INCIDENCE STRUCTURES OF PARTITIONS

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Abstract

In this thesis we study incidence structures arising from *unordered partitions* of a set $\Omega = \{1, 2, \ldots, ab\}$. We denote by $P(a, b)$ the set of unordered partitions of Ω into a parts of cardinality b. A partition α in $P(a, b)$ will be incident with β in $P(b, a)$ if each part of α contains exactly one element from each part of β. The symmetric group $G = Sym(ab)$ acts transitively on $P(a, b)$. We form the permutation module $FP(a, b)$ over the field F of complex numbers and consider some of the irreducible modules in its decomposition using representation theory of the symmetric groups. We construct some of the eigenspaces of MM^T, where M is the incidence matrix of the structure. For the case when $a = 2$ and b is arbitrary, we decompose the module $FP(2, b)$ completely and use this decomposition to obtain the complete set of eigenvalues of MM^T. Moreover, we show for $b \geq 2$ that all eigenvalues of MM^T are non-zero, hence showing that all irreducible modules in the decomposition of $FP(2, b)$ are in the decomposition of $FP(b, 2)$. Thus we have a simple proof for a special case of a conjecture of Foulkes. When $a = 3$ and b is arbitrary we give a complete decomposition of $FP(3, b)$ and show that $FP(3, b)$ is isomorphic to an FG-submodule of $FP(b, 3)$. This gives a combinatorial proof of the next case of Foulkes' conjecture. We finish by giving some of the eigenvectors and eigenvalues of MM^T for the case when $a = 3$ and b is arbitrary. When $a = 3$ and $1 \leq b \leq 8$ we give the complete set of eigenvalues of MM^T which we show to satisfy a surprising partial ordering property.

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Contents

Chapter 1

Introduction

1.1 Preliminaries

We consider incidence structures arising from certain unordered partitions of a finite set. We study permutation modules induced by the natural action of symmetric groups on such partitions. Thus we shall begin by introducing some basic definitions and results which we will use throughout.

Let $\Omega = \{1, 2, \ldots, n\}$ be a finite set. The symmetric group $Sym(n)$ is the set of all permutations of Ω. If G is any group then G *acts* on a set X if $x1 = x$ for all $x \in X$ (where 1 is the identity element of G) and $(xg)h = x(gh)$ for all $g, h \in G$ and $x \in X$. We will always write actions by group elements on the right. If G acts on X then we say that (G, X) is a *G-space*. The G-space is *transitive* if for all $x, y \in X$ there exists a $g \in G$ such that $xg = y$. The *G-orbit* containing $x \in X$ is $x^G = \{xg \mid g \in G\}$ and the *stabilizer* of x in G is the group $G_x = \{g \in G \mid xg = x\}$. The well known *orbit-stabilizer theorem* says that if (G, X) is a transitive G-space then for $x \in X$ the space (G, X) is G-equivalent to the G-space $cos(G : G_x)$, where $cos(G, G_x)$ is the coset space of the stabilizer G_x in X. Let F be a field then an *FG-module* is a vector space on which the group G acts such that $v \mapsto vg$ is a linear transformation from V into itself. By an *irreducible* FG-module we mean a non-zero FG-module which contains no FG-submodule apart from itself and the zero module. Let $[g]_E$ denote the matrix of the linear transformation $v \mapsto vg$ relative to the basis E. A *representation* *of degree n* of G over F is a homomorphism ρ from G to $GL(n, F)$ (where $GL(n, F)$ denotes the General Linear group, that is the group of invertible $n \times n$ matrices over a field F). To each FG-module V we associate a representation of G over F given by $g \mapsto [g]_E$ and a function $\chi : G \to \mathbb{C}$, called the *character* of V, given by $\chi(g) = tr([g]_E)$ where $tr([g]_E)$ is the trace of the matrix $[g]_E$. The vector space FG with the natural multiplication vg (for $v \in FG$ and $g \in G$) is called the *regular FG-module*. We will denote the *regular representation* associated to this module by R_G. The simplest type of representation is the *trivial representation* given by 1_G which maps each element of G to the identity of the field. We will mainly be concerned with *permutation modules*, that is FG-modules which permute the basis elements of V. The *permutation character* of the permutation module is given by $\pi(g) = |Fix(g)|$, where $Fix(g)$ is the set of elements fixed by g. If (G, X) is a G-space we can construct the permutation module FX given by

$$FX = \left\{ \sum_{x \in X} a_x x \mid a_x \in F \right\},$$

with $g : \sum a_x x \to \sum a_x xg$ for $g \in G$. If ρ is a representation of a subgroup H of G we obtain the *induced representation* $\rho \uparrow^G$ of G in the following way. Consider the decomposition of G into right cosets of H

$$G = \bigcup_{i=1}^{|G/H|} Hg_i.$$

Then we form the blocked matrix $(\rho \uparrow^G)(g)$ with the ijth block equal to $\rho(g_i g g_j^{-1})$ if $g_i g g_j^{-1} \in H$ and the zero matrix otherwise. Since $Hg_i g = Hg_j$ if and only if $g_i g g_j^{-1} \in H$, the representation $1_H \uparrow^G$ of G induced by the identity representation 1_H of H is equivalent to the representation of G which corresponds to the action of G on the right cosets of H in G. If (G, X) is a transitive G-space then from the orbit-stabilizer theorem the action of G on the right cosets of G_x in G is equivalent to the action of G on X. Thus the permutation representation of the module FX is equivalent to the induced representation $1_{G_x} \uparrow^G$. We will always be working over a field of characteristic zero. In this case Maschke's Theorem (see for example [15]) tells us that FX can be decomposed (in a unique way up to isomorphism) into a direct sum of irreducible FG-modules. In the same way, the permutation character or permutation representation can be decomposed into a sum of irreducible characters or irreducible representations respectively. We call a decomposition of FX (or equivalently of the character or representation) into irreducibles a *complete decomposition*. We will use the phrase "U *appears* in FX" to mean that U is isomorphic to a module in the complete decomposition of FX. We will say that "U appears in FX with *multiplicity* m" to mean that there are m irreducible FG-modules in the decomposition of FX which are isomorphic to U. In a similar way, we will say that χ_i (or ρ_i) is a *constituent* of a character χ (or a representation ρ) if the coefficient of χ_i (or ρ_i) in the complete decomposition of χ (or ρ) is non-zero. Moreover, we say that a character is *multiplicity free* if every constituent has multiplicity one. If $V = c_1X_1 + c_2X_2 + \cdots + c_rX_r$ and $W = d_1X_1 + d_2X_2 + \cdots + d_rX_r$ (with c_i, $d_i \geq 0$) are the complete decompositions of the FG-modules V and W then the multiplicities d_i are greater than or equal to the multiplicities c_i for all $i \in \{1, 2 \ldots, r\}$ if and only if there exists an injective G-homomorphism from V to W. Over a field F of characteristic zero and in the case when G is the symmetric group $Sym(n)$ much is known about the FG-modules which appear in FX. We will give a result in Section 2.2 which shows that all irreducible FG-submodules of FX are isomorphic to certain cyclic FG-modules S^μ, known as *Specht modules* (see definition 2.2.2), each one being indexed by a partition μ of n. If V and W are FG-modules then denote by $Hom_{FG}(V, W)$ the set of FG-homomorphisms from V to W. When $W = \mathcal{M}^{\mu^*}$, the permutation module of $Sym(n)$ acting on the set of μ^*-*tabloids* defined in Section 2.2.2, a basis for $Hom_{FG}(S^\mu, \mathcal{M}^{\mu^*})$ is known and given in Theorem 2.32 in terms of certain *semistandard homomorphisms*.

1.2 Overview

Chapter 2 states some well known definitions and general results of incidence structures and the representation theory of the symmetric groups (see for example [1] and [13] respectively). We also give some basic properties of matrices which can be found for example in [11], [17] or [19].

In Chapter 3 we consider *unordered partitions* of $\Omega = \{1, 2, \ldots, ab\}$ into a parts of cardinality b which we denote by $P(a, b)$. We define an incidence relation between partitions in $P(a, b)$ and $P(b, a)$ by $\alpha \in P(a, b)$ being *incident* with $\beta \in P(b, a)$ if and only if each part of α has exactly one element in common with each part of β. We denote this incidence structure by $\mathcal{P}_{a,b}$. The permutation module $FP(a, b)$ of $Sym(ab)$ on $P(a, b)$ (where F is the field of complex numbers) is isomorphic to an $FSym(ab)$-submodule of $\mathcal{M}^{(b^a)}$. We use the known basis of $Hom_{FSym(ab)}(S^\mu, \mathcal{M}^{(b^a)})$ to construct a set of $FSym(ab)$-homomorphisms from S^μ to $FP(a, b)$ and using these homomorphisms we construct some of the irreducible $FSym(ab)$-modules in the decomposition of $FP(a, b)$ for general a and b. Denoting the incidence matrix of the structure $\mathcal{P}_{a,b}$ by $M^{a,b}$, we use the $FSym(ab)$-modules which appear in $FP(a, b)$ with multiplicity one to find some of the eigenvectors and eigenvalues of $M^{a,b}(M^{a,b})^T$.

In Chapter 4 we study the structure $\mathcal{P}_{2,k}$ and the module $FP(2, k)$ in more detail. Using the representation theory of the symmetric groups we give a complete decomposition of $FP(2, k)$ and show that all $FSym(2k)$-modules which appear in $FP(2, k)$ have multiplicity one. We use this decomposition to find the complete set of eigenvectors and eigenvalues of $M^{2,k}(M^{2,k})^T$. Apart from determining these eigenvalues explicitly we give also a simple proof to show that all eigenvalues of $M^{2,k}(M^{2,k})^T$ are non-zero. We express the eigenvalues firstly in summation form and show then how they can be reduced to a neat closed form. These eigenvalues have also been constructed in [8] using association schemes. However, our method is probably more appropriate as it does extend at least partially to the case of general a and b. Finally, we give a short combinatorial argument, using semistandard homomorphisms, to show that all $FSym(2k)$-modules which appear in $FP(2, k)$ also appear in $FP(k, 2)$.

In Chapter 5 we mimic the ideas of Chapter 4 to decompose the module $FP(3, k)$ into irreducibles. Again using semistandard homomorphisms and results from the representation theory of the symmetric groups, we decompose the module $FP(3, k)$ completely.

In Chapter 6 we show that if S^μ is a $FSym(3k)$-module which appears in $FP(3, k)$ with multiplicity m then S^μ appears in $FP(k, 3)$ with multiplicity at least m. This settles the next case of a conjecture of H.O. Foulkes from 1950 [9] which states that $FP(a, b)$ is isomorphic to an FG-submodule of $F(b, a)$ if $a \leq b$.

Using the complete decomposition of $FP(3,k)$ and a computer program we are able to find the complete set of eigenvalues of $M^{3,k}(M^{3,k})^T$ for $1 \leq k \leq 8$. These examples unveil a surprising partial ordering property which such eigenvalues seem to satisfy. In the final section we give some of the eigenvalues of $M^{3,k}(M^{3,k})^T$ explicitly for general k. We also give an inductive proof for bounding below some of the eigenvalues of $M^{3,k}(M^{3,k})^T$.

1.3 Motivation

The ideas behind this thesis stem from a private communication between my supervisor Johannes Siemons and Otto Wagner in 1987 [27] and the paper of S.C. Black and R.J. List "A Note on Plethysm" [3]. In Black and List's paper the incidence matrices $M^{a,b}$ were considered and the question of these matrices having full rank is related to the conjecture due to H.O. Foulkes [9]. This conjecture was originally stated in terms of certain representations of Sym(ab) known as *plethysms* (or the "new multiplication" as it was called at the time) but the equivalent form of the conjecture is to show that $FP(a,b)$ is isomorphic to an $FSym(ab)$-submodule of $FP(b,a)$ for all integers $b \geq a$. Black and List showed that if the rank of $M^{a,b}$ can be shown to be $|P(a,b)|$ for all integers $1 < a \leq b$ then Foulkes' conjecture holds for the pair of integers r, b with $1 \leq r \leq a$. In Section 2.2.6 we look at Foulkes' conjecture in more detail showing how the original statement of the conjecture can be interpreted in the way we have described above. In Section 3.1.3 we explain the link between Foulkes' conjecture and the rank of $M^{a,b}$.

We conclude this chapter with a brief description of the known results relating to Foulkes' conjecture. Only a few papers have been published on this question and only few references are made to it in books on representation theory. These include [3], [4], [8], [9], [12], [14], [16], [18], [20], [22] and [26]. Let α be an unordered partition in $P(a,b)$. Then $G = Sym(ab)$ acts transitively on the elements of $P(a,b)$ and we will show in Section 3.1.3 that the stabilizer in G of α is isomorphic to $Sym(b) \wr Sym(a)$ where $\wr$ denotes the *wreath product*. Thus the permutation representation of G on $P(a,b)$ is equivalent to $1_{Sym(b) \wr Sym(a)} \uparrow^G$. The results in Thrall's paper [26] (1942) give the complete decompositions of the permutation representations $1_{Sym(b)\wr Sym(2)}\uparrow^G$ and $1_{Sym(2)\wr Sym(b)}\uparrow^G$. Thrall also decomposed the permutation representation $1_{Sym(b)\wr Sym(3)}\uparrow^G$ into irreducibles. In [18] (1944) Littlewood went on to give the complete decompositions of the representations $1_{Sym(3)\wr Sym(b)}\uparrow^G$ for $b \leq 6$ and $1_{Sym(4)\wr Sym(b)}\uparrow^G$ for $b \leq 5$. In [9] (1950) Foulkes decomposes completely $1_{Sym(5)\wr Sym(b)}\uparrow^G$ for $b \leq 4$ and $1_{Sym(6)\wr Sym(b)}\uparrow^G$ for $b \leq 4$. Using Littlewood's known decomposition of $1_{Sym(4)\wr Sym(5)}\uparrow^G$, Foulkes noticed that every term in the decomposition of $1_{Sym(5)\wr Sym(4)}\uparrow^G$ is included in $1_{Sym(4)\wr Sym(5)}\uparrow^G$. Foulkes then made his conjecture that every term in the complete decomposition of $1_{Sym(b)\wr Sym(a)}\uparrow^G$ is included in the complete decomposition of $1_{Sym(a)\wr Sym(b)}\uparrow^G$ for $a \leq b$. In [14] (1981) (and also in [16] (1991)) the result given by Theorem 5.4.34 on page 227 covers the case when $a = 2$ and b is arbitrary. It says that if μ is a partition of ab with at most two rows then every irreducible representation which can be indexed by μ and is in the decomposition of $1_{Sym(b)\wr Sym(a)}\uparrow^G$ is included in the complete decomposition of $1_{Sym(a)\wr Sym(b)}\uparrow^G$. It is known that all irreducible representations which are included in $1_{Sym(b)\wr Sym(2)}\uparrow^G$ can be indexed by partitions μ with at most two rows (see Theorem 2.22), thus Foulkes' conjecture for the case when $a = 2$ and b is arbitrary follows straight from the result in [14]. Foulkes' conjecture for the case when $a = 2$ and b is arbitrary can also be confirmed using Thrall's results. In [3] (1989) Black and List came up with the alternative method of proving Foulkes' conjecture by showing that the incidence matrix $M^{a,b}$ of $\mathcal{P}_{a,b}$ has full rank. In [8] (1993) Coker considered the eigenvalues of $M^{2,b}(M^{2,b})^T$ and by showing that these eigenvalues are all non-zero for $2 \leq b$ this gave an alternative proof that all constituents of $1_{Sym(b)\wr Sym(2)}\uparrow^G$ are included in $1_{Sym(2)\wr Sym(b)}\uparrow^G$ for $b \geq 2$. Coker also showed that all constituents of $1_{Sym(b)\wr Sym(3)}\uparrow^G$ are included in $1_{Sym(3)\wr Sym(b)}\uparrow^G$ for $2 \leq b \leq 6$. These results can be confirmed by Littlewood's paper. In [4] (1993) Brion shows that Foulkes' conjecture holds for integers a and b with b 'large enough' with respect to a. [1]

Our results in Chapter 4 give an alternative proof that every constituent of $1_{Sym(b)\wr Sym(2)}\uparrow^G$ is included in $1_{Sym(2)\wr Sym(b)}\uparrow^G$ for all integers $b \geq 2$. Further, the work we have done in Chapters 5 and 6 give an original proof of Foulkes' conjecture for the case when $a = 3$ and b is arbitrary.

[1] R.J. List (unpublished) has shown that Foulkes' conjecture holds for $a = 4$ with $b = 6$ and $a = 3$ with $b = 8$ (pers. comm. R.T. Curtis).

Chapter 2

Incidence Structures and Representation Theory

In this chapter we introduce some basic definitions and properties of incidence structures and matrices. Following this we give some definitions and important results from the representation theory of the symmetric groups. We introduce the definition of a plethysm in order to state Foulkes' conjecture in its original form. In the final section we look at centralizer algebras and the link between the multiplicities of the irreducible components of the permutation character and the dimension of the algebra.

2.1 Incidence Structures

A *(finite) incidence structure* consists of two finite sets $\mathcal{P}$, $\mathcal{B}$ and a subset $\mathcal{I}$ of $\mathcal{P}\times\mathcal{B}$. We will denote the incidence structure by $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$. The members of $\mathcal{P}$ are called *points* and the members of $\mathcal{B}$ are called *blocks*. If the ordered pair (p, B) is in $\mathcal{I}$ we say that p is *incident* with B. We will call the pair $(p, B) \in \mathcal{I}$ a *flag*. Let $\mathcal{S}$ $= (\mathcal{P}, \mathcal{B}, \mathcal{I})$ with $|\mathcal{P}| = s$ and $|\mathcal{B}| = b$. Label the points $\{p_1, p_2, \ldots, p_s\}$ and the blocks $\{B_1, B_2, \ldots, B_b\}$. An *incidence matrix* for $\mathcal{S}$ is the $s \times b$ monomial matrix $M = (m_{ij})$

where

$$m_{ij} = \begin{cases} 1 & \text{if } (p_i, B_j) \in \mathcal{I} \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality let $s \leq b$ (if not relabel the points and blocks) and consider the matrix MM^T which is a square, symmetric matrix, indexed by points. The set of eigenvalues of this matrix forms the *spectrum* of $\mathcal{S}$ denoted by $spec(\mathcal{S}) = spec(MM^T)$. Since MM^T is a symmetric matrix, all of its eigenvalues are non-negative real numbers (to see this consider $\bar{x}^T MM^T x$ for an eigenvector x, where $\bar{x}$ is the vector with all of the entries of x changed to their complex conjugate). As they are eigenvalues of an integer matrix they are algebraic integers and so the spectrum consists of integers or groups of conjugate algebraic integers. If the spectrum of $\mathcal{S}$ does not contain zero, then M has full rank. This fact follows from the following well known result.

Lemma 2.1 *The rank of the product of two matrices A and B cannot exceed the rank of either matrix:*

$$rank(AB) \leq min\{rank(A), rank(B)\}.$$

Lemma 2.2 *For $s \leq b$ the spectrum of $M^T M$ is the spectrum of MM^T with $b - s$ zeros appended.*

Proof: Let $x_1, x_2, \ldots, x_s$ be a set of orthonormal eigenvectors associated to the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_s$ of MM^T. Then

$$\bar{x}_j^T MM^T x_i = \lambda_i \bar{x}_j^T x_i = \lambda_i \delta_{ij}.$$

However, $\bar{x}_j^T MM^T x_i = (\overline{M^T x_j})^T (M^T x_i)$ and so $(\overline{M^T x_i})^T (M^T x_i) = \lambda_i$. Therefore, $M^T x_i$ is zero if and only if λ_i is zero. Since

$$M^T M(M^T x_i) = M^T (MM^T x_i) = \lambda_i (M^T x_i),$$

we see that for $\lambda_i \neq 0$, the vector $M^T x_i$ is an eigenvector of $M^T M$ with eigenvalue λ_i. Thus, the non-zero eigenvalues of MM^T are eigenvalues of $M^T M$. Interchanging the roles of M and M^T shows that the non-zero eigenvalues of M^TM are eigenvalues of MM^T. This completes the proof. □

An incidence structure $\mathcal{S}$ is called *regular* with parameters r and t if each point is incident with r blocks and if each block is incident with t points. A point p is *point connected* to a point p' if there is a sequence $p = p_0, B_1, p_1, B_2, \ldots, p_{c-1}, B_c, p_c = p'$ in which consecutive elements are incident with each other. We call such a sequence a *trail*. The *length* of the trail between p and p' is c, the number of blocks in the sequence. In a similar way we can define a block B being *block connected* to a block B'. If a structure is point and block connected then we will say that it is *connected*. Further we will call the length of the shortest path between two points (or blocks) the *distance* between these points. The *diameter* of $\mathcal{S}$ is the maximum distance between points (or blocks). The following is a standard result which comes from the theorem of Perron and Frobenius (see Section 2 in volume 2 of [10] or page 40 of [6]). It says that a non-negative matrix has an eigenvector with positive components. If λ is the eigenvalue associated with this eigenvector, then $|\lambda^*| \leq \lambda$ for all eigenvalues λ^*. If $\mathcal{S}$ is regular, then MM^T has constant row and column sum. This sum is rt which follows since

$$MM^T(1,1,\ldots,1)^T = M(M^T(1,1,\ldots,1)^T) = tM(1,1,\ldots,1)^T = rt(1,1,\ldots,1)^T.$$

Thus rt is an eigenvalue of MM^T and, by the Perron-Frobenius theorem, this is the maximal eigenvalue. Arrange the remaining values of $spec(\mathcal{S})$ in descending order, so $rt = \lambda_0 > \lambda_1 > \ldots > \lambda_m \geq 0$ and denote by m_i the multiplicity of λ_i. Thus $\sum_{0\leq i\leq m} m_i\lambda_i = rs = bt$ is the trace of MM^T and M^TM. From the Perron-Frobenius theorem it is easy to show that:

Theorem 2.3 *The number of connected components of a regular incidence structure is the multiplicity of the maximal value in its spectrum.*

Corollary 2.4 *Let $\mathcal{S}$ be a connected, regular incidence structure with parameters r and t. Then $\lambda_0 = rt$ is the maximal value in $spec(\mathcal{S})$ and has multiplicity 1. The eigenspace of MM^T for λ_0 is spanned by $(1,1,\ldots,1)^T$.*

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure and let $Aut(\mathcal{S})$ denote the automorphism group of $\mathcal{S}$, defined as follows:

$$Aut(\mathcal{S}) := \{(g, h) \in Sym(\mathcal{P}) \times Sym(\mathcal{B}) \,|\, (p, B) \in \mathcal{I} \Leftrightarrow (pg, Bh) \in \mathcal{I} \,\forall\, p \in \mathcal{P}, B \in \mathcal{B}\}.$$

Thus $Aut(\mathcal{S})$ is the group of permutations of $\mathcal{P}$ and $\mathcal{B}$ which preserve incidence. If $Aut(\mathcal{S})$ acts transitively on flags $(p, B) \in \mathcal{I}$ then the structure is called *flag transitive.* Let $[g]_{\mathcal{P}}$ and $[h]_{\mathcal{B}}$ be the permutation matrices relative to the point and block sets respectively then the condition of being an automorphism is precisely

$$[g]_{\mathcal{P}} M = M[h]_{\mathcal{B}}.$$

From this we have that

$$MM^T[g]_{\mathcal{P}} = [g]_{\mathcal{P}} MM^T. \tag{1}$$

Proposition 2.5 *Let $\mathcal{S}$ be an incidence structure and let H be a subgroup of $Aut(\mathcal{S})$. Then the entries of MM^T are constant on the H-orbits on $\mathcal{P} \times \mathcal{P}$.*

Proof: For p and p^* in $\mathcal{P}$, the (p, p^*)-entry of MM^T is the number of blocks B incident with both p and p^*. This is equal to the number of blocks B such that $(ph_1, Bh_2) \in \mathcal{I}$ and $(p^*h_1, Bh_2) \in \mathcal{I}$ for $(h_1, h_2) \in H$. □

2.2 Representation Theory of Symmetric Groups

2.2.1 Partitions, Tableaux and Tabloids

The definitions given in this section will closely follow the book [13] by G.D. James. Let n be a positive integer and denote by $Sym(n)$ the symmetric group on the set $\{1, 2, \ldots, n\}$ of n elements.

Definition 2.6 $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ is a *partition* of n, written $\mu \vdash n$, if $\mu_1, \mu_2, \ldots, \mu_l$ are non-negative integers with $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_l$ and $\sum_{i=1}^{l} \mu_i = n$.

We will use power notation to abbreviate partitions in which two or more of the μ_i are the same and remove any zeros at the end.

Example 2.7 *We will write the partition* $\mu = (5,3,2,2,2,1,1,0)$ *of* 16 *as* $\mu = (5,3,2^3,1^2)$.

The set of partitions can be partially ordered by the dominance relation:

Definition 2.8 If μ^* and μ are partitions of n, we say that μ^* *dominates* μ and write $\mu^* \trianglerighteq \mu$, provided that for all j

$$\sum_{i=1}^{j} \mu_i^* \geq \sum_{i=1}^{j} \mu_i.$$

If $\mu^* \trianglerighteq \mu$ and $\mu^* \neq \mu$ then we write $\mu^* \triangleright \mu$.

Definition 2.9 If $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ is a partition of n, then the *diagram* $[\mu]$ is $\{(i,j) | i,j \in \mathbb{N},\ 1 \leq i \leq l, 1 \leq j \leq \mu_i\}$. If $(i,j) \in [\mu]$ then (i,j) is called a *node* of $[\mu]$.

We will draw diagrams in the following way. The diagram for $\mu = (4,3,2^2)$ is:

$$[\mu] = \begin{matrix} \times & \times & \times & \times \\ \times & \times & \times & \\ \times & \times & & \\ \times & \times & & \end{matrix} .$$

Definition 2.10 If $[\mu]$ is a diagram for μ, then the *conjugate* diagram $[\mu']$ is obtained by interchanging the rows and columns in $[\mu]$. We call μ' the *conjugate partition.*

Definition 2.11 A *μ-tableau* is a diagram with each node replaced by one of the integers $1, 2, \ldots, n$ with no repeats.

So for each diagram $[\mu]$ with n nodes there are $n!$ different μ-tableaux.

Example 2.12 *Two examples of* $(4,3,2^2)$*-tableaux are:*

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & 9 & & \\ 10 & 11 & & \end{matrix} \quad \textit{and} \quad \begin{matrix} 3 & 9 & 4 & 6 \\ 8 & 7 & 2 & \\ 5 & 11 & & \\ 1 & 10 & & \end{matrix} .$$

The symmetric group $Sym(n)$ acts in the natural way on a tableau t by acting on its entries.

Definition 2.13 Let t be a tableau. Then the *row-stabilizer* R_t of t is the subgroup of $Sym(n)$ fixing all rows set-wise, i.e.

$$R_t := \{g \in Sym(n) | \; \forall\, i, \text{ the elements } i \text{ and } ig \text{ belong to the same row of } t\}.$$

Similarly, the *column-stabilizer* C_t of the tableau t is the subgroup of $Sym(n)$ fixing all columns set-wise, so

$$C_t := \{g \in Sym(n) | \; \forall\, i, \text{ the elements } i \text{ and } ig \text{ belong to the same column of } t\}.$$

Definition 2.14 Define an *equivalence relation* on the set of μ -tableaux by $t_1 \sim t_2$ if and only if $t_1 g = t_2$ for some $g \in R_{t_1}$. The *tabloid* $\{t\}$ containing t is the equivalence class of t under this equivalence relation.

The tabloid $\{t\}$ can therefore be thought of as a tableau with unordered row entries. When writing down tabloids, to distinguish them from tableaux, we will draw lines between the rows.

Example 2.15 *If t is the following $(4,3,2)$-tableau*

$$t = \begin{array}{l} 1\,2\,3\,4 \\ 5\,6\,7 \\ 8\,9 \end{array}$$

then

$$\{t\} = \begin{array}{l} \hline 1\,2\,3\,4 \\ \hline 5\,6\,7 \\ \hline 8\,9 \\ \hline \end{array} = \begin{array}{l} \hline 2\,1\,4\,3 \\ \hline 5\,7\,6 \\ \hline 9\,8 \\ \hline \end{array} = \cdots.$$

2.2.2 The Module $\mathcal{M}^\mu$ and the Specht Module

We continue now with some more definitions and results which can be found in [13]. To each partition μ of n we associate a *Young subgroup* S_μ of $Sym(n)$ by taking

$$S_\mu := Sym(\{1,2,\ldots,\mu_1\}) \times Sym(\{\mu_1+1,\ldots,\mu_1+\mu_2\}) \times \ldots \times Sym(\{n-\mu_l+1,\ldots,n\}).$$

Definition 2.16 Let F be an arbitrary field and let $\mathcal{M}^\mu$ be the vector space over F whose basis elements are the μ-tabloids.

The symmetric group $Sym(n)$ acts transitively on the set of μ-tabloids in the natural way by $\{t\}g = \{tg\}$. This action can be extended linearly on $\mathcal{M}^\mu$ to turn $\mathcal{M}^\mu$ into an $FSym(n)$-module. The stabilizer of a tabloid is S_μ. Thus $\mathcal{M}^\mu$ can be thought of as the permutation module of $Sym(n)$ on the cosets of S_μ. It is a cyclic $FSym(n)$-module, generated by any one tabloid.

Definition 2.17 Let t be a tableau. The *signed column sum* κ_t is the element of the group algebra $FSym(n)$ obtained by summing the elements in the column stabilizer of t, attaching the signature to each permutation. Denoting the sign of a permutation g by $sgn(g)$ which has value 1 if g is an even permutation and -1 otherwise, we can write κ_t as follows:

$$\kappa_t = \sum_{g \in C_t} sgn(g)g.$$

Definition 2.18 The *polytabloid* e_t associated with the tableau t is given by

$$e_t = \{t\}\kappa_t.$$

A polytabloid therefore depends on the tableau t and not just the tabloid $\{t\}$. If $v \in \mathcal{M}^\mu$ then v can be written as a linear combination of tabloids. We will say that the tabloid $\{t\}$ is *involved* in v if its coefficient in v is non-zero.

Definition 2.19 The *Specht module* S^μ for the partition μ of n is the $FSym(n)$-submodule of $\mathcal{M}^\mu$ spanned by the polytabloids e_t, where t is a μ-tableau.

We have that $\kappa_t g = g\kappa_{tg}$, so $e_t g = e_{tg}$. This shows that S^μ is a cyclic module, generated by any one polytabloid.

The following are very important and useful results which can be found in [13].

Theorem 2.20 *For a group G, let S be an irreducible $\mathbb{C}G$-module and U be any $\mathbb{C}G$-module. Then S appears in U with multiplicity equal to $dim(Hom_{\mathbb{C}G}(S,U))$.*

Theorem 2.21 *Over $\mathbb{Q}$ the Specht modules are irreducible and give all the irreducible representations of $Sym(n)$.*

Theorem 2.22 *If char $F = 0$, then the irreducible $FSym(n)$-submodules of $\mathcal{M}^\mu$ are S^μ (once) and modules isomorphic to S^{μ^*} with $\mu^* \rhd \mu$ (possibly with repeats).*

2.2.3 Semistandard Homomorphisms

The results in this section and their proofs can again be found in [13]. We will start by defining tableaux with repeated entries. To distinguish these from tableaux with distinct entries, we will denote them by capital letters.

Definition 2.23 A *tableau T of type* $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ is a diagram with each node replaced by an integer i between 1 and r such that i occurs μ_i times in T.

Remark: The tableaux defined in Definition 2.11 are tableaux of type (1^n). To be consistent with this definition we will always denote tableaux of type (1^n) using lower case letters.

Example 2.24 *The following is a* $(6,3,2)$*-tableau of type* $(6,5)$*:*

$$T = \begin{matrix} 1\,2\,1\,2\,1\,2 \\ 2\,1\,2 \\ 1\,1 \end{matrix}.$$

Let μ and μ^* be partitions of n. Denote by $\mathcal{T}(\mu^*, \mu)$ the set of μ^*-tableaux of type μ. If T is in $\mathcal{T}(\mu^*, \mu)$, then we can define an action of $Sym(n)$ on $\mathcal{T}(\mu^*, \mu)$. We label the place numbers of T according to positions of the numbers $1, 2, \ldots, n$ in t. For $i = 1, \ldots, n$ we let $T(i)$ be the entry in T which occurs in the same position as i occurs in t. Then $Tg(i) = T(ig^{-1})$. We will say that T_1 and T_2 are *row equivalent* if $T_2 = T_1 g$ for some permutation g in the row stabilizer of the given tableau t.

Example 2.25 *If T and t are given respectively by*

$$T = \begin{matrix} 2\,2\,1\,2\,1 \\ 1\,1\,2 \\ 1\,1 \end{matrix} \quad \textit{and} \quad t = \begin{matrix} 1\,4\,7\,9\,10 \\ 2\,5\,8 \\ 3\,6 \end{matrix}$$

then for $g = (1\ 4\ 10)$ *the tableau* Tg *is row equivalent to* T *and is given by:*

$$Tg = \begin{matrix} 1\,2\,1\,2\,2 \\ 1\,1\,2 \\ 1\,1 \end{matrix}\ .$$

Definition 2.26 If $T \in \mathcal{T}(\mu^*, \mu)$, define the map Θ_T by

$$\Theta_T : \{t\} \mapsto \sum\{T_1 | T_1 \text{ is row equivalent to } T\}.$$

We can extend the above definition to general elements of $\mathcal{M}^{\mu^*}$ which will be of the form $\{t\}s$ for $s \in FSym(n)$. We do this in the natural way by mapping $\{t\}$ by Θ_T and letting s act on the result in a linear way. Given a pair T and t such that t is a μ^*-tableau and T is a μ^*-tableau of type μ, we can write down a μ-tabloid in the following way. We label the rows of the tabloid to be constructed $1, 2, \ldots r$ where r is the number of distinct entries in T. Put the entries of t into the rows of the tabloid according to the entries of T, that is put the ij-entry of t into the row number given by the ij-entry of T. It is easy to see that given a μ^*-tableau t there is a one-to-one correspondence between μ^*-tableaux T of type μ and μ-tabloids. The action of $Sym(n)$ on tableaux is well defined, meaning that if T is in correspondence with $\{t^*\}$ then Tg is in correspondence with $\{t^*g\}$ for $g \in Sym(n)$.

Example 2.27 *If* T *and* t *are the following* $(5,3,2)$*-tableaux*

$$T = \begin{matrix} 2\,2\,1\,2\,1 \\ 1\,1\,2 \\ 1\,1 \end{matrix} \quad \textit{and} \quad t = \begin{matrix} 1\,4\,7\,9\,10 \\ 2\,5\,8 \\ 3\,6 \end{matrix}\ ,$$

then the corresponding tabloid is

$$\begin{matrix} \overline{\underline{2\,3\,5\,6\,7\,10}} \\ \underline{1\,4\,8\,9} \end{matrix}\ .$$

The way in which we have defined the map Θ_T means that it is an $FSym(n)$-homomorphism, in other words $(\Theta_T\{t\})g = \Theta_T\{tg\}$. Thus, Θ_T is an $FSym(n)$-homomorphism from $\mathcal{M}^{\mu^*}$ into $\mathcal{M}^{\mu}$. When we write down the image of a tabloid under Θ_T, for a fixed tableau t, we will sometimes choose to express it in terms of the tableaux of type μ and other times it will be easier to express it in terms of the μ-tabloids.

Example 2.28 *If T and t are $(4,1)$-tableaux given by*

$$T = \begin{matrix} 2\,1\,1\,2 \\ 1 \end{matrix} \quad \text{and} \quad t = \begin{matrix} 1\,3\,4\,5 \\ 2 \end{matrix}$$

then we have

$$\Theta_T\{t\} = \begin{matrix} 2\,1\,1\,2 \\ 1 \end{matrix} + \begin{matrix} 2\,1\,2\,1 \\ 1 \end{matrix} + \begin{matrix} 2\,2\,1\,1 \\ 1 \end{matrix} + \begin{matrix} 1\,2\,1\,2 \\ 1 \end{matrix} + \begin{matrix} 1\,2\,2\,1 \\ 1 \end{matrix} + \begin{matrix} 1\,1\,2\,2 \\ 1 \end{matrix}.$$

As an element of $\mathcal{M}^{(3,2)}$ this can be written as

$$\frac{\overline{2\,3\,4}}{\underline{1\,5}} + \frac{\overline{2\,3\,5}}{\underline{1\,4}} + \frac{\overline{2\,4\,5}}{\underline{1\,3}} + \frac{\overline{1\,2\,4}}{\underline{3\,5}} + \frac{\overline{1\,2\,5}}{\underline{3\,4}} + \frac{\overline{1\,2\,3}}{\underline{4\,5}}.$$

Define the restriction of Θ_T to S^{μ^*} by $\hat{\Theta}_T$. We can express the image of a polytabloid e_t under $\hat{\Theta}_T$ in the form

$$\sum_{g_2 \in C_t} \sum_{g_1 \in R_t} sgn(g_2) T g_1 g_2.$$

Remark: It is easy to see that $Tg_1\kappa_t = 0$ (with $g_1 \in R_t$) if and only if some column of Tg_1 contains two identical numbers. Thus we only need to consider those Tg_1 which have distinct entries in each column.

Definition 2.29 A tableau T is *semistandard* if the numbers are non-decreasing along the rows of T and strictly increasing down the columns of T. Let $\mathcal{T}_o(\mu^*, \mu)$ be the set of semistandard tableaux in $\mathcal{T}(\mu^*, \mu)$.

Example 2.30 *If $\mu^* = (4,2,1)$ and $\mu = (3,3,1)$, then $\mathcal{T}_o(\mu^*, \mu)$ consists only of the tableau*

$$\begin{matrix} 1\,1\,1\,2 \\ 2\,2 \\ 3 \end{matrix}.$$

Definition 2.31 The homomorphisms $\hat{\Theta}_T$ with T in $\mathcal{T}_o(\mu^*, \mu)$ are called *semistandard homomorphisms.*

When we are working over a field of characteristic zero the semistandard homomorphisms actually give all $FSym(n)$-homomorphisms from S^{μ^*} to $\mathcal{M}^{\mu}$ and we have the following results which can be found in [13].

Theorem 2.32 *Over a field of characteristic zero,* $\{\hat{\Theta}_T | T \in \mathcal{T}_o(\mu^*, \mu)\}$ *is a basis for* $Hom_{FSym(n)}(S^{\mu^*}, \mathcal{M}^\mu)$.

Corollary 2.33 *Over a field of characteristic zero,* $dim(Hom_{FSym(n)}(S^{\mu^*}, \mathcal{M}^\mu))$ *is the number of semistandard* μ^**-tableaux of type* μ.

2.2.4 Hooks and Dimension of the Specht Module

Definition 2.34 Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition of n. Then the (i,j)-*hook* of the diagram $[\mu]$ consists of the (i,j)-node along with the $\mu_i - j$ nodes to the right of it and the $\mu'_j - i$ nodes below it. The *length* of the (i,j)-hook is $h_{ij} = \mu_i + \mu'_j + 1 - i - j$.

So, the (i,j)-hook is the intersection of a Γ-shape (with (i,j)-node at its corner) with the diagram and the length of the hook is simply the number of nodes in the hook. A very powerful result which can be found in [13] is the following due to Frame, Robinson and Thrall. It gives us the dimension of a given Specht module S^μ simply by taking the product of all of the hook lengths of μ.

Theorem 2.35 (Frame, Robinson, Thrall) *The dimension of the Specht module* S^μ *is given by*

$$dim(S^\mu) = \frac{n!}{\prod(hook\ lengths\ in\ [\mu])}.$$

Example 2.36 *The diagram for* $\mu = (4, 3, 2^2)$ *is*

$$\mu = \begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & \times & \\ \times & \times & & \\ \times & \times & & \end{array},$$

so

$$dim(S^\mu) = \frac{11!}{7.6.3.5.4.3.2.2} = 1320.$$

2.2.5 Plethysms

In this section we look at the permutation representations induced from Young subgroups. We then go on to define the *plethysm* of two representations. The results of this section can be found in Chapters 1 and 5 of [16] or Chapters 2, 4 and 5 of [14].

For a field F and a partition μ of n, we consider two one-dimensional representations of Young subgroups S_μ. The first one is the *identity representation* $1_{S_\mu} : S_\mu \to GL(1,F)$ and the second is the *alternating representation* $A_{S_\mu} : S_\mu \to GL(1,F)$, where σ is mapped to sign of σ times the identity in F:

$$A_{S_\mu} : \sigma \mapsto sgn(\sigma) \cdot 1.$$

We can induce these representations up to the symmetric group $Sym(n)$ obtaining the representations

$$1_{S_\mu} \uparrow^{Sym(n)} \quad \text{and} \quad A_{S_\mu} \uparrow^{Sym(n)} .$$

The following result can be found on page 35 of [14] or on page 173 of [16].

Theorem 2.37 *For each partition μ of n the induced representations $1_{S_\mu} \uparrow^{Sym(n)}$ and $A_{S_{\mu'}} \uparrow^{Sym(n)}$ (where μ' denotes the conjugate partition) have exactly one irreducible representation as a common constituent. They both contain this irreducible constituent with multiplicity one.*

We denote this constituent by $[\mu]$ and extending the use of the intersection symbol we write:

$$[\mu] := 1_{S_\mu} \uparrow^{Sym(n)} \cap A_{S_{\mu'}} \uparrow^{Sym(n)} .$$

When μ is the partition (n) it is clear that $1_{S_{(n)}} \uparrow^{Sym(n)} = 1_{Sym(n)}$, the identity representation. Since μ' is the partition (1^n) we have $A_{S_{(1^n)}} \uparrow^{Sym(n)} = R_{Sym(n)}$, the regular representation. Thus

$$[n] = 1_{S_{(n)}} \uparrow^{Sym(n)} \cap A_{S_{(1^n)}} \uparrow^{Sym(n)} = 1_{Sym(n)} \cap R_{Sym(n)} = 1_{Sym(n)}.$$

The next result follows from Theorem 2.1.11 and Lemma 7.1.4 of [14].

Theorem 2.38 *The set $\{[\mu] \mid \mu \vdash n\}$ is a complete set of irreducible representations of $Sym(n)$ over a field of characteristic zero. Moreover, the Specht module S^μ affords the irreducible representation $[\mu]$ of $Sym(n)$.*

Denote by Y^X the set of all mappings from X into Y:

$$Y^X := \{f \mid f : X \to Y\}.$$

Definition 2.39 Let H and G be groups such that G acts on the set X. Let $H \wr G$ be the group given by

$$H \wr G := H^X \times G = \{(f, g) \mid f : X \to H \text{ and } g \in G\}.$$

$H \wr G$ is the *wreath product* of H and G.

The following lemma (see page 133 of [14] or Lemma 1.2.12 of [16]) gives us a normal subgroup of the wreath product $H \wr G$.

Lemma 2.40 *Let H and G be groups such that G acts on the set X. Then the wreath product $H \wr G$ has the following subgroups:*

1. *The normal subgroup*

$$H^* := \{(\psi, 1) \mid \psi \in H^X\} \trianglelefteq H \wr G,$$

is called the base group *and for $x \in X$ it is isomorphic to a direct product of $|X|$ copies of*

$$H^x := \{(\psi, 1) \mid \psi \in H^X \text{ and } \forall x' \neq x \text{ we have } \psi(x') = 1\} \cong H.$$

2. *The subgroup $G' := \{(\imath, g) \mid g \in G\} \cong G$, where $\imath$ denotes the identity element of H^X, is a* complement *of H^*. Since H^* is a normal subgroup of $H \wr G$ and the intersection of H^* and G' is just the identity element $(\imath, 1)$ of $H \wr G$, the wreath product of H and G is the semidirect product of H^* and G' so*

$$H \wr G = H^* \cdot G'.$$

We consider the case when H is $Sym(m)$ and G is $Sym(n)$. Then $Sym(m) \wr Sym(n)$ embeds naturally into $Sym(mn)$: The map $\delta : Sym(m) \wr Sym(n) \hookrightarrow Sym(mn)$ takes an element of $Sym(m) \wr Sym(n)$ to the permutation given by

$$\delta : (\psi, g) \mapsto \begin{pmatrix} (j-1)m + i \\ (jg-1)m + i(\psi(jg)) \end{pmatrix}, \tag{2}$$

where i and j run through elements of $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$ respectively. It is easy to see that as j runs through $\{1, \ldots, n\}$ and i runs through $\{1, \ldots, m\}$ then $(j-1)m + i$ and $(jg-1)m + i(\psi(jg))$ run through $\{1, 2, \ldots, mn\}$. The base group $Sym(m)^*$ of $Sym(m) \wr Sym(n)$ is isomorphic to the direct product of $Sym(m)^j$ for $j \in \{1, 2, \ldots, n\}$. The image $\delta[Sym(m)^l]$ is the set of permutations which take $(j-1)m + i$ to $(j-1)m + i$ for $j \neq l$ and $(j-1)m + i(\psi j)$ for $j = l$ where $\psi \in H^X$, $j \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, m\}$. Therefore the image $\delta[Sym(m)^j]$ acts on the set $\{(j-1)m + 1, \ldots, jm\}$ as $Sym(m)$ does on $\{1, 2, \ldots, m\}$. The image of the complement $Sym(n)'$ of the base group is the set of permutations which take $(j-1)m + i$ to $(jg-1)m + i$. Thus $\delta[Sym(n)']$ acts on the set of the n subsections $\{(j-1)m + 1, \ldots, jm\}$ of length m of the set $\{1, 2, \ldots, mn\}$ as $Sym(n)$ does on $\{1, 2, \ldots, n\}$. The image of $Sym(m) \wr Sym(n)$ under δ:

$$Sym(m) \odot Sym(n) := \delta[Sym(m) \wr Sym(n)] \tag{3}$$

is called the *plethysm* of $Sym(m)$ and $Sym(n)$. We extend the use of plethysms to representations (see page 272 of [16]) and define a *plethysm* of $[m]$ and $[n]$ to be the representation of $Sym(mn)$ as follows:

$$[m] \odot [n] = 1_{(Sym(m) \odot Sym(n))} \uparrow^{Sym(mn)} . \tag{4}$$

2.2.6 Foulkes' Conjecture

We dedicate this section to a conjecture of Foulkes originally proposed in [9]. Using the same notation as in the previous section, Foulkes conjectured the following (see page 277 of [16] or page 227 of [14]):

If $m \leq n$ then

$$([m] \odot [n], [\mu]) \geq ([n] \odot [m], [\mu]) \tag{5}$$

for all $\mu \vdash mn$. Here $([m]\odot[n],[\mu])$ is the multiplicity of the irreducible representation $[\mu]$ in $[m]\odot[n]$.

The conjecture has been proved for the case when μ is a two-row partition and we have the following result (see page 227 of [14] or page 277 of [16] for the proof):

Theorem 2.41 *For two-rowed diagrams $[\mu]$, where $\mu \vdash mn$, we have:*

$$([n]\odot[m],[\mu]) = ([m]\odot[n],[\mu]).$$

The conjecture of Foulkes in general is still open. We can rewrite Foulkes' conjecture using more standard notation of representation theory in the following way. Denote by $\pi_{m,n} : Sym(mn) \to \mathbb{C}$ the permutation character of $Sym(mn)$ acting on the cosets of $Sym(n) \wr Sym(m)$ in $Sym(mn)$. Then the permutation character $\pi_{m,n}$ can be written as a linear combination of the irreducible characters of $Sym(mn)$:

$$\pi_{m,n} = \sum a_\mu^{m,n}\chi_\mu,$$

where the $a_\mu^{m,n}$ are non-negative integers, χ_μ runs over the irreducible characters of $Sym(mn)$ which can be indexed by partitions μ of mn. From equations (3) and (4) of Section 2.2.5 we see that the representation $[n]\odot[m]$ corresponds to the action of $Sym(mn)$ on the cosets of $Sym(n)\wr Sym(m)$ in $Sym(mn)$. Thus, using Theorem 2.38 Foulkes' Conjecture can be interpreted in the following way:

Foulkes' Conjecture: If we write the permutation characters $\pi_{m,n}$ and $\pi_{n,m}$ as linear combinations of the irreducible characters of $Sym(mn)$,

$$\pi_{m,n} = \sum a_\mu^{m,n}\chi_\mu \text{ and } \pi_{n,m} = \sum a_\mu^{n,m}\chi_\mu,$$

then we have $a_\mu^{m,n} \leq a_\mu^{n,m}$ for $n \geq m$ and for all partitions μ of mn.

Thus Theorem 2.41 can be written in the following way:

Theorem 2.42 *If we write the permutation characters $\pi_{m,n}$ and $\pi_{n,m}$ as linear combinations of the irreducible characters of $Sym(mn)$,*

$$\pi_{m,n} = \sum a_\mu^{m,n}\chi_\mu \text{ and } \pi_{n,m} = \sum a_\mu^{n,m}\chi_\mu,$$

then we have $a_\mu^{m,n} \leq a_\mu^{n,m}$ for $n \geq m$ where μ is any two-rowed partition of mn.

2.3 Centralizer Algebras of Permutation Representations

In this section we introduce the definition of a centralizer algebra and give some results on these algebras (see Chapter 2 of [2] or Chapter 2 of [5]). Let G be a transitive permutation group on a set X of n elements, where as before we will write actions on the right. G acts naturally on $X \times X$, the orbits of which are called *orbitals* and we will denote them by $\Lambda_0, \Lambda_1, \ldots, \Lambda_d$, where $\Lambda_0 = \{(x,x) \mid x \in X\}$. If $\Lambda_i^T = \{(x,y) \mid (y,x) \in \Lambda_i\}$, then Λ_i^T is also an orbit of G and so there exists some j such that $\Lambda_i^T = \Lambda_j$. Denote this index j by i'. Let $\Lambda_i(x) = \{y \in X \mid (x,y) \in \Lambda_i\}$ and $k_i = |\Lambda_i(x)|$. This is independent of the choice of x and so $k_i = k_{i'}$. To each orbital Λ_i we associate an *adjacency matrix* A_i indexed by the elements of X with (x,y)-entry given by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in \Lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

So, in particular, A_0 is the identity matrix I. It is easy to check that the adjacency matrices satisfy the following properties. The sum of the $d+1$ adjacency matrices equals the matrix, J, of all 1's. This shows that the A_i are linearly independent over $\mathbb{C}$. For any i satisfying $0 \leq i \leq d$ the adjacency matrix for $\Lambda_{i'}$ is the transpose of the adjacency matrix for Λ_i. Every column or row of A_i contains exactly k_i ones and for $i, j \in \{0, 1, \ldots, d\}$ we have $tr(A_{j'}A_i) = nk_i\delta_{ij}$ (where δ_{ij} is the Kronecker delta function).

To each $g \in G$ there corresponds a permutation matrix $[g]_X$ where

$$([g]_X)_{xy} = \begin{cases} 1 & \text{if } y = xg, \\ 0 & \text{otherwise.} \end{cases}$$

The *centralizer algebra* of this permutation representation over $\mathbb{C}$ is the algebra $\mathcal{A}$ of all complex matrices A commuting with $[g]_X$ for all $g \in G$. Since

$$\begin{aligned} ([g]_X A [g]_X^{-1})_{xy} &= \textstyle\sum_{u,v} \delta_{xg\, u} A_{uv} \delta_{vg^{-1}\, y} \\ &= A_{xg\, yg}, \end{aligned}$$

A commutes with all $[g]_X$ ($g \in G$) if and only if $A_{xy} = A_{xg\, yg}$ for all $g \in G$ and $x, y \in X$. In particular, each matrix A_i commutes with all $[g]_X$, for $g \in G$. Since $\sum_i A_i = J$, every matrix which commutes with all $[g]_X$ is a $\mathbb{C}$-linear combination of $A_0, A_1, \ldots, A_d$. So in particular the A_i are a basis for $\mathcal{A}$ and $dim_{\mathbb{C}}\mathcal{A} = d + 1$.

Let $\pi : G \to \mathbb{C}$ be the permutation character of G on X. For $1 \leq i \leq r$ let χ_i be the irreducible characters of G over $\mathbb{C}$ so that $\pi = 1_G + e_1\chi_1 + \cdots + e_r\chi_r$ is the decomposition of the permutation character. The following result follows from the well known Schur's lemma and its proof can be found on page 49 of [2].

Theorem 2.43 *Let G be a transitive permutation group on X and let π be the permutation character. Let $e_0 = 1, e_1, \ldots, e_r$ be the multiplicities of the irreducible components of π and $\mathcal{A}$ the centralizer algebra of the permutation representation. Then*

1. *$dim_{\mathbb{C}}\mathcal{A} = \sum_{i=0}^{r} e_i^2 = d + 1$ where $d + 1$ is the number of orbits of G_x on X for $x \in X$.*

2. *$\mathcal{A}$ is commutative if and only if $e_i = 1$ for all $i \in \{0, 1, \ldots, r\}$.*

Lemma 2.44 *If $i' = i$ for all $i = 0, 1, \ldots, d$ then $\mathcal{A}$ is commutative. The condition $i = i'$ holds if and only if for any $(x, y) \in \Lambda_i$ there exists $g \in G$ such that $xg = y$ and $yg = x$.*

Proof: If $i = i'$ then each element of $\mathcal{A}$ is a symmetric matrix since $A_i^T = A_{i'} = A_i$ holds for the basis $A_0, A_1, \ldots, A_d$ of $\mathcal{A}$. In particular A_iA_j is symmetric for all i, j. So

$$A_jA_i = A_j^T A_i^T = (A_iA_j)^T = A_iA_j.$$

Therefore $\mathcal{A}$ is commutative. The rest of the lemma is trivial by the definition of A_i. □

When the matrices A_i of the centralizer algebra are symmetric, the above lemma tells us that these matrices commute pairwise. It is well known that symmetric, pairwise commutative matrices can be simultaneously diagonalized, i.e. there is a matrix S such that for all $A \in \mathcal{A}$ there is a diagonal matrix D_A such that:

$$S^{-1}AS = D_A.$$

Chapter 3

An Incidence Structure of Partitions

Certain *unordered partitions* of the set $\{1, 2, \ldots, ab\}$ are the main topic of this thesis. We start by introducing these partitions and define an incidence relation between two particular kinds of partition. Some results are given about the regularity and connectedness of this structure. We describe a way of working out the number of $Sym(ab)$-orbits on $\mathcal{P} \times \mathcal{P}$, where $\mathcal{P}$ is the set of points of the structure. We consider the permutation module of $Sym(ab)$ acting on the points of the structure over the field of complex numbers, so unless otherwise stated let $F = \mathbb{C}$. Using the representation theory of symmetric groups and in particular semistandard homomorphisms, we determine some of the irreducible modules in the decomposition of this permutation module. Denoting the incidence matrix of the structure by M, we use our knowledge of the irreducible modules in the decomposition of the permutation module to construct some eigenvectors of MM^T.

3.1 Preliminaries and Notation

3.1.1 Ordered and Unordered Partitions

Let a and b be positive integers, $n = ab$ and $\Omega = \{1, 2, \ldots, n\}$. Let the set of *unordered (a, b)-partitions* of Ω into a subsets of cardinality b be given by

$$P(a,b) = \{\{\Delta_1, \Delta_2, \ldots, \Delta_a\} \mid \Delta_i \subseteq \Omega,\ \Delta_i \cap \Delta_j = \emptyset \text{ if } i \neq j,\ |\Delta_i| = b\}.$$

The set of *ordered (a, b)-partitions* of Ω into a subsets of cardinality b is

$$P^o(a,b) = \{(\Delta_1, \Delta_2, \ldots, \Delta_a) \mid \Delta_i \subseteq \Omega,\ \Delta_i \cap \Delta_j = \emptyset \text{ if } i \neq j,\ |\Delta_i| = b\}.$$

So for example, when $n = 6$ the partitions $(\{1,2,3\}, \{4,5,6\})$ and $(\{4,5,6\}, \{1,2,3\})$ are the same in $P(2,3)$ but different in $P^o(2,3)$. It is easy to see that there is a one-to-one correspondence between ordered (a,b)-partitions $P^o(a,b)$ and the (b^a)-tabloids introduced in Section 2.2.1, so we will not distinguish between them. Moreover, it is straightforward to check that

$$|P(a,b)| = \binom{n}{b}\binom{n-b}{b}\cdots\binom{b}{b}\frac{1}{a!} = \frac{n!}{(b!)^a a!}$$

and

$$|P^o(a,b)| = a!\,|P(a,b)|.$$

In most of the following we will be concerned with unordered (a,b)-partitions, which we will simply refer to as (a,b)-partitions. We will refer to the subsets in a partition as *parts* and in general we will use lower case Greek letters to denote unordered (a,b)-partitions and tabloid notation or Greek letters with superscript 'o' to denote ordered (a,b)-partitions. Let $d(a,b) = |P(a,b)|$. If a and b satisfy $1 \leq a \leq b$ then

$$\begin{aligned}
(b!)^{a-1} &= [b(b-1)\ldots(a+1)]^{a-1}(a!)^{a-1} \\
&= (b)^{a-1}(b-1)^{a-1}\ldots(a+1)^{a-1}(a!)^{a-1} \\
&\geq (a!)^{b-a}(a!)^{a-1} \quad (\text{since } (d)^{a-1} \geq a!\ \forall\, d \geq a) \\
&= (a!)^{b-1}.
\end{aligned}$$

Thus

$$\frac{n!}{(b!)^{a-1}} \le \frac{n!}{(a!)^{b-1}}$$

and so

$$d(a,b) \le d(b,a).$$

We will abuse notation slightly and write $\alpha = \dot\bigcup_{i=1}^{a} \alpha_i$ and $\beta = \dot\bigcup_{i=1}^{b} \beta_i$ for (a,b)-partition and (b,a)-partitions respectively. So the α_i are disjoint subsets of Ω of cardinality b and the β_i are disjoint subsets of Ω of cardinality a.

3.1.2 An Incidence Structure

We will define an incidence structure between (a,b)-partitions and (b,a)-partitions, denoting the incidence matrix of the structure by M. We will show that the structure is regular and connected which, using results of Section 2.1, gives us the maximum eigenvalue of MM^T. Throughout the rest of this chapter, G will always denote the symmetric group $Sym(ab)$ on the set $\{1,2,\ldots,ab\}$.

We say that α and β, as defined above, *intersect nicely* if each part of β has exactly one element in common with each part of α, that is $|\alpha_r \cap \beta_s| = 1$ for all r,s satisfying $1 \le r \le a$, $1 \le s \le b$. For $a \le b$, we visualize elements of $P(a,b)$ as 'row vectors' and elements of $P(b,a)$ as 'column vectors' so

$$\alpha = \left.\begin{matrix} (*, & *, & \cdots, & *) \\ (*, & *, & \cdots, & *) \\ \vdots & \vdots & & \vdots \\ (*, & *, & \cdots, & *) \end{matrix}\right\}_{a\ rows} \in P(a,b),$$

$$\beta = \underbrace{\begin{pmatrix} * \\ * \\ \vdots \\ * \end{pmatrix}\begin{pmatrix} * \\ * \\ \vdots \\ * \end{pmatrix} \cdots \begin{pmatrix} * \\ * \\ \vdots \\ * \end{pmatrix}}_{b\ columns} \in P(b,a).$$

Writing the elements in this way means that if $\alpha \in P(a,b)$ and $\beta \in P(b,a)$ intersect nicely then we can write Ω as an $a \times b$ array such that its rows are the parts of α and its columns are the parts of β. We call this the *array form of* Ω *with respect to* α *and* β and denote it by $\Omega_{\alpha,\beta}$.

Example 3.1 *Let $a = 3, b = 4$, (so $n = 12$) and*

$$\alpha = \begin{matrix}(1, & 2, & 3, & 4)\\(5, & 6, & 7, & 8)\\(9, & 10, & 11, & 12)\end{matrix} \in P(3,4).$$

Two examples of $(4,3)$-partitions which intersect nicely with α are

$$\begin{pmatrix}1\\5\\9\end{pmatrix}\begin{pmatrix}2\\6\\10\end{pmatrix}\begin{pmatrix}3\\7\\11\end{pmatrix}\begin{pmatrix}4\\8\\12\end{pmatrix} \quad \textit{and} \quad \begin{pmatrix}1\\6\\12\end{pmatrix}\begin{pmatrix}3\\8\\9\end{pmatrix}\begin{pmatrix}2\\11\\5\end{pmatrix}\begin{pmatrix}10\\4\\7\end{pmatrix}$$

while

$$\begin{pmatrix}1\\5\\9\end{pmatrix}\begin{pmatrix}2\\3\\10\end{pmatrix}\begin{pmatrix}6\\7\\11\end{pmatrix}\begin{pmatrix}4\\8\\12\end{pmatrix}$$

does not intersect nicely with α.

Consider the incidence structure $\mathcal{P}_{a,b} = (P(a,b), P(b,a); \mathcal{I})$, where $(\alpha, \beta) \in \mathcal{I}$ if and only if α intersects nicely with β. Let $M^{a,b}$ denote the $d(a,b) \times d(b,a)$ incidence matrix of $\mathcal{P}_{a,b}$. When the context is clear we will omit the superscripts and just write M.

G acts transitively on the point and block sets of this structure by acting on the elements in each part of the partition in the natural way. Given an (a,b)-partition α and a (b,a)-partition β and any $g \in G$, it is easy to see that α intersects nicely with β if and only if αg intersects nicely with βg. The structure is flag transitive. This is seen by letting α be an (a,b)-partition which intersects nicely with the (b,a)-partition β and considering the array form $\Omega_{\alpha,\beta}$ of Ω with respect to α and β. Doing the same thing for any other pair of partitions γ and ν in $P(a,b)$ and $P(b,a)$ respectively which intersect nicely and mapping the ij-entry of $\Omega_{\alpha,\beta}$ to the ij-entry of $\Omega_{\gamma,\nu}$ by an element g of G shows that g maps α to γ and at the same time β to ν.

Proposition 3.2 *The incidence structure $\mathcal{P}_{a,b}$ is regular with $r = (b!)^{a-1}$ blocks per point and $t = (a!)^{b-1}$ points per block.*

Proof: Let α be an (a,b)-partition. Then the number of (b,a)-partitions β which intersect nicely with α is the number of ways we can choose b disjoint sets which contain exactly one element from each part of α. Since our partitions are unordered, we first assign each element in one of the parts of α to the different parts of β and then fill the parts of β in all possible ways so that any one part of α has its elements in different parts of β. Thus a 'point' is incident with $(b!)^{a-1}$ 'blocks'. Similarly a 'block' is incident with $(a!)^{b-1}$ 'points'. □

Thus the row sum and column sum of M are constant and equal $(b!)^{a-1}$ and $(a!)^{b-1}$ respectively. Therefore the trace of the square symmetric matrix MM^T (which is the same as the trace of M^TM) equals $n!/(b!\,a!)$ (see Section 2.1). By Proposition 2.5 the entries of the square symmetric matrix MM^T are constant on the G-orbits on $P(a,b) \times P(a,b)$.

Proposition 3.3 *The incidence structure $\mathcal{P}_{a,b}$ is connected.*

Proof: Let α and γ be any two (a,b)-partitions. Fix the parts of α and γ in some arbitrary way. Consider the array form $\Omega_{\alpha,\beta}$ of Ω with respect to α and β where β is any (b,a)-partition which intersects nicely with α. Similarly consider the array form $\Omega_{\gamma,\nu}$ of Ω with respect to γ and ν where ν is any (b,a)-partition which intersects nicely with γ. The (a,b)-partitions which intersect nicely with β are the rows of $(\Omega_{\alpha,\beta})g$, where g is an element of $Sym(n)$ which set-wise fixes the columns of $\Omega_{\alpha,\beta}$. The (b,a)-partitions which intersect nicely with αg are the columns of $(\Omega_{\alpha,\beta})gh$, where h is an element of $Sym(n)$ which set-wise fixes the rows of $(\Omega_{\alpha,\beta})g$. Thus to show the structure is point connected, we need to show that we can alternately permute the elements in the rows and columns of $\Omega_{\alpha,\beta}$ until we get to $\Omega_{\gamma,\nu}$. An easy construction is to locate the digits in $\Omega_{\alpha,\beta}$ which make up the top row of $\Omega_{\gamma,\nu}$. Permute the elements in the rows of $\Omega_{\alpha,\beta}$ in an arbitrary way so that the 'located digits' are all in different columns. Next, permute the elements in the columns so that all the 'located digits' are in the top row. Denote the resulting array by $\Omega'_{\alpha,\beta}$. Repeat the above steps for the second row of $\Omega_{\gamma,\nu}$, leaving the top row unchanged. Continuing in this way means that we will eventually arrive at $\Omega_{\gamma,\nu}$. We repeat this method after interchanging the roles of points and blocks to show that the structure is block connected. □

Remark: The method used in the proof of the above proposition gives a path between any two points or blocks, but it may not be the shortest path. To find the shortest path or diameter of the structure is not such an easy task. We show that the diameters of $\mathcal{P}_{2,k}$ and $\mathcal{P}_{3,k}$ are both one in Chapters 4 and 5 respectively.

Example 3.4 *Let $\alpha, \gamma \in P(a,b)$ be given by*

$$\alpha = \begin{matrix} (1, & 2, & 3, & 4) \\ (5, & 6, & 7, & 8) \\ (9, & 10, & 11, & 12) \end{matrix} \quad \text{and} \quad \gamma = \begin{matrix} (4, & 8, & 10, & 11) \\ (1, & 5, & 7, & 9) \\ (2, & 3, & 6, & 12) \end{matrix} .$$

Then $\Omega_{\alpha,\beta}$ and $\Omega_{\gamma,\nu}$ can be written, for example, as

$$\Omega_{\alpha,\beta} = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{matrix} \quad \text{and} \quad \Omega_{\gamma,\nu} = \begin{matrix} 4 & 8 & 10 & 11 \\ 1 & 5 & 7 & 9 \\ 2 & 3 & 6 & 12 \end{matrix}$$

where of course β is the $(4,3)$-partition with its parts equal to the columns of $\Omega_{\alpha,\beta}$ and ν is the $(4,3)$-partition with its parts equal to the columns of $\Omega_{\gamma,\nu}$. Then we have a path from α to γ given by

$$\begin{matrix} 1 & 2 & 3 & \mathbf{4} \\ 5 & 6 & 7 & \mathbf{8} \\ 9 & \mathbf{10} & \mathbf{11} & 12 \end{matrix} \rightsquigarrow \begin{matrix} 1 & 2 & 3 & \mathbf{4} \\ 5 & 6 & \mathbf{8} & 7 \\ \mathbf{11} & \mathbf{10} & 9 & 12 \end{matrix} \rightsquigarrow \begin{matrix} \mathbf{11} & \mathbf{10} & \mathbf{8} & \mathbf{4} \\ \mathbf{5} & 6 & 3 & \mathbf{7} \\ \mathbf{1} & 2 & \mathbf{9} & 12 \end{matrix} \rightsquigarrow$$

$$\begin{matrix} 4 & 8 & 10 & 11 \\ \mathbf{5} & 6 & 3 & \mathbf{7} \\ 2 & \mathbf{1} & \mathbf{9} & 12 \end{matrix} \rightsquigarrow \begin{matrix} 4 & 8 & 10 & 11 \\ \mathbf{5} & \mathbf{1} & \mathbf{9} & \mathbf{7} \\ 2 & 6 & 3 & 12 \end{matrix} \rightsquigarrow \begin{matrix} 4 & 8 & 10 & 11 \\ 1 & 5 & 7 & 9 \\ 2 & 3 & 6 & 12 \end{matrix} .$$

Thus we have (cf. Corollary 2.4).

Corollary 3.5 *The maximum value of $spec(MM^T)$ is $(a!)^{b-1}(b!)^{a-1}$. This eigenvalue has multiplicity one and the eigenspace is spanned by $(1, 1, \ldots, 1)^T$.*

The eigenvalues of $M^{a,b}(M^{a,b})^T$ are not known in general. In Theorem 4.14 we will give the complete set of eigenvalues for $M^{2,k}(M^{2,k})^T$ and in Theorem 6.12 we will give some of the eigenvalues of $M^{3,k}(M^{3,k})^T$, where k is arbitrary.

3.1.3 Foulkes' Conjecture: An Alternative Approach

In the following we will explain how computing the eigenvalues of our matrix MM^T can be used to prove the conjecture due to Foulkes which we discussed in Section 2.2.6.

We have seen that $\mathcal{P}_{a,b}$ is a regular, transitive incidence structure. The stabilizer of an (a, b)-partition α is the set of elements in G (where as usual $G = Sym(ab)$) which stabilize set-wise the elements in the parts of α and at the same time may permute the parts of α. Thus the stabilizer in G of α can be written in the form

$$G_\alpha = \{(f, \sigma) | f : \{1, 2, \ldots, a\} \to Sym(b),\ \sigma \in Sym(a)\} \cong Sym(b) \wr Sym(a).$$

By the orbit-stabilizer theorem, the G-sets $(G, P(a, b))$ and $(G, P(b, a))$ are G-equivalent to $cos(G : G_\alpha)$ and $cos(G : G_\beta)$ respectively, where $\alpha \in P(a, b)$ and $\beta \in P(b, a)$. Let $FP(a, b)$ and $FP(b, a)$ be the permutation modules of G acting on the sets $P(a, b)$ and $P(b, a)$ respectively. Thus to prove Foulkes' conjecture we need to show that in the complete decompositions $FP(a, b) = c_1X_1 + c_2X_2 + \cdots + c_rX_r$ and $FP(b, a) = d_1X_1 + d_2X_2 + \cdots + d_rX_r$ of $FP(a, b)$ and $FP(b, a)$ respectively, we have $d_i \geq c_i$ for all i. In [3] Black and List describe the following approach to solve Foulkes' conjecture which was also suggested in [27]. Let $M^{a,b} = (m_{\alpha\beta})$ be the incidence matrix of $\mathcal{P}_{a,b}$ and let $\{v_\alpha \mid \alpha \in P(a, b)\}$ and $\{w_\beta \mid \beta \in P(b, a)\}$ be the natural bases for $FP(a, b)$ and $FP(b, a)$ respectively. Then the map ϕ from $FP(a, b)$ to $FP(b, a)$ given by

$$\phi(v_\alpha) = \sum_{\beta \in P(b,a)} m_{\alpha\beta} w_\beta$$

is an FG-homomorphism since

$$\phi(v_\alpha g) = \phi(v_{\alpha g}) = \sum_{\beta \in P(b,a)} m_{\alpha g\, \beta} w_\beta = \sum_{\beta \in P(b,a)} m_{\alpha\, \beta g^{-1}} w_\beta$$

$$= \sum_{\beta \in P(b,a)} m_{\alpha\beta} w_{\beta g} = \sum_{\beta \in P(b,a)} m_{\alpha\beta} w_\beta g = (\phi(v_\alpha))g$$

for all $g \in G$. If we show that $M^{a,b}$ has full rank (that is, $rank(M^{a,b}) = d(a, b)$) then we have an injective FG-homomorphism from $FP(a, b)$ to $FP(b, a)$, or in other words $d_i \geq c_i$ for all i. Thus showing that $M^{a,b}$ has full rank for all integers $b \geq a \geq 1$ proves Foulkes' conjecture.

Remark: If we show for some particular values of a and b satisfying $a \leq b$ that the matrix $M^{a,b}$ does not have full rank then this does not disprove Foulkes' conjecture.

When $a = 2$ and b is arbitrary we will use this alternative way of proving Foulkes' conjecture (see Section 4.4). When $a = 3$ with b arbitrary we show how it is no longer a straightforward process to compute the complete set of eigenvalues of $M^{3,b}(M^{3,b})^T$ but instead we give a direct proof of the conjecture (see Section's 6.1 and 6.2).

3.2 The Modules $FP^o(a, b)$ and $FP(a, b)$

We study in more details the permutation modules $FP^o(a, b)$ and $FP(a, b)$. We show that $FP(a, b)$ is isomorphic to a FG-submodule of $FP^o(a, b)$. Since we know what irreducible FG-modules in the decomposition of $FP^o(a, b)$ look like, we can use them to find some of the irreducible FG-modules in the decomposition of $FP(a, b)$.

3.2.1 Modules in the Decomposition of $FP^o(a, b)$

The module $FP^o(a, b)$ is clearly isomorphic to the module $\mathcal{M}^{(b^a)}$ defined in Section 2.2.2. From Theorems 2.21 and 2.22 we know that over a field of characteristic zero the Specht modules S^μ form a complete set of irreducible modules of $\mathcal{M}^{\mu^*}$, where μ and μ^* are partitions of n satisfying $\mu \trianglerighteq \mu^*$. By Theorem 2.32 we know that the semistandard homomorphisms $\hat{\Theta}_T$, where T is a semistandard μ-tableau of type μ^*, form a basis for $Hom_{FG}(S^\mu, \mathcal{M}^{\mu^*})$. Moreover, by Theorem 2.20 the multiplicity of S^μ in the decomposition of $\mathcal{M}^{\mu^*}$ equals the number of semistandard μ-tableaux of type μ^*. In particular, the partition (b^a) is dominated by all partitions of n with not more than b parts. Thus if μ has no more than b parts then S^μ appears in $FP^o(a, b)$ with multiplicity equal to the number of semistandard μ-tableaux of type (b^a). Therefore, the irreducible modules in the decomposition of $FP^o(a, b)$ are images of certain Specht modules under linear combinations of these semistandard homomorphisms.

Define an equivalence relation on tabloids in $\mathcal{M}^{(b^a)}$ by $\{t^*\} \sim \{t'\}$ if and only if $\{t'\}$ can be obtained from $\{t^*\}$ by permuting its rows. For example, the tabloids given by

$$\begin{array}{c}\hline 1\ 2\ \ldots\ldots\ldots\ b \\ \hline b+1\ \ldots\ldots\ 2b \\ \hline 2b+1\ \ldots\ldots\ 3b \\ \hline \vdots \\ \hline (a-1)b+1\ldots ab \\ \hline \end{array} \quad \text{and} \quad \begin{array}{c}\hline b+1\ \ldots\ldots\ 2b \\ \hline 1\ 2\ \ldots\ldots\ldots\ b \\ \hline 2b+1\ \ldots\ldots\ 3b \\ \hline \vdots \\ \hline (a-1)b+1\ldots ab \\ \hline \end{array}$$

are $\sim$-equivalent. Thus if $\alpha \in P(a,b)$ and $\{t^*\}$ is a tabloid which has the parts of α as its rows, then α can be thought of as a *representative* of the $\sim$-equivalence class which contains $\{t^*\}$. We will denote by $\{\bar{t}^*\}$ a representative of the $\sim$-equivalence class which contains $\{t^*\}$. It is clear that each $\sim$-equivalence class will contain $a!$ elements. Since G acts transitively on (a,b)-partitions, for any fixed (b^a)-tableau t' the FG-module $FP(a,b)$ is isomorphic to the cyclic FG-submodule of $FP^o(a,b)$ spanned by the element

$$\sum_{\{t^*\}\sim\{t'\}} \{t^*\}.$$

We showed in Section 2.2.3 that given a partition μ of n and a μ-tableau t, there is a one-to-one correspondence between a (b^a)-tabloid $\{t^*\}$ and a tableau T of type (b^a).

Definition 3.6 Let T and T^* be μ-tableaux of type (b^a) and let t be a μ-tableau. For $i = 1, \ldots, a$ let Δ_i^T be the set which contains the entries of t that occur in the same position as i does in T. We will say that T and T^* have the *same pattern* if for all i satisfying $1 \leq i \leq a$ there exists $j \in \{1, 2, \ldots, a\}$ such that $\Delta_i^T = \Delta_j^{T^*}$.

For example, T and T^* given by

$$T = \begin{array}{l} 1\ 1\ 1\ 2\ 3 \\ 2\ 2\ 3 \\ 3 \end{array} \quad \text{and} \quad T^* = \begin{array}{l} 2\ 2\ 2\ 1\ 3 \\ 1\ 1\ 3 \\ 3 \end{array}$$

have the same pattern. It is easy to see that if T is a tableau of type (b^a) then there are $a!$ tableaux with the same pattern as T.

Remark: If T and T^* have the same pattern then they will always have the same pattern regardless of the choice of the tableau t which just acts as a labelling set for the positions of the digits in T.

Therefore this equivalence relation induces a relation on tableaux of type (b^a) in the following way. If μ is a partition of n and t is a μ-tableau then the μ-tableaux T and T^* of type (b^a) are $\sim$-equivalent if and only if T and T^* have the same pattern. We will denote by $\bar{T}$ a representative for the $\sim$-equivalence class containing T.

3.2.2 Modules in the Decomposition of $FP(a, b)$

We have shown that $FP(a, b)$ is isomorphic to a FG-submodule of $FP^o(a, b)$. Since we know what the irreducible FG-modules in the decomposition of $FP^o(a, b)$ look like, we can use them to find some of the irreducible FG-modules in the decomposition of $FP(a, b)$. In particular, FG-modules which appear in $FP(a, b)$ will be isomorphic to certain Specht modules. We use the semistandard homomorphisms to construct FG-homomorphisms from S^μ to $FP(a, b)$. Using an inductive proof we will show that for every $FSym(a(b-2))$-module which appears in $FP(a, b-2)$ with multiplicity m, we can construct an FG-module which appears in $FP(a, b)$ which will also have multiplicity m. In a similar way we will show that some of the $FSym(a(b-1))$-modules which appear in $FP(a, b-1)$ can be used to construct irreducible FG-modules in the decomposition of $FP(a, b)$ again with the same multiplicity. In general, the results we obtain only partially decompose the module $FP(a, b)$. However when $a = 2$ and $a = 3$ with b arbitrary we can decompose the modules $FP(2, b)$ and $FP(3, b)$ completely (see Chapters 4 and 5 respectively). In the remainder of this work, when we talk about modules, submodules and homomorphisms we will always mean FG-modules, FG-submodules and FG-homomorphisms where throughout $G = Sym(ab)$.

Define a linear map $^-$: $FP^o(a, b) \to FP(a, b)$ which maps $\{t^*\}$ to a representative $\{\bar{t}^*\}$ of its $\sim$-equivalence class. Or equivalently, for a fixed μ-tableau t the map which takes the semistandard μ-tableau T of type (b^a) to a representative $\bar{T}$ of its $\sim$-equivalence class. We will denote by $\bar{\Theta}_T$ the compositions of the maps $^-$ and $\hat{\Theta}_T$ from S^μ to $FP(a, b)$. It is easy to verify that $^-$ is invariant under G or in other words $^-(\{t\}g) = (\{\bar{t}\})g$ for $g \in G$. Thus $\bar{\Theta}_T$ is a homomorphism and an element involved in $\bar{\Theta}_T\{t\}\kappa_t$ can be written in the form $\bar{T}g_1g_2$ for $g_1 \in R_t$ and $g_2 \in C_t$ (see Section 2.2.3).

We are interested in the Specht modules which appear in $FP(a,b)$. Therefore for the different semistandard μ-tableaux T of type (b^a) we consider the image of S^μ under $\bar{\Theta}_T$. As S^μ is a cyclic module generated by any one polytabloid and $\bar{\Theta}_T$ is a homomorphism, we only need to consider the image of a fixed polytabloid. Unless otherwise stated we will always choose t to be the μ-tableau with the digits $1, 2, \ldots, n$ placed in increasing order down the columns of t. For example, when $\mu = (3,2,1)$ we choose t to be the tableau

$$t = \begin{matrix} 1\,4\,6 \\ 2\,5 \\ 3 \end{matrix} .$$

In the remainder of this work T will always denote a semistandard tableau of type (b^a), that is a tableau in which each of the integers $1, 2, \ldots, a$ occur b times. It is straightforward to write down the image of a tabloid $\{t\}$ under $\hat{\Theta}_T$ in terms of tableaux of type (b^a). Moreover, there is an easy way to write down the image of a tabloid $\{t\}$ under $\hat{\Theta}_T$ directly in terms of tabloids. We do this by mapping $\{t\}$ to the sum of all tabloids $\{t^*\}$ such that the number of digits from the ith row of t which are in the jth row of $\{t^*\}$ equals the number of j's in the ith row of T. Since $\hat{\Theta}_T(\{t\}\kappa_t) = (\hat{\Theta}_T\{t\})\kappa_t$, we apply κ_t to the result. Taking $\bar{\ }$ of this gives us the image of the polytabloid $\{t\}\kappa_t$ under $\bar{\Theta}_T$. To help understand this construction we give an example.

Example 3.7 *Let T and t be given by*

$$T = \begin{matrix} 1\,1\;1\,2 \\ 2\,2 \end{matrix} \quad \textit{and } t = \begin{matrix} 1\,3\,5\,6 \\ 2\,4 \end{matrix} .$$

Then

$$\begin{aligned} \hat{\Theta}_T\{t\} &= \begin{matrix} 1\,1\;1\,2 \\ 2\,2 \end{matrix} + \begin{matrix} 1\,1\;2\,1 \\ 2\,2 \end{matrix} + \begin{matrix} 1\,2\;1\,1 \\ 2\,2 \end{matrix} + \begin{matrix} 2\,1\;1\,1 \\ 2\,2 \end{matrix} \\ &= \frac{\overline{1\,3\,5}}{\underline{2\,4\,6}} + \frac{\overline{1\,3\,6}}{\underline{2\,4\,5}} + \frac{\overline{1\,5\,6}}{\underline{2\,4\,3}} + \frac{\overline{3\,5\,6}}{\underline{2\,4\,1}} \end{aligned}$$

and so

$$\begin{aligned}\hat{\Theta}_T\{t\}\kappa_t &= \frac{\overline{1\,3\,5}}{\underline{2\,4\,6}} + \frac{\overline{1\,3\,6}}{\underline{2\,4\,5}} + \frac{\overline{1\,5\,6}}{\underline{2\,4\,3}} + \frac{\overline{3\,5\,6}}{\underline{2\,4\,1}} - \frac{\overline{2\,3\,5}}{\underline{1\,4\,6}} - \frac{\overline{2\,3\,6}}{\underline{1\,4\,5}} - \frac{\overline{2\,5\,6}}{\underline{1\,4\,3}} - \frac{\overline{3\,5\,6}}{\underline{1\,4\,2}} \\ &\quad + \frac{\overline{2\,4\,5}}{\underline{1\,3\,6}} + \frac{\overline{2\,4\,6}}{\underline{1\,3\,5}} + \frac{\overline{2\,5\,6}}{\underline{1\,3\,4}} + \frac{\overline{4\,5\,6}}{\underline{1\,3\,2}} - \frac{\overline{1\,4\,5}}{\underline{2\,3\,6}} - \frac{\overline{1\,4\,6}}{\underline{2\,3\,5}} - \frac{\overline{1\,5\,6}}{\underline{2\,3\,4}} - \frac{\overline{4\,5\,6}}{\underline{2\,3\,1}} \\ &= \frac{\overline{1\,3\,5}}{\underline{2\,4\,6}} + \frac{\overline{1\,3\,6}}{\underline{2\,4\,5}} - \frac{\overline{2\,3\,5}}{\underline{1\,4\,6}} - \frac{\overline{2\,3\,6}}{\underline{1\,4\,5}} + \frac{\overline{2\,4\,5}}{\underline{1\,3\,6}} + \frac{\overline{2\,4\,6}}{\underline{1\,3\,5}} - \frac{\overline{1\,4\,5}}{\underline{2\,3\,6}} - \frac{\overline{1\,4\,6}}{\underline{2\,3\,5}}.\end{aligned}$$

Writing this as an element of $FP(a,b)$ we get

$$\bar{\Theta}_T\{t\}\kappa_t = 2\genfrac{}{}{0pt}{}{(1\,3\,5)}{(2\,4\,6)} + 2\genfrac{}{}{0pt}{}{(1\,3\,6)}{(2\,4\,5)} - 2\genfrac{}{}{0pt}{}{(2\,3\,5)}{(1\,4\,6)} - 2\genfrac{}{}{0pt}{}{(2\,3\,6)}{(1\,4\,5)}.$$

The remark following Example 2.28 tells us that if some column of T^* contains two identical numbers (where T^* is a tableau row equivalent to T) then $T^*\kappa_t = 0$. Thus, we can cut out some unnecessary work by only looking at those tableaux row equivalent to T which have distinct entries in each column.

We need to introduce some notation which will be useful when writing down general elements of $FP^o(a,b)$ and $FP(a,b)$. We will say that α is *involved* in $x \in FP(a,b)$ (or $FP^o(a,b)$) if α has non-zero coefficient in x. It will often not be convenient to write down every element explicitly (for example, when writing down the elements involved in $\hat{\Theta}_T\{t\}\kappa_t$), but instead we will use 'union' notation to help abbreviate. If

$$\{t\} = \begin{array}{ccc}\hline 1 & 2 & 3 \\ \hline 6 & 7 & 8 \\ \hline 11 & 12 & 13 \\ \hline\end{array} \text{ and } \{t^*\} = \begin{array}{cc}\hline 4 & 5 \\ \hline 9 & 10 \\ \hline 14 & 15 \\ \hline\end{array}$$

then

$$\{t\} \cup \{t^*\} = \begin{array}{ccccc}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline\end{array}.$$

We use similar notation for tableaux so by $T \cup T^*$ (where the rows of T are of equal length) we simply mean join the ith row of T to the ith row of T^*. If we write

$$\hat{\Theta}_{T_1}\{t_1\}\kappa_{t_1} \cup \hat{\Theta}_{T_2}\{t_2\}\kappa_{t_2}$$

we mean "take each tabloid involved in $\hat{\Theta}_{T_1}\{t_1\}\kappa_{t_1}$ and 'union' them with all tabloids involved in $\hat{\Theta}_{T_2}\{t_2\}\kappa_{t_2}$, multiplying together the appropriate coefficients". We can take $^-$ of this result to find the corresponding element in $FP(a,b)$. Moreover, if $\hat{\Theta}_{T_1}\{t_1\}\kappa_{t_1}$ (or $\hat{\Theta}_{T_2}\{t_2\}\kappa_{t_2}$) has the property that if $\{t^*\}$ is involved in $\hat{\Theta}_{T_1}\{t_1\}\kappa_{t_1}$ with coefficient c, then all tabloids $\{t'\}$ which are $\sim$-equivalent to $\{t^*\}$ are also involved in $\hat{\Theta}_{T_1}\{t_1\}\kappa_{t_1}$ with coefficient c then in $\hat{\Theta}_{T_1}\{t_1\}\kappa_{t_1} \cup \hat{\Theta}_{T_2}\{t_2\}\kappa_{t_2}$ we join each part of $\{t^*\}$ to each part of those tabloids involved in $\hat{\Theta}_{T_2}\{t_2\}\kappa_{t_2}$. In this case we will write

$$^-(\hat{\Theta}_{T_1}\{t_1\}\kappa_{t_1} \cup \hat{\Theta}_{T_2}\{t_2\}\kappa_{t_2}) = \bar{\Theta}_{T_1}\{t_1\}\kappa_{t_1} \cup \bar{\Theta}_{T_2}\{t_2\}\kappa_{t_2},$$

where by $\alpha \cup \gamma$ (with α and γ involved in $\bar{\Theta}_{T_1}\{t_1\}\kappa_{t_1}$ and $\bar{\Theta}_{T_2}\{t_2\}\kappa_{t_2}$ respectively) we mean "join each part of α to each part of γ and multiplying together the appropriate coefficients".

We extend the use of union a bit further. If β and ζ are given by

$$\beta = \begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}4\\5\\6\end{pmatrix} \text{ and } \zeta = \begin{pmatrix}7\\8\\9\end{pmatrix}\begin{pmatrix}10\\11\\12\end{pmatrix}\begin{pmatrix}13\\14\\15\end{pmatrix}$$

then $\beta \cup \zeta$ will be the partition in $P(5,3)$ formed by simply amalgamating the parts of β with the parts of ζ.

Theorem 3.8 *Let $\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_a^*) \vdash a(b-2)$ and suppose that S^{μ^*} appears in $FP(a, b-2)$ with multiplicity $m \geq 0$. Let $\mu = (\mu_1^*+2, \mu_2^*+2, \ldots, \mu_a^*+2) \vdash ab$. Then S^μ appears in $FP(a,b)$ with multiplicity m.*

Proof: Let $T_1^*, T_2^*, \ldots, T_r^*$ be the semistandard μ^*-tableaux of type $((b-2)^a)$. Let t^* be the usual μ^*-tableau with the digits $1, 2, \ldots, a(b-2)$ placed in increasing order down its columns. For $1 \leq i \leq r$ consider the semistandard μ-tableaux T_i and the μ-tableau t given by $T_i = T' \cup T_i^*$ and $t = t' \cup t^*$ where

$$T' = \begin{matrix}1 & 1\\ 2 & 2\\ \vdots & \vdots \\ a & a\end{matrix} \quad \text{and} \quad t' = \begin{matrix}a(b-2)+1 & ab-a+1\\ a(b-2)+2 & ab-a+2\\ \vdots & \vdots\\ ab-a & ab\end{matrix}.$$

In $\hat{\Theta}_{T_i}\{t\}\kappa_t$, since we only need to consider those tableaux row equivalent to T_i which have distinct entries in each column (see Section 3.2.2), we can write

$$\hat{\Theta}_{T_i}\{t\}\kappa_t = \hat{\Theta}_{T'}\{t'\}\kappa_{t'} \cup \hat{\Theta}_{T_i^*}\{t^*\}\kappa_{t^*}.$$

Thus to compute $\bar{\Theta}_{T_i}\{t\}\kappa_t$, we take every (a,b)-partition α involved in $\bar{\Theta}_{T_i^*}\{t^*\}\kappa_{t^*}$ and we adjoin (in all possible ways) a pair of elements, one from each of the sets $\{a(b-2)+1, a(b-2)+2, \ldots, ab-a\}$ and $\{ab-a+1, ab-a+2, \ldots, ab\}$ to each part of α, attaching the appropriate sign. Therefore we can write

$$\bar{\Theta}_{T_i}\{t\}\kappa_t = \bar{\Theta}_{T'}\{t'\}\kappa_{t'} \cup \bar{\Theta}_{T_i^*}\{t^*\}\kappa_{t^*}.$$

From the above expression for $\bar{\Theta}_{T_i}\{t\}\kappa_t$ it is clear that for $1 \leq i \leq r$ the number of linearly independent homomorphisms $\bar{\Theta}_{T_i^*}$ is equal to the number of linearly independent homomorphisms $\bar{\Theta}_{T_i}$. Since $T_1, T_2, \ldots, T_r$ give all semistandard μ-tableaux, we have from Theorem 2.20 and Theorem 2.32 that the multiplicity of S^{μ^*} in $FP(a, b-2)$ is the same as the multiplicity of S^{μ} in $FP(a,b)$. □

Corollary 3.9 *The Specht module* $S^{(ab-c(a-1),c^{(a-1)})}$ *(for* $0 \leq c \leq b$*) appears in* $FP(a,b)$ *if and only if* c *is even. When* c *is even* $S^{(ab-c(a-1),c^{(a-1)})}$ *appears in* $FP(a,b)$ *with multiplicity one.*

Proof: Let μ be the partition $(ab-c(a-1), c^{(a-1)})$ of ab. So the top row of μ is of length $ab-c(a-1)$ and the other parts are all of length c.

Assume that c is even then we can write $\mu = ((ab-2s(a-1), (2s)^{(a-1)})$ where $c = 2s$. Let $\mu^* = (ab-ac)$. Applying Theorem 3.8 s times we see that the multiplicity of $S^{(a(b-c))}$ in $FP(a, b-c)$ equals the multiplicity of $S^{(ab-c(a-1),c^{(a-1)})}$ in the decomposition of $FP(a,b)$. The only semistandard $(a(b-c))$-tableau T of type $((b-c)^a)$ is

$$T = 1\,1\ldots1\,2\,2\ldots2\ldots\ldots a\,a\ldots a.$$

Let t be given by

$$t = 1\,2\ldots\ldots\ldots a(b-c).$$

Then clearly κ_t is the identity element and it is easy to see that $\bar{\Theta}_T\{t\}\kappa_t$ equals $a!$ times the sum of all $(a, b-c)$-partitions. Since there is only one semistandard $(ab-ac)$-tableau (namely T given above) and $\bar{\Theta}_T(S^{(ab-ac)})$ is non-zero, the multiplicity of $S^{(a(b-c))}$ in $FP(a, b-c)$ is one.

Now assume that c is odd then we can write $\mu = (ab-(2s+1)(a-1), (2s+1)^{(a-1)})$ where $c = 2s+1$. Let $\mu^* = (ab-ac+1, 1^{(a-1)})$. Again applying Theorem 3.8 s times tells us that the multiplicity of $S^{(a(b-c)+1,1^{(a-1)})}$ in $FP(a, b-c+1)$ equals the multiplicity of $S^{(ab-c(a-1),c^{(a-1)})}$ in the decomposition of $FP(a,b)$. The only semistandard $(ab-ac+1, 1^{(a-1)})$-tableau of type $(b-c+1)^a$ is

$$T = \begin{matrix} 1 & 1 & \ldots & 1 & \ldots & a \ldots a \\ 2 \\ \vdots \\ a \end{matrix} .$$

Let t be given by

$$t = \begin{matrix} 1 & a+1 & \ldots \ldots & a(b-c+1) \\ 2 \\ \vdots \\ a \end{matrix} .$$

It is easy to see that all tabloids involved in $\hat{\Theta}_T\{t\}\kappa_t$ have coefficient ± 1. Moreover, if $\{t^*\}$ is a tabloid involved in $\hat{\Theta}_T\{t\}\kappa_t$ then half of the tabloids $\sim$-equivalent to $\{t^*\}$ will have coefficient $+1$ and half will have coefficient -1. Therefore $\bar{\Theta}_T\{t\}\kappa_t = 0$. This completes the proof. □

Remark: It is easy to see from the way in which we have constructed the homomorphism from $S^{(ab-c(a-1),c^{(a-1)})}$ to $FP(a,b)$ that when c is even, every partition involved in $\bar{\Theta}_T\{t\}\kappa_t$ (where T is the (only) semistandard $(ab-c(a-1), c^{(a-1)})$-tableau of type (b^a) and t is the usual tableau) has coefficient $\pm a!$.

In a similar way, we can relate Specht modules in the decomposition of $FP(a, b-1)$ to Specht modules in the decomposition of $FP(a,b)$.

Theorem 3.10 *Let $\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_a^*) \vdash a(b-1)$ be such that $\mu_2^* \leq b-1$ and suppose S^{μ^*} appears in $FP(a, b-1)$ with multiplicity $m \geq 0$. Let $\mu = (\mu_1^*+a, \mu_2^*, \ldots, \mu_a^*) \vdash ab$. Then S^μ appears in $FP(a,b)$ with multiplicity m.*

Proof: Let $T_1^*, T_2^*, \ldots, T_r^*$ be the semistandard μ^*-tableaux of type $((b-1)^a)$. Let t^* be the usual semistandard μ^*-tableau with the digits $1, 2, \ldots, a(b-1)$ filled in down its columns in increasing order. For $1 \leq i \leq r$ consider the semistandard μ-tableaux T_i and the μ-tableau t given by

$$T_i = T_i^* \cup 1 \; 2 \cdots a \text{ and } t = t^* \cup a(b-1)+1 \;\; a(b-1)+2 \cdots ab .$$

Clearly we need to reshuffle the elements in the top row of T_i so that it really is semistandard and the digits increase along the rows. As we chose μ_2^* to be less than or equal to $b-1$, this 'reshuffling' will only affect the columns of T_i^* with one element. The signed column sums κ_{t^*} and κ_t will be the same as they only affect the first μ_2^* columns of t^* and t respectively. For all tableaux T_i' row equivalent to T_i^* we can write $T_i' = T_i'' \cup T_i'''$ where T_i'' is the tableau consisting of the first μ_2^* columns of T_i' and T_i''' is a tableau with one row. Thus we can write

$$\hat{\Theta}_{T_i^*}\{t^*\}\kappa_{t^*} = \sum_{T_i'' \,:\, T_i'=T_i''\cup T_i'''} T_i''\kappa_{t^*} \cup T_i'''. \tag{6}$$

For each $T_i' = T_i'' \cup T_i'''$ row equivalent to T_i^* we can write down a set of tableaux row equivalent to T_i which are of the form $T_i'' \cup T_i^{**}$ where T_i^{**} runs over all tableaux row equivalent to $T_i''' \cup 1\ 2\ \ldots\ a$. Thus for the same possibilities for T_i'' as in (6) we can write

$$\hat{\Theta}_{T_i}\{t\}\kappa_t = \sum_{T_i''} T_i''\kappa_{t^*} \;\cup \sum_{\substack{T_i^{**} \text{ row equivalent} \\ \text{to } T_i'''\cup 1\,2\,\ldots,a}} T_i^{**}. \tag{7}$$

Let Δ^* be the set consisting of the elements from $\{1, 2, \ldots, a(b-1)\}$ which are not in the first μ_2^* columns of t^*. Then $(a, b-1)$-partitions involved in $\bar{\Theta}_{T_i^*}\{t^*\}\kappa_{t^*}$ will be of the form $^-(\sum_{T_i''} T_i''\kappa_{t^*})$ and we fill up the remaining 'space' in each $(a, b-1)$-partition involved in the expression with digits from Δ^*. Similarly, (a, b)-partitions involved in $\bar{\Theta}_{T_i}\{t\}\kappa_t$ will be of the form $^-(\sum_{T_i''} T_i''\kappa_{t^*})$ and we fill up the remaining 'space' with digits from $\Delta = \Delta^* \cup \{a(b-1)+1, a(b-1)+2, \ldots, ab\}$. Thus we can write

$$\bar{\Theta}_{T_i}\{t\}\kappa_t = \frac{1}{|Sym(\Delta^*)|}\sum_{g\in Sym(\Delta)} \bar{\Theta}_{T_i^*}\{t^*\}\kappa_{t^*} \cup \begin{pmatrix} (a(b-1)+1) \\ (a(b-1)+2) \\ \vdots \\ ab \end{pmatrix} g.$$

Thus it is clear that for $1 \le i \le r$ the number of linearly independent homomorphisms $\bar{\Theta}_{T_i^*}$ is equal to the number of linearly independent homomorphisms $\bar{\Theta}_{T_i}$. As $T_1, \ldots, T_r$ give all semistandard μ tableaux then by Theorem 2.20 and Corollary 2.32 S^μ appears in $FP(3,k)$ with multiplicity m. □

3.2.3 Intersection Types

Let $FP(a,b) = c_1X_1 + c_2X_2 + \cdots + c_rX_r$ be the complete decomposition of the module $FP(a,b)$. Then $\sum_{i=1}^{r} c_i^2$ equals the number of G-orbits on $P(a,b) \times P(a,b)$. We show that the G-orbits on $P(a,b) \times P(a,b)$ can be viewed as equivalence classes of certain *intersection arrays* between pairs of ordered (a,b)-partitions. If the equivalence class of the arrays all contain a symmetric array then we show that $c_i = 1$ for all i. It is possible to count the number of G-orbits on $P(a,b) \times P(a,b)$ when $a = 2$ or $a = 3$ and b is arbitrary which we will do in Chapters 4 and 5 respectively.

Let $d+1$ be the number of G-orbits on $P(a,b) \times P(a,b)$ and let A_i (for $i = 0, 1, \ldots, d$) be the adjacency matrices of G on $P(a,b) \times P(a,b)$. Consider the centralizer algebra $PS(a,b)$ spanned by $A_0, A_1, \ldots, A_d$ (see Section 2.3). $PS(a,b)$ can be viewed as (non-commutative) association scheme. Although we will not formally define association schemes or use any properties of them, we will choose to adopt the name *partition scheme* for our algebra. G acts transitively on $P(a,b)$ so by Theorem 2.43 the sum of the squares of the multiplicities of the irreducible modules in the decomposition of $FP(a,b)$ equals $d+1$. We introduce some notation to help calculate the number of G-orbits on $P(a,b) \times P(a,b)$.

For $\alpha, \gamma \in P(a,b)$ let $\alpha^o = \uplus_{i=1}^{a} \alpha_i^o$ and $\gamma^o = \uplus_{j=1}^{a} \gamma_j^o$ be elements in $P^o(a,b)$ which are contained in the $\sim$-equivalence classes represented by α and γ respectively. Form the $a \times a$ *intersection array* $I(\alpha^o, \gamma^o)$ such that the (i,j)-cell of the array is given by

$$I(\alpha^o, \gamma^o)_{(i,j)} = i_{ij} = |\alpha_i^o \cap \gamma_j^o|$$

and so

$$I(\alpha^o, \gamma^o) = \begin{bmatrix} i_{11} & i_{12} & \cdots & i_{1a} \\ i_{21} & i_{22} & \cdots & i_{2a} \\ \vdots & \vdots & \ddots & \vdots \\ i_{a1} & i_{a2} & \cdots & i_{aa} \end{bmatrix}.$$

It is clear that the row and column sum of $I(\alpha^o, \gamma^o)$ is equal to b as each part of α^o and γ^o are of cardinality b and each digit in each part of α^o (respectively γ^o) is in exactly one of the parts of γ^o (respectively α^o). As α^o and γ^o run through the elements in the $\sim$-equivalence classes represented by α and γ respectively, we can write down $(a!)^2$

arrays, some of which may be the same. This amounts to writing down $\{I(\alpha^o, \gamma^o)h\}$ for $h = (h_1, h_2) \in Sym(a) \times Sym(a)$ where h_1 permutes the rows of the array and h_2 permutes the columns of the array. Define an equivalence relation on these arrays by $I(\alpha^o, \gamma^o)$ is *equivalent* to $I(\delta^o, \eta^o)$ (with $\alpha^o, \gamma^o, \delta^o, \eta^o \in P^o(a,b)$) if and only if $I(\delta^o, \eta^o)$ can be obtained from $I(\alpha^o, \gamma^o)$ by permuting its rows and columns independently. Thus, in particular, $I(\alpha^o, \gamma^o)$ is equivalent to $I(\alpha'^o, \gamma'^o)$ for all α'^o $\sim$-equivalent to α^o and γ'^o $\sim$-equivalent to γ^o. Denote by $\bar{I}(\alpha, \gamma)$ the equivalence class containing $I(\alpha^o, \gamma^o)$, where of course α^o and γ^o are in the $\sim$-equivalence classes represented by α and γ respectively.

Example 3.11 *Let α and γ be $(3,3)$-partitions given by*

$$\alpha = \begin{matrix} (1, & 2, & 3) \\ (4, & 5, & 6) \\ (7, & 8, & 9) \end{matrix} \quad \text{and } \gamma = \begin{matrix} (1, & 2, & 6) \\ (4, & 8, & 9) \\ (3, & 5, & 7) \end{matrix} .$$

Then two examples of intersection arrays in $\bar{I}(\alpha, \gamma)$ are

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} .$$

In a natural way the equivalence relation on the intersection arrays induces an equivalence relation on $P(a,b) \times P(a,b)$. So we will write $(\alpha, \gamma) \approx (\delta, \eta)$ if and only if $\bar{I}(\alpha, \gamma) = \bar{I}(\delta, \eta)$. We show now that the $\approx$-equivalence classes of $P(a,b) \times P(a,b)$ are just the G-orbits on $P(a,b) \times P(a,b)$. If we permute the digits in α^o and γ^o by the same element of G this will not change how the ordered (a,b)-partitions intersect with each other so $I(\alpha^o, \gamma^o) = I(\alpha^o g, \gamma^o g)$ for any $g \in G$. Thus, if $(\alpha g, \gamma g) = (\delta, \eta)$ then $\bar{I}(\alpha, \gamma) = \bar{I}(\delta, \eta)$ and so $(\alpha, \gamma) \approx (\delta, \eta)$. Conversely, if $(\alpha, \gamma) \approx (\delta, \eta)$ then we can find α^o, γ^o, δ^o and η^o in the respective classes so that $I(\alpha^o, \gamma^o) = I(\delta^o, \eta^o)$. Now 'split up' each part of γ^o into a sets according to which part of α^o they intersect with. Do the same thing with η^o according to which part of δ^o they intersect with. Now map the 'sets' in γ^o to the corresponding sets in η^o by an element of G. This element of G maps α to δ and at the same time γ to η.

Thus we will make the following definition.

Definition 3.12 Let α, γ, δ and η be (a,b)-partitions. Then we will say that (α,γ) is in the same *intersection class* as (δ,η) if (α,γ) is in the same G-orbit as (δ,η). Moreover, we will call the intersection class $\bar{I}(\alpha,\gamma)$ *symmetric* if (α,γ) is in the same intersection class as (γ,α). We will *index* each intersection class by one of the arrays in the class.

Remark: It is easy to verify that if the intersection class contains a symmetric array then the intersection class will be symmetric.

Example 3.13 *Let α and γ be the $(3,6)$-partitions given by*

$$\alpha = \begin{matrix}(1, & 2, & 3, & 4, & 5, & 6)\\(7, & 8, & 9, & 10, & 11, & 12)\\(13, & 14, & 15, & 16, & 17, & 18)\end{matrix} \quad \text{and } \gamma = \begin{matrix}(1, & 2, & 3, & 4, & 7, & 13)\\(8, & 9, & 10, & 14, & 15, & 16)\\(5, & 6, & 11, & 12, & 17, & 18)\end{matrix}.$$

Then the arrays given by

$$\begin{bmatrix} 4 & 0 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 1 & 1 \\ 0 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

will be in $\bar{I}(\alpha,\gamma)$ and $\bar{I}(\gamma,\alpha)$ respectively. It is clear that $\bar{I}(\alpha,\gamma) \neq \bar{I}(\gamma,\alpha)$ since the rows and columns of the first array cannot be permuted to give the second array. Thus the intersection class is not symmetric.

The following definition can be found in Chapter 1 of [25].

Definition 3.14 The square matrix N with entries in the natural numbers is called *integer stochastic* if its row sum and column sum are constant. Denote by $H_m(r)$ the number of $m \times m$ matrices with entries in the natural numbers and row and column sum equal to r.

The expressions for the numbers $H_m(r)$ when $m = 1, 2$ and 3 can be found on page 30 of [25] and are given by:

$$H_1(r) = 1,$$
$$H_2(r) = r + 1,$$
$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}. \tag{8}$$

For general values of $m \geq 3$, little appears to be known about $H_m(r)$. Since any $a \times a$ integer stochastic matrix with row and column sum b corresponds to an intersection array, we can define the same equivalence relation on the set of integer stochastic matrices by saying that N_1 and N_2 are *equivalent* if and only if N_2 can be obtained from N_1 by permuting its rows and columns independently. The number of different intersection classes of $P(a, b) \times P(a, b)$ will therefore equal the number of H-orbits on the set of $a \times a$ integer stochastic matrices with row and column sum b, where $H = Sym(a) \times Sym(a)$. Trivially when $a = 1$ the number of intersection classes is $H_1(b) = 1$. In Lemma 4.4 and Theorem 5.4 respectively we will give the number of intersection classes of $P(2, k) \times P(2, k)$ and $P(3, k) \times P(3, k)$, where k is an arbitrary positive integer. In general the number of intersection classes is extremely difficult to calculate.

When the intersection classes are symmetric it is clear that the adjacency matrices A_i are symmetric. Thus, when the intersection classes are symmetric, by Lemma 2.44, $PS(a, b)$ is commutative and moreover by Theorem 2.43 each irreducible module in the decomposition of $FP(a, b)$ has multiplicity one. In general $PS(a, b)$ is not symmetric although trivially when $a = 1$ the partition scheme is symmetric and we will show in Chapter 4 that $PS(2, b)$ is symmetric.

3.3 Eigenvectors of MM^T: General Results

We will use irreducible modules which appear in $FP(a, b)$ with multiplicity one to construct some of the eigenvectors of MM^T.

Under the symmetric matrix MM^T the FG-module $FP(a,b)$ decomposes into eigenspaces

$$FP(a,b) = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_r}.$$

From (1) of Section 2.1 we have

$$MM^T(vg) = (MM^T v)g = (\lambda v)g = \lambda vg \text{ for all } g \in G$$

and so:

Lemma 3.15 *The eigenspaces of MM^T are FG-submodules of $FP(a,b)$.*

Maschke's theorem tells us that the decomposition of $FP(a,b)$ into irreducibles is unique up to isomorphism. Thus each eigenspace is isomorphic to a direct sum of certain Specht modules. It is clear that if S^μ appears in precisely one eigenspace of MM^T and T is a semistandard μ-tableau of type (b^a) then either $\bar{\Theta}_T(S^\mu)$ is zero or it is contained in an eigenspace of MM^T. In general we can not expect such nice properties of eigenspaces. However, when $a = 1$ or 2 and when $a = 3$ with $b \leq 5$ we will show that every irreducible module in the decomposition of $FP(a,b)$ appears with multiplicity one. We can use this fact to find eigenvectors and eigenvalues of MM^T. The case $a = 1$ is trivial as M is the 1×1 identity matrix with eigenvalue equal to 1. The cases when $a = 2$ or $a = 3$ with b small are studied in Chapters 4 and 6 respectively.

We will say that the partition μ is *directly associated* to the eigenvalue λ or the eigenspace E_λ (or we say μ is in *direct association* with λ or E_λ) if there exists a semistandard μ-tableau T such that $\bar{\Theta}_T(S^\mu) \subseteq E_\lambda$. If we require linear combinations of the maps $\bar{\Theta}_T$ (for different semistandard μ-tableaux T) to map the Specht module S^μ into an eigenspace E_λ, we will say that μ is *indirectly associated* to λ or E_λ. When the eigenspace decomposes into two or more non-isomorphic irreducible subspaces, more than one partition may be associated to an eigenvalue. If S^μ appears in $FP(a,b)$ with multiplicity *one* and so for some semistandard tableau T we have $\bar{\Theta}_T(S^\mu) \subseteq E_\lambda$, we will say that μ is *simply associated* to λ (or μ is in *simple association* with λ). In this case it is not necessary to use the word 'directly' as a simple association must be direct. When μ is directly associated to λ we will sometimes associate the semistandard μ-tableau T to the eigenvalue rather than μ itself. Since every eigenspace of MM^T decomposes completely into a direct sum of modules isomorphic to Specht modules, once we know the decomposition we can calculate the dimensions of the eigenspace by summing the dimensions of the appropriate Specht modules (these can be calculated using by hook lengths, see Theorem 2.35).

We consider here some of the semistandard tableaux which are simply associated to eigenvalues of MM^T. The most straightforward case is the partition $(ab - c(a-1), c^{(a-1)})$. When c is even we know from Corollary 3.9 that $S^{(ab-c(a-1),c^{(a-1)})}$ appears in $FP(a,b)$ with multiplicity one and so must be simply associated to an eigenvalue of MM^T. Thus we have the following result.

Theorem 3.16 *If $\mu = (ab - 2s(a-1), (2s)^{(a-1)})$ and if T and t are μ-tableaux given by*

$$T = \begin{array}{l} 1\,1 \ldots 1 \ldots 1\,2 \ldots 2\ \ldots b \ldots b \\ 2\,2 \ldots 2 \\ \vdots\ \vdots\ \ \ddots\ \vdots \\ \underbrace{a\,a \ldots a}_{2s} \end{array} \qquad t = \begin{array}{llll} 1 & a+1 & \ldots\ 2as-a+1 & 2as+1 \ldots ab \\ 2 & a+2 & \ldots\ 2as-a+2 & \\ \vdots & \vdots & \qquad\quad \vdots & \\ a & 2a & \ldots \qquad 2as & \end{array}$$

then $\bar{\Theta}_T\{t\}\kappa_t$ is an eigenvector of MM^T.

Remark: The result above gives an alternative method of finding the maximum eigenvalue of MM^T which is given in Corollary 3.5 and can be stated as follows.

Proposition 3.17 *The eigenvalue of MM^T associated to the semistandard tableau*

$$T = 1\,1 \ldots 1\,2\,2 \ldots 2 \ldots \ldots a\,a \ldots a$$

is $(a!)^{b-1}(b!)^{a-1}$. This eigenvalue is the largest eigenvalue of MM^T and has multiplicity 1.

In general it is not very easy to write down the eigenvalue associated to $(ab - 2s(a-1), (2s)^{(a-1)})$ explicitly in a neat closed form. When $a = 2$ and b is arbitrary we can write down these eigenvalues explicitly for all values of s which we will do in Section 4.3.3. When $a = 3$ and b is arbitrary we can write down these eigenvalues explicitly for the case when $s = 0$ or 1 (see Theorem 6.12).

As a direct corollaries of Theorems 3.8 and 3.10 we have.

Proposition 3.18 *Let M^* be the incidence matrix of $\mathcal{P}_{a,b-2}$ and M be the incidence matrix of $\mathcal{P}_{a,b}$. For $\mu^* = (\mu_1^*, \ldots, \mu_a^*)$, let $\mu = (\mu_1^* + 2, \mu_2^* + 2, \ldots, \mu_a^* + 2)$. If S^{μ^*} has multiplicity one in the decomposition of $FP(a, b-2)$ then S^μ has multiplicity one in the decomposition of $FP(a, b)$. Moreover, there exists a semistandard μ-tableau T and a μ-tableau t such that $\bar{\Theta}_T\{t\}\kappa_t$ is non-zero and hence $\bar{\Theta}_T\{t\}\kappa_t$ is an eigenvector of MM^T.*

Proposition 3.19 *Let M^* be the incidence matrix of $\mathcal{P}_{a,b-1}$ and M be the incidence matrix of $\mathcal{P}_{a,b}$. For $\mu^* = (\mu_1^*, \ldots, \mu_a^*)$ with $\mu_2^* \leq b-1$, let $\mu = (\mu_1^* + a, \mu_2^*, \ldots, \mu_a^*)$. If S^{μ^*} has multiplicity one in the decomposition of $FP(a, b-1)$ then S^μ has multiplicity one in the decomposition of $FP(a, b)$. Moreover, there exists a semistandard μ-tableau T and a μ-tableau t such that $\bar{\Theta}_T\{t\}\kappa_t$ is non-zero and hence $\bar{\Theta}_T\{t\}\kappa_t$ is an eigenvector of MM^T.*

In general, constructing eigenvectors of MM^T is not such an easy task. This is illustrated in Section 6.2 when we construct some of the eigenvectors of $M^{3,k}(M^{3,k})^T$. When we know the eigenvectors of MM^T we can calculate the corresponding eigenvalues in the following way. Let v be an eigenvector of MM^T with corresponding eigenvalue λ. Let α be an (a, b)-partition involved in v and denote by e_α the standard basis vector then we can write

$$e_\alpha^T MM^T v = (e_\alpha^T M)(M^T v) = \lambda e_\alpha^T v$$

and so

$$\lambda = \frac{(e_\alpha^T M)(M^T v)}{e_\alpha^T v}.$$

Therefore we multiply the β-entry of $e_\alpha^T M$ by the β-entry of $M^T v$ and sum over $\beta \in P(b, a)$. To find λ we divide this result by the coefficient of α in v. The β-entry of $e_\alpha^T M$ is one if β intersects nicely with α and zero otherwise. Thus, we only need to consider those β which intersect nicely with α. To show that the eigenvalue corresponding to an eigenvector v is non-zero is an even easier process. We write

$$\lambda = \frac{(M^T v)^T (M^T v)}{v^T v}.$$

Since v is a non-zero vector we know that $v^T v$ is non-zero so there is no problem dividing by this factor. Moreover, λ is non-zero if and only if $M^T v$ has a least one non-zero entry. Thus to show that λ is non-zero we just need to exhibit an element β of $P(b, a)$ such that the β-entry of $M^T v$ is non-zero. We will use these methods in Chapters 4 and 6 when calculating eigenvalues of MM^T, where M is the incidence matrix of $\mathcal{P}_{2,k}$ and $\mathcal{P}_{3,k}$ respectively.

Chapter 4

$(2, k)$-Partitions

In this chapter we study the permutation module $FP(2,k)$ of $Sym(2k)$ acting on $P(2,k)$ using the same notation as in the previous chapter. We begin with some basic properties of the incidence structure $\mathcal{P}_{2,k}$ and give examples of the incidence matrix M of this structure. For α and γ in $FP(2,k)$ we give an expression for the (α,γ)-entry of MM^T. Using representation theory of symmetric groups we determine the complete decomposition of $FP(2,k)$ and show that all irreducible modules in this decomposition have multiplicity one. We use this decomposition to find the eigenvectors of the matrix MM^T. Using these eigenvectors we show that all eigenvalues of MM^T are non-zero which gives a simple proof of Foulkes' conjecture (see Section 2.2.6) for $m = 2$ and n arbitrary. We then determine the eigenvalues explicitly first in a summation form and then in reduced form. In the final section of this chapter we show that all modules which appear in $FP(2,k)$ appear in $FP(k,2)$, this gives a direct proof of Foulkes' conjecture for the case when $m = 2$ and n is arbitrary.

4.1 The Incidence Structure $\mathcal{P}_{2,k}$

In the following example we give the sizes of the matrix $M^{2,k}$ for small values of k which gives an indication of just how quickly they grow in size. When $k = 1$ the incidence matrix $M^{2,1}$ is trivially the 1×1 identity matrix so we will just give $M^{2,k}$ explicitly for $k = 2$ and 3.

Example 4.1 *The following table gives the size of $M^{2,k}$ for $k = 1, 2, 3, \ldots, 10$.*

k	$\|P(2,k)\|$	$\|P(k,2)\|$
1	1	1
2	3	3
3	10	15
4	35	105
5	126	945
6	462	10395
7	1716	135135
8	6435	2027025
9	24310	34459425
10	92378	654729075

The incidence matrices of $M^{2,2}$ and $M^{2,3}$ are given by

$$M^{2,2} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$M^{2,3} = \begin{pmatrix} 0&0&0&0&0&0&0&1&1&0&1&1&0&1&1 \\ 0&0&0&0&1&1&0&0&0&1&0&1&1&0&1 \\ 0&0&0&1&0&1&1&0&1&0&0&0&1&1&0 \\ 0&0&0&1&1&0&1&1&0&1&1&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0&1&1&0&1&1&0 \\ 1&0&1&0&0&0&1&1&0&0&0&0&1&0&1 \\ 1&1&0&0&0&0&1&0&1&1&0&1&0&0&0 \\ 1&1&0&1&1&0&0&0&0&0&0&0&0&1&1 \\ 1&0&1&1&0&1&0&0&0&0&1&1&0&0&0 \\ 0&1&1&0&1&1&0&1&1&0&0&0&0&0&0 \end{pmatrix}.$$

Definition 4.2 Let α and γ be two $(2,k)$-partitions with $\alpha = \alpha_1 \dot\cup \alpha_2$ and $\gamma = \gamma_1 \dot\cup \gamma_2$. Define a function, called the *maximum intersection* of α and γ, from $P(2,k) \times P(2,k)$ to $\mathbb{N}$ by

$$m(\alpha, \gamma) := max\{\, |\alpha_r \cap \gamma_s| \;|\, r, s \in \{1, 2\}\}.$$

Thus the maximum intersection of α and γ is the biggest 'overlap' between the parts of α and γ respectively.

Remark: The maximum intersection of two $(2,k)$-partitions is an integer $k-v$ for some $v \in \{0,1,\ldots,[k/2]\}$ (where the square brackets mean we take the integer part of $k/2$). This restriction on v is quite clear: Let $|\alpha_1 \cap \gamma_1| = s$ for some $s \in \{0,1,\ldots,k\}$, then $|\alpha_1 \cap \gamma_2| = k-s$. The largest of these two values is the maximum intersection of α and γ. Therefore $v = min\{s, k-s\}$ cannot exceed $[k/2]$. Thus we could define the maximum intersection as

$$m(\alpha,\gamma) := max\{\,|\alpha_1 \cap \gamma_s| \mid s \in \{1,2\}\}.$$

Example 4.3 *Let α and γ be $(2,5)$-partitions given by*

$$\alpha = \begin{matrix}(1,2,3,4,5\ ) \\ (6,7,8,9,10)\end{matrix} \ \textit{and}\ \gamma = \begin{matrix}(1,3,5,7,9\ ) \\ (2,4,6,8,10)\end{matrix} .$$

Then α and γ have maximum intersection 3, that is $m(\alpha,\gamma) = 3$.

For α and γ in $P(2,k)$ the intersection class $\bar{I}(\alpha,\gamma)$ can be indexed by an array of the form

$$\begin{bmatrix} k-i & i \\ i & k-i \end{bmatrix}$$

with $i \in \{0,1,\ldots,[k/2]\}$. Thus we have the following result.

Lemma 4.4 *The number of intersection classes of $P(2,k) \times P(2,k)$ is $[k/2]+1$.*

Proposition 4.5 *Let α and γ be any (2,k)-partitions. Then the (α,γ)-entry of MM^T is $(k-v)!v!$ where $v = k - m(\alpha,\gamma)$.*

Proof: The (α,γ)-entry of MM^T is the number of 1's in the same positions of row α and row γ of M. Row α of M has 1's in the positions corresponding to those $(k,2)$-partitions which intersect nicely with α. So in other words, the (α,γ)-entry of MM^T is the number of $(k,2)$-partitions which intersect nicely with both α and γ. Since $Sym(2k)$ acts transitively on the elements of $P(2,k)$, without loss of generality we can choose α and γ to be:

$$\alpha = \begin{matrix}(\ 1, & \ldots, & k-v, & k-v+1, & \ldots, & k\ ) \\ (\ k+1, & \ldots, & 2k-v, & 2k-v+1, & \ldots, & 2k\ )\end{matrix}$$

$$\gamma = \begin{pmatrix} 1, & \ldots, & k-v, & 2k-v+1, & \ldots, & 2k \\ k+1, & \ldots, & 2k-v, & k-v+1, & \ldots, & k \end{pmatrix}$$

so that $m(\alpha, \gamma) = k - v$ for some $v \in \{0, 1, \ldots, [k/2]\}$. Consider those $(k, 2)$-partitions which intersect nicely with both α and γ. Let β be one such $(k, 2)$-partition. So β has k parts of size 2, where each part has one element from each part of α and γ. Thus elements in $\{1, \ldots, k - v\}$ can only be 'paired off' with elements of the set $\{k + 1, \ldots, 2k - v\}$ and elements of $\{k - v + 1, \ldots, k\}$ can only be paired off with elements of $\{2k - v + 1, \ldots, 2k\}$. Any other type of pairing prevents a nice intersection between α and β or γ and β or both. There are $(k - v)!v!$ such pairings, so $(k - v)!v!$ different $(k, 2)$-partitions which intersect nicely with both α and γ. □

Corollary 4.6 *The diameter of $\mathcal{P}_{2,k}$ equals one.*

Proof: The diameter of $\mathcal{P}_{2,k}$ is one if for all α and γ in $P(2, k)$ there exists a β in $P(k, 2)$ which intersects nicely with both α and γ. The (α, γ)-entry of MM^T counts the number of $(k, 2)$-partitions which intersect nicely with both α and γ. Since this entry is always non-zero the proof is complete. □

Thus MM^T can be written as a linear combination of $l+1$ (where $l = [k/2]$) adjacency matrices:

$$MM^T = k!\,0!\,A_0 + (k-1)!\,1!\,A_1 + (k-2)!\,2!\,A_2 + \cdots + (k-l)!\,l!\,A_l \qquad (9)$$

where

$$(A_i)_{\alpha\gamma} = \begin{cases} 1 & \text{if } m(\alpha, \gamma) = k - i, \\ 0 & \text{otherwise.} \end{cases}$$

Since each intersection class can be indexed by a symmetric array we know that the partition scheme $PS(2, k)$ is commutative or in other words, $A_iA_j = A_jA_i$ for all $i, j \in \{0, 1, \ldots, l\}$. This means that the permutation character of $Sym(2k)$ on $P(2, k)$ is multiplicity free and so the permutation module $FP(2, k)$ decomposes into a direct sum of $l + 1$ irreducible modules all with multiplicity one. Before we study the permutation module $FP(2, k)$ in more detail we will give a couple of examples of the matrix MM^T and the corresponding eigenvalues.

Example 4.7 *The two simplest (non-trivial) examples are when $k = 2$ and $k = 3$.*

$$M^{2,2}(M^{2,2})^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$M^{2,3}(M^{2,3})^T = \begin{pmatrix} 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 \end{pmatrix}.$$

Example 4.8 *The eigenvalues of the matrices $M^{2,k}(M^{2,k})^T$ for $1 \le k \le 3$ can be calculated using, for example, the mathematical package Pari. The results are tabulated below.*

	Eigenvalue	*Multiplicity*
$k = 1$	1	1
$k = 2$	4	1
	1	2
$k = 3$	24	1
	4	9

4.2 Decomposition of $FP(2, k)$ and Eigenvectors of $M^{2,k}(M^{2,k})^T$

We can use results of the previous chapter to construct the complete decomposition of $FP(2, k)$. We use the methods introduced in Section 3.3 to construct a generating set for the eigenvectors of MM^T. Before doing this, we briefly mention the irreducible modules in the decomposition of $FP^o(2, k)$. We know from Section 3.2.1 that if μ is a partition of $2k$ with at most two parts then the multiplicity of S^μ in the decomposition of $FP^o(2, k)$ is the number of semistandard μ-tableaux of type (k^2). It is easy to see that for any given partition μ of $2k$ with at most two parts (which we can write in the form $\mu = (2k-i, i)$ for some $i \in \{0, 1, \ldots, k\}$) we can construct exactly one semistandard μ-tableau of type (k^2). This semistandard tableau will be of the form

$$T = \begin{matrix} 1\,1 \ldots\ldots 1\,2\,2 \ldots 2 \\ \underbrace{2\,2 \ldots 2}_{i} \end{matrix}.$$

Thus, the modules in the complete decomposition of $FP^o(2,k)$ will be $S^{(2k-i,i)}$ with $i \in \{0, 1, \ldots, k\}$ and each of these will appear in $FP^o(2,k)$ with multiplicity equal to one. Since $FP(2,k)$ is isomorphic to a submodule of $FP^o(2,k)$, we already know quite a lot about the modules in the complete decomposition of $FP(2,k)$ and in particular that each module which appears in $FP(2,k)$ will have multiplicity equal to one.

By Corollary 3.9 we know that for $0 \leq c \leq k$ the Specht module $S^{(2k-c,c)}$ appears in $FP(2,k)$ precisely when c is even. This gives us $l+1$ (where $l = [k/2]$) irreducible modules in the decomposition of $FP(2,k)$, each one being isomorphic to $S^{(2k-2s,2s)}$ for $s \in \{0, 1, \ldots, l\}$. We already know that $FP(2,k)$ decomposes into exactly $l+1$ irreducible modules so these modules must form the complete set. That is

$$FP(2,k) \simeq S^{\mu^0} \oplus S^{\mu^1} \oplus \cdots \oplus S^{\mu^l}$$

where $\mu^s = (2k-2s, 2s)$ for $s = 0, 1, \ldots, l$.

Thus, the eigenvectors given in Theorem 3.16 are a generating set for the eigenvectors of MM^T. This gives us the following result.

Proposition 4.9 *Let M be the incidence matrix of $\mathcal{P}_{2,k}$. For $0 \leq s \leq l$ let t_s be the $(2k-2s, 2s)$-tableau with the digits $1, 2, \ldots, 2k$ placed in increasing order down its columns and T_s be the semistandard $(2k-2s, 2s)$-tableau of type (k^2) given by*

$$T_s = \begin{matrix} 1\,1 \ldots\ldots 1\,2\,2 \ldots 2 \\ \underbrace{2\,2 \ldots 2}_{2s} \end{matrix}.$$

If s runs over the elements of $\{0, 1, \ldots, l\}$ then $\bar{\Theta}_{T_s}\{t_s\}\kappa_{t_s}$ runs over a generating set for the eigenvectors of MM^T.

It is easy to see from the definition of $\hat{\Theta}_{T_s}\{t_s\}\kappa_{t_s}$ that we can write

$$\hat{\Theta}_{T_s}\{t_s\}\kappa_{t_s} = \sum_{g \in C_{t_s}} \{t'_s\}g\ sgn(g) \cup \sum_{h \in Sym(\Delta)} \{t^*_s\}h.$$

where $\Delta = \{4s+1, 4s+2, \ldots, 2k\}$,

$$\{t'_s\} = \frac{\overline{1\,3 \ldots 4s-1}}{\underline{2\,4 \ldots \ \ 4s}}$$

and $\{t^*_s\}$ is any $((k-2s)^2)$-tabloid with entries from Δ with no repeats. To find the coefficients of a $(2,k)$-partition α involved in $\bar{\Theta}_{T_s}\{t_s\}\kappa_{t_s}$ we add together the coefficients in $\hat{\Theta}_{T_s}\{t_s\}\kappa_{t_s}$ of tabloids $\sim$-equivalent to $\{t^{**}_s\}$, where $\{t^{**}_s\}$ is a tabloid with the parts of α making up its rows. It is easy to see that any $(2,k)$-partition involved in $\bar{\Theta}_{T_s}\{t_s\}\kappa_{t_s}$ has coefficient $+2$ if it contains an even number of elements from $\{1, 2, \ldots, 4s\}$ in each part and -2 otherwise.

Remark: It can be shown that the eigenvectors $\bar{\Theta}_{T_s}\{t_s\}\kappa_{t_s}$ simultaneously diagonalize the adjacency matrices A_i, for $i = 0, 1, \ldots, [k/2]$, given in equation (9). Thus, the eigenvalues of MM^T could be calculated by first finding the eigenvalues of the matrices A_i and taking the appropriate linear combination.

4.3 Eigenvalues of $M^{2,k}(M^{2,k})^T$

4.3.1 Lower Bounds for the Eigenvalues of $M^{2,k}(M^{2,k})^T$

Now that we have the linearly independent eigenvectors of the matrix MM^T it is a straightforward process to show that all eigenvalues are non-zero. We have two different methods of proof to show that the eigenvalues are non-zero. The first proof is direct and the second is inductive on k.

Theorem 4.10 *Let M be the incidence matrix of $\mathcal{P}_{2,k}$. Then all eigenvalues of MM^T are non-zero.*

Proof 1: For a fixed value s satisfying $0 \leq 2s \leq k$ let T be the semistandard $(2k-2s, 2s)$-tableau of type (k^2) with 2 occurring $2s$ times in the second row and t

the $(2k-2s, 2s)$-tableau with the digits $1, 2, \ldots, 2k$ placed in increasing order down its columns. Let v denote the eigenvector $\bar{\Theta}_T\{t\}\kappa_t$ corresponding to the eigenvalue λ. From the results at the end of Section 3.3 to show that λ is non-zero we need to find a $(k, 2)$-partition β indexing a non-zero entry of $M^T v$. Let β be the $(k, 2)$-partition given by

$$\beta = \binom{2}{4}\binom{6}{8}\binom{10}{12}\cdots\binom{4s-2}{4s}\binom{1}{3}\cdots\binom{4s-3}{4s-1}\binom{4s+1}{4s+2}\cdots\binom{2k-1}{2k}.$$

Consider all $(2, k)$-partitions which intersect nicely with β and are involved in v. Recall that the coefficient of a $(2, k)$-partition involved in v is $+2$ if each part of the $(2, k)$-partition has an even number of even numbers from $\{1, 2, \ldots, 4s\}$ and -2 otherwise (see the explanation following Proposition 4.9). Any $(2, k)$-partition which intersects nicely with β must have s even numbers from $\{1, 2, \ldots, 4s\}$ in each of its parts. As s is fixed, all $(2, k)$-partitions involved in v which intersect nicely with β must have the same coefficient in v as each other. For example, the $(2, k)$-partition α given by

$$\alpha = \begin{matrix}(1, 4, 5, 8, \ldots, 4s-3, & 4s & , 4s+1, \ldots, 2k-1) \\ (2, 3, 6, 7, \ldots, 4s-2, & 4s-1 & , 4s+2, \ldots, \quad 2k \quad)\end{matrix}$$

intersects nicely with β and is involved in v. Thus $M^T v$ has at least one non-zero entry and so λ must be non-zero. □

Proof 2: Let M^* be the incidence matrix of $\mathcal{P}_{2,k-2}$ and let M be the incidence matrix of $\mathcal{P}_{2,k}$. By Proposition 4.9 we know that for s satisfying $1 \leq s \leq [k/2]$ the partitions $(2k-2s-2, 2s-2)$ and $(2k-2s, 2s)$ are associated to eigenvalues of M^*M^{*T} and MM^T respectively. We will use induction on k to show that if the eigenvalue of M^*M^{*T} associated to $(2k-2s-2, 2s-2)$ is non-zero then the eigenvalue of MM^T associated to $(2k-2s, 2s)$ is larger and so also non-zero. There are two base steps for this inductive proof as we go up in steps of two. We have already completed these base steps in Example 4.8 where we showed that all eigenvalues of $M^{2,k}(M^{2,k})^T$ for $1 \leq k \leq 3$ are non-zero. So for $k-2 \geq 1$ assume that the eigenvalues of M^*M^{*T} are all non-zero. Let s be an integer satisfying $1 \leq s \leq [k/2]$, let t^* be the $(2k-2s-2, 2s-2)$-tableau with the digits placed in increasing order down its columns and let T^* be the semistandard $(2k-2s-2, 2s-2)$-tableau given by

$$T^* = \begin{matrix} 1\,1 \ldots\ldots 1\,2\,2 \ldots 2 \\ \underbrace{2\,2 \ldots 2}_{2s-2} \end{matrix} .$$

Form the $(2k-2s, 2s)$-tableau t and the semistandard $(2k-2s, 2s)$-tableau T by setting $t = t' \cup t^*$ and $T = T' \cup T^*$ where

$$t' = \begin{matrix} 2k-3 & 2k-1 \\ 2k-2 & 2k \end{matrix} \quad \text{and} \quad T' = \begin{matrix} 1\,1 \\ 2\,2 \end{matrix}.$$

Thus from the proof of Theorem 3.8 we can write

$$\bar{\Theta}_T\{t\}\kappa_t = \bar{\Theta}_{T'}\{t'\}\kappa_{t'} \cup \bar{\Theta}_{T^*}\{t^*\}\kappa_{t^*}. \tag{10}$$

Let λ^* be the eigenvalue of M^*M^{*T} associated to $(2k-2s-2, 2s-2)$ with eigenvector $v^* = \bar{\Theta}_{T^*}\{t^*\}\kappa_{t^*}$ and let λ be the eigenvalue of MM^T associated to $(2k-2s, 2s)$, with eigenvector $v = \bar{\Theta}_T\{t\}\kappa_t$. Since v^* and λ^* are non-zero, we can write

$$\lambda^* = \frac{(M^{*T}v^*)^T(M^{*T}v^*)}{v^{*T}v^*} \neq 0.$$

Therefore $M^{*T}v^*$ must have at least one non-zero entry. Let β^* be the $(k-2, 2)$-partition which indexes this non-zero entry. Now let β be given by

$$\beta = \binom{2k-3}{2k}\binom{2k-2}{2k-1} \cup \beta^*.$$

We claim that the β-entry of $M^T v$ is non-zero which, by the results at the end of Section 3.3, shows that λ is non-zero. We know that if α^* is involved in v^* then the $(2, k)$-partition α formed by joining one element from each of the sets $\{2k-3, 2k-2\}$ and $\{2k-1, 2k\}$ to each part of α^* is involved in v. Moreover, all $(2, k)$-partitions involved in v can be constructed in this way. Let $\alpha^* = \alpha_1^* \cup \alpha_2^*$ be a $(2, k-2)$-partition involved in v^* which intersects nicely with β^*. Denote by α^1 the $(2, k)$-partition formed by adjoining $\{2k-3, 2k-1\}$ to α_1^* and $\{2k-2, 2k\}$ to α_2^*. Denote by α^2 the $(2, k)$-partition formed by adjoining $\{2k-2, 2k\}$ to α_1^* and $\{2k-3, 2k-1\}$ to α_2^*. Then it is easy to see that α^* intersects nicely with β^* and is involved in v^* if and only if α^1 and α^2 intersect nicely with β and are involved in v. Moreover, the coefficients of

α^1 and α^2 in v are the same as the coefficient of α^* in v^*. Thus, the β-entry of $M^T v$ is twice the β^*-entry of $M^{*T} v^*$ and so is non-zero as claimed. We have shown that the eigenvalues associated to $(2k-2s, 2s)$ are non-zero for $1 \leq s \leq [k/2]$. Thus, the only partition we haven't considered is $(2k)$. We have shown in Proposition 3.17 that $(2k)$ is associated to the largest eigenvalue of MM^T and this eigenvalue is non-zero. Therefore we have shown that all eigenvalues of MM^T are non-zero. This completes the proof. □

Remark: In our opinion the first proof is probably the shortest known proof of Foulkes' conjecture for the case when $m = 2$ and n is arbitrary.

4.3.2 The Eigenvalues of $M^{2,k}(M^{2,k})^T$ in Summation Form

In the previous section we constructed a set of linearly independent eigenvectors of MM^T which yield the complete set of eigenvalues. We can use these eigenvectors to determine the eigenvalues explicitly.

Proposition 4.11 *Let M be the incidence matrix of $\mathcal{P}_{2,k}$. For $s \in \{0, 1, \ldots, [k/2]\}$ the eigenvalues of MM^T are given by*

$$\lambda_s = \frac{1}{2}\sum_{i=0}^{k}(k-i)!i!\sum_{b=0}^{i}(-1)^b \binom{2s}{b}\binom{k-2s}{i-b}^2.$$

Proof: Let s be an element of $\{0, 1, \ldots, [k/2]\}$ and as usual T_s and t_s be given by

$$T_s = \begin{matrix} 1\,1\ldots\ldots 1\,2\,2\ldots 2 \\ \underbrace{2\,2\ldots 2}_{2s} \end{matrix} \quad \text{and} \quad t_s = \begin{matrix} 1\,3\ldots\ldots 4s-1\;4s+1\;4s+2\ldots 2k \\ \underbrace{2\,4\ldots 4s}_{2s} \end{matrix}.$$

Let $v = \bar{\Theta}_{T_s}\{t_s\}\kappa_{t_s}$ be the eigenvector of MM^T associated to $(2k-2s, 2s)$ and let α be the $(2,k)$-partition with the odd numbers $\{1, 3, \ldots, 2k-1\}$ in one part (which we will label α_1) and the even numbers $\{2, 4, \ldots, 2k\}$ in the other part. α has $2s$ even numbers from $\{1, 2, \ldots, 4s\}$ in α_2 and none in α_1. Thus the coefficient of α in v is $+2$. For $i \in \{0, 1, \ldots, [k/2]\}$ let $\gamma = \gamma_1 \cup \gamma_2$ be a $(2,k)$-partition involved in v

such that $|\alpha_1 \cap \gamma_1| = k - i$. For some b satisfying $0 \leq b \leq 2s$ then γ_1 will contain $2s - b$ elements from $\{1, 3, \ldots, 4s - 1\}$ and the remaining b elements from this set will be in γ_2. As γ has non-zero entry in v, this determines the b elements from $\{2, 4, \ldots, 4s\}$ which must be in γ_1 and the remaining $2s - b$ elements from this set must be in γ_2. To ensure that $|\gamma_1 \cap \alpha_1| = k - i$, we must fill γ_1 with $k - 2s - i + b$ elements from the set $\{4s + 1, 4s + 3, \ldots, 2k - 1\}$ and $i - b$ elements from the set $\{4s + 2, 4s + 4, \ldots, 2k\}$. The remaining elements will fill γ_2. The coefficient of γ in v will be $+2$ if b is even and -2 otherwise. If b runs over $0, 1, \ldots, 2s$ then γ runs over all $(2, k)$-partitions such that $|\gamma_1 \cap \alpha_1| = k - i$. Thus we are able to calculate the number of $\gamma = \gamma_1 \uplus \gamma_2$ such that $|\gamma_1 \cap \alpha_1| = k - i$. If $i = k/2$ we have counted γ twice so we need to take half of the result. We have shown in Proposition 4.5 that the (α, γ)-entry of MM^T is $(k - i)!i!$ where $max\{k - i, i\} = m(\alpha, \gamma)$. To find the α-entry of $MM^T v$ we multiply the (α, γ)-entry MM^T by the γ-entry of v for all γ with $|\gamma_1 \cap \alpha_1| = k - i$ and sum over all values of i between 0 and $[k/2]$. Equivalently (and using the fact that $|\alpha_1 \cap \gamma_1| = k - i$ implies $|\alpha_1 \cap \gamma_2| = i$) we can sum over i from 0 to k and take half of the result. Thus, we can write down the α-entry of $MM^T v$ as follows:

$$\frac{1}{2} \sum_{i=0}^{k} (k - i)!i! \sum_{b=0}^{i} 2(-1)^b \binom{2s}{2s - b} \binom{k - 2s}{k - 2s + b - i} \binom{k - 2s}{i - b}$$
$$= \sum_{i=0}^{k} (k - i)!i! \sum_{b=0}^{i} (-1)^b \binom{2s}{b} \binom{k - 2s}{i - b}^2.$$

To find the eigenvalue corresponding to v we need to divide this entry by 2 which is the coefficient of α in v. This gives the required result. □

Remark: Since we have constructed the eigenvalues given in the proposition using a generating set of all eigenvectors of MM^T, as s runs over $0, 1, \ldots, [k/2]$ we have the complete set of eigenvalues of MM^T. In the next section we will show that the eigenvalues λ_s, for $s \in \{0, \ldots, [k/2]\}$, are actually distinct.

We give here a slight variation on the expression for the eigenvalues given in the proposition which will be useful in the next section. We do this by first writing out the binomial coefficient explicitly and rearranging the terms, then we make the substitution $j = i - b$. Next we interchange the sums and finally make a substitution for $i-j$:

$$
\begin{aligned}
\lambda_s &= \frac{1}{2}\sum_{i=0}^{k}(k-i)!i!\sum_{b=0}^{i}(-1)^b\binom{2s}{b}\binom{k-2s}{i-b}^2 \\
&= \frac{1}{2}\sum_{i=0}^{k}(k-2s)!(2s)!\sum_{b=0}^{i}(-1)^b\binom{i}{i-b}\binom{k-2s}{i-b}\binom{k-i}{2s-b} \\
&= \frac{1}{2}(k-2s)!(2s)!\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\binom{k-2s}{j}\binom{k-i}{2s+j-i} \\
&= \frac{1}{2}(k-2s)!(2s)!\sum_{j=0}^{k}\sum_{i=j}^{k}(-1)^{i-j}\binom{i}{j}\binom{k-2s}{j}\binom{k-i}{2s+j-i} \\
&= \frac{1}{2}(k-2s)!(2s)!\sum_{j=0}^{k}\sum_{t=0}^{k-j}(-1)^{t}\binom{t+j}{j}\binom{k-2s}{j}\binom{k-t-j}{2s-t}. \qquad (11)
\end{aligned}
$$

4.3.3 The Eigenvalues of $M^{2,k}(M^{2,k})^T$ in Product Form

The eigenvalues of MM^T given in the previous section can be written in a closed form. In order to do this, we first state and prove a more general equality.

Theorem 4.12 *If s and k are integers satisfying $0 \le s \le [k/2]$ and $k \ge 1$, then*

$$\sum_{j=0}^{k-2s}\sum_{t=0}^{k-j}\binom{k-2s}{j}\binom{t+j}{j}\binom{k-t-j}{2s-t}(-1)^t = 2^{k-2s}\binom{k-s}{s}. \qquad (12)$$

To prove this we need the following:

Lemma 4.13 *The function $f(k,s)=\binom{k-s}{s}2^{k-2s}$ satisfies the following recurrence for $k-s, s \ge 0$ with s and $k-s$ not both zero:*

$$f(k,s) = 2f(k-1,s)+f(k-2,s-1), \qquad (13)$$

where as usual we will take $\binom{a}{b}$ to be zero if $b<0$ or $b>a$.

Proof: We will use a basic equality which can be found on page 1 of [23] and that holds provided that n and k are not both zero:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \qquad (14)$$

Thus we can write

$$\begin{aligned} f(k,s) &= \binom{k-s}{s}2^{k-2s} = 2^{k-2s}\left\{\binom{k-s-1}{s} + \binom{k-s-1}{s-1}\right\} \\ &= 2.2^{k-1-2s}\binom{k-1-s}{s} + 2^{k-2-2(s-1)}\binom{k-2-(s-1)}{s-1} \\ &= 2f(k-1,s) + f(k-2,s-1) \end{aligned}$$

as required. □

Proof of Theorem 4.12: We need to show that

$$X = \sum_{j=0}^{k-2s}\sum_{t=0}^{k-j}\binom{k-2s}{j}\binom{t+j}{j}\binom{k-t-j}{2s-t}(-1)^t \tag{15}$$

satisfies the recurrence (13) for a general k and s satisfying $0 \leq s < k/2$ and that the theorem holds for $k = 2s$. We also need to show that the theorem holds for small values of k and s so that we are able to apply an inductive proof. We begin by verifying the theorem for $k = 2s$. Here we have

$$\sum_{t=0}^{2s}(-1)^t = 1 = 2^0\binom{s}{0}$$

and so the theorem holds when $k = 2s$. When $s = 0$ and k is arbitrary we have

$$\sum_{j=0}^{k}\sum_{t=0}^{k-j}\binom{k}{j}\binom{t+j}{j}\binom{k-t-j}{-t}(-1)^t = \sum_{j=0}^{k}\binom{k}{j} = 2^k = \binom{k}{k}2^k,$$

as required. When $k = 1$ we have $s = t = 0$ and the statement of the theorem is trivial. We can also verify that the theorem holds when $k = 2$ or $k = 3$ and s is arbitrary. We do this using Example 4.8 and the equation for the eigenvalues of MM^T given in (11). Thus all there is left to do is to show that X satisfies (13) for $2s < k$. To do this we need to show that:

$$\begin{aligned} X &= 2\sum_{j=0}^{k-1-2s}\sum_{t=0}^{k-1-j}\binom{k-1-2s}{j}\binom{t+j}{j}\binom{k-1-t-j}{2s-t}(-1)^t \\ &\quad + \sum_{j=0}^{k-2-2(s-1)}\sum_{t=0}^{k-2-j}\binom{k-2-2(s-1)}{j}\binom{t+j}{j}\binom{k-2-t-j}{2(s-1)-t}(-1)^t \end{aligned}$$

or equivalently,

$$
\begin{aligned}
& 2\sum_{j=0}^{k-2s-1}\sum_{t=0}^{k-j-1}\binom{k-2s-1}{j}\binom{t+j}{j}\binom{k-t-j-1}{2s-t}(-1)^t \\
= & \sum_{j=0}^{k-2s}\sum_{t=0}^{k-j}\binom{k-2s}{j}\binom{t+j}{j}\binom{k-t-j}{2s-t}(-1)^t \\
& -\sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s}{j}\binom{t+j}{j}\binom{k-t-j-2}{2s-t-2}(-1)^t.
\end{aligned} \tag{16}
$$

To show that the recurrence relation holds we will repeatedly use (14). Denote the right-hand side of (16) by Y and let

$$
Z=\sum_{j=0}^{k-2s-1}\sum_{t=0}^{k-j-1}\binom{k-2s-1}{j}\binom{t+j}{j}\binom{k-t-j-1}{2s-t}(-1)^t.
$$

So, we would like to show that $Y=2Z$. We can write

$$
\begin{aligned}
Y = & \sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s}{j}\binom{t+j}{j}(-1)^t\left\{\binom{k-t-j}{2s-t}-\binom{k-t-j-2}{2s-t-2}\right\} \\
& +\sum_{j=0}^{k-2s}\binom{k-2s}{j}\left\{(-1)^{k-j-1}\binom{k-1}{j}\binom{1}{k-2s-j}\right. \\
& \left.+(-1)^{k-j}\binom{k}{j}\binom{0}{k-2s-j}\right\}.
\end{aligned}
$$

Using (14) on $\binom{n}{k}$ and $\binom{n-1}{k-1}$ we can write

$$
\binom{n}{k}-\binom{n-2}{k-2}=\binom{n-1}{k}+\binom{n-2}{k-1}.
$$

We can apply this result to the first sum (since $t\neq k-j$ or $k-j-1$) to give:

$$
\begin{aligned}
Y = & \sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s}{j}\binom{t+j}{j}(-1)^t\left\{\binom{k-t-j-1}{2s-t}+\binom{k-t-j-2}{2s-t-1}\right\} \\
& +\binom{k-2s}{k-2s-1}\binom{k-1}{k-2s-1}(-1)^{2s}+\binom{k-2s}{k-2s}\binom{k-1}{k-2s}(-1)^{2s-1} \\
& +\binom{k-2s}{k-2s}\binom{k}{k-2s}(-1)^{2s}
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s}{j}\binom{t+j}{j}(-1)^t\binom{k-t-j-1}{2s-t} \\
& +\sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s}{j}(-1)^t\binom{t+j}{j}\binom{k-2-t-j}{2s-t-1} \\
& +(k-2s)\binom{k-1}{k-2s-1}-\binom{k-1}{k-2s}+\binom{k}{k-2s}.
\end{aligned}
$$

Using (14) on the first sum (as $k \neq 2s$):

$$
\begin{aligned}
Y = & \sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j}\binom{t+j}{j}(-1)^t\binom{k-t-j-1}{2s-t} \\
& +\sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j-1}\binom{t+j}{j}(-1)^t\binom{k-t-j-1}{2s-t} \\
& +\sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s}{j}(-1)^t\binom{t+j}{j}\binom{k-2-t-j}{2s-t-1} \\
& +(k-2s)\binom{k-1}{k-2s-1}+\binom{k-1}{k-2s-1}.
\end{aligned}
$$

Turning the first sum into the form we want it for the right hand side:

$$
\begin{aligned}
Y = & \sum_{j=0}^{k-2s-1}\sum_{t=0}^{k-j-1}\binom{k-2s-1}{j}\binom{t+j}{j}(-1)^t\binom{k-t-j-1}{2s-t} \\
& -\sum_{j=0}^{k-2s-1}\binom{k-2s-1}{j}\binom{k-1}{j}\binom{0}{2s-k+j+1}(-1)^{k-j-1} \\
& +\sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j-1}\binom{t+j}{j}(-1)^t\binom{k-t-j-1}{2s-t} \\
& +\sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s}{j}(-1)^t\binom{t+j}{j}\binom{k-2-t-j}{2s-t-1} \\
& +(k-2s)\binom{k-1}{k-2s-1}+\binom{k-1}{k-2s-1}.
\end{aligned}
$$

Using (14) on the fourth sum (since $k \neq 2s$):

$$
Y = Z+\sum_{j=0}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j-1}\binom{t+j}{j}(-1)^t\binom{k-t-j-1}{2s-t}
$$

$$-\binom{k-1}{2s}+\sum_{j=0}^{k-2s-1}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j}(-1)^t\binom{t+j}{j}\binom{k-2-t-j}{2s-t-1}$$
$$+\sum_{j=1}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j-1}(-1)^t\binom{t+j}{j}\binom{k-2-t-j}{2s-t-1}$$
$$+(k-2s+1)\binom{k-1}{k-2s-1}.$$

Using the substitution $t^*=t+1$ in the third sum:

$$\begin{aligned}
Y &= Z+\sum_{j=1}^{k-2s}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j-1}\binom{t+j}{j}(-1)^t\binom{k-t-j-1}{2s-t}\\
&+\sum_{j=0}^{k-2s-1}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j}(-1)^t\binom{t+j}{j}\binom{k-2-t-j}{2s-t-1}\\
&+(k-2s)\binom{k-1}{k-2s-1}\\
&+\sum_{j=1}^{k-2s}\sum_{t^*=1}^{k-j-1}\binom{k-2s-1}{j-1}(-1)^{t^*-1}\binom{t^*+j-1}{j}\binom{k-t^*-j-1}{2s-t^*}.
\end{aligned}$$

Combining the first and last sums gives:

$$\begin{aligned}
Y &= Z+\sum_{j=1}^{k-2s}\sum_{t=0}^{k-j-1}\binom{k-2s-1}{j-1}(-1)^t\binom{k-t-j-1}{2s-t}\left\{\binom{t+j}{j}-\binom{t+j-1}{j}\right\}\\
&-\sum_{j=1}^{k-2s}\binom{k-2s-1}{j-1}(-1)^{k-j-1}\binom{k-1}{j}\binom{0}{2s-k+j+1}\\
&+\sum_{j=0}^{k-2s-1}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j}(-1)^t\binom{t+j}{j}\binom{k-2-t-j}{2s-t-1}\\
&+\binom{k-1}{k-2s-1}(k-2s).
\end{aligned}$$

Using (14) on first sum (since $j\neq 0$) and then substituting $j^*=j-1$:

$$\begin{aligned}
Y &= Z+\sum_{j^*=0}^{k-2s-1}\sum_{t=0}^{k-j^*-2}\binom{k-2s-1}{j^*}(-1)^t\binom{k-t-j^*-2}{2s-t}\binom{t+j^*}{j^*}\\
&+\sum_{j=0}^{k-2s-1}\sum_{t=0}^{k-j-2}\binom{k-2s-1}{j}(-1)^t\binom{t+j}{j}\binom{k-2-t-j}{2s-t-1}\\
&+\binom{k-1}{k-2s-1}(k-2s)-\binom{k-1}{k-2s-1}(k-2s-1).
\end{aligned}$$

Combining the first and second sums:

$$\begin{aligned} Y &= Z + \sum_{j=0}^{k-2s-1} \sum_{t=0}^{k-j-2} (-1)^t \binom{k-2s-1}{j} \binom{t+j}{j} \left\{ \binom{k-t-j-2}{2s-t} + \right. \\ & \left. \binom{k-t-j-2}{2s-t-1} \right\} + \binom{k-1}{k-2s-1} \{(k-2s)-(k-2s-1)\}. \end{aligned}$$

Using (14) on the remaining sum (since $t \neq k-j-1$):

$$\begin{aligned} Y &= Z + \sum_{j=0}^{k-2s-1} \sum_{t=0}^{k-j-2} \binom{k-2s-1}{j} (-1)^t \binom{t+j}{j} \binom{k-t-j-1}{2s-t} \\ & + \binom{k-1}{k-2s-1} \\ &= 2Z - \sum_{j=0}^{k-2s-1} \binom{k-2s-1}{j} (-1)^{k-j-1} \binom{k-1}{j} \binom{0}{2s-k+j+1} \\ & + \binom{k-1}{k-2s-1} \\ &= 2Z. \end{aligned}$$

This proves the required result for $2s < k$ and the proof is now complete. □

We can now write down the eigenvalues of MM^T in a much simpler way. We remark that the eigenvalues of MM^T have already been written in this form in [8] although the method used by Coker to find these eigenvalues was different from the method we used.

Theorem 4.14 *Let M be the incidence matrix for $\mathcal{P}_{2,k}$. Then the complete set of distinct eigenvalues of the matrix MM^T are given by*

$$\lambda_s = (2s)!(k-2s)! \binom{k-s}{k-2s} 2^{k-2s-1} = (k-s)!\ s!\ 2^{k-2s-1} \binom{2s}{s}$$

for $s = 0, 1, \ldots, [k/2]$. Moreover, the multiplicity of each eigenvalue λ_s is

$$\binom{2k}{2s} - \binom{2k}{2s-1}.$$

Proof: The formula for the eigenvalues follows immediately from (11) and Theorem 4.12. It is easy to see that the eigenvalues are distinct as s runs over the integers between 0 and $[k/2]$. Thus the eigenvalue λ_s is associated to precisely one partition of $2k$, namely $\mu^s = (2k-2s, 2s)$ and so E_{λ_s} is isomorphic to S^{μ^s}. We can use this fact to find the multiplicity of λ_s as an eigenvalue of MM^T which will be the dimension of the the Specht module S^{μ^s}. The dimension of Specht modules are known over fields of characteristic zero and can be calculated using hook lengths (see Theorem 2.35). So the multiplicity of λ_s as an eigenvalue of MM^T is

$$\frac{(2k)!(2k-4s+1)}{(2k-2s+1)!(2s)!} = \frac{(2k)!}{(2s-1)!(2k-2s)!}\left\{\frac{1}{2s} - \frac{1}{(2k-2s+1)}\right\}$$
$$= \binom{2k}{2s} - \binom{2k}{2s-1}.$$

This completes the proof. □

Remark: After completing this proof Professor Paule [21] suggested the following alternative way of reducing the eigenvalues given by equation (11) to the form given in Theorem 4.14. This was done by his diploma student Kurt Wegschaider who has implemented algorithms in MATHEMATICA which deliver recurrences for given multi-sums (the underlying theory of which is described in the paper [28]). Calling the summand term $F(k,t,j)$ of equation (11) (including the terms outside the sum), we get the following recurrence for it.

$$-2(k-s)(k-2s)F[k-1,t-1,j]+(k-2s)F[k,t-1,j] =$$
$$\Delta_t((k-2s)(t+k-2s-1)F[k-1,t-1,j]+(t-j)F[k,t-1,j]-F[k,t-1,j+1]) + \Delta_j(-F[k,t-1,j]+(1+t-j)F[k,t,j])$$

(where Δ is the (forward) difference operator $\Delta_t f(t,j,...) := f(t+1,j,...) - f(t,j,...)$ or $\Delta_j f(t,j,...) := f(t,j+1,...) - f(t,j,...)$). It easily seen that the double sum is a standard sum, i.e. we can sum t and j over all integer values. Then the sum fulfills the recurrence

$$-2(k-s)SUM[k-1] + SUM[k] = 0 \tag{17}$$

which too is satisfied by the expression for the eigenvalues given in Theorem 4.14. So after checking the initial value $k = 2s$ the identity is proved. To prove the equality in this requires a similar amount of work and rearranging of the terms in the sum.

Example 4.15 *We list the eigenvalues λ_i of MM^T for $k = 1, 2, \ldots, 10$ and $i = 0, \ldots, [k/2]$. Underneath we list the corresponding multiplicities m_i of λ_i (for $i = 0, \ldots, [k/2]$) in the same fashion.*

	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
$k = 1$	1					
$k = 2$	4	1				
$k = 3$	24	4				
$k = 4$	192	24	12			
$k = 5$	1920	192	72			
$k = 6$	23040	1920	576	360		
$k = 7$	322560	23040	5760	2880		
$k = 8$	5160960	322560	69120	28800	20160	
$k = 9$	92897280	5160960	967680	345600	201600	
$k = 10$	1857945600	92897280	15482880	4838400	2419200	1814400

	m_0	m_1	m_2	m_3	m_4	m_5
$k = 1$	1					
$k = 2$	1	2				
$k = 3$	1	9				
$k = 4$	1	20	14			
$k = 5$	1	35	90			
$k = 6$	1	54	275	132		
$k = 7$	1	77	637	1001		
$k = 8$	1	104	1260	3640	1430	
$k = 9$	1	135	2244	9996	11934	
$k = 10$	1	170	3705	23256	48450	16796

Remark: It is easy to see that the eigenvalues of MM^T fall into natural order of magnitude with $\lambda_0 > \lambda_1 > \cdots > \lambda_{[k/2]}$. Moreover this ordering is the same as the ordering for the partitions associated to these eigenvalues, that is $\lambda_i > \lambda_j$ if and only if $(2k - 2i, 2i) \rhd (2k - 2j, 2j)$.

4.4 Modules in the Decomposition of $FP(k,2)$

In this section we show that each module which appears in $FP(2,k)$ appears in $FP(k,2)$. We use two different methods to do this, the first is just a direct consequence of the results given in the previous section and the second is a direct approach. These results verify Foulkes' conjecture (see Section 2.2.6) for the case when $m = 2$ and n is arbitrary.

In the previous section we showed that all eigenvalues of MM^T are non-zero which means that M has full rank. Since $FP(2,k) \simeq S^{\mu^0} \oplus S^{\mu^1} \oplus \cdots \oplus S^{\mu^{[k/2]}}$ where $\mu^s = (2k-2s, 2s)$ it follows then that $FP(k,2) \simeq S^{\mu^0} \oplus S^{\mu^1} \oplus \cdots \oplus S^{\mu^{[k/2]}} \oplus V$ for some module V. This is what Foulkes' conjectured for the case when $m = 2$ and $n = k$. An alternative way to show that $FP(k,2)$ can be written in the form $FP(k,2) \simeq S^{\mu^0} \oplus S^{\mu^1} \oplus \cdots \oplus S^{\mu^{[k/2]}} \oplus V$ is to show that for each $s \in \{0, 1, \ldots, [k/2]\}$ there is a semistandard μ^s-tableau T of type (2^k) such that $\bar{\Theta}_T\{t\}\kappa_t$ is non-zero for some μ^s-tableau t. If this is the case then S^{μ^s} must appear in $FP(k,2)$.

Proposition 4.16 *For $s = 0, 1, \ldots, [k/2]$ the module $S^{(2k-2s,2s)}$ appears in $FP(k,2)$.*

Proof: Let T be the semistandard $(2k-2s, 2s)$-tableau and t the $(2k-2s)$ tableau given by

$$T = \begin{matrix} 1\,1\,3\,3 \ldots 2s-1\;2s-1\;2s+1\;2s+1 \ldots k\;k \\ 2\,2\,4\,4 \ldots \;2s \quad\;\; 2s \end{matrix}$$

and

$$t = \begin{matrix} 1\,3\,5\,7 \ldots 4s-3\;4s-1\;4s+1\;4s+2 \ldots 2k \\ 2\,4\,6\,8 \ldots 4s-2\;4s \end{matrix}.$$

It is easy to see that the coefficients in $\hat{\Theta}_T\{t\}\kappa_t$ of all tableaux $\sim$-equivalent to T are positive. Thus the coefficient of $\bar{T}$ in $\bar{\Theta}_T\{t\}\kappa_t$ is non-zero and so from Theorem 2.20 $S^{(2k-2s,2s)}$ appears in $FP(k,2)$. □

Remark: If F is a field of characteristic p where p does not divide any of the eigenvalues of MM^T given in Theorem 4.14 then M will have full rank over F. In the expression for the eigenvalues of MM^T we see that each term is less than or equal to $k!$. Since $\lambda_0 = k!\,2^{k-1}$ is always an eigenvalue of MM^T, it is clear that the eigenvalues of MM^T are non-zero over F precisely when $p > k$. Thus, for any field of characteristic p

where $p > k$ the module $FP(2, k)$ is isomorphic to a submodule of $FP(k, 2)$. This shows that the modular version of Foulkes' conjecture holds for all but a few primes.

Remark: The complete decomposition of $FP(k, 2)$ with k arbitrary can be found in [26]. This decomposition can be written as follows

$$FP(k, 2) \simeq \sum S^{(2\mu_1, 2\mu_2, \ldots, 2\mu_r)}$$

where the sum runs over all partitions μ of k. For each Specht module S^μ which appears in the decomposition it is easy to construct a semistandard μ-tableau T of type (2^k) (for example the tableau with the pairs of integers between 1 and k placed in increasing order along the rows of the tableau) such that $\bar{\Theta}_T\{t\}\kappa_t$ is non-zero for some μ-tableau t and hence showing that S^μ appears in $FP(k, 2)$.

Chapter 5

$(3, k)$-Partitions

We will use the same notation as in Chapter 3. Recall that if μ is a partition of $3k$ then T will always denote a semistandard μ-tableau of type (k^3) and unless otherwise stated t will be the μ-tableau with the digits $1, 2, \ldots, 3k$ placed in increasing order down its columns. In the first section we will consider the incidence structure $\mathcal{P}_{3,k}$ and show that this has diameter equal to one. We then compute the number of intersection classes of $P(3, k) \times P(3, k)$. It is this result that we use in calculating the number of irreducible modules in the decomposition of $FP(3, k)$. We study the different semistandard tableaux of type (k^3) and use them to construct semistandard homomorphisms from S^μ to $FP^o(3, k)$. The modules which appear in $FP(3, k)$ comprise of a subset of the modules which appear in $FP^o(3, k)$ and we show precisely which ones they are. We use our knowledge of the modules which appear in $FP(3, k)$ to construct some of the eigenvectors and eigenvalues of MM^T, where M is the incidence matrix for $\mathcal{P}_{3,k}$, which we will do in Sections 5.4.3 and 6.3 respectively.

5.1 The Diameter of $\mathcal{P}_{3,k}$

We begin by giving an indication of the size of the incidence matrix of $\mathcal{P}_{3,k}$ for some small values of k before showing that this structure has diameter equal to one.

Example 5.1 *For $1 \leq k \leq 5$ the number of elements in $P(3, k)$ and $P(k, 3)$ are given in the table.*

k	$\|P(3,k)\|$	$\|P(k,3)\|$
1	1	1
2	15	10
3	280	280
4	5775	15400
5	126126	1401400
6	2858856	190590400
7	66512160	36212176000
8	1577585295	9161680527750

Proposition 5.2 *The incidence structure $\mathcal{P}_{3,k}$ has diameter equal to one.*

Proof: We need to show that for any pair of $(3,k)$-partitions α and γ there exists a $(k,3)$-partition which intersects nicely with both α and γ. It is easy to see that there exists $\alpha^o = \uplus_{i=1}^{3} \alpha_i^o$ and $\gamma^o = \uplus_{j=1}^{3} \gamma_j^o$ in $P^o(3,k)$ which are contained in the $\sim$-equivalence classes represented by α and γ respectively and such that the intersection array $I(\alpha^o, \gamma^o)$ is of the form

$$I(\alpha^o, \gamma^o) = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}$$

with $r \geq max\{s,t,u,v,w,x,y,z\}$, $s \geq t$ and $u \geq x$. Since the row and column sum of the array is k we have $t = k - r - s$, $w = k - u - v$, $x = k - r - u$, $y = k - s - v$ and $z = r + s + u + v - k$. Without loss of generality assume that $s \leq u$ (otherwise change the roles of α and γ). It is clear that a $(k,3)$-partition β intersects nicely with α if and only if each part of β contains one element from each part of α^o. Thus it will be sufficient to find a $(k,3)$-partition whose parts have one element in common with each part of α^o and one element in common with each part of γ^o. We can arrange the digits in each part of α^o into three sets according to which part of γ^o they intersect with. On a 'diagram' for α^o we illustrate, using three different fonts, which part of γ^o the elements in each part of α^o are in. That is we will use italic font to represent the size of the sets in the parts of α^o which are contained in γ_1^o. Similarly, we use bold face font to represent the size of the sets in the parts of α^o which are contained in γ_2^o and typewriter font to represent the size of the sets in the parts of α^o which are contained in γ_3^o. Thus to find a $(k,3)$-partition β which intersects nicely with α and γ we need to arrange the sets in each part of α^o so that 'looking' down the columns of α^o we see each font exactly once. We illustrate such a 'solution' diagrammatically, drawing vertical lines between the sets in each part of α^o (note that the sets are not drawn to scale):

r	$\mathtt{s}$		$\mathtt{t}$
$\mathtt{w}$	$\mathbf{r-w}$	u	$\mathtt{t-u+s}$.
$\mathbf{w}$	$\mathtt{r-w+s}$	$\mathbf{u-s}$	$t-u+s$

All we need to check is that the size of each set is non-negative and for all i and j with $1 \leq i,j \leq 3$ that $\alpha_i^o \cap \gamma_j^o$ is the correct size. It is clear by our original assumptions that all the set sizes are non-negative. Since $r-w+t-u+s=v$, $t-u+s=k-r-u=x$, $w+u-s=k-v-s=y$ and $r-w+s=z$ this shows that α_i^o intersects γ_j^o in a set of the correct size. The proof is now complete. □

Remark: There is not necessarily a unique element which intersects nicely with both α and γ, the above proof merely proves existence of at least one such element.

Since the diameter of the structure equals one and the (α,γ)-entry of MM^T counts the number of elements which intersect nicely with α and γ, we know that all entries of MM^T are non-zero.

5.2 Number of Intersection Classes of $P(3,k) \times P(3,k)$

In the following example we list the different arrays indexing the intersection classes of $P(3,k) \times P(3,k)$ with $1 \leq k \leq 4$.

Example 5.3 *The intersection classes of $P(3,1) \times P(3,1)$ and $P(3,2) \times P(3,2)$ can be indexed respectively by*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

For $P(3,3) \times P(3,3)$ the classes can be indexed by:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

For $P(3,4) \times P(3,4)$ the classes can be indexed by:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

Remark: When $a = 3$ and $1 \leq b \leq 5$ it is easy to check that the intersection classes can all be indexed by symmetric arrays. However, for $a = 3$ and $b = 6$ the intersection classes are not all symmetric. In particular there is not an intersection class which contains both

$$\begin{bmatrix} 4 & 0 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 1 & 1 \\ 0 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}.$$

Theorem 5.4 *The number of intersection classes of $P(3,k) \times P(3,k)$ is*

$$\begin{cases} \frac{1}{288}\{k^4 + 6k^3 + 64k^2 + 192k + 160 + 128c\} & \text{if } k \text{ is even} \\ \frac{1}{288}\{k^4 + 6k^3 + 64k^2 + 138k + 79 + 128c\} & \text{if } k \text{ is odd.} \end{cases}$$

where

$$c = \begin{cases} 1 & \text{if 3 divides } k \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The number of intersection classes of $P(3,k) \times P(3,k)$ is the number of orbits of $Sym(3) \times Sym(3)$ on the set of 3×3 integer stochastic matrices with row and column sum k (see Section 3.2.3). Let $H = Sym(3) \times Sym(3)$. Then H acts on the set X of 3×3 integer stochastic matrices with row and column sum k, by respectively permuting the rows and columns of the matrix. That is if $(h_1, h_2) \in H$ then h_1 permutes the rows of the matrix and h_2 permutes the columns of the matrix. Since H is a finite group and X is a finite set, the number of orbits of H on X is given by the orbit counting theorem (Cauchy and Frobenius):

$$\frac{1}{|H|} \sum_{h \in H} |Fix(h)|,$$

where of course $Fix(h)$ is the set of all elements of X fixed by h. We note that by equation (8) of Section 3.2.3 the cardinality of X is

$$H_3(k) = \binom{k+4}{4} + \binom{k+3}{4} + \binom{k+2}{4}.$$

We begin by listing the different types of elements in H and the number of each type.

Element Type	No.
$h_1 = id \times id$	1
$h_2 = id \times$ 2-cycle	3
$h_3 = id \times$ 3-cycle	2
$h_4 =$ 2-cycle $\times\, id$	3
$h_5 =$ 2-cycle $\times$ 2-cycle	9
$h_6 =$ 2-cycle $\times$ 3-cycle	6
$h_7 =$ 3-cycle $\times\, id$	2
$h_8 =$ 3-cycle $\times$ 2-cycle	6
$h_9 =$ 3-cycle $\times$ 3-cycle	4

For a given integer stochastic matrix x in X, all matrices which can be obtained from x by permuting its rows and columns are also in X. Hence it is enough to consider one element of each type from H, that is without loss of generality we can choose which rows and columns an element from each type permutes. We study each case separately below.

Case 1: The identity element fixes all integer stochastic matrices, so fixes $H_3(k)$ matrices.

Case 2: An element $h_2 = id \times (1\,2)$ fixes matrices of the form:

$$\begin{pmatrix} r & r & k-2r \\ u & u & k-2u \\ k-r-u & k-r-u & 2r+2u-k \end{pmatrix}.$$

Each of the entries of the matrix must be greater than or equal to zero. Therefore the number of matrices which h_2 fixes is the number of non-negative integers r, u, x such that $r, u, x \leq \frac{k}{2}$ and $r + u + x = k$. When k is odd, the possibilities for r, u and x are given in the table below.

r	u	x	r	u	x	$\cdots$	r	u	x
$\frac{k-1}{2}$	$\frac{k-1}{2}$	1	$\frac{k-3}{2}$	$\frac{k-1}{2}$	2		1	$\frac{k-1}{2}$	$\frac{k-1}{2}$
$\frac{k-1}{2}$	$\frac{k-3}{2}$	2	$\frac{k-3}{2}$	$\frac{k-3}{2}$	3				
$\frac{k-1}{2}$	$\frac{k-5}{2}$	3	$\frac{k-3}{2}$	$\frac{k-5}{2}$	4				
	$\vdots$			$\vdots$					
$\frac{k-1}{2}$	2	$\frac{k-3}{2}$	$\frac{k-3}{2}$	2	$\frac{k-1}{2}$				
$\frac{k-1}{2}$	1	$\frac{k-1}{2}$							

Thus when k is odd, the number of matrices fixed by h_2 is

$$\frac{k-1}{2} + \frac{k-3}{2} + \frac{k-5}{2} + \cdots + 1 = \frac{1}{2}\left(\frac{k+1}{2}\right)\left(\frac{k-1}{2}\right) = \frac{1}{8}(k+1)(k-1).$$

Similarly when k is even, the number of matrices fixed by h_2 is

$$\frac{k+2}{2} + \frac{k}{2} + \cdots + 1 = \frac{1}{2}\left(\frac{k+4}{2}\right)\left(\frac{k+2}{2}\right) = \frac{1}{8}(k+4)(k+2).$$

Case 3: The element $h_3 = id \times (1\,2\,3)$ fixes matrices which have all entries the same and equal to $k/3$. The entries of the matrix must be integer so if k is not divisible by 3 then there are no elements fixed by h_3. If k is divisible by 3 then there is one matrix fixed by h_3.

Case 4: The element $h_4 = (1\,2) \times id$ fixes the same number of matrices as h_2 does.

Case 5: The element $h_5 = (1\,2) \times (1\,2)$ fixes matrices of the form

$$\begin{pmatrix} r & s & k-r-s \\ s & r & k-r-s \\ k-r-s & k-r-s & 2r+2s-k \end{pmatrix}.$$

The number of matrices of this type is the number of non-negative integers r, s, t such that $t \leq k/2$ and $r + s + t = k$.

r	s	t	r	s	t	$\cdots$	r	s	t
0	k	0	0	$k-1$	1		0	$k - \left[\frac{k}{2}\right]$	$\left[\frac{k}{2}\right]$
1	$k-1$	0	1	$k-2$	1		1	$k - \left[\frac{k}{2}\right] - 1$	$\left[\frac{k}{2}\right]$
2	$k-2$	0	2	$k-3$	1			$\vdots$	
	$\vdots$			$\vdots$			$\left[\frac{k+1}{2}\right]$	0	$\left[\frac{k}{2}\right]$
$k-1$	1	0	$k-1$	0	1				
k	0	0							

Therefore the number of matrices fixed by h_5 is

$$(k+1) + k + (k-1) + \cdots + \left(\left[\frac{k+1}{2}\right] + 1\right) = \frac{1}{2}\left(\left[\frac{k}{2}\right] + 1\right)\left(\left[\frac{k+1}{2}\right] + k + 2\right)$$

$$= \begin{cases} \frac{1}{8}(k+2)(3k+4) & \text{if } k \text{ is even} \\ \frac{1}{8}(k+1)(3k+5) & \text{if } k \text{ is odd.} \end{cases}$$

Case 6: The element $h_6 = (1\,2) \times (1\,2\,3)$ fixes matrices of the form

$$\begin{pmatrix} r & r & r \\ r & r & r \\ k-2r & k-2r & k-2r \end{pmatrix}.$$

Thus we must have $r = k - 2r = k/3$ so the number of matrices fixed by h_6 is the same as the number of matrices fixed by h_3.

Case 7: The element $h_7 = (1\,2\,3) \times id$ fixes the same number of matrices as h_3.

Case 8: The element $h_8 = (1\,2\,3) \times (1\,2)$ fixes the same number of matrices as h_6 does.

Case 9: The element $h_9 = (1\,2\,3) \times (1\,2\,3)$ fixes matrices of the form

$$\begin{pmatrix} r & s & k-r-s \\ k-r-s & r & s \\ s & k-r-s & r \end{pmatrix}.$$

The number of matrices of this type is the number of non-negative integers r, s, t such that $r+s+t=k$. By drawing up a table of possible values for r, s and t, it is easy to check that the number of matrices fixed by h_9 is

$$1+2+3+\cdots+k+(k+1) = \frac{(k+2)(k+1)}{2}.$$

Therefore, by the orbit counting theorem, the number of orbits of H on X when k is odd is

$$\frac{1}{36}\{\binom{k+4}{4} + \binom{k+3}{4} + \binom{k+2}{4} + \frac{3}{4}(k+1)(k-1) + \frac{9}{8}(k+1)(3k+5)$$

$$+2(k+2)(k+1) + 16c\}$$

$$= \frac{1}{288}\left\{k^4 + 6k^3 + 64k^2 + 138k + 79 + 128c\right\}.$$

When k is even the number of orbits is

$$\frac{1}{36}\{\binom{k+4}{4} + \binom{k+3}{4} + \binom{k+2}{4} + \frac{3}{4}(k+4)(k+2) + \frac{9}{8}(k+2)(3k+4)$$

$$+2(k+2)(k+1) + 16c\}$$

$$= \frac{1}{288}\left\{k^4 + 6k^3 + 64k^2 + 192k + 160 + 128c\right\}$$

where

$$c = \begin{cases} 1 & \text{if 3 divides } k \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof. □

Example 5.5 *The number of intersection classes of $P(3,k) \times P(3,k)$ for small values of k are listed in the following table:*

k	*no. of classes*
1	1
2	3
3	5
4	9
5	13
6	22
7	30
8	45
9	61
10	85

Remark: The sequence of numbers given in the above table is a known sequence (see sequence 973 of [24]) given by an expansion of a generating function.

5.3 Semistandard Tableaux of Type (k^3)

As semistandard tableaux of type (k^3) play an important part in finding irreducible modules in the decomposition of $FP^{o}(3,k)$ and $FP(3,k)$, for a partition μ of $3k$ we will consider the different kinds of semistandard μ-tableaux of type (k^3).

A partition μ of $3k$ can be written in the form $\mu = (3k-s-2r, s+r, r)$ for some non-negative integers r and s satisfying $2s+3r \leq 3k$. Thus, in general, if $r+s \leq k$ a semistandard tableau of type (k^3) is of the form

$$\begin{array}{l} 11\ldots1\,11\ldots1\,11\ldots1\,\underbrace{11\ldots1\,22\ldots2\,33\ldots3}_{3k-2s-3r} \\ 22\ldots2\,\underbrace{22\ldots2}_{s-w}\,\underbrace{33\ldots3}_{w} \\ \underbrace{33\ldots3}_{r} \end{array}.$$

If $r+s > k$ a semistandard tableau of type (k^3) is of the form

$$\begin{array}{l} 11\ldots1\,11\ldots1\,11\ldots1\,22\ldots2\,\underbrace{22\ldots2\,33\ldots3}_{3k-2s-3r} \\ 22\ldots2\,\underbrace{22\ldots2}_{s-w}\,\underbrace{33\ldots3\,33\ldots3}_{w} \\ \underbrace{33\ldots3}_{r} \end{array}.$$

If $r + s \leq k$ then $0 \leq w \leq s$ and if $r + s > k$ then $2(s + r - k) \leq w \leq k - r$. Thus, in general w is a non-negative integer satisfying $max\{0, 2(s + r - k)\} \leq w \leq min\{s, k - r\}$. For a fixed w in the appropriate range we will denote by T_w the semistandard $(3k - s - 2r, s + r, r)$-tableau with 3 appearing w times in its second row.

Example 5.6 *The semistandard tableaux of type* (3^3) *are as follows:*

111222333	11122333 2	11122233 3	1112333 22	1112233 23
1112223 33	111333 222	111233 223	111223 233	111222 333
11123 2233	11122 2333	1112233 2 3	111233 22 3	111223 23 3
11133 222 3	11123 223 3	11122 233 3	1112 2233 3	11123 22 33
1113 222 33	1112 223 33	111 222 333		

Lemma 5.7 *For fixed non-negative integers r and s satisfying $2s + 3r \leq 3k$, let $\mu = (3k - s - 2r, s + r, r)$. Then the number of semistandard μ-tableaux of type (k^3) is $s + 1$ if $k \geq r + s$ and $3k - 2s - 3r + 1$ if $k < r + s$.*

Proof: All semistandard $(3k - s - 2r, s + r, r)$-tableaux of type (k^3) are of the form T_w for some w in the appropriate range. Moreover, for each value of w we have a distinct semistandard tableau. Hence the result follows straight from the restrictions on w. □

5.4 The Decomposition of $FP^o(3,k)$ and $FP(3,k)$

5.4.1 Modules in the Decomposition of $FP^o(3,k)$

From Section 3.2.1 we know that all irreducible modules in the decomposition of $FP^o(3,k)$ are isomorphic to Specht modules S^μ where $\mu = (\mu_1, \mu_2, \mu_3)$ is a partition of $3k$ with at most three parts. Since the semistandard homomorphisms $\hat{\Theta}_T$ form a basis for $Hom_{FSym(3k)}(S^\mu, FP^o(3,k))$ and $dim(Hom_{FSym(3k)}(S^\mu, FP^o(3,k)))$ equals the number of semistandard μ-tableaux of type (k^3) we can calculate precisely how many irreducible modules are in the decomposition of $FP^o(3,k)$. We state the result here but remark that it may already be a known result.

Lemma 5.8 *The number of irreducible modules in the decomposition of $FP^o(3,k)$ is*

$$\begin{cases} \frac{(k+2)}{8}\{2k^2+5k+4\} & \text{if } k \text{ is even} \\ \frac{(k+1)}{8}\{2k^2+7k+7\} & \text{if } k \text{ is odd.} \end{cases}$$

Proof: Let $\mu = (3k-s-2r, s+r, r)$. Then from Lemma 5.7 the multiplicity of S^μ in $FP^o(3,k)$ is $s+1 = \mu_2 - \mu_3 + 1$ if $k \geq r+s$ and $3k-2s-3r+1 = \mu_1 - \mu_2 + 1 = 3k - 2\mu_2 - \mu_3 + 1$ if $k < r+s$. To calculate the number of irreducible modules in the decomposition of $FP^o(3,k)$ we sum over the possibilities for μ. Thus the number of irreducible modules in the decomposition of $FP^o(3,k)$ is:

$$\sum_{\mu_2=0}^{k}\sum_{\mu_3=0}^{\mu_2}(\mu_2-\mu_3+1) + \sum_{\mu_3=0}^{k-2}\sum_{\mu_2=k+1}^{\left[\frac{3k-\mu_3}{2}\right]}(3k-2\mu_2-\mu_3+1).$$

The first sum is

$$\begin{aligned} & 1+(1+2)+(1+2+3)+\cdots+(1+2+3+\cdots+(k+1)) \\ = & \frac{1}{2}\{1\cdot 2+2\cdot 3+3\cdot 4+4\cdot 5+\cdots+(k+1)(k+2)\} \\ = & \frac{1}{6}(k+1)(k+2)(k+3). \end{aligned}$$

The second sum is

$$\begin{cases} 1+(1+2+3)+(1+2+3+4+5)+\cdots+(1+\cdots+(k-1)) & \text{if } k \text{ is even} \\ (1+2)+(1+2+3+4)+\cdots+(1+2+3+\cdots+(k-1)) & \text{if } k \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{1}{2}\{1\cdot 2+3\cdot 4+5\cdot 6+\cdots+(k-1)k\} & \text{if } k \text{ is even} \\ \frac{1}{2}\{2\cdot 3+4\cdot 5+6\cdot 7+\cdots+(k-1)k\} & \text{if } k \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{k(k+2)}{24}\{2k-1\} & \text{if } k \text{ is even} \\ \frac{(k-1)(k+1)}{24}\{2(k-1)+5\} & \text{if } k \text{ is odd.} \end{cases}$$

Adding the two sums together we get that the number of irreducible modules in the decomposition of $FP^o(3,k)$ is

$$\begin{cases} \frac{(k+2)}{8}\{2k^2+5k+4\} & \text{if } k \text{ is even} \\ \frac{(k+1)}{8}\{2k^2+7k+7\} & \text{if } k \text{ is odd} \end{cases}$$

as required. □

The dimension of each of these modules can be calculated using hook lengths (as defined in Section 2.2.4).

Lemma 5.9 *If* $\mu = (\mu_1, \mu_2, \mu_3)$ *is a partition of* $3k$ *then the dimension of* S^μ *is given by*

$$\frac{(3k)!\,(\mu_1-\mu_3+2)\,(\mu_1-\mu_2+1)\,(\mu_2-\mu_3+1)}{(\mu_1+2)!\,(\mu_2+1)!\,\mu_3!}.$$

5.4.2 The Complete Decomposition of $FP(3,k)$

In this section we will completely decompose the module $FP(3,k)$. From Corollary 3.9 we know that for $c \in \{0,1,\ldots,k\}$ the module $S^{(3k-2c,c,c)}$ appears in $FP(3,k)$ precisely when c is even and in this case the multiplicity of this module in $FP(3,k)$ is one. Thus we have already found $[k/2]+1$ irreducible modules in the decomposition of $FP(3,k)$. Theorem 3.8 also provides a source of modules in the decomposition of $FP(3,k)$: If $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*) \vdash 3(k-2)$ and S^{μ^*} appears in $FP(3,k-2)$ with multiplicity $m \geq 0$ then S^μ appears in $FP(3,k)$ with multiplicity $m \geq 0$, where $\mu = (\mu_1^*+2, \mu_2^*+2, \mu_3^*+2) \vdash 3k$. To use this inductive way of finding irreducible modules in the decomposition of $FP(3,k)$ we need to find the complete decompositions of $FP(3,1)$ and $FP(3,2)$. Trivially, $FP(3,1)$ is isomorphic to $S^{(3)}$ so we only need to consider $FP(3,2)$. We know from Corollary 3.9 that $S^{(6)}$ and $S^{(2,2,2)}$ appear in

$FP(3,2)$. Moreover, the results of Section 4.4 show that $S^{(4,2)}$ appears in $FP(3,2)$. A simple calculation summing the dimension of these modules shows that they all have multiplicity one and form a complete set of irreducible modules in the decomposition of $FP(3,2)$.

Remark: The fact that every irreducible module in the decomposition of $FP(3,1)$ and $FP(3,2)$ has multiplicity one is not a surprise to us. By the remark following Example 5.3 we know that when $a=3$ and $1 \le k \le 5$, the partition scheme $PS(3,k)$ is commutative and so all irreducible modules in the decomposition of the permutation module $FP(3,k)$ will have multiplicity one (see the end of Section 3.2.3).

Next we consider the partitions with two rows, that is partitions of the form $(3k-s,s)$ for $s=0,1,\ldots,k+[k/2]$. In the following we give a lower bound for the multiplicity of $S^{(3k-s,s)}$ in the decomposition of $FP(3,k)$.

Proposition 5.10 *For $s \in \{0,1,\ldots,k+[k/2]\}$ let d be the non-negative integer satisfying $0 \le d \le 2$ so that 3 divides $2s-d$ when s is even and 3 divides $2s-d-2$ when s is odd. Then the Specht module $S^{(3k-s,s)}$ appears in $FP(3,k)$ with multiplicity greater than or equal to $m_{k,s}$ which is given by:*

1. *If $s \le k$ then*

$$m_{k,s} = \begin{cases} \frac{2s-d}{3} - \frac{s}{2} + 1 & \text{if } s \text{ is even} \\ \frac{2s-d-2}{3} - \frac{s-1}{2} + 1 & \text{if } s \text{ is odd and } d \neq 0 \\ \frac{2s-d-2}{3} - \frac{s-1}{2} & \text{if } s \text{ is odd and } d = 0. \end{cases}$$

2. *If $s > k$ with k even then*

$$m_{k,s} = \begin{cases} \frac{2s-d}{3} - \frac{2s-k}{2} + 1 & \text{if } s \text{ is even} \\ \frac{2s-d-2}{3} - \frac{2s-k}{2} + 1 & \text{if } s \text{ is odd and } d \neq 0 \\ \frac{2s-d-2}{3} - \frac{2s-k}{2} & \text{if } s \text{ is odd and } d = 0. \end{cases}$$

3. *If $s > k$ with k odd then*

$$m_{k,s} = \begin{cases} \frac{2s-d}{3} - \frac{2s-k+1}{2} + 1 & \text{if } s \text{ is even} \\ \frac{2s-d-2}{3} - \frac{2s-k-1}{2} + 1 & \text{if } s \text{ is odd and } d \neq 0 \\ \frac{2s-d-2}{3} - \frac{2s-k-1}{2} & \text{if } s \text{ is odd and } d = 0. \end{cases}$$

Before we prove this result we will introduce some notation to enable us to write down the coefficients of certain elements involved in $\bar{\Theta}_T\{t\}\kappa_t$ for given semistandard tableau T and tableau t. Let T_w be the semistandard $(3k-s,s)$-tableau with 3 appearing w times in the second row. So if $s \leq k$ for some w satisfying $0 \leq w \leq s$ then the tableau T_w will be of the form

$$\begin{array}{l} 1\cdots 1\; 1\cdots 1\; 1\cdots 1\; 2\cdots 2\; 3\cdots 3 \\ \underbrace{2\cdots 2}_{s-w}\; \underbrace{3\cdots 3}_{w} \end{array}.$$

If $s > k$ for some w satisfying $2(s-k) \leq w \leq k$ then the tableau T_w will be of the form

$$\begin{array}{l} 1\cdots 1\; 1\cdots 1\; 2\cdots 2\; 2\cdots 2\; 3\cdots 3 \\ \underbrace{2\cdots 2}_{s-w}\; \underbrace{3\cdots 3\; 3\cdots 3}_{w} \end{array}.$$

As usual t will be the semistandard $(3k-s,s)$-tableau given by

$$\begin{array}{l} 1\; 3 \cdots 2s-1\; 2s+1 \cdots 3k \\ 2\; 4 \cdots 2s \end{array}.$$

The image of $\{t\}\kappa_t$ under $\hat{\Theta}_{T_w}$ can be written in terms of $(3k-s,s)$-tableaux of type (k^3) (see Section 3.2.2). The effect of applying the map $^-$ to $\hat{\Theta}_{T_w}\{t\}\kappa_t$ is to add together the coefficients of those tableaux which are $\sim$-equivalent (i.e. those with the same pattern). Below we calculate the coefficient of $\bar{T}^*$ in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ where

$$T^* = \begin{array}{l} \overbrace{1\ldots 1\; 1\ldots 1}^{x}\; 3\ldots 3\; 1\ldots 1\; 2\ldots 2\; 3\ldots 3 \\ 3\ldots 3\; \underbrace{2\ldots 2\; 2\ldots 2}_{y} \end{array}. \qquad (18)$$

Let z be the number of 3's in the first s columns of T^*, so $x+y+z=2s$. Denote the coefficient of $\bar{T}^*$ in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ by $b^w_{x,y,z}$. It is easy to see from the definition of $\hat{\Theta}_{T_w}$ that if T' is a $(3k-s,s)$-tableau of type (k^3) with 1 appearing x times in the top row of the first s columns, 2 appearing y times in the first s columns of the second row, 3 appearing z times in the first s columns (such that each column contains distinct entries) and the remaining columns the same as T^* then the coefficient of T' in $\hat{\Theta}_{T_w}\{t\}\kappa_t$ will be the same as the coefficient of T^* in $\hat{\Theta}_{T_w}\{t\}\kappa_t$. Thus the coefficient of $\bar{T}^*$ in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ will be the same as the coefficient of $\bar{T}'$ in $\bar{\Theta}_{T_w}\{t\}\kappa_t$. Hence, to calculate the coefficient of $\bar{T}^*$ in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ we can calculate the sum of the coefficients of each of the $\bar{T}'$ in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ and divide by the number of different T'. The tableaux $\sim$-equivalent to T^* are

$$\begin{array}{l} 1\ldots1\,1\ldots1\,3\ldots3\,1\ldots1\,2\ldots2\,3\ldots3 \\ 3\ldots3\,2\ldots2\,2\ldots2 \end{array}, \quad \begin{array}{l} 2\ldots2\,2\ldots2\,3\ldots3\,2\ldots2\,1\ldots1\,3\ldots3 \\ 3\ldots3\,1\ldots1\,1\ldots1 \end{array},$$

$$\begin{array}{l} 1\ldots1\,1\ldots1\,2\ldots2\,1\ldots1\,3\ldots3\,2\ldots2 \\ 2\ldots2\,3\ldots3\,3\ldots3 \end{array}, \quad \begin{array}{l} 3\ldots3\,3\ldots3\,2\ldots2\,3\ldots3\,1\ldots1\,2\ldots2 \\ 2\ldots2\,1\ldots1\,1\ldots1 \end{array},$$

$$\begin{array}{l} 2\ldots2\,2\ldots2\,1\ldots1\,2\ldots2\,3\ldots3\,1\ldots1 \\ 1\ldots1\,3\ldots3\,3\ldots3 \end{array}, \quad \begin{array}{l} 3\ldots3\,3\ldots3\,1\ldots1\,3\ldots3\,2\ldots2\,1\ldots1 \\ 1\ldots1\,2\ldots2\,2\ldots2 \end{array}.$$

Thus $\binom{s}{x}\binom{x}{y+x-s}b^w_{x,y,z}$ is equal to

$$\binom{s}{w}\left\{\binom{w}{y-s+w}\binom{s-w}{y-s+x}(-1)^{y-s+w}+\binom{w}{x-s+w}\binom{s-w}{y-s+x}(-1)^{2s-w-x}\right.$$
$$+\binom{s-w}{y-w}\binom{w}{x-s+y}(-1)^{y-w}+\binom{s-w}{x-w}\binom{w}{y-s+x}(-1)^{w+s-x}$$
$$\left.+\binom{w}{x-s+w}\binom{s-w}{y-w}(-1)^{s-w}+\binom{w}{y-s+w}\binom{s-w}{x-w}(-1)^{w}\right\}.$$

By multiplying out the terms of the binomial coefficients this expression can be rearranged into the form

$$\binom{s}{x}\binom{s-x}{y-s+w}\binom{x}{s-y}(-1)^{y-s+w}+\binom{s}{y}\binom{s-y}{x-s+w}\binom{y}{s-x}(-1)^{w+x}$$
$$+\binom{s}{x}\binom{s-x}{y-w}\binom{x}{s-y}(-1)^{y-w}+\binom{s}{y}\binom{s-y}{x-w}\binom{y}{s-x}(-1)^{w+s-x}$$
$$+\binom{s}{x+y-s}\binom{x+y-s}{y-w}\binom{2s-x-y}{s-x}(-1)^{s-w}$$
$$+\binom{s}{x+y-s}\binom{x+y-s}{x-w}\binom{2s-x-y}{s-x}(-1)^{w}.$$

Now on noting that

$$\binom{s}{x}\binom{x}{x+y-s}=\binom{s}{y}\binom{y}{s-x}=\binom{s}{x+y-s}\binom{2s-x-y}{s-x},$$

we have that

$$\begin{aligned} b^w_{x,y,z} &= (-1)^w \left\{ \binom{s-x}{y-s+w}(-1)^{s+y} + \binom{s-x}{y-w}(-1)^y + \binom{s-y}{x-s+w}(-1)^x \right. \\ &\quad \left. + \binom{s-y}{x-w}(-1)^{s+x} + \binom{x+y-s}{y-w}(-1)^s + \binom{x+y-s}{x-w} \right\} \\ &= (-1)^w \left\{ \binom{s-x}{z-w}(-1)^{s+y} + \binom{s-x}{y-w}(-1)^y + \binom{s-y}{z-w}(-1)^x \right. \qquad (19) \\ &\quad \left. + \binom{s-y}{x-w}(-1)^{s+x} + \binom{s-z}{y-w}(-1)^s + \binom{s-z}{x-w} \right\}. \end{aligned}$$

It is easy to see that $b^w_{x,y,z} = \pm b^w_{x,z,y} = \pm b^w_{y,x,z} = \pm b^w_{y,z,x} = \pm b^w_{z,x,y} = \pm b^w_{z,y,x}$ so without loss of generality we can restrict x, y and z so that $0 \leq x \leq y \leq z$. We use this way of finding the coefficients of certain elements in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ in the following proof of the proposition.

Proof: Let T_w be the semistandard $(3k-s,s)$-tableau of type (k^3) defined above and let t be the usual $(3k-s,s)$-tableau with the digits increasing down its columns. We will construct a subset I of the even integers between 0 and s such that $\{\bar{\Theta}_{T_i} \mid i \in I\}$ are linearly independent. To show linearly independence we will show for $i \in I$ that there is a $(3,k)$-partition involved in $\bar{\Theta}_{T_i}\{t\}\kappa_t$ which is not involved in $\bar{\Theta}_{T_j}\{t\}\kappa_t$ for all $j \in I$ with $j < i$.

Suppose s is even and let T^* be the tableau with 1 appearing x times in the first s columns of the first row and 2 appearing x times in the first s columns of the second row such that $z = 2s - 2x \geq x$ and the entries in the columns of T^* are distinct. The coefficient of $\bar{T}^*$ in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ is $b^w_{x,x,z}$ which is given in equation (19). We look at the possible values that x can take. It is easy to see that z must be less than or equal to the minimum of k and s. Let d be the integer satisfying $0 \leq d \leq 2$ such that 3 divides $2s-d$. Using (18) we see that when $k \geq s$, the smallest value that x can take is $s/2$ and the largest value is $(2s-d)/3$. When $k < s$ with $3k \neq 2s+1$, the smallest value that x can take is $\lceil (2s-k+1)/2 \rceil$ and the largest value that x can take is $(2s-d)/3$. When $k < s$ with $3k = 2s+1$ then 3 divides $2s+1$ and so $d = 2$. In this case $(2s-k+1)/2 = k$ and so $2s - 2x = k - 1$ which contradicts $z \geq x$. Also $(2s-d)/3 = k-1$ and so $2s - 2x = k+1$ which contradicts $z \leq k$.

Therefore when $3k = 2s + 1$ the number of different values x can take is zero. Thus in general, the number of values x can take is $(2s - d)/3 - s/2 + 1$ when $s \leq k$ and $(2s - d)/3 - [(2s - k + 1)/2] + 1$ when $k < s$ (note that this is consistent with the case $3k = 2s + 1$). As x runs over the values in the appropriate range, let I be the set of integers of the form $2x - s$. Then $|I|$ is the value $m_{k,s}$ given in the proposition. From equation (19)

$$b^w_{x,x,2s-2x} = 2(-1)^w \left\{ \binom{s-x}{2s-2x-w}(-1)^x + \binom{s-x}{x-w}(-1)^x + \binom{2x-s}{x-w} \right\} \tag{20}$$

$$= 2(-1)^w \left\{ \binom{s-x}{w-(s-x)}(-1)^x + \binom{s-x}{w-(2x-s)}(-1)^x + \binom{2x-s}{w-(s-x)} \right\}.$$

Since $z = 2s - 2x \geq x$ we have $s - x \geq 2x - s$. Fix x in the appropriate range and let $w = 2x - s \in I$. Therefore w is less than $s - x$ unless $3x = 2s$. So for $3x \neq 2s$

$$b^w_{x,x,2s-2x} = 2(-1)^x \neq 0.$$

If $3x = 2s$ and so $2x - s = s - x$ then

$$b^w_{x,x,x} = 6 \neq 0.$$

When $i < w$ then $i < s-x$ and so $b^i_{x,x,2s-2x} = 0$. Thus $\bar{T}^*$ is involved in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ but is not involved in $\bar{\Theta}_{T_i}\{t\}\kappa_t$ for $i < w$. Since x was arbitrary we have constructed a set of $m_{k,s}$ linearly independent homomorphisms $\{\bar{\Theta}_{T_w} \mid w \in I\}$ from $S^{(3k-s,s)}$ to $FP(3,k)$. By Theorem 2.20 the multiplicity of $S^{(3k-s,s)}$ in the decomposition of $FP(3,k)$ is greater than or equal to $m_{k,s}$.

Now suppose s is odd. We use a similar method to construct a set of $m_{k,s}$ linearly independent homomorphisms from $S^{(3k-s,s)}$ to $FP(3,k)$. Let T^* be a tableau with 1 occurring x times in the first s columns of the first row and 2 occurring $x+1$ times in the first s columns of the second row with the property that $z = 2s - 2x - 1 \geq x + 1$ and the columns of T^* contain distinct entries. The coefficient of $\bar{T}^*$ in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ is $b^w_{x,x+1,2s-2x-1}$ which is given in equation (19). We look at the possible values that x can take. Again it is easy to see that z must be less than or equal to the minimum of s and k. Let d be the integer satisfying $0 \leq d \leq 2$ such that 3 divides $2s - d - 2$.

Using (18) we see that when $k \geq s$, the smallest value that x can take is $(s-1)/2$ and the largest value is $(2s-d-2)/3$. When $k < s$ with $3k \neq 2s$ or $3k \neq 2s+2$, the smallest value that x can take is $[(2s-k)/2]$ and the largest value that x can take is $(2s-d-2)/3$. When $k < s$ with $3k = 2s$ or $3k = 2s+2$ then x cannot be $(2s-k)/2$ for otherwise we have $2s-2x-1 < x+1$ which contradicts $z \geq x+1$ and x cannot be $(2s-d-2)/3$ for otherwise $2s-2x-1 > k$. Therefore x cannot take any values when $3k = 2s$ or $3k = 2s+2$. Thus in general, the number of values x can take is $(2s-d-2)/3-(s-1)/2+1$ when $s \leq k$ and $(2s-d-2)/3-[(2s-k)/2]+1$ when $k < s$ (again this is consistent with the cases $3k = 2s$ and $3k = 2s+2$). When $d = 0$ one of the values that x can take is $(2s-2)/3$. In this case $z = 2s-2x-1 = x+1 = y$ and it is easy to see from equation (19) (using the fact that x is even and s is odd) that $b^w_{x,x+1,z} = 0$ for any value of w. When $d \neq 0$ it is clear that $(2s-2)/3$ is not an integer and so x cannot take this value. As x runs over the values in the appropriate range with $x \neq (2s-2)/3$, let I be the set of even integers of the form $2x-s+1$. From equation (19)

$$\begin{aligned}
b^w_{x,x+1,2s-2x-1} &= (-1)^w \Bigg\{ \binom{s-x}{2s-2x-1-w}(-1)^x + \binom{s-x}{x+1-w}(-1)^{x+1} \\
&\quad + \binom{s-x-1}{2s-2x-1-w}(-1)^x + \binom{s-x-1}{x-w}(-1)^{x+1} \\
&\quad + \binom{2x+1-s}{x+1-w}(-1) + \binom{2x+1-s}{x-w} \Bigg\} \\
&= (-1)^w \Bigg\{ \binom{s-x}{w-(s-x-1)}(-1)^x + \binom{s-x}{w-(2x-s+1)}(-1)^{x+1} \\
&\quad + \binom{s-x-1}{w-(s-x)}(-1)^x + \binom{s-x-1}{w-(2x-s+1)}(-1)^{x+1} \\
&\quad + \binom{2x+1-s}{w-(s-x)}(-1) + \binom{2x+1-s}{w-(s-x-1)} \Bigg\}. \qquad (21)
\end{aligned}$$

As $2s-2x-1 \geq x+1$ we have $2x-s+1 \leq s-x-1$. Fix x in the appropriate range and let $w = 2x-s+1 \in I$. Thus w is less than $s-x-1$ and so

$$b^w_{x,x+1,2s-2x-1} = 2(-1)^{x+1} \neq 0.$$

When $i \in I$ with $i < w$ we have $i < s-x-1$ and so $b^i_{x,x+1,2s-2x-1} = 0$. Thus $\bar{T}^*$ is involved in $\bar{\Theta}_{T_w}\{t\}\kappa_t$ but is not involved in $\bar{\Theta}_{T_i}\{t\}\kappa_t$ for $i \in I$ with $i < w$. Since x was arbitrary we have constructed a set of $m_{k,s}$ linearly independent homomorphisms $\{\bar{\Theta}_{T_w} \mid w \in I\}$ from $S^{(3k-s,s)}$ to $FP(3,k)$. Again using Theorem 2.20, the multiplicity of $S^{(3k-s,s)}$ in the decomposition of $FP(3,k)$ is greater than or equal to $m_{k,s}$. This completes the proof. □

For $s \in \{0,1,\ldots,k+[k/2]\}$ the expression for the multiplicities $m_{k,s}$ given in Proposition 5.10 can be simplified in the following way.

Lemma 5.11 *Let $s \in \{0,1,\ldots,k+[k/2]\}$. When $s \neq 1$ write $s = 6c+r$ with $r \in \{0,2,3,4,5,7\}$. Then the multiplicity $m_{k,s}$ of Proposition 5.10 can be written as*

$$m_{k,s} = \begin{cases} 0 & \text{if } s = 1 \\ c+1 & \text{if } s \neq 1 \text{ and } s \leq k \\ c - \left[\frac{s-k+1}{2}\right] + 1 & \text{if } s \neq 1 \text{ and } s > k. \end{cases}$$

Proof: There are twelve cases in Proposition 5.10 to consider which we will work through in turn but first we look at what happens when $s = 1$. When $s = 1$ we have $d = 0$ and it is clear from Proposition 5.10 that $m_{k,1} = 0$.

Assume s is even. Then

$$3|2s-d \Leftrightarrow 6|4s-2d \Leftrightarrow 6|s-2d$$

and so $s = 6c+2d$ for some non-negative integer c. Thus $r = 2d$. When $s \leq k$ then

$$\frac{2s-d}{3} - \frac{s}{2} + 1 = \frac{s-r}{6} + 1 = c+1.$$

When $s > k$ and k is even we have

$$\frac{2s-d}{3} - \frac{2s-k}{2} + 1 = \frac{3k-2s-2d}{6} + 1 = c - \frac{s-k}{2} + 1.$$

When $s > k$ and k is odd we have

$$\frac{2s-d}{3} - \frac{2s-k+1}{2} + 1 = \frac{3k-2s-2d-3}{6} + 1 = c - \frac{s-k+1}{2} + 1.$$

Now assume s is odd in which case

$$3|2s-d-2 \Leftrightarrow 6|4(s-1)-2d \Leftrightarrow 6|s-1-2d$$

and so $s = 6c + 2d + 1$ for some non-negative integer c. Thus $r = 2d + 1$ when $d \neq 0$ and $r = 6 + 2d + 1$ when $d = 0$. For $s \leq k$ if $d \neq 0$ then

$$\frac{2s-d-2}{3} - \frac{s-1}{2} + 1 = c + 1$$

and if $d = 0$

$$\frac{2s-d-2}{3} - \frac{s-1}{2} = c + 1.$$

When $s > k$ and k is even with $d \neq 0$ we have

$$\frac{2s-d-2}{3} - \frac{2s-k}{2} + 1 = \frac{3k-2s-2d-4}{6} + 1 = c - \frac{s-k+1}{2} + 1$$

and when $d = 0$ then

$$\frac{2s-d-2}{3} - \frac{2s-k}{2} = \frac{3k-2s-2d-4}{6} = c - \frac{s-k+1}{2} + 1.$$

When $s > k$ and k is odd with $d \neq 0$ we have

$$\frac{2s-d-2}{3} - \frac{2s-k-1}{2} + 1 = \frac{3k-2s-2d-1}{6} + 1 = c - \frac{s-k}{2} + 1$$

and when $d = 0$ then

$$\frac{2s-d-2}{3} - \frac{2s-k-1}{2} = \frac{3k-2s-2d-1}{6} = c - \frac{s-k}{2} + 1.$$

This completes the proof. □

For small values of k we constructed the modules in the complete decompositions of $FP(3,k)$ using semistandard homomorphisms. We've shown that if $S^{(\mu_1,\mu_2,\mu_3)}$ appears in $FP(3,k-2)$ with multiplicity m then $S^{(\mu_1+2,\mu_2+2,\mu_3+2)}$ appears in $FP(3,k)$ also with multiplicity m. The examples that we constructed suggested that the only other modules which appear in $FP(3,k)$ are $S^{(3k-s,s)}$ and $S^{(3k-s-5,s+4,1)}$ with multiplicities $m_{k,s}$ and $m_{k-3,s}$ respectively (with $m_{k,s}$ given in Lemma 5.11). We will prove that these are in fact all the modules which appear in $FP(3,k)$ for general k.

Proposition 5.12 *For $k \geq 3$ and $0 \leq s \leq (k-3) + [(k-3)/2]$ the Specht module $S^{(3k-s-5,s+4,1)}$ appears in $FP(3,k)$ with multiplicity at least $m_{k-3,s}$ (given in Lemma 5.11).*

Proof: For $k \geq 3$ and $0 \leq s \leq (k-3) + [(k-3)/2]$ let $\hat{T}_w$ be the semistandard $(3k-s-5, s+4, 1)$-tableaux with 3 appearing $w+2$ times in the middle row and 2 appearing at least twice in the middle row. So for some w satisfying $max\{0, 2(s-k+3)\} \leq w \leq min\{s, k-3\}$ the tableau $\hat{T}_w$ will be of the form

$$\begin{matrix} 1\,1\,1\,1\,1\ldots 1\,1\ldots 1\,1\ldots 1 2\ldots 2\,3\ldots 3 \\ 2\,\underbrace{2\,2\,2\,2\ldots 2}_{s-w+1}\,\underbrace{3\ldots 3}_{w+2} \\ 3 \end{matrix} \quad \text{or} \quad \begin{matrix} 1\,1\,1\,1\,1\ldots 1\,1\ldots 1\,2\ldots 2\,2\ldots 2\,3\ldots 3 \\ 2\,\underbrace{2\,2\,2\,2\ldots 2}_{s-w+1}\,\underbrace{3\ldots 3\,3\ldots 3}_{w+2} \\ 3 \end{matrix} .$$

Let $t = t_1 \cup t_2$ where

$$t_1 = \begin{matrix} 1\,4\,6\,8 \\ 2\,5\,7\,9 \\ 3 \end{matrix} \quad \text{and} \quad t_2 = \begin{matrix} 10\ldots 2s+8\ldots 3k \\ 11\ldots 2s+9 \end{matrix} .$$

We will use a method similar to the proof of Proposition 5.10 and construct a set I with the property that if $w \in I$ then there is a $(3,k)$-partition involved in $\bar{\Theta}_{\hat{T}_w}\{t\}\kappa_t$ which is not involved in $\bar{\Theta}_{\hat{T}_i}\{t\}\kappa_t$ for $i \in I$ with $i < w$.

Consider the tableaux in $\bar{\Theta}_{\hat{T}_w}\{t\}\kappa_t$ which are $\sim$-equivalent to the tableau $T^\circ = T^{**} \cup T^*$ given by

$$T^{**} = \begin{matrix} 1\,1\,1\,2 \\ 2\,2\,3\,3 \\ 3 \end{matrix} \quad \text{and} \quad T^* = \begin{matrix} \overbrace{1\ldots 1\,1\ldots 1}^{x}\,2\ldots 2\,1\ldots 1\,2\ldots 2\,3\ldots 3 \\ 2\ldots 2\,\underbrace{3\ldots 3\,3\ldots 3}_{y} \end{matrix} .$$

Let the coefficient of $\bar{T}^\circ$ in $\bar{\Theta}_{\hat{T}_w}\{t\}\kappa_t$ be $d^w_{x+3,y+3,z+3}$ where $z = 2s - x - y$. Any tableau $\sim$-equivalent to T° which is involved in $\hat{\Theta}_{\hat{T}_w}\{t\}\kappa_t$ can be written in the form $T''g_1 \cup T'g_2$ for $g_1 \in C_{t_1}$, $g_2 \in C_{t_2}$ and $T'' \cup T'$ row equivalent to $\hat{T}_w$ (where clearly $T''g_1$ is $\sim$-equivalent to T^{**} and $T'g_2$ is $\sim$-equivalent to T^*). The tableau T'' must contain three of each of the digits $1, 2, 3$ and hence will be of the form

$$\begin{matrix} 1\,1\,1\,2 \\ 2\,2\,3\,3 \\ 3 \end{matrix}\, g_3 \quad \text{or} \quad \begin{matrix} 1\,1\,1\,3 \\ 2\,2\,3\,2 \\ 3 \end{matrix}\, g_4 \quad \text{for } g_3, g_4 \in R_{t_1}.$$

Thus T' must contain w or $w+1$ (respectively) 3's in its second row. We consider the possibilities for $T''g_1$. These will be

$$\begin{matrix} 1\,1\,1\,2 \\ 2\,2\,3\,3 \\ 3 \end{matrix}, \quad \begin{matrix} 1\,1\,1\,3 \\ 3\,3\,2\,2 \\ 2 \end{matrix}, \quad \begin{matrix} 2\,2\,2\,1 \\ 1\,1\,3\,3 \\ 3 \end{matrix}, \quad \begin{matrix} 2\,2\,2\,3 \\ 3\,3\,1\,1 \\ 1 \end{matrix}, \quad \begin{matrix} 3\,3\,3\,1 \\ 1\,1\,2\,2 \\ 2 \end{matrix}, \quad \begin{matrix} 3\,3\,3\,2 \\ 2\,2\,1\,1 \\ 1 \end{matrix} .$$

It is easy to check that the sign of g_1 in C_{t_1} is $+1$ if T'' has two 2's and two 3's in its middle row and -1 if T'' has three 2's and one 3 in its middle row. Moreover, there is only one way of constructing each of these formulations for the first four columns of a tableau involved in $\hat{\Theta}_{\hat{T}_w}\{t\}\kappa_t$. Therefore

$$d^w_{x+3,y+3,z+3} = b^w_{x,y,z} - b^{w+1}_{x,y,z}$$

where the $b^w_{x,y,z}$ are coefficients of the $(3(k-3)-s,s)$-tableaux given in (19).

Assume s is even and let d be the integer satisfying $0 \le d \le 2$ such that 3 divides $2s-d$. For $3(k-3) \neq 2s+1$ let I be the set of integers of the form $2x-s$ with $s/2 \le x \le (2s-d)/3$ if $k-3 \ge s$ and $[(2s-(k-3)+1)/2] \le x \le (2s-d)/3$ if $k-3 < s$ (note that we have already shown in the proof of Proposition 5.10 that x can run over these values). We will now show for $w \in I$ that $d^w_{x+3,x+3,z+3}$ is zero if and only if the coefficient $b^w_{x,x,z}$ of the $(3(k-3)-s,s)$-tableaux given in (20) is zero. From equation (20) we have $b^{w+1}_{x,x,2s-x}$ equals

$$2(-1)^{w+1}\left\{(-1)^x\left(\binom{s-x}{w+1-(s-x)}+\binom{s-x}{w+1-(2x-s)}\right)+\binom{2x-s}{w+1-(s-x)}\right\}.$$

Fix x in the appropriate range and let $w = 2x-s \in I$. If $2x-s < s-x$ then $b^w_{x,x,2s-x} = 2(-1)^x$. When $2x-s+1 < s-x$ we have from the above

$$b^{w+1}_{x,x,2s-2x} = 2(-1)^{x+1}(s-x)$$

and thus

$$d^w_{x+3,x+3,2s-2x+3} = 2(-1)^x(s-x+1) \neq 0 \quad (\text{since } s \ge x).$$

If $2x-s+1 = s-x$ then $3x = 2s-1$ (so x is odd) and we have

$$b^{w+1}_{x,x,2s-2x} = 2(-1)\{(-1)+(s-x)(-1)+1\} = 2(s-x)$$

and thus

$$d^w_{x+3,x+3,2s-2x+3} = -2-2(s-x) = -2(s-x+1) \neq 0.$$

When $2x-s = s-x$ (so $3x = 2s$) then $b^w_{x,x,2s-x} = 6$ and

$$\begin{aligned} b^{w+1}_{x,x,2s-2x} &= 2(-1)\{(s-x)+(s-x)+(2x-s)\} \\ &= 2(-1)s, \end{aligned}$$

so we have

$$d^w_{x,x,2s-2x} = 6 + 2s \neq 0.$$

If $i \in I$ is less than w then $i+1$ must also be less than w (as i and w are even) and since $2x - s \leq s - x$ we have that $b^i_{x,x,2s-2x}$, $b^{i+1}_{x,x,2s-2x}$ and $d^i_{x+3,x+3,2s-2x+3}$ are all zero. Therefore when $w = 2x - s$ we have shown that $d^w_{x+3,x+3,2s-2x+3}$ is non-zero and $d^i_{x+3,x+3,2s-2x+3}$ is zero for $i \in I$ with $i < w$. In other words, $\bar{T}^\circ$ is involved in $\bar{\Theta}_{\hat{T}_w}\{t\}\kappa_t$ but is not involved in $\bar{\Theta}_{\hat{T}_i}\{t\}\kappa_t$ for $i \in I$ with $i < w$. Hence we have a set of $m_{k-3,s}$ linearly independent $FSym(3k)$-homomorphisms from $S^{(3k-s-5,s+4,1)}$ to $FP(3,k)$, one for each value of w in I.

Now assume that s is odd and let d be the integer satisfying $0 \leq d \leq 2$ such that 3 divides $2s - d - 2$. When $3(k-3) \neq 2s$ or $2s+2$ let x be such that $(s-1)/2 \leq x \leq (2s-d-2)/3$ if $k-3 \geq s$ and $[2s-(k-3)/2] \leq x \leq (2s-d-2)/3$ if $k-3 < s$ (note that we have already shown in the proof of Proposition 5.10 that x can run over these values). For x in the appropriate range discluding $x = (2s-2)/3$ let I be the set of integers of the form $2x - s + 1$. We will now show when s is odd and $w \in I$ that $d^w_{x+3,x+4,z+3}$ is zero if and only if the coefficient $b^w_{x,x+1,z}$ of the $(3(k-3)-s,s)$-tableau given in (21) is zero. From equation (21) $b^{w+1}_{x,x+1,2s-2x-1}$ equals

$$(-1)^{w+1}\left\{\binom{s-x}{w+1-(s-x-1)}(-1)^x + \binom{s-x}{w+1-(2x-s+1)}(-1)^{x+1} + \binom{s-x-1}{w+1-(s-x)}(-1)^x + \binom{s-x-1}{w+1-(2x-s+1)}(-1)^{x+1} + \binom{2x+1-s}{w+1-(s-x)}(-1) + \binom{2x+1-s}{w+1-(s-x-1)}\right\}. \qquad (22)$$

Fix x in the appropriate range and let $w = 2x - s + 1 \in I$. If $w + 1 < s - x - 1$ then $b^w_{x,x+1,2s-2x-1} = 2(-1)^{x+1}$ and

$$\begin{aligned} b^{w+1}_{x,x+1,2s-2x-1} &= (-1)\left\{(-1)^{x+1}(s-x) + (-1)^{x+1}(s-x-1)\right\} \\ &= (-1)^x(2s-2x-1). \end{aligned}$$

Thus

$$d^w_{x+3,x+4,2s-2x+2} = (-1)^{x+1}(2s-2x+1) \neq 0 \quad (\text{since } s \geq x).$$

If $2x - s + 2 = s - x - 1$ (so $3x = 2s - 3$) then $b^w_{x,x+1,2s-2x-1} = 2$ and

$$b^{w+1}_{x,x+1,2s-2x-1} = (-1)\{(-1) + (s-x) + (s-x-1) + 1\} = (-1)(2s-2x-1).$$

Therefore we have that

$$d^w_{x+3,x+4,2s-2x+2} = 2s - 2x + 1 \neq 0.$$

If $i \in I$ is less than w then $i+1$ must also be less than w (as w and i are even) and as $2s-2x-1 \geq x+1$ (so $2x-s+1 \leq s-x-1$) we have that $b^i_{x,x+1,2s-2x-1}$, $b^{i+1}_{x,x+1,2s-2x-1}$ and $d^i_{x+3,x+4,2s-2x+2}$ are all zero. Therefore we have shown that $\bar{T}^\circ$ is involved in $\bar{\Theta}_{\hat{T}_w}\{t\}\kappa_t$ but is not involved in $\bar{\Theta}_{\hat{T}_i}\{t\}\kappa_t$ for $i \in I$ with $i < w$. Hence, we have a set of $m_{k-3,s}$ linearly independent $FSym(3k)$-homomorphisms from $S^{(3k-s-5,s+4,1)}$ to $FP(3,k)$, one for each value of w in I. This completes the proof. □

The following theorem gives us an inductive way of finding the complete decomposition of $FP(3,k)$. Using the known decompositions of $FP(3,1)$ and $FP(3,2)$ we give, as a corollary to this theorem, a simple and direct way of writing down the modules in the complete decomposition of $FP(3,k)$.

Theorem 5.13 *The complete decomposition of $FP(3,k)$ into irreducibles is given as follows:*

1. *For $k \geq 1$ with $0 \leq s \leq k + [k/2]$ the modules $S^{(3k-s,s)}$ appear in $FP(3,k)$ with multiplicity $m_{k,s}$ (given in Lemma 5.11).*
2. *For $k \geq 3$ with $0 \leq s \leq k - 3 + [(k-3)/2]$ the modules $S^{(3k-s-5,s+4,1)}$ appear in $FP(3,k)$ with multiplicity $m_{k-3,s}$ (given in Lemma 5.11).*
3. *For $k \geq 3$, if $S^{(\mu_1,\mu_2,\mu_3)}$ appears in $FP(3,k-2)$ with multiplicity m then the module $S^{(\mu_1+2,\mu_2+2,\mu_3+2)}$ appears in $FP(3,k)$ with multiplicity m.*

Proof: We have already shown in Theorem 3.8 that if $\mu = (\mu_1, \mu_2, \mu_3)$ is a partition of $3(k-2)$ then the multiplicity of $S^{(\mu_1+2,\mu_2+2,\mu_3+2)}$ in $FP(3,k)$ is the same as the multiplicity of $S^{(\mu_1,\mu_2,\mu_3)}$ in $FP(3,k-2)$. We have also shown that $S^{(3k-s,s)}$ and $S^{(3k-s-5,s+4,1)}$ appear in $FP(3,k)$ with multiplicity greater than or equal to $m_{k,s}$ and

$m_{k-3,s}$ respectively. Thus we need to show that these give all of the irreducible modules in the decomposition of $FP(3,k)$ and so the multiplicity of $S^{(3k-s,s)}$ and $S^{(3k-s-5,s+4,1)}$ are precisely $m_{k,s}$ and $m_{k-3,s}$ respectively.

Let the sum of the squares of the multiplicities of all modules which appear in $FP(3,k)$ be n_k and let d_k be the sum of the squares of the multiplicities $m_{k,s}$ for $s \in \{0,1,\ldots,k+[k/2]\}$. A fundamental result of Section 3.2.3 is that n_k is equal to the number of intersection classes of $P(3,k) \times P(3,k)$. We will use this to show that

$$n_k = n_{k-2} + d_k + d_{k-3}. \tag{23}$$

We will write

$$d_k = \sum_i a_i i^2$$

where a_i equals the number of $s \in \{0,1,\ldots,k+[k/2]\}$ with $m_{k,s} = i$. There are twelve cases to consider, one for each value of i in $\{0,1,\ldots,11\}$ such that $12|k-i$. We will work through the first case in detail, the other cases can be constructed in an analogous way. For each s in $\{0,1,\ldots,k+[k/2]\}$ we work out the multiplicity $m_{k,s}$ given by Lemma 5.11 and tabulate the results. When $k \geq s$ we have

$m_{k,s}$	s
0	1
1	0, 2, 3, 4, 5, 7
2	6, 8, 9, 10, 11, 13
$\vdots$	$\vdots$
$k/6-1$	$k-12,\ k-10,\ k-9,\ k-8,\ k-7,\ k-5$
$k/6$	$k-6,\ k-4,\ k-3,\ k-2,\ k-1$
$k/6+1$	k

and when $k < s$ we have

$m_{k,s}$	s
$k/6$	$k+2$
$k/6-1$	$k+1,\ k+3,\ k+4,\ k+6$
$k/6-2$	$k+5,\ k+8$
$\vdots$	$\vdots$
1	$k+k/2-5,\ k+k/2-3,\ k+k/2-2,\ k+k/2$
0	$k+k/2-1$

Therefore, when $12|k$ we have

$$d_k = 10\cdot 1^2 + 8\cdot 2^2 + 10\cdot 3^2 + \cdots + 10\left(\frac{k}{6}-1\right)^2 + 6\left(\frac{k}{6}\right)^2 + 1\left(\frac{k}{6}+1\right)^2.$$

More generally, for $i = 0, 1, \ldots, 11$ (listed in this order) with $12|k-i$ we can write d_k as:

1. $10\cdot 1^2 + 8\cdot 2^2 + 10\cdot 3^2 + \cdots + 10\left(\frac{k}{6}-1\right)^2 + 6\left(\frac{k}{6}\right)^2 + 1\left(\frac{k}{6}+1\right)^2.$

2. $9\cdot 1^2 + 9\cdot 2^2 + 9\cdot 3^2 + \cdots + 9\left(\frac{k-1}{6}-1\right)^2 + 8\left(\frac{k-1}{6}\right)^2 + 1\left(\frac{k-1}{6}+1\right)^2.$

3. $8\cdot 1^2 + 10\cdot 2^2 + 8\cdot 3^2 + \cdots + 8\left(\frac{k-2}{6}-1\right)^2 + 9\left(\frac{k-2}{6}\right)^2 + 2\left(\frac{k-2}{6}+1\right)^2.$

4. $9\cdot 1^2 + 9\cdot 2^2 + 9\cdot 3^2 + \cdots + 9\left(\frac{k-3}{6}-1\right)^2 + 9\left(\frac{k-3}{6}\right)^2 + 3\left(\frac{k-3}{6}+1\right)^2.$

5. $10\cdot 1^2 + 8\cdot 2^2 + 10\cdot 3^2 + \cdots + 10\left(\frac{k-4}{6}-1\right)^2 + 8\left(\frac{k-4}{6}\right)^2 + 5\left(\frac{k-4}{6}+1\right)^2.$

6. $9\cdot 1^2 + 9\cdot 2^2 + 9\cdot 3^2 + \cdots + 9\left(\frac{k-5}{6}-1\right)^2 + 9\left(\frac{k-5}{6}\right)^2 + 6\left(\frac{k-5}{6}+1\right)^2.$

7. $8\cdot 1^2 + 10\cdot 2^2 + 8\cdot 3^2 + \cdots + 10\left(\frac{k-6}{6}\right)^2 + 6\left(\frac{k-6}{6}+1\right)^2 + 1\left(\frac{k-6}{6}+2\right)^2.$

8. $9\cdot 1^2 + 9\cdot 2^2 + 9\cdot 3^2 + \cdots + 9\left(\frac{k-7}{6}\right)^2 + 8\left(\frac{k-7}{6}+1\right)^2 + 1\left(\frac{k-7}{6}+2\right)^2.$

9. $10\cdot 1^2 + 8\cdot 2^2 + 10\cdot 3^2 + \cdots + 8\left(\frac{k-8}{6}\right)^2 + 9\left(\frac{k-8}{6}+1\right)^2 + 2\left(\frac{k-8}{6}+2\right)^2.$

10. $9\cdot 1^2 + 9\cdot 2^2 + 9\cdot 3^2 + \cdots + 9\left(\frac{k-9}{6}\right)^2 + 9\left(\frac{k-9}{6}+1\right)^2 + 3\left(\frac{k-9}{6}+2\right)^2.$

11. $8\cdot 1^2 + 10\cdot 2^2 + 8\cdot 3^2 + \cdots + 10\left(\frac{k-10}{6}\right)^2 + 8\left(\frac{k-10}{6}+1\right)^2 + 5\left(\frac{k-10}{6}+2\right)^2.$

12. $$9\cdot 1^2+9\cdot 2^2+9\cdot 3^2+\cdots+9\left(\tfrac{k-11}{6}\right)^2+9\left(\tfrac{k-11}{6}+1\right)^2+6\left(\tfrac{k-11}{6}+2\right)^2.$$

We can reduce the above to six cases:

1. If $6|k$ then $d_k = d_{k-4}+1\left(\frac{k}{6}-1\right)^2+4\left(\frac{k}{6}\right)^2+1\left(\frac{k}{6}+1\right)^2$.
2. If $6|k-1$ then $d_k = d_{k-4}+5\left(\frac{k-1}{6}\right)^2+1\left(\frac{k-1}{6}+1\right)^2$.
3. If $6|k-2$ then $d_k = d_{k-4}+4\left(\frac{k-2}{6}\right)^2+2\left(\frac{k-2}{6}+1\right)^2$.
4. If $6|k-3$ then $d_k = d_{k-4}+3\left(\frac{k-3}{6}\right)^2+3\left(\frac{k-3}{6}+1\right)^2$.
5. If $6|k-4$ then $d_k = d_{k-4}+2\left(\frac{k-4}{6}\right)^2+4\left(\frac{k-4}{6}+1\right)^2$.
6. If $6|k-5$ then $d_k = d_{k-4}+1\left(\frac{k-5}{6}\right)^2+5\left(\frac{k-5}{6}+1\right)^2$.

Let $\star_k = d_k + d_{k-3} - d_{k-4} - d_{k-7}$. Then we have:

1. If $6|k$ ($\Rightarrow 6|(k-3)-3$) then $\star_k = 4\left(\frac{k}{6}-1\right)^2+7\left(\frac{k}{6}\right)^2+1\left(\frac{k}{6}+1\right)^2$.
2. If $6|k-1$ ($\Rightarrow 6|(k-3)-4$) then $\star_k = 2\left(\frac{k-1}{6}-1\right)^2+9\left(\frac{k-1}{6}\right)^2+1\left(\frac{k-1}{6}+1\right)^2$.
3. If $6|k-2$ ($\Rightarrow 6|(k-3)-5$) then $\star_k = 1\left(\frac{k-2}{6}-1\right)^2+9\left(\frac{k-2}{6}\right)^2+2\left(\frac{k-2}{6}+1\right)^2$.
4. If $6|k-3$ ($\Rightarrow 6|(k-3)-0$) then $\star_k = 1\left(\frac{k-3}{6}-1\right)^2+7\left(\frac{k-3}{6}\right)^2+4\left(\frac{k-3}{6}+1\right)^2$.
5. If $6|k-4$ ($\Rightarrow 6|(k-3)-1$) then $\star_k = 7\left(\frac{k-4}{6}\right)^2+5\left(\frac{k-4}{6}+1\right)^2$.
6. If $6|k-5$ ($\Rightarrow 6|(k-3)-2$) then $\star_k = 5\left(\frac{k-5}{6}\right)^2+7\left(\frac{k-5}{6}+1\right)^2$.

It is easy to check by multiplying out terms that

$$\begin{aligned}\tfrac{1}{3}k^2-k+5 &= 4\left(\tfrac{k}{6}-1\right)^2+7\left(\tfrac{k}{6}\right)^2+\left(\tfrac{k}{6}+1\right)^2\\ &= \left(\tfrac{k-3}{6}-1\right)^2+7\left(\tfrac{k-3}{6}\right)^2+4\left(\tfrac{k-3}{6}+1\right)^2\end{aligned}$$

and

$$\begin{aligned}\tfrac{1}{3}k^2-k+\tfrac{11}{3} &= 2\left(\tfrac{k-1}{6}-1\right)^2+9\left(\tfrac{k-1}{6}\right)^2+\left(\tfrac{k-1}{6}+1\right)^2\\ &= \left(\tfrac{k-2}{6}-1\right)^2+9\left(\tfrac{k-2}{6}\right)^2+2\left(\tfrac{k-2}{6}+1\right)^2\\ &= 7\left(\tfrac{k-4}{6}\right)^2+5\left(\tfrac{k-4}{6}+1\right)^2\\ &= 5\left(\tfrac{k-5}{6}\right)^2+7\left(\tfrac{k-5}{6}+1\right)^2.\end{aligned}$$

Therefore, if 3 divides k then $\star_k = \frac{1}{3}k^2 - k + 5$ and if 3 does not divide k then $\star_k = \frac{1}{3}k^2 - k + \frac{11}{3}$. We can now prove (23) by induction. For $1 \leq k \leq 8$ we have constructed the modules which appear in $FP(3, k)$ using Propositions 5.10 and 5.12 and Theorem 3.8. Using Theorem 5.4 and as an extra verification calculating and adding together the dimensions of the modules, we have shown that these give all modules which appear in $FP(3, k)$. The results of this exercise can be seen in Section 6.2 and in the Appendix. From these results it is easy to see that (23) holds for small values of k. Assume that (23) holds for all $k^* < k$. When 3 divides k we can write

$$\begin{aligned} d_k + d_{k-3} &= d_{k-4} + d_{k-7} + \frac{1}{3}k^2 - k + 5 \\ &= n_{k-4} - n_{k-6} + \frac{1}{3}k^2 - k + 5 \quad \text{(by (23))}. \end{aligned}$$

Similarly, when 3 does not divide k then

$$d_k + d_{k-3} = n_{k-4} - n_{k-6} + \frac{1}{3}k^2 - k + \frac{11}{3}.$$

Therefore we need to prove that

$$n_k = n_{k-2} + n_{k-4} - n_{k-6} + \star_k. \tag{24}$$

By Theorem 5.4 we have

$$n_k = \begin{cases} \frac{1}{288}\{k^4 + 6k^3 + 64k^2 + 138k + 79 + 128c\} & \text{if } k \text{ is odd} \\ \frac{1}{288}\{k^4 + 6k^3 + 64k^2 + 192k + 160 + 128c\} & \text{if } k \text{ is even.} \end{cases}$$

where

$$c = \begin{cases} 1 & \text{if 3 divides } k \\ 0 & \text{otherwise.} \end{cases}$$

To complete the inductive step we need to show that n_k satisfies equation (24). So let k be odd and such that 3 divides k (so 3 does not divide $k-2$ or $k-4$ but 3 divides $k-6$) then

$$\begin{aligned} n_{k-2} + n_{k-4} - n_{k-6} = \frac{1}{288}\{&(k-2)^4 + (k-4)^4 - (k-6)^4 + 6(k-2)^3 + 6(k-4)^3 \\ &-6(k-6)^3 + 64(k-2)^2 + 64(k-4)^2 - 64(k-6)^2 \end{aligned}$$

$$
\begin{aligned}
&+138(k-2)+138(k-4)-138(k-6)+79-128\} \\
&= \frac{1}{288}\left\{k^4+6k^3-32k^2+426k-1233\right\} \\
&= \frac{1}{288}\left\{k^4+6k^3+64k^2+138k+79+128\right. \\
&\left. -96k^2+288k-1440\right\} \\
&= n_k - \left(\frac{1}{3}k^2-k+5\right).
\end{aligned}
$$

When k is odd and 3 does not divide k then either 3 divides $k-2$ or 3 divides $k-4$. Therefore

$$
\begin{aligned}
n_{k-2}+n_{k-4}-n_{k-6} &= \frac{1}{288}\left\{k^4+6k^3-32k^2+426k-977\right\} \\
&= \frac{1}{288}\left\{k^4+6k^3+64k^2+138k+79-96k^2+288k-1056\right\} \\
&= n_k - \left(\frac{1}{3}k^2-k+\frac{11}{3}\right).
\end{aligned}
$$

Similarly, when k is even and 3 divides k we have

$$
\begin{aligned}
n_{k-2}+n_{k-4}-n_{k-6} &= \frac{1}{288}\left\{k^4+6k^3-32k^2+480k-1152\right\} \\
&= \frac{1}{288}\left\{k^4+6k^3+64k^2+192k+160+128\right. \\
&\quad \left. -96k^2+288k-1440\right\} \\
&= n_k - \left(\frac{1}{3}k^2-k+5\right)
\end{aligned}
$$

and when k is even and 3 does not divide k then

$$
\begin{aligned}
n_{k-2}+n_{k-4}-n_{k-6} &= \frac{1}{288}\left\{k^4+6k^3-32k^2+480k-896\right\} \\
&= \frac{1}{288}\left\{k^4+6k^3+64k^2+192k+160-96k^2+288k-1056\right\} \\
&= n_k - \left(\frac{1}{3}k^2-k+\frac{11}{3}\right).
\end{aligned}
$$

Hence $n_k = n_{k-2}+n_{k-4}-n_{k-6}+\star_k$ and so $n_k = n_{k-2}+d_k+d_{k-3}$ as required. □

Corollary 5.14 *The modules which appear in $FP(3,k)$ are $S^{(3(k-2u)-s+2u,2u+s,2u)}$ with multiplicity $m_{k-2u,s}$ and $S^{(3(k-2u)-s-5+2u,s+2u+4,2u+1)}$ with multiplicity $m_{k-2u-3,s}$. These form a complete set of irreducible modules in the decomposition of $FP(3,k)$.*

Proof: By construction and using the known complete decompositions of $FP(3,1)$ and $FP(3,2)$, all partitions μ such that S^μ appears in $FP(3,k)$ have diagrams of the form

$$\begin{array}{l}
\times\ \times \ldots \times\ \times\ \times \ldots \times \underbrace{\times\ \times \ldots\ldots \times\ \times}_{3(k-2u)-2s} \\
\times\ \times \ldots \times \underbrace{\times\ \times \ldots \times}_{s} \\
\underbrace{\times\ \times \ldots \times}_{2u}
\end{array}$$

or

$$\begin{array}{l}
\times\ \times \ldots \times \times \times \times \times \times\ \times \ldots \times \underbrace{\times\ \times \ldots\ldots \times\ \times}_{3(k-2u)-2s-9} \\
\times\ \times \ldots \times \times \times \times \times \underbrace{\times\ \times \ldots \times}_{s} \\
\underbrace{\times\ \times \ldots \times}_{2u} \times
\end{array}$$

The multiplicity of the modules are given by $m_{k-2u,s}$ and $m_{k-2u-3,s}$ respectively. $\square$

5.4.3 Modules in the Decomposition of $FP(3,k)$ with Multiplicity One

Using the results of Section 3.3, for any module S^μ which appears in $FP(3,k)$ with multiplicity one there exists a semistandard μ-tableau T of type (k^3) such that $\bar{\Theta}_T(S^\mu)$ is contained in an eigenspace of MM^T. Therefore we can use our knowledge of the complete decomposition of $FP(3,k)$ to find some of the eigenvectors of MM^T. The semistandard tableaux which we require to construct these eigenvectors are given in the proofs of Propositions 5.10, 5.12 and Theorem 3.8. We start by constructing the set of modules which appear in $FP(3,k)$ with multiplicity one or in other words we look at the partitions which are simply associated to eigenvectors of MM^T. By Lemma 5.11 and Corollary 5.14 the modules which appear in $FP(3,k)$ with multiplicity one are given below.

1. For $s \in \{0,1,\ldots,k-2u+[(k-2u)/2]\}$ let c and r be non-negative integers with $r \in \{0,2,3,4,5,7\}$ such that $s = 6c+r$. Then the module $S^{(3(k-2u)-s+2u,s+2u,2u)}$ appears in $FP(3,k)$ with multiplicity one if $s \in \{0,2,3,4,5,7\}$ and $s \leq k-2u$. This module also appears with multiplicity one if $3(k-2u) = 2s+r$ or $3(k-2u) = 2s+r+3$ and $s > k-2u$.

2. For $s \in \{0, 1, \ldots, k-2u-3+[(k-2u-3)/2]\}$ let c and r be the non-negative integers with $r \in \{0,2,3,4,5,7\}$ such that $s = 6c + r$. Then the module $S^{3(k-2u-3)-s+2u+4,s+2u+4,2u+1)}$ appears in $FP(3,k)$ with multiplicity one if $s \in \{0,2,3,4,5,7\}$ and $s \leq k-2u-3$. This module also appears with multiplicity one if $3(k-2u-3) = 2s+r$ or $3(k-2u-3) = 2s+r+3$ with $s > k-2u-3$.

We will use these modules in Section 6.2 and Section 6.3 to construct some of the eigenvalues of MM^T. The modules which appear in $FP(3,k)$ with multiplicity greater than one do not give rise so readily to eigenvectors of MM^T. We can use the linearly independent homomorphisms from S^μ to $FP(3,k)$ which we constructed in the proofs of Propositions 5.10, 5.12 and Theorem 3.8. Taking an appropriate linear combination of these homomorphisms we can map S^μ into an eigenspace of MM^T. It is not clear how to do this in general, but in Section 6.2 we work out all eigenvectors and eigenvalues of $M^{3,k}(M^{3,k})^T$ for small values of k.

Chapter 6

Foulkes' Conjecture for $(3, k)$-Partitions

We will use the results of Chapter 5 to prove Foulkes' conjecture for the case when $m = 3$ and $n = k$ is arbitrary. As we discussed in Sections 2.2.6 and 3.1.3, there are two different methods which can be used to prove the conjecture. The first method is a 'direct approach' where we show for $k \geq 3$ that if S^μ appears in $FP(3, k)$ with multiplicity m then S^μ appears in $FP(k, 3)$ with multiplicity greater than or equal to m. The other approach is to show that all eigenvalues of the matrix MM^T, where M is the incidence matrix of $\mathcal{P}_{3,k}$, are non-zero. We consider the 'direct approach' first and show how we can in fact prove Foulkes' conjecture using this method. Next we study the eigenvalues of MM^T. The eigenvalue approach was relatively straight forward when we were considering $(2, k)$-partitions but this method begins to fail for the $(3, k)$-partitions with $k \geq 6$. For $1 \leq k \leq 8$ we give the complete set of eigenvalues of MM^T with the aid of a MAGMA [7] computer program but as k increases the computing time increases rapidly making it infeasible to construct many more examples. From the examples we constructed we noticed a surprising partial ordering property of the eigenvalues: The eigenvalues satisfy the same partial ordering as the partitions associated to these eigenvalues do. If this 'nice' ordering property could be shown to hold in general we would have an alternative and simple proof of Foulkes' conjecture. We prove in an inductive way that some of the eigenvalues associated to partitions of

$3k$ can be bounded below by eigenvalues associated to partitions of $3(k-2)$. In the final section we give some explicit eigenvalues of MM^T in neat closed form.

6.1 Modules in the Decomposition of $FP(k,3)$

We use the decomposition of $FP(3,k)$ in Chapter 5 to show for $k \geq 3$ that if S^{μ} appears in $FP(3,k)$ with multiplicity m then S^{μ} appears in $FP(k,3)$ with multiplicity greater than or equal to m.

Firstly we recall the decomposition of $FP(3,k)$. From Corollary 5.14 the modules which appear in $FP(3,k)$ are S^{μ} where μ is the partition $(3(k-2u)-s+2u, s+2u, 2u)$ with u and s non-negative integers satisfying $3k \geq 6u+2s$ or μ is the partition $(3(k-2u-3)-s+2u+4, s+2u+4, 2u+1)$ with u and s non-negative integers satisfying $3k \geq 6u+2s+9$. Moreover, $S^{(3(k-2u)-s+2u,s+2u,2u)}$ appears in $FP(3,k)$ with multiplicity $m_{k-2u,s}$ and $S^{(3(k-2u-3)-s+2u+4,s+2u+4,2u+1)}$ appears in $FP(3,k)$ with multiplicity $m_{k-2u-3,s}$. The $(3(k-2u)-s+2u, s+2u, 2u)$-tableaux of type (3^k) are of the following shape:

$$\begin{array}{l} * \ * \ \ldots \ * \ * \ * \ \ldots \ * \ * \ \ldots \ * \\ * \ * \ \ldots \ * \ * \ \underbrace{* \ \ldots \ *}_{s} \\ \underbrace{* \ * \ \ldots \ * \ *}_{2u} \end{array}.$$

The $(3(k-2u-3)-s+2u, s+2u, 2u)$-tableaux of type (3^k) are of the following shape:

$$\begin{array}{l} * \ * \ \ldots \ * \ * \ * \ * \ * \ * \ * \ \ldots \ * \ * \ \ldots \ * \\ * \ * \ \ldots \ * \ * \ * \ * \ * \ * \ \underbrace{* \ \ldots \ *}_{s} \\ \underbrace{* \ * \ \ldots \ * \ *}_{2u} \ * \end{array}.$$

For each of the cases we will use the same method to prove for $k \geq 3$ that S^{μ} appears in the decomposition of $FP(k,3)$ with multiplicity greater than or equal to the multiplicity of S^{μ} in $FP(3,k)$. So we begin by explaining this method.

Step 1: We construct μ-tableaux T^* of type (3^k) and for each T^* we form a semistandard μ-tableau T. We construct T^* out of unions of 'blocks' of tableaux T^{*1}, T^{*2},

$\ldots,\ T^{*10}$ which have the same pattern respectively as

$$(1)\ \begin{array}{llllll}1&1&1&2&2&2\\4&3&3&3&4&4\\6&6&5&5&5&6\end{array}\quad (2)\ \begin{array}{llll}1&1&1&2\\2&2&3&3\\3&4&4&4\end{array}\quad (3)\ \begin{array}{lllll}1&1&1&2&3\\2&2\\3&3\end{array}\quad (4)\ \begin{array}{llll}1&1&1&2\\2&2&3&3\\3\end{array}$$

$$(5)\ \begin{array}{llllll}1&1&2&3&1&4\\2&2&4&4\\3&3\end{array}\quad (6)\ \begin{array}{llllllll}1&1&1&2&2&2&3&5\\3&3&4&4&4\\5&5\end{array}\quad (7)\ \begin{array}{llllll}1&1&2&2&2&1\\3&3&3&4&4&4\end{array}\quad (8)\ \begin{array}{llllll}1&1&2&1&2&2\\3&3&3\end{array}$$

$$(9)\ \begin{array}{llll}1&1&1&2\\2&2\end{array}\quad \text{and}\quad (10)\ \ 1\ 1\ 1\,.$$

Each block that we use in the construction we label with a unique set of consecutive numbers with the property that in $T^{*i} \cup T^{*j}$ the labelling set for T^{*j} will be the next set of (increasing) consecutive numbers after the labelling set for T^{*i} (the first tableau in the union will always be labelled from 1). We will write

$$T^* = \bigcup_{i=1}^{10} a_i T^{*i}$$

where a_i is the number of tableaux T^{*i} in the construction. Similarly we write t in 'block form' as

$$t = \bigcup_{i=1}^{10} a_i t^i$$

where the t^i are the same shape as the T^{*i} and we fill in the numbers $1, 2, \ldots, 3k$ consecutively down the columns of the t^i. It might be necessary to reorder the columns of the tableaux T^* and t so that columns of the same length are together. For example if we have

$$T^{*5} \cup T^{*6}$$

where the labeling set for T^{*5} is $1, 2, 3, 4$ and for T^{*6} it is $5, 6, 7, 8, 9$ then we write

$$T^{*5} \cup T^{*6} = \begin{array}{llllll}1&1&2&3&1&4\\2&2&4&4\\3&3\end{array} \ \cup\ \begin{array}{llllllll}5&5&5&6&6&6&7&9\\7&7&8&8&8\\9&9\end{array}$$

as

$$\begin{array}{llllllllllllll}1&1&5&5&2&3&5&6&6&1&4&6&7&9\\2&2&7&7&4&4&8&8&8\\3&3&9&9\end{array}\ .$$

Similarly we write

$$t^5 \cup t^6 = \begin{matrix} 1\ 4\ 7\ \ 9\ \ 11\ 12 \\ 2\ 5\ 8\ 10 \\ 3\ 6 \end{matrix} \cup \begin{matrix} 13\ 16\ 19\ 21\ 23\ 25\ 26\ 27 \\ 14\ 17\ 20\ 22\ 24 \\ 15\ 18 \end{matrix}$$

as

$$\begin{matrix} 1\ 4\ 13\ 16\ 7\ \ 9\ \ 19\ 21\ 23\ 11\ 12\ 25\ 26\ 27 \\ 2\ 5\ 14\ 17\ 8\ 10\ 20\ 22\ 24 \\ 3\ 6\ 15\ 18 \end{matrix} .$$

From T^* we form the semistandard tableau T by permuting the elements in the rows of T^* until the numbers increase along the rows (we will automatically have the property that the digits strictly increase down the columns).

Step 2: For each T^* constructed in Step 1 we show that the coefficient of $\bar{T}^*$ in $\bar{\Theta}_T\{t\}\kappa_t$ is non-zero. To do this we consider all tableaux which are involved in $\hat{\Theta}_T\{t\}\kappa_t$ and are $\sim$-equivalent to T^*. For some $g \in C_t$ and T' row equivalent to T, these can be written in the form

$$T'g = \bigcup_{i=1}^{10} a_i T'^i g_i$$

where $g_i \in C_{t^i}$ and $T'^i g_i$ is $\sim$-equivalent to T^{*i}. We consider the possibilities for T'^i. Each digit which is in T'^i must occur three times in T'^i. So in other words, the same digit can not occur in more than one 'block'. We will use the terminology 'triplet', 'pair' and 'singleton' to indicate how many of a particular digit (out of a maximum of three) appear in a certain row of a tableau. The T^{*i} have the property that the third row (if there is one) contains triplets or it contains pairs with the remaining digit in the first row or it contains singletons with the remaining pairs in the second row. The second row of the T^{*i} are made up only of triplets and pairs and the first row contains only triplets and singletons. Since T'^1 and T'^2 are the only tableaux which have more than three elements in their third row, the third rows of T'^1 and T'^2 must contain triplets. Hence there must also be a singleton in the third row of T'^2. It is clear that T'^4 must have a singleton in its third row. Thus all triplets and singletons from the third row of T have been accounted for. Hence the third rows of T'^3, T'^5 and T'^6 must contain the pairs. It is clear that the second rows of T'^2, T'^3, T'^4, T'^5 and T'^9 must contain only pairs and the second row of T'^6 must contain a pair and a triplet. Therefore the second rows of T'^1, T'^7 and T'^8 must contain only triplets. The singletons in the first row of the T'^i have been determined by the digits used in the lower rows hence we fill the remaining 'space' in the top row of T'^i with the appropriate number of triplets. Thus we have shown that T'^i must have the same pattern as a tableau row equivalent to T^{*i}. Let T^i be the semistandard tableau row equivalent to T'^i. So to find the coefficient of $\bar{T}^*$ in $\bar{\Theta}_T\{t\}\kappa_t$ we take the product of the coefficients of $\bar{T}^{*i}$ in $\bar{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$ and we multiply by the number of possible choices for the entries of the T^i such that $\cup_{i=1}^{10} a_i T^i$ is row equivalent to T. Hence to show that the coefficient of $\bar{T}^*$ in $\bar{\Theta}_T\{t\}\kappa_t$ is non-zero we show for any labelling set for the T^i and for all i that the coefficient of $\bar{T}^{*i}$ in $\bar{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$ is non-zero.

Step 3: The final step is to construct the appropriate number of non-zero linearly independent homomorphisms from S^μ to $FP(k,3)$ so that as a direct consequence of Theorem 2.20 the multiplicity of S^μ in $FP(k,3)$ is greater than or equal to the required value. We construct a set $\{T_0^*, T_1^*, \ldots, T_q^*\}$ of μ-tableaux which are made up of unions of the 'blocks' $T^{*1}, T^{*2}, \ldots, T^{*10}$ and we show that $\bar{T}_i^*$ is involved in $\bar{\Theta}_{T_i}\{t\}\kappa_t$ but not in $\bar{\Theta}_{T_j}\{t\}\kappa_t$ for $j \in \{0, 1, \ldots, q\}$ with $j < i$. This shows that $\bar{\Theta}_{T_0}, \bar{\Theta}_{T_1}, \ldots, \bar{\Theta}_{T_q}$ are linearly independent.

The following result shows that the coefficient of $\bar{T}^{*i}$ in $\bar{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$ is non-zero. This shows that for every T^* constructed in Step 1 above, the coefficient of $\bar{T}^*$ in $\bar{\Theta}_T\{t\}\kappa_t$ is non-zero and hence does Step 2 of the above method.

Lemma 6.1 *For $i = 1, 2, \ldots, 10$ the coefficient of $\bar{T}^{*i}$ in $\bar{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$ (where T^{*i}, T^i and t^i are defined as above) is non-zero.*

Proof: Without loss of generality we can choose the labelling set for each of the tableaux T^{*i} to be $\{1, 2, \ldots\}$ and t^i to be the usual tableau with the digits $1, 2, 3, \ldots$ placed in increasing order down its columns. For each value of i between 1 and 10 we write down all tableaux $\sim$-equivalent to T^{*i} which are involved in $\hat{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$. Calculating the coefficient of each of these tableaux in $\hat{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$ and adding together these coefficients gives us the coefficient of $\bar{T}^{*i}$ in $\bar{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$.

1. Let T^{*1} be the tableau given by

$$T^{*1} = \begin{matrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 4 & 3 & 3 & 3 & 4 & 4 \\ 6 & 6 & 5 & 5 & 5 & 6 \end{matrix}.$$

The rows of the tableaux $\sim$-equivalent to T^{*1} which are involved in $\hat{\Theta}_{T^1}\{t^1\}\kappa_{t^1}$ will always have the same pair of triplets in each row as T^{*1} does. Thus the tableaux $\sim$-equivalent to T^{*1} which are involved in $\hat{\Theta}_{T^1}\{t^1\}\kappa_{t^1}$ are

$$\begin{array}{c}1\,1\,1\,2\,2\,2\\4\,3\,3\,3\,4\,4\\6\,6\,5\,5\,5\,6\end{array},\quad \begin{array}{c}1\,1\,1\,2\,2\,2\\3\,4\,4\,4\,3\,3\\6\,6\,5\,5\,5\,6\end{array},\quad \begin{array}{c}2\,2\,2\,1\,1\,1\\4\,3\,3\,3\,4\,4\\6\,6\,5\,5\,5\,6\end{array},\quad \begin{array}{c}2\,2\,2\,1\,1\,1\\4\,3\,3\,3\,4\,4\\6\,6\,5\,5\,5\,6\end{array},\quad \begin{array}{c}4\,4\,4\,3\,3\,3\\2\,1\,1\,1\,2\,2\\6\,6\,5\,5\,5\,6\end{array},\quad \begin{array}{c}3\,3\,3\,4\,4\,4\\2\,1\,2\,2\,1\,1\\6\,6\,5\,5\,5\,6\end{array}\ldots$$

It is clear that all tableaux $\sim$-equivalent to T^{*1} have coefficient $+1$ in $\hat{\Theta}_{T^1}\{t^1\}\kappa_{t^1}$. So the coefficient of $\bar{T}^{*1}$ in $\bar{\Theta}_{T^1}\{t^1\}\kappa_{t^1}$ is $3!\,2^3 = 48 \neq 0$.

2. Let T^{*2} be the tableau given by

$$\begin{array}{c}1\,1\,1\,2\\2\,2\,3\,3\\3\,4\,4\,4\end{array}$$

then the tableaux $\sim$-equivalent to T^{*2} are

$$\begin{array}{c}1\,1\,1\,2\\2\,2\,3\,3\\3\,4\,4\,4\end{array},\ \begin{array}{c}1\,1\,1\,2\\2\,2\,4\,4\\4\,3\,3\,3\end{array},\ \begin{array}{c}1\,1\,1\,3\\3\,3\,2\,2\\2\,4\,4\,4\end{array},\ \begin{array}{c}1\,1\,1\,3\\3\,3\,4\,4\\4\,2\,2\,2\end{array},\ \begin{array}{c}1\,1\,1\,4\\4\,4\,3\,3\\3\,2\,2\,2\end{array},\ \begin{array}{c}1\,1\,1\,4\\4\,4\,2\,2\\2\,3\,3\,3\end{array},$$

$$\begin{array}{c}2\,2\,2\,1\\1\,1\,3\,3\\3\,4\,4\,4\end{array},\ \begin{array}{c}2\,2\,2\,1\\1\,1\,4\,4\\4\,3\,3\,3\end{array},\ \begin{array}{c}2\,2\,2\,3\\3\,3\,1\,1\\1\,4\,4\,4\end{array},\ \begin{array}{c}2\,2\,2\,3\\3\,3\,4\,4\\4\,1\,1\,1\end{array},\ \begin{array}{c}2\,2\,2\,4\\4\,4\,3\,3\\3\,1\,1\,1\end{array},\ \begin{array}{c}2\,2\,2\,4\\4\,4\,1\,1\\1\,3\,3\,3\end{array},$$

$$\begin{array}{c}3\,3\,3\,2\\2\,2\,1\,1\\1\,4\,4\,4\end{array},\ \begin{array}{c}3\,3\,3\,2\\2\,2\,4\,4\\4\,1\,1\,1\end{array},\ \begin{array}{c}3\,3\,3\,1\\1\,1\,2\,2\\2\,4\,4\,4\end{array},\ \begin{array}{c}3\,3\,3\,1\\1\,1\,4\,4\\4\,2\,2\,2\end{array},\ \begin{array}{c}3\,3\,3\,4\\4\,4\,1\,1\\1\,2\,2\,2\end{array},\ \begin{array}{c}3\,3\,3\,4\\4\,4\,2\,2\\2\,1\,1\,1\end{array},$$

$$\begin{array}{c}4\,4\,4\,1\\1\,1\,3\,3\\3\,2\,2\,2\end{array},\ \begin{array}{c}4\,4\,4\,1\\1\,1\,2\,2\\2\,3\,3\,3\end{array},\ \begin{array}{c}4\,4\,4\,3\\3\,3\,1\,1\\1\,2\,2\,2\end{array},\ \begin{array}{c}4\,4\,4\,3\\3\,3\,2\,2\\2\,1\,1\,1\end{array},\ \begin{array}{c}4\,4\,4\,2\\2\,2\,3\,3\\3\,1\,1\,1\end{array},\ \begin{array}{c}4\,4\,4\,2\\2\,2\,1\,1\\1\,3\,3\,3\end{array}.$$

All of these tableaux are involved in $\hat{\Theta}_{T^2}\{t^2\}\kappa_{t^2}$ and have coefficient $+1$. So the coefficient of $\bar{T}^{*2}$ in $\bar{\Theta}_{T^2}\{t^2\}\kappa_{t^2}$ is $24 \neq 0$.

3. Let T^{*3} be the tableau given by

$$T^{*3} = \begin{array}{l}1\,1\,1\,2\,3\\2\,2\\3\,3\end{array}$$

then the tableaux $\sim$-equivalent to T^{*3} are

$$\begin{array}{l}1\,1\,1\,2\,3\\2\,2\\3\,3\end{array},\quad \begin{array}{l}1\,1\,1\,3\,2\\3\,3\\2\,2\end{array},\quad \begin{array}{l}2\,2\,2\,1\,3\\1\,1\\3\,3\end{array},\quad \begin{array}{l}2\,2\,2\,3\,1\\3\,3\\1\,1\end{array},\quad \begin{array}{l}3\,3\,3\,1\,2\\1\,1\\2\,2\end{array},\quad \begin{array}{l}3\,3\,3\,2\,1\\2\,2\\1\,1\end{array}.$$

Again, all of these tableaux are all involved in $\hat{\Theta}_{T^3}\{t^3\}\kappa_{t^3}$ and have coefficient $+1$. So the coefficient of $\bar{T}^{*3}$ in $\bar{\Theta}_{T^3}\{t^3\}\kappa_{t^3}$ is $6 \neq 0$.

4. Let T^{*4} be the tableau given by

$$T^{*4} = \begin{array}{l} 1\ 1\ 1\ 2 \\ 2\ 2\ 3\ 3 \\ 3 \end{array}$$

then the tableaux $\sim$-equivalent to T^{*4} which are involved in $\hat{\Theta}_{T^4}\{t^4\}\kappa_{t^4}$ are

$$\begin{array}{l} 1\ 1\ 1\ 2 \\ 2\ 2\ 3\ 3 \\ 3 \end{array}, \quad \begin{array}{l} 1\ 1\ 1\ 3 \\ 3\ 3\ 2\ 2 \\ 2 \end{array}, \quad \begin{array}{l} 2\ 2\ 2\ 1 \\ 1\ 1\ 3\ 3 \\ 3 \end{array}, \quad \begin{array}{l} 2\ 2\ 2\ 3 \\ 3\ 3\ 1\ 1 \\ 1 \end{array}, \quad \begin{array}{l} 3\ 3\ 3\ 1 \\ 1\ 1\ 2\ 2 \\ 2 \end{array}, \quad \begin{array}{l} 3\ 3\ 3\ 2 \\ 2\ 2\ 1\ 1 \\ 1 \end{array}.$$

Once again all of these tableaux have coefficient $+1$ in $\hat{\Theta}_{T^4}\{t^4\}\kappa_{t^4}$. Thus the coefficient of $\bar{T}^{*4}$ in $\bar{\Theta}_{T^4}\{t^4\}\kappa_{t^4}$ is $6 \neq 0$.

5. Let T^{*5} be the tableau given by

$$T^{*5} = \begin{array}{l} 1\ 1\ 2\ 3\ 1\ 4 \\ 2\ 2\ 4\ 4 \\ 3\ 3 \end{array}$$

then the tableaux $\sim$-equivalent to T^{*5} which are involved in $\hat{\Theta}_{T^5}\{t^5\}\kappa_{t^5}$ are

$$\begin{array}{l} 1\ 1\ 2\ 3\ 1\ 4 \\ 2\ 2\ 4\ 4 \\ 3\ 3 \end{array}, \quad \begin{array}{l} 1\ 1\ 3\ 2\ 1\ 4 \\ 3\ 3\ 4\ 4 \\ 2\ 2 \end{array}, \quad \begin{array}{l} 1\ 1\ 4\ 3\ 1\ 2 \\ 4\ 4\ 2\ 2 \\ 3\ 3 \end{array}, \quad \begin{array}{l} 1\ 1\ 3\ 4\ 1\ 2 \\ 3\ 3\ 2\ 2 \\ 4\ 4 \end{array}, \quad \begin{array}{l} 2\ 2\ 1\ 3\ 2\ 4 \\ 1\ 1\ 4\ 4 \\ 3\ 3 \end{array},$$

$$\begin{array}{l} 2\ 2\ 3\ 1\ 2\ 4 \\ 3\ 3\ 4\ 4 \\ 1\ 1 \end{array}, \quad \begin{array}{l} 4\ 4\ 1\ 3\ 4\ 2 \\ 1\ 1\ 2\ 2 \\ 3\ 3 \end{array}, \quad \begin{array}{l} 4\ 4\ 3\ 1\ 4\ 2 \\ 3\ 3\ 2\ 2 \\ 1\ 1 \end{array}, \quad \begin{array}{l} 3\ 3\ 1\ 4\ 3\ 2 \\ 1\ 1\ 2\ 2 \\ 4\ 4 \end{array}, \quad \begin{array}{l} 3\ 3\ 4\ 1\ 3\ 2 \\ 4\ 4\ 2\ 2 \\ 1\ 1 \end{array},$$

$$\begin{array}{l} 3\ 3\ 1\ 2\ 3\ 4 \\ 1\ 1\ 4\ 4 \\ 2\ 2 \end{array}, \quad \begin{array}{l} 3\ 3\ 2\ 1\ 3\ 4 \\ 2\ 2\ 4\ 4 \\ 1\ 1 \end{array}, \quad \begin{array}{l} 3\ 3\ 2\ 4\ 3\ 1 \\ 2\ 2\ 1\ 1 \\ 4\ 4 \end{array}, \quad \begin{array}{l} 3\ 3\ 4\ 2\ 3\ 1 \\ 4\ 4\ 1\ 1 \\ 2\ 2 \end{array}.$$

The first twelve tableaux are involved in $\hat{\Theta}_{T^5}\{t^5\}\kappa_{t^5}$ with coefficient $+1$ and the last two are involved in $\hat{\Theta}_{T^5}\{t^5\}\kappa_{t^5}$ with coefficient -2. Therefore the coefficient of $\bar{T}^{*5}$ in $\bar{\Theta}_{T^5}\{t^5\}\kappa_{t^5}$ is $8 \neq 0$.

6. Let T^{*6} be the tableau given by

$$T^{*6} = \begin{array}{l} 1\ 1\ 1\ 2\ 2\ 2\ 3\ 5 \\ 3\ 3\ 4\ 4\ 4 \\ 5\ 5 \end{array}$$

then the tableaux $\sim$-equivalent to T^{*6} which are involved in $\hat{\Theta}_{T^6}\{t^6\}\kappa_{t^6}$ are

$$\begin{matrix}1&1&1&2&2&2&3&5\\3&3&4&4&4&&&\\5&5&&&&&&\end{matrix}\;,\quad \begin{matrix}1&1&1&2&2&2&5&3\\5&5&4&4&4&&&\\3&3&&&&&&\end{matrix}\;,\quad \begin{matrix}2&2&2&1&1&1&3&5\\3&3&4&4&4&&&\\5&5&&&&&&\end{matrix}\;,\quad \begin{matrix}2&2&2&1&1&1&5&3\\5&5&4&4&4&&&\\3&3&&&&&&\end{matrix}\;,$$

$$\begin{matrix}3&3&3&1&1&1&2&5\\2&2&4&4&4&&&\\5&5&&&&&&\end{matrix}\;,\quad \begin{matrix}3&3&3&1&1&1&5&2\\5&5&4&4&4&&&\\2&2&&&&&&\end{matrix}\;,\quad \begin{matrix}3&3&3&2&2&2&1&5\\1&1&4&4&4&&&\\5&5&&&&&&\end{matrix}\;,\quad \begin{matrix}3&3&3&2&2&2&5&1\\5&5&4&4&4&&&\\1&1&&&&&&\end{matrix}\;,$$

$$\begin{matrix}5&5&5&1&1&1&2&4\\2&2&4&4&4&&&\\3&3&&&&&&\end{matrix}\;,\quad \begin{matrix}5&5&5&1&1&1&3&2\\3&3&4&4&4&&&\\2&2&&&&&&\end{matrix}\;,\quad \begin{matrix}5&5&5&2&2&2&1&4\\1&1&4&4&4&&&\\3&3&&&&&&\end{matrix}\;,\quad \begin{matrix}5&5&5&2&2&2&3&1\\3&3&4&4&4&&&\\1&1&&&&&&\end{matrix}\;,$$

$$\begin{matrix}4&4&4&3&3&3&2&5\\2&2&1&1&1&&&\\5&5&&&&&&\end{matrix}\;,\quad \begin{matrix}4&4&4&3&3&3&5&2\\5&5&1&1&1&&&\\5&5&&&&&&\end{matrix}\;,\quad \begin{matrix}4&4&4&3&3&3&1&5\\1&1&2&2&2&&&\\5&5&&&&&&\end{matrix}\;,\quad \begin{matrix}4&4&4&3&3&3&5&1\\5&5&2&2&2&&&\\1&1&&&&&&\end{matrix}\;,$$

$$\begin{matrix}4&4&4&1&1&1&2&5\\2&2&3&3&3&&&\\5&5&&&&&&\end{matrix}\;,\quad \begin{matrix}4&4&4&1&1&1&5&2\\5&5&3&3&3&&&\\2&2&&&&&&\end{matrix}\;,\quad \begin{matrix}4&4&4&2&2&2&1&5\\1&1&3&3&3&&&\\5&5&&&&&&\end{matrix}\;,\quad \begin{matrix}4&4&4&2&2&2&5&1\\5&5&3&3&3&&&\\1&1&&&&&&\end{matrix}\;.$$

The first twelve tableaux are involved in $\hat{\Theta}_{T^6}\{t^6\}\kappa_{t^6}$ with coefficient +1 and the last eight are involved in $\hat{\Theta}_{T^6}\{t^6\}\kappa_{t^6}$ with coefficient −1. Therefore the coefficient of $\bar{T}^{*6}$ in $\bar{\Theta}_{T^6}\{t^6\}\kappa_{t^6}$ is $4 \neq 0$.

7. Let T^{*7} be the tableau given by

$$T^{*7} = \begin{matrix}1&1&2&2&2&1\\3&3&3&4&4&4\end{matrix}.$$

By a similar argument to that for T^{*1} it is easy to show that the coefficient of $\bar{T}^{*7}$ in $\bar{\Theta}_{T^7}\{t^7\}\kappa_{t^7}$ is $2!\,2^2 = 8 \neq 0$.

8. Let T^{*8} be the tableau given by

$$T^{*8} = \begin{matrix}1&1&2&1&2&2\\3&3&3&&&\end{matrix}$$

then the only tableaux $\sim$-equivalent to T^{*8} which are involved in $\hat{\Theta}_{T^8}\{t^8\}\kappa_{t^8}$ are

$$\begin{matrix}1&1&2&1&2&2\\3&3&3&&&\end{matrix}\;,\quad \begin{matrix}2&2&1&2&1&1\\3&3&3&&&\end{matrix}\;.$$

These tableaux both have coefficient +1 in $\hat{\Theta}_{T^8}\{t^8\}\kappa_{t^8}$. Hence the coefficient of $\bar{T}^{*8}$ in $\bar{\Theta}_{T^8}\{t^8\}\kappa_{t^8}$ is $2 \neq 0$.

9. Let T^{*9} be the tableau given by

$$T^{*9} = \begin{matrix} 1\,1\,1\,2 \\ 2\,2 \end{matrix}$$

then

$$\begin{matrix} 1\,1\,1\,2 \\ 2\,2 \end{matrix} \quad \text{and} \quad \begin{matrix} 2\,2\,2\,1 \\ 1\,1 \end{matrix}$$

are the only tableaux which are $\sim$-equivalent to T^{*9} and are involved in $\hat{\Theta}_{T^9}\{t^9\}\kappa_{t^9}$. They both have coefficient $+1$ in $\hat{\Theta}_{T^9}\{t^9\}\kappa_{t^9}$. Thus the coefficient of $\bar{T}^{*9}$ in $\bar{\Theta}_{T^9}\{t^9\}\kappa_{t^9}$ is $2 \neq 0$.

10. Trivially, when

$$T^{*10} = 1\,1\,1$$

then the coefficient of $\bar{T}^{*10}$ in $\bar{\Theta}_{T^{10}}\{t^{10}\}\kappa_{t^{10}}$ is $1 \neq 0$.

Thus the proof is complete. □

Remark: As a direct consequence of the lemma, we have that $S^{(6,6,6)}$ appears in $FP(6,3)$; the module $S^{(8,5,2)}$ appears in $FP(5,3)$; the modules $S^{(4,4,4)}$, $S^{(6,4,2)}$ and $S^{(6,6)}$ appear in $FP(4,3)$; the modules $S^{(5,2,2)}$, $S^{(4,4,1)}$ and $S^{(6,3)}$ appear in $FP(3,3)$; the module $S^{(4,2)}$ appears in $FP(2,3)$ and the module $S^{(3)}$ appears in $FP(1,3)$.

Proposition 6.2 *For $s = 0, 1, \ldots, k + [k/2]$ the multiplicity of $S^{(3k-s,s)}$ in the decomposition of $FP(k,3)$ is greater than or equal to the multiplicity of $S^{(3k-s,s)}$ in the decomposition of $FP(3,k)$.*

Proof: We know that the multiplicity of $S^{(3k-s,s)}$ in $FP(3,k)$ is $m_{k,s}$ which is given in Lemma 5.11. Thus we will show that the multiplicity of $S^{(3k-s,s)}$ in $FP(k,3)$ is greater than or equal to $m_{k,s}$. We will construct a set of $m_{k,s}$ linearly independent $FSym(3k)$-homomorphisms from $S^{(3k-s,s)}$ to $FP(k,3)$ using the method described at the beginning of this section. We consider the case when $s = 1$ separately. In this case $m_{k,s} = 0$ so we have nothing to prove. So assume that s is a non-negative integer with $s \neq 1$ then this can be written in the form $s = 6c+r$ for a non-negative integer c and $r \in \{0, 2, 3, 4, 5, 7\}$. When $k < s-2c$ then $m_{k,s} = c-[(s-k+1)/2]+1 \leq 0$ (where $[\,]$ denotes the integer part) so we only need to consider the case when $k \geq s-2c$. In this case let $c = e + f$ for some non-negative integers e, f satisfying $0 \leq e, f \leq c$ and let $r = 2r_1 + 3r_2$ with $r_1 \in \{0, 1, 2\}$ and $r_2 \in \{0, 1\}$ (note that there is a unique way of writing r in this form).

Step 1: Denote by T_f^* the $(3k - s, s)$-tableau of type (3^k) given by

$$T_f^* = \underbrace{T^{*7} \cup \cdots \cup T^{*7}}_{e} \cup \underbrace{T^{*9} \cup \cdots \cup T^{*9}}_{3f} \cup \underbrace{T^{*8}}_{r_2} \cup \underbrace{T^{*9} \cup T^{*9}}_{r_1} \cup \underbrace{T^{*10} \cup \cdots \cup T^{*10}}_{k-s+2e}$$

where the T^{*i} are the tableaux with pattern given at the beginning of this section. As $k \geq s - 2c$ and $e \leq c$ we can always construct a tableau of this sort by taking e large enough so that $k \geq s - 2e$. Let t be the $(3k - s, s)$-tableau constructed from blocks t^7, t^8, t^9 and t^{10} in the way described at the beginning of this section and let T_f be the semistandard tableau row equivalent to T_f^*. The number of blocks T^{*9} which we can use in T_f^* depends on the size of k. If $k \geq s$ then f can take any value between 0 and c as $k - s + 2e$ will always be non-negative. If $k < s$ then for $k - s + 2e = k - s + 2c - 2f$ to be non-negative we require $0 \leq f \leq c - (s - k)/2$. Thus f can take $c + 1$ different values if $s \leq k$ and $c - [(s - k + 1)/2] + 1$ different values if $s > k$.

Step 2: By Lemma 6.1 we know that the coefficient of $\bar{T}_f^*$ in $\bar{\Theta}_{T_f}\{t\}\kappa_t$ is non-zero.

Step 3: To show linearly independence of the $\bar{\Theta}_{T_f}$ we show, for a fixed f in the appropriate range, that $\bar{T}_f^*$ is not involved in $\bar{\Theta}_{T_i}\{t\}\kappa_t$ for $i < f$. To do this we only need to look at the number of pairs in the second row of T_i. There will be less pairs (and more triplets) in the second row of T_i than there is in the second row of T_f. Therefore there are not enough pairs in the second row of T_i to make all of the tableaux $\sim$-equivalent to the T^{*9}. This means that the coefficient of $\bar{T}_f^*$ in $\bar{\Theta}_{T_i}\{t\}\kappa_t$ is zero. Hence we have constructed a set of $c + 1$ or $c - [(s - k + 1)/2] + 1$ (for $k \geq s$ and $k < s$ respectively) linearly independent homomorphisms $\bar{\Theta}_{T_f}$ from $S^{(3k-s,s)}$ to $FP(k, 3)$, one for each value of f in the appropriate range. From Theorem 2.20 and Lemma 5.11 the proof is complete. □

Proposition 6.3 *For u and s non-negative integers satisfying $k - 2u \geq (2s)/3$ the multiplicity of $S^{(3(k-2u)-s+2u,s+2u,2u)}$ in the decomposition of $FP(k, 3)$ is greater than or equal to the multiplicity of $S^{(3(k-2u)-s+2u,s+2u,2u)}$ in the decomposition of $FP(3, k)$.*

Proof: We have done the case when $u = 0$ above so we only need to consider the case when $u > 0$. We know that the multiplicity of $S^{(3(k-2u)-s+2u,s+2u,2u)}$ in $FP(3,k)$ is $m_{k-2u,s}$ which is given in Lemma 5.11. Using the method described at the beginning of this section we construct a set of $m_{k-2u,s}$ linearly independent homomorphisms from $S^{(3(k-2u)-s+2u,s+2u,2u)}$ to $FP(k,3)$. When $s = 1$ or $k - 2u < s - 2c$ again $m_{k-2u,s} = 0$ for any value of u so there is nothing to prove, thus we assume that $s \neq 1$ and $k - 2u \geq s - 2c$. As before we write s in the form $s = 6(e + f) + 2r_1 + 3r_2$, where $c = e + f$ and $r = 2r_1 + 3r_2 \in \{0, 2, 3, 4, 5, 7\}$ with $r_1 \in \{0, 1, 2\}$ and $r_2 \in \{0, 1\}$.

Step 1: We split this step into four cases.

Case 1: Consider $u \neq 1$ then we can write $2u$ (in a unique way) in the form $2u = 6u_1 + 4u_2$ where u_1 and u_2 are non-negative integers with $u_1 \in \{0, 1\}$. Denote by T_f^{**} the $((3k - 2u) - s + 2u, s + 2u, 2u)$-tableau of type (3^k) given by

$$T_f^{**} = \underbrace{T^{*1}}_{u_1} \cup \underbrace{T^{*2} \cup \cdots \cup T^{*2}}_{u_2} \cup T_f^*$$

where T_f^* is given in the proof of the previous proposition but with the appropriate labelling set and clearly we can only use $k - s - 2u + 2e$ of the blocks T^{*10} in T_f^*. Let t be the $(3(k - 2u) - s + 2u, s + 2u, 2u)$-tableau constructed in the way described at the beginning of the section and let T_f be the semistandard tableau row equivalent to T_f^{**}. If $s \leq k - 2u$ then $k - s - 2u + 2e$ will always be non-negative and so f can take any value between 0 and c. When $s > k - 2u$ then $k - s - 2u + 2e = k - s - 2u + 2(c - f)$ will be non-negative for $0 \leq f \leq c - (s - k + 2u)/2$. Thus f can take $c + 1$ different values if $s \leq k - 2u$ and $c - [(s - k + 2u + 1)/2] + 1$ different values if $s > k - 2u$.

Case 2: If $u = 1$ and $k - 2 > s$ then let T_f^{**} be given by

$$T^{*3} \cup \underbrace{T^{*7} \cup \cdots \cup T^{*7}}_{e} \cup \underbrace{T^{*9} \cup \cdots \cup T^{*9}}_{3f} \cup \underbrace{T^{*8}}_{r_2} \cup \underbrace{T^{*9} \cup T^{*9}}_{r_1} \cup \underbrace{T^{*10} \cup \cdots \cup T^{*10}}_{k-s-3+2e}. \quad (*)$$

Let t be the $(3(k - 2u) - s + 2u, s + 2u, 2u)$-tableau constructed in the way described at the beginning of the section and let T_f be the semistandard tableau row equivalent to T_f^{**}. It is clear that f can take any value between 0 and c as $k - s - 3 + 2e = k - s - 3 + 2(c - f)$ is always non-negative for any value of f in this range.

Case 3: If $u = 1$ and $k - 2 \leq s$ with $c \geq 1$ then if $f \neq c$ (that is, $e \neq 0$) instead of using the block $T^{*3} \cup T^{*7}$ in $(*)$ we use $T^{*4} \cup T^{*4}$, so T_f^{**} is given by

$$T^{*4} \cup T^{*4} \cup \underbrace{T^{*7} \cup \cdots \cup T^{*7}}_{e-1} \cup \underbrace{T^{*9} \cup \cdots \cup T^{*9}}_{3f} \cup \underbrace{T^{*8}}_{r_2} \cup \underbrace{T^{*9} \cup T^{*9}}_{r_1} \cup \underbrace{T^{*10} \cup \cdots \cup T^{*10}}_{k-s-2+2e}$$

and if $f = c$ (so $e = 0$) we use the block $T^{*5} \cup T^{*9} \cup T^{*9}$ in $(*)$ instead of $T^{*3} \cup T^{*9} \cup T^{*9} \cup T^{*9}$, so T_c^{**} is given by

$$T^{*5} \cup T^{*9} \cup T^{*9} \cup \underbrace{T^{*9} \cup \cdots \cup T^{*9}}_{3(c-1)} \cup \underbrace{T^{*8}}_{r_2} \cup \underbrace{T^{*9} \cup T^{*9}}_{r_1} \cup \underbrace{T^{*10} \cup \cdots \cup T^{*10}}_{k-s-2}.$$

Let t be the $(3(k-2u) - s + 2u, s + 2u, 2u)$-tableau constructed in the way described at the beginning of the section and let T_f be the semistandard tableau row equivalent to T_f^{**}. $k - s - 2 + 2e = k - s - 2 + 2(c - f)$ is non-zero when $0 \leq f \leq c - (s - k + 2)/2$.

Case 4: If $u = 1$ and $k - 2 \leq s$ with $c = 0$ then $r \neq 0$ (since $k \geq 3$). If $r_2 = 1$ then we replace $T^{*3} \cup T^{*8}$ in $(*)$ with T^{*6} and let

$$T_0^{**} = T^{*6} \cup \underbrace{T^{*9} \cup T^{*9}}_{r_1} \cup \underbrace{T^{*10} \cup \cdots \cup T^{*10}}_{k-r-2}.$$

Otherwise, if $r_2 = 0$ then we replace $T^{*3} \cup T^{*9}$ in $(*)$ with T^{*5} and let

$$T_0^{**} = T^{*5} \cup \underbrace{T^{*9}}_{r_1 - 1} \cup \underbrace{T^{*10} \cup \cdots \cup T^{*10}}_{k-r-2}.$$

Let t be the $(3(k-2u) - s + 2u, s + 2u, 2u)$-tableau constructed in the way described at the beginning of the section and let T_f be the semistandard tableau row equivalent to T_f^{**}. We can always construct the above tableaux since $k - r - 2 = k - s - 2$ (as $c = 0$) and this is non-zero by assumption. As c is zero then the only value that f can take is 0.

Step 2: By Lemma 6.1 we know that the coefficient of $\bar{T}_f^{**}$ in $\bar{\Theta}_{T_f}\{t\}\kappa_t$ is non-zero for each of the above T_f^{**}.

Step 3: For each T_f^{**} we need to show that as f runs over the values in the appropriate range then the $\bar{\Theta}_{T_f}$ are linearly independent. The first two cases follow in the same way as in the proof of Proposition 6.2, that is for $i < f$ we just need to count the number of pairs in the second row of T_i and we see that $\bar{T}_f^{**}$ is not involved in $\bar{\Theta}_{T_i}\{t\}\kappa_t$. In case 3 again it is easy to see by counting pairs in the middle row of T_i that $\bar{T}_f^{**}$ (for $f < c$) is not involved in $\bar{\Theta}_{T_i}\{t\}\kappa_t$. Thus $\bar{\Theta}_{T_0}, \bar{\Theta}_{T_1}, \ldots, \bar{\Theta}_{T_q}$ are linearly independent, where $q = c - [(s-k+3)/2]$ if $k \neq s+2$ and $q = c-1$ if $k = s+2$. As e can only be zero when $k = s+2$ we will consider this case separately. Since T_c has a pair in its bottom row, this pair will always be in the first two columns of an element involved in $\hat{\Theta}_{T_c}\{t\}\kappa_t$. However, the first two columns of $\bar{T}_i^{**}$ with $i < c$ have distinct entries. Thus it can be seen that $\bar{T}_o^{**}, \bar{T}_1^{**}, \ldots, \bar{T}_{c-1}^{**}$ are not involved in $\bar{\Theta}_{T_c}\{t\}\kappa_t$. As $\bar{\Theta}_{T_c}\{t\}\kappa_t \neq 0$ then $\bar{\Theta}_{T_0}, \bar{\Theta}_{T_1}, \ldots, \bar{\Theta}_{T_c}$ are linearly independent homomorphisms from $S^{(3(k-2u)-s+2u,s+2u,2u)}$ to $FP(k,3)$. Thus in Case 3 we will always have a set of $c - [(s+2u-k+1)/2]+1$ linearly independent homomorphisms. The final case that we look at is when $u = 1$ with $k-2 \leq s$ and $c = 0$. We know from Lemma 5.11 that $m_{k-2,r} \leq 1$ and since we have have constructed one non-zero homomorphism from $S^{(3(k-2)-r+2,s+2,2)}$ to $FP(k,3)$ there is nothing more to prove. Comparing the number of values that f can take with the multiplicities $m_{k-2u,s}$ given in Lemma 5.11 completes the proof. □

Proposition 6.4 *The multiplicity of the module $S^{(3(k-2u-3)-s+2u+4,s+2u+4,2u+1)}$ in the decomposition of $FP(k,3)$ is greater than or equal to the multiplicity of the module $S^{(3(k-2u-3)-s+2u+4,s+2u+4,2u+1)}$ in the decomposition of $FP(3,k)$.*

Proof: The proof follows in a completely analogous way to the proof of Proposition 6.3. For each T_f^{**} which we constructed in the last proof, we 'replace' three of the T^{*10} by T^{*4}. Therefore f can take $c+1$ different values if $k-2u-3 > s$ and $c - [(s-(k-2u+3)+1)/2]+1$ different values if $k-3-2u \leq s$. This is the number of linearly independent homomorphisms from $S^{(3(k-2u-3)-s+2u+4,s+2u+4,2u+1)}$ to $FP(k,3)$. □

Theorem 6.5 *Foulkes' conjecture holds when $a = 3$ and b is arbitrary.*

Proof: From Corollary 5.14 and Propositions 6.2, 6.3 and 6.4 we have that all modules S^{μ} which appear in $FP(3,k)$ with multiplicity m appear in $FP(k,3)$ with multiplicity greater than or equal to m. This proves the conjecture. □

6.2 Eigenvalues and Eigenvectors of $M^{3,k}(M^{3,k})^T$

We use the decomposition of $FP(3,k)$ given in Section 5.4.2 to calculate the eigenvalues of $M^{3,k}(M^{3,k})^T$ for small values of k. The proofs of Propositions 5.10, 5.12 and Theorem 3.8 give us a set of linearly independent homomorphisms from S^μ to $FP(3,k)$ for each module S^μ which appears in $FP(3,k)$. When this set of linearly independent homomorphisms contains more than one element, we will use the notation for the semistandard tableaux which we introduced in Section 5.4.2. We will begin by considering the modules S^μ in the decomposition of $FP(3,k)$ which appear with multiplicity one. These are the easiest cases to consider since there is a semistandard μ-tableaux T and a μ-tableau t (given in the proofs of Propositions 5.10, 5.12 or Theorem 3.8) such that $\bar{\Theta}_T\{t\}\kappa_t$ is an eigenvector of MM^T. Using the method described at the end of Section 3.3, to find the eigenvalue corresponding to $\bar{\Theta}_T\{t\}\kappa_t$ we calculate the α-entry of MM^Tv for a $(3,k)$-partition α involved in v and divide by the coefficient of α in v. We can write a MAGMA program to calculate these eigenvalues for particular values of k. When S^μ appears in $FP(3,k)$ with multiplicity greater than one, it may be necessary to take a linear combination of our linearly independent homomorphisms in order to construct an eigenvector of MM^T. There doesn't appear to be an obvious way to construct such eigenvectors, the best we can do is to construct them case by case for small values of k.

We consider the eigenvalues of MM^T for $1 \leq k \leq 8$. When $1 \leq k \leq 5$ we know that all modules in the complete decomposition of $FP(3,k)$ appear with multiplicity one so we can use our program to calculate the complete set of eigenvalues of MM^T. For $6 \leq k \leq 8$ the module $S^{(3k-6,6)}$ appears in $FP(3,k)$ with multiplicity two. The homomorphisms $\bar{\Theta}_{T_0}$ and $\bar{\Theta}_{T_2}$ given in the proof of Proposition 5.10 are linearly independent. It is easy to see from looking at the results of our original program that neither $\bar{\Theta}_{T_0}\{t\}\kappa_t$ or $\bar{\Theta}_{T_2}\{t\}\kappa_t$ are eigenvectors of MM^T. Thus for some $c \in F$ the vector $v = \bar{\Theta}_{T_0}\{t\}\kappa_t + c\bar{\Theta}_{T_2}\{t\}\kappa_t$ will be an eigenvector of MM^T. We adapt our original program slightly so that we can construct the α and γ entries of both $MM^T\bar{\Theta}_{T_0}\{t\}\kappa_t$ and $MM^T\bar{\Theta}_{T_2}\{t\}\kappa_t$ where α and γ are distinct $(3,k)$-partitions involved in both $\bar{\Theta}_{T_0}\{t\}\kappa_t$ and $\bar{\Theta}_{T_2}\{t\}\kappa_t$. Using these results we calculate the α and

γ-entries of

$$MM^T(\bar{\Theta}_{T_0}\{t\}\kappa_t + c\bar{\Theta}_{T_2}\{t\}\kappa_t)$$

and

$$\bar{\Theta}_{T_0}\{t\}\kappa_t + c\bar{\Theta}_{T_2}\{t\}\kappa_t.$$

These give us a pair of simultaneous equations to solve for c and the eigenvalue λ. Solving for c first gives

$$c = \begin{cases} \frac{\lambda - 2^{11}3^35\cdot 29}{2^{12}3^45\cdot 11} & \text{if } k=6 \\ \frac{\lambda - 2^{14}3^55^213}{2^{14}3^65^211} & \text{if } k=7 \\ \frac{\lambda - 2^{17}3^65^5}{2^{17}3^75^323} & \text{if } k=8. \end{cases}$$

We then solve for λ which gives

$$\lambda = \begin{cases} 2^{10}3^3(13\cdot 17 \pm \sqrt{/}5641) & \text{if } k=6 \\ 2^{13}3^55(3^211 \pm \sqrt{/}1401) & \text{if } k=7 \\ 2^{16}3^65^2(191 \pm 79) & \text{if } k=8. \end{cases}$$

For $k = 6$ and 7 these complete the set of eigenvalues of MM^T. When $k = 8$ the Specht modules $S^{(16,8)}$ and $S^{(14,8,2)}$ appear in the decomposition of $FP(3,8)$ also with multiplicity two. Using our 'improved' program and the linearly independent homomorphisms given in the proofs of Proposition 5.10 and Theorem 3.8 we can calculate the eigenvalues associated to these partitions. When $\mu = (16,8)$ the vector $v = \bar{\Theta}_{T_0}\{t\}\kappa_t + c\bar{\Theta}_{T_2}\{t\}\kappa_t$ where

$$c = \frac{\lambda - 2^{21}3^45\cdot 7\cdot 17}{2^{19}3^45^37\cdot 11}$$

is an eigenvector of MM^T with eigenvalue

$$\lambda = 2^{17}3^65(3^217 \pm \sqrt{/}(13\cdot 293)).$$

When $\mu = (14,8,2)$ the vectors $\bar{\Theta}_{T_0}\{t\}\kappa_t$ and $\bar{\Theta}_{T_2}\{t\}\kappa_t$ are linearly independent and are already eigenvectors of MM^T. They both correspond to the eigenvalue

$$\lambda = 4777574400 = 2^{18}3^65^2.$$

In the Appendix we give the complete set of eigenvalues of $M^{3,k}(M^{3,k})^T$ for $4 \leq k \leq 8$ and below we list the eigenvalues of $M^{3,k}(M^{3,k})^T$ for $1 \leq k \leq 3$. Alongside the eigenvalue we list the dimension of the corresponding Specht module, the sum of which confirms that we do indeed have the complete set of eigenvalues of MM^T.

Example 6.6 *The eigenvalues of $M^{3,k}(M^{3,k})^T$ with $1 \leq k \leq 3$ are given in the following tables.*

$k = 1$

Tableau	*Partition*	*Eigenvalue*	*Dimension*
$T =$ 123	(3)	1	1
Sum of dimensions			1

$k = 2$

Tableau	*Partition*	*Eigenvalue*	*Dimension*
$T =$ 112233	(6)	$24 = 2^3 \cdot 3$	1
$T =$ 1133 22	(4, 2)	$4 = 2^2$	9
$T =$ 11 22 33	(2, 2, 2)	0	5
Sum of dimensions			15

$k = 3$

Tableau	*Partition*	*Eigenvalue*	*Dimension*
$T =$ 111222333	(9)	$1296 = 2^4 \cdot 3^4$	1
$T =$ 1112333 22	(7, 2)	$144 = 2^4 \cdot 3^2$	27
$T =$ 111222 222	(6, 3)	$64 = 2^6$	48
1112 $T =$ 2233 3	(4, 4, 1)	$16 = 2^4$	84
11123 $T =$ 22 33	(5, 2, 2)	$4 = 2^2$	120
Sum of dimensions			280

Proposition 6.7 *For $3 \leq k \leq 8$, the matrix $M^{3,k}$ has full rank.*

Proof: Using the tables in the Appendix and the table above for $k = 3$, we see that all eigenvalues of MM^T are non-zero. Using Lemma 2.1, this means that the matrix M must have full rank. □

The results we have obtained for small values of k indicate the following surprising property: Eigenvalues which are directly associated to partitions can be partially ordered (in terms of size) in the same way as the natural partial ordering on the partitions (see Definition 2.8). Moreover, the eigenvalues which are 'indirectly associated' with partitions join in this partial ordering if we sum the eigenvalues associated to a particular partition. If we could prove this partial ordering property in general then to prove Foulkes' conjecture it would be sufficient to show for $k \geq 3$ that the eigenvalues associated to the 'smallest' partitions are non-zero. We will show that the eigenvalues associated to the 'smallest' partitions are non-zero in Theorem 6.11. The partial ordering information can be expressed more clearly in the form of partial ordering diagrams. When $k = 1$ the diagram will just be a single dot so we will not illustrate it. The diagrams for $k = 2$ and $k = 3$ can be found overleaf and the diagrams for $4 \leq k \leq 8$ can be found in the Appendix (see figures 5 to 14).

6.3 Construction of Non-Zero Eigenvalues

We look more generally at eigenvalues of $M^{3,k}(M^{3,k})^T$ which are simply associated to partitions of $3k$. We show that certain partitions of $3k$ are associated to non-zero eigenvalues of $M^{3,k}(M^{3,k})^T$. Using an inductive proof similar to that of Theorem 4.10 we show that certain non-zero eigenvalues of $M^{3,k-2}(M^{3,k-2})^T$ correspond to non-zero eigenvalues of $M^{3,k}(M^{3,k})^T$. At the end of the section we give some of the eigenvalues of $M^{3,k}(M^{3,k})^T$ explicitly in a closed form.

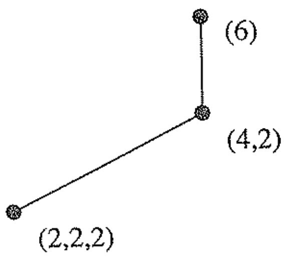

Figure 1: Partial Ordering of the Partitions of 6

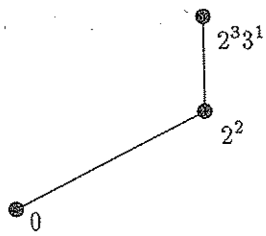

Figure 2: Partial Ordering of the Eigenvalues of $M^{3,2}(M^{3,2})^T$

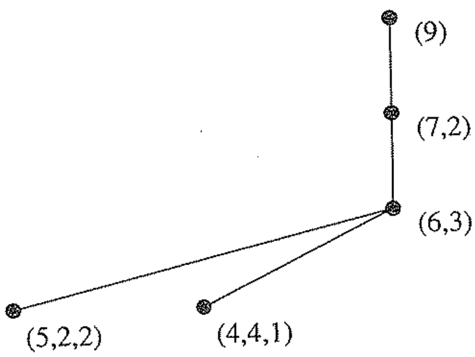

Figure 3: Partial Ordering of the Partitions of 9

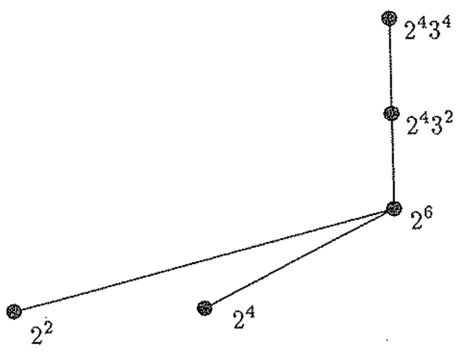

Figure 4: Partial Ordering of the Eigenvalues of $M^{3,3}(M^{3,3})^T$

Proposition 6.8 *For $s \in \{0, 2, 3, 4, 5, 7\}$ with $k \geq s$ the eigenvalue associated to $(3k - s, s)$ is strictly positive.*

Proof: We have already proved the proposition for the case when $s = 0$ so assume that s is an element of $\{2, 3, 4, 5, 7\}$ with $k \geq s$. Let $\mu = (3k - s, s)$. We know from the result of Section 5.4.3 that S^μ appears in $FP(3, k)$ with multiplicity one (or in other words, μ is simply associated to an eigenvalue λ of MM^T). Thus, we only need to show that λ is non-zero. Let T be the semistandard μ-tableau and t be the μ-tableau given by

$$T = \begin{matrix} 1\,1\ldots1\,1\ldots1\,2\ldots2\,3\ldots3 \\ \underbrace{2\,2\ldots2}_{s} \end{matrix} \quad \text{and} \quad t = \begin{matrix} 1\,3\ldots2s-1\;2s+1\;\ldots\;3k \\ 2\,4\ldots2s \end{matrix}.$$

We know from the proof of Proposition 5.10 that $\bar{\Theta}_T\{t\}\kappa_t$ is an eigenvector of MM^T corresponding to λ. Let $v = \bar{\Theta}_T\{t\}\kappa_t$. We will construct a $(k, 3)$-partition β indexing a non-zero entry of $M^T v$. From the results at the end of Section 3.3 this is enough to show that λ is non-zero. Let $\beta = \cup_{i=1}^k \beta_i$ be the $(k, 3)$-partition constructed as follows. Begin by placing 1 and 3 in β_1 and 2 and 4 in β_2. Fill up the parts of β by putting the even numbers $2j$ with $6 \leq 2j \leq 2s$ into β_j and the odd numbers between 5 and $2s - 1$ into the first part which has 'available space'. Fill the rest of the space in the β_i with the digits $\{2s + 1, \ldots, 3k\}$ in some arbitrary way. Thus β looks like

$$\begin{pmatrix}1\\3\\5\end{pmatrix}\begin{pmatrix}2\\4\\7\end{pmatrix}\begin{pmatrix}6\\9\\*\end{pmatrix}\begin{pmatrix}8\\*\\*\end{pmatrix}\begin{pmatrix}10\\*\\*\end{pmatrix}\begin{pmatrix}*\\*\\*\end{pmatrix}\cdots\begin{pmatrix}*\\*\\*\end{pmatrix}.$$

The β-entry of $M^T v$ is the sum of the coefficients in v of those $(3, k)$-partitions which are involved in v and intersect nicely with β. As T has 2 occurring s times in its second row, any tableaux involved in $\hat{\Theta}_T\{t\}\kappa_t$ must have 2 appearing s times in its first s columns. Thus any $(3, k)$-partition α involved in $\bar{\Theta}_T\{t\}\kappa_t$ will have s of the digits $\{1, 2, \ldots, 2s\}$ in one of its parts (call this part α_2) with the property that exactly one of the pairs $\{1, 2\}$, $\{3,4\}, \ldots, \{2s - 1, 2s\}$ is in α_2. We consider the possibilities for α_2. Either α_2 will contain the digits $\{1, 4, 6, 8, \ldots 2s\}$ or it will contain the digits $\{2, 3, 6, 8, \ldots, 2s\}$. As each part of β contains at most two odd numbers we put one of each of these odd numbers into the remaining parts of α. It is easy to see that there will always be a $(3,k)$-partition α which intersects nicely with β and moreover this partition will always have coefficient -2 in v. Therefore the β-entry of M^Tv is non-zero. Hence

$$\frac{(M^Tv)^T M^T v}{v^T v} = \lambda \neq 0.$$

□

Remark: The proof of the above proposition also gives us the following result that for any s with $s \leq k$ if

$$T = \begin{array}{l} 1\,1\ldots1\,1\ldots1\,2\ldots2\,3\ldots3 \\ \underbrace{2\,2\ldots2}_{s} \end{array}$$

is directly associated to an eigenvalue of MM^T then this eigenvalue is non-zero.

We use an inductive proof to show that if we have a non-zero eigenvalue of $M^{3,k-2}(M^{3,k-2})^T$ which is simply associated to a partition of $3(k-2)$ then we can construct a non-zero eigenvalue of $M^{3,k}(M^{3,k})^T$ which is simply associated to a partition of $3k$. Moreover we give a lower bound for the size of the eigenvalue which we construct for $M^{3,k}(M^{3,k})^T$.

Theorem 6.9 *Let M^* be the incidence matrix of $\mathcal{P}_{3,k-2}$ and M be the incidence matrix of $\mathcal{P}_{3,k}$. Let $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*)$ be a partition of $3(k-2)$ which is simply associated to a non-zero eigenvalue of M^*M^{*T} then $\mu = (\mu_1^*+2, \mu_2^*+2, \mu_3^*+2)$ is associated to a non-zero eigenvalue of MM^T. Moreover, the eigenvalue associated to μ is greater than or equal to $4(k-2)$ times the eigenvalue associated to μ^*.*

Proof: We know from Proposition 3.18 that if $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*)$ is in simple association with an eigenvalue of M^*M^{*T} then $\mu = (\mu_1^*+2, \mu_2^*+2, \mu_3^*+2)$ is in simple association with an eigenvalue of MM^T. Let T^* be the semistandard μ^*-tableau and t^* be the μ-tableau such that $v^* = \bar{\Theta}_{T^*}\{t^*\}\kappa_{t^*}$ is an eigenvector of M^*M^{*T}. Let $T = T' \cup T^*$ and $t = t' \cup t^*$ where

$$T' = \begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{array} \quad \text{and} \quad t' = \begin{array}{cc} 3k-5 & 3k-2 \\ 3k-4 & 3k-1 \\ 3k-3 & 3k \end{array}$$

then we know from Proposition 3.8 that $v = \bar{\Theta}_T\{t\}\kappa_t$ is an eigenvector of MM^T. Since the eigenvalue associated to μ^* is non-zero, we know that there exists a $(k-2,3)$-partition β^* which indexes a non-zero entry of $M^{*T}v^*$. Let $*_1, *_2, *_3$ make up one part of β^* and denote by β' the result after removing this part from β^*. Form the $(k,3)$-partition β given by

$$\beta = \begin{pmatrix} 3k-5 \\ 3k-1 \\ *_1 \end{pmatrix} \begin{pmatrix} 3k-4 \\ 3k \\ *_2 \end{pmatrix} \begin{pmatrix} 3k-3 \\ 3k-2 \\ *_3 \end{pmatrix} \cup \beta'.$$

We claim that the β-entry of $M^T v$ is non-zero. We know that if α^* is involved in v^* then the $(3,k)$-partition α formed by joining one element from each of the sets $\{3k-5, 3k-4, 3k-3\}$ and $\{3k-2, 3k-1, 3k\}$ to each part of α^* is involved in v. Moreover, all $(3,k)$-partitions involved in v can be constructed in this way. Thus, any $(3,k)$-partition which intersects nicely with β and is involved in v must have $*_1, *_2, *_3$ in distinct parts. Let α^* be a $(3, k-2)$-partition involved in v^* which intersects nicely with β^*. Denote by α^1 the $(3,k)$-partition formed by adjoining $\{3k-5, 3k\}$ to the part of α^* containing $*_3$, adjoining $\{3k-4, 3k-2\}$ to the part of α^* containing $*_1$ and adjoining $\{3k-3, 3k-1\}$ to the part of α^* containing $*_2$. Denote by α^2 the $(3,k)$-partition formed by adjoining $\{3k-5, 3k-2\}$ to the part of α^* containing $*_2$, adjoining $\{3k-4, 3k-1\}$ to the part of α^* containing $*_3$ and adjoining $\{3k-3, 3k\}$ to the part of α^* containing $*_1$. Then it is easy to see that α^* intersects nicely with β^* and is involved in v^* if and only if α^1 and α^2 intersect nicely with β and are involved in v. Moreover, the coefficients of α^1 and α^2 in v are the same as the coefficient of α^* in v^*. Thus, the β-entry of $M^T v$ is twice the β^*-entry of $M^{*T}v^*$ and so is non-zero. Thus, the eigenvalue associated to μ is non-zero. To give a lower bound on the size of this eigenvalue, for each β^* indexing a non-zero entry of $M^{*T}v^*$ we construct a family of $(k,3)$-partitions indexing a non-zero entry of $M^T v$. There are $(k-2)$ different choices for $*_1, *_2, *_3$ and 36 arrangements of the integers $3k-5, 3k-4, 3k-3$ and $3k-2, 3k-1, 3k$ in the parts of β. So, for every β^* indexing a non-zero entry of $M^{*T}v^*$ we construct $36(k-2)$ different $(k,3)$-partitions β. We chose β arbitrarily so for each of these $(k,3)$-partitions, the corresponding entry of $M^T v$ will be ± 2 times the β^*-entry of $M^{*T}v^*$. Therefore,

$$(M^Tv)^T(M^Tv) \geq 4(k-2)\cdot 36(M^{*T}v^*)^T(M^{*T}v^*).$$

We can see that $v^{*T}v^* = 36v^Tv$ and so

$$\lambda = \frac{(M^Tv)^T(M^Tv)}{v^Tv} \geq 4(k-2)\frac{(M^{*T}v^*)^T(M^{*T}v^*)}{v^{*T}v^*} = 4(k-2)\lambda^*$$

as required. □

Remark: The lower bound which we gave for the size of the eigenvalue λ is largely an under estimate and can probably be improved by considering different types of $(k,3)$-partitions indexing non-zero entries of M^Tv.

Using the results from Section 5.4.3 and Proposition 6.8 which tell us that for $k \geq s$ with $s \in \{0,2,3,4,5,7\}$ the partition $(3k-s,s)$ of $3k$ is simply associated to a non-zero eigenvalue of $M^{3,k}(M^{3,k})^T$ we have the following result.

Corollary 6.10 *For non-negative integers u and s satisfying $2u+s \leq k$ with $s \in \{0,2,3,4,5,7\}$ the partition $\mu = (3k-4u-s, 2u+s, 2u)$ is associated to a non-zero eigenvalue of $M^{3,k}(M^{3,k})^T$.*

We are now in a position to prove for $k \geq 3$ that if S^μ appears in $FP(3,k)$ with the property that all other modules S^{μ^*} which appear in $FP(3,k)$ satisfy $\mu^* \rhd \mu$ then the eigenvalue associated to μ is non-zero. We prove this result below and remark that, as discussed earlier, if we could show that the eigenvalues of $M^{3,k}(M^{3,k})^T$ can be partially ordered in terms of size in the same way that the partitions associated to these eigenvalues are partially ordered then we would have an alternative method for proving Foulkes' conjecture for this case.

Theorem 6.11 *Let $k \geq 3$. If we partially order the partitions μ (see Definition 2.8) such that S^μ appears in $FP(3,k)$ then the eigenvalues associated to the 'smallest' partitions (in this partial ordering) are non-zero.*

Proof: When k is even we have shown that $S^{(k,k,k)}$ appears in $FP(3,k)$ with multiplicity one. It is easy to see using Theorem 2.22 that any other module S^{μ^*} which appears in $FP(3,k)$ satisfies $\mu^* \rhd \mu$. We know that $S^{(4,4,4)}$ appears in $FP(3,4)$ and the eigenvalue associated to $(4,4,4)$ is non-zero. Hence by applying Theorem 6.9 the appropriate number of times, the eigenvalue of $M^{3,k}(M^{3,k})^T$ associated to (k,k,k) is non-zero when k is even. When k is odd $S^{(k,k,k)}$ does not appear in $FP(3,k)$. It is easy to verify from the result at the beginning of Section 5.4.3 that $S^{(k+2,k-1,k-1)}$ and $S^{(k+1,k+1,k-2)}$ appear in $FP(3,k)$ both with multiplicity one. Moreover, $(k+2,k-1,k-1)$ and $(k+1,k+1,k-2)$ both dominate (k,k,k) and are incomparable. Any other partition which dominates (k,k,k) must dominate one of $(k+2,k-1,k-1)$ or $(k+1,k+1,k-2)$ (except for $(k+1,k,k-1)$ but $S^{(k+1,k,k-1)}$ does not appear in $FP(3,k)$ by Corollary 5.14). Thus, when k is odd we have two 'smallest' partitions in this partial ordering. We can write $(k+2,k-1,k-1)$ as $(k-1+3,k-1,k-1)$ and since the partition (3) appears in $FP(3,1)$ such that the eigenvalue associated to this partition is non-zero, we can apply Theorem 6.9 the appropriate number of times to show that the eigenvalue associated to $(k+2,k-1,k-1)$ is non-zero. Similarly we can write $(k+1,k+1,k-2)$ as $(k-3+4,k-3+4,k-3+1)$ and since $(4,4,1)$ appears in $FP(3,3)$ and the eigenvalue associated to $(4,4,1)$ is non-zero then by Theorem 6.9 the eigenvalue associated to $(k+1,k+1,k-2)$ is non-zero. □

For certain types of partitions we can calculate the eigenvalues associated to these partitions explicitly in neat closed form. The following theorem gives us these eigenvalues. We prove each case separately in the same long-winded fashion.

Theorem 6.12 *The following table lists some of the eigenvalues of the matrix MM^T (where M is the incidence matrix of $\mathcal{P}_{3,k}$) for $k \geq 3$ and the partitions associated to them.*

Partition	*Eigenvalue*
$(3k)$	$(k!)^2(3!)^{k-1}$
$(3k-2,2)$	$2k!(k-1)!(3!)^{k-2}$
$(3k-3,3)$	$2^4((k-1)!)^2(3!)^{k-3}$
$(3k-4,2,2)$	$2(k-1)!(k-2)!(k-2)(3!)^{k-3}$

Proof: From Lemma 5.11 it is easy to see that for $k \geq 3$ the partitions $(3k)$, $(3k-2,2)$, $(3k-3,3)$ and $(3k-4,2,2)$ are all simply associated to eigenvalues of $M^{3,k}(M^{3,k})^T$. For each of these partitions μ we can find a semistandard μ-tableau T of type (k^3) and a μ-tableau t (which is given in the proof of Proposition 5.10 or Theorem 3.8) such that $\bar{\Theta}_T\{t\}\kappa_t$ is an eigenvector of MM^T. Letting $v = \bar{\Theta}_T\{t\}\kappa_t$ we can find the eigenvalue λ corresponding to v using the method described at the end of Section 3.3. For a $(3,k)$-partition α involved in v, let e_α be the is the standard basis vector then

$$\lambda = \frac{(e_\alpha^T M)(M^T v)}{e_\alpha^T v}.$$

To find $(e_\alpha^T M)(M^T v)$ we multiply the β-entry of $e_\alpha^T M$ by the β-entry of $M^T v$ and sum over all β in $P(k,3)$. Since the β-entry of $e_\alpha^T M$ is one if α and β intersect nicely and zero otherwise, we only need consider those β which intersect nicely with α. Thus to find the eigenvalue λ we divide $(e_\alpha^T M)(M^T v)$ by the coefficient of α in v. We consider each of the partitions $(3k)$, $(3k-2,2)$, $(3k-3,3)$ and $(3k-4,2,2)$ in turn.

1. The proof of the first type of partition has been done in Proposition 3.17.

2. Let T and t be given by

$$T = \begin{matrix} 1\,1\,1\ldots1\,2\ldots2\,3\ldots3 \\ 2\,2 \end{matrix} \quad \text{and } t = \begin{matrix} 1\;3\;5\ldots3k \\ 2\;4 \end{matrix}.$$

Then $\bar{\Theta}_T\{t\}\kappa_t$ can be written in the form

$$\sum 4\begin{pmatrix} 13*\ldots* \\ 24*\ldots* \\ ***\ldots* \end{pmatrix} - 4\begin{pmatrix} 14*\ldots* \\ 23*\ldots* \\ ***\ldots* \end{pmatrix} + 2\begin{pmatrix} 1**\ldots* \\ 24*\ldots* \\ 3**\ldots* \end{pmatrix} - 2\begin{pmatrix} 2**\ldots* \\ 14*\ldots* \\ 3**\ldots* \end{pmatrix} + 2\begin{pmatrix} 2**\ldots* \\ 13*\ldots* \\ 4**\ldots* \end{pmatrix} - 2\begin{pmatrix} 1**\ldots* \\ 23*\ldots* \\ 4**\ldots* \end{pmatrix}$$

where the asterisks run over all possible values between 5 and $3k$. Let $v = \bar{\Theta}_T\{t\}\kappa_t$. Then to make it easier for reference, we will write

$$v = 4v_1 - 4v_2 + 2v_3 - 2v_4 + 2v_5 - 2v_6,$$

where

$$v_1 = \sum \begin{pmatrix} 13*\ldots* \\ 24*\ldots* \\ ***\ldots* \end{pmatrix}, \quad v_2 = \sum \begin{pmatrix} 14*\ldots* \\ 23*\ldots* \\ ***\ldots* \end{pmatrix} \quad \text{etc.}$$

Let α be the $(3,k)$-partition given by

$$\alpha = \begin{pmatrix} 1\ 3\ 7 \dots 3k-2 \\ 2\ 4\ 8 \dots 3k-1 \\ 5\ 6\ 9 \dots \quad 3k \end{pmatrix}.$$

It is easy to see that α is involved in v with coefficient $+4$. If β has 1 and 2 or 3 and 4 together in one of its parts then the β-entry of $M^T v$ will be zero due to the alternating signs of $(3,k)$-partitions involved in v. Similarly if 1, 2, 3 and 4 are all in different parts of β then the β-entry of $M^T v$ will be zero. Thus we only need to consider the $(k,3)$-partitions of the following kinds:

(a) Let β be of the form

$$\beta = \begin{pmatrix} 1 \\ 4 \\ *_1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ *_2 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \end{pmatrix} \cdots \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

where $*_1$ and $*_2$ are fixed elements of $\{5,6,\dots,3k\}$ and the remaining $*$'s are fixed with the property that each part of β has one element in common with each part of α. Then β intersects nicely with α and with $(3,k)$-partitions involved in v_1, v_3 and v_5. There are $(3!)^{k-2}$ different $(3,k)$-partitions involved in each of v_1, v_3 and v_5 which intersect nicely with β. Thus the β-entry of $M^T v$ is $8(3!)^{k-2}$.

(b) Let β be of the form

$$\beta = \begin{pmatrix} 1 \\ 4 \\ *_1 \end{pmatrix} \begin{pmatrix} 2 \\ *_2 \\ *_3 \end{pmatrix} \begin{pmatrix} 3 \\ *_4 \\ *_5 \end{pmatrix} \cdots \begin{pmatrix} * \\ * \\ * \end{pmatrix}.$$

Then β intersects nicely with $2^2(3!)^{k-3}$ of those $(3,k)$-partitions involved in v_1, v_3, v_5 and v_6. Thus the β-entry of $M^T v$ is $8 \cdot 2^2(3!)^{k-3} - 2 \cdot 2^2(3!)^{k-3} = 2^2(3!)^{k-2}$.

(c) Similarly, when β is of the form

$$\beta = \begin{pmatrix} 2 \\ 3 \\ *_1 \end{pmatrix} \begin{pmatrix} 1 \\ *_2 \\ *_3 \end{pmatrix} \begin{pmatrix} 4 \\ *_4 \\ *_5 \end{pmatrix} \cdots \begin{pmatrix} * \\ * \\ * \end{pmatrix}.$$

The β-entry of $M^T v$ will be $2^2(3!)^{k-2}$.

To compute $(e_\alpha^T M)(M^T v)$ we need to count the number of β of each kind there are which intersect nicely with α. When β is of the first kind, there are k choices for

$*_1$, there are $k-1$ choices for $*_2$ and there are $((k-2)!)^2$ choices for the remaining $*$'s. When β is of the second or third kinds there are k choices for $*_1$, $(k-1)$ choices for $*_2$, $(k-2)$ choices for $*_3$, $*_4$ and $*_5$ and $((k-3)!)^2$ choices for the remaining $*$'s. Therefore

$$\begin{aligned}(e_\alpha^T M)(M^T v) &= 8(3!)^{k-2}k(k-1)((k-2)!)^2) \\ &\quad +2^3(3!)^{k-2}k(k-1)(k-2)^3((k-3)!)^2 \\ &= 8(3!)^{k-2}k!(k-1)!.\end{aligned}$$

Dividing by the coefficient of α in v gives us the required eigenvalue of

$$2k!(k-1)!(3!)^{k-2}.$$

3. Let T and t be given by

$$T = \begin{matrix}1\,1\,1\,1 \ldots 1\,2 \ldots 2\,3 \ldots 3 \\ 2\,2\,2 \qquad\qquad\qquad\quad\end{matrix} \text{ and } t = \begin{matrix}1\,3\,5\,7 \ldots 3k \\ 2\,4\,6 \qquad\quad\end{matrix}.$$

Let $v = \bar{\Theta}_T\{t\}\kappa_t$. A $(3,k)$-partition α involved in v will have one element from each of the sets $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$ in one of its parts (call this part α_2) and of the remaining three digits two of them will be together in another part of α (call this α_3) and the last digit will be in α_1. The coefficient of α in v will be $+2$ if α_2 contains an odd number of the digits $\{2,4,6\}$ and -2 if it contains an even number of the digits $\{2,4,6\}$. Let α be the $(3,k)$-partition

$$\alpha = \begin{matrix}(1\;7\;8\;\ldots\;3k-2) \\ (2\;4\;6\;\ldots\;3k-1) \\ (3\;5\;9\;\ldots\;\quad 3k\;\;\,)\end{matrix}.$$

Then α has coefficient $+2$ in v. The $(k,3)$-partitions which have non-zero entry in $M^T v$ and intersect nicely with α must have three digits from $\{1,2,3,4,5,6\}$ together in one of its parts where exactly two of them are from $\{1,3,5\}$. Thus, of these $(k,3)$-partitions the ones which intersect nicely with α will be of the following kinds.

(a) Let β be of the form

$$\beta = \begin{pmatrix}1\\4\\5\end{pmatrix}\begin{pmatrix}2\\3\\*\end{pmatrix}\begin{pmatrix}6\\*\\*\end{pmatrix}\cdots\begin{pmatrix}*\\*\\*\end{pmatrix}.$$

In this case β intersects nicely with $4 \cdot 2(3!)^{k-3}$ different $(3,k)$-partitions involved in v, all of which have coefficient $+2$ in v. If β is as above but with 2 and 6 interchanged then the β-entry of $M^T v$ will also be $8 \cdot 2(3!)^{k-3}$. We have the same result when β is the $(k,3)$-partition with 1, 3 and 6 together in one part and either 2 and 5 or 4 and 5 together in another part.

(b) Let β be the $(k,3)$-partition of the form

$$\beta = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \begin{pmatrix} 2 \\ * \\ * \end{pmatrix} \begin{pmatrix} 3 \\ * \\ * \end{pmatrix} \begin{pmatrix} 6 \\ * \\ * \end{pmatrix} \cdots \begin{pmatrix} * \\ * \\ * \end{pmatrix}.$$

Then β intersects nicely with $6 \cdot 2^3(3!)^{k-4}$ of the $(3,k)$-partitions involved in v these all have coefficient $+2$ in v. In a similar way when β has 1, 6 and 3 in one part and 2, 5 and 4 in distinct parts, the β-entry of $M^T v$ is also $12 \cdot 2^3(3!)^{k-4}$.

There are $(k-1)(k-2)^2((k-3)!)^2$ different $(k,3)$-partitions of the first kind which intersect nicely with α and $(k-1)(k-2)^2(k-3)^3((k-4)!)^2$ of the second kind. Therefore

$$\begin{aligned}(e_\alpha^T M)(M^T v) &= 4 \cdot 8 \cdot 2(3!)^{k-3}(k-1)(k-2)^2((k-3)!)^2 \\ &\quad + 4 \cdot 6 \cdot 2^3(3!)^{k-4}(k-1)(k-2)^2(k-3)^3((k-4)!)^2 \\ &= (3!)^{k-3}2^5(k-1)!(k-2)!\{2+(k-3)\} \\ &= (3!)^{k-3}2^5(k-1)!^2.\end{aligned}$$

Since α has coefficient $+2$ in v, the eigenvalue associated to v is $(3!)^{k-3}2^4(k-1)!^2$.

4. Let T and t be given by

$$T = \begin{matrix} 1\,1\,1\ldots1\,2\ldots2\,3\ldots3 \\ 2\,2 \\ 3\,3 \end{matrix} \qquad \text{and } t = \begin{matrix} 1\,4\,7\ldots3k \\ 2\,5 \\ 3\,6 \end{matrix}.$$

Let $v = \bar{\Theta}_T\{t\}\kappa_t$. Using the proof of Theorem 3.8 we can write

$$v = \sum \sum_{g \in Sym(\{4,5,6\})} 6\, sgn(g) \begin{matrix} (1 & 4 & * & \ldots & *) \\ (2 & 5 & * & \ldots & *) \\ (3 & 6 & * & \ldots & *) \end{matrix}\, g$$

where the $*$'s run over the elements of $\{7, 8, \ldots, 3k\}$. Let α be the $(3, k)$-partition given by

$$\alpha = \begin{pmatrix} 1 & 4 & 7 & \ldots & 3k-2 \\ 2 & 5 & 8 & \ldots & 3k-1 \\ 3 & 6 & 9 & \ldots & 3k \end{pmatrix}.$$

Again we locate those $(k, 3)$-partitions β which intersect nicely with α and have non-zero entry in $M^T v$. It is easy to check that these $(k, 3)$-partitions can not have any two of $\{1, 2, 3\}$ or any two of $\{4, 5, 6\}$ together in one part or any two of them on their own (where by "on their own" we mean not with any other elements from $\{1, 2, 3, 4, 5, 6\}$) in distinct parts, for otherwise the β-entry of $M^T v$ will be zero. Thus the $(k, 3)$-partitions which we need to consider can be described in the following way.

(a) The first kind are those β which have the digits 1, 2 and 3 in distinct parts and we pair them off with the digits 4, 5 and 6 so that no two of the pairs $\{1, 4\}$, $\{2, 5\}$ and $\{3, 6\}$ are together in one part (otherwise we don't have a nice intersection with α). So, for example β may be of the form

$$\begin{pmatrix} 1 \\ 5 \\ * \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ * \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ * \end{pmatrix} \cdots \begin{pmatrix} * \\ * \\ * \end{pmatrix}.$$

Each β of this kind intersects nicely with $2(3!)^{k-3}$ of the $(3, k)$-partitions involved in v and these $(3, k)$-partitions always have coefficient $+6$ in v.

(b) The other case is when β has 1, 2, 3 in distinct parts and we just pair off two of them with digits from $\{4, 5, 6\}$. The two digits which we haven't 'paired off' may either be from the same part of α or from distinct parts. So for example β may be of the form

$$\beta = \begin{pmatrix} 1 \\ 5 \\ * \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ * \end{pmatrix} \begin{pmatrix} 3 \\ * \\ * \end{pmatrix} \begin{pmatrix} 6 \\ * \\ * \end{pmatrix} \cdots \begin{pmatrix} * \\ * \\ * \end{pmatrix} \quad (i)$$

or

$$\beta = \begin{pmatrix} 1 \\ 5 \\ * \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ * \end{pmatrix} \begin{pmatrix} 3 \\ * \\ * \end{pmatrix} \begin{pmatrix} 4 \\ * \\ * \end{pmatrix} \cdots \begin{pmatrix} * \\ * \\ * \end{pmatrix}. \quad (ii)$$

There are two other $(k, 3)$-partitions similar to (i), one with 5 and 6 interchanged and also 2 and 3 interchanged, the the other with 1 and 3 interchanged and also 4 and 6 interchanged. There are five other $(k, 3)$-partitions similar to (ii), one for each choice of digits in the third and fourth parts (discluding the ones that we constructed in the first case). The $(k,3)$-partition given in (i) intersects nicely with the following kinds of $(3,k)$-partition involved in v:

$$\begin{pmatrix} 1 & 4 & * & \dots & * \\ 2 & 5 & * & \dots & * \\ 3 & 6 & * & \dots & * \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & * & \dots & * \\ 2 & 6 & * & \dots & * \\ 3 & 5 & * & \dots & * \end{pmatrix}, \quad \begin{pmatrix} 1 & 6 & * & \dots & * \\ 2 & 5 & * & \dots & * \\ 3 & 4 & * & \dots & * \end{pmatrix}.$$

The first kind has coefficient $+6$ in v and the last two kinds have coefficient -6 in v. β intersects with $2^2(3!)^{k-4}$ different $(3,k)$-partitions of each kind. Thus, the β-entry of M^Tv is $-6 \cdot 2^2(3!)^{k-4}$. Similarly, when β is the $(k,3)$-partition given in (ii) the β-entry of M^Tv can be shown to be $+6 \cdot 2^2(3!)^{k-4}$.

Therefore

$$\begin{aligned}
(e_\alpha^T M)(M^T v) &= 6 \cdot 2^2(3!)^{k-3}(k-2)^3((k-3)!)^2 \\
&\quad +6(6-3)2^2(3!)^{k-4}(k-2)^3(k-3)^3((k-4)!)^2 \\
&= 3^2 \cdot 2^3(3!)^{k-4}((k-2)!)^2(k-2)\{2+(k-3)\} \\
&= 2(3!)^{k-2}(k-1)!(k-2)!(k-2).
\end{aligned}$$

Since α has coefficient $+6$ in v, the eigenvalue associated to v is $2(3!)^{k-3}(k-1)!(k-2)!(k-2)$.

This completes the proof. □

Appendix A

Eigenvalues of $M^{3,k}(M^{3,k})^T$ for Small k

In this section, we give the complete set of eigenvalues of the matrix $M^{3,k}(M^{3,k})^T$ for $3 \leq k \leq 8$. We list the eigenvalues in decreasing order of magnitude and for each eigenvalue λ we give the partition μ of $3k$ associated to it. We also give the dimension of the Specht module S^μ, from which we can work out the multiplicity of λ as an eigenvalue of $M^{3,k}(M^{3,k})^T$. The last column of the table gives the multiplicity of S^μ in the complete decomposition of $FP(3,k)$. After each table we give the diagrams for the partial ordering of the partitions of $3k$ and using the same diagram we show that when we replace the partition of $3k$ with the eigenvalue associated to it, the eigenvalues also satisfy the same partial ordering.

Partition	Eigenvalue	Dimension	Mult.
(12)	$124416 = 2^9 \cdot 3^5$	1	1
(10, 2)	$10368 = 2^7 \cdot 3^4$	54	1
(9, 3)	$3456 = 2^7 \cdot 3^3$	154	1
(8, 4)	$2304 = 2^8 \cdot 3^2$	275	1
(6^2)	$768 = 2^8 \cdot 3$	132	1
(7, 4, 1)	$576 = 2^6 \cdot 3^2$	1408	1
$(8, 2^2)$	$288 = 2^5 \cdot 3^2$	616	1
(6, 4, 2)	$128 = 2^7$	2673	1
(4^3)	$96 = 2^5 \cdot 3$	462	1
	Sum of dimensions	5775	

Table 1: Eigenvalues of $M^{3,4}(M^{3,4})^T$

Partition	Eigenvalue	Dimension	Mult.
(15)	$18662400 = 2^{10} \cdot 3^6 \cdot 5^2$	1	1
(13, 2)	$1244160 = 2^{10} \cdot 3^5 \cdot 5$	90	1
(12, 3)	$331776 = 2^{12} \cdot 3^4$	350	1
(11, 4)	$207360 = 2^9 \cdot 3^4 \cdot 5$	910	1
(10, 5)	$110592 = 2^{12} \cdot 3^3$	1638	1
(9, 6)	$41472 = 2^9 \cdot 3^4$	2002	1
(10, 4, 1)	$41472 = 2^9 \cdot 3^4$	7007	1
(11, 2, 2)	$31104 = 2^7 \cdot 3^5$	1925	1
(8, 6, 1)	$18432 = 2^{11} \cdot 3^2$	11583	1
(9, 4, 2)	$8832 = 2^7 \cdot 3 \cdot 23$	22113	1
(8, 5, 2)	$6912 = 2^8 \cdot 3^3$	32032	1
(7, 4, 4)	$3456 = 2^7 \cdot 3^3$	25025	1
$(6^2, 3)$	$2304 = 2^8 \cdot 3^2$	21450	1
	Sum of dimensions	126126	

Table 2: Eigenvalues of $M^{3,5}(M^{3,5})^T$

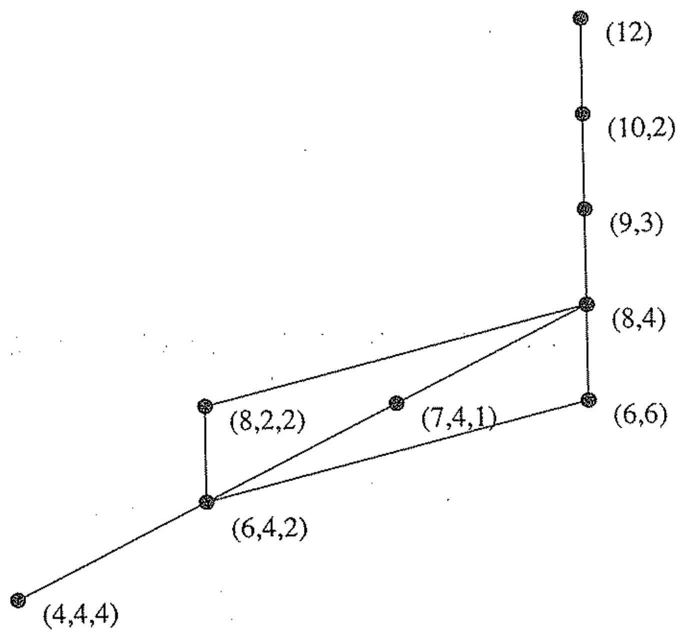

Figure 5: Partial Ordering of the Partitions of 12

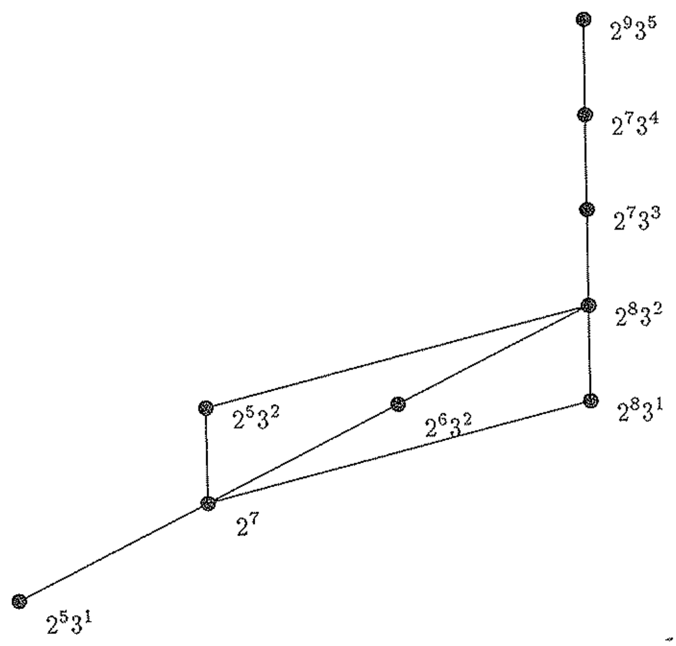

Figure 6: Partial Ordering of the Eigenvalues of $M^{3,4}(M^{3,4})^T$

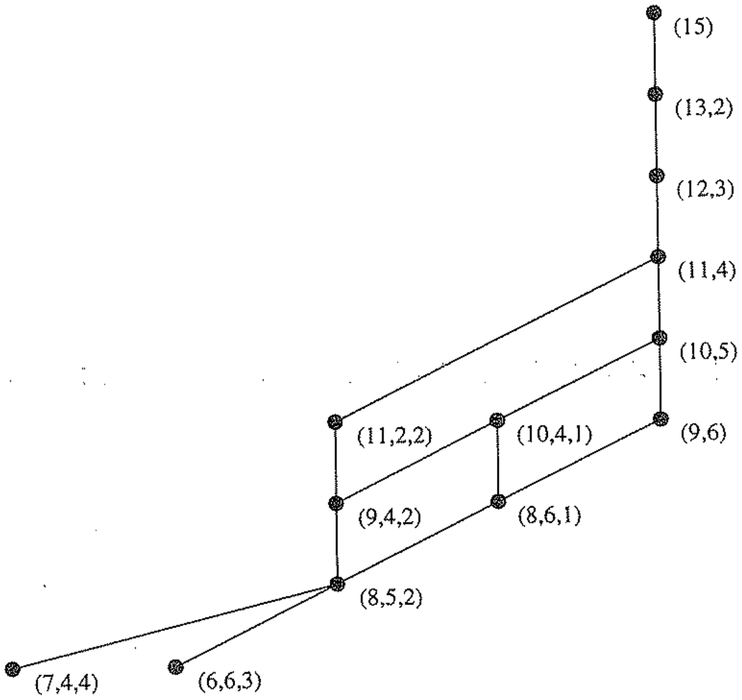

Figure 7: Partial Ordering of the Partitions of 15

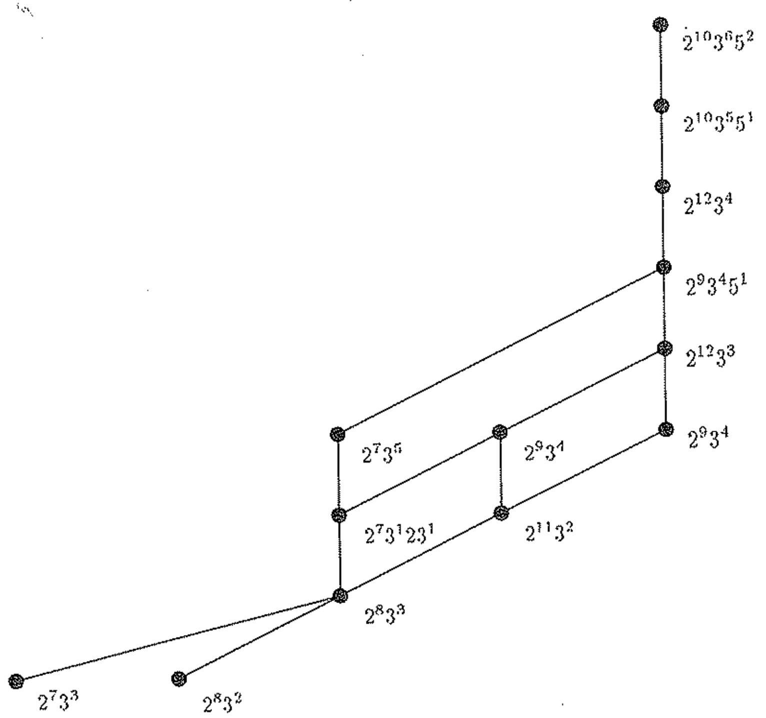

Figure 8: Partial Ordering of the Eigenvalues of $M^{3,5}(M^{3,5})^T$

Partition	Eigenvalue	Dimension	Mult.
(18)	$4031078400 = 2^{13} \cdot 3^9 \cdot 5^2$	1	1
$(16,2)$	$223948800 = 2^{12} \cdot 3^7 \cdot 5^2$	135	1
$(15,3)$	$49766400 = 2^{13} \cdot 3^5 \cdot 5^2$	663	1
$(14,4)$	$29859840 = 2^{13} \cdot 3^6 \cdot 5$	2244	1
$(13,5)$	$13271040 = 2^{15} \cdot 3^4 \cdot 5$	5508	1
$(12,6)$	$2^{10} \cdot 3^3(13 \cdot 17 \pm \sqrt{5641})$ *	9996	2
$(14,2,2)$	$4976640 = 2^{12} \cdot 3^5 \cdot 5$	4641	1
$(13,4,1)$	$4976640 = 2^{12} \cdot 3^5 \cdot 5$	22848	1
$(10,8)$	$1990656 = 2^{13} \cdot 3^5$	11934	1
$(11,6,1)$	$1658880 = 2^{12} \cdot 3^4 \cdot 5$	88128	1
$(10,7,1)$	$1105920 = 2^{13} \cdot 3^3 \cdot 5$	102102	1
$(12,4,2)$	$1022976 = 2^{10} \cdot 3^3 \cdot 37$	99144	1
$(11,5,2)$	$691200 = 2^{10} \cdot 3^3 \cdot 5^2$	219912	1
$(10,6,2)$	$442368 = 2^{14} \cdot 3^3$	331500	1
$(8,8,2)$	$331776 = 2^{12} \cdot 3^4$	136136	1
$(10,4,4)$	$262656 = 2^9 \cdot 3^3 \cdot 19$	259896	1
$(9,6,3)$	$172800 = 2^8 \cdot 3^3 \cdot 5^2$	678912	1
$(8,6,4)$	$124416 = 2^9 \cdot 3^5$	787644	1
$(6,6,6)$	$46080 = 2^{10} \cdot 3^2 \cdot 5$	87516	1
	Sum of dimensions	2858856	

* Sum of eigenvalues associated to $(12,6)$ is $12220416 = 2^{11} \cdot 3^3 \cdot 13 \cdot 17$.

Table 3: Eigenvalues of $M^{3,6}(M^{3,6})^T$

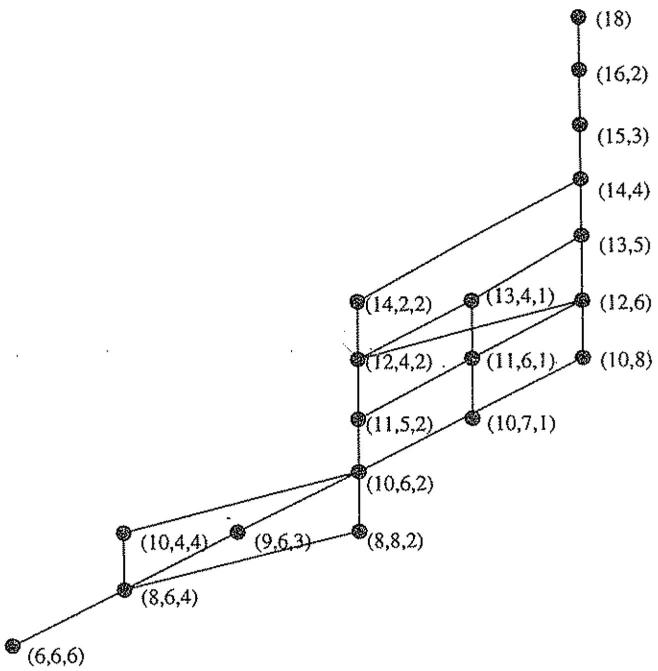

Figure 9: Partial Ordering of the Partitions of 18

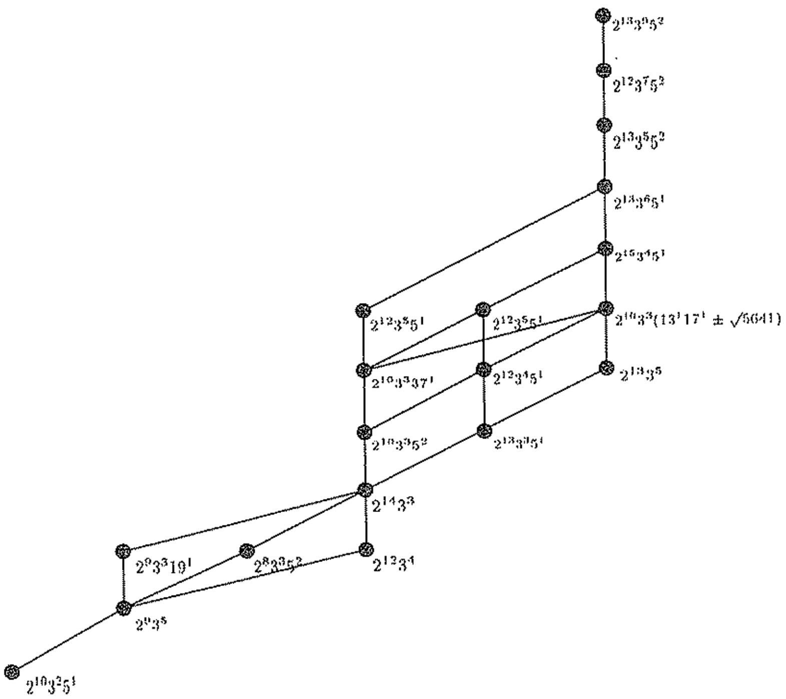

Figure 10: Partial Ordering of the Eigenvalues of $M^{3,6}(M^{3,6})^T$

Partition	Eigenvalue	Dimension	Mult.
(21)	$1185137049600 = 2^{14} \cdot 3^{10} \cdot 5^2 \cdot 7^2$	1	1
(19, 2)	$56435097600 = 2^{14} \cdot 3^9 \cdot 5^2 \cdot 7$	189	1
(18, 3)	$10749542400 = 2^{16} \cdot 3^8 \cdot 5^2$	1120	1
(17, 4)	$6270566400 = 2^{14} \cdot 3^7 \cdot 5^2 \cdot 7$	4655	1
(16, 5)	$2388787200 = 2^{17} \cdot 3^6 \cdot 5^2$	14364	1
(15, 6)	$2^{13} \cdot 3^5 \cdot 5(3^2 \cdot 11 \pm \sqrt{1401})$ *	33915	2
(17, 2, 2)	$1119744000 = 2^{12} \cdot 3^7 \cdot 5^3$	9520	1
(16, 4, 1)	$895795200 = 2^{14} \cdot 3^7 \cdot 5^2$	58786	1
(14, 7)	$796262400 = 2^{17} \cdot 3^5 \cdot 5^2$	62016	1
(13, 8)	$238878720 = 2^{16} \cdot 3^6 \cdot 5$	87210	1
(14, 6, 1)	$238878720 = 2^{16} \cdot 3^6 \cdot 5$	392445	1
(15, 4, 2)	$179159040 = 2^{14} \cdot 3^7 \cdot 5$	323190	1
(12, 9)	$159252480 = 2^{17} \cdot 3^5 \cdot 5$	90440	1
(13, 7, 1)	$132710400 = 2^{16} \cdot 3^4 \cdot 5^2$	664734	1
(14, 5, 2)	$106168320 = 2^{18} \cdot 3^4 \cdot 5$	949620	1
(12, 8, 1)	$99532800 = 2^{14} \cdot 3^5 \cdot 5^2$	839800	1
(10, 10, 1)	$59719680 = 2^{14} \cdot 3^6 \cdot 5$	293930	1
(13, 6, 2)	$58890240 = 2^{11} \cdot 3^4 \cdot 5 \cdot 71$	2015520	1
(12, 7, 2)	$40697856 = 2^{16} \cdot 3^3 \cdot 23$	3139560	1
(13, 4, 4)	$33592320 = 2^{10} \cdot 3^8 \cdot 5$	1492260	1
(11, 8, 2)	$31850496 = 2^{17} \cdot 3^5$	3481940	1
(12, 6, 3)	$21067776 = 2^{11} \cdot 3^4 \cdot 127$	5969040	1
(11, 6, 4)	$12358656 = 2^{10} \cdot 3^4 \cdot 149$	10988460	1
(10, 8, 3)	$11943936 = 2^{14} \cdot 3^6$	7936110	1
(10, 7, 4)	$8626176 = 2^{13} \cdot 3^4 \cdot 13$	14108640	1
(8, 8, 5)	$4976640 = 2^{12} \cdot 3^5 \cdot 5$	6466460	1
(9, 6, 6)	$3939840 = 2^9 \cdot 3^4 \cdot 5 \cdot 19$	7054320	1
	Sum of dimensions	66512160	

* Sum of eigenvalues associated to (15, 6) is $1970749440 = 2^{14} \cdot 3^7 \cdot 5 \cdot 11$.

Table 4: Eigenvalues of $M^{3,7}(M^{3,7})^T$

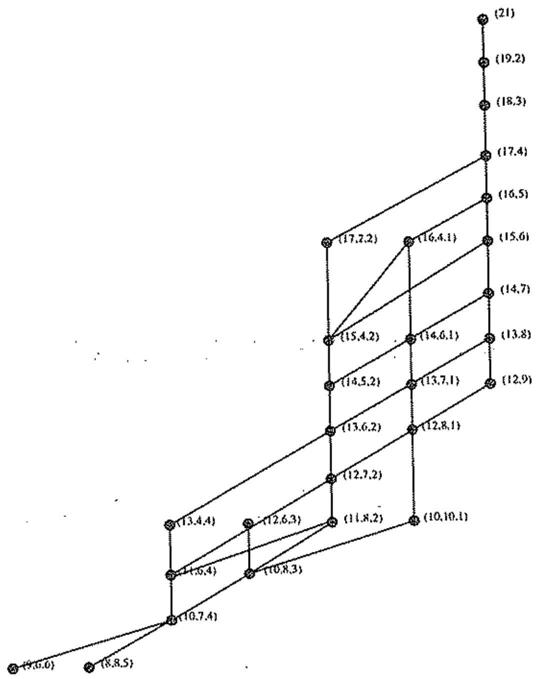

Figure 11: Partial Ordering of the Partitions of 21

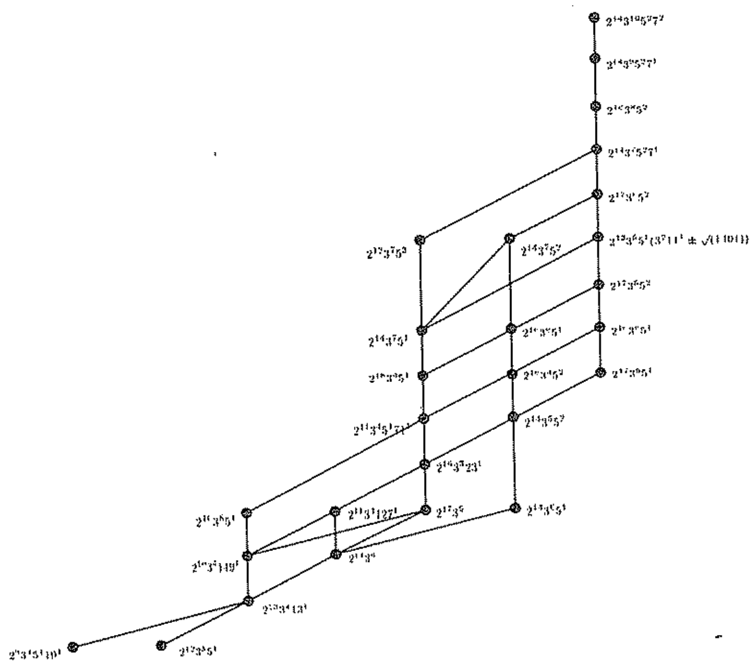

Figure 12: Partial Ordering of the Eigenvalues of $M^{3,7}(M^{3,7})^T$

Partition	Eigenvalue	Dimension	Mult.
(24)	$455092627046400 = 2^{21} \cdot 3^{11} \cdot 5^2 \cdot 7^2$	1	1
(22, 2)	$18962192793600 = 2^{18} \cdot 3^{10} \cdot 5^2 \cdot 7^2$	252	1
(21, 3)	$3160365465600 = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7^2$	1748	1
(20, 4)	$1805923123200 = 2^{19} \cdot 3^9 \cdot 5^2 \cdot 7$	8602	1
(19, 5)	$601974374400 = 2^{19} \cdot 3^8 \cdot 5^2 \cdot 7$	31878	1
(18, 6)	$2^{16} \cdot 3^6 \cdot 5^2(191 \pm 79)$ *	92092	2
(20, 2, 2)	$338610585600 = 2^{15} \cdot 3^{10} \cdot 5^2 \cdot 7$	17480	1
(19, 4, 1)	$225740390400 = 2^{16} \cdot 3^9 \cdot 5^2 \cdot 7$	129536	1
(17, 7)	$167215104000 = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7$	211508	1
(16, 8)	$2^{17} \cdot 3^6 \cdot 5(3^2 \cdot 17 \pm \sqrt{(13 \cdot 293)})$ **	389367	2
(17, 6, 1)	$50164531200 = 2^{17} \cdot 3^7 \cdot 5^2 \cdot 7$	1311552	1
(18, 4, 2)	$44192563200 = 2^{16} \cdot 3^6 \cdot 5^2 \cdot 37$	860706	1
(15, 9)	$23887872000 = 2^{18} \cdot 3^6 \cdot 5^3$	572033	1
(16, 7, 1)	$23887872000 = 2^{18} \cdot 3^6 \cdot 5^3$	2860165	1
(17, 5, 2)	$23290675200 = 2^{15} \cdot 3^7 \cdot 5^2 \cdot 13$	3131128	1
(14, 10)	$19110297600 = 2^{20} \cdot 3^6 \cdot 5^2$	653752	1
(15, 8, 1)	$16721510400 = 2^{17} \cdot 3^6 \cdot 5^2 \cdot 7$	4922368	1
(16, 6, 2)	$11585617920 = 2^{15} \cdot 3^6 \cdot 5 \cdot 97$	8460320	1
(12, 12)	$11466178560 = 2^{20} \cdot 3^7 \cdot 5$	208012	1
(14, 9, 1)	$11147673600 = 2^{18} \cdot 3^5 \cdot 5^2 \cdot 7$	6619239	1
(15, 7, 2)	$7184056320 = 2^{18} \cdot 3^3 \cdot 5 \cdot 7 \cdot 29$	17521515	1
(13, 10, 1)	$7166361600 = 2^{17} \cdot 3^7 \cdot 5^2$	6656384	1
(16, 4, 4)	$6449725440 = 2^{16} \cdot 3^9 \cdot 5$	6124118	1
(14, 8, 2)	$4777574400 = 2^{18} \cdot 3^6 \cdot 5^2$	28029617	2
(15, 6, 3)	$3822059520 = 2^{20} \cdot 3^6 \cdot 5$	32303040	1
(12, 10, 2)	$2853273600 = 2^{15} \cdot 3^4 \cdot 5^2 \cdot 43$	28883952	1
(14, 6, 4)	$1924300800 = 2^{15} \cdot 3^4 \cdot 5^2 \cdot 29$	79430868	1
(13, 8, 3)	$1672151040 = 2^{16} \cdot 3^6 \cdot 5 \cdot 7$	94140288	1
(12, 9, 3)	$1313832960 = 2^{15} \cdot 3^6 \cdot 5 \cdot 11$	100677808	1
(13, 7, 4)	$1174487040 = 2^{14} \cdot 3^5 \cdot 5 \cdot 59$	151016712	1
(12, 8, 4)	$883851264 = 2^{15} \cdot 3^6 \cdot 37$	204297500	1
(10, 10, 4)	$597196800 = 2^{15} \cdot 3^6 \cdot 5^2$	75716368	1
(12, 6, 6)	$510105600 = 2^{11} \cdot 3^5 \cdot 5^2 \cdot 41$	109830336	1
(11, 8, 5)	$507617280 = 2^{13} \cdot 3^6 \cdot 5 \cdot 17$	292880896	1
(10, 8, 6)	$298598400 = 2^{14} \cdot 3^6 \cdot 5^2$	267711444	1
(8, 8, 8)	$209018880 = 2^{13} \cdot 3^6 \cdot 5 \cdot 7$	23371634	1
	Sum of dimensions	1520190267	

* Sum of eigenvalues associated to (18, 6) is $456258355200 = 2^{17} \cdot 3^6 \cdot 5^2 \cdot 191$.
** Sum of eigenvalues associated to (16, 8) is $146193776640 = 2^{18} \cdot 3^8 \cdot 5 \cdot 17$.

Table 5: Eigenvalues of $M^{3,8}(M^{3,8})^T$

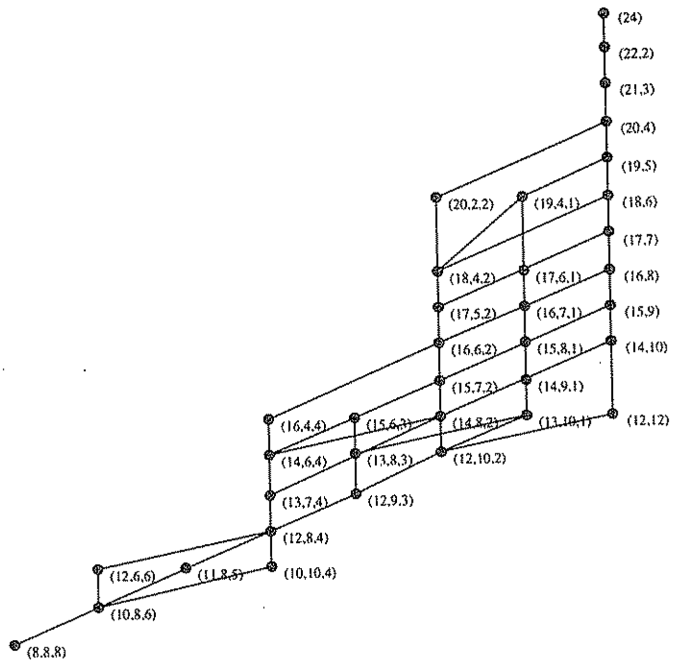

Figure 13: Partial Ordering of the Partitions of 24

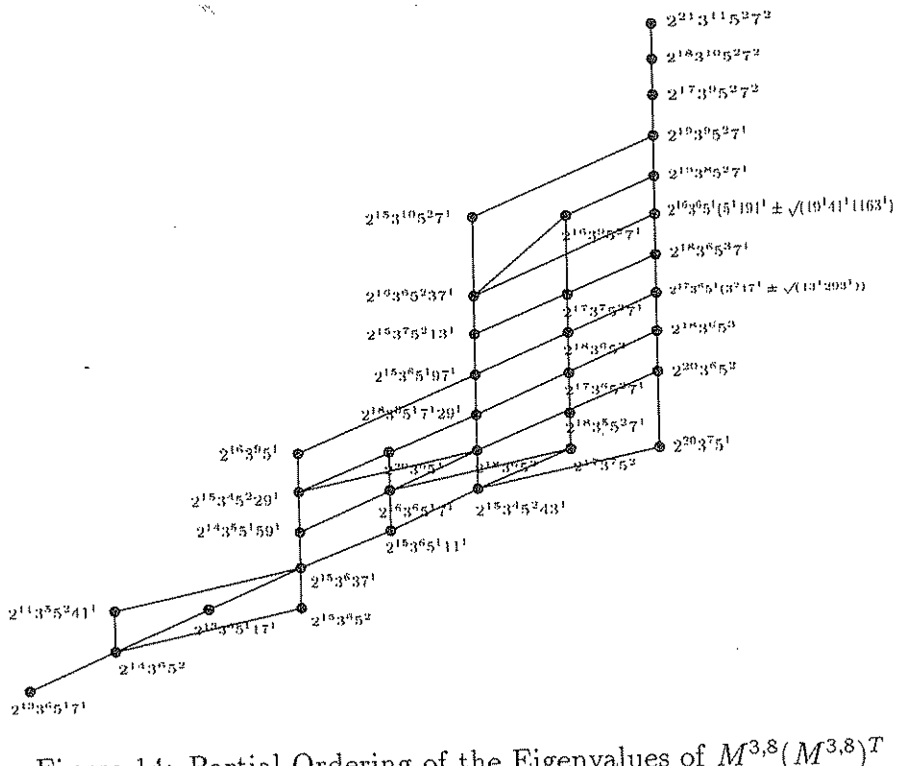

Figure 14: Partial Ordering of the Eigenvalues of $M^{3,8}(M^{3,8})^T$

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