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DOES PRIMITIVITY ON LINES IMPLY PRIMITIVITY ON POINTS

by

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Abstract

Let $D = (P,L,I)$ be a design with points P, lines L, and incidence I. Assume that G is a group of automorphisms of D acting primitively, of rank-3 on L, and transitively, rank-3 on P. Assume further that the characters of the actions $G|P$, $G|L$ are χ_P and χ_L with $\chi_P = \chi_L = 1 + \chi + \psi$, where χ, and ψ are irreducibles. We show that almost always the action $G|P$ is also primitive. The argument uses an appropriate commuting algebra C for G to show that, except for the case where a specific numeric condition holds, a certain matrix in C is invertible. It is believed that the argument can be generalized to arbitrary rank for $G|L$.

Introduction

If $G|\Omega$ is a group action, we denote by $\nu(G|\Omega)$ the number of G-orbits on Ω. Thus, $G|\Omega$ is transitive if and only if $\nu(G|\Omega) = 1$. If $G|\Omega$ is transitive, then the <u>rank</u> $\rho(G|\Omega)$ is defined to be the number $\nu(G|\Omega \times \Omega)$ of G-orbits on $\Omega \times \Omega$. Thus, for $|\Omega| > 1$, $\rho(G|\Omega) \geq 2$, and $\rho(G|\Omega) = 2$ if and only if $G|\Omega$ is doubly transitive.

If $G|\Omega$ is a transitive action, a non-empty set $\Delta \subseteq \Omega$ is called a <u>block</u> of <u>imprimitivity</u> if and only if $\Delta^g \cap \Delta = \Delta$ or $\emptyset$ for each $g \in G$. A transitive action $G|\Omega$ is called <u>primitive</u> if and only if the only blocks of imprimitivity are the singleton subsets $\{x\}$, $x \in \Omega$, and Ω itself. If $G|\Omega$ is 2-transitive, then it is primitive. $G|\Omega$ is primitive if and only if the stabilizer G_α is a maximal subgroup of G.

By a <u>design</u> we mean an incidence structure $D = (P,L,I)$ which is a t-design for $t \geq 2$ [1]. Here, P is the set of points, L the set of lines, and I ($\subseteq P \times L$) the incidence between points and lines. If G is an automorphism group of a design $D = (P,L,I)$, then two analogs of Fisher's inequality hold. The first is that

$$\nu(G|P) \leq \nu(G|L) \qquad (1)$$

Thus, if $G|L$ is transitive, so is $G|P$. The second analog is that if $G|L$ is transitive, then:

$$\rho(G|P) \leq \rho(G|L) \tag{2}$$

A consequence of (2) is, of course, that if G is 2-transitive on lines, then it must be 2-transitive on points.

The rank-3 case

The question has been frequently asked: does primitivity on lines imply primitivity on points? There are some theorems known [1] about when primitivity of $G|P$ can be deduced from other conditions, but the general question is still open. The rank-3 case which we investigate here was proposed by A. Wagner whom we thank for many inspiring discussions. We prove the following theorem:

Theorem: Suppose that G is a group of automorphisms of the design $D = (P,L,I)$ acting primitively and of rank-3 on L. Assume, furthermore, that in case $\rho(G|P) = 3$, the characters χ_P, χ_L of the actions $G|P$, $G|L$ satisfy $\chi_P = \chi_L = 1 + \chi + \psi$, where χ, ψ are irreducible characters of G. Then, $G|P$ is almost always primitive.

Proof: If $\rho(G|P) = 2$ then G is doubly transitive on points and therefore primitive. Suppose now that $\rho(G|P) = 3$, and $\chi(1) \leq \psi(1)$. If we were to suppose that $G|P$ is not primitive, then for $p \in P$, $G_p \lneqq H \lneqq G$ for some subgroup H of G and there is a non-trivial system of imprimitivity $B = \{\beta_1,\dots,\beta_r\}$ in the action $G|P$, with $1 < r < v$, $r|v$, and $H = G_\beta$. Since $G_p \lneqq H$, the character of $G|B$ is $\chi_B = 1 + \chi$ and $r = 1 + \chi(1)$ divides $\psi(1)$. It follows that $\chi(1) < \psi(1)$ and $\chi \neq \psi$.

Since $\chi_B = 1 + \chi$, $\chi_P = \chi_L = 1 + \chi + \psi$, we have that $(\chi_B,\chi_P) = (\chi_B,\chi_L) = (\chi_B,\chi_B) = 2$ and $(\chi_P,\chi_P) = (\chi_P,\chi_L) = (\chi_L,\chi_L) = 3$. This implies, among other, that G is doubly transitive on B, hence $|G|$ is even, that G has exactly two orbits Λ and Θ on $L \times B$, and exactly three orbits I, Δ, Γ on $L \times L$. Here, I is the diagonal orbit in $L \times L$ and since $|G|$ is even, Δ and Γ are symmetric orbitals ($\Delta^{-1} = \Gamma$).

Each of Λ and Θ define an incidence structure between L and B, while I, Δ, Γ define graphs with vertex set L. We identify Λ, Θ, I,

Δ, Γ with the incidence (adjacency) matrices of the corresponding incidence structures or graphs. As matrices Λ, Θ are of size $v \times r$, while I, Δ, Γ are of size $v \times v$. We let $k, \ell, \lambda, \mu, d, s, t, f, g$ be the rank-3 parameters [3] of the graph defined by Δ. For lines ℓ_0, ℓ_1, ℓ_2 such that $(\ell_0, \ell_1) \in \Delta$ and $(\ell_0, \ell_2) \in \Gamma$ we define the integer parameters $h_0 = |\Lambda(\ell_0)|$, $h_1 = |\Lambda(\ell_0) \cap \Lambda(\ell_1)|$ and $h_2 = |\Lambda(\ell_0) \cap \Lambda(\ell_2)|$. Then,

$$\Lambda\Lambda^T = h_0 I + h_1 \Delta + h_2 \Gamma \tag{3}$$

and,

$$\text{rank}(\Lambda\Lambda^T) \leq \text{rank}(\Lambda) \leq r \leq v \tag{4}$$

If we can show that $\Lambda\Lambda^T$ is nonsingular, then $\text{rank}(\Lambda\Lambda^T) = v$, and we would have shown that $r = v$, a contradiction to $G_p \neq H$. We normalize, and consider the matrix

$$M = (1/h_0)\Lambda\Lambda^T = I + x\Delta + y\Gamma \tag{5}$$

where, $0 \leq y = (h_2/h_0) \leq x = (h_1/h_0) \leq 1$. Furthermore, we seek conditions under which an inverse for M exists in the commuting algebra $C = \{aI + b\Delta + c\Gamma : a,b,c \in Q\}$.

Recall that $I + \Delta + \Gamma = J$, the all 1's $v \times v$ matrix, and that $\Delta\Delta^T = kI + \lambda\Delta + \mu\Gamma$. Thus, $\Delta^2 = \Delta\Delta^T = kI + \lambda\Delta + \mu(J-\Delta-I) = (k-\mu)I + (\lambda-\mu)\Delta + \mu J$. We further have that $\Delta J = J\Delta = kJ$, and $J^2 = vJ$. Now, changing bases from $\{I,\Delta,\Gamma\}$ to $\{I,\Delta,J\}$ yields $M = I + x\Delta + y(J-I-\Delta) = (1-y)I + (x-y)\Delta + yJ$.

Since in any non-trivial case $y \neq 1$, we normalize again and seek an inverse for $M^* = (1/1-y)M = I + u\Delta + wJ$ in C, where

$$u = (x-y)/(1-y) \text{ , and } \quad w = y/(1-y) \tag{6}$$

There exist $a,b,c \in Q$ such that

$$(I + u\Delta + wJ)(aI + b\Delta + cJ) = I \tag{7}$$

if and only if $aI + b\Delta + cJ + ua\Delta + ub\Delta^2 + uc\Delta J + waJ + wbJ\Delta + wcJ^2 = I$ from which we get $(a + bu(k-\mu))I + (b + ua + ub(\lambda-\mu))\Delta + (c + ub\mu + uck + wbk + wa + wcv)J = I$. Since I, Δ, and J form a basis for C over Q we have:

$$\begin{aligned} a &+ u(k-\mu)b &+ 0c &= 1 \\ ua &+ (1 + u(\lambda-\mu))b &+ 0c &= 0 \\ wa &+ (u\mu + wk)b &+ (1 + uk + wv)c &= 0 \end{aligned} \tag{8}$$

M fails to have an inverse in C if and only if the determinant of the coefficients in system (8) vanishes, which is equivalent to

$$(1 + uk + wv)((\mu-k)u^2 + (\lambda-\mu)u + 1) = 0 \tag{9}$$

Now, u, w, k, v are all non-negative, hence (9) can hold iff

$$(\mu-k)u^2 + (\lambda-\mu)u + 1 = 0 \tag{10}$$

Since $k > \mu$, we have that $\sqrt{d} = \sqrt{(\lambda-\mu)^2 + 4(k-\mu)} > |\lambda-\mu|$ therefore, $(\mu - \lambda + \sqrt{d})/2(\mu-k) < 0$, and the only possible solution $u > 0$ to equation (10) is

$$u = (\mu - \lambda - \sqrt{d})/2(\mu-k) = s/(k-\mu) \tag{11}$$

where s is the positive eigenvalue of Δ [3]. Since $u = (h_1-h_2)/(h_0-h_2)$ we have proved that $G|P$ is primitive unless $(h_0-h_2)(k-\mu) = s(h_1-h_2)$.

References

[1] Dembowski, P., Finite Geometries, Ergebnisse der Mathematik, Band **44**, Springer Verlag, 1968

[2] Gorenstein, D., Finite Groups, Harper and Row, N.Y. (1968).

[3] Higman, D.G., Finite permutation groups of rank 3, Math. Zeitschrift **186**, (1964) 145-156.

[4] Hughes, D.R., Collineations and generalized incidence matrices, Trans. Amer. Math. Soc. **86**, 284-296.

[5] Kantor, W.M., Automorphism groups of designs, Math. Zeitschrift **71**, (1968)

[6] Parker, E.T., On Collineations of symmetric designs, Proc. Amer. Math. Soc. **8**, (1957) 350-351.

[7] Wielandt, H., Finite Permutation Groups, Academic Press, N.Y. (1964).