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Permutation groups on unordered sets I

By

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I. Introduction. Let G be a permutation group on a finite or infinite set S. Consider the system X_k of all k-element subsets of S and the natural action of G on X_k. The numbers n_k of G-orbits on X_k form a non-decreasing sequence for $k \leqq \frac{1}{2} \cdot |S|$, but little else is known apart from this fact. See [1, 3].

In this note we examine the growth of n_k (if these numbers are finite) in terms of the groups induced by G on subsets of S. If G is $(k-1)$-fold homogeneous on S and $l \geqq k$, a rough estimate for the growth rate is $\binom{n_k}{l-k+1} \leqq \binom{l}{k-1} \cdot n_l$. Much sharper results are obtained if the action induced on subsets is rich.

The notation used is standard. The setwise and pointwise stabilizers of a subset Y of S are denoted by $G_{\{Y\}}$ and $G_{(Y)}$ respectively. The group $G^Y = G_{\{Y\}}/G_{(Y)}$ always is considered as a permutation group on Y. The orbits of G on X_k are denoted by $X_k(G)$ and $n_k = |X_k(G)|$.

II. Arrangements. Let H be a group acting on a set Y of finite size l and let $x\,(\neq Y)$ be a subset of Y. We allow x to be empty. An *arrangement* is a collection $\{x; y_1, y_2, \ldots, y_t\}$ such that a) all y_i have size $k = |x| + 1$ and contain x, b) $Y = \cup\, y_i$ and c) for $i \neq j$, y_i and y_j belong to different H-orbits. The set x is called the *centre* of the arrangement. Clearly $t = l - k + 1$. A second arrangement $A' = \{x'; y'_1, y'_2, \ldots, y'_t\}$ is *isomorphic* to $A = \{x;\ y_1, y_2, \ldots, y_t\}$ if there is some h in H such that $A^h = A'$. Notice that two arrangements are isomorphic if and only if their centres belong to the same H-orbit. The total number of non-isomorphic arrangements with centre size $k-1$ is denoted by $m(H, k)$. Clearly $m(H, k) \leqq \binom{l}{k-1}$ and equality holds if and only if H is the identity on Y. We determine the structure of groups for which arrangements exist and determine the numbers $m(H, k)$ for some small values of k.

Theorem 2.1. *Let $H \neq 1$ be a permutation group on a set Y of size l and let $k \leqq l$. Suppose that $x = \{\alpha, \beta, \ldots\}$ is the centre of an arrangement with $|x| = k - 1$. Then*

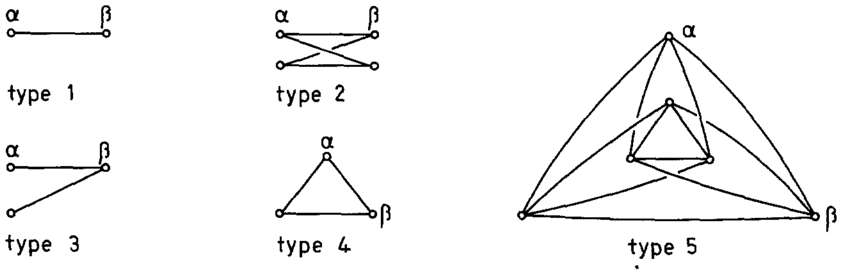
i) $k > 1$. (*In fact $m(H, 1) = 0$ if $H \neq 1$ and $m(H, 1) = 1$ if $H = 1$.*)

ii) *If $k = 2$, then H is an elementary abelian 2-group and $m(H, 2)$ is the number of H-orbits on the points of Y that have length $|H|$.*

*) Questo lavoro è stato fatto mentre ero all'Università di Milano per un anno. Vorrei ringraziare tutti per l'eccellente ospitalita.

iii) *If* $k = 3$, *then* $|H_{\{x\}}| \leqq 2$. *If* $|H_{\{x\}}| = 2$, *then* $H = \mathrm{Sym}(2)$ *and* $m(H, 3) = l - 1$ *or* $H = \mathrm{Sym}(3)$ *and* $m(H, 3) = 1$

iv) *If* $k = 3$ *and* $|H_{\{x\}}| = 1$, *then* $|H_\alpha|$ *and* $|H_\beta|$ *are at most* 2. *Let* O_α *and* O_β *be the orbits of* α *and* β *respectively. Then the graph on* $O_\alpha \cup O_\beta$ *with edge set* x^H *has the following connected components*: type 1 *for* $|H_\alpha| = |H_\beta| = 1$ *and* $O_\alpha \neq O_\beta$, type 2 *for* $|H_\alpha| = |H_\beta| = 2$ *and* $O_\alpha \neq O_\beta$, type 3 *for* $|H_\alpha| = 1$, $|H_\beta| = 2$ *and* $O_\alpha \neq O_\beta$, type 4 *for* $|H_\alpha| = 1$ *and* $O_\alpha = O_\beta$, *or* type 5 *for* $|H_\alpha| = 2$ *and* $O_\alpha = O_\beta$.



P r o o f. First we note that $H_{\{x\}}$ acts as the identity on $Y - x$ if x is a centre of an arrangement. This in particular proves the statement i). If $k = 2$, let O be the orbit of α. If $h \neq 1$ is in H, then also $\beta = \alpha^h$ is a centre and $\beta \in \{\alpha, \beta\} \cap \{\alpha, \beta\}^h$ implies that these two sets are the same. Therefore $\beta^h = \alpha$, $h^2 = 1$ and H is an elementary abelian 2-group of order $|H| = |O|$. Vice versa, if H is an elementary abelian 2-group and if γ belongs to an orbit of length $|H|$, then γ is the centre of an arrangement. For if $\gamma \in \{\gamma, \delta\} \cap \{\gamma, \delta\}^h$ for some h in H, then either $\gamma^h = \gamma$ and $h = 1$ or $\gamma^h = \delta$ and $\gamma = \delta^h$. In both cases $\{\gamma, \delta\}$ is fixed by h and so γ is a centre. This proves ii).

Now we assume that $x = \{\alpha, \beta\}$ is a centre of size $k - 1 = 2$. By the initial remark, $|H_{\{x\}}|$ has size at most 2. Consider the case $|H_{\{x\}}| = 2$. Let O be the orbit containing α and β. If $O = x$, $H = \mathrm{Sym}(2)$. If $O \neq x$, then any H-image is a centre again and as there is a transposition $(\alpha, \beta)(.)\ldots(.)$, the images must intersect x in a point. Counting these images we obtain $|x^H| = \frac{1}{2} \cdot |H| = (|O| - 2) \cdot 2 + 1$, or $|O| \cdot (4 - |H_\alpha|) = 6$. Therefore $|O| = 3$, $|H_\alpha| = 2$ and H is the symmetric group on O. As H is generated by transpositions fixing all points in $Y - x$, H acts as the identity on $Y - x$ and the only centres are the three isomorphic pairs in O. Therefore $m(H, 3) = 1$ which proves iii).

Secondly consider the case $|H_{\{x\}}| = 1$. Suppose that k in H_α displaces β i.e. $k: \gamma \to \beta \to \delta$. As $\{\alpha, \beta, \gamma\}$ and $\{\alpha, \beta, \gamma\}^k$ both contain x we conclude that $\gamma = \delta$. Therefore $|H_\alpha| \leqq 2$ and similarly $|H_\beta| \leqq 2$. Consider the graph on the vertices $O_\alpha \cup O_\beta$ with edge set x^H. If $O_\alpha \neq O_\beta$ it is bipartite with respective degrees $d_\alpha = |H_\alpha|$ and $d_\beta = |H_\beta|$. This results in the components of type 1–3. If $O_\alpha = O_\beta$, the degree is $d_\alpha = 2 \cdot |H_\alpha| = 2$ or 4. If $h = (\alpha, \beta, \gamma, \ldots, \delta)\ldots(\ldots)$ maps α onto β, then $\{\alpha, \beta, \delta\}$ and $\{\alpha. \beta, \delta\}^h$ both contain x. Therefore $\gamma = \delta$ and h has order 3. If $|H_\alpha| = 1$, the edges x, $\{\alpha, \gamma\}$ and $\{\gamma, \beta\}$ form a component of the graph. This is type 4. If $|H_\alpha| = 2$, there is some $k = (\alpha)(\beta, \xi)\ldots$ in H_α with $\xi \neq \gamma$ and ξ must be displaced by $h = (\alpha, \beta, \gamma)(\xi, \theta, \eta)\ldots$ From this one conludes that $k = (\alpha)(\beta, \xi)(\gamma, \theta)(\eta)\ldots$ The resulting images of x form a component of type 5. This completes the proof.

We suppose now that for any subset Y_i of Y some group H_i acting on Y_i is given. Denote this collection of groups by $\mathscr{H} = \{H_i\}$. Let x be a given set of size $k-1$ and $\mathscr{Y} = \{x; y \mid x \subset y$ and $y \subseteqq Y$ has size $k\}$. We say that $\mathscr{Y}$ is a *flag arrangement* for $\mathscr{H}$, if the following is true: Whenever $A = \{x; y_1, y_2, \ldots, y_i\} \subseteqq \mathscr{Y}$, then A is an arrangement in $Y_i = y_1 \cup y_2 \cup \ldots \cup y_i$ for the group H_i. Two flag arrangements with centres x and x' are *isomorphic* if $x^h = x'$ for some $h \in H$, the group on Y. Let $m(\mathscr{H}, k)$ be the number non-isomorphic flag arrangements for $\mathscr{H}$.

III. The growth of the sequence n_k. Let G be a permutation group on a finite or infinite set S. If $X_l(G) = \{O_1, \ldots, O_j, \ldots\}$ are the orbits on l-element subsets we define $m_j(l, k) = m(\mathscr{H}, k)$ where $\mathscr{H}$ is the collection of groups G^{Y_t} induced by G on the subsets $Y_t \subseteqq Y$ for some fixed Y in O_j. It is clear that the definition does not depend upon the choice of Y in O_j.

Theorem 3.1. *Suppose that G acts $(k-1)$-fold homogeneously on a set S with a finite number of orbits on X_k for some k. If $l \geqq k$ let $t = l - k + 1$. Then*

$$\binom{n_k}{t} \leqq \sum_{i=1,\ldots,n_l} m_i(l, k).$$

Proof. Let $Q_1, \ldots, Q_{n_k}$ be all orbits of G on X_k and select some set x of size $k-1$. For any t distinct orbits $Q_1, \ldots, Q_t$, we select y_i in Q_i for $i = 1, \ldots, t$ such that $x \subset y_i$. This is possible because G is $k-1$ homogeneous. Then $\mathscr{Y} = \{x; y_1, \ldots, y_t\}$ is a flag arrangement for $\mathscr{H} = \{G^{Y_i} \mid Y_i \subseteq Y\}$ where $Y = y_1 \cup y_2 \cup \ldots \cup y_t$. This is a consequence of the fact that the y_i belong to distinct G-orbits on X_k. We label the collection $Q_1, \ldots, Q_t$ by j if Y belongs to O_j. (Of course the label is not necessarily uniquely determined). In all we require $\binom{n_k}{t}$ labels where a label may be used several times.

Suppose therefore that also the sequence $Q'_1, Q'_2, \ldots, Q'_t$ obtains the label j. Then there are $y'_i \supset x$, $y'_i \in Q'_i$ for $i = 1, \ldots, t$ such that $Y' = y'_1 \cup y'_2 \cup \ldots \cup y'_t$ belongs to the same orbit as Y. Let therefore g in G be such that $Y'^g = Y$. Then $\{x; y_1, \ldots, y_t\}$ and $\{x'^g; y'^g_1, \ldots, y'^g_t\}$ are flag arrangements for $\mathscr{H}$. However, they are not isomorphic as $\{Q_1, \ldots, Q_t\} \neq \{Q'_1, \ldots, Q'_t\}$. Therefore a label j may be used at most $m_j(l, k)$ times. This gives the required inequality.

We note several consequences of the theorem:

Corollary 3.2. *Let G be a transitive permutation group on a set S with a finite number n_2 of orbits on X_2. For a given $l \geqq 3$ let $n_{l,1}$ be the number of orbits O for which $G^Y = 1$, $Y \in O$ and let $n_{l,2}$ be the number of orbits O' for which G^Y is an elementary abelian 2-group, $Y \in O'$. Then $\binom{n_2}{l-1} \leqq l \cdot n_{l,1} + l/2 \cdot n_{l,2}$.*

Corollary 3.3. *Suppose that G acts doubly homogeneously on a set S with a finite number n_3 of orbits on X_3. Let $n_{4,j}$ be the number of orbits O for which $|G^Y| = j$, $Y \in O$ and $j = 1, 2, 3$, or 6. Then $n_3(n_3 - 1) \leqq 12 \cdot n_{4,1} + 6 \cdot n_{4,2} + 2 \cdot (n_{4,3} + n_{4,6})$.*

We also note the following theorem which gives a bound for n_2 if the action induced on subsets is sufficiently rich:

Corollary 3.4. *Let G be transitive on a finite or infinite set S. Suppose there is a value l such that the following holds: Whenever $Y \subseteqq S$ has size l and $s \in Y$ then there is a subset Y', $s \in Y' \subseteqq Y$ with the following properties* a) $G^{Y'} \neq 1$ *and* b) *if $G^{Y'}$ is an elementary abelian 2-group, then the orbit of s under $G^{Y'}$ has length different from $|G^{Y'}|$. Then $n_2 < l - 1$.*

Proof of 3.2. If $G^Y = 1$ on Y then $m_i(l, 2) \leqq l$ for the orbit containing Y and if G^Y is an elementary abelian 2-group on Y, then $m_i(l, k) \leqq l/2$ for the orbit containing Y by theorem 2.1. The conclusion now follows from theorem 3.1.

Proof of 3.3. Using theorem 2.1 we get the bounds $m_i(4, 3) \leqq 6$ if $G^Y = 1$, $m_i(4, 3) \leqq 3$ if $|G^Y| = 2$ and $m_i(4, 3) \leqq 1$ if $|G^Y| = 3$ or 6. In all other cases $m_i(4, 3) = 0$. The conclusion now follows from theorem 3.1.

Proof of 3.4. The hypothesis together with theorem 2.1 implies that no element of Y is the center of a flag arrangement. Therefore $m_i(l, 2) = 0$ for all orbits and so $n_2 < l - 1$ by theorem 3.1.

A simple but useful fact on orbits on X_k and X_l in general is

Theorem 3.5. *Let G be a permutation group on a finite or infinite set with finite numbers n_k and n_l of orbits on X_k and X_l for some $k < l$. Let $E = O_1 \cup O_2 \cup \ldots \cup O_s$ be a union of distinct orbits of G on X_l and let r_i denote the number of orbits of G^{Y_i} on the k-element subsets of $Y_i \in O_i$. Suppose the following holds about E: If Q_1 and Q_2 are any given G-orbits on X_k, then there exist $x_1, y_1, \ldots, y_t, x_2$ such that $x_1 \subset y_1$, $|y_i \cap y_{i+1}| \geqq k$ for $i = 1, \ldots, t-1$, $y_t \supset x_2$ with $x_1 \in O_1$, $x_2 \in O_2$ and $y_i \in E$. Then*

$$n_k \leqq \sum_{i=1..s} \binom{r_i}{2} + 1 .$$

Proof. We consider the graph whose vertices are the orbits $X_k(G)$. Two distinct orbits Q and Q' are linked by an edge e if there are $x \in Q$ and $x' \in Q'$ such that $x \cup x' \subseteqq y \in E$. We label this edge by j if y belongs to O_j. The condition on E implies that this graph is connected. Therefore the total number of edges is at least $n_k - 1$. On the other hand, a label j may be used at most $\binom{r_j}{2}$ times. This yields the inequality.

We conclude with the following inequalities obtained from a theorem on orbits in graphs [4].

Theorem 3.6. *Let G be a permutation group on a finite set S. Suppose that X_2 is a disjoint union $E_1 \cup E_2 \cup \ldots \cup E_r$ where each E_i is a union of G-orbits on X_2.*

a) *If each graph (S, E_i), $(i = 1, \ldots, r)$, is connected then $n_1 \leqq r^{-1} \cdot n_2 + 1$.*
b) *If every connected component of (S, E_i) contains a circular path of odd length for all $i = 1, \ldots, r$, then $n_1 \leqq r^{-1} \cdot n_2$.*

Proof. Let Γ_i be the graph with vertices S and edge set E_i. Then G is a group of automorphisms of Γ_i and we denote the number of orbits of G on E_i by $|E_i(G)|$. By theorems 3.1 and 3.2 in [4] we have $n_1 \leqq |E_i(G)| + 1$ and as $n_2 = \sum |E_i(G)|$ the assertion a) follows. If all connected components of Γ_i contain a cycle of odd length, then

$n_1 \leqq |E_i(G)|$ as a consequence of theorem 2.1 and the proof of theorem 3.1 in [4]. This yields b).

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