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REGULARITY IN A GRAPH

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Let G be a finite undirected graph that has no multiple edges nor loops. The vertex set will be denoted by V and the edge set by E. We consider a partition Q of E into disjoint classes $Q_1,\dots,Q_j,\dots$. For a vertex v we define the degree $d(v,j)$ as the number of edges in the class Q_j that have v as an end vertex. Similarly, if P is a partition of V into disjoint classes $P_1,\dots,P_i,\dots,$ we define $d(e,i)$ to be the number of end vertices of the edge e that belong to the class P_i. A pair (P,Q) is called a *colouring* of G.

DEFINITION : A colouring (P,Q) is *regular* if a) for any two vertices v,v' in the same class the degrees $d(v,j)$ and $d(v',j)$ are equal for all j; and if b) for any two edges e,e' in the same class the degrees $d(e,i)$ and $d(e',i)$ are equal for all i.

In this note I shall report on the properties of regular colourings and their role in the investigation of graph symmetries. Detailed information about the results presented here is available in [3]. Note that the definition of regularity here is equivalent to the notion of tactical decompositions in the framework of general incidence structures. In the literature on graphs also the term "equitable" or "feasible" is used. Colourings are ordered in a natural way by $(P,Q) \leq (P',Q')$ if every class of the first colouring is a union of classes of the second one. The fundamental fact about regular colourings is the following:

THEOREM 1 : *Every colouring of* G *has a unique, minimal, regular refinement. This is to say: for an arbitrary* (P,Q) *there is a regular colouring* $(P',Q') \geq (P,Q)$ *such that* $(P',Q') \leq (P'',Q'')$ *whenever* $(P'',Q'') \geq (P,Q)$ *is a* *regular colouring.*

Details of the proof of this theorem and an algorithm to produce the regular refinement may be found in [2], in particular, theorem 2.3.

GRAPH SYMMETRIES : Let Γ be a group of automorphisms of G. One verifies quite easily that the orbits of Γ on vertices and edges satisfy the regularity condition. Hence any group of automorphisms produces a regular colouring of G. An immediate consequence of theorem 1 is

COROLLARY 1 : *Let* C = (P,Q) *be a colouring of* G *and* Γ *a group of automorphisms which leave each class of* C *invariant. Then* Γ *also leaves every class of the regular refinement* C' *of* C *invariant.*

In particular, the regular refinement C_0 of the trivial colouring $\mathbf{0}$ = (V,E) consisting of one vertex and one edge class, is invariant under the full automorphism group of G. C_0 is the coarsest regular colouring and yields a decomposition of G into regular subgraphs which are linked among each other in a regular fashion. This decomposition is unique by theorem 1 and we shall call it the *regular decomposition* of G. The subgroups spanned by the vertex classes in C_0 will be called the *regularity components* of G. Hence the regularity components are invariant under automorphisms and the automorphism group of G is contained in the direct product of the automorphism groups of all its regularity components.

Regular colourings also allow to introduce a notion of rigidity. Let R be a set of vertices $v_1,\dots,v_r$ and edges $e_1,\dots,e_s$ and let C be the colouring where each class either consists of a single element of R or of all the remaining vertices or edges not in R. We shall call R *rigid* provided the regular refinement of C is the trivial colouring $\mathbb{1}$ whose classes consist of individual vertices or edges. A regular component G_1 is called *faithful* if its vertex set is rigid.

COROLLARY 2 : *Suppose* R *is a rigid set in* G. *Then every automorphism is uniquely determined by the images of all elements in* R. *If* G_1 *is a faithful regularity component then the restriction of automorphisms of* G *to* G_1 *is an* *endomorphism from* Aut(G) *into* Aut(G_1).

As an example, consider the graph in figure (a) below. The same graph is shown again in figure (b), this time with its regularity components G_1, G_2, G_3 and G_4. The subgraph G_2 happens to be faithful and therefore Aut(G) has order at most two. (In fact, all regularity components except G_1 are faithful.) Now it remains to show that the automorphism of G_2 can be extended to the whole of G and so the automorphism group has order 2.

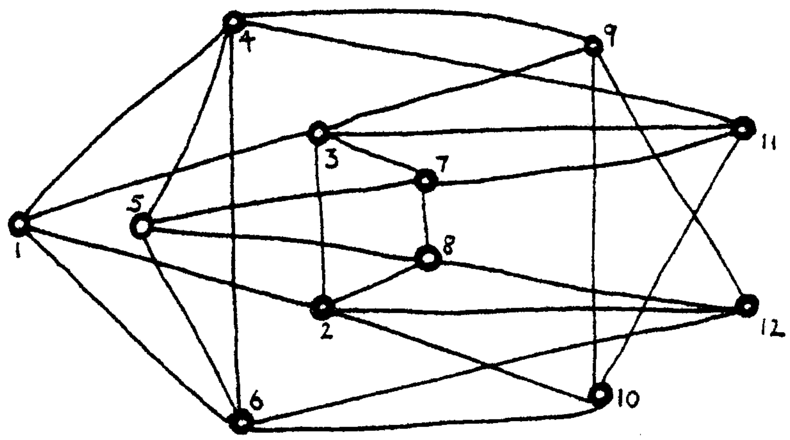

Figure (a)

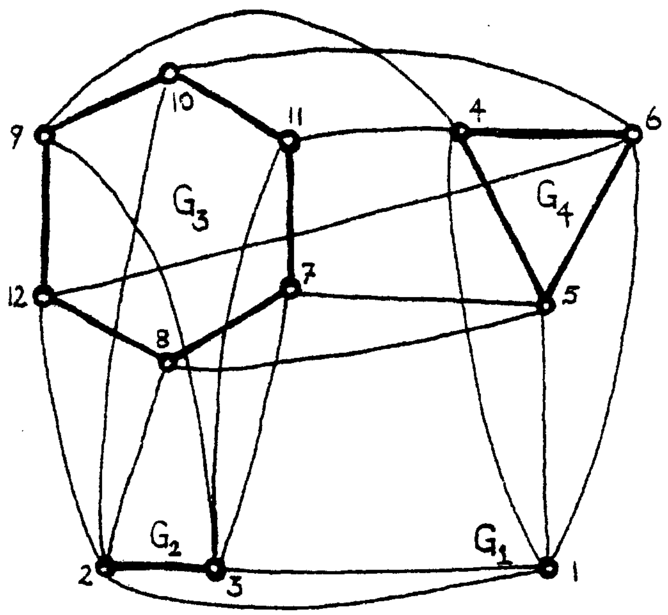

Figure (b)

GRAPH SPECTRA : Let P be the vertex partition of a regular colouring C and let $d_{i,j}$ be the number of edges from a vertex v in P_i to vertices in P_j. By definition, this number does not depend on v but only on the class that contains v. The *spectrum* of C, $\mathrm{Spec}(C)$, is the set of eigenvalues of the matrix $D = (d_{i,j})$. If C is the trivial colouring $\mathbb{1}$ then D is just the adjacency matrix of G. In this case $\mathrm{Spec}(C)$ is the *spectrum* of G. The spectrum is an important invariant of graphs that often has interesting physical interpretations. A molecule, for instance, can be regarded as a graph whose vertices are the atoms and whose edges are the bonds between atoms. For certain molecules, the graph spectrum coincides with the energy levels of free electrons. (See for instance chapter 8 in [1].)

The theorem of Petersdorf and Sachs states that the spectrum of a regular colouring is contained in the graph spectrum, cf. chapter 4 in [1]. More generally, we prove

PROPOSITION 1 : *Let* $C \leq C'$ *be regular colourings. Then* $\mathrm{Spec}(C) \subset \mathrm{Spec}(C')$.

PROOF : Suppose that $D = (d_{ij})$ is the degree matrix of C and let λ be an eigenvalue of D with eigenvector $x = (x_1, x_2, \ldots, x_j, \ldots)^T$. The degree matrix D' of C' is a blocked matrix (D'_{ij}) in which the block row $D'_{i1}, \ldots, D'_{ij}, \ldots$ corresponds to the C'-classes that are contained in the vertex class P_i of C. As C' is regular, the row sum in each block D'_{ij} is constant and equal to d_{ij}. Now consider the blocked vector $x' = (x'_1, x'_2, \ldots, x'_j, \ldots)^T$ in which $x'_j = (x_j, x_j, \ldots, x_j)$ has length n_j, the size of D'_{jj}. It now follows immediately that $D'.x' = \lambda.x'$ and hence λ is an eigenvalue of D'.

ORBIT THEOREMS : The vertex classes and the edge classes of a regular colouring are related only via the condition on the degrees $d(v,j)$ and $d(e,i)$. It turns out that this requirement yields a rather strong interdependence.

THEOREM 2A : *Let* G *be a connected graph that contains at least one cycle of odd length and suppose* (P,Q) *is a regular colouring. Then* Q *determines* P *completely and* Q *has at least as many classes as* P.

An important conclusion for automorphism groups is the fact that the orbits on edges always determine the vertex orbits, independently of the particular group under consideration. Similar results are the theorem of Livingstone & Wagner on permutation groups and the Dembowski-Parker-Hughes theorem on 2-designs. They rest upon the fact that the linear rank of the relevant incidence matrix is maximal (cf.[2]). This too is the case for the graphs in theorem 2A (compare theorem 3.1 and theorem 3.2 in [3]). For a bipartite graph (thus containing no cycles of odd length) the situation is slightly different:

THEOREM 2B : *Let* G *be a connected graph that contains no cycle of odd length and suppose* (P,Q) *is a regular colouring. Then* Q *determines* P *completely unless* G *is regular and* P *has at most* 2 *classes of equal size. The number of classes in* P *is less or equal to the number of classes in* Q *plus* 1.

As an example of the exceptional case, consider for instance the graph $K_{m,n}$. Its automorphism group is transitive on edges and has two orbits on vertices if $m \neq n$. If $m = n$, G is regular and the two vertex classes can be interchanged by an automorphism of order 2.

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