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Closure Properties of the Special Linear Groups

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1. Introduction.

A permutation group G on a finite set Ω has a natural action on the set $\Omega^{\{k\}}$ of all k-element subsets of Ω. The k-closure of G is the largest subgroup $G^{\{k\}}$ of $Sym(\Omega)$ that has the same orbits as G on $\Omega^{\{k\}}$, and G is said to be k-closed if $G = G^{\{k\}}$. Cameron, Neumann and Saxl show in [2] that any primitive group $G \not\geq Alt(n)$ of sufficiently large degree n is $[\frac{n}{2}]$-closed. The general question of k-closure was studied earlier by Inglis [6], Siemons [9] and Siemons and Wagner [10]. In [7] and [8] the present authors examined the k-closure of collineation groups of projective and affine geometries over finite fields. In this paper we obtain minimal values of k for k-closure of the special linear groups (see Theorem 3.5).

We remark that for geometric dimension at least two of our results on k-closure are independent of the classification theorem for finite simple groups. For the line, the lack of geometry has forced us to rely on classification. Wherever the classification theorem is assumed, we will note it by C.T.

The organization of this paper is as follows: in Section 2 we give terminology and definitions, together with statements of earlier results required for our theorems; Section 3 proves the main theorem (Theorem 3.5) by a sequence of lemmas and theorems, together with computations, most of which involved the use of the Cayley package of J. Cannon [3] on the Birmingham University computer.

2. Notation and Assumed Results.

The terminology and notation used is the same as that in [7], and is mostly that of Dembowski [4] and Wielandt [11].

If Ω is a set and $|\Omega| = n$ then the symmetric and alternating groups on Ω are denoted by $Sym(\Omega)$ and $Alt(\Omega)$ respectively, or S_n and

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A_n. If $G \leq Sym(\Omega)$, and $\Delta \subset \Omega$, then $G_{\{\Delta\}}$ denotes the set stabilizer of Δ (the global stabilizer), and $G_{(\Delta)}$ denotes the pointwise stabilizer. We will refer to $G_{\{\Delta\}}/G_{(\Delta)}$ as the restriction of G to Δ, written G^{Δ}. For $k \leq n$, G acts in a natural way on the set $\Omega^{\{k\}}$ of all k-element subsets of Ω. The k-closure $G^{\{k\}}$ of G is the largest subgroup of $Sym(\Omega)$ having the same orbits on $\Omega^{\{k\}}$ as G. G is k-closed if $G = G^{\{k\}}$. When speaking of k-closure we will always assume that $k \leq [\frac{n}{2}]$. Members of $\Omega^{\{k\}}$ will be called k-sets. For $\Delta \subset \Omega$, the set of all images of Δ under G is denoted by Δ^G.

Definition 2.1. Let $G \leq Sym(\Omega)$. A subset Δ of Ω is called a base set for G if $G < H \leq Sym(\Omega)$ implies that $\Delta^G \neq \Delta^H$.

Clearly if G has a base set of size k then G is k-closed, but the converse is not generally true.

Let $V_{d+1}(q)$ denote the vector space of dimension $d+1$ over the Galois field $GF(q)$. Then the projective geometry of dimension d over $GF(q)$ is denoted by $PG(d,q)$. Its full automorphism group is $P\Gamma L(d+1,q)$ for $d \geq 2$, by the fundamental theorem of projective geometry. The affine geometry of dimension d over $GF(q)$ (formed by deleting a hyperplane H and all its points from $PG(d,q)$) is denoted by $AG(d,q)$. It has full automorphism group $A\Gamma L(d,q)$ for $d \geq 2$, by the fundamental theorem of affine geometry. We will always consider the permutation action of $P\Gamma L(d+1,q)$ on points of $PG(d,q)$, and that of $A\Gamma L(d,q)$ on points of $AG(d,q)$. Clearly $A\Gamma L(d,q) \cong (P\Gamma L(d+1,q))_{\{H\}}$.

If $\{e_1,e_2,\ldots,e_r\}$ is a set of r linearly independent vectors in $V_d(q)$ and if $E_i = \langle e_i \rangle$, $i = 1,2,\ldots,r$, then the set of points $\{E_1,E_2,\ldots,E_r\}$ is called an r-simplex in $PG(d-1,q)$. If $r = d$ we will refer to a d-simplex in $PG(d-1,q)$ as a simplex. If $\{e_1,e_2,\ldots,e_d\}$ is the standard basis for $V_d(q)$ then the corresponding simplex will be called the unit simplex, with unit point $E = \langle e \rangle$ where $e = \sum_{i=1}^{d} e_i$, and unit hyperplane $\langle e^t \rangle$, in terms of homogeneous co-ordinates.

We will need the following results, which are easily deduced from the reference given in each case.

Result 2.1. (Key and Siemons, Theorems A and B [7]).

(I): In the action on the points of $PG(d-1,q)$, $d \geq 3$, $PGL(d,q)$ is 4-closed if $q \notin \{4,8,9,16\}$; $PGL(3,4)$ is 10-closed but not k-closed for $k \leq 9$; $PGL(d,4)$ is 8-closed but not k-closed for $k \leq 6$, when $d \geq 4$; $PGL(d,8)$ and $PGL(d,9)$ are 6-closed but not 5-closed for $d \geq 3$; $PGL(d,16)$ is 6-closed for $d \geq 3$.

(II): (C.T.) In the action on the points of $PG(1,q)$, $q \geq 7$, $P\Gamma L(2,q)$ is 4-closed if and only if $q \notin \{8,32\}$; $P\Gamma L(2,32)$ is 5-closed; $PGL(2,q)$ is 4-closed if and only if $q \notin \{8,9,16\}$; $PGL(2,16)$ is 6-closed but not 5-closed.

(III): In the action on the points of $AG(d,q)$, $d \geq 2$, $AGL(d,q)$ is 3-closed if $q \notin \{2,4,8,9,16\}$; $AGL(d,2)$ is 4-closed but not 3-closed for $d \geq 3$; $AGL(d,4)$ is 6-closed but not 5-closed for $d \geq 2$; $AGL(d,q)$ for $q \in \{8,9,16\}$ is 4-closed but not 3-closed for $d \geq 2$.

(IV): (C.T.) In the action on the points of $AG(1,q)$, $q \geq 7$, $A\Gamma L(1,q)$ is 3-closed if and only if $q \notin \{8,9,16,32\}$; $A\Gamma L(1,8)$ is not 4-closed; $A\Gamma L(1,q)$ is 4-closed for $q \in \{9,16,32\}$; $AGL(1,q)$ is 3-closed if and only if $q \notin \{8,9,16\}$; $AGL(1,16)$ is 4-closed; $AGL(1,8)$ and $AGL(1,9)$ are not 4-closed.

Result 2.2. (Siemons [9], Theorem 5.1, p. 399). Let $G \leq Sym(\Omega)$ where $|\Omega| = n$. Let n^* satisfy $\frac{n-1}{2} \leq n^* \leq \frac{n}{2}$. Then if $1 \leq k \leq n^*$

$$G \leq G^{\{n^*\}} \leq \cdots \leq G^{\{k\}} \leq G^{\{k-1\}} \leq \cdots \leq G^{\{1\}} \leq Sym(\Omega).$$

Further, if $n^* \geq \ell \geq k$ then

(i) if G is k-closed then G is ℓ-closed;

(ii) if $H \leq Sym(\Omega)$ and G and H have distinct orbits on k-sets, then G and H have distinct orbits on ℓ-sets;

(iii) if $G \leq H$ then $G^{\{k\}} \leq H^{\{k\}}$.

Result 2.3. (Key and Siemons [8]).

A. Let $PGL(d,q)$ act on the points of $PG(d-1,q)$ for $d \geq 2$, $q \geq 3$. If $H \leq PGL(d,q)$ then H is k-closed according to the entries in the table below:

$d = 2$:

q	$13 \leq q \leq 17$	≥ 19
k	6	5

(C.T.)

$d = 3$:

q	4	5	≥ 7
k	10	7	6

$d = 4$:

q	3	4	≥ 5
k	8	8	7

$d \geq 5$, d odd:

q	$3 \leq q < d$	$d = q \in \{5,7\}$	$d \leq q$
k	$2d$	$2d$	$d+3$

$d \geq 6$, d even:

q	$3 \leq q < d$	$d \leq q$
k	$2d-1$	$d+3$

B. Let $AGL(d,q)$ act on the points of $AG(d,q)$ for $d \geq 1$, $q \geq 3$. If $H \leq AGL(d,q)$ then H is k-closed according to the entries in the table below:

$d = 1$:

q	11	13	16	≥ 17
k	3	3	4	3

(C.T.)

$d = 2$:

q	4	5	7	≥ 8
k	6	5	5	4

$d = 3$:

q	3	4	≥ 5
k	8	8	7

$d = 4$, d even:

q	$q < d$	$q = d = 4$	$q = d+1 \in \{5,7\}$	$q \geq d$
k	$2(d+1)$	$2(d+1)$	$2(d+1)$	$d+4$

$d \geq 5$, d odd:

q	$q < d$	$q = d \in \{5,7\}$	$q \geq d$
k	$2d+1$	$2d+1$	$d+4$

(These results are deduced from Results 2.1 and 2.2 above and Theorems in [8], in particular Theorems 3.1, 4.1 and 5.1, which show the existence of regular sets for $P\Gamma L(d,q)$ and $A\Gamma L(d,q)$ for $q \geq 3$.)

3. *k*-Closure.

Our results on the k-closure of the special linear groups are collected together in Theorem 3.5, which is proved by the lemmas and theorems below, together with computations for the exceptional values of q.

Lemma 3.1. *Let Δ be a simplex in $PG(d-1,q)$, $d \geq 2$. Then $(PSL(d,q))^{\Delta} = Sym(\Delta)$.*

Proof. We can suppose that Δ is the unit simplex. Then the matrix

$$A = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ a & 0 & \\ \hline \multicolumn{2}{c|}{0} & I_{d-2} \end{array}\right]$$

with $a = \pm 1$ suitably chosen belongs to $PSL(d,q)$ and induces a transposition on Δ. □

Definition 3.1. Let Δ be a simplex in $PG(d-1,q)$ and E the unit point for Δ. Define $\Lambda_d = \Delta \cup \{E\}$.

Lemma 3.2. *Suppose that -1 is a dth root in $GF(q)$, $d \geq 2$. Then $(PSL(d,q))^{\Lambda_d} = Sym(\Lambda_d)$ where Λ_d is defined as above.*

Proof. We take Δ to be the unit simplex. Then the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & 1 \\ -1 & -1 & -1 & . & . & -1 & -1 \end{bmatrix}$$

induces a $(d+1)$-cycle on the points of Λ_d and has determinant $(-1)^d \cdot (-1)$; thus it belongs to $PSL(d,q)$. In the proof of Lemma 3.1 entry a of A may be chosen $a = 1$. Thus A fixes Λ_d and the result follows. □

Definition 3.2. If $G < K \leq Sym(\Omega)$ and $\Lambda \subseteq \Omega$, then Λ is a K-base set for G if $G < H \leq K$ implies that $\Lambda^G \subset \Lambda^H$.

Evidently from our Definition 2.1 in Section 2, if $K = Sym(\Omega)$ then a K-base set is a base set.

Lemma 3.3. *In the action on the points of $PG(d-1,q)$, $d \geq 2$, any group H with $PSL(d,q) \leq H < PGL(d,q)$ has the same orbits on d-sets as $PGL(d,q)$. If -1 is a dth root in $GF(q)$ then Λ_d is a $(PGL(d,q))$-base set for H, where Λ_d is as in Definition 3.1 above.*

Proof. Let ϕ be a d-set in $PG(d-1,q)$. If ϕ is contained in a hyperplane then the ϕ-images under $PSL(d,q)$ are the same as the ϕ-images under $PGL(d,q)$. Otherwise ϕ is a simplex and as $PSL(d,q)$ permutes simplices transitively, any group H with $PSL(d,q) \leq H \leq PGL(d,q)$ has the same orbits on d-sets. Suppose (-1) is a dth root in $GF(q)$. As $(PGL(d,q))_{(\Lambda_d)} = 1$ and $(PSL(d,q))^{\Lambda_d} = Sym(\Lambda_d)$ by Lemma 3.2, it follows that $H_{\{\Lambda_d\}} = PGL(d,q)_{\{\Lambda_d\}}$ and hence that Λ_d is a $PGL(d,q)$-base set for H. □

Theorem 3.1. *Let H be a group with $PSL(d,q) \leq H < PGL(d,q)$ in the natural action on the points of $PG(d-1,q)$. Suppose that $d \geq 3$, (-1) is a dth root in $GF(q)$ and that $q \notin \{4,8,9,16\}$. Then H is $(d+1)$-closed but not d-closed.*

Proof. By Lemma 3.3 H is not d-closed. By Result 2.2 the $(d+1)$-closure of H is contained in the $(d+1)$-closure of $PGL(d-1,q)$. As $d + 1 \geq 4$ and $q \notin \{4,8,9,16\}$ Result 2.1 applies and the result follows from Lemma 3.3. □

We remark that Λ_d is a base set for H only if q is a prime.

We now consider the affine special linear groups acting on $AG(d-1,q)$, $d \geq 2$. For a simplex Δ in $PG(d-1,q)$ let S be the unit hyperplane. Then the points of $AG(d-1,q)$ are the points of $PG(d-1,q)$ not in S so that Δ is a subset of $AG(d-1,q)$. The analogue of Lemma 3.3 is as follows:

Lemma 3.4. *In the action on the points of $AG(d-1,q)$, $d \geq 2$, any group H with $ASL(d-1,q) \leq H < AGL(d-1,q)$ has the same orbits on $(d-1)$-sets as $AGL(d-1,q)$. If -1 is a dth root in $GF(q)$ then Δ is a $(AGL(d-1,q))$-base set for H.*

Proof. The first part of the lemma is standard. If $K = AGL(d-1,q)$, then $K_{(\Delta)} = 1$ as K also fixes the unit hyperplane. As in the projective case one shows that $H_{\{\Delta\}} = Sym(\Delta)$. Thus $K_{\{\Delta\}} = H_{\{\Delta\}}$ and Δ is a K-base set for H. □

Theorem 3.2. *Let H be a group with $ASL(d-1,q) \leq H < AGL(d-1,q)$ in the natural action on the points of $AG(d-1,q)$. Suppose that $d \geq 3$, (-1) is a dth root in $GF(q)$ and that $q \notin \{4,8,9,16\}$. Then H is d-closed but not $(d-1)$-closed.*

Proof. As before this follows from Lemma 3.4 above, Result 2.2 and Result 2.1. □

As in the projective case, Δ is a base set for H only if q is a prime.

Now we consider the natural action of $PSL(2,q)$ and $ASL(1,q)$ on the projective and affine line. The case of even characteristic is dealt with in Result 2.1.

Lemma 3.5. *If q is odd and $q \geq 7$ and $q \neq 11$ then $PSL(2,q)$ and $PGL(2,q)$ have different orbits on 4-sets.*

Proof. If $q \equiv 1 \pmod 4$ then $PSL(2,q)$ has 2 orbits on 3-sets. The result now follows from Result 2.2. Now suppose that $q \equiv 3 \pmod 4$. In $PGL(2,q)$ the sets $\{\infty,0,1,a\}$ and $\{\infty,0,1,a^{-1}\}$ $(a \neq \infty,0,1)$ belong to the same orbit. A case by case analysis shows that these sets are not in the same $PSL(2,q)$-orbit if a is chosen such that a is a non-square, $a-1$ is a square in $GF(q)$ and $a \neq -1,2,1/2$. □

Theorem 3.3 (C.T.). *Let $G = PSL(2,q)$ act on the projective line in the natural way, q odd.*

(i): G is 3-closed if and only if $q \equiv 1 \pmod 4$ and q is a prime, $q \geq 5$.

(ii): If $q \geq 7$, $q \notin \{9,11\}$ then G is 4-closed.

Proof. (i) Let G^* be the 3-closure of G. If $q \equiv 3 \pmod 4$ then $PSL(2,q)$ is 3 homogeneous, thus $G^* = Sym(q+1)$.

If $q \equiv 1 \pmod 4$, G^* has 2-orbits on 3-sets and it follows easily from Result 2.5 of [7] that $G^* = PSL(2,q)\cdot\langle\sigma\rangle$ where σ generates the automorphism group of $GF(q)$. Thus $G = G^*$ if and only if q is a prime.

(ii) Let G^* be the 4-closure of G. By Result 2.2 G^* is contained in the 4-closure of $PGL(2,q)$. If $q \neq 9,11$ Result 2.1 and Lemma 3.5 above imply the result. □

The case of affine one-dimensional special linear groups can be treated in an analogous way.

Lemma 3.6. *If $q \geq 7$ is odd, then $ASL(1,q)$ and $AGL(1,q)$ have distinct orbits on 3-sets.*

Proof. If $q \equiv 1 \pmod 4$ then $ASL(1,q)$ has 2 orbits on 2-sets and by Result 2.2 $ASL(1,q)$ and $AGL(1,q)$ have distinct orbits on 3-sets. If $q \equiv 3 \pmod 4$ consider $\{0,1,a\}$ and $\{0,1,a^{-1}\}$, $a \notin \{0,1\}$. These sets are in the same AGL-orbit. Case by case analysis shows that these sets are in distinct ASL-orbits if a is chosen to be a non-square in $GF(q)\backslash\{0,1,2,1/2,-1\}$, a non-empty set for $q \geq 7$. □

Theorem 3.4. *Let $G = ASL(1,q)$ act on the affine line in the natural way, q odd.*

(i): G is 2-closed if and only if $q \equiv 1 \pmod 4$ and $q \geq 5$ is a prime.

(ii): If $9 \neq q \geq 7$ then G is 3-closed.

Proof. (i) Let G^* be the 2-closure of G. If $q \equiv 3 \pmod 4$ then G is 2-homogeneous and so $G^* = Sym(q)$. If $q \equiv 1 \pmod 4$, then G^* has 2 orbits on 2-sets. Therefore G^* is not 2-transitive. If q is not a prime, $G \cdot <\sigma>$ (σ the field automorphism of $GF(q)$) is contained in G^*. Therefore q is a prime and a theorem of Burnside (Theorem 11.7 in [11]) implies that G^* is contained in $AGL(1,q)$. Thus $G^* = G$.

(ii) Let G^* be the 3-closure of G. By Result 2.2 G^* is contained in the 3-closure of $AGL(1,q)$. By Result 2.1 $G^* \leq AGL(1,q)$ and the result follows from Lemma 3.6 above. □

Our theorem on the k-closure of the special linear groups can now be proved by using the lemmas and theorems above, some of the results stated in Section 2, and by direct computation. The latter involved the use of the Cayley language [3].

Theorem 3.5. *Let G be $PSL(d,q)$ or $ASL(d,q)$ acting on the points of $PG(d-1,q)$ or $AG(d,q)$ respectively. Assume that $G \neq PGL(d,q)$ and $G \neq AGL(d,q)$. Then G is k-closed according to the entries given in the tables below, where ω is a primitive element for $GF(q)$, and where an entry of the form "–" indicates that the group is not k-closed for any k.*

$PSL(d,q)$:

$d = 2$

q	3	5	7	9	11	q prime, $q \equiv 1 \pmod 4$, $q \geq 13$	$q \geq 19$ (C.T.)
k	–	3	4	5	6	3	4

$d = 3$: $q \neq 16$, $k = 4$; $q = 16$, $k = 6$.

$d = 4$: $q \equiv 1 \pmod 8$, $q \geq 3$, $k = 7$; $q \equiv 1 \pmod 8$, $q > 9$, $k = 5$; $q = 9$, $k = 6$.

$d \geq 5$ (d odd): $k = d + 1$ all q.

$d = 6$:

q	3	4	$q \geq 5, -1 \in <\omega^6>$	$q \geq 7, -1 \notin <\omega^6>$
k	11	8	7	9

$d \geq 8$ (d even):

q	$-1 \in <\omega^d>$	$-1 \notin <\omega^d>$ $3 \leq q < d$	$-1 \notin <\omega^d>$ $q \geq d$
k	$d+1$	$2d-1$	$d+3$

$ASL(d,q)$:

$d = 1$:

q	3	5	7	9	q prime, $q \equiv 1 \pmod 4$, $q \geq 13$	$q \geq 11$
k	–	2	3	4	2	3

(C.T.)

$d = 2$: $q \neq 3,16$, $k = 3$; $q = 16$, $k = 4$.

$d = 3$: $q \not\equiv 1 \pmod 8$, $q \geq 5$, $k = 6$; $q = 3$, $k = 7$; $q \equiv 1 \pmod 8$, $q > 9$, $k = 4$; $q = 9$, $k = 4$.

$d \geq 4$ (d even): $k = d + 1$ all q.

$d \geq 5$ (d odd):

q	$-1 \in <\omega^{d+1}>$	$-1 \notin <\omega^{d+1}>$ $3 \leq q < d$	$q \in \{5,7\}$, $-1 \notin <\omega^{d+1}>$	$-1 \notin <\omega^{d+1}>, q \geq d$
k	$d+1$	$2d+1$	$2d+1$	$d+4$

Proof. We restrict attention to $PSL(d,q) \neq PGL(d,q)$ and $ASL(d,q) \neq AGL(d,q)$ as the general linear groups have already been dealt with in Results 2.1. First we deal with the projective groups.

$d = 2$: We use Theorem 3.3 for all $q \neq 9$ or 11. For $q = 9$ we have 4-closure of $P\Gamma L(2,9)$ by Result 2.1. By computation we found that $P\Sigma L(2,9)$ (i.e. $PSL(2,9)<\sigma>$ where σ is the field automorphism) has distinct orbits on 4-sets from $P\Gamma L(2,9)$ and is thus 4-closed by Result 2.2. Also by computation we found that $PSL(2,9)$ has distinct orbits on 5-sets from $P\Sigma L(2,9)$, and thus, by Results 2.1 and 2.2, is 5-closed. For $q = 11$ we have 4-closure for $PGL(2,11)$ and by computation we found that $PSL(2,11)$ has distinct orbits on 6-sets and, for the same reason as above, is thus 6-closed.

$d = 3$: Since -1 is a third root, Theorem 3.1 applies, except when $q \in \{4,8,9,16\}$; i.e. when $q = 4$ or 16. For $PSL(3,4)$, using the 3-closure of $P\Gamma L(3,4)$, we proved that $P\Gamma L(3,4)$ and $PSL(3,4)$ are both 4-closed by computing that they all have distinct orbits on 4-sets, and using the results quoted for $d = 2$. For $PSL(3,16)$, we have that $PGL(3,16)$ is 6-closed (by Result 2.1) and, by Lemma 3.3, $PSL(3,16)$ has distinct orbits from $PGL(3,16)$ on 4-sets and hence on 6-sets by Result 2.2, and is hence also 6-closed.

$d = 4$: Here -1 is a 4th root if and only if $q \equiv 1 \pmod 8$. Thus by Theorem 3.1 if $q \neq 9$, G is 5-closed. If $q = 9$ we use Lemma 3.3 and Result 2.1 to obtain 6-closure. If $q \not\equiv 1 \pmod 8$ we use Result 2.3 and by computation improve on $k = 8$ for $q = 3$ to get 7-closure for $G = PSL(4,3)$.

$d \geq 5$, d odd: Here -1 is a dth root so that Theorem 3.1 applies when $q \neq 4,8,9,16$. But for these values of q (except 9), it is easy to see that Result 2.1 always gives $(d+1)$-closure.

$d = 6$: If -1 is a 6th-root then Theorem 3.1 applies except when $q = 4$, 9 or 16. For $q = 9$ and 16, $PGL(6,q)$ is 6-closed and hence 7-closed, and thus so is G. For $q = 4$, $PGL(6,4)$ is 8-closed, and by Lemma 3.3, G has distinct orbits from $PGL(6,4)$ on 7-sets and hence on 8-sets by Result 2.2 and thus G is 8-closed by Result 2.2. If -1 is not a 6th-root then Result 2.3 must be used, and here 3 is the only value of $q < 6$ such that -1 is not a 6th root.

$d \geq 8$, d even: This follows from Result 2.1, Theorem 3.1 and Results 2.2.

For the affine groups the reasoning is similar. □

Corollary. *Let G be $PSL(d,q)$ or $ASL(d-1,q)$ acting in the natural way on $PG(d-1,q)$ or $AG(d-1,q)$ for $d \geq 2$. Suppose that G does not contain the alternating group of the same degree and that G is not k-closed for any k. Then $G = ASL(1,8)$, $PSL(2,8)$ or $ASL(2,3)$.*

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