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REGULAR SETS AND GEOMETRIC GROUPS

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If G is a permutation group acting on a set Ω, a subset Λ of Ω is called a regular set for G if the set-stabilizer of Λ in G is the identity subgroup. We show here that the projective and affine semi-linear groups acting in the natural way as permutation groups on their respective finite geometries, have, in general, for all finite dimensions and all finite fields, regular sets of points. The exceptions to this are found, and an extension of the results to infinite fields is discussed.

1. INTRODUCTION

In a geometry a set of elements is regular if only the identity automorphism leaves these elements invariant as a set. In this paper we shall show that there are such regular sets in affine and projective spaces over finite fields of at least three elements. Exceptions do occur for some low-dimensional spaces over small fields. These results are given in Theorems 3.1 and 4.1. For permutation groups the notion of a regular set is similar: a set is regular for a group G if only the identity in G leaves the set invariant. Cameron, Neumann and Saxl [2] showed that all finite primitive groups of sufficiently large degree, not containing the alternating group, have regular sets. From our results here it is immediate that any collineation group of an affine or projective space over a finite field of at least three elements, has regular sets, apart from some finite number of exceptions. Recently, Dalla Volta [4] has shown that the same is true for the field of two elements. There are related results on regular sets in [6] and [11].

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A permutation group is geometric if there is an incidence structure on the same set of points such that the group is the *full* automorphism group of this structure. This yields a concept related to regular sets and is due to Betten [1]. In Section 5 of this paper we show that - apart from some exceptions - all collineation groups of affine or projective spaces are geometric (Theorem 5.1).

In [8], [9], [12] and [13] a permutation group G is called k-closed if the largest group having the same orbits as G in the natural action on the system of k-element subsets coincides with G. The results of this paper are used to show that a permutation group of odd degree containing an elementary abelian normal and transitive subgroup is k-closed and bounds for k are given by logarithmic functions of the degree. The results in this paper are independent of the classification theorem of finite simple groups.

The paper is organized as follows: Section 2 deals with definitions and notation, with some preliminary lemmas; Sections 3 and 4 deal with regular sets for the projective and affine cases respectively, leading to the main theorems on regular sets Theorems 3.1 and 4.1; Section 5 establishes k-geometric properties of subgroups of the semi-linear groups; Section 6 deals with analogous results for infinite fields.

2. PRELIMINARIES

The terminology and notation used is mostly that of Dembowski [5] and Wielandt [15], with exceptions as noted below. The symmetric and alternating groups on a set Ω are denoted by $\mathrm{Sym}(\Omega)$ and $\mathrm{Alt}(\Omega)$ respectively, or S_n and A_n if $|\Omega| = n$. If $G \leqslant \mathrm{Sym}(\Omega)$, and $\Delta \subseteq \Omega$, then $G_{\{\Delta\}}$ denotes the set stabilizer of Δ and $G_{(\Delta)}$ denotes the pointwise stabilizer. We will refer to $G_{\{\Delta\}}/G_{(\Delta)}$ as the restriction of G to Δ, written as G^{Δ}. For $k \leqslant n$, G acts in a natural way on the set $\Omega^{\{k\}}$ of all k-element subsets of Ω. The *k-closure* $G^{\{k\}}$ of G is the largest subgroup of $\mathrm{Sym}(\Omega)$ having the same orbits on $\Omega^{\{k\}}$ as G. G is *k-closed* if $G = G^{\{k\}}$. When speaking of k-closure we will always assume that $k \leqslant \frac{n}{2}$. Members of $\Omega^{\{k\}}$ will be called k-sets. For $\Delta \subseteq \Omega$, the set of all images of Δ under G is denoted by Δ^G. All sets and groups are finite, except in Section 6 where we extend some of the results to infinite fields and groups.

DEFINITION 2.1: Let $G \leqslant Sym(\Omega)$. A subset Δ of Ω is called a *regular set* for G if $G_{\{\Delta\}} = 1$. If $|\Delta| = k$ then Δ will also be called a *regular k-set* for G .

For any collection B of subsets of Ω , (Ω,B) denotes the incidence structure whose elements are the "points" in Ω and the "blocks" in B with incidence given by set inclusion. An automorphism is a permutation g in $Sym(\Omega)$ such that $b \in B$ implies $b^g \in B$. The *full* automorphism group of (Ω,B) is denoted by $Aut(\Omega,B)$.

DEFINITION 2.2 (cf. [1]): Let $G \leqslant Sym(\Omega)$. Then G is *geometric* if there is a collection B of subsets of Ω such that $G = Aut(\Omega,B)$. If all blocks in B have size k for some k , $1 \leqslant k \leqslant |\Omega|$, then G is said to be *k-geometric*.

Thus to be geometric is a property of permutation groups in a specific representation. It is immediately clear that a k-geometric permutation group is k-closed.

Let $V_{d+1}(q)$ denote the vector space of dimension $d + 1$ over the Galois field GF(q) . Then the projective geometry of dimension d over GF(q) is denoted by PG(d,q) . Its full automorphism group is $P\Gamma L(d+1,q)$ for $d \geqslant 2$, by the fundamental theorem of projective geometry. The affine geometry of dimension d over GF(q) (formed by deleting a hyperplane H and all its points from PG(d,q)) is denoted by AG(d,q) . It has full automorphism group $A\Gamma L(d,q)$ for $d \geqslant 2$, by the fundamental theorem of affine geometry. We will always consider the permutation action of $P\Gamma L(d+1,q)$ on points of PG(d,q) , and that of $A\Gamma L(d,q)$ on points of AG(d,q) . Clearly $A\Gamma L(d,q) = (P\Gamma L(d+1,q))_{\{H\}}$. If $\{e_1, e_2, \ldots, e_r\}$ is a set of r linearly independent vectors in $V_d(q)$ and if $E_i = \langle e_i \rangle$, $i = 1,2, \ldots, r$, then the set of points $\{E_1, E_2, \ldots, E_r\}$ is called an *r-simplex* in PG(d-1,q) . If $r = d$ we will refer to a d-simplex in PG(d-1,q) as a *simplex*. If $\{e_1, e_2, \ldots, e_d\}$ is a basis for $V_d(q)$ then the corresponding simplex will be called the *unit simplex*, with *unit point* $E = \langle e\rangle$ where

$$e = \sum_{i=1}^{d} e_i$$

and *unit hyperplane* $\langle e^t\rangle$, in terms of homogeneous co-ordinates. We will need the following lemmas:

LEMMA 2.1: Let $G \leqslant \mathrm{Sym}(\Omega)$ act semi-regularly on Ω where $|\Omega| = n$. Let k be any integer such that $1 \leqslant k \leqslant n-1$. Then either

(i) there is a regular k-set for G, or

(ii) $k = 2$, G is elementary abelian of order $2^m = |\Omega|$ and $|G_{\{\Delta\}}| = 2$ for any subset Δ of Ω of size 2.

PROOF: Let k satisfy $1 < k \leqslant \frac{n}{2}$ and suppose that there is no regular k-set for G. Since G acts semi-regularly on Ω, each $g \in G$ has n/m cycles of length m, where $|g| = m$. If a k-set Δ is not a regular set for G then there is at least one non-identity element of G such that some of g's cycles lie in Δ. Let $m_1, m_2, \ldots, m_t$ be the distinct orders of all those elements of G for which $m_i \mid k$, and $1 \neq m_i$ for $1 \leqslant i \leqslant t$. Let $M_i = \{g \in G \mid |g| = m_i\}$ for $1 \leqslant i \leqslant t$. Let $n_i = n/m_i$ and $k_i = k/m_i$ for $1 \leqslant i \leqslant t$. Further, let $c_i = \binom{n_i}{k_i}$. Then an element $g \in M_i$ stabilizes c_i k-sets, so that M_i fixes at most $c_i|M_i|$ k-sets. Since this holds for each i such that $1 \leqslant i \leqslant t$, if the assumption (i) is false, then we must have

$$\sum_{i=1}^{t} c_i|M_i| \geqslant \binom{n}{k} . \tag{1}$$

If m_1 is the smallest value of m_i for $1 \leqslant i \leqslant t$, then, since $k \leqslant \frac{n}{2}$, and $n_i \leqslant n_1$, $k_i \leqslant k_1$ for $1 \leqslant i \leqslant t$, we have also $c_i \leqslant c_1$. Thus from (1) we have

$$c_1(n-1) \geqslant c_1 \sum_{i=1}^{t} |M_i| \geqslant \sum_{i=1}^{t} c_i|M_i| \geqslant \binom{n}{k}, \quad \text{i.e. with } m = m_1 ,$$

$$\frac{n(n-m)(n-2m)\ \ldots\ (n-k+m)(n-1)}{k(k-m)(k-2m)\ \ldots\ m} \geqslant \frac{n(n-1)\ \ldots\ (n-k+1)}{k(k-1)\ \ldots\ 1} .$$

Since $k \leqslant \frac{n}{2}$, the above implies that $k = 2$, and $m = m_1 = 2$.

In this case G is an elementary abelian 2-group, and case (ii) holds. □

LEMMA 2.2: Let G be $PGL(2,q)$ acting on the points Ω of the projective line $PG(1,q)$. Suppose that $\Gamma \subset \Omega$ is a set of 3 points and d is a given integer such that $1 \leqslant d \leqslant q-3$. If $q > 7$ then there is a set Δ of size d disjoint from Γ such that $G_{\{\Gamma\}} \cap G_{\{\Delta\}} = 1$. If $q = 7$ the same holds when $d = 3$ or $d = 2$.

PROOF: We may take Γ to be the set $\{0,1,\infty\}$. Then $G^{\Gamma} = S_3$. As $G_{(\Gamma)} = 1$ it follows that $G_{\{\Gamma\}}$ has order 6. On $\Omega\setminus\Gamma$ the group $G_{\{\Gamma\}}$ has at most one orbit of odd length and at most one orbit of size two. It follows that for $q > 7$ there is at least one orbit Λ of size 6 and $G_{\{\Gamma\}}$ acts regularly on Λ. For a given value d', $1 \leqslant d' \leqslant 5$ the lemma 2.1 shows that there is a set $\Lambda' \subset \Lambda$, $|\Lambda'| = d'$ such that $G_{\{\Gamma\}} \cap G_{\{\Lambda'\}} = 1$. Now select any set Δ' in $\Omega\setminus\{\Gamma \cup \Lambda\}$ of size $d - d'$ and define $\Delta = \Delta' \cup \Lambda'$. It follows that $G_{\{\Gamma\}} \cap G_{\{\Delta\}} \subseteq G_{\{\Gamma\}} \cap G_{\{\Lambda'\}} = 1$. For $q = 7$, $G_{\{\Gamma\}}$ has an orbit Γ_2 of length 2 and an orbit Γ_3 of length 3 on $\Omega\setminus\Gamma$. Here we take one point in Γ_2 and one or two points in Γ_3 to form Δ. □

COROLLARY: If G and Γ are as in the lemma, and if $q > 7$ and $1 \leqslant d \leqslant q$, then there is a subset Δ of Ω of size d such that $G_{\{\Gamma\}} \cap G_{\{\Delta\}} = 1$. If $q = 7$ the same holds for $2 \leqslant d \leqslant 6$; if $q = 5$ the same holds for $3 \leqslant d \leqslant 4$.

PROOF: We allow $\Delta \cap \Gamma \neq \phi$, so the proof is immediate from the lemma for $q > 7$. For $q = 5$ or 7 trivial computations give the result. □

LEMMA 2.3: Let G be $P\Gamma L(2,q)$ acting on the points Ω of the projective line $PG(1,q)$ where q is a prime power, not a prime. Let $\Gamma \subset \Omega$ be a set of 3 points, and let d satisfy $1 \leqslant d \leqslant q-3$. If $q > 16$ then there is a set Δ of size d, $\Delta \subseteq \Omega$, $\Delta \cap \Gamma = \phi$ such that $G_{\{\Gamma\}} \cap G_{\{\Delta\}} = 1$. The same holds for $q = 16$ and $2 \leqslant d \leqslant 12$, for $q = 9$ and $3 \leqslant d \leqslant 4$ or for $q = 8$ and $d = 3$.

PROOF: Again, we may take Γ to be the set $\{0,1,\infty\}$. The lemma follows immediately from lemma 2.1 once we have shown that $G_{\{\Gamma\}}$ has at least one regular orbit on the points not in Γ. For an element a in $GF(q)$ we define $C(a) = \{a^{\pm 1}, (1-a)^{\pm 1}, (1-a^{-1})^{\pm 1}\}$. If $a \notin \Gamma$ then $C(a)$ is the set of cross-ratios obtained by arranging $\{0,1,\infty,a\}$ in all possible ways. The following facts are easily established:

(i) Two elements a and a' are in the same orbit of $G_{\{\Gamma\}} \cap PGL(2,q)$ if and only if $C(a) = C(a')$. In particular the sets $C(a)$ for a in $GF(q)$ form a partition of $GF(\overline{q}) \cup \{\infty\}$ for every subfield of $GF(q)$.

(ii) If $a \notin \Gamma$ then $|C(a)| = 1$ or 3 if and only if $C(a) = C(-1)$.

(iii) If $a \neq -1$ then $|C(a)| = 2$ if and only if $a^2 - a + 1 = 0$ so that a is a 6th root of unity.

So, in all other cases $|C(a)| = 6$. The sets $C(a)$ are permuted by a field automorphism σ of $GF(q)$ according to $(C(a))^\sigma = C(a^\sigma)$. If $C(a) = (C(a))^\sigma$ for some a then this element satisfies one of the equations $a = a^{\pm\sigma}$, $a = (1-a)^{\pm\sigma}$ or $a = (1-a^{-1})^{\pm\sigma}$. If $q > 16$ then lemma 3.5 in [8] can be used to show that there is some a in $GF(q)$ with $|C(a)| = 6$ that satisfies none of these equations for any field automorphism of $GF(q)$. It follows that $C = \cup\, C(a^\sigma)$ (where σ runs over all field automorphisms) is an orbit of $G_{\{\Gamma\}}$ of length $6n$ where $q = p^n$. On the other hand $|G_{\{\Gamma\}}| = 6n$ and hence C is a regular orbit. For $q = 8, 9$ or 16 the result follows by computation. □

COROLLARY: If G, Γ and q are as in the lemma then for $q > 16$ and $1 \leqslant d \leqslant q$ there is a d-subset Δ of Ω such that $G_{\{\Delta\}} \cap G_{\{\Gamma\}} = 1$. If $q = 8, 9$ or 16, the same holds for $2 \leqslant d \leqslant q$.

PROOF: From the lemma for $q > 16$, since we allow $\Delta \cap \Gamma \neq \phi$, and by computation for $q = 8, 9$ or 16. □

3. REGULAR SETS IN PROJECTIVE SPACE

In this section we establish the existence of regular sets in the projective spaces over Galois fields of at least 3 elements. Our main result follows at the end of this section and is proved in the general case by a sequence of lemmas. For some exceptional parameters direct computation is required. In the following all groups act on the points of projective space in the natural way, and all sets and groups are finite.

LEMMA 3.1: If $q \geqslant 16$, $q \neq 17$, then any 4-set of points of $PG(1,q)$ can be extended to a regular 5-set for $PGL(2,q)$. For $q < 16$, or $q = 17$, there are no regular 5-sets for $PGL(2,q)$.

PROOF: Let $G = PGL(2,q)$. Comparing the order of G to the total number of 5-sets shows that q must be at least 16. Computation shows that there is no regular 5-set for $q = 17$. Without loss of generality we may assume that $\Gamma = \{\infty,0,1,a\}$, where a is arbitrarily chosen. Let b be a further point, $\Lambda = \Gamma \cup \{b\}$ and define $\Lambda_x = \Lambda\setminus\{x\}$ for x in Λ . We now calculate cross-ratios: $c_b = (\infty,0;\ 1,a) = a$, $c_a = (\infty,0;\ 1,b) = b$, $c_1 = (\infty,0;\ a,b) = b/a$, $c_0 = (\infty,1;\ a,b) = (b-1)/(a-1)$ and $c_\infty = a.(b-1)/b.(a-1)$. When the points in Λ_x are taken in all possible orders the cross-ratios obtained are

$$C_x = \left\{c_x^{\pm1},\ (1-c_x)^{\pm1},\ (1-c_x^{-1})^{\pm1}\right\}$$

for every x in Λ .

Let g be a collineation in $G_{\{\Lambda\}}$ with $x^g = x'$. As cross-ratios are preserved and as $\Lambda_x^g = \Lambda_{x'}$, it follows that $c_x \in C_{x'}$ and $c_{x'} \in C_x$. We now choose $b \notin \{0,\infty,1,a\}$ such that

(i) $c_a = b \notin C_b \cup C_0 \cup C_1 \cup C_\infty$,

(ii) $c_1 = \frac{b}{a} \notin C_b \cup C_0 \cup C_\infty$ and

(iii) b is not a third root of -1 .

The conditions (i) and (ii) after some simplifications give a total of 26 equations, 11 of which are quadratic, that must not be satisfied by b . Thus in total at most 44 values are excluded. Therefore, if $q \geqslant 43$ a value of b may be chosen so that b satisfies conditions (i) - (iii) above. Condition (i) implies that a is fixed by every collineation in $G_{\{\Lambda\}}$. Condition (ii) implies that also 1 is fixed. The remaining points $\{\infty,0,b\}$ can be displaced in $G_{\{\Lambda\}}$ only when b satisfies $b^2 - b + 1 = 0$, hence is a third root of -1 . Therefore condition (iii) implies that $G_{\{\Lambda\}} = 1$. The remaining cases for $16 \leqslant q \leqslant 43$ have been considered by exhaustive computer search. □

NOTE: It is shown in [10] that $P\Gamma L(2,q)$ has a regular 5-set for any $q \geqslant 32$.

LEMMA 3.2:

(1) Let $d \geqslant 3$, $q \geqslant 5$ and $q \geqslant d$, where $d = 4$ if $q = 5$ and $d \leqslant 6$ if $q = 7$. Then $PGL(d,q)$ has a regular $(d+3)$-set.

(2) Let $d \geqslant 3$, $q \geqslant 8$ and $q \geqslant d$, where $d < q$ if $q = 8, 9$ or 16 . Then $P\Gamma L(d,q)$ has a regular $(d+3)$-set.

PROOF: (1) Let $G = PGL(d,q)$. For a given dimension d and a basis of $V_d(q)$ let ℓ be the line through the points $A_1 = \langle(1,0,0, \dots ,0)\rangle$ and $A_2 = \langle(0,1,0, \dots ,0)\rangle$. For a set Γ of three points on ℓ we choose a set $\overline{\Delta}$ of d points on ℓ such that $G_{\{\Gamma\},\{\overline{\Delta}\}}$ fixes every point of ℓ , where if $d = 3$ we choose $\overline{\Delta}$ such that $\Gamma \cap \overline{\Delta} = \phi$. Such a set can be chosen by lemma 2.2 and its corollary. Without loss of generality we may assume that $\overline{\Delta}$ contains the points A_1 , A_2 and $A_3 = \langle(1,1,0,0, \dots ,0)\rangle$. Let a_i , $i = 4, \dots ,d$ be such that the remaining points in $\overline{\Delta}$ are of the form $A_i = \langle(1,a_i,0,0, \dots ,0)\rangle$. For the following matrix

$$H = \begin{bmatrix} 0 & 1 & -1 & -a_4 & \dots & -a_d \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

let H_i be the hyperplane in $PG(d-1,q)$ whose homogeneous coordinate is the i^{th} column vector of H . As H has rank d , any d-1 hyperplanes H_i intersect in a point and the points so obtained form a simplex

$\Delta = \{B_1, B_2, \ldots, B_d\}$ in $PG(d-1,q)$, where $B_i = \bigcap_{j \neq i} H_j$. We observe the following:

(a) A_i is on H_i for every $i = 1, 2, \ldots, d$;

(b) if A_i is on H_j then $i = j$;

(c) a point on ℓ not in $\overline{\Delta}$ is not on any of the H_i .

It follows that

(i) if $B \neq B'$ are in Δ and D is in Γ then B , B', D are not collinear, and

(ii) no point of Δ is on ℓ .

Let $\Lambda = \Gamma \cup \Delta$. As Γ is the only set of 3 collinear points in Λ any collineation g fixing Λ as a set fixes both Γ and Δ setwise. Therefore in particular $\ell^g = \ell$ and g permutes the hyperplanes H_i forming our simplex Δ . Thus g permutes the points $H_i \cap \ell = A_i$ so that g belongs to $G_{\{\Gamma\},\{\overline{\Delta}\}}$. By construction g fixes every point on ℓ . As distinct hyperplanes intersect ℓ in distinct points, every hyperplane H_i is fixed as a set. Therefore every simplex point is fixed. As no point of Δ is on ℓ it follows that g is the identity on $PG(d-1,q)$. The proof of (2) follows similarly, using lemma 2.3. □

LEMMA 3.3: Let $d \geq 5$ and $q \geq 3$. Then $P\Gamma L(d,q)$ has a regular set of size $2d - 1$ if d is even or of size $2d$ if d is odd.

PROOF: For a given dimension d and a basis of $V_d(q)$ let $E_1 = \langle(1,0,0, \ldots ,0)\rangle$, $E_2 = \langle(0,1,0, \ldots ,0)\rangle , \ldots , E_d = \langle(0,0,0, \ldots ,1)\rangle$. Define Δ to be the unit simplex $\{E_1, \ldots ,E_d\}$. Let $d^* = \frac{1}{2}d - 1$ if d is even and $d^* = \frac{1}{2}(d - 1)$ if d is odd. We define further points P and P_i, Q_i for every $1 \leq i \leq d^*$. The coordinates of P , P_i and Q_i are the first d entries in the following tables in which w is any primitive field element:

	1	2	3	4	5
P	w	w	1	1	1
Q_1	1	0	1	1	w
P_1	w-1	w	0	0	1-w
Q_2	w-1	1	1	0	1
P_2	1	w-1	0	1	0

Table 1: $d = 5,\quad q \geq 3$

	1	2	•3	4	5	6	7	8	9	10	11	12	13	14	15	...
P	w	1	1	1	1	1	1	1	1	1	1	1	1	1	1	...
Q_1	w	0	1	w	w	w	w	w	w	w	w	w	w	w	w	...
P_1	0	1	0	1-w	1-w	1-w	1-w	1-w	1-w	1-w	1-w	1-w	1-w	1-w	1-w	...
Q_2	0	1	1	0	1	w	w	w	w	w	w	w	w	w	w	...
P_2	w	0	0	1	0	1-w	1-w	1-w	1-w	1-w	1-w	1-w	1-w	1-w	1-w	...
Q_3	1	1	1	1	w	0	1	w	w	w	w	w	w	w	w	...
P_3	w-1	0	0	0	1-w	1	0	1-w	1-w	1-w	1-w	1-w	1-w	1-w	1-w	...
Q_4	0	1	1	1	1	w	w	0	1	w	w	w	w	w	w	...
P_4	w	0	0	0	0	1-w	1-w	1	0	1-w	1-w	1-w	1-w	1-w	1-w	...
Q_5	1	1	1	1	1	1	w	w	w	0	1	w	w	w	w	...
P_5	w-1	0	0	0	0	0	1-w	1-w	1-w	1	0	1-w	1-w	1-w	1-w	...
Q_6	0	1	1	1	1	1	1	w	w	w	w	0	1	w	w	...
P_6	w	0	0	0	0	0	0	1-w	1-w	1-w	1-w	1	0	1-w	1-w	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 2: $d \geq 6,\quad q \geq 3$

Let $\Delta^* = \{P, P_i, Q_i, \mid 1 \leq i \leq d^*\}$ and $\Lambda = \Delta \cup \Delta^*$. Note that

(a) $\{P, P_i, Q_i\}$ is a collinear set for every $i \leq d^*$.

(b) If $E \in \Delta$ and X, $Y \in \Delta^*$, then $\{E, X, Y\}$ is not a collinear set.

(c) If E, $E' \in \Delta$ and $X \in \Delta^*$, then $\{E, E', X\}$ is not a collinear set.

(d) If Q_j is on the line PP_i then $i = j$.

Thus Λ consists of the vertices of the simplex Δ and triples of collinear points centered at P . From (a) - (c) it follows that 3 collinear points in Λ must be contained in Δ^* . Therefore a collineation g of $PG(d-1,q)$ fixing Λ as a set fixes both Δ and Δ^* as sets. Therefore g is represented by a monomial matrix M , and thus X and X^g have the same number of zero coordinates for any point X . It follows that

(i) $P^g = P$;

(ii) $P_i^g = P_i$, for all $i = 1, \ldots, d^*$;

(iii) $Q_i^g = Q_j$ and (d) above implies $i = j$ for every $i \leqslant d^*$.

Thus g fixes Δ^* pointwise and an easy inductive argument shows that g also fixes Δ pointwise. Now having chosen w to be primitive, the fact that g fixes P and Q_1 will force g to be the identity. □

The preceeding lemmas show that for generic parameters (d,q) there are sets of points in $PG(d-1,q)$ that are regular sets for $P\Gamma L(d,q)$. For sufficiently large fields, a regular set as constructed consists of either

A : the vertices of a d-simplex together with three points on a line avoiding these vertices and not on a line through any two vertex points, when $d \leqslant q$; or,

B : the vertices of a d-simplex together with a set of d (or $d-1$) points consisting of collinear points centered at a point belonging to the regular set, when $d \geqslant q$.

Clearly a regular set for $P\Gamma L(d,q)$ is regular for every group of collineations of $PG(d-1,q)$. The following theorem states the existence and gives the size of regular sets in projective space.

THEOREM 3.1: For $d \geqslant 2$ and $q \geqslant 3$, $P\Gamma L(d,q)$ in its natural action on the points of projective space $PG(d-1,q)$ has a regular k-set according to the entries given in the table below. An entry of the form "-" indicates that there is no regular set of any size for $P\Gamma L(d,q)$.

d = 2 :

q	⩽ 11	13	16	17	19	23	25	27	29 ⩽ q
k	-	6	-	6	5	5	6	6	5*

;

d = 3 :

q	3	4	5	⩾ 7
k	-	-	7	6

; d = 4 :

q	3	4	⩾ 5
k	8	9	7

;

d ⩾ 5, d odd :

q	q < d	d=q∈{5,7,9}	d ⩽ q
k	2d	2d	d+3

d ⩾ 6, d even :

q	q < d	d=q∈{8,16}	d ⩽ q
k	2d-1	2d-1	d+3

.

Note: (i) For $d = 2$ the entry * follows from [10].

(ii) The values of k given in the table are not necessarily minimal. For example, it can be shown that for $d = 3$ and $q \geqslant 29$, a regular set of size $k = 5$ can be found (in fact, a 5-arc).

PROOF: The only parameters in the table not covered by lemmas 3.1 to 3.3 are $(d,q) = (2,\bar{q})$ for $\bar{q} \leqslant 17$, $(d,q) = (3,\bar{q})$ for $\bar{q} \leqslant 5$ and $(d,q) = (4,\bar{q})$ for $\bar{q} \leqslant 4$. In these cases the results were obtained by direct computation. □

COROLLARY 1: In its natural action on the points of $PG(d-1,q)$ the group $PGL(d,q)$ has a regular k-set where $k = 5$ for $(d,q) = (2,16)$ and $k = 8$ for $(d,q) = (3,4)$. Furthermore, $PGL(d,q)$ has a regular k-set with k less than the value given for $P\Gamma L(d,q)$, for the following parameters

(d,q)	(2,25)	(2,27)	(4,4)	(8,8)	(9,9)	(16,16)
k	5	5	8	11	12	19

.

COROLLARY 2: Let G be a subgroup of $P\Gamma L(d,q)$ for $d \geqslant 2$ and $q \geqslant 3$ acting primitively on the points of $PG(d-1,q)$ without containing the alternating group of the same degree. If there is no regular k-set for G for any value of k, then (i) $(d,q) = (2,\bar{q})$, $\bar{q} \in \{4,5,7,8,9,11,16\}$ and G is one of D_{10}; $PSL(2,5)$, $PGL(2,5)$; $PSL(2,7)$, $PGL(2,7)$; $PGL(2,8)$, $P\Gamma L(2,8)$; $Sym(5)$, $PSL(2,9)$, $PGL(2,9)$, M_{10}, $PSL(2,9).C_2$, $P\Gamma L(2,9)$; $PGL(2,11)$; $PGL(2,16).C_2$, $P\Gamma L(2,16)$; or

(ii) $(d,q) = (3,\bar{q})$, $\bar{q} \in \{3,4\}$ and G is $PGL(3,3)$ or $P\Gamma L(3,4)$. (Here $D_{10} < P\Gamma L(2,4) \approx Sym(5)$ and is the dihedral group; $PSL(2,9).C_2$ is $PSL(2,9)$ extended by the field automorphism; $PGL(2,16).C_2$ is $PGL(2,16)$ extended by $\langle\sigma^2\rangle$ where σ is the field automorphism of order 4).

PROOF: As a regular set of $P\Gamma L(d,q)$ is a regular set for any subgroup, only the parameters $(d,q) = (2,\bar{q})$ for $\bar{q} \in \{3,4,5,7,8,9,11,16\}$ and $(d,q) = (3,3)$ or $(3,4)$ need to be considered, by theorem 3.1. In these cases, all primitive subgroups not containing the alternating group of the same degree can be found, for example by using the list of Sims [14]. □

4. REGULAR SETS IN AFFINE SPACE

It is possible to obtain regular sets for the affine semi-linear group $A\Gamma L(d,q)$ acting on the points of $AG(d,q)$ for $d \geq 1$, $q \geq 3$, in a manner directly analogous to that in section 3 for the projective groups. Since the derivation of these affine regular sets is so similar (but not the same) we merely state our results here, and indicate the nature of the geometrical configurations involved.

THEOREM 4.1: For $d \geq 1$ and $q \geq 3$, $A\Gamma L(d,q)$ in its natural action on the points of affine space $AG(d,q)$ has a regular k-set according to the entries given in the table below. An entry of the form "-" indicates that there is no regular set of any size for $A\Gamma L(d,q)$.

$d = 1$:

q	$3 \leq q \leq 9$	$q \geq 11,\ q \neq 16$	16
k	-	3	4

;

$d = 2$:

q	3	4	$5 \leq q \leq 9$	$q \geq 11,\ q \neq 16$	16
k	-	-	5	4	5

;

$d = 3$:

q	3	4	≥ 5
k	8	9	7

;

$d \geq 4$, d even :

q	$q < d$	$q=d\in\{4,8,16\}$	$q=d+1\in\{5,7,9\}$	$q \geq d$
k	$2(d+1)$	$2(d+1)$	$2(d+1)$	$d+4$

;

$d \geq 5$, d odd :

q	$q < d$	$q=d\in\{5,7,9\}$	$q=d+1\in\{8,16\}$	$q \geq d$
k	$2d+1$	$2d+1$	$2d+1$	$d+4$

.

PROOF: As in Theorem 3.1, for some of the smaller parameters direct computations had to be made. For the general cases, lemmas for the dimensions 1, 2 and greater than 3 , established the regular sets, as in the projective case.

For $d = 2$ the parameters quoted in the table refer to regular sets consisting of three points on a line, and one or two points off the line. This is a somewhat better (i.e. smaller) configuration than that obtained in the projective plane, where the configuration of a triangle of points and three collinear points was given as a special case of such a configuration for higher dimensions. Such a configuration will also apply to the affine case.

For $d \geq 3$, there are, as in the projective case, essentially two types of configuration, viz. A or B as described before Theorem 3.1. The proof of this is quite similar to the proofs of lemmas 3.2 and 3.3. For example, the tables in lemma 3.3 can be used directly in the affine case by taking the hyperplane at infinity to be $\langle(1,-1,1,-1, \ldots)'\rangle$ in table 2, with some restrictions on d and q . Three other tables were constructed to cover all the fields with $q \geq 3$ and all dimensions $d \geq 2$. □

COROLLARY 1: In its natural action on the points of $AG(d,q)$, the group $AGL(d,q)$ has a regular k-set where $k = 3$ for $(d,q) = (1,9)$ and $k = 6$ for $(d,q) = (2,4)$. Furthermore, $AGL(d,q)$ has a regular k-set with k less than the value given for $A\Gamma L(d,q)$ for the following parameters:

(d,q)	(1,16)	(2,8)	(2,9)	(2,16)	(3,4)	(7,8)	(8,8)	(8,9)	(9,9)	(15,16)	(16,16)
k	3	4	4	4	8	11	12	12	13	19	20

COROLLARY 2: Let G be a subgroup of $A\Gamma L(d,q)$ for $d \geq 1$ and $q \geq 3$ that acts primitively on the points of $AG(d,q)$ without containing the alternating group of the same degree. If there is no regular k-set for G for any value of k , then

(i) $(d,q) = (1,\bar{q})$, $\bar{q} \in \{5,7,8,9\}$ and G is $ASL(1,5)$, $AGL(1,5)$, $AGL(1,7)$, $A\Gamma L(1,8)$, $A\Sigma L(1,9)$ or $A\Gamma L(1,9)$ or

(ii) $(d,q) = (2,\bar{q})$, $\bar{q} \in \{3,4\}$ and G is $ASL(2,3)$, $AGL(2,3)$ or $A\Gamma L(2,4)$.

PROOF: As for theorem 3.1, corollary 2. □

5. GEOMETRIC GROUPS

Let G be any group of collineations of S where S is a finite projective or affine space of geometric dimension $d \geqslant 2$. Throughout, the points of S will be denoted by Ω. In this section we prove that G acts k-geometrically on Ω where k depends only on the dimension of S. It follows, as we remarked in section 2, that G is, in particular, k-closed in its action on Ω.

For generic parameters (d,q) let Λ be a regular set for $P\Gamma L(d+1,q)$ or $A\Gamma L(d,q)$, respectively, as constructed in sections 3 and 4. For reasons of uniformity in our argument we take Λ to be a line segment of 3 points and a triangle for the affine plane AG(2,q), as in lemma 3.2. Thus Λ is a configuration of type A or of type B as described in the paragraph preceding theorem 3.1. When k is the size of Λ let d^* be the lowest dimension of a projective, respectively affine, space over GF(q) containing at least k + 1 points. Let $\mathcal{B}_S$ be the collection of all k-sets of Ω that are contained in some d^*-dimensional subspace or coset, respectively. Now define

$$\mathcal{B}_G = \{\Lambda^g \mid g \in G\} \cup \mathcal{B}_S .$$

It is clear that G is a group of automorphisms of the block structure $(\Omega,\mathcal{B}_G)$. The following theorem shows that all its automorphisms are in G.

THEOREM 5.1: Let S be the projective space PG(d,q) or the affine space AG(d,q) where $d \geqslant 2$, $q \geqslant 3$ and let G be any group of collineations of S. Then $G = \mathrm{Aut}(\Omega,\mathcal{B}_G)$ and G is k-geometric where k is as given below:

$d = 2$, $q \geqslant 7$: $k = 6$; $d = 3$, $q \geqslant 5$: $k = 7$;
$d \geqslant 4$, $q \geqslant 3$: $k \in \{d+4, 2(d+1)\}$ if d is even, $k \in \{d+4, 2d+1\}$ if d is odd, and $k = d+4$ if $q \geqslant d$ in both cases.

The parameters in the theorem guarantee that there is a regular set for S of standard type A or B. (The reader may verify this in lemmas 3.2 to 3.3.) For some of the exceptional small fields similar constructions yield similar results. We also note that the value given for k is not necessarily minimal. However, the result on special linear groups in [9] shows there are collineation groups of S that are *not* d-geometric. This remark shows that for sufficiently large fields the values given for k are close to being smallest possible.

PROOF: Let A be the automorphism group of $(\Omega, \mathcal{B}_G)$ for a given group G. Then $G \subseteq A$ as G is a group of collineations of S. As Λ is a regular set in S the proof will be completed once we have shown that A permutes the blocks in Λ^G and that A consists of collineations of S. We assume first that S is a projective space.

As above, let d^* be the lowest dimension of a subspace containing at least $|\Lambda| + 1$ points. From the values for $|\Lambda|$ given in theorems 3.1 and 4.1, it follows that $d^* < d$ so that $\mathcal{B}_G$ does not consist of *all* k-sets in Ω. Notice that $k \geqslant d^* + 1$. The proof now follows in two steps.

(i) Let Θ be a subset of S such that every k-subset of Θ, $k \geqslant d^* + 1$, is contained in a subspace of S of dimension at most d^*. Then Θ is contained in a subspace of dimension d^*.

PROOF of (i): Let S' be the space spanned by Θ, of dimension d'. By elementary arguments from linear algebra it follows that there are $d' + 1$ independent points in Θ that span S'. If $d' > d^*$ then some k points in Θ will span a subspace of dimension bigger than d^*.

We shall say that an m-set Θ is an *m-clique* or simply a *clique* if all its k-subsets belong to $\mathcal{B}_G$, $m \geqslant k$.

(ii) Suppose that Θ is a clique containing a k-set from Λ^G. Then $|\Theta| = k$. The cliques of maximal size are the d^*-dimensional subspaces of S.

PROOF of (ii): It is sufficient to show that no $(k+1)$-clique contains a k-set from Λ^G. From the construction of regular sets in sections 3 and 4 it follows that no $(k-1)$-subset of a regular set is contained in a hyperplane of S. Suppose that $\Theta = \Lambda' \cup \{\rho\}$ is a $(k+1)$-clique with Λ' in Λ^G. As the other k-subsets of Θ are of the form $(\Lambda' \setminus \{\lambda\}) \cup \{\rho\}$ with λ in Λ' it follows that all k-subsets of Θ belong to Λ^G. This is easily seen to be impossible.

To prove the second statement of (ii), observe that every d^*-dimensional subspace S^* of S is a clique. Thus let Θ be a clique of maximal size $|\Theta| \geq |S^*|$. From the first part it follows that every k-subset of Θ is in a d^*-dimensional subspace of S. The step (i) now implies that Θ is in fact a d^*-dimensional subspace.

We now complete the proof of the theorem. Let g be an automorphism of $(\Omega, \mathcal{B}_G)$. Then g will map maximal cliques onto maximal cliques. Hence g preserves the d^*-dimensional subspaces of S and therefore is a collineation of S. Considering the action of g on pairs (Δ, S^*) where Δ belongs to $\mathcal{B}_G$ and S^* is a d^*-dimensional subspace of S we see that $\Delta \in \Lambda^G$ implies $\Delta^g \in \Lambda^G$ since g cannot map a non-incident antiflag onto a flag. This completes the proof of the theorem when S is a projective space. Very similar arguments establish the result in the affine case. □

COROLLARY 1: If G is given as in the theorem then G is k-closed for the value of k given in terms of the parameters defining G.

Notice that for all values of the parameters d and q, the existence of a regular k-set implies k-closure. We do not give the details here but the results can be deduced easily from theorems 3.1 and 4.1 and theorems A and B of [8]. We state the following deduction which to a minor extent depends on the classification theorem of finite simple groups.

COROLLARY 2 (C.T): Let G be a primitive subgroup of $P\Gamma L(d+1,q)$ or $A\Gamma L(d,q)$, $d \geq 1$, $q \geq 3$, such that G does not contain the alternating group of the same degree and is not k-closed for any value of k. Then G is one of the following groups: C_5, $AGL(1,5)$, $PGL(2,5)$, $AGL(1,8)$, $A\Gamma L(1,8)$, $PGL(2,8)$, $P\Gamma L(2,8)$, $AGL(1,9)$, $PGL(2,9)$ or $ASL(2,3)$ in their usual representations.

For the general theory of permutation groups the results in this paper give the following theorem on closure.

THEOREM 5.2: Let G be a transitive permutation group of odd degree with a normal elementary abelian subgroup of order $p^d > 9$ where p is a prime.

Then there is a value of k satisfying $k \leqslant 2(d+1)$ for which G is k-closed.

With the help of [4] this result also holds for groups of degree $2^d \geqslant 32$ and $k \approx 2^{d-1}$. Before we prove the theorem, we need a lemma:

LEMMA 5.1: If p is a prime and $p \geqslant 11$, then $AGL(1,p)$ is 3-closed.

PROOF: Let H be the 3-closure of $G = AGL(1,p)$, and suppose that $H \neq G$. Since G is sharply 2-transitive of prime degree, theorem 27.1 of [15] implies that H is 3-transitive. Since H has the same orbits on 3-sets as G, G is 3-homogeneous. But the number of 3-sets is

$$p(p-1)(p-2)/6 > |G| = p(p-1) \quad \text{for} \quad p \geqslant 11 .$$

Thus we have a contradiction, so that $G = H$. □

PROOF of theorem: The assumption implies that G may be identified with a subgroup of $AGL(d,p)$. When d is at least 4, theorem 5.1 implies that G is k-geometric, hence in particular k-closed with the given range for k. Thus we need only deal with the case $d \leqslant 3$. Here we select a regular k-set Λ for $AGL(d,p)$ where $3 \leqslant k \leqslant 2(d+1)$ by theorem 4.1. Hence Λ is regular for G. When $G^{\{k\}} = H$ is the k-closure of G then $G^{\{k\}} \subseteq AGL(d,p)^{\{k\}} \subseteq AGL(d,p)^{\{3\}} = AGL(d,p)$ by theorem 5.1 of [12], theorem A of [8] and lemma 5.1. As Λ is a regular set we have $|\Lambda^H| = |H|$ and $|\Lambda^G| = |G|$. This shows that $G = H$. □

6. INFINITE FIELDS

In this final section we discuss the existence of regular sets in finite dimensional projective and affine spaces over infinite fields. Let Γ denote the full automorphism group of the infinite field F; for any subgroup Γ' of Γ the extension of $PGL(d+1,F)$ or $AGL(d,F)$ by the automorphisms in Γ' will be denoted by $P\Gamma'L(d+1,F)$ or $A\Gamma'L(d,F)$ respectively. As before, we consider the natural action of these groups on the set of points Ω of the projective or affine space of finite dimension d over the field F.

THEOREM 6.1: Let G be $P\Gamma'L(d+1,F)$ or $A\Gamma'L(d,F)$ acting on Ω where $d \geq 2$. Suppose that, in the natural action on the points of the line, there is a set $\overline{\Delta}$ of $d+1$ points, and a finite set Δ of at least 3 points such that $P\Gamma'L(2,F)_{\{\Delta\},\{\overline{\Delta}\}} = 1$ or $A\Gamma'L(1,F)_{\{\Delta\},\{\overline{\Delta}\}} = 1$, respectively. Then there is a regular set for G of size $k = d + 1 + |\Delta|$. If the assumption applies to $\Gamma' = \Gamma$, then every subgroup of $P\Gamma L(d+1,F)$ or $A\Gamma L(d,F)$ is k-geometric in its natural action on Ω.

PROOF: Identify the set Δ with a set of points on a line ℓ in $PG(d,F)$ or $AG(d,F)$ so that the points of $\overline{\Delta}$ are also on this line. As in sections 3 and 4 we construct a simplex Δ' whose faces intersect ℓ in distinct points $A_1, \ldots, A_{d+1}$ such that $\overline{\Delta} = \{A_1, \ldots, A_{d+1}\}$. As before, it follows that the vertex points of Δ' together with Δ form a regular set for G. This proves the first part of the theorem.

Suppose now that Δ and $\overline{\Delta}$ exist as required with $P\Gamma L(d+1,F)_{\{\Delta\},\{\overline{\Delta}\}} = 1$ or $A\Gamma L(d,F)_{\{\Delta\},\{\overline{\Delta}\}} = 1$ respectively. Then there is a regular set Λ for $P\Gamma L(d+1,F)$ or $A\Gamma L(d,F)$ by the above argument. For any subgroup H of one of these groups we define $\mathcal{B}_H$ as in section 5. From the arguments in that section, and the fundamental theorem of geometry, it follows that H is the full automorphism group of the incidence structure $(\Omega,\mathcal{B}_H)$. □

NOTE: The existence of a finite regular set in a space over an infinite field implies that the set of points of that space and its group are of the same cardinality. For this reason it is clear that $P\Gamma L(d+1,F)$ and $A\Gamma L(d,F)$ have no regular sets if $|\Gamma| > |F|$.

The hypothesis of theorem 6.1 on the other hand is satisfied for an arbitrary field when Γ' is a finite group. This can be seen just as in the case of finite fields. It follows therefore that regular sets exist in finite dimensional spaces over fields such as $\mathbb{R}$, $\mathbb{Q}$ or finite extensions of $\mathbb{Q}$. As a consequence any collineation group of such a space is geometric.

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