Homology Modules in Projective Space

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Abstract

For a Boolean algebra one may consider the set of subsets of size k to be the basis for a permutation module over a field which has prime characteristic p. By considering an *inclusion map* one can obtain sequences of maps on these modules. It turns out that these maps and sequences have certain homological properties. This theory goes back to the topologist W.Mayer in the 1940's.

One can extend this idea to projective space by taking the set of k-dimensional subspaces over a field of size q to be the basis for such permutation modules where q is a prime power not dividing p. It turns out that such sequences are also homological in a certain way.

The homology modules in this case are modules over the finite general linear groups $GL(n,q)$. We prove a decomposition formula for these homology modules in terms of $GL(n-1,q)$-modules. Such a decomposition formula is known for the case of the Boolean algebra from S.Bell, P.Jones, and J.Siemons (*J. Algebra* 199, 1996, 556-580) and matches the result we prove here when we put $q = 1$.

We prove some interesting results as direct consequences of this decomposition formula. These include showing that the homological sequence discussed above is almost exact, proving a rank formula for certain incidence matrices and giving a condition which ensures that the homology module is irreducible. We also look at representations associated to these homology modules and give an explicit description for an irreducible representation of $PGL(2,q)$.

Contents

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1 Introduction

We begin by giving some general background to the questions we will be addressing in this thesis. Our starting point is to consider modular homology in terms of the Boolean algebra and this leads naturally to consider the same issues in terms of projective space.

1.1 Modular Homology in the Boolean Algebra

Modular homology appears to first be mentioned in two papers [2] by Mayer in 1942. More recently this theory has been the subject of papers such as those by Kapranov [10] and Dubois-Violette [6]. In [14, 17] Mnukhin, Siemons, Bell and Jones considered the idea of modular homology in the Boolean algebra. This concept can be briefly described in the following way. For a finite set Ω of size n, we consider the subsets of Ω. The lattice of these subsets is the Boolean algebra. We let R be a ring of prime characteristic p. Now if we take a fixed integer $0 \leq k \leq n$ then we may consider the set of k-element subsets to be the basis elements for an R-module M_k^n.

The *inclusion map* ∂ is the linear map which maps a k-element subset Δ to the sum of all the $(k-1)$-element subsets of Δ. This leads to a chain of inclusion maps

$$\mathcal{M} : 0 \xleftarrow{\partial} M_0^n \xleftarrow{\partial} M_1^n \xleftarrow{\partial} M_2^n \xleftarrow{\partial} M_3^n \xleftarrow{\partial} M_4^n \xleftarrow{\partial} \cdots .$$

If we consider a k-element subset Δ and apply the inclusion map p times then we get each of the $(k-p)$-element subsets of Δ occurring $p!$ many times. Since R has characteristic p it is therefore apparent that ∂^p is the zero map. One can then consider homological properties of such a sequence $\mathcal{M}$. For fixed integers k^* and $0 < i^* < p$ we consider the subsequence

$$\mathcal{M}_{k^*,i^*} : 0 \leftarrow M^n_{k^*} \leftarrow M^n_{k^*+i^*} \leftarrow M^n_{k^*+p} \leftarrow M^n_{k^*+i^*+p} \leftarrow M^n_{k^*+2p} \leftarrow \cdots$$

where each arrow represents the relevant power of ∂. Since ∂^p is the zero map, then whatever our starting point in this sequence, if we "take two steps along this sequence" we get zero. So the sequence $\mathcal{M}_{k^*,i^*}$ is homological. Thus for any three consecutive terms

$$M^n_{k-i} \xleftarrow{\partial^i} M^n_k \xleftarrow{\partial^{p-i}} M^n_{k+p-i}$$

in this sequence we can define a corresponding *homology module* $H^n_{k,i}$.

Many interesting results for these homology modules were obtained in [17]:

Theorem 1.1. *[17, Thm 3.2]*
$H^n_{k,i} \neq 0$ if and only if $n < 2k + p - i < n + p$.

Theorem 1.2. *[17, Thm 6.2]*
Let $\alpha \in \Omega$, then $H^n_{k,i} \cong H^{n-1}_{k,i+1} \oplus H^{n-1}_{k-1,i-1}$ as Sym$(\Omega \setminus \alpha)$-modules.

Theorem 1.3. *[17, Thm 6.4]*
If $2k - i = n - 1$ then $H^n_{k,i}$ is irreducible.

It is also noted in [17] that for the case when $p = 5$ the dimensions of the homology modules $H^n_{k,i}$ are Fibonacci numbers and the corresponding modules are precisely the Fibonacci representations of the symmetric group described in Ryba's paper [12].

1.2 Modular Homology in Projective Space

The ideas discussed above were extended in the paper [13] of Mnukhin and Siemons to consider modular homology in projective space. We think of projective space as follows. Take V to be a vector space of dimension n over the field $GF(q)$ where q is a prime power. Then we can consider the lattice of all subspaces of this vector space. This is the projective space of dimension $n-1$ over the field $GF(q)$ which we shall denote by $PG(n-1,q)$. Note that there is a change in dimensions - a d-dimensional projective space is a $(d+1)$-dimensional vector space. For a more detailed discussion about projective spaces see for example the notes of Cameron [4].

In the same way as for the Boolean algebra we can introduce some natural permutation modules for projective spaces. Consider the projective space $PG(n-1,q)$ and let F be a field of characteristic p where p is a prime which does not divide q. For a given number $0 \leq k \leq n$ we take the collection of all k-dimensional subspaces of V as a basis for a new vector space over F. We denote this space by M_k^n and we may consider it to be a module over the ring $FGL(n,q)$. It is clear that such a module M_k^n is a permutation module.

In this context the *inclusion map* is the homomorphism ∂ which maps any k-dimensional subspace to the sum of all the $(k-1)$-dimensional subspaces contained in it. This is a linear map $\partial : M_k^n \to M_{k-1}^n$ and thus we may obtain a sequence of the permutation modules

$$\mathcal{M} : 0 \stackrel{\partial}{\longleftarrow} M_0^n \stackrel{\partial}{\longleftarrow} M_1^n \stackrel{\partial}{\longleftarrow} \cdots \stackrel{\partial}{\longleftarrow} M_{k-1}^n \stackrel{\partial}{\longleftarrow} M_k^n \stackrel{\partial}{\longleftarrow} \cdots .$$

The inclusion map ∂ is said to be nilpotent if $\partial^\pi = 0$ for some $\pi > 1$. Recall that for the Boolean algebra we have that ∂^p is the zero map. It therefore makes sense to look for a least such π which gives the zero map for any n, however large (of course $\pi > n$ always works). For fixed p and q we are able to define $\pi(p, q)$ in such a way so that it is the least power of ∂ which always gives zero. When $\pi = 2$ we have that the above sequence $\mathcal{M}$ is a homological sequence in the usual sense. When $\pi > 2$ the classical homology theory does not apply. However, Mnukhin and Siemons [13] introduced the notion of a sequence being π-*homological* and this allows us to find a sequence about which we can consider homological properties. Thus, in the same way as for the Boolean algebra, we are able to make our definition of the homology module. We denote this module by $H^n_{k,i}$ and it is dependent on n, k and i, the number of times we apply our map ∂. In Chapter 2 we give formal definitions of some of the notions discussed above.

We aim to obtain analogous results for homology modules in projective space to those proven for the Boolean algebra. In particular we wish to find answers to two main questions:

- Can we find a homology module decomposition for $H^n_{k,i}$ in terms of $FGL(n-1, q)$-modules?
- Is there a condition which ensures that the homology module $H^n_{k,i}$ is irreducible as an $FGL(n, q)$-module?

We will show that we can provide positive answers to these two questions as long as we make an assumption about our field F. In [9] James gives a branching rule which is valid for any $FGL(n, q)$-module where F is a field of prime characteristic p with p not dividing q and where F is large enough to allow for the existence of q distinct linear characters $(F, +) \to GF(q)^{\times}$. If we make the assumption that our field F satisfies this condition by adjoining a primitive root of unity then we are able to use this branching rule of James to assist with finding a decomposition for $H^n_{k,i}$. This process is discussed in Chapter 5 and we are able to prove an $FGL(n-1, q)$-module decomposition of $H^n_{k,i}$ for such a field F. We find that if we "put $q = 1$" into this result then this matches the decomposition result already known for the Boolean algebra, as stated in Theorem 1.2.

This correspondence between homology in the Boolean algebra and in projective space is connected to the well known relationship between the representation theory of the symmetric and general linear groups. It has been observed that there is a close connection between representations of $FGL(n, q)$ and representations of the symmetric group $\mathcal{G}_n$ over the same field F. In the case of the Boolean algebra the module $H^n_{k,i}$ is a module over the symmetric group on n elements, while in projective space the corresponding homology module is a module over the finite general linear group $GL(n, q)$. In [9] it is noted that representation theory of the symmetric group $\mathcal{G}_n$ (see [11]) appears in some way to correspond to the representations of finite general linear groups $GL_n(q)$ if we "put $q = 1$". Many of the results for the case of the symmetric groups remain true if formulated for $GL(n, q)$ with q set equal to one. However it should also be emphasized that the proofs usually do not hold and this is also the case for the results which we will give here.

One other question which arises is: What do these modules tell us about representations of general linear groups? Each irreducible homology module has an irreducible representation associated to it. In Chapter 7 we consider the low-dimensional case when $n = 2$. Here it is relatively easy to construct the corresponding homology modules explicitly. Each such homology module gives a representation of $PGL(2, q)$ and this allows us to describe explicitly an irreducible representation of $PGL(2, q)$.

1.3 Summary of Main Results

Our main result is the homology module decomposition as discussed above. This decomposition formula is given in Theorem 5.11. In Chapter 6 we use this decomposition theorem to prove several other results about the homology module. In Theorem 6.2 we give a new and more direct proof of a result from [13] that our sequence of permutation modules is almost exact. Similarly we use the homology module decomposition to give a new proof for the rank formula of certain incidence matrices given in Theorem 6.3. We also prove a different and new expression for the rank formula in 6.4 and we illustrate the connection between these two formulae.

In Section 6.4 we obtain a result which gives an answer to our second main question which asked about the possible irreducibility of $H^n_{k,i}$. We prove in Theorem 6.8 that the same condition which ensures that $H^n_{k,i}$ is irreducible in the Boolean algebra (Theorem 1.3) also ensures irreducibility of the homology module as an $FGL(n, q)$-module in projective space. Each such homology module gives a representation of $PGL(2, q)$ and thus in Theorem 7.1 we give an explicit form for an irreducible representation $PGL(2,q) \rightarrow GL(q,p)$, subject to the condition that p does not divide $q+1$.

The structure of the forthcoming chapters is as follows. In Chapter 2 we give some formal definitions and we also state the result [13, Thm 3.1] of Mnukhin and Siemons which tells us that the homological sequence we are working with is almost exact. We will assume this result of almost exactness in Chapter 3 where we consider the dimensions of the homology modules.

From Chapter 4 onwards our work will not depend on the result that the sequence is almost exact. Indeed as we have stated above we will give an independent proof of almost exactness in Theorem 6.2. However, in Chapter 4 and in subsequent chapters we will need the assumption that our field F contains at least one non-trivial $p'th$ root of unity where q is a power of the prime p'. In Theorem 4.20 we prove a decomposition of the permutation module M_k^n for this larger field which is a specific case of a result of James. We then go on in Chapter 5 to discuss the homology module decomposition before looking in Chapter 6 at the results noted above which are consequences of this decomposition.

The result of Theorem 7.1 which gives a representation of $PGL(2,q)$ is independent of any of our earlier results and in fact requires no assumptions about p and q, so we also allow for the possibility that $p = q$. It is only in Section 7.3 when we relate this representation to the irreducible homology modules that we again require the assumption that our field contains an extra root of unity. We conclude in Chapter 8 with three different examples which are useful in illustrating the theory of the earlier chapters.

2 Modular Homology

We begin by formally setting up the construction of the homology modules we will be working with. As discussed in the introduction we will look at sequences of permutation modules over some field F from which we can construct a homological sequence. From this homological sequence we may give a precise definition of what we mean by a 'homology module'. In the second part of this chapter we give an account of some of the results already known for homology modules in projective space. These results come from the paper [13] of Mnukhin and Siemons.

2.1 A Homological Sequence

First of all we need to give precise definitions of some of the notions we have described in the introduction. We then describe a sequence of $GL(n, q)$-modules which contains a subsequence with homological properties.

Let $\mathbb{F}$ be the field $GF(q)$ and take V to be the n-dimensional vector space over $\mathbb{F}$. Then we consider the projective space $PG(n-1, q)$ to be the lattice of subspaces of V. For a fixed integer $0 \leq k \leq n$ we let L_k^n denote the collection of k-dimensional subspaces of V. Now let F be a finite field of characteristic p where p is a prime which does not divide q. We denote by M_k^n the vector space over F with basis L_k^n, so $M_k^n = FL_k^n$. We also define the following homomorphism which will be of great importance to us.

Definition 2.1. The linear map

$$\partial : M_k^n \to M_{k-1}^n$$

is defined on the basis L_k^n by

$$\partial(x) := \sum y$$

where the sum runs over all subspaces of co-dimension 1 in $x \in L_k^n$. We call this map the *inclusion* map.

So this homomorphism takes a k-dimensional subspace x and maps it to the sum of all the $(k-1)$-dimensional subspaces contained in x and this sum is regarded as an element of M_{k-1}^n. From this map we obtain the sequence

$$\mathcal{M} : 0 \xleftarrow{\partial} M_0^n \xleftarrow{\partial} M_1^n \xleftarrow{\partial} \cdots \xleftarrow{\partial} M_{k-1}^n \xleftarrow{\partial} M_k^n \xleftarrow{\partial} \cdots .$$

Remark 2.2. We shall use the convention of writing all of our sequences in this 'backwards' way $(0 \longleftarrow M_0^n \leftarrow \cdots)$ as a matter of convenience since it is usually the beginning of the sequence which is of interest to us. This convention will be adopted throughout the remaining chapters.

Now suppose that $\pi > 1$ is some fixed arbitrary integer and select positive integers i^*, k^* satisfying $k^* + i^* < \pi$. We consider the new sequence

$$\mathcal{M}_{k^*,i^*} : 0 \overset{\partial^*}{\leftarrow} M_{k^*} \overset{\partial^*}{\leftarrow} M_{k^*+i^*} \overset{\partial^*}{\leftarrow} M_{k^*+\pi} \overset{\partial^*}{\leftarrow} M_{k^*+i^*+\pi} \overset{\partial^*}{\leftarrow} M_{k^*+2\pi} \overset{\partial^*}{\leftarrow} \cdots$$

where ∂^* is the appropriate power of ∂. This is the sequence we shall use to construct our homology modules.

We now give the definitions of homological and (almost) exact sequences.

Definition 2.3. If $\rho : A \to B$ and $\sigma : B \to C$ are homomorphisms then the sequence

$$C \leftarrow B \leftarrow A$$

is *homological* at B if $\ker(\sigma) \supseteq \rho(A)$ and it is *exact* if $\ker(\sigma) = \rho(A)$. A sequence

$$\cdots \leftarrow A_{k-1} \leftarrow A_k \leftarrow A_{k+1} \leftarrow \cdots$$

is homological (resp. exact) if it has that property at every A_i. The sequence is said to be *almost exact* if at most one of the A_i is not exact.

This definition of a homological sequence is extended in [13] to define the notion of when a sequence such as $\mathcal{M}$ described above is called π-homological.

Definition 2.4. Let $\pi > 1$ be a fixed integer. If $\mathcal{M}_{k^*,i^*}$ is homological for every choice of $i^* < \pi$ and k^* with $k^* + i^* < \pi$ then $\partial^\pi = 0$ and $\mathcal{M}$ is said to be *π-homological.* We call a π-homological sequence $\mathcal{M}$ *almost π-exact* if each $\mathcal{M}_{k^*,i^*}$ is almost exact.

If $\partial^\pi = 0$ then at any point in the sequence $\mathcal{M}_{k^*,i^*}$, if we take two steps along, we always get zero. Thus for every three consecutive modules in the sequence we have that the image of the map from the first module to the second module must be contained in the kernel of the map from the second module to the third. Thus the sequence $\mathcal{M}_{k^*,i^*}$ is homological.

The following combinatorial functions will be useful to us. We define

$$[i]_q := 1 + q + q^2 + \cdots + q^{i-1}$$

for any non-negative integer i. Notice that $[0]_q = 0$, $[1]_q = 1$ and $[m]_q = \frac{q^m-1}{q-1}$ for $m \geq 2$. Using this we may define the *q-factorial function* by

$$(i!)_q := [1]_q[2]_q \cdots [i]_q.$$

Definition 2.5. For co-prime integers p and q we define $\pi(p,q) > 0$ to be the least integer π for which $[\pi]_q \equiv 0 \mod p$.

We see that $\pi \leq p$ and $\pi = p$ if $q \equiv 1 \mod p$. If p does not divide $q-1$ then π is the order of q modulo p because $1 + q + q^2 + \cdots + q^{\pi-1} = \frac{q^\pi-1}{q-1}$. So $q^\pi \equiv 1 \mod p$ for all cases.

We consider again the inclusion map ∂. For $x \in L_k^n$ and $i \leq k$, if we apply this map i many times then we have

$$\partial^i(x) = c \sum y$$

where the sum runs over all $y \in L_{k-i}^n$ with $y \subset x$. The number c is the number of saturated chains $y = y_0 \subset y_1 \subset \cdots \subset y_i = x$. To calculate this number we take a $(k-i)$-dimensional subspace y which is contained in the k-dimensional subspace x. We are looking for saturated chains and so y_1 must be a $(k-i+1)$-dimensional subspace which contains y and is contained in x. So we must add one vector to the basis for y, and this must be a vector from x. We have $k-(k-i) = i$ many basis vectors of x which are not already in y, and hence we have $\frac{q^i-1}{q-1} = 1 + q + q^2 + \cdots + q^{i-1}$ many choices for the vector we add. Continuing this process we then have $\frac{q^{i-1}-1}{q-1}$ many choices for the additional vector for y_2 and so on. Once we have defined y_{i-2} we only have $\frac{q^2-1}{q-1} = q+1$ choices for the vector we add to get y_{i-1}. Hence we must have that

$$c = 1(1+q)\cdots(1+q+q^2+\cdots+q^{i-1})$$

giving us

$$\partial^i(x) = (i!)_q \sum y.$$

For $\pi \geq 2$ we have $((\pi-1)!)_q \not\equiv 0 \mod p$ while $(\pi!)_q \equiv 0 \mod p$ and therefore as long as $\pi \leq n$ we can say that ∂^π is the least power of ∂ for which $\partial^\pi = 0$. So for given p and q we can define such a number π in the manner stated to ensure that we have a homological sequence.

2.2 The Homology Module

Having made some initial definitions and constructed a homological sequence we are now able to give the formal definition of the *homology module* for such a sequence.

Definition 2.6. For integers $0 < i < \pi$ and k we put

$$K^n_{k,i} := \ker \partial^i \cap M^n_k$$

and

$$I^n_{k,i} := \partial^{\pi-i}(M^n_{k+\pi-i}).$$

Since $\partial^\pi = 0$ it is clear that $I^n_{k,i} \subseteq K^n_{k,i}$ and so we are able to define

$$H^n_{k,i} := K^n_{k,i}/I^n_{k,i}$$

to be the homology module of the sequence

$$M^n_{k-i} \xleftarrow{\partial^i} M^n_k \xleftarrow{\partial^{\pi-i}} M^n_{k+\pi-i}.$$

In later chapters we will also need to consider how these modules are defined for $i = 0$ and $i = \pi$. For $i = 0$ we let $K^n_{k,0} = I^n_{k,0} = H^n_{k,0} = 0$ while for $i = \pi$ we have that $K^n_{k,\pi} = M^n_k$ (since $\partial^\pi \equiv 0$) and $I^n_{k,\pi} = M^n_k$ (since ∂^0 is the identity map). Hence we also define $H^n_{k,\pi} = 0$. For an element $f \in K^n_{k,i}$ we denote $[f] = f + I^n_{k,i}$ to be an element of the homology module $H^n_{k,i}$.

The following result shows that the above sequence is exact unless k and i satisfy a certain condition.

Theorem 2.7. *[13, Thm 3.1] Let $i < \pi := \pi(p, q)$ and k be positive integers. Then $H^n_{k,i} = 0$ unless*

$$n < 2k + \pi - i < n + \pi.$$

If a pair of integers k and $i < \pi$ satisfy the condition of Theorem 2.7 then we call these parameters (k, i) a *middle term* for the sequence. There may be no middle term for some $\mathcal{M}_{k^*,i^*}$ but if one does exist it can be shown to be unique and thus we have

Corollary 2.8. *The sequence $\mathcal{M}_{k^*,i^*}$ is almost exact.*

Remark 2.9. In Chapter 6 we will give a different and more exciting proof of this result to the one given in [13], making use of the main branching theorem which will be discussed in Chapter 5.

3 Betti Numbers

Our first step towards obtaining a decomposition of homology modules is to consider their dimensions. These dimensions are called *Betti numbers*. In the first part of this chapter we introduce combinatorial functions called *Gaussian polynomials*. These are useful to us as they give the dimensions of the modules M_k^n. The Euler-Poincaré Equation gives a relation between alternating sums of Betti numbers and alternating sums of the dimensions of the permutation modules M_k^n. If we make the assumption that our sequence is almost exact then we can use the trace formula and Gaussian polynomials to derive a formula for calculating the Betti numbers. In the last part of this chapter we establish a relation for Gaussian polynomials which, combined with the Betti number formula, leads to a result giving certain relations between Betti numbers. From this we can derive the equivalent result for the dimensions of the kernel $K_{k,i}^n$ and image $I_{k,i}^n$.

3.1 Gaussian Polynomials

The usual binomial coefficients are the numbers $\binom{n}{k}$ which tell us the number of different ways we may choose k different elements from a set of n elements. They can be determined by the formula

$$\binom{n}{k} := \frac{n!}{(n-k)!k!}.$$

The q-analogues of these binomial coefficients are known as the Gaussian polynomials. It can be shown that these numbers give the number of k-

dimensional subspaces of an n-dimensional vector space. Hence any relation between Gaussian polynomials gives us a relation for the dimensions of the modules M_k^n.

Recall from the previous section the definition of $[i]_q = 1 + q + \cdots + q^{i-1}$ and let n be any non-negative integer. Then for $0 \leq k \leq n$ we define the q-binomial function, or Gaussian polynomial, $\binom{n}{k}_q$ by

$$\binom{n}{k}_q := \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q \cdot [k-1]_q \cdots [1]_q}.$$

Gaussian polynomials have the following properties:

Theorem 3.1. *[3, Thm 3.2] Let $0 \leq k \leq n$ be integers. The Gaussian polynomial $\binom{n}{k}_q$ is a polynomial of degree $k(n-k)$ in q that satisfies the relations*

$$\binom{n}{0}_q = \binom{n}{n}_q = 1 \tag{3.1}$$

$$\binom{n}{k}_q = \binom{n}{n-k}_q \tag{3.2}$$

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q \tag{3.3}$$

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^{k}\binom{n-1}{k}_q \tag{3.4}$$

$$\lim_{q \to 1} \binom{n}{k}_q = \frac{n!}{k!(n-k)!} = \binom{n}{k}. \tag{3.5}$$

We now prove that the number $\binom{n}{k}_q$ is indeed equal to the number of k-dimensional subspaces of an n-dimensional vector space and hence is equal to the dimension of M_k^n.

Theorem 3.2. *[9, Thm 3.1] Let V_1 and V_2 be subspaces of V with $\dim V_1 = a$, $\dim V_2 = b$ and $V_1 \cap V_2 = 0$. The number of k-dimensional subspaces W of V such that $W \cap V_1 = 0$ and $W \supseteq V_2$ is*

$$q^{a(k-b)} \binom{n-a-b}{k-b}_q.$$

Proof. Suppose we are given a basis for V_2. Then the number of ways of choosing $k-b$ further linearly independent vectors from V such that no linear combination of them lies in $V_1 \oplus V_2$ is

$$(q^n - q^{a+b})(q^n - q^{a+b+1}) \cdots (q^n - q^{a+k-1}).$$

This number can be expressed as

$$(q-1)^{k-b} q^{\frac{1}{2}(k-b)(k+b+2a-1)} [n-a-b]_q [n-a-b-1]_q \cdots [n-a-k+1]_q.$$

However, any k-dimensional subspace which contains V_2 has the following number of bases which extend the given basis of V_2:

$$(q^k - q^b)(q^k - q^{b+1}) \cdots (q^k - q^{k-1}).$$

We can rewrite this number as

$$(q-1)^{k-b} q^{\frac{1}{2}(k-b)(k+b-1)} [k-b]_q [k-b-1]_q \cdots [1]_q.$$

Now if we take the quotient of these two numbers we get the required answer. □

If we put $a = b = 0$ into this theorem, so that $V_1 = V_2 = 0$ then this result gives the number of k-dimensional subspaces of V and so we have

Corollary 3.3.

$$\binom{n}{k}_q = |L_k^n| = \dim M_k^n.$$

3.2 The Betti Number

The fact that the Gaussian polynomials give the dimensions of the modules M_k^n means that they are useful in calculating the dimension of the homology module. Recall from the previous chapter that we have defined $H_{k,i}^n$ to be the homology module of the sequence

$$M_{k-i}^n \xleftarrow{\partial^i} M_k^n \xleftarrow{\partial^{\pi-i}} M_{k+\pi-i}^n$$

where $\pi := \pi(p,q)$ is defined to be the smallest power of ∂ which gives $\partial^\pi \equiv 0$. We make the following definition:

Definition 3.4. The dimension of the homology module $H_{k,i}^n$ is called the *Betti number* of $H_{k,i}^n$, denoted by

$$\beta_{k,i}^n := \dim H_{k,i}^n.$$

The Euler-Poincaré Equation (see [1, Ch.XX.3] or [8, Ch.IX.4]) for a homological sequence relates the alternating sum of its Betti numbers to its characteristic, the alternating sum of its dimensions.

Theorem 3.5. *(Euler-Poincaré Equation) Suppose that R is a ring of prime characteristic p and let G be a group whose order is not divisible by p. If we have a homological sequence of RG-modules*

$$\mathcal{A} : 0 \leftarrow A_0 \leftarrow \cdots \leftarrow A_{k-1} \leftarrow A_k \leftarrow A_{k+1} \leftarrow \cdots \leftarrow 0$$

with homology modules H_k then

$$\sum_i (-1)^i \dim H_i = \sum_i (-1)^i \dim A_i.$$

We can use this to prove a formula for calculating the Betti numbers:

Theorem 3.6. *If k and $0 < i < \pi$ are integers such that (k, i) is the middle term of $\mathcal{M}_{k^*,i^*}$ then*

$$\beta^n_{k,i} = \sum_{t\in\mathbb{Z}} \binom{n}{k-\pi t}_q - \binom{n}{k-i-\pi t}_q.$$

Proof. Since the sequence $\mathcal{M}_{k^*,i^*}$ is almost π-exact then at most one of the homology modules is non-zero and thus applying the above Euler-Poincaré Equation to this sequence gives

$$\dim H^n_{k,i} = \sum_{t\in\mathbb{Z}} \dim M^n_{k-\pi t} - \dim M^n_{k-i-\pi t}$$

when (k, i) is the middle term. Since the dimension of M^n_k is $\binom{n}{k}_q$ we have the required formula. □

Remark 3.7. (1) If we consider the function $\phi^n_{k,i} := \sum_{t\in\mathbb{Z}} \binom{n}{k-\pi t}_q - \binom{n}{k-i-\pi t}_q$ for general integers n, k and i then $\beta^n_{k,i}$ agrees with $\phi^n_{k,i}$ when (k, i) is the middle term. If (k, i) is not the middle term then $\beta^n_{k,i} = 0$.

(2) If we consider the case when $\pi = 2$ and take n to be odd then we may observe that $\phi^n_{k,i} = 0$.

(3) If p is a prime power not dividing q then $\pi(p, q)$ is the order of q modulo p. If we take $q = 1$ then $\pi(p, 1) = p$ and $\binom{n}{k}_1$ is the usual binomial

coefficient. Thus we can still define the Betti number in this way and an interesting observation is that if we put $q = 1$ and $p = 5$ then $\phi^n_{k,i}$ (and hence the non-zero Betti numbers) are $(n-1)^{th}$, n^{th} or $(n+1)^{th}$ Fibonacci numbers, as noted in [17].

The following table gives some values for the Betti numbers $\beta^n_{k,i}$ when π, n, k and i are small. Note that as π is a function of p and q then if we fix π we may express the number in terms of q only.

		Values of (k,i)			
π	n	(1,1)	(1,2)	(2,1)	(2,2)
2	1	0	0	0	0
	2	$q-1$	0	0	0
	3	0	0	0	0
	4	0	0	q^4-q^3-q+1	0
3	1	0	1	0	0
	2	q	q	0	0
	3	q^2+q-1	0	0	q^2+q-1
	4	0	0	q^4+q^2-1	q^4+q^2-1
4	1	0	1	0	0
	2	q	$q+1$	0	0
	3	q^2+q	q^2+q	0	q^2+q
	4	q^3+q^2+q-1	0	q^4+q^2	$q^4+q^3+2q^2+q-1$

Table 1: THE FUNCTION $\beta^n_{k,i}$

3.3 Some Betti Number Relations

We can now use the formula for calculating $\beta^n_{k,i}$ to examine some relations for the Betti numbers. We begin with a result which gives a relation for Gaussian polynomials and hence for the dimension of the modules M^n_k.

Lemma 3.8. *For* $0 \leq k \leq n$*,*

$$\dim M^n_k = \dim M^{n-1}_k + \dim M^{n-1}_{k-1} + (q^{n-1} - 1) \dim M^{n-2}_{k-1}.$$

Proof. The dimension of M^n_k is the number of k-dimensional subspaces in V which is given by the q-binomial function $\binom{n}{k}_q$. So we are required to show that

$$\binom{n}{k}_q = \binom{n-1}{k}_q + \binom{n-1}{k-1}_q + (q^{n-1} - 1)\binom{n-2}{k-1}_q. \tag{3.6}$$

It is easy to check that this relation holds when $k = 0, 1$ or n. For $1 < k < n$ we may use the definition of the Gaussian polynomial to express the right hand side as

$$\frac{[n-1]_q \cdots [n-k]_q}{[k]_q \cdots [1]_q} + \frac{[n-1]_q \cdots [n-k+1]_q}{[k-1]_q \cdots [1]_q} + (q^{n-1} - 1)\frac{[n-2]_q \cdots [n-k]_q}{[k-1]_q \cdots [1]_q}.$$

This may be simplified to

$$\binom{n}{k}_q \left(\frac{[n-k]_q}{[n]_q} + \frac{[k]_q}{[n]_q} + (q^{n-1} - 1)\frac{[k]_q[n-k]_q}{[n]_q[n-1]_q} \right).$$

Now for integer $m \geq 1$ we have $[m]_q = \frac{q^m - 1}{q-1}$ and thus our expression is equal to

$$
\begin{aligned}
&\binom{n}{k}_q \left(\frac{(q^{n-k}-1)+(q^k-1)+(q^k-1)(q^{n-k}-1)}{q^n-1}\right)\\
&= \binom{n}{k}_q \left(\frac{q^{n-k}-1+q^k-1+q^n-q^k-q^{n-k}+1}{q^n-1}\right)\\
&= \binom{n}{k}_q \left(\frac{q^n-1}{q^n-1}\right) = \binom{n}{k}_q .
\end{aligned}
$$

□

We can now use this relation for Gaussian polynomials together with the formula for calculating $\beta^n_{k,i}$ to investigate some relations for the Betti numbers. Here we prove two relations which have the same shape as the result in the previous lemma for M^n_k.

Lemma 3.9. *(1) If $0 < i < \pi$ and $n+1 \leq 2k+\pi-i \leq n+\pi-1$ then*

$$\beta^n_{k,i} = \beta^{n-1}_{k,i+1} + \beta^{n-1}_{k-1,i-1} + (q^{n-1}-1)\beta^{n-2}_{k-1,i}$$

(2) If $0 < i < \pi$ and $n < 2k+\pi-i < n+\pi$ then

$$\beta^n_{k,i} = \beta^{n-1}_{k,i} + \beta^{n-1}_{k-1,i} + (q^{n-1}-1)\beta^{n-2}_{k-1,i}.$$

Proof. (1) We consider the expression (3.6) for Gaussian polynomials,

$$\binom{n}{k}_q = \binom{n-1}{k}_q + \binom{n-1}{k-1}_q + (q^{n-1}-1)\binom{n-2}{k-1}_q .$$

Substituting this into the formula for calculating $\beta^n_{k,i}$ gives

$$\begin{aligned}\beta^n_{k,i} =&\sum_{t\in\mathbb{Z}} \binom{n-1}{k-\pi t}_q + \binom{n-1}{k-\pi t-1}_q + (q^{n-1}-1)\binom{n-2}{k-\pi t-1}_q \\ &- \binom{n-1}{k-i-\pi t}_q - \binom{n-1}{k-i-\pi t-1}_q \\ &- (q^{n-1}-1)\binom{n-2}{k-i-\pi t-1}_q \\ =&\sum_{t\in\mathbb{Z}} \binom{n-1}{k-\pi t}_q - \binom{n-1}{k-i-\pi t-1}_q \\ &+ \sum_{t\in\mathbb{Z}} \binom{n-1}{k-\pi t-1}_q - \binom{n-1}{k-i-\pi t}_q \\ &+ (q^{n-1}-1)\sum_{t\in\mathbb{Z}} \binom{n-2}{k-\pi t-1}_q - \binom{n-2}{k-i-\pi t-1}_q \\ =&\beta^{n-1}_{k,i+1} + \beta^{n-1}_{k-1,i-1} + (q^{n-1}-1)\beta^{n-2}_{k-1,i}.\end{aligned}$$

For (2) we notice that

$$\begin{aligned}\beta^{n-1}_{k-1,i-1} + \beta^{n-1}_{k,i+1} =&\sum_{t\in\mathbb{Z}} \binom{n-1}{k-1-\pi t}_q - \binom{n-1}{k-i-\pi t}_q \\ &+ \sum_{t\in\mathbb{Z}} \binom{n-1}{k-\pi t}_q - \binom{n-1}{k-i-1-\pi t}_q\end{aligned}$$

and rearranging gives

$$\begin{aligned}\beta^{n-1}_{k-1,i-1} + \beta^{n-1}_{k,i+1} =&\sum_{t\in\mathbb{Z}} \binom{n-1}{k-\pi t}_q - \binom{n-1}{k-i-\pi t}_q \\ &+ \sum_{t\in\mathbb{Z}} \binom{n-1}{k-1-\pi t}_q - \binom{n-1}{k-i-1-\pi t}_q \\ =&\ \beta^{n-1}_{k,i} + \beta^{n-1}_{k-1,i}.\end{aligned}$$

So the second Betti number relation follows immediately by substituting this into (1). □

Again we note the connection with homology in the Boolean algebra. In [17, Thm 4.5] it is shown that in the Boolean algebra case the Euler-Poincaré Equation leads to the dimension formula

$$\beta_{k,i}^n = \sum_{t \in \mathbb{Z}} \binom{n}{k - pt} - \binom{n}{k - i - pt}$$

and [17, Cor 4.6] gives two relations

$$\begin{aligned} \beta_{k,i}^n &= \beta_{k,i}^{n-1} + \beta_{k-1,i}^{n-1} \\ &= \beta_{k,i+1}^{n-1} + \beta_{k-1,i-1}^{n-1}. \end{aligned}$$

These are precisely the relations we obtain if we put $q = 1$ in Theorem 3.6 and Lemma 3.9.

We can also prove that exactly the same relations hold for the kernels $K_{k,i}^n$ which we defined earlier.

Corollary 3.10. *For integers k and $0 < i < \pi$ we have*

$$\dim K_{k,i}^n = \dim K_{k,i+1}^{n-1} + \dim K_{k-1,i-1}^{n-1} + (q^{n-1} - 1) \dim K_{k-1,i}^{n-2}.$$

Proof. Recall that the homology module is defined by

$$H_{k,i}^n := K_{k,i}^n / I_{k,i}^n$$

where

$$I_{k,i}^n := \partial^{\pi - i}(M_{k+\pi-i}^n).$$

So we have

$$\dim K_{k,i}^n = \dim H_{k,i}^n + \dim(\partial^{\pi-i}(M_{k+\pi-i}^n)).$$

By the rank + nullity formula we know that

$$\dim(\partial^{\pi-i}(M^n_{k+\pi-i})) = \dim M^n_{k+\pi-i} - \dim K^n_{k+\pi-i,\pi-i}$$

and substituting this into our previous expression gives us

$$\dim K^n_{k,i} = \dim H^n_{k,i} + \dim M^n_{k+\pi-i} - \dim K^n_{k+\pi-i,\pi-i}.$$

We can now prove the result using induction on k, starting with a base step of $k \geq n$ and then proving the result for all smaller values of k.

For the case of $k \geq n$ we notice that since $M^n_\alpha = 0$ for $\alpha > n$, we must have that $K^n_{k+\pi-i,\pi-i} = 0$. So we can ignore the last term in our expression for $\dim K^n_{k,i}$:

$$\dim K^n_{k,i} = \dim H^n_{k,i} + \dim M^n_{k+\pi-i}.$$

From the two previous lemmas we have that this relation holds for both $\dim H^n_{k,i}$ and $\dim M^n_k$, so the relation holds for $\dim K^n_{k,i}$ when $k \geq n$. We now assume that the result holds for all $k^* > k$ and prove that from this it holds for k.

Since $\pi - i > 0$, from our inductive step we may assume that

$$\begin{aligned}\dim K^n_{k+\pi-i,\pi-i} = &\dim K^{n-1}_{k+\pi-i,\pi-i+1} \\ &+ \dim K^{n-1}_{k+\pi-i-1,\pi-i-1} + (q^{n-1}-1)\dim K^{n-2}_{k+\pi-i-1,\pi-i}.\end{aligned}$$

So by the formula

$$\dim K^n_{k,i} = \dim H^n_{k,i} + \dim M^n_{k+\pi-i} - \dim K^n_{k+\pi-i,\pi-i}$$

we have that the result holds for all k. □

By the definition of the homology module $H^n_{k,i} = K^n_{k,i}/I^n_{k,i}$ we have

$$\dim H^n_{k,i} = \dim K^n_{k,i} - \dim I^n_{k,i}.$$

Thus any relations which hold for $\dim H^n_{k,i}$ and $\dim K^n_{k,i}$ must also hold for the dimension of the image.

Corollary 3.11. *For integers k and $0 < i < \pi$ we have*

$$\dim I^n_{k,i} = \dim I^{n-1}_{k-1,i-1} + \dim I^{n-1}_{k,i+1} + (q^{n-1} - 1) \dim I^{n-2}_{k-1,i}.$$

The result in Lemma 3.9 giving a relation for the dimension of homology modules helps us in predicting a potential module decomposition for $H^n_{k,i}$. In Chapter 5 we will investigate this further and prove such a branching rule. However it is worth noting that this will be done independently of Lemma 3.9 and the proof will not rely on this fact that the dimensions satisfy the corresponding branching rule.

4 Decomposing the module M_k^n

In this chapter we consider the permutation module M_k^n which consists of the formal F-linear combinations of the k-dimensional subspaces of a vector space V of dimension n. Our aim is to construct a decomposition of this $FGL(n,\mathbb{F})$-module in terms of $FGL(n-1,\mathbb{F})$-modules. If our field F is extended by adjoining a primitive $p^{prime}th$ root of unity then we may use a result of James [9, Thm 10.5] which gives a branching rule for the $FGL(n,\mathbb{F})$-permutation module M_λ corresponding to a composition λ of n. A *composition* is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of non-negative integers whose sum is n. When we take λ to be the two-part composition $(n-k, k)$ this module becomes M_k^n. This branching rule splits M_k^n into two parts by using two idempotent elements which we shall define later. By considering each of these two parts in turn we can obtain two $FGL(n-1,q)$-isomorphisms which lead to the module decomposition we desire.

4.1 A Branching Rule for $\bar{F}GL(n,\mathbb{F})$-modules

When our field F is extended to include a $p'th$ root of unity there is a result of James [9, Thm 9.11] which gives a branching rule for **any** $FGL(n,q)$-module. This branching rule involves some idempotent elements and in this section we give a precise definition of how to construct these idempotents and state the branching rule itself.

We begin by defining some notation for the general linear group and some particular subgroups. We shall use the notation $G_n = GL(n,\mathbb{F})$ to denote the general linear group of degree n over $\mathbb{F}$. We may think of G_{n-1} as a subgroup of G_n in the following sense:

$$G_{n-1} = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & g & \\ 0 & & & \end{pmatrix} : g \in GL(n-1, \mathbb{F}) \right\}.$$

Similarly we define

$$G_{n-2} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & g & \\ 0 & 0 & & & \end{pmatrix} : g \in GL(n-2, \mathbb{F}) \right\}.$$

We also define another subgroup G^*_{n-1} of G_n to be the group of affine linear transformations on $\mathbb{F}^{n-1}$:

$$G^*_{n-1} = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_2 & & & \\ \vdots & & g & \\ a_n & & & \end{pmatrix} : a_i \in \mathbb{F}, g \in G_{n-1} \right\}$$

and similarly we define the subgroup

$$H^*_{n-1} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2 & 1 & 0 & \cdots & 0 \\ a_3 & b_3 & & & \\ \vdots & \vdots & & h & \\ a_n & b_n & & & \end{pmatrix} : a_i, b_j \in \mathbb{F}, h \in G_{n-2} \right\}.$$

It is clear that these are indeed subgroups of G_n. Notice that we have

$$G_{n-2} \subseteq H^*_{n-1} \subseteq G^*_{n-1} \subseteq G_n.$$

The order of G_n is given by

$$|G_n| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}).$$

Hence we have

$$\left|G^*_{n-1}\right| = q^{n-1} \left|G_{n-1}\right| = q^{n-1}(q^{n-1} - 1)(q^{n-1} - q) \cdots (q^{n-1} - q^{n-2})$$

and

$$\left|H^*_{n-1}\right| = q^{n-1}q^{n-2} \left|G_{n-2}\right| = q^{n-1}q^{n-2}(q^{n-2} - 1)(q^{n-2} - q) \cdots (q^{n-2} - q^{n-3}).$$

Thus we can compute the index

$$\left|G^*_{n-1} : H^*_{n-1}\right| = \frac{\left|G^*_{n-1}\right|}{\left|H^*_{n-1}\right|} = q^{n-1} - 1.$$

Before going any further we need to make the assumption that our field F contains a primitive $p'th$ root of unity. Hence we make the following definitions:

Definition 4.1. Let F be a field of characteristic p and take q to be a power of the prime p' with $p' \neq p$. Then we define

(i) $\bar{F}$ is the extension field obtained by adjoining a primitive $(p')th$ root of unity to F,

(ii) $\bar{M}^n_k = \bar{F}L^n_k$ is the permutation module over $\bar{F}$ with basis L^n_k.

Extending the field in this way allows us to construct some idempotent elements of $\bar{F}G_n$. These idempotents can then be used to give a general branching rule for any $\bar{F}G_n$-module.

Definition 4.2. A linear $\bar{F}$-character χ of $(\mathbb{F},+)$ is a homomorphism

$$\chi : (\mathbb{F},+) \to \bar{F}^{\times}.$$

If q is a power of the prime p' then every element of $(\mathbb{F},+)$ has order p' and thus $(\mathbb{F},+)$ is an elementary abelian p'-group. Since $\bar{F}$ contains a primitive $p'th$ root of unity there exists q distinct linear $\bar{F}$-characters of $(\mathbb{F},+)$. These can be used to define some idempotent elements as is done in [9].

Definition 4.3. Let $\chi_1, \cdots, \chi_q$ be the distinct linear $\bar{F}$-characters of $(\mathbb{F},+)$ with χ_1 equal to the trivial character. Denote $\phi^- = \{(i,j) : 1 \le j < i \le n\}$. Now suppose that $1 \le r \le n$. We make the following definitions:

(i) $\Gamma(r) = \{(i,j) \in \phi^- : j \le r\}$.

(ii) The function $c_r : \Gamma(r) \to \{1, 2, \cdots, q\}$ is defined by

$$(j+1,j)c_r = 2 \text{ if } j < r,$$
$$(i,j)c_r = 1 \text{ for all other } (i,j) \in \Gamma(r).$$

(iii) The group $G(\Gamma(r))$ consists of all matrices which have every diagonal entry is 1, arbitrary elements of $\mathbb{F}$ in positions $(i,j) \in \Gamma(r)$ and zeros everywhere else.

(iv) For a matrix $g \in G(\Gamma(r))$ we define the function $\chi_{c_r} : G(\Gamma(r)) \to \bar{F}$ by

$$\chi_{c_r} : g = (\alpha_{ij}) \mapsto \prod_{i,j} \chi_{(i,j)c_r}(\alpha_{ij})$$

where α_{ij} is the (i,j)-entry of g.

(v) Let E_r be the following element in $\bar{F}G_n$:

$$E_r = \frac{1}{q^{|\Gamma(r)|}} \sum_{g \in G(\Gamma(r))} \chi_{c_r}(g)g.$$

Example:

If $n = 4$ then

$$E_1 = \frac{1}{q^3} \sum_{\alpha_i \in \mathbb{F}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_2 & 1 & 0 & 0 \\ \alpha_3 & 0 & 1 & 0 \\ \alpha_4 & 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \frac{1}{q^5} \sum_{\alpha_i, \beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_2 & 1 & 0 & 0 \\ \alpha_3 & \beta_3 & 1 & 0 \\ \alpha_4 & \beta_4 & 0 & 1 \end{pmatrix},$$

$$E_3 = \frac{1}{q^6} \sum_{\alpha_i, \beta_j, \gamma_k \in \mathbb{F}} \chi_2(\alpha_2)\chi_2(\beta_3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_2 & 1 & 0 & 0 \\ \alpha_3 & \beta_3 & 1 & 0 \\ \alpha_4 & \beta_4 & \gamma_4 & 1 \end{pmatrix},$$

$$E_4 = \frac{1}{q^6} \sum_{\alpha_i, \beta_j, \gamma_k \in \mathbb{F}} \chi_2(\alpha_2)\chi_2(\beta_3)\chi_2(\gamma_4) \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_2 & 1 & 0 & 0 \\ \alpha_3 & \beta_3 & 1 & 0 \\ \alpha_4 & \beta_4 & \gamma_4 & 1 \end{pmatrix}.$$

It is worth noting that we consider these idempotents as elements of $\bar{F}G_n$ rather than as matrix sums. It can be shown that if $1 \leq r \leq n$ then we have $E_r E_r = E_r$ and hence these elements E_r are idempotent elements of $\bar{F}G_n$. Given any $\bar{F}G_n$-module M we apply the idempotents E_r by multiplication on the right. So ME_1 is the space

$$\langle xE_1 : x \in M \rangle_{\bar{F}}.$$

It is clear that $E_1 \in \bar{F}G^*_{n-1}$. Also every $g \in G^*_{n-1}$ commutes with E_1 and so we have $E_1 \cdot g = g \cdot E_1 \in \bar{F}G_n E_1$. Since M is an $\bar{F}G_n$-module we have that $x \cdot g \in ME_1$ for all $x \in ME_1$ and hence ME_1 is a right G^*_{n-1}-module. For each idempotent E_r we define the space

$$ME_r G^*_{n-1} = \langle xE_r g : x \in M, g \in G^*_{n-1} \rangle_{\bar{F}}$$

which is clearly also a right G^*_{n-1}-module. We now give the following theorem of James which gives an $\bar{F}G^*_{n-1}$-decomposition of any $\bar{F}G_n$-module.

Theorem 4.4. *[9, Thm 9.11]*
If M is any $\bar{F}G_n$-module then

(i) $M = ME_1 \oplus \left(\sum_{r=2}^{n} ME_r G^*_{n-1}\right)$ *as $\bar{F}G^*_{n-1}$-modules, and*

(ii) $\dim M = \sum_{r=1}^{n} (q^{n-1} - 1)(q^{n-2} - 1) \cdots (q^{n-r+1} - 1) \dim(ME_r)$.

4.2 Standard Matrix Form

The elements of the permutation module $\bar{M}^n_k$ are $\bar{F}$-linear combinations of the k-dimensional subspaces of V. It would therefore be useful when considering

$\bar{M}_k^n$ as an $\bar{F}G_n$-module to be able to express these subspaces in matrix form. We now describe how this can be done so that every k-dimensional subspace of the n-dimensional vector space corresponds to a unique $k \times n$ matrix over the field $\mathbb{F}$.

To express any k-dimensional subspace in matrix form we first need to fix a basis $\{v_1, \cdots, v_n\}$ of V. We then take any k-dimensional subspace $x = \langle x_1, \cdots, x_k \rangle$ and express it as a $k \times n$ matrix in the following way. We index the columns of the matrix by $v_1, \cdots, v_n$ and the rows by $x_1, \cdots, x_k$. Each vector x_i can be expressed as a linear combination of the basis vectors $v_1, \cdots, v_n$ and so

$$x_i = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n$$

with $a_{ij} \in \mathbb{F}$. In the (i, j)-position of the matrix we insert the coefficient a_{ij}. Thus we have a $k \times n$ matrix which represents x. This matrix is not yet unique however, as there are many bases we can choose for our space x giving different vectors x_i and hence different matrix entries.

We observe that we can apply the following row operations to this matrix without changing the subspace it represents:

(i) Divide all entries in a row by a non-zero element of $\mathbb{F}$.

(ii) Subtract a multiple of one row from another.

(iii) Rearrange the order of the rows.

By applying (i) and (ii) we can arrange that the last non-zero entry in each row is a 1 and that all other entries in the column containing this 1 are zero. We call this entry a leading one. For each row i we denote the column containing the leading one by a_i. Using (iii) we can interchange rows so that $1 \le a_1 < a_2 < \cdots < a_k \le n$. This means that we have a matrix of the following 'standard form':

$$x = \begin{pmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Remark 4.5. (1) This process of row-reducing to standard form is the usual Gauss-Jordan elimination, but where we start from bottom-right rather than top-left.

(2) Note that when we say that we have a $k \times n$ matrix in standard form then we are assuming that the matrix has rank k. If the row operations produce a row of zeros then we remove this row from the matrix. Here we are dealing with k-dimensional subspaces and so it is clear that the matrices must have rank k.

We can prove that this 'standard matrix form' is unique.

Lemma 4.6. *There is a bijection between the set of all $k \times n$ matrices in 'standard form' and the set of k-dimensional subspaces of V.*

Proof. To prove this we show that the number of matrices in 'standard form' is equal to $\binom{n}{k}_q$, the number of k-dimensional subspaces of V. We do this by induction on n, the dimension of V. For small n it is easy to verify that this is the case. So we assume that the result holds for $n-1$. Now we observe that any standard $k \times n$ matrix must be obtained in one of two ways:

- From a $(k-1)\times(n-1)$ matrix M^* with an additional top row of zeros and then an additional left hand column with a one in the top position and zeros in every position below

$$\begin{pmatrix} 1 & 0 \\ 0 & M^* \end{pmatrix}.$$

- From a $k \times (n-1)$ matrix M^{**} with an additional left hand column containing arbitrary elements of $\mathbb{F}$

$$\begin{pmatrix} * & \\ : & M^{**} \\ * & \end{pmatrix}.$$

These give all possible standard $k \times n$ matrices and so the total number of matrices is the number of standard $(k-1)\times(n-1)$ matrices plus q^k times the number of standard $k \times (n-1)$ matrices.

Now we recall the formula for Gaussian polynomials from Theorem 3.1 which gives the relation (3.4):

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q.$$

Then by induction we must have that for any n the number of matrices is equal to the number of subspaces. Since every subspace corresponds to some matrix and since this correspondence is injective, we have a bijection and hence this matrix representation must be unique. □

Remark 4.7. In the above proof we have counted the number of objects on both sides to prove that the bijection exists. However it should also be noted that we have the same bijection regardless of whether the field $\mathbb{F}$ is finite or not.

We also have the following result which says that if we multiply together two matrices in standard form then the resulting matrix is already in this form also, with no need to apply further row operations. This will be useful to us in later chapters.

Lemma 4.8. *The product of two 'standard' matrices is itself a 'standard' matrix.*

Proof. Let A be a $k \times l$ matrix and let B be an $l \times m$ matrix with both A and B in 'standard' form. Then the two matrices A and B have the following properties:

1. The last non-zero entry in each row is a 1 (the leading one).

2. If we denote the position of the leading one in row i by $l(i)$ then

$$l(1) < l(2) < \cdots < l(n)$$

where n denotes the number of rows in the matrix.

3. Each column which contains a leading one has zero in every other position.

We shall denote the (i, j)-entries of the two matrices A and B by a_{ij} and b_{ij} respectively. We denote the rows of A by $s_1, \cdots, s_k$ and the rows of B by $r_1, \cdots, r_l$. Now if we put $C = AB$ with (i, j)-entry c_{ij} and rows $t_1, \cdots, t_k$ then we see that

$$t_i = a_{i1}r_1 + a_{i2}r_2 + \cdots + a_{il}r_l$$

where we add any two row vectors by adding together their corresponding entries. Denote the position of the leading one in row i of A by $l(s_i)$. Then $a_{ij} = 0$ for all $j > l(s_i)$ and $a_{il(s_i)} = 1$. So

$$t_i = a_{i1}r_1 + a_{i2}r_2 + \cdots + a_{il(s_i)-1}r_{l(s_i)-1} + r_{l(s_i)}.$$

Now denote the position of the leading one in row i of B by $l(r_i)$. Since B is in 'standard' form we have

$$l(r_1) < l(r_2) < \cdots < l(r_{l(s_i)})$$

and hence

$$l(t_i) = l(r_{l(s_i)}).$$

Since A is in 'standard' form we know that if $i < j$ then $l(s_i) < l(s_j)$ and so

$$l(t_1) < l(t_2) < \cdots < l(t_l).$$

So C satisfies the first two properties of a 'standard' matrix.

Now note that since A is in 'standard' form then $a_{jl(s_i)} = 0$ for $j \neq i$. This means that the coefficient in column position $l(s_i)$ is zero in s_j for $j \neq i$.

Hence only row t_i has a non-zero entry in column $l(s_i)$. This means that C also satisfies Property 3.

So C is in 'standard' form. □

Another useful observation which can be made relating to how G_n acts on the set of k-dimensional subspaces is the following lemma which shows that it is a transitive action.

Lemma 4.9. *The group G_n acts transitively on L_k^n.*

Proof. Let

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

be the matrix representing any k-dimensional subspace. If we take

$$y = \begin{pmatrix} & 0 & \cdots & 0 \\ I_k & \vdots & & \vdots \\ & 0 & \cdots & 0 \end{pmatrix}$$

then selecting any element of the form

$$g = \begin{pmatrix} x \\ \bar{g} \end{pmatrix} \in G_n$$

gives us $y \cdot g = x$. Thus $\{y \cdot g : g \in G_n\} = L_k^n$. □

4.3 The Permutation Module $\bar{M}_k^n$

The module $\bar{M}_k^n$ is the $\bar{F}G_n$-module consisting of all $\bar{F}$-linear combinations of the k-dimensional subspaces. We may now apply the branching rule of Theorem 4.4 to this module, allowing us to split the module into two parts. We can then consider each part separately and define $\bar{F}G_{n-1}$-isomorphisms which will lead us to a decomposition of this permutation module.

Recall that a composition of n is sequence λ of non-negative integers whose sum is n. If we take $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_h)$ such that $\lambda_i \neq 0$ for $1 \leq i \leq h$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_h = n$ then we call λ an h-part composition of n. For each such composition James defined a permutation module M_λ which is an $\bar{F}G_n$-module. The significance of these modules to us here is that if we take λ to be the two-part composition $\lambda = (n-k, k)$ then M_λ is the module $\bar{M}_k^n$ which we have defined above. It is shown in the proof of [9, Thm 10.5] that $M_\lambda E_r = 0$ for $r > h$, where h is the number of parts of the composition λ. So for $\lambda = (n-k, k)$ we have

$$\bar{M}_k^n E_r = 0 \text{ for } r > 2$$

Thus if we apply the branching rule of Theorem 4.4 to our module $\bar{M}_k^n$ then the decomposition is greatly simplified and involves only two idempotents. We have the following result which is given in greater generality in [9].

Theorem 4.10. *[9, Thm 10.5] We have*

(i) $\bar{M}_k^n = \bar{M}_k^n E_1 \oplus \bar{M}_k^n E_2 G_{n-1}^*$ *as* G_{n-1}^**-modules,*

(ii) $\dim \bar{M}_k^n = \dim \bar{M}_k^n E_1 + (q^{n-1} - 1) \dim \bar{M}_k^n E_2$.

We now consider the two parts of the module decomposition in $4.9(i)$ with the intention of constructing $\bar{F}G_{n-1}$-isomorphisms for each of these parts to lead us to a decomposition of $\bar{M}_k^n$ in terms of $\bar{F}G_{n-1}$-modules. We must first make some definitions. Given two subspaces $a = \langle a_1, \cdots, a_r \rangle$ and $b = \langle b_1, \cdots, b_s \rangle$ we define

$$a \vee b = \langle a_1, \cdots, a_r, b_1, \cdots, b_s \rangle.$$

We may extend this linearly to elements of $\bar{M}_k^n$. If we let $f = \sum f_x x$ and $h = \sum h_y y$ then $f \vee h = \sum_{x,y} f_x h_y (x \vee y)$. If we fix a basis $\{v_1, v_2, \cdots, v_n\}$ of V then we may define two subsets of L_k^n. We denote the subset of those k-dimensional subspaces containing the first fixed basis vector v_1 by

$$L_{k,1}^n = \{x \in L_k^n : v_1 \in x\}$$

and we denote the set of subspaces which do not contain v_1 by

$$L_{k,2}^n = \{x \in L_k^n : v_1 \notin x\}$$

Thus we define $\bar{M}_{k,1}^n = \bar{F}L_{k,1}^n$ and $\bar{M}_{k,2}^n = \bar{F}L_{k,2}^n$. So we have $L_k^n = L_{k,1}^n \cup L_{k,2}^n$ and $\bar{M}_k^n = \bar{M}_{k,1}^n \oplus \bar{M}_{k,2}^n$. Both $\bar{M}_{k,1}^n$ and $\bar{M}_{k,2}^n$ are $\bar{F}G_{n-1}$-modules. Since $E_1 \in \bar{F}G_{n-1}^*$ we have $\bar{M}_k^n E_1 = \bar{M}_{k,1}^n E_1 \oplus \bar{M}_{k,2}^n E_1$ as $\bar{F}G_{n-1}^*$-modules.

We consider the standard matrix form for subspaces in these two sets. If $x \in L_{k,1}^n$ then the standard form matrix has shape

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ : & & x^{(1)} & \\ 0 & & & \end{pmatrix}$$

where $x^{(1)}$ is a $(k-1) \times (n-1)$ standard form matrix. Otherwise we have $x \in L^n_{k,2}$ with standard form matrix shape

$$\begin{pmatrix} a_1 & \\ \vdots & x^{(2)} \\ a_k & \end{pmatrix}$$

where $x^{(2)}$ is a $k \times (n-1)$ standard form matrix and $a_1, \cdots, a_k$ are elements of $\mathbb{F}$. So the subspace represented by the submatrix $x^{(1)}$ is a $(k-1)$-dimensional subspace of an $(n-1)$-dimensional space whilst the space represented by $x^{(2)}$ is a k-dimensional subspace of an $(n-1)$-dimensional space.

We can now define two maps $\phi_1 : L^n_{k,1} \to L^{n-1}_{k-1}$ and $\phi_2 : L^n_{k,2} \to L^{n-1}_k$ by

$$\phi_1 : x \mapsto x^{(1)},$$

$$\phi_2 : x \mapsto x^{(2)}.$$

These maps may be extended linearly to $\bar{M}^n_k$ so that $\phi_1 : \bar{M}^n_{k,1} \to \bar{M}^{n-1}_{k-1}$ and $\phi_2 : \bar{M}^n_{k,2} \to \bar{M}^{n-1}_k$. Since x is in standard form the submatrices $x^{(1)}$ and $x^{(2)}$ are uniquely determined and they are also in standard form. Hence both maps ϕ_1 and ϕ_2 are well-defined and linear. Indeed it is clear that ϕ_1 is also a bijection. The following result shows that the first idempotent E_1 leaves a subspace unchanged if it contains the first basis vector v_1.

Lemma 4.11. *If $x \in L^n_{k,1}$ then $xE_1 = x$.*

Proof. If we express x in standard matrix form and multiply on the right by

E_1 we get the following

$$\begin{aligned} xE_1 &= \frac{1}{q^{n-1}} \sum_{a^\top \in \mathbb{F}^{n-1}} \begin{pmatrix} 1 & 0 \\ 0 & x^{(1)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & I_{n-1} \end{pmatrix} \\ &= \frac{1}{q^{n-1}} \sum_{a^\top \in \mathbb{F}^{n-1}} \begin{pmatrix} 1 & 0 \\ x^{(1)}a & x^{(1)} \end{pmatrix}. \end{aligned}$$

Since $x^{(1)}$ is already in standard form each of the matrices in the sum row reduces to

$$\begin{pmatrix} 1 & 0 \\ 0 & x^{(1)} \end{pmatrix}$$

and hence we must have

$$xE_1 = x$$

whenever $v_1 \in x$. □

A direct consequence of this result is that if we apply E_1 to $\bar{M}^n_{k,1}$ then this module remains unchanged, so we have $\bar{M}^n_{k,1}E_1 = \bar{M}^n_{k,1}$. For this first part of the branching of $\bar{M}^n_k E_1$ we can describe the following isomorphism.

Lemma 4.12. *We have*

$$\bar{M}^n_{k,1} \cong \bar{M}^{n-1}_{k-1}$$

as $\bar{F}G_{n-1}$-modules.

Proof. If we consider the map ϕ_1 as defined above we have

$$\phi_1 : \bar{M}^n_{k,1}E_1 \to \bar{M}^{n-1}_{k-1}.$$

For $x \in L^n_{k,1}$ if we right multiply by some element $g \in G_{n-1}$ this gives

$$x \cdot g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & x^{(1)} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & g & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & x^{(1)} \cdot g & \\ 0 & & & \end{pmatrix}.$$

Thus $\phi_1(x \cdot g) = x^{(1)} \cdot g = \phi_1(x) \cdot g$ and so ϕ_1 is a G_{n-1}-isomorphism. □

We now consider $\bar{M}^n_{k,2}E_1$ and look at what happens if we take a subspace not containing v_1 and multiply on the right by E_1. We are able to show that the map ϕ_2 defined earlier gives us an isomorphism for this module.

Lemma 4.13. *We have*

$$\bar{M}^n_{k,2}E_1 \cong \bar{M}^{n-1}_k$$

as $\bar{F}G_{n-1}$-modules.

Proof. The subspaces $x \in L^n_{k,2}$ do not contain v_1. So the standard form matrix for x has shape

$$x = \begin{pmatrix} a & x^{(2)} \end{pmatrix}$$

where a is a column vector of height k over $\mathbb{F}$ and where the submatrix $x^{(2)}$ is a uniquely determined $k \times (n-1)$ matrix in standard form. Now

$$xE_1 = \frac{1}{q^{n-1}} \sum_{b^\top \in \mathbb{F}^{n-1}} \begin{pmatrix} a & x^{(2)} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ b & & I_{n-1} & \\ & & & \end{pmatrix}$$

and multiplying this out we have

$$xE_1 = \frac{1}{q^{n-1}} \sum_{b^\top \in \mathbb{F}^{n-1}} \begin{pmatrix} a + x^{(2)}b & x^{(2)} \end{pmatrix}.$$

The matrix for $x^{(2)}$ has full rank k and so as we sum over all column vectors b of height $n-1$ over $\mathbb{F}$ then the products $x^{(2)}b$ run through all column vectors of height k over $\mathbb{F}$ (with multiplicity q^{n-1-k}). Thus the column vector a makes no difference when we calculate xE_1 and we have

$$\begin{aligned} xE_1 &= \frac{1}{q^{n-1}} \sum_{b^\top \in \mathbb{F}^{n-1}} \begin{pmatrix} x^{(2)}b & x^{(2)} \end{pmatrix} \\ &= \frac{q^{n-1-k}}{q^{n-1}} \sum_{b_i \in \mathbb{F}} \begin{pmatrix} b_1 & \\ \vdots & x^{(2)} \\ b_k & \end{pmatrix} \\ &= \frac{1}{q^k} \sum_{b_i \in \mathbb{F}} \begin{pmatrix} b_1 & \\ \vdots & x^{(2)} \\ b_k & \end{pmatrix}. \end{aligned}$$

So $\phi_2(xE_1) = x^{(2)}$ and thus

$$\phi_2 : \bar{M}^n_{k,2}E_1 \to \bar{M}^{n-1}_k.$$

The map is clearly surjective. It is also injective because if $x^{(2)} = y^{(2)}$ then $xE_1 = yE_1$. So ϕ_2 is a bijection. To show that this map is an $\bar{F}G_{n-1}$-

homomorphism we take $x \in L^n_{k,2}$ and right multiply by any $g \in G_{n-1}$. Then

$$x \cdot g = \begin{pmatrix} a_1 & & \\ \vdots & & x^{(2)} \\ a_k & & \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & g & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} a_1 & & \\ \vdots & & x^{(2)} \cdot g \\ a_k & & \end{pmatrix}.$$

This shows that

$$\phi_2(x \cdot g) = x^{(2)} \cdot g = \phi_2(x) \cdot g$$

for all $g \in G_{n-1}$. □

Combining these last two results gives us:

Lemma 4.14. *We have*

$$\bar{M}^n_k E_1 \cong \bar{M}^{n-1}_k \oplus \bar{M}^{n-1}_{k-1}$$

as $\bar{F}G_{n-1}$-modules.

We now consider $\bar{M}^n_k E_2$, the other part of the decomposition for $\bar{M}^n_k$, and show that we can obtain an $\bar{F}G_{n-2}$-isomorphism for this module. First we define the subset $(L^n_k)^\dagger$ of L^n_k to be the set of all subspaces in L^n_k whose standard form matrix has shape:

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ a_k & 0 & & & \end{pmatrix}$$

where $x^\dagger$ is a standard $(k-1)\times(n-2)$ matrix and $a_1, \cdots, a_k \in \mathbb{F}$. The following result is also a specific case of the result [9, Thm 10.2], but the proof given here is independent.

Lemma 4.15. *If* $x \in L_k^n$ *then we have* $xE_2 = 0$ *unless the standard form matrix for* x *is in* $(L_k^n)^\dagger$.

Proof. We look at three possible types of standard form matrices and deal with each of these individually:

Case 1:

First of all we consider the situation when $v_1 \in x$. When we apply E_2 to the standard form matrix for a subspace of this type this gives

$$xE_2 = \frac{1}{q^{2n-3}} \sum_{\alpha_i,\beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & x^{(1)} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 1 & 0 & \cdots & 0 \\ \alpha_3 & \beta_3 & & & \\ \vdots & \vdots & & I_{n-2} & \\ \alpha_n & \beta_n & & & \end{pmatrix}$$

$$= \frac{1}{q^{2n-3}} \sum_{\alpha_i,\beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & & & \\ \vdots & \vdots & & x^{(1)} & \\ \gamma_k & \delta_k & & & \end{pmatrix}$$

where

$$\gamma_i = x_{i2}\alpha_2 + x_{i3}\alpha_3 + \cdots + x_{in}\alpha_n,$$

$$\delta_i = x_{i2} + x_{i3}\beta_3 + \cdots + x_{in}\beta_n$$

where x_{ij} denotes the (i,j)-entry of the standard matrix for x. Now since $x^{(1)}$ is in standard form, if we row reduce these matrices into standard form we get

$$xE_2 = \frac{1}{q^{2n-3}} \sum_{\alpha_i,\beta_j\in\mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \delta_2 & & & \\ \vdots & \vdots & & x^{(1)} & \\ 0 & \delta_k & & & \end{pmatrix}.$$

The crucial thing to note is that the matrices in the sum have no dependence on α_2. Since $\sum_{\alpha_2\in\mathbb{F}} \chi_2(\alpha_2) = 0$ then we must have $xE_2 = 0$.

Case 2:

The second case we consider is when x is in $(L_k^n)^\dagger$, so the standard matrix has shape:

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ a_k & 0 & & & \end{pmatrix}.$$

When we apply E_2 we get

$$xE_2 = \frac{1}{q^{2n-3}} \sum_{\alpha_i,\beta_j\in\mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ a_k & 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 1 & 0 & \cdots & 0 \\ \alpha_3 & \beta_3 & & & \\ \vdots & \vdots & & I_{n-2} & \\ \alpha_n & \beta_n & & & \end{pmatrix}$$

which we can express as

$$\frac{1}{q^{2n-3}} \sum_{\alpha_i,\beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} a_1+\alpha_2 & 1 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & & & \\ \vdots & \vdots & & x^\dagger & \\ \gamma_k & \delta_k & & & \end{pmatrix}$$

where

$$\gamma_i = a_i + x_{i3}\alpha_3 + \cdots + x_{in}\alpha_n,$$

$$\delta_i = x_{i3}\beta_3 + \cdots + x_{in}\beta_n$$

where again x_{ij} denotes the (i,j)-entry of x. By row-reducing these matrices we get

$$\frac{1}{q^{2n-3}} \sum_{\alpha_i,\beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} a_1+\alpha_2 & 1 & 0 & \cdots & 0 \\ \gamma_2 - \delta_2(a_1+\alpha_2) & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ \gamma_k - \delta_k(a_1+\alpha_2) & 0 & & & \end{pmatrix}.$$

We note that this time there is a dependency on α_2 in the matrices in the sum and so we cannot say that xE_2 is zero.

Case 3:

The final case to consider is all other possible standard form matrices. These must have the following shape:

$$\begin{pmatrix} a_1 & b_1 & \\ \vdots & \vdots & x^* \\ a_k & b_k & \end{pmatrix}$$

where x^* is a standard $k \times (n-2)$ matrix. When we apply E_2 to a matrix of this shape we get

$$xE_2 = \frac{1}{q^{2n-3}} \sum_{\alpha_i, \beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} a_1 & b_1 & \\ \vdots & \vdots & x^* \\ a_k & b_k & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 1 & 0 & \cdots & 0 \\ \alpha_3 & \beta_3 & & & \\ \vdots & \vdots & & I_{n-2} & \\ \alpha_n & \beta_n & & & \end{pmatrix}.$$

This gives

$$xE_2 = \frac{1}{q^{2n-3}} \sum_{\alpha_i, \beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} \gamma_1 & \delta_1 & \\ \vdots & \vdots & x^* \\ \gamma_k & \delta_k & \end{pmatrix}$$

where

$$\gamma_i = a_i + b_i\alpha_2 + x_{i3}\alpha_3 + \cdots + x_{in}\alpha_n,$$

$$\delta_i = b_i + x_{i3}\beta_3 + \cdots + x_{in}\beta_n.$$

Since the sum is over all α_i and β_j in $\mathbb{F}$ then if we fix α_2 and define

$$\gamma_i^* = a_i + x_{i3}\alpha_3 + \cdots + x_{in}\alpha_n.$$

we have that for every fixed i each γ_i^* corresponds to some γ_i. Hence we may express xE_2 as

$$xE_2 = \frac{1}{q^{2n-3}} \sum_{\alpha_2 \in \mathbb{F}} \chi_2(\alpha_2) \sum_{\alpha_3, \cdots, \alpha_n, \beta_j \in \mathbb{F}} \begin{pmatrix} \gamma_1^* & \delta_1 & \\ \vdots & \vdots & x^* \\ \gamma_k^* & \delta_k & \end{pmatrix}.$$

Since $\sum_{\alpha_2 \in \mathbb{F}} \chi_2(\alpha_2) = 0$ then we must have $xE_2 = 0$. $\square$

Any $x \in (L_k^n)^\dagger$ has standard form matrix shape

$$x = \begin{pmatrix} x_{11} & 1 & 0 & \cdots & 0 \\ x_{21} & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ x_{k1} & 0 & & & \end{pmatrix}$$

and thus we define $x^\ddagger$ to be the space with standard matrix form

$$x^\ddagger = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ 0 & 0 & & & \end{pmatrix}.$$

From the above we see that if $x \in (L_k^n)^\dagger$ then

$$xE_2 = \frac{1}{q^k} \sum_{\alpha_i, \gamma_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} x_{11} + \alpha_2 & 1 & 0 & \cdots & 0 \\ \gamma_2 & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ \gamma_k & 0 & & & \end{pmatrix}$$

and similarly

$$x^\ddagger E_2 = \frac{1}{q^k} \sum_{\alpha_i, \gamma_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} \alpha_2 & 1 & 0 & \cdots & 0 \\ \gamma_2 & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ \gamma_k & 0 & & & \end{pmatrix}.$$

Since χ_2 is a linear character of $(\mathbb{F}, +)$ we have

$$\chi_2(x_{11} + \alpha_2) = \chi_2(x_{11}) \cdot \chi_2(\alpha_2).$$

giving

$$\chi_2(x_{11}) \cdot xE_2 = x^{\ddagger}E_2.$$

Thus for any $f = \sum_{x \in L_k^n} f_x x \in \bar{M}_k^n$ we have

$$fE_2 = \sum_{x \in (L_k^n)^{\dagger}} f_x x E_2 = \sum_{x \in (L_k^n)^{\dagger}} \frac{f_x}{\chi_2(x_{11})} x^{\ddagger} E_2.$$

For every such element $f \in \bar{M}_k^n$ we can define

$$f^{\ddagger} = \sum_{x \in (L_k^n)^{\dagger}} \frac{f_x}{\chi_2(x_{11})} x^{\ddagger}$$

so that $fE_2 = f^{\ddagger}E_2$. If we let

$$(L_k^n)^{\ddagger} = \left\{ \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ : & : & & x^{\dagger} & \\ 0 & 0 & & & \end{pmatrix} : x^{\dagger} \in L_{k-1}^{n-2} \right\}$$

and put $(M_k^n)^{\ddagger} = \bar{F}(L_k^n)^{\ddagger}$ then we have

$$\bar{M}_k^n E_2 = (M_k^n)^{\ddagger} E_2.$$

We now use this information to construct an isomorphism for the module $\bar{M}_k^n E_2$.

Lemma 4.16. *We have*

$$\bar{M}_k^n E_2 \cong \bar{M}_{k-1}^{n-2}$$

as $\bar{F}G_{n-2}$-modules.

Proof. By our previous discussion we have shown that $\bar{M}_k^n E_2 = (M_k^n)^\ddagger E_2$ and that for any $f \in \bar{M}_k^n$ we have $fE_2 = f^\ddagger E_2$ where $f^\ddagger \in (M_k^n)^\ddagger$ is defined as above. We define a map $\phi_3 : (M_k^n)^\ddagger E_2 \to \bar{M}_{k-1}^{n-2}$ by mapping any such element

$$f^\ddagger E_2 = \sum_{x \in (L_k^n)^\ddagger} f_x (v_2 \vee x^\dagger) E_2 \in (M_k^n)^\ddagger E_2$$

to

$$\phi_3(f^\ddagger E_2) = \sum_{x \in (L_k^n)^\ddagger} f_x x^\dagger.$$

The map ϕ_3 is clearly surjective. If $\sum_{x \in (L_k^n)^\ddagger} f_x x^\dagger = \sum_{x \in (L_k^n)^\ddagger} h_x x^\dagger$ then $f^\ddagger E_2 = h^\ddagger E_2$ so ϕ_3 is also injective. For each $x \in (L_k^n)^\ddagger$ in standard form the space $x^\dagger$ is uniquely defined and in standard form. So ϕ_3 is well-defined. It is also an $\bar{F}G_{n-2}$-homomorphism for any $g \in G_{n-2}$ commutes with E_2 and so $(f^\ddagger E_2) \cdot g = (f^\ddagger \cdot g)E_2$. If we multiply any $x \in (L_k^n)^\ddagger$ on the right by any $g \in G_{n-2}$ then we have

$$x \cdot g = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ : & : & & x^\dagger & \\ 0 & 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ : & : & & g & \\ 0 & 0 & & & \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ : & : & & x^\dagger \cdot g & \\ 0 & 0 & & & \end{pmatrix}.$$

So for any $g \in G_{n-2}$ we have

$$\phi_3(f^\ddagger E_2 \cdot g) = \phi_3(f^\ddagger \cdot gE_2) = \phi(f^\ddagger E_2) \cdot g$$

showing that ϕ_3 is an $\bar{F}G_{n-2}$-isomorphism. □

Combining these results with the branching rule of Theorem 4.10 gives us the following decomposition for the permutation module $\bar{M}_k^n$. This result corresponds to the much more general result [9, Cor 10.16] of James for the permutation module associated to arbitrary compositions.

Theorem 4.17. *We have*

$$\bar{M}_k^n \cong \bar{M}_k^{n-1} \oplus \bar{M}_{k-1}^{n-1} \oplus \bar{M}_{k-1}^{n-2} G_{n-1}^*$$

as $\bar{F}G_{n-1}$-modules.

Recall that we have defined the following subgroups

$$G_{n-2} \subseteq H_{n-1}^* \subseteq G_{n-1}^*$$

of the general linear group G_n with the index

$$\left| G_{n-1}^* : H_{n-1}^* \right| = q^{n-1} - 1.$$

Then we can make the following definition

Definition 4.18. Let M be any $\bar{F}H_{n-1}^*$-module. We define $M(q^{n-1}-1)$ to be the module induced from M to an $\bar{F}G_{n-1}^*$-module.

We can show that the module $\bar{M}_k^n E_2 G_{n-1}^*$ is isomorphic to the module induced from the $\bar{F}H_{n-1}^*$-module $\bar{M}_k^n E_2$ to an $\bar{F}G_{n-1}^*$-module.

Lemma 4.19. *We have a G_{n-1}-isomorphism*

$$\bar{M}_k^n E_2 G_{n-1}^* \cong \bar{M}_k^n E_2 (q^{n-1} - 1).$$

Proof. Our first step is to show that $\bar{M}_k^n E_2$ is indeed an $\bar{F}H_{n-1}^*$-module. Take any matrix $h \in H_{n-1}^*$. We claim that this commutes with E_2. The matrix for h has shape

$$h = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2 & 1 & 0 & \cdots & 0 \\ a_3 & b_3 & & & \\ \vdots & \vdots & & g & \\ a_n & b_n & & & \end{pmatrix}$$

where $a_i, b_j \in \mathbb{F}$ and $g \in G_{n-2}$. If we take E_2 and multiply on the right by h then we have

$$E_2 h = \frac{1}{q^{2n-3}} \sum_{\alpha_i, \beta_j \in \mathbb{F}} \chi(\alpha_2) \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 1 & 0 & \cdots & 0 \\ \alpha_3 & \beta_3 & & & \\ \vdots & \vdots & & I_{n-2} & \\ \alpha_n & \beta_n & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2 & 1 & 0 & \cdots & 0 \\ a_3 & b_3 & & & \\ \vdots & \vdots & & g & \\ a_n & b_n & & & \end{pmatrix}$$

which we may express as

$$E_2 h = \frac{1}{q^{2n-3}} \sum_{\alpha_2, a_2, * \in \mathbb{F}} \chi(\alpha_2) \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_2 + a_2 & 1 & 0 & \cdots & 0 \\ * & * & & & \\ \vdots & \vdots & & g & \\ * & * & & & \end{pmatrix}.$$

The $*$ in the resultant matrix are $\mathbb{F}$-sums of the α_i, β_j, a_i and b_j's. But since we are summing over all such possible entries from $\mathbb{F}$ then in the resultant matrix we are also summing over all possible entries $* \in \mathbb{F}$. If we multiply on the left by h we get

$$hE_2 = \frac{1}{q^{2n-3}} \sum_{\alpha_2, a_2, * \in \mathbb{F}} \chi(\alpha_2) \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_2 + \alpha_2 & 1 & 0 & \cdots & 0 \\ * & * & & & \\ \vdots & \vdots & & g & \\ * & * & & & \end{pmatrix}.$$

We see that E_2 commutes with the group H^*_{n-1}. Thus $E_2 H^*_{n-1} = H^*_{n-1} E_2$ and so we have

$$\bar{M}^n_k E_2 H^*_{n-1} = \bar{M}^n_k E_2$$

So $\bar{M}^n_k E_2$ is an $\bar{F} H^*_{n-1}$-module.

Since $\left|G^*_{n-1} : H^*_{n-1}\right| = q^{n-1} - 1$ we can express

$$G^*_{n-1} = \bigcup_{i=1}^{q^{n-1}-1} H^*_{n-1} \cdot g_i$$

for a set $\{g_1, g_2, \cdots, g_{q^{n-1}-1}\}$ of right coset representatives for H^*_{n-1} in G^*_{n-1}.

Now if we consider our module $\bar{M}^n_k E_2 G^*_{n-1}$ we have

$$\begin{aligned} \bar{M}^n_k E_2 G^*_{n-1} &= \sum_{i=1}^{q^{n-1}-1} \bar{M}^n_k E_2 H^*_{n-1} \cdot g_i \\ &= \sum_{i=1}^{q^{n-1}-1} \bar{M}^n_k E_2 \cdot g_i. \end{aligned}$$

We are now required to show that this right hand side is a direct sum, which we may do by considering the dimensions of both sides. By Theorem 4.10 (ii) we have

$$\dim \bar{M}^n_k = \dim M^n_k E_1 + (q^{n-1} - 1) \dim \bar{M}^n_k E_2$$

which shows that $\dim \bar{M}_k^n E_2 G_{n-1}^* = (q^{n-1} - 1) \dim \bar{M}_k^n E_2$ and since both sides have the same dimension then $\sum_{i=1}^{q^{n-1}-1} \bar{M}_k^n E_2 \cdot g_i$ is a direct sum as required. □

We conclude the following result:

Theorem 4.20. *There is an $\bar{F}G_{n-1}$-isomorphism*

$$\bar{M}_k^n \cong \bar{M}_k^{n-1} \oplus \bar{M}_{k-1}^{n-1} \oplus \bar{M}_{k-1}^{n-2}(q^{n-1} - 1).$$

Note that by taking dimensions of this branching rule we have a formula which corresponds with the result of Lemma 3.8.

In Chapter 8 we give two examples which describe how this isomorphism works in certain low-dimensional cases and help to illustrate some of the theory in this chapter.

5 The Homology Module

We now consider the homology module $H^n_{k,i}$ and look at how we can obtain a branching rule similar to the one obtained for the permutation module $\bar{M}^n_k$ in the previous chapter. If we take an element which is contained in the kernel $K^n_{k,i}$ then this is still a linear combination of k-dimensional subspaces. An important tool in being able to construct a decomposition for these kernels is having a good understanding of the inclusion map, ∂. In the first part of this chapter we identify this inclusion map in terms of matrices. The homology module is itself an FG_n-module and thus, as before, if our field F is extended to contain a primitive $p'th$ root of unity then we may apply the general branching rule of James for any $\bar{F}G_n$-module. Using this branching rule we can obtain a decomposition for $H^n_{k,i}$ into $\bar{F}G_{n-1}$-modules. Taking the dimensions of these modules, the branching rule coincides with one of the Betti number relations found in Lemma 3.9.

5.1 Some Definitions Revisited

Before considering the homology module itself we need to look at how the inclusion map ∂ as defined in Chapter 2 can be represented in terms of matrices. Consider the formal combination of matrices

$$\partial_k := \sum_i B_i$$

where the B_i's are all 'standard' $(k-1) \times k$ matrices over $\mathbb{F}$. If we take such a matrix B_i and multiply on the right by the standard form matrix of a k-dimensional subspace x, then we see that each matrix product $B_i \cdot x$ is the product of a $(k-1) \times k$ 'standard' matrix and a $k \times n$ 'standard' matrix. In Lemma 4.8 we have proven that the product of two standard matrices is itself a standard matrix. So $B_i \cdot x$ is a $(k-1) \times n$ matrix in standard form. For a given subspace x the matrix corresponding to this subspace has rank k and hence has some right inverse z. So we have $x \cdot z = I_k$ (the $k \times k$ identity matrix) and thus if $B_i \cdot x = B_j \cdot x$ then $B_i \cdot x \cdot z = B_j \cdot x \cdot z$ giving $B_i = B_j$. This means that the matrix products $B_i \cdot x$ are pairwise distinct. In Lemma 4.6 we have shown that there is a bijection between the set of $(k-1) \times n$ matrices in standard form and the set of $(k-1)$-dimensional subspaces of an n-dimensional space. Therefore each matrix product $B_i \cdot x$ corresponds to a unique subspace y contained in x with dimension $k-1$. Since there are $\binom{k}{k-1}_q$ such subspaces y contained in x and exactly this number of matrices B_i then we have the following result which identifies ∂_k with the inclusion map ∂ we defined in Chapter 2.

Lemma 5.1. *For any k-dimensional subspace $x \in L_k^n$ we have*

$$\partial_k \cdot x = \partial(x) = \sum_{y \subset x, y \in L_{k-1}^n} y.$$

The subscript k denotes that the operator ∂_k is being applied to k-dimensional subspaces, so we have $(k-1) \times k$ matrices in the right hand side. Clearly the size of the matrices in the operator will depend on the size of the subspaces they are going to be applied to. This identification of the inclusion map ∂ in terms of matrices clearly illustrates why the map is an

$\bar{F}G_n$-homomorphism since $\partial_k \cdot (x \cdot g) = (\partial_k \cdot x) \cdot g$ by the associative law for matrix products.

We can split the matrices in ∂_k into two types, those with top row $(1, 0, \cdots, 0)$ and those with different top row. In the first case we have matrices of shape

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ : & & x^{(1)} & \\ 0 & & & \end{pmatrix}$$

where the matrices $x^{(1)}$ are $(k-2) \times (k-1)$ matrices also in standard form. If we take the sum of all such matrices $x^{(1)}$ these give ∂_{k-1}, the ∂-map applied to subspaces of one less dimension. In the second case the matrices have shape

$$\begin{pmatrix} a_1 & & \\ : & & x^{(2)} \\ a_{k-1} & & \end{pmatrix}.$$

This time the matrices $x^{(2)}$ are standard $(k-1) \times (k-1)$ matrices. For these matrices no leading 1 appears in the left hand column and so each of the remaining columns contains a leading 1. Hence there is in fact only one such matrix $x^{(2)}$, the $(k-1) \times (k-1)$ identity matrix. We define

$$\partial_k^{(1)} := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ : & & \partial_{k-1} & \\ 0 & & & \end{pmatrix}$$

which means that if we take an element of ∂_{k-1} which is a formal combination of $(k-2) \times (k-1)$ matrices then each of the matrices in this combination is made into a $(k-1) \times k$ matrix by adding a left hand column of zeros and then a top row $(1, 0, \cdots, 0)$. We also define

$$\partial_k^{(2)} := \sum_{a_i \in \mathbb{F}} \begin{pmatrix} a_1 & \\ : & I_{k-1} \\ a_{k-1} & \end{pmatrix}$$

then we have

$$\partial_k = \partial_k^{(1)} + \partial_k^{(2)}.$$

We now consider what happens when we apply the ∂-map a given number i many times. Recall that for $x \in L_k^n$ we have

$$\partial^i(x) = (i!)_q \sum y$$

where the sum runs over all $y \in L_{k-i}^n$ with $y \subset x$. Define

$$\partial_{k,i} := (i!)_q \sum_j C_j$$

where the C_j's are all standard $(k-i) \times k$ matrices over $\mathbb{F}$. Similarly to above we define

$$\partial_{k,i}^{(1)} := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ : & & \partial_{k-1,i} & \\ 0 & & & \end{pmatrix}$$

to be the formal combination of $(k-i-1) \times (k-1)$ matrices in $\partial_{k-1,i}$ extended to a combination of $(k-i) \times k$ matrixes by adding a left hand column of zeros and a top row $(1, 0, \cdots, 0)$. We also define

$$\partial_{k,i}^{(2)} := \sum_{a_i \in \mathbb{F}} \begin{pmatrix} a_1 & \\ \vdots & \partial_{k-1,i-1} \\ a_{k-i} & \end{pmatrix}$$

to be the combination of matrices obtained by taking the formal combination of all $(k-i) \times (k-1)$ standard form matrices, and then for each possible column vector $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-i} \end{pmatrix}$ where $a_i \in \mathbb{F}$, extending these matrices by adding the column vector as the first column of a $(k-i) \times k$ matrix.

Using such elements as above we are able to obtain the following useful lemma which identifies the homomorphism ∂^i with $\partial_{k,i}$.

Lemma 5.2. *We have the relation*

$$\partial^i = \partial_{k,i} = \partial_{k,i}^{(1)} + [i]_q \partial_{k,i}^{(2)}.$$

Proof. By the same argument as for the 1-step map we have that each matrix product $C_j \cdot x$ is a standard $(k-i) \times n$ matrix and that these matrices are pairwise distinct. So each matrix product $C_j \cdot x$ corresponds to a unique subspace $y \subset x$ with $y \in L_{k-i}^n$. Since there are $\binom{k}{k-i}_q$ such subspaces y we have that if $x \in L_k^n$ then

$$\partial_{k,i} \cdot x = \partial^i(x).$$

Again we can split the matrices in $\partial_{k,i}$ into two types: those which have top row $(1, 0, \cdots, 0)$ and those which do not. The first case gives matrices of shape

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & x^{(1)} & \\ 0 & & & \end{pmatrix}$$

where the matrices $x^{(1)}$ are all standard $(k-i-1)\times(k-1)$ matrices. These correspond to the matrices in $\partial_{k-1,i}$. The second case gives matrices of shape

$$\begin{pmatrix} a_1 & & \\ \vdots & & x^{(2)} \\ a_{k-i} & & \end{pmatrix}$$

where the matrices $x^{(2)}$ are all standard $(k-i)\times(k-1)$ matrices and hence correspond to the matrices in $\partial_{k-1,i-1}$. So we have

$$\partial_{k,i} = (i!)_q \left(\sum_{x^{(1)}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & x^{(1)} & \\ 0 & & & \end{pmatrix} + \sum_{a_i \in \mathbb{F}, x^{(2)}} \begin{pmatrix} a_1 & & \\ \vdots & & x^{(2)} \\ a_{k-i} & & \end{pmatrix} \right)$$

where the first sum runs through all standard form matrices $x^{(1)}$ of size $(k-i-1)\times(k-1)$ and the second sum runs through all standard form matrices $x^{(2)}$ of size $(k-i)\times(k-1)$. If we recall the definition of $\partial_{k,i}^{(1)}$ we have

$$\partial_{k,i}^{(1)} := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \partial_{k-1,i} & \\ 0 & & & \end{pmatrix} = (i!)_q \sum_{B_j} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B_j & \\ 0 & & & \end{pmatrix}$$

where the B_j run through all standard form $(k-i-1)\times(k-1)$ matrices. Similarly we have

$$\partial^{(2)}_{k,i} := \sum_{a_i\in\mathbb{F}} \begin{pmatrix} a_1 & \\ \vdots & \partial_{k-1,i-1} \\ a_{k-i} & \end{pmatrix} = ((i-1)!)_q \sum_{a_i\in\mathbb{F},C_i} \begin{pmatrix} a_1 & \\ \vdots & C_j \\ a_{k-i} & \end{pmatrix}$$

where the matrices C_j run through all standard form $(k-i)\times(k-1)$ matrices. Thus we can see that we have

$$\begin{aligned}\partial_{k,i} &= (i!)_q \left(\frac{1}{(i!)_q}\partial^{(1)}_{k,i} + \frac{1}{((i-1)!)_q}\partial^{(2)}_{k,i}\right)\\ &= \partial^{(1)}_{k,i} + [i]_q\partial^{(2)}_{k,i}\end{aligned}$$

as required. □

We can use this matrix form $\partial_{k,i}$ of the homomorphism ∂^i to look at what happens when the map is applied to different types of subspaces and hence to different shapes of standard form matrices. Since we have used the notation L^n_k to denote the set of k-dimensional subspaces of $V = \langle v_1, v_2, \cdots, v_n\rangle$ we will now use L^{n-1}_t to denote the set of t-dimensional subspaces of $\langle v_2, \cdots, v_n\rangle$. Similarly we shall use the notation L^{n-2}_t to denote the set of t-dimensional subspaces of $\langle v_3, \cdots, v_n\rangle$. We now prove the following two results which will be useful to us later in the chapter.

Lemma 5.3. *If $x = \langle v_1\rangle \vee x^{(1)}$ with $x^{(1)} \in L^{n-1}_{k-1}$ then*

$$\partial_{k,i}\cdot(\langle v_1\rangle \vee x^{(1)})\cdot E_1 = (\langle v_1\rangle \vee \partial_{k-1,i}\cdot x^{(1)}) + q^{k-i}[i]_q\partial_{k-1,i-1}\cdot x^{(1)}\cdot E_1.$$

Note: To make sense of the right hand side of this statement in the Lemma we must say that we may apply E_1 to such a $k \times (n-1)$ matrix $x^{(1)}$ by considering $x^{(1)}$ to be a $k \times n$ matrix with left hand column consisting of zeros. We can then apply the idempotent $E_1 \in \bar{F}G_n$ by right multiplication in the usual way.

Proof. By Lemma 5.2 we have that $\partial_{k,i} = \partial_{k,i}^{(1)} + [i]_q \partial_{k,i}^{(2)}$. Clearly

$$
\begin{aligned}
\partial_{k,i}^{(1)} \cdot (\langle v_1 \rangle \vee x^{(1)}) &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \partial_{k-1,i} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & x^{(1)} & \\ 0 & & & \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \partial_{k-1,i} \cdot x^{(1)} & \\ 0 & & & \end{pmatrix} \\
&= \langle v_1 \rangle \vee \partial_{k-1,i} \cdot x^{(1)}.
\end{aligned}
$$

Recall that we have proven in Lemma 4.11 that if we multiply matrices of this shape on the right by E_1 then they are left unchanged. So we have

$$(\langle v_1 \rangle \vee \partial_{k-1,i} \cdot x^{(1)}) \cdot E_1 = \langle v_1 \rangle \vee \partial_{k-1,i} \cdot x^{(1)}.$$

Applying $\partial^{(2)}_{k,i}$ to $\langle v_1\rangle \vee x^{(1)}$ gives

$$\partial^{(2)}_{k,i} \cdot (\langle v_1\rangle \vee x^{(1)}) = \sum_{a_i\in\mathbb{F}} \begin{pmatrix} a_1 & \\ \vdots & \partial_{k-1,i-1} \\ a_{k-i} & \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & x^{(1)} & \\ 0 & & & \end{pmatrix}$$

and multiplying this out we get

$$\partial^{(2)}_{k,i} \cdot (\langle v_1\rangle \vee x^{(1)}) = \sum_{a_i\in\mathbb{F}} \begin{pmatrix} a_1 & \\ \vdots & \partial_{k-1,i-1} \cdot x^{(1)} \\ a_{k-i} & \end{pmatrix}.$$

Now we apply the idempotent E_1. In the proof of Lemma 4.13 we have shown that

$$\begin{pmatrix} a_1 & \\ \vdots & x^{(1)} \\ a_k & \end{pmatrix} E_1 = \begin{pmatrix} 0 & \\ \vdots & x^{(1)} \\ 0 & \end{pmatrix} E_1.$$

So each matrix in the above sum gives the same as $(\partial_{k-1,i-1} \cdot x^{(1)}) \cdot E_1$ (where we consider $\partial_{k-1,i-1} \cdot x^{(1)}$ to be a $(k-i) \times n$ matrix with left hand column consisting of zeros). Hence we have

$$\partial^{(2)}_{k,i} \cdot (\langle v_1\rangle \vee x^{(1)})E_1 = q^{k-i}\partial_{k-1,i-1} \cdot x^{(1)} \cdot E_1.$$

Adding the two parts together gives the required result. □

Lemma 5.4. *If $x^\dagger \in L^{n-2}_{k-1}$ then*

$$\partial_{k,i} \cdot (\langle v_2\rangle \vee x^\dagger) \cdot E_2 = (\langle v_2\rangle \vee \partial_{k-1,i} \cdot x^\dagger) \cdot E_2.$$

Note that in the same way as we have remarked above for E_1, we may also apply E_2 to elements such as $\langle v_2 \rangle \vee \partial^i(x^\dagger)$ etc. by considering these to be $k \times n$ matrices with a left hand column of zeros.

Proof. Again we use the result of Lemma 5.2 that $\partial_{k,i} := \partial_{k,i}^{(1)} + [i]_q \partial_{k,i}^{(2)}$. We consider what happens when we take each part and multiply on the right by the matrix representing $\langle v_2 \rangle \vee x^\dagger$. The first part gives

$$\partial_{k,i}^{(1)} \cdot (\langle v_2 \rangle \vee x^\dagger) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \partial_{k-1,i} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ 0 & 0 & & & \end{pmatrix}$$

and multiplying this out we have

$$\begin{aligned} \partial_{k,i}^{(1)} \cdot (\langle v_2 \rangle \vee x^\dagger) &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \partial_{k-1,i} \cdot x^\dagger & \\ 0 & 0 & & & \end{pmatrix} \\ &= \langle v_2 \rangle \vee \partial_{k-1,i} \cdot x^\dagger. \end{aligned}$$

For the second part,

$$\partial_{k,i}^{(2)} \cdot (\langle v_2 \rangle \vee x^\dagger) = \sum_{a_i \in \mathbb{F}} \begin{pmatrix} a_1 & & \\ \vdots & & \partial_{k-1,i-1} \\ a_{k-i} & & \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & x^\dagger & \\ 0 & 0 & & & \end{pmatrix}$$

and hence

$$\partial^{(2)}_{k,i} \cdot (\langle v_1 \rangle \vee x^\dagger) = \sum_{a_i \in \mathbb{F}} \begin{pmatrix} 0 & a_1 & \\ \vdots & \vdots & \partial_{k-1,i-1} \cdot x^\dagger \\ 0 & a_{k-i} & \end{pmatrix}.$$

Since $\partial_{k-1,i-1} \cdot x^\dagger$ must have rank $k-i$ it is clear that when we row-reduce the matrices in $\partial^{(2)}_{k,i} \cdot (\langle v_2 \rangle \vee x^\dagger)$ they are not in the set $(L^n_k)^\dagger$ as described in the previous chapter. Hence if we apply E_2 by right multiplication then by Lemma 4.15 we have that

$$\partial^{(2)}_{k,i} \cdot (\langle v_2 \rangle \vee x^\dagger) \cdot E_2 = 0$$

and hence

$$\partial_{k,i} \cdot (\langle v_2 \rangle \vee x^\dagger) \cdot E_2 = (\langle v_2 \rangle \vee \partial_{k-1,i} \cdot x^\dagger) \cdot E_2$$

as required. □

5.2 The Decomposition of $H^n_{k,i}$

These initial results concerning how the map ∂ interacts with different types of subspaces are useful to us as we now aim to construct a decomposition for the homology module $H^n_{k,i}$. In the same way as we did in the previous chapter for the permutation module $\bar{M}^n_k$ we must look at how $H^n_{k,i}$ decomposes when we apply the idempotents E_1 and E_2.

To be able to apply the branching rule of James we again have to ensure that our field is large enough. Recall the definition of $\bar{F}$ in Definition 4.1 to be the field F extended by adjoining a primitive $p'th$ root of unity. We define $\bar{K}^n_{k,i}$, $\bar{I}^n_{k,i}$ and $\bar{H}^n_{k,i}$ to be the corresponding kernel, image and homology module over this larger field, $\bar{F}$. So we have

$$\bar{K}^n_{k,i} := \ker \partial^i \cap \bar{M}^n_k,$$

$$\bar{I}^n_{k,i} := \partial^{\pi-i}(\bar{M}^n_{k+\pi-i}),$$

and

$$\bar{H}^n_{k,i} := \bar{K}^n_{k,i}/\bar{I}^n_{k,i}.$$

This homology module $\bar{H}^n_{k,i}$ is an $\bar{F}G_n$-module and so we can apply the branching rule of Theorem 4.4 to give us a G^*_{n-1}-decomposition

$$\bar{H}^n_{k,i} = \bar{H}^n_{k,i}E_1 \oplus \sum_{r=2}^{n} \bar{H}^n_{k,i}E_r G^*_{n-1}.$$

This can be simplified since we know that $\bar{M}^n_k E_r = 0$ for $r > 2$ which implies that we must also have $\bar{H}^n_{k,i}E_r = 0$ for $r > 2$. Hence

$$\bar{H}^n_{k,i} = \bar{H}^n_{k,i}E_1 \oplus \bar{H}^n_{k,i}E_2 G^*_{n-1}.$$

We have similar decompositions for both $\bar{K}^n_{k,i}$ and $\bar{I}^n_{k,i}$ as these are also $\bar{F}G_n$-modules. So we have

$$\bar{K}^n_{k,i} = \bar{K}^n_{k,i}E_1 \oplus \bar{K}^n_{k,i}E_2 G^*_{n-1}$$

and

$$\bar{I}^n_{k,i} = \bar{I}^n_{k,i}E_1 \oplus \bar{I}^n_{k,i}E_2 G^*_{n-1}.$$

We consider the first part of each of these two decompositions where we apply the idempotent element E_1 to the modules $\bar{K}^n_{k,i}$ and $\bar{I}^n_{k,i}$. The following result will be useful.

Lemma 5.5. *If* $f = \sum_{x \in L_k^{n-1}} f_x x$ *where* $f_i \in \bar{F}$ *then*

$$fE_1 = 0 \text{ if and only if } f = 0.$$

Proof. We consider the matrices in the sum for f as $k \times n$ standard form matrices with left hand column consisting of zeros. Then we may apply the idempotent E_1 to give

$$fE_1 = \frac{1}{q^{n-1}} \sum_{x \in L_k^{n-1}} f_x \begin{pmatrix} 0 & \\ \vdots & x \\ 0 & \end{pmatrix} \sum_{a^\top \in \mathbb{F}^{n-1}} \begin{pmatrix} 1 & 0 \quad \cdots \quad 0 \\ & \\ a & I_{n-1} \\ & \end{pmatrix}$$

$$= \frac{1}{q^{n-1}} \sum_{x \in L_k^{n-1}} f_x \left(\sum_{a^\top \in \mathbb{F}^{n-1}} \begin{pmatrix} x \cdot a & x \end{pmatrix} \right).$$

Since the matrix for x has rank k then as we sum over all column vectors a of height $n-1$ over $\mathbb{F}$ the products $x \cdot a$ give all column vectors of height k over $\mathbb{F}$ with multiplicity q^{n-1-k}. Hence

$$fE_1 = \frac{q^{n-1-k}}{q^{n-1}} \sum_{x \in L_k^{n-1}} f_x \left(\sum_{b^\top \in \mathbb{F}^k} \begin{pmatrix} b & x \end{pmatrix} \right)$$

$$= \frac{1}{q^k} \sum_{b^\top \in \mathbb{F}^k} \left(\sum_{x \in L_k^{n-1}} f_x \begin{pmatrix} b & x \end{pmatrix} \right).$$

As x is a standard form matrix it is clear that every matrix in the above sum is already in standard form without the need to apply any row operations. If $fE_1 = 0$ then the inner sum equals zero for any fixed column vector b. So if we fix $b = 0$ we get

$$\sum_x f_x \begin{pmatrix} 0 & & \\ : & x & \\ 0 & & \end{pmatrix} = 0$$

and hence we have $f = 0$.

The other way is obvious by associativity of matrix multiplication. □

In the previous chapter we showed how we can express $\bar{M}^n_k = \bar{M}^n_{k,1} \oplus \bar{M}^n_{k,2}$ where $\bar{M}^n_{k,1}$ consists of $\bar{F}$-linear combinations of subspaces containing v_1 and $\bar{M}^n_{k,2}$ consists of linear combinations of those subspaces which do not. Thus any element $f \in \bar{M}^n_k$ is a sum of elements from $\bar{M}^n_{k,1}$ and $\bar{M}^n_{k,2}$. For any element $f = \sum_x f_x x \in \bar{M}^n_k$ we define the *support* of f to be the subspace generated by all x for which $f_x \neq 0$ and we denote this by *supp*(f). Hence we can express any $f \in \bar{M}^n_k$ uniquely as

$$f = (\langle v_1 \rangle \vee f_v) + l$$

so that v_1 does not belong to any space in *supp*$(f_v) \vee$ *supp*(l). We can now use some of the results from earlier in this chapter to give a decomposition for the module obtained by applying the idempotent E_1 to the kernel $\bar{K}^n_{k,i}$.

Lemma 5.6. *If* $0 < k \leq n$ *and* $0 < i < \pi$ *then*

$$\bar{K}^n_{k,i} E_1 \cong \bar{K}^{n-1}_{k,i+1} \oplus \bar{K}^{n-1}_{k-1,i-1}$$

as $\bar{F}G_{n-1}$-modules.

Proof. Let $f = (\langle v_1\rangle \vee f_v) + l$ be an element of $\bar{K}^n_{k,i}$. So $\partial^i(f) = 0$. By Lemma 4.11 we know that $xE_1 = x$ for all x containing v_1 and hence if we apply the idempotent E_1 to such an element f we have

$$fE_1 = (\langle v_1\rangle \vee f_v) + lE_1.$$

If $v_1 \notin x$ the standard matrix for x has shape

$$\begin{pmatrix} a_1 & \\ \vdots & x^{(1)} \\ a_k & \end{pmatrix}$$

with $a_i \in \mathbb{F}$ and in the proof of Lemma 4.13 we have shown that for matrices of this shape we have $xE_1 = x^{(1)}E_1$. For $l = \sum_{v_1 \notin x} f_x x$ we define

$$l^{(1)} = \sum_{v_1 \notin x} f_x x^{(1)} \in \bar{M}^{n-1}_k$$

and thus we have that $lE_1 = l^{(1)}E_1$. Since $\partial^i(f) = 0$ we have $\partial^i(f)E_1 = 0$ and by associativity of matrix multiplication this gives $\partial^i(fE_1) = 0$. Hence

$$\partial^i(\langle v_1\rangle \vee f_v)E_1 + \partial^i(l^{(1)})E_1 = 0.$$

By Lemma 5.3

$$(\langle v_1\rangle \vee \partial^i(f_v)) + q^{k-i}[i]_q \partial^{i-1}(f_v)E_1 + \partial^i(l^{(1)})E_1 = 0$$

and hence we have

$$\langle v_1\rangle \vee \partial^i(f_v) = 0 \tag{5.1}$$

and

$$(q^{k-i}[i]_q \partial^{i-1}(f_v) + \partial^i(l^{(1)}))E_1 = 0.$$

By the previous lemma the second of these equations implies that

$$q^{k-i}[i]_q \partial^{i-1}(f_v) + \partial^i(l^{(1)}) = 0. \tag{5.2}$$

Equation 5.1 gives us

$$\partial^i(f_v) = 0$$

and by applying ∂ to 5.2 we see that

$$\begin{aligned} 0 &= q^{k-i}[i]_q \partial^i(f_v) + \partial^{i+1}(l^{(1)}) \\ &= \partial^{i+1}(l^{(1)}). \end{aligned}$$

We also have, directly from (5.2), the following:

$$\partial^{i-1}(q^{k-i}[i]_q f_v + \partial(l^{(1)})) = 0.$$

Hence we have

$$l^{(1)} \in \bar{K}^{n-1}_{k,i+1}$$

and

$$q^{k-i}[i]_q f_v + \partial(l^{(1)}) \in \bar{K}^{n-1}_{k-1,i-1}.$$

So we are able to define a map $\psi : \bar{K}^n_{k,i}E_1 \to \bar{K}^{n-1}_{k,i+1} \oplus \bar{K}^{n-1}_{k-1,i-1}$ by

$$\psi(fE_1) = (l^{(1)}, q^{k-i}[i]_q f_v + \partial(l^{(1)})).$$

This map ψ is surjective. To show this we let $(h_1, h_2) \in \bar{K}^{n-1}_{k,i+1} \oplus \bar{K}^{n-1}_{k-1,i-1}$. If we put

$$m_v = \frac{1}{q^{k-i}[i]_q}(h_2 - \partial(h_1))$$

and

$$m = (\langle v_1 \rangle \vee m_v) + h_1$$

then $\psi(mE_1) = (h_1, h_2)$. So we need to show that mE_1 is an element of $\bar{K}^n_{k,i}E_1$. Since $h_1 \in \bar{K}^{n-1}_{k,i+1}$ and $h_2 \in \bar{K}^{n-1}_{k-1,i-1}$ we have $\partial^{i+1}(h_1) = \partial^{i-1}(h_2) = 0$ and thus we also have $\partial^i(m_v) = 0$. Applying Lemma 5.3 gives

$$\begin{aligned}\partial^i(mE_1) = \partial^i(m)E_1 &= (\langle v_1 \rangle \vee \partial^i(m_v)) + q^{k-i}[i]_q \partial^{i-1}(m_v)E_1 + \partial^i(h_1)E_1 \\ &= (q^{k-i}[i]_q \partial^{i-1}(m_v) + \partial^i(h_1))E_1.\end{aligned}$$

By the definition of m_v we see that $\partial^{i-1}(m_v) = \frac{1}{q^{k-i}[i]_q}(\partial^{i-1}(h_2) - \partial^i(h_1))$ and since $\partial^{i-1}(h_2) = 0$ this gives

$$\partial^i(mE_1) = (-\partial^i(h_1) + \partial^i(h_1))E_1 = 0.$$

So mE_1 is an element of $\bar{K}^n_{k,i}E_1$ and thus we have surjectivity.

To show that the map is also injective we take two elements f and f^* with $f = (\langle v_1 \rangle \vee f_v) + l$ and $f^* = (\langle v_1 \rangle \vee f^*_v) + l^*$ such that $\psi(fE_1) = \psi(f^*E_1)$. This gives

$$(l^{(1)}, q^{k-i}[i]_q f_v + \partial(l^{(1)})) = (l^{*(1)}, q^{k-i}[i]_q f^*_v + \partial(l^{*(1)})).$$

and thus $l^{(1)} = l^{*(1)}$ and $f_v = f^*_v$. So we have

$$fE_1 = (\langle v_1 \rangle \vee f_v) + l^{(1)}E_1 = (\langle v_1 \rangle \vee f^*_v) + l^{*(1)}E_1 = f^*E_1$$

showing that the map is injective.

We are also required to show that ψ is an $\bar{F}G_{n-1}$-homomorphism and that it is well-defined. Every element of G_{n-1} commutes with E_1 so if we take any $g \in G_{n-1}$ then $fE_1 \cdot g = f \cdot gE_1$. Since

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ : & & x^{(1)} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ : & & g & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ : & & x^{(1)} \cdot g & \\ 0 & & & \end{pmatrix}$$

we have $(\langle v_1 \rangle \vee f_v) \cdot g = \langle v_1 \rangle \vee (f_v \cdot g)$ and we also have $\partial(l^{(1)}) \cdot g = \partial(l^{(1)} \cdot g)$ by associativity of matrix multiplication. Hence

$$\begin{aligned} \psi(fE_1 \cdot g) &= \psi((\langle v_1 \rangle \vee (f_v \cdot g)) + l^{(1)} \cdot gE_1) \\ &= (l^{(1)} \cdot g, q^{k-i}[i]_q f_v \cdot g + \partial(l^{(1)} \cdot g)) \\ &= (l^{(1)} \cdot g, q^{k-i}[i]_q (f_v + \partial(l^{(1)})) \cdot g) \\ &= \psi(fE_1) \cdot g \end{aligned}$$

for any $g \in G_{n-1}$ and so ψ is an $\bar{F}G_{n-1}$-homomorphism.

We can also show that the map ψ is well-defined. If we have $fE_1 = f^*E_1$ with $f = (\langle v_1 \rangle \vee f_v) + l$ and $f^* = (\langle v_1 \rangle \vee f_v^*) + l^*$ then $(\langle v_1 \rangle \vee f_v) + lE_1 = (\langle v_1 \rangle \vee f_v^*) + l^*E_1$ and so $f_v = f_v^*$ and $l^{(1)} = l^{*(1)}$. Thus we can see that $\psi(fE_1) = \psi(f^*E_1)$ and so ψ is a well-defined $\bar{F}G_{n-1}$-isomorphism. □

Next we consider the module obtained when E_1 is applied to the image $\bar{I}^n_{k,i}$. We have an analogous result to the one in the previous lemma for the kernel.

Lemma 5.7. *For $0 < k \leq n$ and $0 < i < \pi$ we have*

$$\bar{I}^n_{k,i} E_1 \cong \bar{I}^{n-1}_{k,i+1} \oplus \bar{I}^{n-1}_{k-1,i-1}$$

as $\bar{F}G_{n-1}$*-modules.*

Proof. We will use the same map ψ as in the previous lemma. We take f to be an element of $\bar{I}^n_{k,i} := \partial^{\pi-i}(\bar{M}^n_{k+\pi-i})$ so that $f = \partial^{\pi-i}(m)$ for some $m \in \bar{M}^n_{k+\pi-i}$. Thus $fE_1 = \partial^{\pi-i}(m)E_1 = \partial^{\pi-i}(mE_1)$. We now express $f = (\langle v_1\rangle \vee f_v) + l$ and $m = (\langle v_1\rangle \vee m_v) + s$ where v_1 is not contained in any space in $supp(f_v) \vee supp(l)$ or $supp(m_v) \vee supp(s)$. Using Lemma 5.3 we have

$$
\begin{aligned}
(\langle v_1\rangle \vee f_v) + l^{(1)}E_1 = fE_1 = \partial^{\pi-i}(m)E_1 \\
= (\langle v_1\rangle \vee \partial^{\pi-i}(m_v)) + q^k[\pi-i]_q\partial^{\pi-i-1}(m_v)E_1 + \partial^{\pi-i}(s^{(1)})E_1.
\end{aligned}
$$

This gives us two equations

$$
\begin{aligned}
f_v &= \partial^{\pi-i}(m_v) && (5.3)\\
l^{(1)} &= q^k[\pi-i]_q\partial^{\pi-i-1}(m_v) + \partial^{\pi-i}(s^{(1)}). && (5.4)
\end{aligned}
$$

where we have used Lemma 5.5 to get the second of the two equations. Equation 5.4 gives

$$
l^{(1)} = \partial^{\pi-i-1}(q^k[\pi-i]_q m_v + \partial(s^{(1)}))
$$

and so $l^{(1)} \in \bar{I}^{n-1}_{k,i+1} := \partial^{\pi-i-1}(\bar{M}^{n-1}_{k+\pi-i-1})$. Using 5.3 and 5.4 we also have

$$
\begin{aligned}
q^{k-i}[i]_q f_v + \partial(l^{(1)}) &= q^{k-i}[i]_q\partial^{\pi-i}(m_v) + q^k[\pi-i]_q\partial^{\pi-i}(m_v) + \partial^{\pi-i+1}(s^{(1)}) \\
&= (q^{k-i}[i]_q + q^k[\pi-i]_q)\partial^{\pi-i}(m_v) + \partial^{\pi-i+1}(s^{(1)}).
\end{aligned}
$$

Recall $[m]_q = \frac{q^m - 1}{q-1}$. Then we see that

$$\begin{aligned} q^{k-i}[i]_q + q^k[\pi - i]_q &= q^{k-i}([i]_q + q^i[\pi - i]_q) \\ &= q^{k-i}\left(\frac{q^i - 1 + q^i(q^{\pi-i} - 1)}{q-1}\right) \\ &= q^{k-i}\left(\frac{q^\pi - 1}{q-1}\right) = q^{k-i}[\pi]_q. \end{aligned}$$

and since $[\pi]_q \equiv 0 \mod p$ this gives

$$q^{k-i}[i]_q + q^k[\pi - i]_q \equiv 0 \mod p. \tag{5.5}$$

So $q^{k-i}[i]_q f_v + \partial(l^{(1)}) = \partial^{\pi-i+1}(s^{(1)})$ showing that

$$q^{k-i}[i]_q f_v + \partial(l^{(1)}) \in \bar{I}^{n-1}_{k-1,i-1}.$$

Thus if $f \in \bar{I}^n_{k,i}$ and we take the same map ψ as defined in the previous lemma we have

$$\psi : \bar{I}^n_{k,i}E_1 \to \bar{I}^{n-1}_{k,i+1} \oplus \bar{I}^{n-1}_{k-1,i-1}.$$

To show that this map is surjective we take $h_1 \in \bar{I}^{n-1}_{k,i+1}$ and $h_2 \in \bar{I}^{n-1}_{k-1,i-1}$. So there exists $m_1 \in \bar{M}^{n-1}_{k+\pi-i-1}$ and $m_2 \in \bar{M}^{n-1}_{k+\pi-i}$ such that $h_1 = \partial^{\pi-i-1}(m_1)$ and $h_2 = \partial^{\pi-i+1}(m_2)$. Now, as in the previous lemma, if we put

$$f = \left(\langle v_1 \rangle \vee \frac{1}{q^{k-i}[i]_q}(h_2 - \partial(h_1))\right) + h_1$$

then $\psi(fE_1) = (h_1, h_2)$. We need to show that $fE_1 \in \bar{I}^n_{k,i}E_1$. That is, there exists some $m \in \bar{M}^n_{k+\pi-i}$ such that $\partial^{\pi-i}(mE_1) = fE_1$. If we consider

$$m = \left(\langle v_1 \rangle \vee \frac{q^{i-k}}{[i]_q}(\partial(m_2) - m_1)\right) + m_2$$

then by Lemma 5.3

$$\begin{aligned}\partial^{\pi-i}(m)E_1 &= \langle v_1\rangle \vee \partial^{\pi-i}\left(\frac{q^{i-k}}{[i]_q}(\partial(m_2)-m_1)\right)\\&\quad+\left(q^k[\pi-i]_q\partial^{\pi-i-1}\left(\frac{q^{i-k}}{[i]_q}(\partial(m_2)-m_1)\right)+\partial^{\pi-i}(m_2)\right)E_1\\&= \langle v_1\rangle \vee \frac{q^{i-k}}{[i]_q}(\partial^{\pi-i+1}(m_2)-\partial^{\pi-i}(m_1))\\&\quad+\left(\frac{q^i[\pi-i]_q}{[i]_q}(\partial^{\pi-i}(m_2)-\partial^{\pi-i-1}(m_1))+\partial^{\pi-i}(m_2)\right)E_1.\end{aligned}$$

Now by (5.5) we have $[i]_q + q^i[\pi-i]_q \equiv 0 \mod p$ and from this we get

$$\frac{q^i[\pi-i]_q}{[i]_q} \equiv -1 \mod p.$$

So

$$\begin{aligned}\partial^{\pi-i}(m)E_1 &= \left(\langle v_1\rangle \vee \frac{1}{q^{k-i}[i]_q}(h_2-\partial(h_1))\right)+(-\partial^{\pi-i}(m_2)+h_1+\partial^{\pi-i}(m_2))E_1\\&= \left(\langle v_1\rangle \vee \frac{1}{q^{k-i}[i]_q}(h_2-\partial(h_1))\right)+h_1E_1\\&= fE_1.\end{aligned}$$

So ψ is surjective on $\bar{I}^n_{k,i}$. Also, in the same way as for $\bar{K}^n_{k,i}$ in Lemma 5.6, the map ψ is injective, well-defined and an $\bar{F}G_{n-1}$-homomorphism. Thus ψ is an $\bar{F}G_{n-1}$-isomorphism. □

We now consider the effect of applying the second idempotent element E_2 to these modules. In Chapter 4 we defined the subset

$$(L^n_k)^{\ddagger} = \left\{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & & & \\ \vdots & \vdots & & x^{\dagger} & \\ 0 & 0 & & & \end{pmatrix} : x^{\dagger} \in L^{n-2}_{k-1}\right\} \subseteq L^n_k$$

and the corresponding module $(M_k^n)^\ddagger = \bar{F}(L_k^n)^\ddagger$. We showed that for any element $f = \sum_{x \in L_k^n} f_x x \in \bar{M}_k^n$ we could uniquely define

$$f^\ddagger = \sum_{x \in (L_k^n)^\ddagger} \frac{f_x}{\chi_2(x_{11})}(\langle v_2 \rangle \vee x^\dagger) \in (M_k^n)^\ddagger$$

such that $fE_2 = f^\ddagger E_2$. We can use this new element to define an isomorphism on the module obtained by applying E_2 to $\bar{K}_{k,i}^n$.

Lemma 5.8. *We have*

$$\bar{K}_{k,i}^n E_2 \cong \bar{K}_{k-1,i}^{n-2}$$

as $\bar{F}G_{n-2}$-modules.

Proof. If $f \in \bar{K}_{k,i}^n$ then $\partial^i(fE_2) = \partial^i(f)E_2 = 0$. By the above discussion we have that there exists some unique element $f^\ddagger \in (M_k^n)^\ddagger$ with $fE_2 = f^\ddagger E_2$.

Recall the map $\phi_3 : (M_k^n)^\ddagger E_2 \to \bar{M}_{k-1}^{n-2}$ from Lemma 4.16:

$$\phi_3 : \sum_x f_x(\langle v_2 \rangle \vee x^\dagger)E_2 \mapsto \sum_x f_x x^\dagger.$$

We wish to show that this map restricts to $\bar{K}_{k,i}^n E_2$.

So we have

$$\partial^i(fE_2) = \partial^i(f^\ddagger)E_2 = \sum_{x \in (L_k^n)^\ddagger} f_x \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \partial^i(x^\dagger) & \\ 0 & 0 & & & \end{pmatrix} E_2$$

using the result of Lemma 5.4 that

$$\partial^i(\langle v_2 \rangle \vee x^\dagger)E_2 = (\langle v_2 \rangle \vee \partial^i(x^\dagger))E_2$$

for any $x^\dagger \in L_{k-1}^{n-2}$. This gives that $\partial^i(f^\ddagger)E_2$ is equal to

$$\sum_{x \in (L_k^n)^\ddagger} \frac{f_x}{q^{2n-3}} \sum_{\alpha_i, \beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \partial^i(x^\dagger) & \\ 0 & 0 & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 1 & 0 & \cdots & 0 \\ \alpha_3 & \beta_3 & & & \\ \vdots & \vdots & & I_{n-2} & \\ \alpha_n & \beta_n & & & \end{pmatrix}.$$

If we let α denote the column vector $(\alpha_3, \cdots, \alpha_n)^\top$ and $\beta = (\beta_3, \cdots, \beta_n)^\top$ then multiplying out gives

$$\partial^i(f^\ddagger)E_2 = \sum_{x \in (L_k^n)^\ddagger} f_x \frac{1}{q^{2n-3}} \sum_{\alpha_j, \beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} \alpha_2 & 1 & 0 & \cdots & 0 \\ & & & & \\ \partial^i(x^\dagger)\alpha & \partial^i(x^\dagger)\beta & & \partial^i(x^\dagger) & \\ & & & & \end{pmatrix}.$$

Since every matrix $\partial^i(x^\dagger)$ in the sum has full rank $k-i-1$ then summing over all entries $\alpha_3, \cdots, \alpha_n$ and $\beta_3, \cdots, \beta_n$ gives that the column vectors $\partial^i(x^\dagger)\alpha$ and $\partial^i(x^\dagger)\beta$ run through all column vectors of height $k-i-1$ over $\mathbb{F}$, each occurring with multiplicity $q^{(n-2)-(k-i-1)} = q^{n-k+i-1}$. We can also row reduce the matrices in the sum ensuring that the entries in the second column below the 1 in the top row are all zero. We arrive at

$$\partial^i(f^\ddagger)E_2 = \frac{1}{q^{k-i}} \sum_{\alpha_2, \gamma_j \in \mathbb{F}} \chi_2(\alpha_2) \sum_{x \in (L_k^n)^\ddagger} f_x \begin{pmatrix} \alpha_2 & 1 & 0 & \cdots & 0 \\ \gamma_2 & 0 & & & \\ \vdots & \vdots & & \partial^i(x^\dagger) & \\ \gamma_{k-i} & 0 & & & \end{pmatrix}$$

where each submatrix $\partial^i(x^\dagger)$ is now in standard form. Since $\partial^i(f^\ddagger)E_2 = 0$ if we consider the inner sum for matrices with first column fixed to be zero this gives

$$\sum_{x\in(L_k^n)^\ddagger} f_x \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ : & 0 & & & \\ : & : & & \partial^i(x^\dagger) & \\ 0 & 0 & & & \end{pmatrix} = 0$$

Hence $\langle v_2\rangle \vee \partial^i(\phi_3(f^\ddagger E_2)) = 0$ and so $\partial^i(\phi_3(f^\ddagger E_2)) = 0$. So we have

$$\phi_3 : \bar{K}^n_{k,i}E_2 \to \bar{K}^{n-2}_{k-1,i}.$$

The map ϕ_3 is surjective on $\bar{K}^n_{k,i}E_2$ for if we take any $f^\dagger \in \bar{K}^{n-2}_{k-1,i}$ then the corresponding element $f = \langle v_2\rangle \vee f^\dagger$ gives $\phi_3(fE_2) = f^\dagger$ and

$$\partial^i(fE_2) = \partial^i(f)E_2 = (\langle v_2\rangle \vee \partial^i(f^\dagger))E_2 = 0$$

so that $fE_2 \in \bar{K}^n_{k,i}$ and hence $fE_2 = (fE_2)E_2 \in \bar{K}^n_{k,i}E_2$. By Lemma 4.16 we must have that ϕ_3 is injective, well-defined and an $\bar{F}G_{n-2}$-homomorphism. So ϕ_3 is an $\bar{F}G_{n-2}$-isomorphism. □

We can apply the same map to get an isomorphism for the module $\bar{I}^n_{k,i}E_2$.

Lemma 5.9. *We have*

$$\bar{I}^n_{k,i}E_2 \cong \bar{I}^{n-2}_{k-1,i}$$

as $\bar{F}G_{n-2}$-modules.

Proof. If we take any element $f \in \bar{I}^n_{k,i} := \partial^{\pi-i}(\bar{M}^n_{k+\pi-i})$ then $f = \partial^{\pi-i}(m)$ for some $m \in \bar{M}^n_{k+\pi-i}$. Applying the idempotent E_2 gives $fE_2 = \partial^{\pi-i}(m)E_2 = \partial^{\pi-i}(mE_2) = \partial^{\pi-i}(m^\ddagger E_2)$ for some $m^\ddagger = \langle v_2\rangle \vee m^\dagger \in (M^n_{k+\pi-i})^\ddagger$. Thus by

Lemma 5.4 we have that

$$\begin{aligned} fE_2 &= \partial^{\pi-i}(\langle v_2\rangle \vee m^\dagger)E_2 \\ &= (\langle v_2\rangle \vee \partial^{\pi-i}(m^\dagger))E_2. \end{aligned}$$

This gives us

$$\phi_3(fE_2) = \partial^{\pi-i}(m^\dagger)$$

for $m^\dagger \in \bar{M}^{n-2}_{k+\pi-i-1}$ and so $\phi_3 : \bar{I}^n_{k,i}E_2 \to \bar{I}^{n-2}_{k-1,i}$. We see that the map is surjective on $\bar{I}^n_{k,i}E_2$ for if we take any $f \in \bar{I}^{n-2}_{k-1,i}$ then there exists some $\mu \in \bar{M}^{n-2}_{k+\pi-i}$ such that $\partial^{\pi-i}(\mu) = f$ and hence

$$\begin{aligned} \partial^{\pi-i}(\langle v_2\rangle \vee \mu)E_2 &= (\langle v_2\rangle \vee \partial^{\pi-i}(\mu))E_2 \\ &= (\langle v_2\rangle \vee f)E_2. \end{aligned}$$

So $(\langle v_2\rangle \vee f)E_2 \in \bar{I}^n_{k,i}E_2$ and hence ϕ_3 is surjective on $\bar{I}^n_{k,i}E_2$. The map is injective, well-defined and an $\bar{F}G_{n-2}$-homomorphism by the proof of Lemma 4.16. □

Recall that in Lemma 4.19 we showed that if we take a set of $q^{n-1}-1$ right coset representatives for H^*_{n-1} in G^*_{n-1} then we can use these to construct an $\bar{F}G_{n-1}$-isomorphism $\bar{M}^n_k E_2 G^*_{n-1} \to \bar{M}^n_k E_2(q^{n-1}-1)$. It is clear that this isomorphism restricts to elements in the kernel and image:

$$\bar{K}^n_{k,i}E_2G^*_{n-1} \to \bar{K}^n_{k,i}E_2(q^{n-1}-1)$$

and

$$\bar{I}^n_{k,i}E_2G^*_{n-1} \to \bar{I}^n_{k,i}E_2(q^{n-1}-1).$$

Putting these results together gives

Lemma 5.10. *For $0 < k \leq n$ and $0 < i < \pi$ we have*

$$\bar{K}^n_{k,i} \cong \bar{K}^{n-1}_{k,i+1} \oplus \bar{K}^{n-1}_{k-1,i-1} \oplus \bar{K}^{n-2}_{k-1,i}(q^{n-1} - 1)$$
$$\bar{I}^n_{k,i} \cong \bar{I}^{n-1}_{k,i+1} \oplus \bar{I}^{n-1}_{k-1,i-1} \oplus \bar{I}^{n-2}_{k-1,i}(q^{n-1} - 1)$$

as $\bar{F}G_{n-1}$-modules.

The two isomorphisms we have described for $\bar{K}^n_{k,i}$ and $\bar{I}^n_{k,i}$ are given by the same map. Since the homology module is defined by $\bar{H}^n_{k,i} := \bar{K}^n_{k,i}/\bar{I}^n_{k,i}$ then this map will also give us an isomorphism for the homology module. Hence we have our main result for this chapter, an $\bar{F}G_{n-1}$-decomposition of $\bar{H}^n_{k,i}$:

Theorem 5.11. *For $0 < k \leq n$ and $0 < i < \pi$ we have*

$$\bar{H}^n_{k,i} \cong \bar{H}^{n-1}_{k,i+1} \oplus \bar{H}^{n-1}_{k-1,i-1} \oplus \bar{H}^{n-2}_{k-1,i}(q^{n-1} - 1)$$

as $\bar{F}G_{n-1}$-modules.

Remark 5.12. (1) In Section 8.3 we give an explicit example of how this isomorphism works to give a decomposition of the homology module in the case when $n = 4$.

(2) If we consider the branching rule of Theorem 5.11 then the dimensions of these homology modules gives us

$$\beta^n_{k,i} = \beta^{n-1}_{k,i+1} + \beta^{n-1}_{k-1,i-1} + \beta^{n-2}_{k-1,i}(q^{n-1} - 1)$$

which is precisely the Betti number relation of Lemma 3.9(1).

(3) The isomorphism $\psi : H^n_{k,i} \to H^{n-1}_{k,i+1} \oplus H^{n-1}_{k-1,i-1} \oplus H^{n-2}_{k-1,i}(q^{n-1}-1)$ which we have described in this chapter is different to the map

$$\phi : M^n_k \to M^{n-1}_k \oplus M^{n-1}_{k-1} \oplus M^{n-2}_{k-1}(q^{n-1}-1)$$

which we discussed in Chapter 4. However, it is clear that ψ can also be extended to M^n_k and would map into the three components in the same way as ϕ:

$$\psi : M^n_k \to M^{n-1}_{k-1} \oplus M^{n-1}_k \oplus M^{n-2}_{k-1}(q^{n-1}-1).$$

6 Consequences of the Homology Module Decomposition

The main result of Chapter 5 is the homology module decomposition given in Theorem 5.11. This result is very useful for if we can decompose the $\bar{F}G_n$-module $\bar{H}^n_{k,i}$ in terms of $\bar{F}G_{n-1}$-modules then we are able to prove some results about these homology modules by using this decomposition and an inductive process. This is what we do in this chapter.

We begin by looking at exactly when the homology module is non-trivial, showing that the sequence $\mathcal{M}_{k^*,i^*}$ in Chapter 2 is *almost π-exact*. We go on to give a formula for the rank of a particular type of incidence matrix and also a formula for the Brauer character associated to the sequence $\mathcal{M}_{k^*,i^*}$. We then prove an important result in Section 6.4 which gives a condition on the parameters n, k and i which ensures that the homology module is irreducible as an $\bar{F}G_n$-module. This is an important result as the irreducible G_n-modules give irreducible representations of the general linear group over a finite field in cross-characteristic.

6.1 Almost Exactness

We first use the homology module decomposition of Theorem 5.11 to give a new proof of the result from [13] that $\bar{H}^n_{k,i}$ is non-trivial if and only if (k, i) is a middle term, as stated in Theorem 2.7. To be able to use the homology module decomposition we must make the assumption that our field $\bar{F}$ is extended by adjoining a primitive $p'th$ root of unity but it is clear that once we have proven the result for $\bar{F}$ it is also true for any field F of characteristic p. The proof given here is independent of any results other than the homology module decomposition 5.11, whereas the proof which is given in [13] depends on the rank formula of [18, Thm 3.1].

Theorem 6.1. *We have* $\bar{H}^n_{k,i} = 0$ *unless*

$$n < 2k + \pi - i < n + \pi.$$

Proof. The homology module decomposition gives

$$\bar{H}^n_{k,i} \cong \bar{H}^{n-1}_{k,i+1} \oplus \bar{H}^{n-1}_{k-1,i-1} \oplus \bar{H}^{n-2}_{k-1,i}(q^{n-1} - 1)$$

as $\bar{F}G_{n-1}$-modules. We prove the result by using induction on n.

If $n = 1$ we only need to consider $k = 1$ as $\bar{H}^n_{k,i} = 0$ for $k = 0$ and $k > n$. For $k = 1$ we have $\bar{H}^n_{k,i} = 0$ for $i = 1$ and if $1 < i < \pi$ then $1 < 2+\pi-i < 1+\pi$ so $(1, i)$ is a middle term in these cases.

For $n = 2$ we again have that $\bar{H}^n_{k,i} = 0$ when $k = 0$ or $k > 2$. For $k = 1$ if $0 < i < \pi$ then $2 < 2 + \pi - i < 2 + \pi$. Similarly for $k = 2$ we have $\bar{H}^n_{k,i} = 0$ for $i = 1, 2$ and if $2 < i < \pi$ then $2 < 4 + \pi - i < 2 + \pi$.

So we are able to assume that our result is true for $n - 2$ and $n - 1$. We now need to prove it for n. So we assume the following:

$$\bar{H}^{n-2}_{k,i} = 0 \text{ unless } n - 2 < 2k + \pi - i < n - 2 + \pi$$

and

$$\bar{H}^{n-1}_{k,i} = 0 \text{ unless } n - 1 < 2k + \pi - i < n - 1 + \pi.$$

We must consider two cases separately.

Case 1: Suppose $2k + \pi - i \leq n$. Then

$$\begin{aligned} 2(k-1) + \pi - (i-1) &= 2k - 2 + \pi - i + 1 \\ &= (2k + \pi - i) - 1 \\ &\leq n - 1. \end{aligned}$$

So $\bar{H}^{n-1}_{k-1,i-1} = 0$. Also

$$2k + \pi - (i+1) = (2k + \pi - i) - 1 \leq n - 1$$

and

$$2(k-1) + \pi - i = (2k + \pi - i) - 2 \leq n - 2$$

so $\bar{H}^{n-1}_{k,i+1} = \bar{H}^{n-2}_{k-1,i} = 0$. Hence by Theorem 5.11 we have that $\bar{H}^{n}_{k,i} = 0$.

Case 2: Suppose $2k + \pi - i \geq n + \pi$. Then

$$2(k-1) + \pi - (i-1) = (2k + \pi - i) - 1 \geq n - 1 + \pi,$$

$$2k + \pi - (i+1) = (2k + \pi - i) - 1 \geq n - 1 + \pi$$

and

$$2(k-1) + \pi - i = (2k + \pi - i) - 2 \geq n - 2 + \pi.$$

So by the inductive hypothesis we have

$$\bar{H}^{n-1}_{k-1,i-1} = \bar{H}^{n-1}_{k,i+1} = \bar{H}^{n-2}_{k-1,i} = 0$$

which again, by Theorem 5.11, implies that $\bar{H}^{n}_{k,i} = 0$.

So we can conclude that $\bar{H}^n_{k,i} = 0$ unless

$$n < 2k + \pi - i < n + \pi.$$

□

Now consider the sequence

$$\mathcal{M}_{k^*,i^*} : 0 \xleftarrow{\partial^*} \bar{M}_{k^*} \xleftarrow{\partial^*} \bar{M}_{k^*+i^*} \xleftarrow{\partial^*} \bar{M}_{k^*+\pi} \xleftarrow{\partial^*} \bar{M}_{k^*+i^*+\pi} \xleftarrow{\partial^*} \bar{M}_{k^*+2\pi} \xleftarrow{\partial^*} \cdots$$

where ∂^* is the appropriate power of ∂. Recall that in Chapter 2 we defined any pair (k, i) which satisfies the condition of Theorem 6.1 to be a *middle term* of this sequence $\mathcal{M}_{k^*,i^*}$. We now show that any middle term is unique and hence that this sequence given here is *almost π-exact* in accordance with Definition 2.4.

Theorem 6.2. *The sequence $\mathcal{M}_{k^*,i^*}$ given above is almost π-exact, for any choice of positive integers k^*, i^* satisfying $k^* + i^* < \pi$.*

Proof. If (k, i) is a middle term for $\mathcal{M}_{k^*,i^*}$ then $n < 2k + \pi - i < n + \pi$. The preceding term in the sequence is $(k - i, \pi - i)$. However,

$$2(k - i) + \pi - (\pi - i) = 2k - i < n$$

so this term is not a middle term. Similarly the following term in the sequence is $(k + \pi - i, \pi - i)$ and since

$$2(k + \pi - i) + \pi - (\pi - i) = (2k + \pi - i) + \pi > n + \pi$$

this term again cannot be a middle term.

So if a sequence has a middle term it must be unique and hence we have almost π-exactness. □

The result here gives almost π-exactness of the sequence $\mathcal{M}_{k^*,i^*}$ under the assumption that our field F contains a primitive $p'th$ root of unity. However, since the alternating sum of the Betti numbers is equal to the alternating sum of the dimensions of the permutation modules and these dimensions are independent of F, then it is clear that this result of almost exactness restricts to any field F.

6.2 A Rank Formula for Incidence Matrices

A further application of the branching rule for the homology module and the subsequent proof of almost exactness of the sequence $\mathcal{M}_{k^*,i^*}$ is that it can be used to compute the rank for certain *incidence matrices*.

Let $t \leq k$ be integers such that $t + k \leq n$. This ensures that $\binom{n}{t}_q \leq \binom{n}{k}_q$. Then we define the matrix $W^n_{t,k}$ to be the matrix whose columns are indexed by elements of L^n_k and whose rows are indexed by elements of L^n_t. To compute the (i, j)-entry of this matrix we take the element $X \in L^n_k$ which indexes the j^{th} column, the element $Y \in L^n_t$ which indexes the i^{th} row and put a 1 if $Y \subseteq X$ and 0 otherwise. So $W^n_{t,j}$ is the incidence matrix for the t-dimensional subspaces with the k-dimensional subspaces. It is clear to see that the matrix $W^n_{k-i,k}$ is the matrix of $\partial^i : \bar{M}^n_k \to \bar{M}^n_{k-i}$ apart from a factor of $(i!)_q$. As we have shown in the first part of this chapter, the sequence $\mathcal{M}_{k^*,i^*}$ is almost exact, and thus the sequence

$$0 \longleftarrow \cdots \overset{\partial^{\pi-i}}{\longleftarrow} \bar{M}^n_{k-\pi-i} \overset{\partial^{i}}{\longleftarrow} \bar{M}^n_{k-\pi} \overset{\partial^{\pi-i}}{\longleftarrow} \bar{M}^n_{k-i} \overset{\partial^{i}}{\longleftarrow} \bar{M}^n_{k}$$

is exact for $2k - i \leq n$. Recall the definitions $\bar{I}^n_{k,i} = \partial^{\pi-i}(\bar{M}^n_{k+\pi-i})$ and

$\bar{K}^n_{k,i} = \ker(\partial^i) \cap \bar{M}^n_k$. Thus the image of $\partial^i : \bar{M}^n_k \to \bar{M}^n_{k-i}$ is $\bar{I}^n_{k-i,\pi-i}$ and by the exactness of this sequence we have $\bar{I}^n_{k-i,\pi-i} = \bar{K}^n_{k-i,\pi-i}$. Now applying the rank + nullity formula gives

$$\begin{aligned}\dim \bar{I}^n_{k-i,i} &= \dim \bar{K}^n_{k-i,\pi-i} \\ &= \dim \bar{M}^n_{k-i} - \dim(\partial^{\pi-i} \cap M^n_{k-\pi}) \\ &= \binom{n}{k-i}_q - \dim \bar{I}^n_{k-\pi,i}.\end{aligned}$$

Again this sequence is exact at this next stage and so $\bar{I}^n_{k-\pi,i} = \bar{K}^n_{k-\pi,i}$ giving

$$\dim \bar{I}^n_{k-i,i} = \binom{n}{k-i}_q - \dim \bar{K}^n_{k-\pi,i}.$$

We can use the rank + nullity formula again and exactness at the next stage of the sequence to give

$$\dim \bar{I}^n_{k-i,i} = \binom{n}{k-i}_q - \binom{n}{k-\pi}_q + \dim \bar{K}^n_{k-\pi-i,\pi-i}.$$

Repeating this process for every stage in the sequence gives us a formula for the dimension of $\bar{I}^n_{k-\pi,i}$ and hence a rank formula for the incidence matrix. This result is already known, see [18, Thm 3.1] but the proof given here is new.

Theorem 6.3. *If k and $0 < i < \pi$ satisfy $2k - i \leq n$ then the p-rank of $W^n_{k-i,k}$ is*

$$\binom{n}{k-i}_q - \binom{n}{k-\pi}_q + \binom{n}{k-\pi-i}_q - \cdots.$$

We can also determine the p-rank inductively using the G_{n-1}-module decomposition for $I^n_{k,i}$ given in Lemma 5.10:

$$I^n_{k,i} \cong I^{n-1}_{k,i+1} \oplus I^{n-1}_{k-1,i-1} \oplus I^{n-2}_{k-1,i}(q^{n-1} - 1).$$

If we let $w^n_{t,k}$ denote the p-rank of the incidence matrix $W^n_{t,k}$ then we have

Theorem 6.4. *For integers k and $0 < i < \pi$ with $2k - i \le n$*

$$w^n_{k-i,k} = w^{n-1}_{k-i-1,k} + w^{n-1}_{k-i,k-1} + (q^{n-1} - 1)w^{n-2}_{k-i-1,k-1}.$$

We can show that these two results match. Denote the alternating sums of q-binomial coefficients in the following manner

$$\gamma^n_{k,i} := \binom{n}{k-i}_q - \binom{n}{k-\pi}_q + \binom{n}{k-\pi-i}_q - \cdots$$

so that

$$\begin{aligned}
\gamma^{n-1}_{k,i+1} &= \binom{n-1}{k-i-1}_q - \binom{n-1}{k-\pi}_q + \binom{n-1}{k-\pi-i-1}_q - \cdots \\
\gamma^{n-1}_{k-1,i-1} &= \binom{n-1}{k-i}_q - \binom{n-1}{k-1-\pi}_q + \binom{n-1}{k-\pi-i}_q - \cdots \\
\gamma^{n-2}_{k-1,i} &= \binom{n-2}{k-1-i}_q - \binom{n-2}{k-1-\pi}_q + \binom{n-2}{k-1-\pi-i}_q - \cdots .
\end{aligned}$$

In Chapter 3 we proved the relation (3.6) for q-binomial coefficients:

$$\binom{n}{k}_q = \binom{n-1}{k}_q + \binom{n-1}{k-1}_q + (q^{n-1} - 1)\binom{n-2}{k-1}_q$$

and thus we can see that

$$\gamma^n_{k,i} = \gamma^{n-1}_{k-1,i-1} + \gamma^{n-1}_{k,i+1} + (q^{n-1} - 1)\gamma^{n-2}_{k-1,i}.$$

So these alternating sums of q-binomial coefficients do indeed satisfy the inductive formula of Theorem 6.3. For small values of n it is easy to check that the numbers $\gamma^n_{k,i}$ are the dimensions of the images $\bar{I}^n_{k,i}$ and so this verifies they are indeed the same throughout.

6.3 The Brauer Character

Another consequence of almost exactness of the sequence $\mathcal{M}_{k^*,i^*}$ is that we are able to give an explicit formula for the Brauer character on each non-trivial homology module derived from $\mathcal{M}_{k^*,i^*}$. We give a brief overview here of the general theory concerning Brauer characters and their connection to ordinary characters before proving the Brauer character formula for our homology modules. For a fuller treatment of this theory see the book [7] of Isaacs. The connection between the Brauer character and the ordinary character can be explained as follows.

In the case of the ordinary characters of representations over fields of characteristic zero we are able to recover the eigenvalues of representations from the character and the power maps. However if the field has prime characteristic p then we can no longer recover the eigenvalues in this way. The Brauer character overcomes this problem by lifting the eigenvalues to a field of characteristic zero. We briefly describe how this lifting process works. Let F be a field of order p^t where p is prime. Choose a generator x for the multiplicative group $F^\times$ so that we can regard x as a primitive $(p^t-1)^{th}$ root of unity. Let $\zeta = \exp\left(\frac{2\pi i}{p^t-1}\right)$ so that ζ is a primitive $(p^t-1)^{th}$ root of unity in $\mathbb{C}$. Now we lift each element x^r in $GF(q)^\times$ to ζ^r in $\mathbb{Z}[\zeta]$. Typically there will be many choices we can make for x and thus many possibilities for this lifting map. Once we have fixed a lifting we may define the Brauer character.

Definition 6.5. Suppose we have a representation $\rho : G \to GL(m, F)$ of a finite group G. Let $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_s$ be the conjugacy classes of elements of G whose order is not divisible by p and let $g_1, g_2, \cdots, g_s$ be the corresponding representatives of these conjugacy classes. Then we can define the **Brauer character** φ of the representation ρ to be the map

$$\varphi : \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_s\} \to \mathbb{C}$$

defined by

$$\varphi(\mathcal{C}_i) = \sum_{j=1}^{n} \epsilon'_j,$$

where $\rho(g_i)$ has eigenvalues $\epsilon_1, \cdots, \epsilon_n$ and ϵ'_j is the lift of ϵ_j.

Tables of these Brauer characters for finite simple groups of order less than 10^9 are given in the Atlas [15].

We now consider Brauer characters in the context of this thesis and give a formula for calculating the Brauer characters on homology modules. As we have shown in 6.1 the sequence $\mathcal{M}_{k^*,i^*}$ is almost exact and thus the Hopf trace formula [5, Thm 22.1] can be used to determine the Brauer character on the non-trivial homology module. The general linear group G_n acts on L_k^n by $x \mapsto xg$ for $x \in L_k^n$ and $g \in G_n$. This action can be extended linearly to $\bar{M}_k^n$. The action commutes with ∂ and so G_n acts on the homology module $\bar{H}_{k,i}^n$. We denote the character of G_n on $\bar{M}_k^n$ by $\chi(g, \bar{M}_k^n)$ and the character of G_n on $\bar{H}_{k,i}^n$ by $\chi(g, \bar{H}_{k,i}^n)$. Since q is co-prime to p then every element of G_n has order co-prime to p and so the Brauer character is defined on all elements of this group.

Theorem 6.6. *[13, Thm 4.5] On each non-trivial homology module $\bar{H}^n_{k,i}$ the Brauer character is given by*

$$\chi(g, \bar{H}^n_{k,i}) = \pm \sum_{t \in \mathbb{Z}} fix(g, \bar{M}^n_{k-\pi t}) - fix(g, \bar{M}^n_{k-i-\pi t}),$$

where $fix(g, \bar{M}^n_k)$ denotes the number of k-dimensional subspaces which are left invariant by $g \in G_n$. (The sign is determined by $\chi(1, \bar{H}^n_{k,i}) \geq 0$.)

Proof. Suppose that we have the following homological sequence of modules

$$\mathcal{A} : 0 \leftarrow A_0 \leftarrow \cdots \leftarrow A_{k-1} \leftarrow A_k \leftarrow A_{k+1} \leftarrow \cdots \leftarrow 0$$

with homology modules H_k such that a group G acts on A_k and H_k with characters $\chi(g, A_k)$ and $\chi(g, H_k)$ respectively.

Since the Brauer character of a quotient A/B is the Brauer character of A minus the Brauer character of B then we have

$$\sum_k (-1)^k \chi(g, H_k) = \sum_k (-1)^k \chi(g, A_k).$$

If the sequence $\mathcal{A}$ is almost exact it has at most one non-trivial homology module. So for this non-trivial module H_k we have

$$\chi(g, H_k) = \pm \sum_k (-1)^k \chi(g, A_k).$$

The $\pm$ sign will depend on the position of the non-trivial homology module in the sequence. As in the statement of the theorem it is chosen to ensure that $\chi(1, H_k) \geq 0$.

If we apply this theory to our sequence $\mathcal{M}_{k^*,i^*}$ then for the non-trivial homology module $\bar{H}^n_{k,i}$ we have

$$\chi(g, \bar{H}^n_{k,i}) = \pm \sum_{t \in \mathbb{Z}} \chi(g, \bar{M}^n_{k-\pi t}) - \chi(g, \bar{M}^n_{k-i-\pi t})$$

where the $\pm$ sign is chosen to ensure that $\chi(1, \bar{H}^n_{k,i}) \geq 0$.

Now let $fix(g, \bar{M}^n_k)$ denote the number of fixed points of g on the set L^n_k of k-dimensional subspaces. It is clear that

$$fix(g, \bar{M}^n_k) = \chi(g, \bar{M}^n_k).$$

Thus substituting this into the above expression for $\chi(g, \bar{H}^n_{k,i})$ gives us the required character formula. $\square$

In Theorem 3.6 we used almost exactness of the sequence to prove a formula for calculating Betti numbers. We can also use the above character formula to give a second, independent proof of this result.

Corollary 6.7. *If k and $0 < i < \pi$ are integers which satisfy the condition for being a middle term then*

$$\dim \bar{H}^n_{k,i} = \beta^n_{k,i} = \sum_{t \in \mathbb{Z}} \binom{n}{k - \pi t}_q - \binom{n}{k - i - \pi t}_q.$$

Proof. Theorem 6.6 gives us a character formula for the homology modules:

$$\chi(g, \bar{H}^n_{k,i}) = \pm \sum_{t \in \mathbb{Z}} fix(x, \bar{M}^n_{k-\pi t}) - fix(x, \bar{M}^n_{k-i-\pi t}).$$

If we put x equal to the identity in this formula then $\chi(1, \bar{H}^n_{k,i}) = \dim \bar{H}^n_{k,i}$. Note that in the statement Theorem 6.6 we have ensured that $\chi(1, \bar{H}^n_{k,i})$ is non-negative. We get

$$\dim \bar{H}^n_{k,i} = \sum_{t \in \mathbb{Z}} \dim \bar{M}^n_{k-\pi t} - \dim \bar{M}^n_{k-i-\pi t}.$$

The dimension of the permutation module $\bar{M}^n_k$ is given by the Gaussian polynomial $\binom{n}{k}_q$. Hence we have the required formula. $\square$

6.4 Irreducibility

The homology module decomposition also enables us to give a condition for when the module $\bar{H}^n_{k,i}$ is an irreducible $\bar{F}G_n$-module. We see in the following theorem that this happens when the pair (k,i) is at the "top-end" of the condition that $n < 2k + \pi - i < n + \pi$ necessary for (k,i) to be the middle term.

Theorem 6.8. *If $0 \le k \le n$ and $0 < i < \pi$ satisfy $n-1 = 2k-i$ then $\bar{H}^n_{k,i}$ is an irreducible $\bar{F}G_n$-module. Furthermore, if (k', i') is another pair of positive integers satisfying the above conditions then $\bar{H}^n_{k,i} \not\cong \bar{H}^n_{k',i'}$ as $\bar{F}G_n$-modules.*

Proof. We prove this result by induction on n, using the decomposition

$$\bar{H}^n_{k,i} \cong \bar{H}^{n-1}_{k,i+1} \oplus \bar{H}^{n-1}_{k-1,i-1} \oplus \bar{H}^{n-2}_{k-1,i}(q^{n-1}-1).$$

Notice that if $n-1 = 2k-i$ holds then we also have that $(n-1)-1 = 2k-(i+1) = 2(k-1)-(i-1)$ and that $(n-2)-1 = 2(k-1)-i$. Hence we will be assuming inductively that the modules $\bar{H}^{n-1}_{k-1,i}$ and $\bar{H}^{n-1}_{k,i+1}$ are irreducible G_{n-1}-modules and that $\bar{H}^{n-2}_{k-1,i}$ is an irreducible G_{n-2}-module.

First we need to show that the result holds for small values of n. For $n = 0$ the condition can only be satisfied by $k = 0, i = 1$ and this gives $\bar{H}^n_{k,i} = 0$. If we take $n = 1$ then we require $2k - i = 0$, so we have to consider the case $\bar{H}^1_{1,2}$. This module is equal to $\bar{M}^1_1$ which is clearly irreducible. If $n = 2$ then there are only two non-trivial modules which satisfy the condition. These are $\bar{H}^2_{1,1}$ and $\bar{H}^2_{2,3}$. If $\pi > 2$ then $\bar{H}^2_{1,1} = \bar{K}^2_{1,1} = \langle \sum_{x \in L^2_1} f_x x : \sum f_x = 0 \rangle$. This is an irreducible G_2-module since G_2 is transitive on L^2_1. When $\pi = 2$ we have

$\bar{H}^2_{1,1} = \bar{K}^2_{1,1}/\bar{I}^2_{1,1} = \langle \sum_{x \in L^2_1} f_x x : \sum f_x = 0 \rangle / \langle \sum_{x \in L^2_1} x \rangle$. Any $g \in G_2$ fixes the element $\sum_{x \in L^2_1} x$ while acting transitively on elements in the kernel $\bar{K}^2_{1,1}$ and so this module is also irreducible as a G_2-module. Finally, $\bar{H}^2_{2,3} = \bar{M}^2_2$ which has dimension one and must therefore be irreducible.

So we assume that the result holds for $n-1$ and $n-2$ and in particular we assume that the modules $\bar{H}^{n-1}_{k-1,i-1}$, $\bar{H}^{n-1}_{k,i+1}$ and $\bar{H}^{n-2}_{k-1,i}$ are non-isomorphic and irreducible over $\bar{F}G_{n-1}$ and $\bar{F}G_{n-2}$ respectively.

Now suppose for a contradiction that there exists a non-trivial $\bar{F}G_n$-submodule U of $\bar{H}^n_{k,i}$. By the branching rule of James we have

$$\bar{H}^n_{k,i} = \bar{H}^n_{k,i}E_1 \oplus \bar{H}^n_{k,i}E_2 G^*_{n-1}.$$

Since U is also an $\bar{F}G_n$-module then we can apply the same branching rule to give

$$U = UE_1 \oplus UE_2 G^*_{n-1}$$

where $UE_1 \subseteq \bar{H}^n_{k,i}E_1$ as $\bar{F}G_{n-1}$-modules and $UE_2 \subseteq \bar{H}^n_{k,i}E_2$ as $\bar{F}G_{n-2}$-modules.

By Theorems 5.8 and 5.9 we have that $\bar{H}^n_{k,i}E_2 \cong \bar{H}^{n-2}_{k-1,i}$. Also, by the inductive hypothesis we can assume that $\bar{H}^{n-2}_{k-1,i}$ is an irreducible $\bar{F}G_{n-2}$-module. So $\bar{H}^n_{k,i}E_2$ must also be irreducible as an $\bar{F}G_{n-2}$-module and so it is clear that either $UE_2 = \bar{H}^n_{k,i}E_2$ or $UE_2 = 0$.

Now we consider UE_1. If we take ψ to be the isomorphism described in Lemmas 5.6 and 5.7 then we have

$$\psi : \bar{H}^n_{k,i}E_1 \to \bar{H}^{n-1}_{k,i+1} \oplus \bar{H}^{n-1}_{k-1,i-1}$$

and thus $\psi(UE_1)$ is an $\bar{F}G_{n-1}$-submodule of $\bar{H}^{n-1}_{k,i+1} \oplus \bar{H}^{n-1}_{k-1,i-1}$.

From our assumption that the modules $\bar{H}^{n-1}_{k,i+1}$ and $\bar{H}^{n-1}_{k-1,i-1}$ are non-isomorphic and irreducible $\bar{F}G_{n-1}$-modules we must have four possibilities for $\psi(UE_1)$:

$$\psi(UE_1) = \begin{cases} 0 \\ \bar{H}^{n-1}_{k-1,i-1} \\ \bar{H}^{n-1}_{k,i+1} \\ \bar{H}^{n-1}_{k,i+1} \oplus \bar{H}^{n-1}_{k-1,i-1} \end{cases}$$

We consider in turn each of the four possibilities for $\psi(UE_1)$ and look to eliminate some of these cases:

- If $\psi(UE_1) = 0$ then $UE_1 = 0$ and so $U = UE_2$ giving $UE_2 \neq 0$. Lemma 4.15 states that $xE_2 = 0$ if $x \notin (L^n_k)^\dagger$ and so there must exist some $[u] = u + \bar{I}^n_{k,i} \in U$ such that $supp(u) \cap (L^n_k)^\dagger \neq \emptyset$. Take $y \in supp(u) \cap (L^n_k)^\dagger$. Lemma 4.9 showed that G_n is transitive on L^n_k and so for any $x \in L^n_k$ we can find some $g \in G_n$ such that $x = y \cdot g$. Thus we can find some $g \in G_n$ so that $supp(u \cdot g) \cap L^n_{k,1} \neq \emptyset$ (Recall that $L^n_{k,1}$ consists of all subspaces $x \in L^n_k$ such that $v_1 \in x$). If we apply E_1 to any space in $L^n_{k,1}$ this space is unchanged and hence $(u \cdot g)E_1 \neq 0$. Since U is a G_n-module we have $U \cdot g = U$ for all $g \in G_n$ and hence $UE_1 \neq 0$ giving $\psi(UE_1) \neq 0$. A contradiction.

- If $\psi(UE_1) = \bar{H}^{n-1}_{k-1,i-1}$ then applying the inverse of the isomorphism ψ we have

$$\psi^{-1}(\bar{H}^{n-1}_{k-1,i-1}) = \left\{[\langle v_1 \rangle \vee f]E_1 : [f] \in \bar{H}^{n-1}_{k-1,i-1}\right\} = UE_1.$$

Note that since $\bar{I}^n_{k,i}E_1 \subseteq \bar{I}^n_{k,i}$ when we apply E_1 to cosets in $\bar{H}^n_{k,i}$ we have $[f]E_1 = [fE_1]$. Since any subspace containing v_1 is fixed by E_1 we have

$$UE_1 = \left\{[\langle v_1\rangle \vee f] : [f] \in \bar{H}^{n-1}_{k-1,i-1}\right\}.$$

Hence when we express U in the form $U = (\langle v_1\rangle \vee U_1) \oplus U_2$ we must have that $U_1 = H^{n-1}_{k-1,i-1}$ and $U_2 = 0$ giving

$$U = \left\{[\langle v_1\rangle \vee f] : [f] \in \bar{H}^{n-1}_{k-1,i-1}\right\}.$$

As a result if we take any $[u] \in U$ then the coset representative u must satisfy $supp(u) \subseteq L^n_{k,1}$. However we may again use the fact that G_n is transitive on L^n_k to find $g \in G_n$ such that $supp(u \cdot g) \not\subseteq L^n_{k,1}$. This contradicts our assumption that U is a G_n-module.

- If $\psi(UE_1) = \bar{H}^{n-1}_{k,i+1}$ then again we can apply the inverse mapping of the isomorphism ψ which gives $\psi^{-1}(\bar{H}^{n-1}_{k,i+1}) = \left\{[f]E_1 : f \in H^{n-1}_{k,i+1}\right\} = UE_1$. Hence in the expression $U = (\langle v_1\rangle \vee U_1) \oplus U_2$ we have $U_1 = 0$. So for any $[u] \in U$ the coset representative u must satisfy the condition $v_1 \notin supp(u)$. Since G_n is transitive on L^n_k there must exist some $g \in G_n$ such that $v_1 \in supp(u \cdot g)$ contradicting $[u] \cdot g \in U$.

- The only possibility which remains is that $\psi(UE_1) = \bar{H}^{n-1}_{k,i+1} \oplus \bar{H}^{n-1}_{k-1,i-1}$. Which means that $UE_1 = \bar{H}^n_{k,i}E_1$.

  Now suppose $UE_2 = 0$. Then $U = UE_1$ and applying ψ^{-1} gives

$$U = \left\{\left[\left(\langle v_1\rangle \vee \frac{q^{i-k}}{[i]_q}(f - \partial(g))\right) + g\right] : f \in \bar{K}^{n-1}_{k-1,i-1}, g \in \bar{K}^{n-1}_{k,i+1}\right\}.$$

We consider what happens when we apply the idempotent E_2 to such an element of U. By Lemma 4.15 we know that any space containing v_1 goes to zero when we apply E_2 and hence $\left(\langle v_1\rangle \vee \frac{q^{i-k}}{[i]_q}(f-\partial(g))\right)E_2 = 0$ for any choice of f and g. However we can find $g \in \bar{K}^{n-1}_{k,i+1}$ which has $supp(g) \cap (L^n_k)^\dagger \neq \emptyset$ and hence has $gE_2 \neq 0$ (for example $g = \langle v_2\rangle \vee h$ for any $h \in \bar{K}^{n-2}_{k-1,i}$). This contradicts our assumption that $UE_2 = 0$. So in this case we must have $UE_2 = \bar{H}^n_{k,i}E_2$ and hence $U = \bar{H}^n_{k,i}$.

This shows that the only $\bar{F}G_n$-submodules of $\bar{H}^n_{k,i}$ are 0 and $\bar{H}^n_{k,i}$ itself. Thus $\bar{H}^n_{k,i}$ is irreducible.

To show the non-isomorphic part we suppose that (k', i') is another pair of integers satisfying $2k' - i'$ and $0 < i' < \pi$. Assume for a contradiction that $\bar{H}^n_{k,i} \cong \bar{H}^n_{k',i'}$. But then this gives that $\bar{H}^n_{k,i}E_2 \cong \bar{H}^n_{k',i'}E_2$ and hence $\bar{H}^{n-2}_{k-1,i} \cong \bar{H}^{n-2}_{k'-1,i'}$, contradicting the inductive hypothesis.

□

7 Irreducible Representations of $PGL(2, q)$

For a field F and group G, any FG-module has an associated representation of G. Thus if we consider the irreducible homology modules these lead to irreducible representations of the general linear groups. In 7.1 we look at some of the work which has been done on representations of linear groups in cross-characteristic for small dimensions. Theorem 6.8 tells us that the module $\bar{H}^n_{k,i}$ is irreducible if the parameters satisfy the relation $n-1 = 2k-i$. Thus if we consider such modules $\bar{H}^n_{k,i}$ which satisfy this condition then the corresponding representations are irreducible. In the second part of this chapter we consider the small example where $n = 2$ and $k = i = 1$. In this case we are able to explicitly construct the homology module and thus we have a result which demonstrates an explicit form of the representation associated to this module.

7.1 Low-dimensional Representations

We start this section by producing a short survey of some of the literature on low-dimensional representations of linear groups in cross characteristic. This survey is by no means complete, but just indicates some of the results in this area in the last fifteen years or so.

Our starting point is the main result of two papers [19] and [20] which give the Landázuri-Seitz-Zalesskii lower bounds for the projective special linear groups in cross characteristic. Here it is shown that a lower bound for

$PSL(n,q)$ for $n = 2$ is

$$l(PSL(2,q)) = \frac{q-1}{gcd(2,q-1)}$$

with exceptions $l(PSL(2,4)) = 2$ and $l(PSL(2,9)) = 3$ and for $n \geq 3$

$$l(PSL(n,q)) = \frac{q^n - 1}{q-1} - n$$

apart from some exceptional cases where we have

$$l(PSL(3,2)) = 2$$

$$l(PSL(3,4)) = 4$$

$$l(PSL(4,2)) = 7$$

$$l(PSL(4,3)) = 26.$$

The paper [21] of Guralnick and Tiep gives several results for $PSL(n,q)$ for small n. The Table 2 below [21, Table III] gives some modular characters for such groups $PSL(n,q)$ when $n \leq 4$. We introduce the following notation. Given an integer m, we let $m_{t'}$ denote the t'-part of m, that is the largest power of t which divides m. Thus in the following table, r is defined to be

$$r = \begin{cases} (q-1)_{2'} & \text{if } 2 = p|\frac{1}{2}(q-1), \\ (q+1)_{2'} & \text{if } 2 = p|\frac{1}{2}(q+1), \\ (q-1)_{p'} & \text{if } 2 \neq p|(q-1), \\ (q+1)_{p'} & \text{if } 2 \neq p|(q+1). \end{cases}$$

Given a finite group X, the number $d_{j,p}(X)$ denotes the j^{th} degree of non-trivial projective irreducible modular representations of X and N_j denotes the number of (equivalence classes of) the corresponding representations.

Group	$(d_{1,p}, N_1)$	$(d_{2,p}, N_2)$	$(d_{3,p}, N_3)$	Conditions
$PSL(2,q)$	$(\frac{1}{2}(q-1), 2)$	$(q-1, \frac{1}{4}(q-1))$	$(q+1, \frac{1}{2}(r-1))$	$2 = p \mid \frac{1}{2}(q-1)$
$2 \nmid q$	$(\frac{1}{2}(q-1), 2)$	$(q-1, \frac{1}{2}(r-1))$	$(q+1, \frac{1}{4}(q-3))$	$2 = p \mid \frac{1}{2}(q+1)$
$q \neq 3, 5, 9$	$(\frac{1}{2}(q-1), 2)$	$(\frac{1}{2}(q+1), 2)$	$(q-1, \frac{1}{2}(q-1))$	$2 \neq p \mid (q-1)$
	$(\frac{1}{2}(q-1), 2)$	$(\frac{1}{2}(q+1), 2)$	$(q-1, \frac{1}{2}r)$	$2 \neq p \mid (q+1)$
	$(\frac{1}{2}(q-1), 2)$	$(\frac{1}{2}(q+1), 2)$	$(q-1, \frac{1}{2}(q-1))$	$p = 0$
$PSL(2,q)$	$(q-1, \frac{1}{2}q)$	$(q, 1)$	$(q+1, \frac{1}{2}(r-1))$	$p \mid (q-1)$
$2 \mid q, q \geq 8$	$(q-1, \frac{1}{2}(r+1))$	$(q+1, \frac{1}{2}q-1)$	-	$p \mid (q+1)$
	$(q-1, \frac{1}{2}q)$	$(q, 1)$	$(q+1, \frac{1}{2}(q-2))$	$p = 0$
$PSL(2,5)$	$(2, 2)$	$(4, 1)$	-	$p = 2$
$PSL(2,4)$	$(2, 2)$	$(3, 2)$	$(4, 1)$	$p = 3$
	$(2, 1)$	$(3, 1)$	$(4, 1)$	$p = 5$
	$(2, 2)$	$(3, 2)$	$(4, 2)$	$p = 0$
$PSL(2,9)$	$(3, 4)$	$(4, 2)$	$(8, 2)$	$p = 2$
	$(3, 2)$	$(4, 2)$	$(5, 2)$	$p = 5$
	$(3, 4)$	$(4, 2)$	$(5, 2)$	$p = 0$
$PSL(3,2)$	$(3, 2)$	$(4, 2)$	$(6, 3)$	$p = 0, 3$
	$(2, 1)$	$(3, 1)$	$(4, 1)$	$p = 7$
$PSL(3,4)$	$(4, 6)$	$(6, 3)$	$(8, 12)$	$p = 3$
	$(6, 2)$	$(8, 6)$	$(10, 6)$	$p = 5$
	$(6, 2)$	$(8, 12)$	$(10, 3)$	$p = 7$
	$(6, 2)$	$(8, 12)$	$(10, 6)$	$p = 0$
$PSL(4,2)$	$(7, 1)$	$(8, 1)$	$(13, 1)$	$p = 3, 5$
	$(7, 1)$	$(8, 1)$	$(14, 1)$	$p = 0, 7$

Table 2: MODULAR CHARACTERS OF THE FIRST THREE DEGREES OF SMALL GROUPS

Also in [21] we have the following result which gives the first three degrees of the irreducible modular representations of $GL(3,q)$ in characteristic p as

$$d_{1,p} = q^2 + q - \kappa_3,$$
$$d_{2,p} = q^2 + q + 1,$$
$$d_{3,p} = (q-1)(q^2-1)$$

where the integer κ_n is defined by

$$\kappa_n = \begin{cases} 1 & \text{if } p | \frac{q^n-1}{q-1} \\ 0 & \text{otherwise.} \end{cases}$$

We can compare this result with the Betti numbers we have for $H^n_{k,i}$ when $n = 3$. By Theorem 6.8 we have that $\bar{H}^n_{k,i}$ is irreducible when $n - 1 = 2k - i$. So for $n = 3$ the only non-trivial irreducible homology module is $\bar{H}^3_{2,2}$ (observe that $\bar{H}^3_{1,0} = \bar{H}^3_{3,4} = 0$). By Table 1 in Chapter 3 we have that

$$\beta^3_{2,2} = \begin{cases} 0 & \text{if } \pi = 2 \\ q^2 + q - 1 & \text{if } \pi = 3 \\ q^2 + q & \text{if } \pi = 4 \end{cases}$$

In fact, using the Betti number formula 3.6 we can show that $\beta^3_{2,2} = q^2 + q$ for $\pi \geq 4$. The definition of κ_n gives that $\kappa_3 = 1$ if p divides $q^3 - 1$ and is zero otherwise. Thus $\kappa_3 = 1$ if $\pi = 3$ and $\kappa_3 = 0$ if $\pi > 3$. We see that the first degree $d_{1,p}$ corresponds to the Betti number $\beta^3_{2,2}$ and hence matches with the degree of the irreducible representation associated to the homology module $\bar{H}^3_{2,2}$.

Following on from this, Brundan [22] gives a formula for the values of irreducible unipotent p-modular Brauer characters of $GL(n, q)$ at unipotent elements, where p is a prime not dividing q. This formula is given in terms of weight multiplicities and certain generic polynomials. However, these weight multiplicities are unknown, they are the modular Kostka numbers. This character formula leads to a degree formula which can be used to give quite powerful lower bounds for the degrees of the irreducible Brauer characters by exploiting a q-analogue of the Premet-Suprunenko bound for the Kostka numbers. The details of this are given in the paper [23] of Brundan and Kleshchev. Here it is shown that if $n \geq 5$ and L is an irreducible $FGL(n, q)$-module then

$$\dim L \leq q^{3n-9}$$

if and only if L is isomorphic to one of a certain list of modules. For each of the modules in this list the degrees of the modules are given and hence this paper gives a complete list of all irreducible $FGL(n, q)$-modules of degree $\leq q^{3n-9}$. Similarly for modules not on the list we have a strong lower bound for the degrees. The homology modules which belong to this list are only those with $n \geq 5$ and $k = 1, 2$. So for $k \geq 3$ we know that the dimension of any irreducible homology module must be bounded below by q^{3n-9}. We check this for the example of $\bar{H}^5_{3,2}$ when $\pi \geq 5$. Using the Betti number formula we have

$$\beta^5_{3,2} = \binom{5}{3}_q - \binom{5}{1}_q$$

and so we have

$$\begin{aligned}\beta^5_{3,2} &= \frac{(q^5-1)(q^4-1)}{(q^2-1)(q-1)} - \frac{q^5-1}{q-1} \\ &= \frac{q^5-1}{q-1}\left(\frac{q^4-1}{q^2-1} - 1\right) \\ &= (q^4+q^3+q^2+q+1)\left(\frac{q^4-q^2}{q^2-1}\right) \\ &= (q^4+q^3+q^2+q+1)q^2 > q^6.\end{aligned}$$

This shows that the module $\bar{H}^5_{3,2}$ satisfies the lower bound condition.

7.2 The Explicit Form of the Representation

We now consider how the irreducible homology module $\bar{H}^2_{1,1}$ can give an irreducible representation of $PGL(2,q)$. In this section we look at the explicit form of this representation with no reference to the homology module from which it is obtained. We are able to prove that we have a representation of $PGL(2,q)$ of degree q independently of any theory concerning homology.

Let q be an odd prime and let

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

where α is a primitive root modulo q. We have from [16, §7.5] a presentation for the projective general linear group when $n=2$ given by

$$PGL(2,q) = \langle S,T,Q : S^q = Q^{q-1} = T^2 = (ST)^3 = 1, Q^{-1}SQ = S^\alpha\rangle.$$

We define the following permutation matrices:

- P_* is the square permutation matrix of size $q-1$ which maps the i^{th} row to the $-\frac{1}{i} \mod q$ row.

- P_α is the square permutation matrix of size $q-1$ which maps the i^{th} row to the $(\alpha i)^{th}$ row modulo q.

Theorem 7.1. *Let $\rho : PGL(2,q) \to GL(q,p)$ be the map defined on the generators S, T and Q by*

$$\rho(S) = \begin{pmatrix} 0 & & & \\ \vdots & & I_{q-1} & \\ 0 & & & \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ -1 & & & \\ \vdots & & P_* & \\ -1 & & & \end{pmatrix},$$

$$\rho(Q) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & P_\alpha & \\ 0 & & & \end{pmatrix}.$$

Then we have

(1) For any choice of p and q the map ρ is a representation of $PGL(2,q)$,

(2) ρ is an irreducible representation if and only if p does not divide $q+1$.

Proof. (1) We must show that the five relations on the generators S, T and Q in the presentation for $PGL(2,q)$ hold for $\rho(S), \rho(T)$ and $\rho(Q)$:

- First observe that $\rho(S)$ just corresponds to the q-cycle and so we clearly have that $\rho(S)$ has order q.

- Since P_* is a permutation matrix which sends i to $-i^{-1}$ then this has order 2. Let z denote the column vector $(-1, -1, \cdots, -1)^\top$ of height $q-1$. Then
$$(\rho(T))^2 = \begin{pmatrix} -1 & 0 \\ z & P_* \end{pmatrix} \begin{pmatrix} -1 & 0 \\ z & P_* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -z + P_* z & P_*^2 \end{pmatrix}.$$
If we take the matrix product of the $q-1$ square matrix P_* with the column vector z then as P_* is a permutation matrix it just permutes the entries of z which are all the same and hence $P_* z = z$. Also P_*^2 is the identity and hence $\rho(T)^2 = 1$.

- Since α is a primitive root modulo q then α has order $q-1$. The matrix P_α takes i to αi so this matrix, and hence also $\rho(Q)$, has order $q-1$.

- To show the relation $\rho(Q)^{-1}\rho(S)\rho(Q) = \rho(S)^\alpha$ we just think of $\rho(Q)$ and $\rho(S)$ as permutations. We see that $\rho(S)(i) = i+1$ and $\rho(Q)(i) = \alpha i$. So if we consider the left hand side:
$$\rho(Q)^{-1}(i) = \alpha^{-1} i,$$
thus
$$\rho(Q)^{-1}\rho(S)(i) = \alpha^{-1} i + 1$$
which leads us to
$$\rho(Q)^{-1}\rho(S)\rho(Q)(i) = \alpha(\alpha^{-1} i + 1) = i + \alpha.$$
We also see that each time we apply $\rho(S)$ it increases the value of i by one and so
$$\rho(S)^\alpha(i) = i + \alpha.$$

Thus $\rho(Q)^{-1}\rho(S)\rho(Q) = \rho(S)^{\alpha}$.

- Finally we must show that $\rho(S)\rho(T)$ has order three. We see that

$$\rho(S)\rho(T) = \begin{pmatrix} -1 & & & \\ : & & P_* & \\ -1 & & & \\ -1 & 0 & \cdots & 0 \end{pmatrix}.$$

We know that P_* is the permutation matrix which sends the i^{th} row to the $\left(-\frac{1}{i}\right)^{th}$ row (modulo q). Now when $i = 1$ this gets sent to the $(q-1)^{th}$ row and vice versa. So the matrix has shape

$$\rho(S)\rho(T) = \begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ -1 & & & & 0 \\ : & & P & & : \\ -1 & & & & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where P is the remaining part of the matrix P_*. Now if we let z this time be the column vector of -1's of height $q-2$ then we get

$$(\rho(S)\rho(T))^3 = \begin{pmatrix} -1 & 0 & 1 \\ z & P & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ z & P & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ z & P & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ -z + Pz & P^2 & z \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ z & P & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Since P is a permutation matrix then as stated earlier, $Pz = z$. Thus

$$(\rho(S)\rho(T))^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & P^2 & z \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ z & P & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ P^2 z - z & P^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since P^2 is also a permutation matrix then $P^2 z = z$ and hence

$$(\rho(S)\rho(T))^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\rho(S)\rho(T)$ has order three if and only if the submatrix P does.

Now this submatrix P is the permutation matrix P_* with the top row and last column removed. We observe that P is the permutation matrix which sends i to $-\frac{1}{i+1}$. We show that this permutation has order 3:

$$P(i) = \frac{-1}{i+1},$$
$$P^2(i) = -\frac{1}{\frac{-1}{i+1}+1} = \frac{1}{\frac{1}{i+1}-1} = -\frac{i+1}{i},$$
$$P^3(i) = -\frac{1}{-\frac{i+1}{i}+1} = \frac{1}{\frac{i+1}{i}-1} = \frac{1}{\frac{i+1-i}{i}} = i.$$

So P has order 3 and hence so does $\rho(S)\rho(T)$.

Hence we have indeed shown that the map ρ we defined is a representation of $PGL(2, q)$ of degree q over a field of characteristic p. Note that we have made no assumptions on q and p.

(2) Now suppose that p does not divide $q+1$. We show that in this case the representation ρ is irreducible. Let M be the $\bar{F}GL(q, p)$-module associated to the representation ρ and suppose that U is an $\bar{F}GL(q, p)$-submodule of M. Let $u = (u_1, u_2, \cdots, u_q) \in U$ with $u \neq 0$. Then

$$u \cdot (\rho(S) + \rho(S)^2 + \cdots + \rho(S)^q) = (u_1 + u_2 + \cdots + u_q) \cdot (1, 1, \cdots, 1)$$

If $u_1 + u_2 + \cdots + u_q \neq 0$ then the row vector $j = (1, 1, \cdots, 1)$ is in U. Thus we have $j \cdot \rho(T) = (-1, 0, \cdots, 0) \in U$ giving $U \cdot \bar{F}GL(q, p) = M$ and thus $U = M$. So we are done unless $u_1 + u_2 + \cdots + u_q = 0$. If this is the case multiply the row vector u by $\rho(T)$. We get

$$u \cdot \rho(T) = (-u_1, -u_1 + u_{\pi(2)}, -u_1 + u_{\pi(3)}, \cdots, -u_1 + u_{\pi(q)})$$

where π is the permutation corresponding to the matrix P_*. If we sum the entries in the row vector $u \cdot \rho(T)$ then we have

$$-q \cdot u_1 + u_{\pi(2)} + u_{\pi(3)} + \cdots + u_{\pi(q)} = -q \cdot u_1 + (u_2 + u_3 + \cdots + u_q)$$

since π just permutes the numbers $2, 3, \cdots, q$. We have assumed that $u_1 + u_2 + \cdots + u_q = 0$ so $u_2 + \cdots + u_q = -u_1$ and hence the sum of the entries is $-(q+1)u_1$. Without loss of generality we can assume that $u_1 \neq 0$ (if $u_1 = 0$ use $\rho(S)$ to permute entries to make it non-zero). Thus if p does not divide $q + 1$ then we have a row vector in U whose entries do not sum to zero and we can use the above argument to show that $U = M$. So we have shown that M is an irreducible $\bar{F}GL(p,q)$-module and consequently the associated representation ρ is also irreducible.

Conversely, if $q \equiv -1 \mod p$ then it is clear that $U = \langle(1, 1, \cdots, 1)\rangle$ is invariant under $\rho(S), \rho(T)$ and $\rho(Q)$ and thus we have a proper non-trivial submodule U of M. So ρ is reducible in this case. $\square$

7.3 Construction from the Homology Module

We now show that the explicit representation $PGL(2,q) \to GL(q,p)$ given in Theorem 7.1 corresponds to the irreducible module $\bar{H}^2_{1,1}$ when p does not divide q.

The homology module $\bar{H}^n_{k,i}$ is irreducible if $n - 1 = 2k - i$. We let $n = 2$ and take $k = i = 1$. We let q be any prime power and take p to be a prime not dividing q such that $\pi(p,q) > 2$. We label the $q + 1$ subspaces of L^2_1 in the following manner:

$$y = (1\ 0), x_0 = (0\ 1), x_1 = (1\ 1), \cdots, x_i = (i\ 1), \cdots, x_{q-1} = (q-1\ 1).$$

Since $\pi > 2$ the image $\bar{I}^2_{1,1} = \partial^{\pi-1}(\bar{M}^2_\pi) = 0$ and hence the homology module is equal to the kernel $\bar{K}^2_{1,1}$. This kernel consists of elements of the form $\sum_{x \in L^2_1} f_x x$ such that $\sum f_x = 0$ and hence has a basis

$$\{y - x_0, y - x_1, \cdots, y - x_{q-1}\}.$$

Label these q basis elements for $\bar{H}^2_{1,1}$ in the following way:

$$\omega_0 = y - x_0,\ \omega_1 = y - x_1,\ \cdots,\ \omega_{q-1} = y - x_{q-1}.$$

We now take these basis elements and multiply on the right by the three generating elements for $PGL(2,q)$.

- For $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ we have that S fixes y and sends x_i to x_j where $j = i+1 \mod q$. Thus $\omega_i \cdot S = \omega_{i+1}$ for $i = 0, \cdots, q-2$ and $\omega_{q-1} \cdot S = \omega_0$. Hence the representation of the generator S is indeed the q-cycle defined by $\rho(S)$ in Theorem 7.1.

- Next we consider the second generator $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If we multiply the subspaces in L_1^2 by this element we find that T swaps y and x_0 while for $i = 1, \cdots, q-1$ we have that x_i is sent to x_j where $j = -\frac{1}{i} \mod q$. Hence $\omega_0 \cdot T = -\omega_0$ and for $i = 1, \cdots, q-1$ we have $\omega_i \cdot T = \omega_j - \omega_0$ where $j = -\frac{1}{i} \mod q$. This shows that the representation for T corresponds with $\rho(T)$.

- Finally we have the third generating element $Q = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ where α is a primitive root mod q. We observe that right multiplication by Q fixes y and x_0 while for $i = 1, \cdots, q-1$ we have that x_i is sent to x_j with $j = i \cdot \alpha \mod q$. When we consider the basis elements we have that ω_0 is fixed by right multiplication by Q while ω_i is sent to ω_j where again $j = i \cdot \alpha \mod q$ for $i = 1, \cdots, q-1$. It is clear that the representation of Q matches the description for $\rho(Q)$ in 7.1.

Hence the representation associated to the homology module $\bar{H}^2_{1,1}$ is indeed the map ρ we have described in Theorem 7.1.

Remark 7.2. (1) For this representation of the homology module we have assumed that $\pi(p,q) > 2$. If $\pi(p,q) = 2$ then $q \equiv -1 \mod p$ and in Theorem 7.1 we have shown that ρ is not irreducible in this case. When $\pi = 2$ the irreducible homology module $\bar{H}^2_{1,1}$ has dimension $q-1$ rather than q (because the dimension of $\bar{I}^2_{1,1}$ is one instead of zero).

(2) Theorem 7.1 does not require the condition that p does not divide q and so the choice of p and q is arbitrary. Thus we have an irreducible representation $\rho : PGL(2,p) \to GL(p, \mathbb{F}_p)$. This may suggest that this representation corresponds to some irreducible homology module for $n = 2$ in the case where $\mathbb{F}$ and F have the same characteristic p.

Example: If we take $q = 3$ then the group $PGL(2,3)$ is generated by

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

while the homology module $\bar{H}^2_{1,1}$ is generated by

$$a = (1\ 0) - (0\ 1), \quad b = (1\ 0) - (1\ 1), \quad c = (1\ 0) - (2\ 1).$$

We now consider how each of the generating matrices for $PGL(2,3)$ act on the generators of $\bar{H}^2_{1,1}$:

$$\begin{aligned} aS &= (1\ 0) - (1\ 1) = b, & bS &= (1\ 0) - (2\ 1) = c, & cS &= (1\ 0) - (0\ 1) = a \\ aT &= (0\ 1) - (1\ 0) = -a, & bT &= (0\ 1) - (2\ 1) = c - a, & cT &= (0\ 1) - (1\ 1) = b - a \\ aQ &= (1\ 0) - (0\ 1) = a, & bQ &= (1\ 0) - (2\ 1) = c, & cQ &= (1\ 0) - (1\ 1) = b. \end{aligned}$$

So we get the representation

$$\rho(S) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \rho(Q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

which matches the description for the representation given in Theorem 7.1 when $q = 3$.

8 Examples

We conclude by giving some examples which demonstrate the theory of the earlier chapters. These examples illustrate how the isomorphisms described in Chapters 4 and 5 work to give decompositions of the permutation module $\bar{M}_k^n$ and the homology module $\bar{H}_{k,i}^n$ in certain low-dimensional cases.

8.1 An Example of How M_k^n Decomposes

We take $\mathbb{F} = GF(3)$ and let F be a field of characteristic 2. In this case it is clear why we have to extend our field F since we do not have enough linear F-characters of $(\mathbb{F}, +)$ to define the idempotent elements E_1 and E_2 as in Chapter 4. So we take $\bar{F} = GF(4)$ to be the extension field of F where $\bar{F}$ consists of elements $a + ib$ with $a \in \{0, 1\}$ and $i^2 + i + 1 = 0$. Now we can define χ_1 to be the trivial character

$$\chi_1(0) = \chi_1(1) = \chi_1(2) = 1$$

and we take χ_2 to be the character

$$\chi_2(0) = 1, \quad \chi_2(1) = i, \quad \chi_2(2) = 1 + i.$$

We take V to be an n-dimensional vector space over $\mathbb{F}$ and denote by L_k^n the set of k-dimensional subspaces of V. Thus if we take $n = 2$ and $k = 1$ we have

$$L_1^2 = \{x_1 = (1\ 0), \quad x_2 = (0\ 1), \quad x_3 = (1\ 1), \quad x_4 = (2\ 1)\}.$$

In this example the two idempotent elements are

$$E_1 = \frac{1}{3}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\right),$$

$$E_2 = \frac{1}{3}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + (1+i)\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\right).$$

The module $\bar{M}_1^2$ is the vector space over $\bar{F}$ with L_1^2 as its basis:

$$\bar{M}_1^2 = \left\{\sum_{i=1}^{4} f_i x_i : f_i \in \bar{F}, x_i \in L_1^2\right\}.$$

The branching rule of Theorem 4.10 gives us

$$\bar{M}_1^2 = \bar{M}_1^2 E_1 \oplus \bar{M}_1^2 E_2 G_1^*$$

where

$$G_1^* = \left\{\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix} : a \in \{0,1,2\}, g \in \{1,2\}\right\}.$$

If we apply our idempotents to elements in L_1^2 we see the following

$$\begin{aligned} x_1 E_1 &= x_1 \\ x_2 E_1 &= \frac{1}{3}\left((01) + (11) + (21)\right) = x_3 E_1 = x_4 E_1 \end{aligned}$$

and

$$\begin{aligned} x_1 E_2 &= 0 \\ x_2 E_2 &= \frac{1}{3}\left((01) + i(11) + (1+i)(21)\right) \\ x_3 E_2 &= \frac{1}{3}\left((11) + i(21) + (1+i)(01)\right) \\ x_4 E_2 &= \frac{1}{3}\left((21) + i(01) + (1+i)(11)\right). \end{aligned}$$

We now note that

$$(1+i)\cdot x_2E_2 = \frac{1}{3}((1+i)(01)+(11)+i(21)) = x_3E_2,$$

$$i\cdot x_2E_2 = \frac{1}{3}(i(01)+(1+i)(11)+(21)) = x_4E_2.$$

Thus we have

$$\bar{M}_1^2E_2 = \langle x_2E_2\rangle_{\bar{F}}.$$

Then

$$\begin{aligned}\bar{M}_1^2 &= \bar{M}_1^2E_1 \oplus \bar{M}_1^2E_2G_1^* \\ &= \langle x_1, x_2E_1\rangle_{\bar{F}} \oplus \langle x_2E_2, x_2E_2\cdot g\rangle_{\bar{F}}\end{aligned}$$

where

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

So we have a new basis for the module $\bar{M}_1^2$:

$$\begin{aligned}X_1 &= x_1,\\ X_2 &= x_2E_1 = \frac{1}{3}(x_2+x_3+x_4),\\ X_3 &= x_2E_2 = \frac{1}{3}(x_2+ix_3+(1+i)x_4),\\ X_4 &= x_2E_2g = \frac{1}{3}(x_2+ix_4+(1+i)x_3).\end{aligned}$$

If we let ϕ denote the isomorphism on $\bar{M}_1^2E_1$ which is described in Chapter 4 then ϕ has the following effect on these new basis elements:

$$\phi : X_1 \mapsto \emptyset, \quad X_2 \mapsto (1), \quad X_3 \mapsto \emptyset, \quad X_4 \mapsto \emptyset.$$

Now we see that we can express the old basis in terms of the new one in the following way

$$\begin{aligned}
x_1 &= X_1, \\
x_2 &= X_2 + X_3 + X_4, \\
x_3 &= X_2 + (1+i)X_3 + iX_4, \\
x_4 &= X_2 + iX_3 + (1+i)X_4.
\end{aligned}$$

So if we take any element $f \in \bar{M}_1^2$ then we have

$$\begin{aligned}
f =& f_1x_1 + f_2x_2 + f_3x_3 + f_4x_4 \\
=& f_1X_1 + f_2(X_2 + X_3 + X_4) + f_3(X_2 + (1+i)X_3 + iX_4) \\
&+ f_4(X_2 + iX_3 + (1+i)X_4) \\
=& f_1X_1 + (f_2 + f_3 + f_4)X_2 + (f_2 + (1+i)f_3 + if_4)X_3 \\
&+ (f_2 + if_3 + (1+i)f_4)X_4.
\end{aligned}$$

Thus when we apply our isomorphism ϕ to such an element we have

$$\begin{aligned}
\phi(f) =& (f_1\emptyset, (f_2 + f_3 + f_4)(1), \\
& (f_2 + (1+i)f_3 + if_4)\emptyset, (f_2 + if_3 + (1+i)f_4)\emptyset) \in \bar{M}_0^1 \oplus \bar{M}_1^1 \oplus \bar{M}_0^0 \cdot 2.
\end{aligned}$$

8.2 A Larger Example

To see what is happening more clearly we now move to a slightly larger example with $n = 3$ and $k = 2$ over the same fields as before. So we consider the set L_2^3 which contains the following $\binom{3}{2}_3 = 13$ spaces:

$$x_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$x_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad x_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$x_7 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_8 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad x_9 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$

$$x_{10} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad x_{11} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$

$$x_{12} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad x_{13} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

The idempotent elements are

$$E_1 = \frac{1}{9} \sum_{a,b \in \{0,1,2\}} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix},$$

$$E_2 = \frac{1}{27} \sum_{b,c \in \{0,1,2\}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} + i \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ b & c & 1 \end{pmatrix} + (1+i) \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ b & c & 1 \end{pmatrix}.$$

Applying these idempotents we see:

$$
\begin{aligned}
x_iE_1 &= x_i \text{ for } i = 1, 2, 3, 4 \\
x_5E_1 &= \frac{1}{9} \sum_{a,b\in\{0,1,2\}} \begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \\
&= x_6E_1 = x_7E_1 = \cdots = x_{13}E_1
\end{aligned}
$$

and so $\bar{M}_2^3 E_1 = \langle x_1, x_2, x_3, x_4, x_5E_1 \rangle_{\bar{F}}$. Also

$$
\begin{aligned}
x_iE_2 &= 0 \text{ for } i = 1, 2, 3, 4 \\
x_5E_2 &= \frac{1}{27} \sum \begin{pmatrix} 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} + i \begin{pmatrix} 1 & 1 & 0 \\ b & c & 1 \end{pmatrix} + (1+i) \begin{pmatrix} 2 & 1 & 0 \\ b & c & 1 \end{pmatrix} \\
x_6E_2 &= \frac{1}{27} \sum \begin{pmatrix} 1 & 1 & 0 \\ b & c & 1 \end{pmatrix} + i \begin{pmatrix} 2 & 1 & 0 \\ b & c & 1 \end{pmatrix} + (1+i) \begin{pmatrix} 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} \\
x_7E_2 &= \frac{1}{27} \sum \begin{pmatrix} 2 & 1 & 0 \\ b & c & 1 \end{pmatrix} + i \begin{pmatrix} 0 & 1 & 0 \\ b & c & 1 \end{pmatrix} + (1+i) \begin{pmatrix} 1 & 1 & 0 \\ b & c & 1 \end{pmatrix}.
\end{aligned}
$$

Now if we take x_8 and apply E_2 we get

$$
\begin{aligned}
x_8E_2 = &\frac{1}{27} \sum \begin{pmatrix} 0 & 1 & 0 \\ b+1 & c & 1 \end{pmatrix} + i \begin{pmatrix} 1 & 1 & 0 \\ b+1 & c & 1 \end{pmatrix} \\
&+ (1+i) \begin{pmatrix} 2 & 1 & 0 \\ b+1 & c & 1 \end{pmatrix}.
\end{aligned}
$$

Since we are summing over all $b \in GF(3)$ then the $+1$ in the bottom left entry makes no difference and so we have $x_8E_2 = x_5E_2$. In a similar fashion

$$x_9E_2 = x_5E_2$$
$$x_{10}E_2 = x_{11}E_2 = x_6E_2$$
$$x_{12}E_2 = x_{13}E_2 = x_7E_2.$$

We also note that since $x_6E_2 = (1+i) \cdot x_5E_2$ and $x_7E_2 = i \cdot x_5E_2$ then we have

$$\bar{M}_2^3E_2 = \langle x_5E_2 \rangle_{\bar{F}}$$

and hence

$$\bar{M}_2^3 = \bar{M}_2^3E_1 \oplus \bar{M}_2^3E_2G_2^*$$

has basis

$$\{x_1, x_2, x_3, x_4, x_5E_1, x_5E_2g_1, x_5E_2g_2, \cdots, x_5E_2g_8\}$$

where the g_i are right coset representatives for G_2^* in H_2^*. We take these coset representatives to be the following elements:

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Thus we may relabel this new basis for $\bar{M}_2^3$

$$\begin{aligned}
&X_1 = x_1, \quad X_2 = x_2, \quad X_3 = x_3, \quad X_4 = x_4,\\
&X_5 = x_5E_1 = \frac{1}{9}(x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13}),\\
&X_6 = x_5E_2g_1 = \frac{1}{9}((x_5 + x_8 + x_9) + i(x_6 + x_{10} + x_{11}) + (1 + i)(x_7 + x_{12} + x_{13})),\\
&X_7 = x_5E_2g_2 = \frac{1}{9}((x_5 + x_8 + x_9) + i(x_7 + x_{12} + x_{13}) + (1 + i)(x_6 + x_{10} + x_{11})),\\
&X_8 = x_5E_2g_3 = \frac{1}{9}((x_5 + x_{12} + x_{11}) + i(x_6 + x_8 + x_{13}) + (1 + i)(x_7 + x_{10} + x_9)),\\
&X_9 = x_5E_2g_4 = \frac{1}{9}((x_5 + x_{10} + x_{13}) + i(x_7 + x_8 + x_{11}) + (1 + i)(x_6 + x_{12} + x_9)),\\
&X_{10} = x_5E_2g_5 = \frac{1}{9}(x_5 + x_{10} + x_{13}) + i(x_6 + x_{12} + x_9) + (1 + i)(x_7 + x_8 + x_{11}),\\
&X_{11} = x_5E_2g_6 = \frac{1}{9}(x_5 + x_{12} + x_{11}) + i(x_7 + x_{10} + x_9) + (1 + i)(x_6 + x_8 + x_{13}),\\
&X_{12} = x_5E_2g_7 = \frac{1}{9}(x_5 + x_6 + x_7) + i(x_{12} + x_8 + x_{10}) + (1 + i)(x_{11} + x_{13} + x_9),\\
&X_{13} = x_5E_2g_8 = \frac{1}{9}(x_5 + x_6 + x_7) + i(x_{11} + x_{13} + x_9) + (1 + i)(x_{12} + x_8 + x_{10}).
\end{aligned}$$

Applying the isomorphism ϕ to these basis elements gives

$$\phi(X_1) = (1\ 0), \quad \phi(X_2) = (0\ 1), \quad \phi(X_3) = (1\ 1), \quad \phi(X_4) = (2\ 1),$$

$$\phi(X_5) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi(X_6) = \phi(X_7) = \cdots = \phi(X_{13}) = (1).$$

We can express the new basis in terms of the old one in the following way:

$$
\begin{aligned}
x_1 &= X_1,\\
x_2 &= X_2,\\
x_3 &= X_3,\\
x_4 &= X_4,\\
x_5 &= X_5 + X_6 + X_7 + X_8 + X_9 + X_{10} + X_{11} + X_{12} + X_{13},\\
x_6 &= X_5 + X_{12} + X_{13} + (1+i)(X_6 + X_8 + X_{10}) + i(X_7 + X_9 + X_{11}),\\
x_7 &= X_5 + X_{12} + X_{13} + (1+i)(X_7 + X_9 + X_{11}) + i(X_6 + X_8 + X_{10}),\\
x_8 &= X_5 + X_6 + X_7 + (1+i)(X_8 + X_9 + X_{12}) + i(X_{10} + X_{11} + X_{13}),\\
x_9 &= X_5 + X_6 + X_7 + (1+i)(X_{10} + X_{11} + X_{13}) + i(X_8 + X_9 + X_{12}),\\
x_{10} &= X_5 + X_9 + X_{10} + (1+i)(X_6 + X_{11} + X_{12}) + i(X_7 + X_8 + X_{13}),\\
x_{11} &= X_5 + X_8 + X_{11} + (1+i)(X_6 + X_9 + X_{13}) + i(X_7 + X_{10} + X_{12}),\\
x_{12} &= X_5 + X_8 + X_{11} + (1+i)(X_7 + X_{10} + X_{12}) + i(X_6 + X_9 + X_{13}),\\
x_{13} &= X_5 + X_9 + X_{10} + (1+i)(X_7 + X_8 + X_{13}) + i(X_6 + X_{11} + X_{12}).
\end{aligned}
$$

Thus if we take any element $f = \sum_{i=1}^{13} f_i x_i \in \bar{M}_2^3$ then

$$\begin{aligned}
f =& f_1X_1 + f_2X_2 + f_3X_3 + f_4X_4 \\
&+ (f_5 + f_6 + f_7 + f_8 + f_9 + f_{10} + f_{11} + f_{12} + f_{13})X_5 \\
&+ (f_5 + f_8 + f_9 + (1+i)(f_6 + f_{10} + f_{11}) + i(f_7 + f_{12} + f_{13}))X_6 \\
&+ (f_5 + f_8 + f_9 + (1+i)(f_7 + f_{12} + f_{13}) + i(f_6 + f_{10} + f_{11}))X_7 \\
&+ (f_5 + f_{11} + f_{12} + (1+i)(f_6 + f_8 + f_{13}) + i(f_7 + f_9 + f_{10}))X_8 \\
&+ (f_5 + f_{10} + f_{13} + (1+i)(f_7 + f_8 + f_{11}) + i(f_6 + f_9 + f_{12}))X_9 \\
&+ (f_5 + f_{10} + f_{13} + (1+i)(f_6 + f_9 + f_{12}) + i(f_7 + f_8 + f_{11}))X_{10} \\
&+ (f_5 + f_{11} + f_{12} + (1+i)(f_7 + f_9 + f_{10}) + i(f_6 + f_8 + f_{13}))X_{11} \\
&+ (f_5 + f_6 + f_7 + (1+i)(f_8 + f_{10} + f_{12}) + i(f_9 + f_{11} + f_{13}))X_{12} \\
&+ (f_5 + f_6 + f_7 + (1+i)(f_9 + f_{11} + f_{13}) + i(f_8 + f_{10} + f_{12}))X_{13}.
\end{aligned}$$

Applying the isomorphism ϕ we get

$$\begin{aligned}
\phi(f) = (&f_1(1\ 0) + f_2(0\ 1) + f_3(1\ 1) + f_4(2\ 1), \\
&f_5' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, f_6'(1), f_7'(1), f_8'(1), f_9'(1), f_{10}'(1), f_{11}'(1), f_{12}'(1), f_{13}'(1)) \\
&\in (\bar{M}_1^2, \bar{M}_2^2, \bar{M}_1^1 \cdot 8)
\end{aligned}$$

where f_i' is the coefficient for X_i in the above expression $f = \sum_{i=1}^{13} f_i' X_i$.

8.3 An Example of How the Homology Module Decomposes

We now look at a different example which will enable us to see how the isomorphism described in Chapter 5 decomposes the homology module $\bar{H}^n_{k,i}$ into G_{n-1}-modules. We consider the specific example where $n = 4$, $k = 2$ and $i = 1$, taking our fields to be $\mathbb{F} = GF(2)$ and F with characteristic $p = 5$.

So we take the module $\bar{M}^4_2$ over $\bar{F}$ which has a basis consisting of the 2-dimensional subspaces of a 4-dimensional vector space V over $GF(2)$. There are $\binom{4}{2}_2 = 35$ such subspaces, represented by the following 2×4 matrices:

$$x_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$x_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad x_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$x_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad x_8 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad x_9 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$x_{10} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad x_{11} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad x_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x_{13} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad x_{14} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_{15} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$x_{16} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad x_{17} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad x_{18} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$x_{19} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad x_{20} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_{21} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$x_{22} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_{23} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad x_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$x_{25} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad x_{26} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad x_{27} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$x_{28} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_{29} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad x_{30} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x_{31} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad x_{32} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad x_{33} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$x_{34} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad x_{35} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Thus for any $f \in \bar{M}_2^4$ we have

$$f = \sum_{i=1}^{35} f_i x_i$$

where $f_i \in \mathbb{F}_5$. As discussed at the beginning of Section 5 we can consider the inclusion map ∂ in terms of matrices and thus in this case we have

$$\partial := (1\ 0) + (0\ 1) + (1\ 1).$$

Labelling the subspaces of L_1^4 in the following way

$$y_1 = (1\ 0\ 0\ 0) \quad y_2 = (1\ 0\ 0\ 1) \quad y_3 = (1\ 0\ 1\ 0)$$

$$y_4 = (1\ 1\ 0\ 0) \quad y_5 = (1\ 0\ 1\ 1) \quad y_6 = (1\ 1\ 0\ 1)$$

$$y_7 = (1\ 1\ 1\ 0) \quad y_8 = (1\ 1\ 1\ 1) \quad y_9 = (0\ 0\ 0\ 1)$$

$$y_{10} = (0\ 0\ 1\ 0) \quad y_{11} = (0\ 1\ 0\ 0) \quad y_{12} = (0\ 0\ 1\ 1)$$

$$y_{13} = (0\ 1\ 0\ 1) \quad y_{14} = (0\ 1\ 1\ 0) \quad y_{15} = (0\ 1\ 1\ 1)$$

we see that

$$\begin{array}{lll}
\partial(x_1) = y_1 + y_4 + y_{11} & \partial(x_2) = y_1 + y_3 + y_{10} & \partial(x_3) = y_1 + y_7 + y_{14} \\
\partial(x_4) = y_1 + y_2 + y_9 & \partial(x_5) = y_1 + y_5 + y_{12} & \partial(x_6) = y_1 + y_6 + y_{13} \\
\partial(x_7) = y_1 + y_8 + y_{15} & \partial(x_8) = y_{10} + y_{11} + y_{14} & \partial(x_9) = y_3 + y_7 + y_{11} \\
\partial(x_{10}) = y_4 + y_7 + y_{10} & \partial(x_{11}) = y_3 + y_4 + y_{14} & \partial(x_{12}) = y_9 + y_{11} + y_{13} \\
\partial(x_{13}) = y_2 + y_6 + y_{11} & \partial(x_{14}) = y_4 + y_6 + y_9 & \partial(x_{15}) = y_2 + y_4 + y_{13} \\
\partial(x_{16}) = y_{11} + y_{12} + y_{15} & \partial(x_{17}) = y_5 + y_8 + y_{11} & \partial(x_{18}) = y_4 + y_8 + y_{12} \\
\partial(x_{19}) = y_4 + y_5 + y_{15} & \partial(x_{20}) = y_4 + y_8 + y_{12} & \partial(x_{21}) = y_2 + y_5 + y_{10} \\
\partial(x_{22}) = y_3 + y_5 + y_9 & \partial(x_{23}) = y_2 + y_3 + y_{12} & \partial(x_{24}) = y_{10} + y_{13} + y_{15} \\
\partial(x_{25}) = y_6 + y_8 + y_9 & \partial(x_{26}) = y_3 + y_8 + y_{13} & \partial(x_{27}) = y_3 + y_6 + y_{15} \\
\partial(x_{28}) = y_9 + y_{14} + y_{15} & \partial(x_{29}) = y_2 + y_8 + y_{14} & \partial(x_{30}) = y_7 + y_8 + y_9 \\
\partial(x_{31}) = y_2 + y_7 + y_{15} & \partial(x_{32}) = y_{12} + y_{13} + y_{14} & \partial(x_{33}) = y_5 + y_6 + y_{14} \\
\partial(x_{34}) = y_5 + y_7 + y_{13} & \partial(x_{35}) = y_6 + y_7 + y_{12}. &
\end{array}$$

This gives that $f \in \bar{K}^4_{2,1}$ if and only if the coefficients f_i satisfy the following system of 15 equations:

$$f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 = 0 \tag{8.1}$$

$$f_4 + f_{13} + f_{15} + f_{21} + f_{23} + f_{29} + f_{31} = 0 \tag{8.2}$$

$$f_2 + f_9 + f_{11} + f_{22} + f_{23} + f_{26} + f_{27} = 0 \tag{8.3}$$

$$f_1 + f_{10} + f_{11} + f_{14} + f_{15} + f_{18} + f_{19} = 0 \tag{8.4}$$

$$f_5 + f_{17} + f_{19} + f_{21} + f_{22} + f_{33} + f_{34} = 0 \tag{8.5}$$

$$f_6 + f_{13} + f_{14} + f_{25} + f_{27} + f_{33} + f_{35} = 0 \tag{8.6}$$

$$
\begin{aligned}
f_3 + f_9 + f_{10} + f_{30} + f_{31} + f_{34} + f_{35} &= 0 && (8.7)\\
f_7 + f_{17} + f_{18} + f_{25} + f_{26} + f_{29} + f_{30} &= 0 && (8.8)\\
f_4 + f_{12} + f_{14} + f_{20} + f_{22} + f_{28} + f_{30} &= 0 && (8.9)\\
f_2 + f_8 + f_{10} + f_{20} + f_{21} + f_{24} + f_{25} &= 0 && (8.10)\\
f_1 + f_8 + f_9 + f_{12} + f_{13} + f_{16} + f_{17} &= 0 && (8.11)\\
f_5 + f_{16} + f_{18} + f_{20} + f_{23} + f_{32} + f_{35} &= 0 && (8.12)\\
f_6 + f_{12} + f_{15} + f_{24} + f_{26} + f_{32} + f_{34} &= 0 && (8.13)\\
f_3 + f_8 + f_{11} + f_{28} + f_{29} + f_{32} + f_{33} &= 0 && (8.14)\\
f_7 + f_{16} + f_{19} + f_{24} + f_{27} + f_{28} + f_{31} &= 0. && (8.15)
\end{aligned}
$$

In this example the idempotent elements are

$$E_1 = \frac{1}{8} \sum_{\alpha_i \in \mathbb{F}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_2 & 1 & 0 & 0 \\ \alpha_3 & 0 & 1 & 0 \\ \alpha_4 & 0 & 0 & 1 \end{pmatrix}$$

and

$$E_2 = \frac{1}{32} \sum_{\alpha_i, \beta_j \in \mathbb{F}} \chi_2(\alpha_2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_2 & 1 & 0 & 0 \\ \alpha_3 & \beta_3 & 1 & 0 \\ \alpha_4 & \beta_4 & 0 & 1 \end{pmatrix}.$$

If we apply E_1 to the elements in L_2^4 we have

$$x_i E_1 = x_i \text{ for } i = 1, \cdots, 7.$$

For $x \in \{x_8, x_9, x_{10}, x_{11}\}$ we get

$$xE_1 = \frac{1}{4} \sum_{\alpha,\beta \in \mathbb{F}} \begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \end{pmatrix}.$$

For $x \in \{x_{12}, x_{13}, x_{14}, x_{15}\}$ we get

$$xE_1 = \frac{1}{4} \sum_{\alpha,\beta \in \mathbb{F}} \begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & 0 & 0 & 1 \end{pmatrix}.$$

For $x \in \{x_{16}, x_{17}, x_{18}, x_{19}\}$ we get

$$xE_1 = \frac{1}{4} \sum_{\alpha,\beta \in \mathbb{F}} \begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & 0 & 1 & 1 \end{pmatrix}.$$

For $x \in \{x_{20}, x_{21}, x_{22}, x_{23}\}$ we get

$$xE_1 = \frac{1}{4} \sum_{\alpha,\beta \in \mathbb{F}} \begin{pmatrix} \alpha & 0 & 1 & 0 \\ \beta & 0 & 0 & 1 \end{pmatrix}.$$

For $x \in \{x_{24}, x_{25}, x_{26}, x_{27}\}$ we get

$$xE_1 = \frac{1}{4} \sum_{\alpha,\beta \in \mathbb{F}} \begin{pmatrix} \alpha & 0 & 1 & 0 \\ \beta & 1 & 0 & 1 \end{pmatrix}.$$

For $x \in \{x_{28}, x_{29}, x_{30}, x_{31}\}$ we get

$$xE_1 = \frac{1}{4} \sum_{\alpha,\beta \in \mathbb{F}} \begin{pmatrix} \alpha & 1 & 1 & 0 \\ \beta & 0 & 0 & 1 \end{pmatrix}.$$

For $x \in \{x_{32}, x_{33}, x_{34}, x_{35}\}$ we get

$$xE_1 = \frac{1}{4} \sum_{\alpha,\beta \in \mathbb{F}} \begin{pmatrix} \alpha & 1 & 1 & 0 \\ \beta & 1 & 0 & 1 \end{pmatrix}.$$

Thus if we take $f = \sum_{i=1}^{35} f_i x_i \in \bar{M}_2^4$ then

$$\begin{aligned} fE_1 =& f_1X_1 + f_2X_2 + f_3X_3 + f_4X_4 + f_5X_5 + f_6X_6 + f_7X_7 \\ &+ (f_8 + f_9 + f_{10} + f_{11})X_8 + (f_{12} + f_{13} + f_{14} + f_{15})X_9 \\ &+ (f_{16} + f_{17} + f_{18} + f_{19})X_{10} + (f_{20} + f_{21} + f_{22} + f_{23})X_{11} \\ &+ (f_{24} + f_{25} + f_{26} + f_{27})X_{12} + (f_{28} + f_{29} + f_{30} + f_{31})X_{13} \\ &+ (f_{32} + f_{33} + f_{34} + f_{35})X_{14} \end{aligned}$$

where

$$\begin{aligned} X_i &= x_i \text{ for } i = 1, \cdots, 7, \\ X_8 &= x_8E_1 = \frac{1}{4}(x_8 + x_9 + x_{10} + x_{11}), \\ X_9 &= x_{12}E_1 = \frac{1}{4}(x_{12} + x_{13} + x_{14} + x_{15}), \\ X_{10} &= x_{16}E_1 = \frac{1}{4}(x_{16} + x_{17} + x_{18} + x_{19}), \\ X_{11} &= x_{20}E_1 = \frac{1}{4}(x_{20} + x_{21} + x_{22} + x_{23}), \\ X_{12} &= x_{24}E_1 = \frac{1}{4}(x_{24} + x_{25} + x_{26} + x_{27}), \\ X_{13} &= x_{28}E_1 = \frac{1}{4}(x_{28} + x_{29} + x_{30} + x_{31}), \\ X_{14} &= x_{32}E_1 = \frac{1}{4}(x_{32} + x_{33} + x_{34} + x_{35}). \end{aligned}$$

Thus we can write $fE_1 = (\langle v_1 \rangle \vee f_v) + l^{(1)}E_1$ where

$$f_v = f_1(1\ 0\ 0) + f_2(0\ 1\ 0) + f_3(1\ 1\ 0) + f_4(0\ 0\ 1) + f_5(0\ 1\ 1) + f_6(1\ 0\ 1) + f_7(1\ 1\ 1)$$

and

$$
\begin{aligned}
l^{(1)} =& g_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + g_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + g_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} + g_4 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&+ g_5 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + g_6 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + g_7 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

where

$$
\begin{aligned}
g_1 &= f_8 + f_9 + f_{10} + f_{11}, & g_2 &= f_{12} + f_{13} + f_{14} + f_{15}, \\
g_3 &= f_{16} + f_{17} + f_{18} + f_{19}, & g_4 &= f_{20} + f_{21} + f_{22} + f_{23}, \\
g_5 &= f_{24} + f_{25} + f_{26} + f_{27}, & g_6 &= f_{28} + f_{29} + f_{30} + f_{31}, \\
g_7 &= f_{32} + f_{33} + f_{34} + f_{35}.
\end{aligned}
$$

The isomorphism ψ on $\bar{K}^n_{k,i}E_1$ given in Chapter 5 is determined by

$$\psi : fE_1 \mapsto ([l^{(1)}], [q^{k-i}[i]_q f_v + \partial(l^{(1)})]).$$

If we consider the second part then we have

$$
\begin{aligned}
q^{k-i}[i]_q f_v + \partial(l^{(1)}) =& 2f_v + \partial(l^{(1)}) \\
=& (2f_1 + g_1 + g_2 + g_3)(1\ 0\ 0) + (2f_2 + g_1 + g_4 + g_5)(0\ 1\ 0) \\
&+ (2f_3 + g_1 + g_6 + g_7)(1\ 1\ 0) + (2f_4 + g_2 + g_4 + g_6)(0\ 0\ 1) \\
&+ (2f_5 + g_3 + g_4 + g_7)(0\ 1\ 1) + (2f_6 + g_2 + g_5 + g_7)(1\ 0\ 1) \\
&+ (2f_7 + g_3 + g_5 + g_6)(1\ 1\ 1).
\end{aligned}
$$

Adding equation (8.4) to (8.11) we get

$$
\begin{aligned}
0 =&(f_1 + f_{10} + f_{11} + f_{14} + f_{15} + f_{18} + f_{19}) \\
&+ (f_1 + f_8 + f_9 + f_{12} + f_{13} + f_{16} + f_{17}) \\
=&2f_1 + (f_8 + f_9 + f_{10} + f_{11}) + (f_{12} + f_{13} + f_{14} + f_{15}) \\
&+ (f_{16} + f_{17} + f_{18} + f_{19}) \\
=&2f_1 + g_1 + g_2 + g_3.
\end{aligned}
$$

and similarly we can show

$$
\begin{aligned}
2f_2 + g_1 + g_4 + g_5 = 0 &\quad \text{from } (8.3) + (8.10) \\
2f_3 + g_1 + g_6 + g_7 = 0 &\quad \text{from } (8.7) + (8.14) \\
2f_4 + g_2 + g_4 + g_6 = 0 &\quad \text{from } (8.2) + (8.9) \\
2f_5 + g_3 + g_4 + g_7 = 0 &\quad \text{from } (8.5) + (8.12) \\
2f_6 + g_2 + g_5 + g_7 = 0 &\quad \text{from } (8.6) + (8.13) \\
2f_7 + g_3 + g_5 + g_6 = 0 &\quad \text{from } (8.8) + (8.15).
\end{aligned}
$$

Hence

$$2f_v + \partial(l^{(1)}) = 0$$

and so in our example we have that

$$\psi(fE_1) = ([l^{(1)}], [0]).$$

Notice that $\partial^2(l^{(1)}) = 0$ since $g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + g_7 = 0$ and so $l^{(1)}$ is indeed an element of $\bar{K}^3_{1,2}$.

When we apply the second idempotent E_2 the only subspaces for which x_iE_2 does not equal zero are $x_8, \cdots, x_{19}$. We have

$$x_8E_2 = x_9E_2 = \frac{1}{8}\sum_{\alpha,\beta,\gamma\in\mathbb{F}} \chi_2(\alpha)\begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & \gamma & 1 & 0 \end{pmatrix}$$

$$x_{10}E_2 = x_{11}E_2 = -\frac{1}{8}\sum_{\alpha,\beta,\gamma\in\mathbb{F}} \chi_2(\alpha)\begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & \gamma & 1 & 0 \end{pmatrix}$$

$$x_{12}E_2 = x_{13}E_2 = \frac{1}{8}\sum_{\alpha,\beta,\gamma\in\mathbb{F}} \chi_2(\alpha)\begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & \gamma & 0 & 1 \end{pmatrix}$$

$$x_{14}E_2 = x_{15}E_2 = -\frac{1}{8}\sum_{\alpha,\beta,\gamma\in\mathbb{F}} \chi_2(\alpha)\begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & \gamma & 0 & 1 \end{pmatrix}$$

$$x_{16}E_2 = x_{17}E_2 = \frac{1}{8}\sum_{\alpha,\beta,\gamma\in\mathbb{F}} \chi_2(\alpha)\begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & \gamma & 1 & 1 \end{pmatrix}$$

$$x_{18}E_2 = x_{19}E_2 = -\frac{1}{8}\sum_{\alpha,\beta,\gamma\in\mathbb{F}} \chi_2(\alpha)\begin{pmatrix} \alpha & 1 & 0 & 0 \\ \beta & \gamma & 1 & 1 \end{pmatrix}.$$

Now

$$\begin{aligned} f^{\ddagger}E_2 =& (f_8 + f_9 - f_{10} - f_{11})x_8E_2 + (f_{12} + f_{13} - f_{14} - f_{15})x_{12}E_2 \\ &+ (f_{16} + f_{17} - f_{18} - f_{19})x_{16}E_2 \end{aligned}$$

and thus

$$f^{\dagger} = h_1(1\ 0) + h_2(0\ 1) + h_3(1\ 1)$$

where $h_1 = f_8 + f_9 - f_{10} - f_{11}$, $h_2 = f_{12} + f_{13} - f_{14} - f_{15}$ and $h_3 = f_{16} + f_{17} - f_{18} - f_{19}$. So $\psi : fE_2 \mapsto f^{\dagger}$. By (8.11) – (8.4) we have $h_1 + h_2 + h_3 = 0$ and so $\partial(f^{\dagger}) = 0$ and hence $f^{\dagger} \in \bar{K}^2_{1,1}$.

Remark 8.1. The fact that we get zero in the second component when looking at $\psi(fE_1)$ fits in with the Betti numbers for this example:

$$\beta^4_{2,1} = 20, \quad \beta^3_{2,2} = 6, \quad \beta^3_{1,0} = 0, \quad \beta^2_{1,1} = 2.$$

Note also that

$$\begin{aligned}\beta^3_{2,2} + \beta^3_{1,0} + (q^{n-1} - 1)\beta^2_{1,1} &= 6 + 0 + 7 \cdot 2 \\ &= 20 = \beta^4_{2,1}\end{aligned}$$

verifying that this example satisfies the relation proved in Section 2.

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