we have that $\mathtt{Q}_{x,1} = \mathtt{S}_{x,1} = (x,1)$ for all values of x with $w < x < k$, and $\mathtt{S}_{x,1} \neq (x,1)$ for all values of x with $k \le x \le m$. This fact will be important when we come to construct homomorphisms.

Being a combination of Lemmas 4.6.3 and 4.6.4 and their proofs, we leave the proof of the following result to the reader.

Lemma 4.6.5. *Let $\mathtt{S} \in \mathcal{T}_0(\mu,\lambda)$ be such that $\mathtt{S}_{m,1} = (1,2)$ for some value of m with $2 \le m \le \rho_1(\lambda)$. If $\mu^{(1)}_m = \mu^{(1)}_l$ for some $1 \le l \le m$ and $\mu^{(2)}_1 = \mu^{(2)}_k$ for some $2 \le k \le \rho_2(\mu)$. Then, $\mathtt{Q}$ and the $\mathtt{P}^i$, as defined in as in (4.25) and (4.24) respectively, are the only tableaux with $\mathtt{W}_z(\mathtt{Q}) = \mathtt{W}_0(\mathtt{S})$ for some value of z and $\mathtt{U}(\mathtt{P}^i) = \mathtt{W}_0(\mathtt{S})$.* □

Finally, we consider the case when $m = 1$. This is nearly identical to the situation in Lemmas 4.6.2 and 4.6.3, with the only difference being that we no longer have to worry about tableaux where $(1,2)$ appears in a node higher than $\mathrm{e}_{m,1}$. This simplifies matters considerably, and so we only provide a statement.

Lemma 4.6.6. *Let $\mathtt{S} \in \mathcal{T}_0(\mu,\lambda)$ be such that $\mathtt{S}_{1,1} = (1,2)$.*

- *If $\mu^{(2)}_1 > \mu^{(2)}_2$, then the only tableau $\mathtt{X}$ with $\mathtt{U}(\mathtt{X}) = \mathtt{W}_0(\mathtt{S})$ is $\mathtt{R}$, as defined in (4.23); and*
- *If $\mu^{(2)}_1 = \mu^{(2)}_l$ for an integer $2 \le l \le \rho_2(\mu)$, then the only tableaux $\mathtt{X}$ with $\mathtt{U}(\mathtt{X}) = \mathtt{W}_0(\mathtt{S})$ are the $\mathtt{P}^i$ defined in (4.24).*

We are now in a position to state our conditions which, taken together with Proposition 4.6.1 concludes this section.

Proposition 4.6.7. *Let $\Theta : M^\lambda \to S^\mu$ be a homomorphism given by*

$$\Theta = \sum_{\mathtt{S} \in \mathcal{T}_0(\mu,\lambda)} f_{\mathtt{S}} \Theta_{\mathtt{S}},$$

for $f_{\mathtt{S}} \in \mathbb{F}$. Then

$$\Theta\left(m_\lambda \mathfrak{l}^{(1)}\right) = 0$$

if and only if the following conditions hold:

1. *Suppose that $\mathtt{S}_{m,1} = (1,2)$ and that there exists $c < m$ be such that $\mathtt{S}_{c,1} = (m,1)$, then:*
   - *If $\mu^{(1)}_m < \mu^{(1)}_{m-1}$ and $\mu^{(2)}_2 < \mu^{(2)}_1$, then*

$$\left(\mathrm{res}_\mu\left(\mathrm{e}_{m,1}\right) - \mathrm{res}_\lambda\left(\mathrm{e}_{1,2}\right)\right) f_{\mathtt{S}} - (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathtt{S} : \begin{smallmatrix}(m,1)\\ \downarrow \\ \mathtt{S}_{1,2}\end{smallmatrix}, \begin{smallmatrix}(1,2)\\ \downarrow \\ (1,2)\end{smallmatrix}\right) + (q-1)\mathrm{res}_\mu\left(\mathrm{e}_{m,1}\right) f\left(\mathtt{S} : \begin{smallmatrix}(c,1)\\ \downarrow \\ (1,2)\end{smallmatrix}, \begin{smallmatrix}(m,1)\\ \downarrow \\ (c,1)\end{smallmatrix}\right) = 0; \text{ and}$$

- *If* $\mu^{(1)}_m < \mu^{(1)}_{m-1}$ *and* $\mu^{(2)}_1 = \mu^{(2)}_l$ *, then*

$$\begin{aligned}
\left(\mathrm{res}_\mu\left(\mathrm{e}_{m,1}\right) - \mathrm{res}_\lambda\left(\mathrm{e}_{1,2}\right)\right) f_{\mathrm{S}} - (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S}: \underset{(2,2)}{\overset{(m,1)}{\downarrow}}, \underset{(1,2)}{\overset{(1,2)}{\downarrow}}\right)& \\
+ (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S}: \underset{(3,2)}{\overset{(m,1)}{\downarrow}}, \underset{(1,2)}{\overset{(1,2)}{\downarrow}}, \underset{(2,2)}{\overset{(2,2)}{\downarrow}}\right)& \\
\vdots\qquad& \\
(-1)^{l-1}(q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S}: \underset{\mathrm{S}_{l,2}}{\overset{(m,1)}{\downarrow}}, \underset{(1,2)}{\overset{(1,2)}{\downarrow}}, \underset{(2,2)}{\overset{(2,2)}{\downarrow}}, \dots, \underset{(l,2)}{\overset{(l,2)}{\downarrow}}\right)& \\
+ (q-1)\mathrm{res}_\mu\left(\mathrm{e}_{m,1}\right) f\left(\mathrm{S}: \underset{(1,2)}{\overset{(c,1)}{\downarrow}}, \underset{(c,1)}{\overset{(m,1)}{\downarrow}}\right) = 0.&
\end{aligned}$$

2. *Suppose that* $\mathrm{S}_{m,1} = (1,2)$ *and* $\mu^{(1)}_m = \mu^{(1)}_l$ *for a given* $1 \le l < m$. *Denote by* c *the unique integer such that* $c < l$ *with* $\mathrm{S}_{c,1} = (k,1)$ *for* $l \le k \le m$.

   - *If* $\mu^{(2)}_2 < \mu^{(2)}_1$*, then*

$$\begin{aligned}
\left(\mathrm{res}_\mu\left(\mathrm{e}_{m,1}\right) - \mathrm{res}_\lambda\left(\mathrm{e}_{1,2}\right)\right) f_{\mathrm{S}} - (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S}: \underset{\mathrm{S}_{1,2}}{\overset{(m,1)}{\downarrow}}, \underset{(1,2)}{\overset{(1,2)}{\downarrow}}\right)& \\
+(q-1)\mathrm{res}_\mu\left(\mathrm{e}_{k,1}\right) f\left(\mathrm{S}: \underset{(1,2)}{\overset{(c,1)}{\downarrow}}, \underset{(k,1)}{\overset{(k,1)}{\downarrow}}, \underset{(k+1,1)}{\overset{(k+1,1)}{\downarrow}}, \dots, \underset{(m,1)}{\overset{(m,1)}{\downarrow}}\right) = 0&
\end{aligned}$$

   - *If* $\mu^{(2)}_1 = \mu^{(2)}_w$ *for an integer* w *with* $1 \le w \le \rho_2(\mu)$*, then*

$$\begin{aligned}
\left(\mathrm{res}_\mu\left(\mathrm{e}_{m,1}\right) - \mathrm{res}_\lambda\left(\mathrm{e}_{1,2}\right)\right) f_{\mathrm{S}} - (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S}: \underset{(2,2)}{\overset{(m,1)}{\downarrow}}, \underset{(1,2)}{\overset{(1,2)}{\downarrow}}\right)& \\
+ (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S}: \underset{(3,2)}{\overset{(m,1)}{\downarrow}}, \underset{(1,2)}{\overset{(1,2)}{\downarrow}}, \underset{(2,2)}{\overset{(2,2)}{\downarrow}}\right)& \\
\vdots\qquad& \\
(-1)^{l-1}(q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S}: \underset{\mathrm{S}_{w,2}}{\overset{(m,1)}{\downarrow}}, \underset{(1,2)}{\overset{(1,2)}{\downarrow}}, \underset{(2,2)}{\overset{(2,2)}{\downarrow}}, \dots, \underset{(w,2)}{\overset{(w,2)}{\downarrow}}\right)& \\
+ (q-1)\mathrm{res}_\mu\left(\mathrm{e}_{k,1}\right) f\left(\mathrm{S}: \underset{(1,2)}{\overset{(c,1)}{\downarrow}}, \underset{(k,1)}{\overset{(k,1)}{\downarrow}}, \underset{(k+1,1)}{\overset{(k+1,1)}{\downarrow}}, \dots, \underset{(m,1)}{\overset{(m,1)}{\downarrow}}\right)& \\
= 0.&
\end{aligned}$$

3. *Suppose that* $\mathrm{S}_{1,1} = (1,2)$*,*

   - *If* $\mu^{(2)}_2 < \mu^{(2)}_1$*, then*

$$\left(\mathrm{res}_\mu\left(\mathrm{e}_{1,1}\right) - \mathrm{res}_\lambda\left(\mathrm{e}_{1,2}\right)\right) f_{\mathrm{S}} - (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S}: \underset{\mathrm{S}_{1,2}}{\overset{(1,1)}{\downarrow}}, \underset{(1,2)}{\overset{(1,2)}{\downarrow}}\right) = 0$$

- *If* $\mu_1^{(2)} = \mu_l^{(2)}$, *then*

$$\begin{aligned}
\left(\mathrm{res}_\mu\left(\mathrm{e}_{1,1}\right) - \mathrm{res}_\lambda\left(\mathrm{e}_{1,2}\right)\right) f_{\mathrm{S}} &- (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S} : \begin{smallmatrix}(m,1)\\ \downarrow\\ (2,2)\end{smallmatrix}, \begin{smallmatrix}(1,2)\\ \downarrow\\ (1,2)\end{smallmatrix}\right)\\
&+ (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S} : \begin{smallmatrix}(m,1)\\ \downarrow\\ (3,2)\end{smallmatrix}, \begin{smallmatrix}(1,2)\\ \downarrow\\ (1,2)\end{smallmatrix}, \begin{smallmatrix}(2,2)\\ \downarrow\\ (2,2)\end{smallmatrix}\right)\\
&\vdots\\
&(-1)^{l-1}(q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f\left(\mathrm{S} : \begin{smallmatrix}(m,1)\\ \downarrow\\ \mathrm{S}_{w,2}\end{smallmatrix}, \begin{smallmatrix}(1,2)\\ \downarrow\\ (1,2)\end{smallmatrix}, \begin{smallmatrix}(2,2)\\ \downarrow\\ (2,2)\end{smallmatrix}, \dots, \begin{smallmatrix}(w,2)\\ \downarrow\\ (w,2)\end{smallmatrix}\right)\\
&= 0.
\end{aligned}$$

Proof. Immediate from Lemmas 4.6.2, 4.6.3, 4.6.4, 4.6.5 and 4.6.6, and Propositions 4.4.6, 4.4.8, and 4.5.15. □

Combining Propositions 4.6.1 and 4.6.7 then provides us with the conditions we need for $\Theta\left(m_\lambda \mathfrak{d}_{d,t}^{(s)}\right)$ and $\Theta\left(m_\lambda \mathfrak{l}^{(1)}\right)$ to be zero for all $(d,t,s) \in \mathrm{def}(\lambda, \mathfrak{d})$. In the next section we use these to construct and explicit homomorphism satisfying these conditions.

4.7 An Example of an Explicit Homomorphism

Building upon the results of the previous section, we now exhibit constants $f_{\mathrm{S}} \in \mathbb{F}$ for each $\mathrm{S} \in \mathcal{T}_0(\mu,\lambda)$ such that the image of $m_\lambda \mathfrak{d}_{d,t}^{(s)}$ under the homomorphism

$$\Theta = \sum_{\mathrm{S} \in \mathcal{T}_0(\mu,\lambda)} f_{\mathrm{S}} \Theta_{\mathrm{S}} \tag{4.26}$$

is sent to zero for every $(d,t,s) \in \mathrm{def}(\lambda, \mathfrak{d})$. We then use these constants to show that the conditions in Proposition 4.6.7 are also satisfied, provided that the residues of certain nodes obey a simple relationship.

Note that the coefficients f_{S} are well defined by virtue of the linear independence of Θ_{S}.

Let i and j be such that $1 \le i \le \rho_1(\lambda)$ and $1 \le j < \rho_2(\lambda) - 1$ and define $\mathrm{S}((i,1)), \mathrm{S}((j,2)) \in \mathbb{F}$ by

$$\mathrm{S}((i,1)) = \begin{cases} 1 & \text{if } \mathrm{S}(\mathrm{e}_{i,1}) \neq (i,1)\\ -q^{-1} & \text{if } \mathrm{S}(\mathrm{e}_{i,1}) = (i,1) \text{ and } \mu_i^{(1)} = \mu_{i+1}^{(i)}\\ \left[\lambda_1^{(1)} - \lambda_i^{(1)} + i\right] & \text{if } \mathrm{S}(\mathrm{e}_{i,1}) = (i,1) \text{ and } \mu_i^{(1)} > \mu_{i+1}^{(i)}. \end{cases}$$

$$\mathrm{S}((j,2)) = \begin{cases} 1 & \text{if } \mathrm{S}(\mathrm{e}_{j,2}) \neq (j,2)\\ -q^{-1} & \text{if } \mathrm{S}(\mathrm{e}_{j,2}) = (j,2) \text{ and } \mu_j^{(2)} = \mu_{j+1}^{(2)}\\ -q^{-X}[X] & \text{if } \mathrm{S}(\mathrm{e}_{j,2}) = (j,2) \text{ and } \mu_j^{(2)} > \mu_{j+1}^{(2)}. \end{cases}$$

where $X = \lambda^{(2)}_j + \rho_2(\lambda) - \lambda^{(2)}_{\rho_2(\lambda)} - j$.

With this notation in place set

$$f_{\mathtt{S}} = \prod_{i=2}^{\rho_1(\lambda)} \mathtt{S}((i,1)) \prod_{j=1}^{\rho_2(\lambda)-1} \mathtt{S}((j,2))$$

in the expression for $\Theta : M^\lambda \to S^\mu$ given in (4.26).

Example 18. *Let $\lambda = ((2,1,1),(2,1,1))$, $\mu = ((3,1,1),(2,1))$, and let $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ be given by*

$$\mathtt{S} = \left(\begin{array}{|c|c|c|} \hline 1_1 & 1_1 & 3_1 \\ \hline 2_1 \\ \cline{1-1} 1_2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1_2 & 2_2 \\ \hline 3_2 \\ \cline{1-1} \end{array} \right).$$

Then

$$f_{\mathtt{S}} = -q^{-1}$$

Proposition 4.7.1. *Let $\Theta = \sum_{\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)} f_{\mathtt{S}} \Theta_{\mathtt{S}}$. Then $\Theta\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right) \in \check{\mathscr{H}}^\mu$ for every $(d,t,s) \in$ $\mathrm{def}(\lambda, \mathfrak{d})$.*

Proof. We show that the homomorphism specified above satisfies the criteria of Proposition 4.6.1.

1. Suppose that both d and s are greater than 1 and that $\lambda^{(s)}_d = \lambda^{(s)}_{d+l}$ for some $1 \leq l \leq \rho_s(\mu)$. By section 4.5, $\Theta_{\mathtt{S}}\left(m_\lambda \mathfrak{d}^{(s)}_{d,t}\right) \in \check{\mathscr{H}}^\mu$ if $\mathtt{S}$ is of any form other than (4.17) or (4.20),. That being the case, let us assume that $\mathtt{S}$ is as in (4.19). We then have that $\Theta_{\mathtt{X}}\left(m_{\lambda \cdot \mathfrak{d}^{(s)}_{d,t}}\right)$, where $\mathtt{X} \in \mathscr{T}_0\left(\mu, \lambda \cdot \mathfrak{d}^{(s)}_{d,t}\right)$ is obtained from $\mathtt{S}$ by relabelling the entry $\mathtt{S}(\mathrm{e}_{d,s}) = (d+1,s)$ as (d,s), appears with a coefficient of

$$q^{\lambda^{(s)}_d} f\left(\mathtt{S} : \begin{smallmatrix} (\check{r}(d,s),s) \\ \downarrow \\ (d+1,s) \end{smallmatrix}, \begin{smallmatrix} (d,s) \\ \downarrow \\ (d,s) \end{smallmatrix}\right) + q^{\lambda^{(s)}_{d+1}-1} f_{\mathtt{S}}.$$

   In calculating the value of this coefficient, we may ignore all values of $\mathtt{S}(i,1)$ and $\mathtt{S}(j,2)$ except those corresponding to $i = d,s$ and $i = \check{r}(d,s)$ since these are by definition the same for both tableaux. Hence for some non-zero $\alpha \in \mathbb{F}$ the above simplifies to

$$\left(q^{\lambda^{(s)}_d}(-q^{-1}) + q^{\lambda^{(s)}_{d+1}-1}\right) C = 0.$$

   where C is the remainder of $\prod_{i=2} \mathtt{S}((i,1)) \prod_{j=1} \mathtt{S}((j,2))$ after factoring out the aforementioned values of $\mathtt{S}(i,1)$ and $\mathtt{S}(j,2)$.

2. (a) Let $s = 1$, and $2 \leq d \leq \rho_1(\lambda)$. If $\lambda^{(1)}_d > \lambda^{(1)}_{d+1}$, then, by the same line of reasoning

as in the previous part, there exists $C \in \mathbb{F}$ such that

$$\begin{aligned}
&C\Big(q^{\lambda^{(1)}_d}\left[\lambda^{(1)}_1-\lambda^{(1)}_d+d\right]+q^{\lambda^{(1)}_{d+1}-1}\left[\lambda^{(1)}_d-\lambda^{(1)}_{d+1}+1\right]\\
&-q^{\lambda^{(1)}_{d+1}-1}(-q^{-1})+\cdots\cdots+(-1)^{-(l-1)}q^{\lambda^{(1)}_{d+1}-1}((-q)^{l-1})\\
&+(-1)^l q^{\lambda^{(1)}_{d+1}-1}\left((-q)^{-(l-1)}\left[\lambda^{(1)}_1-\lambda^{(1)}_{d+l}+d+l\right]\right)\Big)\\
&=C\Big(q^{\lambda^{(1)}_d}\left[\lambda^{(1)}_1-\lambda^{(1)}_d+d\right]\\
&+q^{\lambda^{(1)}_{d+1}-1}\left(\left[\lambda^{(1)}_d-\lambda^{(1)}_{d+1}+1\right]+q^{-(l-1)}[l-1]\right)\\
&+(-1)q^{-l+1}\left[\lambda^{(1)}_1-\lambda^{(1)}_{d+1}+d+l\right]\Big)\\
&=0.
\end{aligned}$$

(b) Let $s=2$ and let $1 \le d \le \rho_2(\lambda)-1$. If $\lambda^{(1)}_d > \lambda^{(1)}_{d+1}$, then the proof in this case is identical to that of the second part of the proof of [37, Theorem 4.2.5]

3. Suppose that $(d,s)=(1,1)$ and that $\lambda^{(1)}_2=\lambda^{(1)}_l$ for some $1 \le l \le \rho_1(\lambda)$, Then there exists an element $C \in \mathbb{F}$ such that

$$\begin{aligned}
&\Big(q^{\lambda^{(1)}_2-1}\left[\lambda^{(1)}_1-\lambda^{(1)}_2+2\right]f_{\mathtt{S}}-f\left(\mathtt{S}:\begin{smallmatrix}(1,1)\\ \downarrow\\ (3,1)\end{smallmatrix},\begin{smallmatrix}(2,1)\\ \downarrow\\ (2,1)\end{smallmatrix}\right)\\
&\qquad\vdots\\
&\qquad+(-1)^{l-1}f\left(\mathtt{S}:\begin{smallmatrix}(1,1)\\ \downarrow\\ (l+1,1)\end{smallmatrix},\begin{smallmatrix}(2,1)\\ \downarrow\\ (2,1)\end{smallmatrix},\begin{smallmatrix}(3,1)\\ \downarrow\\ (3,1)\end{smallmatrix},\ldots,\begin{smallmatrix}(l,1)\\ \downarrow\\ (l,1)\end{smallmatrix}\right)\\
&\qquad+(-1)^{l}f\left(\mathtt{S}:\begin{smallmatrix}(1,1)\\ \downarrow\\ j^{(1)}_{l+1}\end{smallmatrix},\begin{smallmatrix}(2,1)\\ \downarrow\\ (2,1)\end{smallmatrix},\begin{smallmatrix}(3,1)\\ \downarrow\\ (3,1)\end{smallmatrix},\ldots,\begin{smallmatrix}(l+1,1)\\ \downarrow\\ (l+1,1)\end{smallmatrix}\right)\Big)\\
&=C\left(q^{\lambda^{(1)}_2-1}\left[\lambda^{(1)}_1-\lambda^{(1)}_2+2\right]+q^{\lambda^{(1)}_2-l+1}[l-2]-q^{\lambda^{(1)}_l-l+1}\left[\lambda^{(1)}_1-\lambda^{(1)}_l+l\right]\right)\\
&=q^{\lambda^{(1)}_2-l+1}C\left(q^{l-2}\left[\lambda^{(1)}_1-\lambda^{(1)}_2+2\right]+[l-2]-\left[\lambda^{(1)}_1-\lambda^{(1)}_l+l\right]\right)\\
&=q^{\lambda^{(1)}_2-l+1}C\left(\left[\lambda^{(1)}_1-\lambda^{(1)}_l+l\right]-\left[\lambda^{(1)}_1-\lambda^{(1)}_l+l\right]\right)=0.
\end{aligned}$$

□

We now show that $\Theta\left(m_{\lambda}\mathfrak{l}^{(1)}\right)=0$ when Θ is determined by the coefficients just defined, subject to a certain condition being satisfied. In particular, let $\mathbf{a}=\mathrm{e}^{\mu}_{1,1}$ and $\mathbf{r}=\mathrm{e}^{\lambda}_{\rho_2(\lambda),1}$, so that $\mathbf{a}$ is the node added to the first component of λ and $\mathbf{r}$ the node removed from the second component in order to construct μ; we prove that $\Theta\left(m_{\lambda}\mathfrak{l}^{(1)}\right)=0$ if and only if $\mathrm{res}_{\mu}(\mathbf{a})=\mathrm{res}_{\lambda}(\mathbf{r})$.

Let x be an integer with $1 \le i \le \rho_1(\lambda)$ and recall that for each $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ with $\mathtt{S}_{x,1}=(1,2)$ we define $\mathtt{W}_0$ to be the tableau defined by relabelling $\mathtt{S}_{x,1}$ as $(\rho_1(\lambda)+1,1)$. Our strategy is to show that the coefficient with which $\Theta_{\mathtt{W}_0}\left(m_{\lambda\cdot\mathfrak{l}^{(1)}}\right)$ appears in the expression for $\Theta\left(m_{\lambda\cdot\mathfrak{l}^{(1)}}\right)$ given in (4.26) is zero if and only if our claimed condition is satisfied. We begin with a proof of this when $x=1$, since this is relatively straight forward, and serves as a good model for the general proof.

Definition 4.7.1. If $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$, we write $g_{\mathtt{W}_0(\mathtt{S})}$ to denote the coefficient with which $\Theta_{\mathtt{W}_0}\left(m_{\lambda\cdot\mathfrak{t}^{(1)}}\right)$ appears in the expression for $\Theta\left(m_{\lambda\cdot\mathfrak{t}^{(1)}}\right)$.

Although this definition adds brings yet more notation to the fray, it has the advantage of allowing us to avoid some cumbersome phrasing throughout the rest of this chapter.

Proposition 4.7.2. *If* $\mathtt{S} \in \mathscr{T}_0(\mu,\lambda)$ *is such that* $\mathtt{S}_{m,1} = (1,2)$ *for some* $1 \le m \le \rho_1(\lambda)$*, then* $g_{\mathtt{W}_0(\mathtt{S})} = 0$ *whenever* $\mathrm{res}_\mu(\mathbf{a}) = \mathrm{res}_\lambda(\mathbf{r})$.

Proof. In accordance with Proposition 4.6.7, we have six possible cases to check:

1. $1 < m$, $\mu_2^{(2)} < \mu_1^{(2)}$, and $\mu_m^{(1)} < \mu_{m-1}^{(1)}$;
2. $1 < m$, $\mu_2^{(2)} < \mu_1^{(2)}$, and $\mu_m^{(1)} = \mu_l^{(1)}$ for some $1 \le l < m$;
3. $1 < m$, $\mu_1^{(2)} = \mu_w^{(2)}$ for some $2 \le w \le \rho_2(\lambda)$, and $\mu_m^{(1)} < \mu_{m-1}^{(1)}$;
4. $1 < m$, $\mu_1^{(2)} = \mu_w^{(2)}$ for some $2 \le w \le \rho_2(\lambda)$, and $\mu_m^{(1)} = \mu_l^{(1)}$ for some $1 \le l < m$;
5. $m = 1$ and $\mu_2^{(2)} < \mu_1^{(2)}$; and
6. $m = 1$ and $\mu_1^{(2)} = \mu_w^{(2)}$ for some $2 \le w \le \rho_2(\lambda)$.

Of these conditions, we will check only the fourth, our reasoning being that the others are more straightforward in light of this case. Recall that in this case there is some $l \le k \le x$ such that $\mathtt{S}_{c,1} = (k,1)$ with $1 \le c < l$.

For every $s < w$, let $\mathtt{P}^u, \mathtt{Q} \in \mathscr{T}_0(\mu,\lambda)$ be as in (4.24) and (4.25). By Lemmas 4.6.4 and 4.6.5 we know that, other than $\mathtt{S}$ itself, only these tableaux contribute to $g_{\mathtt{W}_0(\mathtt{S})}$, and by Proposition 4.6.7 we need our coefficients to satisfy:

$$\begin{aligned}
\left(\mathrm{res}_\mu\left(\mathrm{e}_{m,1}\right) - \mathrm{res}_\lambda\left(\mathrm{e}_{1,2}\right)\right) f_{\mathtt{S}} - (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f &\left(\mathtt{S} : \begin{smallmatrix}(m,1)\\ \downarrow \\ (2,2)\end{smallmatrix}, \begin{smallmatrix}(1,2)\\ \downarrow \\ (1,2)\end{smallmatrix}\right) \\
+ (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f &\left(\mathtt{S} : \begin{smallmatrix}(m,1)\\ \downarrow \\ (3,2)\end{smallmatrix}, \begin{smallmatrix}(1,2)\\ \downarrow \\ (1,2)\end{smallmatrix}, \begin{smallmatrix}(2,2)\\ \downarrow \\ (2,2)\end{smallmatrix}\right) \\
&\vdots \\
(-1)^w (q-1)\mathrm{res}_\mu(\mathrm{e}_{1,2}) f &\left(\mathtt{S} : \begin{smallmatrix}(m,1)\\ \downarrow \\ \mathtt{S}_{w,2}\end{smallmatrix}, \begin{smallmatrix}(1,2)\\ \downarrow \\ (1,2)\end{smallmatrix}, \begin{smallmatrix}(2,2)\\ \downarrow \\ (2,2)\end{smallmatrix}, \ldots, \begin{smallmatrix}(w,2)\\ \downarrow \\ (w,2)\end{smallmatrix}\right) \\
+ (-1)^{m-k}(q-1)\mathrm{res}_\mu\left(\mathrm{e}_{k,1}\right) f &\left(\mathtt{S} : \begin{smallmatrix}(c,1)\\ \downarrow \\ (1,2)\end{smallmatrix}, \begin{smallmatrix}(k,1)\\ \downarrow \\ (k,1)\end{smallmatrix}, \begin{smallmatrix}(k+1,1)\\ \downarrow \\ (k+1,1)\end{smallmatrix}, \ldots, \begin{smallmatrix}(m,1)\\ \downarrow \\ (m,1)\end{smallmatrix}\right) \\
&= 0.
\end{aligned}$$

Consider

$$f_{\mathtt{P}^u} = f\left(\mathtt{S} : \begin{smallmatrix}(m,1)\\ \downarrow \\ \mathtt{S}_{u,2}\end{smallmatrix}, \begin{smallmatrix}(1,2)\\ \downarrow \\ (1,2)\end{smallmatrix}, \begin{smallmatrix}(2,2)\\ \downarrow \\ (2,2)\end{smallmatrix}, \ldots, \begin{smallmatrix}(u,2)\\ \downarrow \\ (u,2)\end{smallmatrix}\right) \tag{4.27}$$

for an arbitrary u with $1 \le u \le w-1$. We have

$$\prod_{j=1}^{w} \mathtt{P}^u((j,2)) = \left(-q^{-1}\right)^u \quad \text{and} \quad \prod_{j=1}^{w} \mathtt{S}((j,2)) = 1$$

since the fact that S is semistandard means that $\mathtt{S}((j,2)) = 1$ for all $1 \le j \le w$. Therefore, we have that $f_{\mathtt{P}^u} = \left(-q^{-1}\right)^u f_{\mathtt{S}}$, and so $\mathtt{P}^u$ contributes

$$(-1)^u \left(-q^{-1}\right)^u (q-1) q^{\mu_2^{(1)}-1} Q_2 f_{\mathtt{S}} = \left(q^{\mu_2^{(1)}-u-1} - q^{\mu_1^{(2)}-u}\right) Q_2 f_{\mathtt{S}}.$$

Adding together the contribution of all such $\mathtt{P}^u$ then gives us

$$\left(q^{\mu_1^{(2)}-1} - q^{\mu_1^{(2)}-2}\right) Q_2 f_{\mathtt{S}}. \tag{4.28}$$

Now let $u = w$. Since $\mu_{w+1}^{(2)} < \mu_w^{(2)}$ we have

$$\prod_{j=1}^{w} \mathtt{P}^u((j,2)) = \left(-q^{-1}\right)^{w-1} \left(-q^{-\left(\lambda_w^{(2)}+\rho_2(\lambda)-\lambda_{\rho_2(\lambda)}^{(2)}-w\right)}\right) \left[\lambda_w^{(2)} + \rho_2(\lambda) - \lambda_{\rho_2(\lambda)}^{(2)} - w\right]$$

and so the contribution of $\mathtt{P}^w$ to $f_{\mathtt{S}}$ is

$$\begin{aligned}
(-1)^w (q-1)\left(-q^{-1}\right)^{w-1} &\left(-q^{-\left(\lambda_w^{(2)}+\rho_2(\lambda)-\lambda_{\rho_2(\lambda)}^{(2)}-w\right)}\right) \left[\lambda_w^{(2)} + \rho_2(\lambda) - \lambda_{\rho_2(\lambda)}^{(2)} - w\right] q^{\mu_1^{(2)}-1} Q_2 f_{\mathtt{S}} \\
&= -q^{1-w} \left(-q^{-\left(\lambda_w^{(2)}+\rho_2(\lambda)-\lambda_{\rho_2(\lambda)}^{(2)}-w\right)}\right) \left(q^{\lambda_w^{(2)}+\rho_2(\lambda)-\lambda_{\rho_2(\lambda)}^{(2)}-w} - 1\right) q^{\mu_1^{(2)}-1} Q_2 f_{\mathtt{S}} \\
&= q^{\mu_1^{(2)}-w} \left(1 - q^{-\left(\lambda_w^{(2)}+\rho_2(\lambda)-\lambda_{\rho_2(\lambda)}^{(2)}-w\right)}\right) Q_2 f_{\mathtt{S}} \\
&= \left(q^{\mu_1^{(2)}-w} - q^{\lambda_{\rho_2(\lambda)}^{(2)}-\rho_2(\lambda)}\right) Q_2 f_{\mathtt{S}},
\end{aligned} \tag{4.29}$$

where we have used the fact that $\lambda_w^{(2)} = \mu_w^{(2)} = \mu_1^{(2)}$.

Finally, consider

$$f_{\mathtt{Q}} = f\left(\mathtt{S} : \underset{(1,2)}{\overset{(c,1)}{\downarrow}}, \underset{(k,1)}{\overset{(k,1)}{\downarrow}}, \underset{(k+1,1)}{\overset{(k+1,1)}{\downarrow}}, \dots, \underset{(m,1)}{\overset{(m,1)}{\downarrow}}\right).$$

The first thing to notice about this is that $\mu_{m+1}^{(1)} < \mu_m^{(1)}$ and that

$$\prod_{i=k}^{m} \mathtt{Q}((i,1)) = \left(-q^{-1}\right)^{m-k} \left[\lambda_1^{(1)} - \lambda_m^{(1)} + m\right] \quad \text{and} \quad \prod_{i=k}^{m} \mathtt{P}((i,1)) = 1$$

(see Lemma 4.6.4 and the remark that follows it), and so

$$f_{\mathtt{Q}} = (-1)^{m-k} \left(-q^{-1}\right)^{m-k} \left[\lambda_1^{(1)} - \lambda_m^{(1)} + m\right] f_{\mathtt{S}}.$$

Thus, we have that $\mathtt{Q}$ contributes

$$\begin{aligned}
(q-1) q^{\mu_k^{(1)}-k} q^{k-m} \left[\lambda_1^{(1)} - \lambda_m^{(1)} + m\right] Q_1 f_{\mathtt{S}} &= q^{\mu_k^{(1)}-m} \left(q^{\lambda_1^{(1)}-\lambda_m^{(1)}+m} - 1\right) Q_1 f_{\mathtt{S}} \\
&= \left(q^{\mu_1^{(1)}-1} - q^{\mu_m^{(1)}-m}\right) Q_1 f_{\mathtt{S}}
\end{aligned} \tag{4.30}$$

to $g_{\mathtt{W}_0(\mathtt{S})}$, making use of the fact that $\lambda_m^{(1)} = \lambda_k^{(1)} = \mu_k^{(1)}$ in our calculations.

Finally, combining (4.28), (4.29), and (4.30) with $\left(\mathrm{res}_\mu\left(\mathrm{e}_{m,1}\right)-\mathrm{res}_\lambda\left(\mathrm{e}_{1,2}\right)\right)f_{\mathrm{S}}$ gives us

$$\begin{aligned}\Big(q^{\mu_m^{(1)}-m}Q_1 - q^{\lambda_2^{(1)}-1}Q_2 + q^{\mu_1^{(1)}-1}Q_1 - q^{\mu_m^{(1)}-m}Q_1 &\\ -q^{\lambda^{(2)}_{\rho_2(\lambda)}-\rho_2(\lambda)}Q_2 + q^{\mu_1^{(2)}-w}Q_2 - q^{\mu_1^{(2)}-w}Q_2 + q^{\mu_1^{(2)}-1}Q_2\Big)f_{\mathrm{S}}&\\ = \left(q^{\mu_1^{(1)}-1}Q_1 - q^{\lambda^{(2)}_{\rho_2(\lambda)}-\rho_2(\lambda)}Q_2\right)f_{\mathrm{S}}&\\ = (\mathrm{res}_\mu(\mathbf{a})-\mathrm{res}_\lambda(\mathbf{r}))f_{\mathrm{S}}&\end{aligned}$$

as required, completing the proof in this case. The remaining cases, being more straightfoward and sufficiently similar to this case, are omitted. □

This concludes our construction of Θ, thus we have (almost) proved the following, the main result of this chapter.

Theorem 4.7.3. *Let λ and μ be multipartitions such that μ is constructed from λ by the deletion of a removable node $\mathbf{r}$ in a single component x and the adjoining of an addable node $\mathbf{a}$ to component $x-1$. Then there exists a non-zero homomorphism $\hat{\Theta}: S^\lambda \to S^\mu$ whenever* $\mathrm{res}_\mu(\mathbf{a}) = \mathrm{res}_\lambda(\mathbf{r})$.

Proof. Let $\mathrm{S} \in \mathscr{T}_0(\mu,\lambda)$ and suppose that $\mathbf{a}$ appears in row a of $[\mu^{(x-1)}]$ and $\mathbf{r}$ in row b of $[\lambda^{(x)}]$. By Lemma 4.1.1 we have

$$\Theta_{\mathrm{S}}(m_\lambda h) = \left(\check{\mathscr{H}}^\mu + m_\mu T_{d(\mathrm{first}(\mathrm{S}))}\prod_{\mathbf{i}=(i_1,i_2)} \mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathrm{S}}_{\mathbf{i}}\right)\right)h. \tag{4.31}$$

By construction, $\mathrm{S}(i,j,k) = (i,k)$ whenever one of the following is true

- $i < a$, and $k = x-1$;
- $k < x-1$;
- $i = a, k = x-1$, and $j < \lambda_a^{(x)} + 1$;
- $i > b$ and $k = x$; or
- $k > x$.

Therefore, the right hand side of (4.31) has the following properties:

- $d(\mathrm{first}(\mathrm{S}))$ is a permutation in the symmetric group on

  $$\left\{\overline{\lambda}_a^{(x-1)}+1, \overline{\lambda}_a^{(x-1)}+2, \dots, \overline{\lambda}_b^{(x)}\right\};$$

  and
- $\mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1} : \Omega^{\mathrm{S}}_{(i_1,i_2)}\right) = 1$ whenever (i_1,i_2) is such that: $i_1 \le a$ and $i_2 = x-1$; $i_2 < x-1$; $i_1 > b$ and $i_2 = x$; or $i_2 > x$.

We then have that

$$\Theta_{\mathrm{S}}\left(m_{\lambda}\mathfrak{d}^{(s)}_{d,t}(\lambda)\right) = \check{\mathscr{H}}^{\mu} + m_{\mu}\mathfrak{d}^{(s)}_{d,t}(\lambda)T_{d(\mathrm{first}(\mathrm{S}))}\prod_{\mathbf{i}=(i_1,i_2)}\mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1}:\Omega^{\mathrm{S}}_{\mathbf{i}}\right)$$

Whenever $d < a$ and $s = x-1$, $s < x-1$, $d > b$ and $s = x$, or $s > x$. Moreover

$$\mathfrak{d}^{(s)}_{d,t}(\lambda) = \mathrm{C}\left(\overline{\lambda}^{(s)}_{d-1}:\lambda^{(s)}_{d},t\right) = \mathrm{C}\left(\overline{\mu}^{(s)}_{d-1}:\mu^{(s)}_{d},t\right) = \mathfrak{d}^{(s)}_{d,t}(\mu)$$

for these values of d and s and any value of t, in which case

$$\Theta_{\mathrm{S}}\left(m_{\lambda}\mathfrak{d}^{(s)}_{d,t}(\lambda)\right) \in \check{\mathscr{H}}^{\mu}.$$

Additionally, $\Theta_{\mathrm{S}}(m_{\lambda}\mathfrak{l}^{(s)}) \in \check{\mathscr{H}}^{\mu}$ for every $s < x-1$ and $s \geq x$ since

$$\Theta_{\mathrm{S}}\left(m_{\lambda}\mathfrak{l}^{(s)}\right) = \check{\mathscr{H}}^{\mu} + m_{\mu}\mathfrak{l}^{(s)}T_{d(\mathrm{first}(\mathrm{S}))}\prod_{\mathbf{i}=(i_1,i_2)}\mathrm{C}\left(\overline{\lambda}^{(i_2)}_{i_1-1}:\Omega^{\mathrm{S}}_{\mathbf{i}}\right)$$

for these values, in which case

$$m_{\mu}\mathfrak{l}^{(s)} = m_{\mu}\left(L_{\overline{\lambda}^{(s+1)}+1} - Q_{s+1}\right) = m_{\mu}\left(L_{\overline{\mu}^{(s+1)}+1} - Q_{s+1}\right) = m_{\mu}(Q_{s+1} - Q_{s+1}) + \check{\mathscr{H}}^{\mu},$$

by Proposition 2.3.2. This then leaves us with

- $\mathfrak{d}^{(s)}_{d,t}$ when $s = x-1$ and $d \geq a$, or $s = x$ and $d \leq b-1$; and
- $\mathfrak{l}^{(x-1)}$

to consider, but this follows from the work previously done on one node homomorphisms since we are essentially in the same case. □

Chapter 5

Concluding Remarks

The main result of this thesis, Theorem 3.2.1, provides us with necessary and sufficient conditions for a homomorphism $\Theta : M^\lambda \to S^\mu$ to factor through S^λ, thereby providing us with a means to identify when it is possible to do so in order to construct a non-zero homomorphism $\hat{\Theta} : S^\lambda \to S^\mu$. The most notable feature of this is that we are only required to focus upon the behaviour of a finite number of elements in M^λ to identify precisely when this condition is satisfied. The result appearing in Chapter 4 may be thought of as providing a 'proof of concept', applying Theorem 3.2.1 to a restricted class of Specht modules in order to provide a relatively straightforward construction of explicit homomorphisms between them.

There are a number of future developments that could increase the strength of or directly lead from the results we have presented here. Perhaps the most obvious would be the provision of an explicit algorithm for the construction of homomorphisms for a more general class of Specht modules. The most immediate hinderance to developing such an algorithm is that we currently know very little about what results from right multiplying $\Theta_{\mathrm{S}}(m_\lambda)$, where $\Theta_{\mathrm{S}} : M^\lambda \to S^\mu$ is a semistandard homomorphism, by a Jucys-Murphy element L_i.

Despite being elementary in a conceptual sense, solving this problem seems complicated in terms of execution. That said, the author and Sinéad Lyle are currently working towards shedding light upon this matter and have reason to believe that a result will be forthcoming.

In addition to possibly playing a role in direction, the following conjecture, due to the author, may be of independent interest to the reader.

Conjecture 2 (Corlett). *Let λ be a multicomposition of n and let $\mathfrak{t}$ be a λ-tableau. For every $1 \le x \le n$, let*

- $\mathbf{x} \in [\lambda]$ *such that* $\mathfrak{t}^\lambda(\mathbf{x}) = x$*; and*
- n_x *be the number of entries less than* $\mathfrak{t}(\mathbf{x})$ *that in* $\mathfrak{t}$ *occupy nodes lower than* $\mathbf{x}$.

Then

$$T_{d(\mathfrak{t})} = \left(\prod_{j=n}^{n-1+n_n} T_j\right)\left(\prod_{j=n-1}^{n-2+n_{n-1}} T_j\right)\cdots\left(\prod_{j=1}^{n_1} T_j\right)$$

where we take the empty product to be 1.

Example 19. *Let* $\lambda = (5,3,3,2,1)$ *and let* $\mathfrak{t}$ *be given by*

$$\mathfrak{t} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 8 & 10 \\ \hline 4 & 6 & 11 \\ \cline{1-3} 5 & 7 & 9 \\ \cline{1-3} 12 & 14 \\ \cline{1-2} 13 \\ \cline{1-1} \end{array}$$

Then $T_{d(\mathfrak{t})} = T_{13}T_{8,9,10}T_7T_{5,6,7,8,9}T_{4,5,6,7}$

We now present a straight-forward but useful corollary of Conjecture 2.

Corollary 5.0.4. *Let* w *be a permutation in* $\mathfrak{S}_n$*. Then a reduced expression for* w *in terms of simple generators* $s = (i, i+1)$ *can be found by applying Conjecture 2 to the* (1^n)*-tableau* $\mathfrak{t}^\lambda \cdot w$.

Example 20. *Let* $w = (\,4\ 8\ 11\ 9\ 5\ 10\ 7\ 6\,)(\,13\ 14\,)$*. Then* $w = s_{13}s_{8,9,10}s_7s_{5,6,7,8,9}s_{4,5,6,7}$.

Successfully providing such a general construction of homomorphisms will also depend upon generalizing Fayers' recent paper [18] to the setting of the Ariki-Koike algebra and thereby providing an algorithm for expressing homomorphisms $\Theta_{\mathrm{S}} : M^\lambda \to S^\mu$ indexed by arbitrary non-semistandard as a linear combination of semistandard homomorphisms.

Another, related, development would be to identify an analogue of the row and column removal theorems [14] in type A for the Ariki-Koike algebra. Such a result would prove useful in calculating the dimension of homomorphism spaces between Specht modules by identifying an isomorphism between the homomorphism space of certain Specht modules for one Ariki-Koike algebra and that between related Specht modules for an Ariki-Koike algebra that is in some sense 'smaller'. In particular, it should be possible to use the results of this thesis in tandem with a row and column removal theorem to exhibit a number of non-zero homomorphism spaces between Specht modules.

It should also be possible to apply the results of this thesis in a similar capacity to the type A results they generalize; namely, in studying reducible Specht modules, as in [37, Theorem 5.2], or attempting to establish lower bounds upon the dimension of homomorphism spaces between Specht modules, as was achieved in [38, Subsection 2.3].

Finally, a proof of Conjecture 1 would, in addition to being of interest in its own right, be a particularly welcome addition to our knowledge of the Ariki-Koike algebra. As discussed in the Introduction to this thesis, much of the study of homomorphism spaces between Specht modules for the Iwahori-Hecke algebra of type A and the symmetric group has been built upon two results: the kernel intersection theorem and the semistandard homomorphism theorem. Theorem 3.2.1 of this thesis provides half of this foundation, being an analogue of the former, leaving the latter yet to be established.

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