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On the k-Closure of Finite Linear Groups.

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Sunto. — *Se G è un gruppo di permutazioni su un insieme finito Ω di n elementi, allora, per ogni $k \leqslant n$, G agisce sull'insieme $\Omega^{\{k\}}$ dei sottoinsiemi di Ω di k elementi. La k-chiusura di G è il massimo sottogruppo $G^{\{k\}}$ di* Sym(Ω) *che su $\Omega^{\{k\}}$ ha le stesse orbite di G, e G è k-chiuso se $G = G^{\{k\}}$. Si mostra che i gruppi lineari proiettivo e affine, nella loro naturale azione sui punti, sono in generale k-chiusi per certi valori di k, e si determinano le relative eccezioni.*

1. – Introduction.

Let G be a permutation group acting on a set Ω of finite size n. This gives rise to permutation actions $(G, \Omega^{\{k\}})$ where G acts in the natural way on the system $\Omega^{\{k\}}$ of k-element subsets of Ω. Two groups G and H are said to be *k-orbit equivalent*, $G \approx_k H$, if they have the same orbits on $\Omega^{\{k\}}$. The *k-closure* of G is the largest group $G^{\{k\}}$ in the symmetric group on Ω that satisfies $G^{\{k\}} \approx_k G$. We say that G is *k-closed* on Ω if $G = G^{\{k\}}$. These definitions follow Wielandt [22] where the group action on ordered k-tuples is studied. Groups that are k-closed in our definition will in particular be k-closed in the sense of Wielandt.

The relationship between the action of G on $\Omega^{\{k\}}$ and $\Omega^{\{l\}}$ has been studied by many mathematicians. Closure properties have been examined in earlier papers by one of the present authors. In Siemons [18] it is shown that the orbits on $\Omega^{\{k\}}$ determine the orbits on $\Omega^{\{l\}}$, without reference to the group, for all $l < k$ and $2k < n$. This result implies in particular that the k-closure of a group is contained in its l-closure and that l-closure implies k-closure. In Siemons and Wagner [19] and Inglis [8] all primitive groups G are classified in which $|G^{\{n^*\}} : G| \neq 1$, $n - 1 < 2n^* < n$, is

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coprime to the order of G. Under the assumption of the classification theorem of finite simple groups (which we shall refer to as C.T.) Cameron, Neumann and Saxl [3] have shown that any primitive group of sufficiently large degree is n^*-closed or contains the alternating group of the same degree.

It is an open problem to provide a proof independent of the C.T. and to characterise those primitive groups that are not k-closed for any k. Furthermore for particular classes of closed group actions it is desirable to determine the minimal value of k for which the group is k-closed. This paper deals with the linear groups in their natural action on the points of projective and affine spaces. Before we can state our main results we need to consider a related concept. Following Betten [2] a group action G on Ω is *geometric* if there exists a system $\mathfrak{B}$ of Ω-subsets such that G is the full automorphism group of the incidence structure $(\Omega, \mathfrak{B})$. If in addition every set in $\mathfrak{B}$ has cardinality k we shall say that G is *k-geometric* on Ω. It is immediately clear that a k-geometric group is k-closed. For projective or affine spaces of dimension at least that of the plane the semilinear groups are the largest groups preserving the collinearity relation. Thus these groups are in particular 3-geometric and hence 3-closed. For the linear groups we obtain the following:

THEOREM A. – (I) *In the action on the points of* $PG(d-1, q)$, $d \geq 3$,

$PGL(d, q)$ *is* 4-*geometric if* $q \notin \{4, 8, 9, 16\}$;

$PGL(3, 4)$ *is* 10-*geometric but not* k-*closed for* $k \leq 9$;

$PGL(d, 4)$ *is* 8-*geometric but not* k-*closed for* $k \leq 6$, *when* $d \geq 4$;

$PGL(d, 8)$ *and* $PGL(d, 9)$ *are* 6-*geometric but not* 5-*closed for* $d \geq 3$;

$PGL(d, 16)$ *is* 6-*geometric for* $d \geq 3$.

(II) *In the action on the points of* $AG(d, q)$, $d \geq 2$,

$AGL(d, q)$ *is* 3-*geometric if* $q \notin \{2, 4, 8, 9, 16\}$;

$AGL(d, 2)$ *is* 4-*geometric but not* 3-*closed for* $d \geq 3$;

$AGL(d, 4)$ *is* 6-*geometric but not* 5-*closed for* $d \geq 2$;

$AGL(d, q)$ *for* $q = 8, 9$ *and* 16 *are* 4-*geometric but not* 3-*closed for* $d \geq 2$.

This theorem is independent of C.T. In essence it is a consequence of the Fundamental Theorem of projective and affine geometry and some close examination of cross-ratios on the projective line. With the exception of the small fields $GF(q)$, $q = 4, 8, 9$

or 16 the blocks of a geometry for $PGL(d, q)$ or $AGL(d, q)$ may be chosen to be segments of lines in projective or affine space. The exceptional fields are dealt with by computational methods. Some geometrical configurations (which in general are not unique) for the projective and affine linear groups are shown in § 5.

The Fundamental Theorem of affine or projective geometry does not apply to the case of a line. Here we need Result 2.5 which depends on C.T.

THEOREM B (C.T.). – (I) *In the action on the points of* $PG(1, q)$, $q \geqslant 7$,

$P\Gamma L(2, q)$ *is* 4-*closed if and only if* $q \notin \{8, 32\}$; $P\Gamma L(2, 32)$ *is* 5-*closed*;

$PGL(2, q)$ *is* 4-*closed if and only if* $q \notin \{8, 9, 16\}$;

$PGL(2, 16)$ *is* 6-*closed but not* 5-*closed.*

(II) *In the action on the points of* $AG(1, q)$, $q \geqslant 7$,

$A\Gamma L(1, q)$ *is* 3-*closed if and only if* $q \notin \{8, 9, 16, 32\}$;

$A\Gamma L(1, 8)$ *is not* 4-*closed*; $A\Gamma L(1, q)$ *is* 4-*closed for* $q \in \{9, 16, 32\}$;

$AGL(1, q)$ *is* 3-*closed if and only if* $q \notin \{8, 9, 16\}$;

$AGL(1, 16)$ *is* 4-*closed*; $AGL(1, 8)$ *and* $AGL(1, 9)$ *are not* 4-*closed.*

As a corollary to Theorem A and B we can list all those general semilinear and linear groups that are not k-closed for any k:

THEOREM C (C.T.). – *Let G be any of the groups* $P\Gamma L(d, q)$, $PGL(d, q)$, $A\Gamma L(d, q)$ *or* $AGL(d, q)$ *in their natural action, and suppose that G is not k-closed for any k. Then G is one of the following*: $PGL(2, 4)$, $PGL(2, 5)$, $PGL(2, 8)$, $P\Gamma L(2, 8)$, $PGL(2, 9)$, $AGL(1, 4)$, $AGL(1, 5)$, $AGL(1, 8)$, $A\Gamma L(1, 8)$ *or* $AGL(1, 9)$.

Using properties of groups with a regular orbit on k-sets for some k, we show in [12] that this together with $AGL(2, 3) \cap \mathrm{Alt}(9)$ contains the complete list of 2-transitive groups not containing the alternating group of the same degree that act on finite desarguesian geometries over $GF(q)$, $q > 2$, in the natural action that are not k-closed for any k.

The organization of this paper is as follows: in § 2 we give definitions, notation and some assumed results; §§ 3 and 4 deal with the projective and affine cases respectively for the non-exceptional fields; § 5 deals with the case of small fields $GF(q)$, $q = 4, 8, 9$

or 16. Theorem A is a combination of theorems 3.1, 4.1, 5.1 and Lemma 5.4. Theorem B is obtained from theorems 3.2, 4.2, and 5.1.

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2. – Notation and assumed results.

The notation and terminology used is standard and mostly that of Dembowski [5] and Wielandt [21]. Variations from their notation will be given below. Our groups and sets will always be finite.

If Ω is a set and $|\Omega| = n$ then the symmetric and alternating groups on Ω are denoted by $\mathrm{Sym}(\Omega)$ and $\mathrm{Alt}(\Omega)$, or simply by $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$. If G is a permutation group on Ω, and $\Delta \subseteq \Omega$, then $G_{\{\Delta\}}$ denotes the set stabilizer of Δ (global stabilizer) and $G_{(\Delta)}$ denotes the pointwise stablizer. We will refer to $G_{\{\Delta\}}/G_{(\Delta)}$ as the restriction of G to Δ. If $k \leqslant n$ then G acts in a natural way on the set $\Omega^{\{k\}}$ of all k-element subsets of Ω. G is k-homogeneous if it is transitive in its action on $\Omega^{\{k\}}$. The k-closure $G^{\{k\}}$ of G is the largest subgroup of $\mathrm{Sym}(\Omega)$ that has the same orbits on $\Omega^{\{k\}}$ as G. G is k-closed if $G = G^{\{k\}}$. When speaking of k-closure we always assume that $k \leqslant [n/2]$. We may also refer to members of $\Omega^{\{k\}}$ as k-sets. The set of all images of Δ under G is denoted by Δ^G.

DEFINITION 2.1. – Suppose that $G < K \leqslant \mathrm{Sym}(\Omega)$. A subset Λ of Ω will be called a *K-base set* for G if $G < H \leqslant K$ implies $\Lambda^G \neq \Lambda^H$. If $K = \mathrm{Sym}(\Omega)$ then a K-base set of size k is called a *base k-set*.

Thus notice that if there is a base set for G of size k, then G is k-geometric as defined in the introduction. Conversely, a k-geometric group does not necessarily have a base k-set.

A $t-(v, k, \lambda)$ design $\mathfrak{D}$ on Ω, $|\Omega| = v$, is a pair $(\Omega, \mathfrak{B})$ where $\mathfrak{B}$ is a collection of k-subsets of Ω, called blocks, such that any t-subset of Ω is contained in precisely λ blocks.

DEFINITION 2.2. – Let G be a transitive permutation group on Ω and suppose that Λ is a base set for G of size k. The design (Ω, Λ^G) is denoted by $\mathfrak{D}(G, \Lambda)$.

Indeed as G is transitive $\mathfrak{D}(G, \Lambda)$ at least is a 1-design; a 2-design if G is doubly transitive. Since Λ is a base set G is the full automorphism group of $\mathfrak{D}(G, \Lambda)$.

The projective geometry of dimension d over the Galois field $GF(q)$ will be denoted by $PG(d, q)$. Its full automorphism group is $P\Gamma L(d+1, q)$ for $d \geqslant 2$, by the fundamental theorem of projective geometry ([1] p. 88). The affine geometry of dimension d over $GF(q)$ will be denoted by $AG(d, q)$. Its full automorphism group is $A\Gamma L(d, q)$ for $d \geqslant 2$, by the fundamental theorem of affine geometry ([5] p. 32). If S is a hyperplane of $PG(d, q)$ then $A\Gamma L(d, q) = (P\Gamma L(d+1, q))_{\{S\}}$, where $P\Gamma L(d+1, q)$ is taken as a permutation group acting on the points of $PG(d, q)$, and S as a subset of points. In what follows we will always take this permutation action of $P\Gamma L(d+1, q)$, unless otherwise stated. Similarly, $A\Gamma L(d, q)$ will act on the points of $AG(d, q)$.

The following results will be needed for the proofs of the theorems. In some cases the statement of the result has been modified to suit our requirements.

Result 2.1 (Siemons [18], Theorem 5.1, p. 399). – Let G be a permutation group on Ω, where $|\Omega| = n$. Let n^* satisfy $(n-1)/2 \leqslant n^* \leqslant n/2$. Then

$$G \leqslant G^{\{n^*\}} \leqslant \ldots \leqslant G^{\{k\}} \leqslant G^{\{k-1\}} \leqslant \ldots \leqslant G^{\{2\}} \leqslant G^{\{1\}} \leqslant \mathrm{Sym}(\Omega)$$

for any k such that $1 \leqslant k \leqslant n^*$. If G is k-closed then G is l-closed for $n^* \geqslant l \geqslant k$. Further, G is k-homogeneous if and only if $G^{\{k\}} = \mathrm{Sym}(\Omega)$.

Result 2.2 (Kantor [9], Theorem 1, p. 261). – Let G be a group k-homogeneous but not k-transitive on a finite set Ω of n points, where $n \geqslant 2k$. Then, up to permutation isomorphism, one of the following holds:

(i) $k = 2$ and $G \leqslant A\Gamma L(1, q)$ with $n = q \equiv 3 \pmod 4$;

(ii) $k = 3$ and $PSL(2, q) \leqslant G \leqslant P\Gamma L(2, q)$ where $n - 1 = q \equiv 3 \pmod 4$;

(iii) $k = 3$ and $G = AGL(1, 8)$, $A\Gamma L(1, 8)$ or $A\Gamma L(1, 32)$; or

(iv) $k = 4$ and $G = PSL(2, 8)$, $P\Gamma L(2, 8)$ or $P\Gamma L(2, 32)$.

Result 2.3 (Mortimer [15], Main Theorem, p. 445). – If $AGL(d, q) \leqslant G \leqslant \mathrm{Sym}(q^d)$ with $d \geqslant 2$, then either

(i) $G = \mathrm{Alt}(q^d)$ or $\mathrm{Sym}(q^d)$, or

(ii) there exist integers r and b with $q = r^b$ such that

$$ASL(bd, r) \leqslant G \leqslant A\Gamma L(bd, r) \, ,$$

or

(iii) $AGL(2, 4) < G < AGL(4, 2)$ and $G_\alpha \cong \mathrm{Alt}(7)$.

RESULT 2.4 (Kantor and McDonough [10] or List [14]). - Suppose p is a prime, $q = p^n$, and $|\Omega| = (q^d - 1)/(q - 1)$ where $d \geqslant 3$. If H is a subgroup of $\mathrm{Sym}(\Omega)$ containing $PSL(d, q)$, then either $H \leqslant P\Gamma L(d, q)$ or $H \geqslant \mathrm{Alt}(\Omega)$.

RESULT 2.5 (C.T.) A. - Suppose that a permutation group G of degree $q + 1$ for some prime power q contains a subgroup H permutation isomorphic to $PSL(2, q)$. Then

(i) $PSL(2, q) \subseteq G \subseteq P\Gamma L(2, q)$, or

(ii) $G \supseteq \mathrm{Alt}(q + 1)$, or

(iii) $G = M_{11}$ or M_{12} of degree 12 or $G = M_{24}$, $q = 23$, or

(iv) $PSL(2, 7) \subseteq G \subseteq AGL(3, 2)$, $q = 7$.

B) Suppose that a permutation group G of degree $q = p^m$ for some prime p contains $AGL(1, q)$. Then either $G \leqslant AGL(m, p)$ or $G \geqslant \mathrm{Alt}(q)$.

(This result is a well-known consequence of C.T. It can easily be proved by checking through the list of 2-transitive groups given in [16], for example).

3. - The projective groups.

In this section we examine the projective groups $PGL(d, q)$ for k-closure. The main results of the section are Theorems 3.1 and 3.2 which follow from the lemmas below.

In the following $P\Gamma L(d, q)$ acts on the points of the projective space $PG(d - 1, q)$ where $q = p^n$ for a prime p.

LEMMA 3.1. - *If $G < K \leqslant \mathrm{Sym}(\Omega)$ and $k \leqslant [n/2]$ where $|\Omega| = n$, then*

(i) $G^{\{k\}} \leqslant K^{\{k\}}$;

(ii) *if K is k-closed then $G^{\{l\}} \leqslant K$ for any l such that $[n/2] \geqslant l \geqslant k$;*

(iii) *if Λ is a K-base set of k points for G, then if K is k-closed, so is G.*

PROOF. - Immediate from the definitions and Result 2.1.

LEMMA 3.2. - *$P\Gamma L(d, q)$ is 3-closed for $d \geqslant 3$.*

PROOF. – Let G be the 3-closure of $P\Gamma L(d, q)$ in Sym(Ω), where Ω denotes the points of $PG(d-1, q)$. Since G has the same orbits as $P\Gamma L(d, q)$ on 3-sets, G will map collinear triples to collinear triples. As it also preserves incidence by definition, G will preserve the lines of $PG(d, q)$ and hence must be a collineation group of $PG(d, q)$. By the fundamental theorem of projective geometry [1] p. 88, $G \leqslant P\Gamma L(d, q)$. Thus $P\Gamma L(d, q)$ is 3-closed for $d \geqslant 3$.

COROLLARY. – *$P\Gamma L(d, q)$ is k-closed for all k satisfying*

$$3 \leqslant k \leqslant \frac{q^d - 1}{2(q-1)}, \quad d \geqslant 3 .$$

PROOF. – By the above and Result 2.1.

LEMMA 3.3. – *For $d \geqslant 3$, $P\Gamma L(d, q)$ and $PGL(d, q)$ have the same orbits on 4-sets consisting of 4 non-collinear points of $PG(d-1, q)$.*

PROOF. – Sets of 4 non-collinear points of $PG(d-1, q)$ are of the following types:

(i) Exactly 3 collinear (line and point);

(ii) No 3 collinear, but 4 coplanar (quadrangle);

(iii) 4 non-coplanar (tetrahedron).

We show that $PGL(d, q)$ (and hence also $P\Gamma L(d, q)$) is transitive on each of the above types.

Type (i): We show that $PSL(d, q)$ has one orbit of this type. Let $\Delta_1 = \{P_1, P_2, P_3, P_4\}$, $\Delta_2 = \{Q_1, Q_2, Q_3, Q_4\}$ where $l_1 = P_2P_3P_4$ and $l_2 = Q_2Q_3Q_4$ are lines, and the Δ_i, $i = 1, 2$, are two 4-sets of type (i). Since $PSL(d, q)$ is transitive on lines, we can map l_1 onto l_2 in $PSL(d, q)$. Since $d \geqslant 3$, $PSL(d, q)$ induces $PGL(2, q)$ on any line, and this is 3-transitive on points of the line. Thus we need only consider the case where $P_i = Q_i$ for $i = 2, 3, 4$ and $l_1 = l_2 = l$. Let H be a hyperplane of $PG(d-1, q)$ containing l but not containing P_1 or Q_1. Then there is an elation h of $PG(d-1, q)$ such that $P_1h = Q_1$, h has axis H and h has centre $P_1Q_1 \cap H$. Thus $\Delta_1 h = \Delta_2$, and $h \in PSL(d, q)$.

Type (ii): $PSL(d, q)$ is transitive on planes, so we can take the two quadrangles to be in the same plane. If $d \geqslant 4$, then $PGL(3, q)$ is induced on the plane, and this is transitive on quadrangles ([7] Theorem 2.12 p. 32). If $d = 3$, then $PSL(3, q)$ may not be transitive on quadrangles: see, for example [11].

Type (iii): Here $d \geqslant 4$ for such configurations to be present. $PSL(d,q)$ is transitive on projective spaces $PG(3,q)$ inside $PG(d-1,q)$, so we can take the two tetrahedra to be in the same $PG(3,q)$. Then $PSL(4,q)$ is transitive on tetrahedra, and hence so is $PSL(d,q)$ and $PGL(d,q)$.

LEMMA 3.4. – *If G is a subgroup of $P\Gamma L(d,q)$ with $PGL(d,q)$ as a proper subgroup, $d \geqslant 2$, then G and $PGL(d,q)$ have the same orbits on sets of 4 collinear points if and only if*

(i) $q = 4, 8$ *or* 9 *and* $G = P\Gamma L(d,q)$ *or*

(ii) $q = 16$ *and* $G = PGL(d,q)\cdot\langle\sigma^2\rangle$ *where* σ *is a generator of the field automorphism group of* $GF(16)$.

PROOF. – As $PGL(d,q)$ is transitive on the lines of $PG(d-1,q)$ it will be sufficient to consider the action of G induced on a fixed line. Thus it will be sufficient to prove the lemma for $d = 2$. For points on the projective line $PG(2,q)$ we use $GF(q) \cup \{\infty\}$ as parametric coordinates. As $PGL(2,q)$ is triply transitive every orbit on 4-sets contains a representative of the form $\Lambda_a = \{\infty, 0, 1, a\}$ for some a in $GF(q)$, $0 \neq a \neq 1$. Now suppose that G and $PGL(2,q)$ have the same orbits on 4-sets. Let $g\cdot\alpha$ be an element in G where $g \in PGL(2,q)$ and α is some field automorphism of $GF(q)$, $\alpha \neq 1$. Thus $(\Lambda_a)^{g\cdot\alpha}$ lies in the same orbit as Λ_a and in some suitable arrangement $(\Lambda_a)^{g\cdot\alpha}$ yields a cross-ratio a^α. As $PGL(d,q)$ preserves cross-ratios, a^α has to be one of the cross-ratios obtained by all possible arrangements of Λ_a. They are a, a^{-1}, $1-a$, $(1-a)^{-1}$, $1-a^{-1}$ and $(1-a^{-1})^{-1}$, see for instance page 42 in [7].

When $q = p^n$ for a prime p, then α is given by $\alpha\colon x \to x^{(p^i)}$ for some i, $0 \leqslant i \leqslant n-1$. Thus, if G and $PGL(2,q)$ have the same orbits on 4-sets, then every element in $GF(q)$ satisfies at least one of the following equations:

$$EQ(i)\colon \begin{cases} \text{I}\colon a^{(p^i)} - a = 0 \\ \text{II}\colon a^{(p^i+1)} - 1 = 0 \\ \text{III}\colon a^{(p^i)} + a - 1 = 0 \\ \text{IV}\colon a^{(p^i+1)} - a^{(p^i)} + 1 = 0 \\ \text{V}\colon a^{(p^i)} - a + 1 = 0 \\ \text{VI}\colon a^{(p^i+1)} - a^{(p^i)} - a = 0\,. \end{cases}$$

(At a later stage we shall make use of these equations. For this reason we have emphasized the exponent of the field automorphism).

We now count the maximal number r of distinct solutions these equations can have. Equations I and VI are solved by $a = 0$; $a = 1$ solves I and II. Thus

$$r \leqslant \{(p^i - 2) + (p^i) + (p^i) + (p^i + 1) + (p^i + 1) + (p^i)\} + 2 = \\ = 6p^i + 2$$

and our assumption implies that $q = p^n \leqslant 6p^i + 2$. Replacing α by α^{-1} we may assume that $i \leqslant n/2$ and therefore $7 \geqslant p^{(n/2)}$. This leaves the possibilities $q = 4, 8, 9, 16$ or 25.

The Case $q = 4, 8$ *or* 9. There is no group properly between $PGL(2, q)$ and $P\Gamma L(2, q)$, thus conclusion (i) holds.

The Case $q = 16$. Notice that $PGL(2, 16)$ and $P\Gamma L(2, 16)$ do not have the same orbits on 4-sets. For a primitive root ω, for instance, a $P\Gamma L(2, 16)$-image of Λ_ω yields a cross-ratio of ω^2, a value that is not amongst the possible cross-ratios of Λ_ω. Thus $G = PGL(2, 16)\cdot\langle\sigma^2\rangle$ is the only group satisfying the hypotheses.

The Case $q = 25$. Here $x^\alpha = x^5$ for all x in $GF(25)$. We analyse the equations in detail. There are at most 14 distinct solutions of I-III. When a solves IV, then

$$a^6 - a^5 + 1 = 0 \Rightarrow a^{30} - a^{25} + 1 = 0 \Rightarrow a^6 - a + 1 = 0 \Rightarrow a^4 = 1$$

and a solves I. Similarly, if a solves V,

$$a^6 - a + 1 = 0 \Rightarrow a^{30} - a^5 + 1 = 0 \Rightarrow a^4 = 1\,,$$

i.e. a solves I. Thus at most 19 elements in $GF(25)$ are solutions and so the case $q = 25$ is ruled out.

It remains to show the converse conclusions. First we note that $PGL(2, 4) = \mathrm{Alt}(5)$ and $PGL(2, 8)$ are 4-homogeneous. (Result 2.2). Thus there is just one orbit on 4-sets. The group $PGL(2, 9)$ has 2 orbits on 4-sets; they have representatives Λ_{-1} and Λ_ω for a primitive root ω. Both orbits are left invariant by field automorphisms. Thus $P\Gamma L(2, 9)$ has the same orbits on 4-sets. The group $PGL(2, 16)$ has 3 orbits on 4-sets; they are represented by $\Lambda_{\omega^5}, \Lambda_{\omega^2}$ and Λ_ω. The field automorphism $\sigma^2\colon x \to x^4$ preserves these orbits. Therefore $PGL(2, 16)\cdot\langle\sigma^2\rangle$ has the same orbits. The automorphism $\sigma\colon x \to x^2$ finally joins Λ_ω to Λ_{ω^2} so that $P\Gamma L(2, 16)$ has 2 orbits on 4-sets. This completes the proof of the lemma.

For a given field $GF(p^n)$, p a prime, let $m_1, \dots, m_s$ be the distinct prime divisors of n. We define $i_1 = n/m_1, \dots, i', \dots, i_s = n/m_s$ so that these numbers are the maximal divisors of n. Each value i' gives rise to a system $EQ(i')$ of the 6 equations (for exponent i') described in the proof of the previous lemma. Thus we obtain in all $6 \cdot s$ equations $EQ(i_1), \dots, EQ(i_s)$. Under suitable conditions on p^n they are not satisfied for some element in $GF(p^n)$:

LEMMA 3.5. – *If $n \geqslant 2$ and $q = p^n \neq 4, 8, 9, 16$ then $GF(q)$ contains some element a satisfying none of the equations in $EQ(i_1), \dots, \dots, EQ(i_s)$ where $i_1, \dots, i_s$ are the distinct maximal divisors of n.*

PROOF. – As we have shown in the proof of lemma 3.4 each system $EQ(i')$ has at most $6p^{i'} + 2$ solutions. Thus there are at most $r = 6(p^{i_1} + \dots + p^{i_s}) + 2$ distinct solutions. When $s = 1$ (so that n is a prime power) the argument of the proof of lemma 3.4 shows that some element in $GF(q)$ does not solve $EQ(i_1)$ provided $q \neq 4, 8, 9$ or 16.

When $s \geqslant 2$ some elementary considerations show that $r \leqslant q$ requires $p = 2$ and $n = 6$. Thus only the field $GF(64)$ and the equations $EQ(2)$ and $EQ(3)$ need to be considered. Computation shows that precisely 28 elements in $GF(64)$ satisfy at least one of the equations, while a primitive root in $GF(64)$ is not a solution of any of the 12 equations. This completes the proof.

In the space $PG(d-1, q)$ we fix some line l and parametrize its points by $GF(q) \cup \{\infty\}$. When q is not one of the exceptional values 4, 8, 9 or 16 let a be an element in $GF(q)$ satisfying none of the equations $EQ(i_1), \dots, EQ(i_s)$ and let $\Lambda_a = \{\infty, 0, 1, a\}$.

DEFINITION 3.1. – The design $\mathfrak{D}(PGL(d, q), \Lambda_a)$ on the points of $PG(d-1, q)$ is denoted by $\mathfrak{D}_{\mathfrak{J}}(d, q)$.

Here, as PGL acts doubly transitively on the points of $PG(d-1, q)$, $\mathfrak{D}_{\mathfrak{J}}(d, q)$ is a $2-((q^d-1)/(q-1), 4, \lambda)$ design where λ may easily be calculated by considering the 1-dimensional space.

THEOREM 3.1. – *In the natural action on the points of $PG(d-1, q)$ with $d \geqslant 3$ we have:*

(i) *$PGL(d, q)$ is 3-closed if and only if q is a prime, and*

(ii) *$PGL(d, q)$ is 4-closed if and only if $q \notin \{4, 8, 9, 16\}$.*

If q is not a prime, $q \notin \{4, 8, 9, 16\}$ and $d \geqslant 3$ then $PGL(d, q)$ is 4-geometric and the full automorphism group of $\mathfrak{D}_{\mathfrak{J}}(d, q)$.

PROOF. – The parts (i) and (ii) follow from lemmas 3.1 to 3.5. When Λ is a block of the design $\mathfrak{D}_{\mathfrak{J}}(d, q)$ then by definition there is a unique line l of $PG(d-1, q)$ containing Λ; conversely any line of $PG(d-1, q)$ contains some block of the design. It follows that an automorphism of $\mathfrak{D}_{\mathcal{T}}(d, q)$ is an automorphism of $PG(d-1, q)$. Since $d \geqslant 3$, the fundamental theorem of projective geometry (or Result 2.4) implies that $\mathrm{Aut}(\mathfrak{D}_{\mathcal{T}}(d, q)) \subseteq P\Gamma L(d, q)$.

Therefore $PGL(d, q) \subseteq \mathrm{Aut}(\mathfrak{D}_{\mathcal{T}}(d, q)) \subseteq P\Gamma L(d, q)$. If the inclusion on the left is a proper one, let G be a smallest group with $PGL(d, q) \subset G \subseteq \mathrm{Aut}(\mathfrak{D}_{\mathcal{T}}(d, q))$. Then G is the extension of $PGL(d, q)$ by some field automorphism σ^i where the minimality of G implies that i is a maximal divisor of n, $q = p^n$. Let Λ_a be as in Definition 3.1 and consider the restriction G^* of G to Λ_a. Then G^* is the extension of $PGL(2, q)$ by σ^i. In G^* Λ_a is mapped into a set of 4 points with cross ratio $a^{\sigma^i} = a^{(p^i)}$. As a is chosen not to satisfy any of the equations in $EQ(i)$ this is a contradiction. Therefore $\mathrm{Aut}(\mathfrak{D}_{\mathcal{T}}(d, q)) = PGL(d, q)$ and Λ_a is a base set for $PGL(d, q)$. This proves the theorem.

To obtain the analogous result for the projective line we need to use the Classification Theorem, through Result 2.5.

THEOREM 3.2 (C.T.). – *In the natural action on the projective line, $P\Gamma L(2, q)$, $q \geqslant 7$, is 4-closed if and only if $q \notin \{8, 32\}$. The group $PGL(2, q)$, $q \geqslant 7$, is 4-closed if and only if $q \notin \{8, 9, 16\}$. If q is not a prime and $q \geqslant 25$ then $PGL(2, q)$ is 4-geometric and is the full automorphism group of $\mathfrak{D}_{\mathcal{T}}(2, q)$.*

PROOF. – Let H be the 4-closure of $P\Gamma L(2, q)$. We use Result 2.5 to determine H. From the structure of the Mathieu groups (see § 7 of [6]), it cannot be a Mathieu group. Observe also that $PGL(2, 7)$ is not a subgroup of $AGL(3, 2)$. If $H \supseteq \mathrm{Alt}(q+1)$, $P\Gamma L(2, q)$ would have to be 4-fold homogeneous, and by Result 2.2, $q = 2, 4, 5, 8$ or 32. Thus $H = P\Gamma L(2, q)$ for the remaining values of q so that $P\Gamma L(2, q)$ is 4-closed. Conversely, if $q \in \{8, 32\}$ then $H = \mathrm{Sym}(q+1)$, by Result 2.2.

Let K now be the 4-closure of $PGL(2, q)$ when $q \geqslant 7$ and $q \notin \{8, 9, 16\}$. By lemma 3.1 $K \subseteq H$ and so $K \subseteq P\Gamma L(2, q)$. For $q \neq 32$ this follows from the above and for $q = 32$ it follows from Results 2.2 and 2.5. Now lemma 3.4 applies and $K = PGL(2, q)$ so that $PGL(2, q)$ is 4-closed. The converse follows from lemma 3.1 and Result 2.2.

When q is a proper power of some prime $\geqslant 5$ let G be the automorphism group of $\mathfrak{D}_{\mathcal{T}}(2, q)$. As above we see that $G \supseteq \mathrm{Alt}(q+1)$ (which clearly is impossible) or $G \subseteq P\Gamma L(2, q)$. The remainder follows as in the proof of Theorem 3.1.

4. – The affine groups.

In this section we examine the k-closure of the affine groups $AGL(d, q)$. The main results of the section are Theorems 4.1 and 4.2, which follow from the lemmas below.

In the following $A\Gamma L(d, q)$ acts on the q^d points of the affine space $AG(d, q)$ where $q = p^n$ for a prime p.

LEMMA 4.1. – *For $d \geqslant 2$ and $q > 2$ $A\Gamma L(d, q)$ is 3-closed.*

PROOF. – Let G be the 3-closure of $A\Gamma L(d, q)$. Since G and $A\Gamma L(d, q)$ have the same orbits on 3-sets of points and since q is at least three, G maps collinear triples onto collinear triples. Thus G preserves the lines of $AG(d, q)$ and so $G \leqslant A\Gamma L(d, q)$ by the fundamental theorem of affine geometry, p. 23 in [5].

LEMMA 4.2. – *For $d \geqslant 3$ $AGL(d, 2)$ is 4-closed but not 3-closed.*

PROOF. – Let G be the 4-closure of $AGL(d, 2)$. By Result 2.3 we conclude $G = AGL(d, 2)$ or $G \supseteq \mathrm{Alt}(2^d)$. The latter is not possible as $AGL(d, 2)$ is not 4-homogeneous. Hence $AGL(d, 2)$ is 4-closed. As $AGL(d, 2)$ is 3-fold transitive, its 3-closure is $\mathrm{Sym}(2^d)$.

LEMMA 4.3. – *For $d \geqslant 2$ $AGL(d, q)$ is transitive on sets of 3 non-collinear points.*

PROOF. – On ordered bases of the d-dimensional vector space over $GF(q)$ the group $GL(d, q)$ acts transitively.

LEMMA 4.4. – *If G is a subgroup of $A\Gamma L(d, q)$ with $AGL(d, q)$ as a proper subgroup, $d \geqslant 1$, then G and $AGL(d, q)$ have the same orbits on sets of 3 collinear points if and only if*

(i) *$q = 4, 8$ or 9 and $G = A\Gamma L(d, q)$, or*

(ii) *$q = 16$ and $G = AGL(d, 16) \cdot \langle \sigma^2 \rangle$ where σ is a generator of the automorphism group of $GF(16)$.*

PROOF. – As in the proof of lemma 3.4 it will suffice to prove the lemma for the case of an affine line. Thus G acts on the elements of $GF(q)$. As G is doubly transitive, every orbit on 3-sets has a representative of the form $\Lambda_a = \{0, 1, a\}$ for some $a \in GF(q)$, $0 \neq a \neq 1$. Suppose that $\alpha \cdot g$ is an element of G with $g \in AGL(1, q)$ and α a non-identity field automorphism of $GF(q)$. By assumption $(\Lambda_a)^{\alpha \cdot g}$ is in the same AGL-orbit as Λ_a, or equivalently $(\Lambda_a)^\alpha = \{0, 1, a^\alpha\}$ is in the same AGL-orbit as Λ_a.

Some simple calculations show that this is only possible if $a^\alpha = a$, a^{-1}, $1-a$, $1-a^{-1}$, $(1-a)^{-1}$ or $(1-a^{-1})^{-1}$. (These are just the values of the possible cross-ratios we obtained for the projective case in lemma 3.4). The arguments in lemma 3.4 now show that either $q = 4, 8$ or 9 and $G = A\Gamma L(1, q)$ or $q = 16$ and $G = AGL(1, 16)\cdot\langle\sigma^2\rangle$.

To prove the converse we observe that $AGL(1, 4) = \mathrm{Alt}(4)$ and $AGL(1, 8)$ are 3-homogeneous, thus have the same orbits on 3-sets as $A\Gamma L(1, 4)$ and $A\Gamma L(1, 8)$ respectively. See also Result 2.2.

$AGL(1, 9)$ has 2 orbits on 3-sets; they have representatives Λ_{-1} and Λ_ω (ω a primitive root of $GF(9)$). Both orbits are kept invariant under field automorphisms, so that $A\Gamma L(1, 9)$ has the same orbits on 3-sets.

$AGL(1, 16)$ has 3 orbits on 3-sets; they are represented by $\Lambda_{\omega^5}, \Lambda_\omega$ and Λ_{ω^2} (ω a primitive root of $GF(16)$). The field automorphism $\sigma^2\colon x \to x^4$ preserves these orbits. So $AGL(1, 16)\cdot\langle\sigma^2\rangle$ has the same orbits on 3-sets. This completes the proof of the lemma.

In the space $AG(d, q)$ $d \geq 1$ and q not a prime we fix some line l and parametrize its points by $GF(q)$. When q is none of the exceptional values 4, 8, 9 or 16 let a be an element satisfying none of the equations $EQ(i_1), \ldots, EQ(i_s)$ of Section 3 and lemma 3.5. Let $\Lambda_a = \{0, 1, a\}$.

DEFINITION 4.1. – The design $\mathfrak{D}(AGL(d, q), \Lambda_a)$ on the points of $AG(d, q)$ is denoted by $\mathfrak{D}_{\mathcal{A}}(d, q)$.

These designs are $2-(q^d, 3, \lambda)$ designs as $AGL(d, q)$ is doubly transitive.

THEOREM 4.1. – *In the action on the points of $AG(d, q)$, $d \geq 2$, $AGL(d, q)$ is 3-closed if and only if $q \notin \{2, 4, 8, 9, 16\}$. If q is not a prime and $q \notin \{4, 8, 9, 16\}$ then $AGL(d, q)$ is 3-geometric and is the full automorphism group of $\mathfrak{D}_{\mathcal{A}}(d, q)$ for $d \geq 2$.*

PROOF. – The fact that $AGL(d, q)$ is 3-closed follows from lemmas 3.1 and 4.1 to 4.4.

The remainder of the proof is analogous to the proof of theorem 3.1, applying the fundamental theorem of affine geometry.

For the affine line we need Result 2.5 and C.T.

THEOREM 4.2 (C.T.). – *In the action on the affine line $AG(1, q)$, $q \geq 7$, $A\Gamma L(1, q)$ is 3-closed if and only if $q \notin \{8, 9, 16, 32\}$. $A\Gamma L(1, 8)$ is not k-closed for any k, and $A\Gamma L(1, q)$ for $q \in \{9, 16, 32\}$ is 4-closed. $AGL(1, q)$ is 3-closed if and only if $q \notin \{8, 9, 16\}$.*

PROOF. – We take $q\geqslant 7$ since the closure properties are clear for $q<7$. Let $q=p^d$ where p is a prime, and $d=d_1d_2$. Then $A\Gamma L(1,q)\leqslant A\Gamma L(d_1,p^{d_2})$, acting on $AG(d_1,q_1)$ where $q_1=p^{d_2}$. If $d_1\geqslant 2$ then $A\Gamma L(d_1,q_1)$ is transitive on triangles of points of $AG(d_1,q_1)$. If $A\Gamma L(1,q)$ has the same orbits on 3-sets as $A\Gamma L(d_1,q_1)$ then its order must be at least as great as the number of triangles, i.e.

$$q(q-1)\,d\geqslant \tfrac{1}{6}q(q-1)(q-q_1)\,,$$

i.e. $p^d\leqslant 6d+q_1$ where $d_1\geqslant 2$, $d=d_1d_2$ and $q_1=p^{d_2}$. It is easy to see that for $q\geqslant 7$ this can hold only for $q\in\{8,9,16,32\}$.

Now let G be the 3-closure of $A\Gamma L(1,q)$. By result 2.5, $G\leqslant AGL(d,p)$ or $G\leqslant \mathrm{Alt}(q)$. In the latter case G would be 3-homogeneous, which is only the case here for $q=8$ or 32. If $d=1$, then $G=AGL(1,p)$, so that $A\Gamma L(1,p)$ is 3-closed for $q\geqslant 7$. If $d\geqslant 2$, Then $A\Gamma L(1,q)\leqslant A\Gamma L(d_1,q_1)$ as above, and we choose d_1 to be a minimal prime divisor of d. If $q_1\neq 2$ then $A\Gamma L(d_1,q_1)$ is 3-closed by lemma 4.1, so that $G\leqslant A\Gamma L(d_1,q_1)$, and $G\neq A\Gamma L(d_1,q_1)$ for $q\notin\{8,9,16,32\}$ by the above argument. If $q_1=2$, then $AGL(d_1,2)$ is 3-transitive and hence $G=A\Gamma L(1,2^d)$ for $d\geqslant 5$ and d prime.

Thus we have $A\Gamma L(1,q)\leqslant G<A\Gamma L(d_1,q_1)$ with q,d,d_1,d_2,q_1 as above, and $q\notin\{8,9,16,32\}$. As G is 2-transitive with a regular normal subgroup, the possibilities for G have been classified by Hering (see the list given in the appendix to Liebeck [13]). Using this list (which depends on C.T.) it is not difficult to show that $G=A\Gamma L(1,q)$. We remark only that the arguments to eliminate the possibility of one of the infinite classes of groups either involved the possible lengths of orbits on triangles, or the impossibility of the particular imbedding required. For the extraspecial and exceptional cases, where $q\in\{2^4,3^4,3^6,5^2,7^2,11^2,19^2,23^2,29^2,59^2\}$, we argue as follows: (i) if $q=p^2$ where $p\equiv -1 (\mathrm{mod}\ 6)$, then there is a triangle of points Δ in $AG(2,p)$ for which $(A\Gamma L(1,p^2))_{\{\Delta\}}=(AGL(2,p))_{\{\Delta\}}=\mathrm{Sym}(3)$, proving that $G=A\Gamma L(1,p^2)$; (ii) for $q=3^4,7^2,19^2$, we showed by direct computation that $A\Gamma L(1,q)$ is maximal in $A\Gamma L(2,q^{\frac{1}{2}})$; (iii) for $q=3^6$, the imbedding is not the one in $A\Gamma L(2,3^3)$ as required. For $q\in\{8,9,16,32\}$ we obtained the stated results by computation.

For $q\notin\{8,9,16,32\}$ let K be the 3-closure of $AGL(1,q)$. By the above $K\subseteq (A\Gamma L(1,q))^{\{3\}}=A\Gamma L(1,q)$ and lemma 4.4 then implies that $AGL(1,q)$ is 3-closed. It remains to consider $q=32$. Here $K=(AGL(1,32))^{\{3\}}$ is doubly transitive and has 5 orbits on 3-sets. Let $M\neq 1$ be a minimal normal subgroup of K. Among the simple groups only $PSL(2, 31)$ could occur; this group, however, is 3-homogeneous. Thus M is the elementary abelian 2 group of order 32 and $K \subseteq AGL(5, 2)$. Theorem 1.1 in [20] implies that $K \subseteq A\Gamma L(1, 32)$ and now lemma 4.4 implies the result.

5. – The exceptional cases of small fields.

In this section we examine the projective and affine groups over the Galois fields of 4, 8, 9 and 16 elements. The lemmas 5.1, 5.2 and 5.3 show that the closure properties of these groups depend on the affine and projective geometries of dimension at most 3. These geometries are examined case by case and K-base sets are given in each instance where $K \leqslant P\Gamma L(d, q)$ or $A\Gamma L(d, q)$. With these results we establish k-closure in both the projective and the affine case whenever the space has the dimension of at least that of a plane: in the projective case we obtain k-closure for projective dimension $d \geqslant 2$ for some $k \geqslant 5$; in the affine case we obtain k-closure for affine dimension $d \geqslant 2$ for some $k \geqslant 4$. In most cases we have obtained the minimal value for k.

NOTATION. – In lemmas 5.1 to 5.3 we use the following notation: let K_d be any group satisfying

$$PGL(d, q) < K_d \leqslant P\Gamma L(d, q) .$$

If $d' \geqslant d$ then $K_{d'}$ will denote an extension of K_d to $PG(d' - 1, q)$ such that $K_{d'}$ restricted to $PG(d - 1, q)$ is K_d, and

$$PGL(d', q) < K_{d'} \leqslant P\Gamma L(d', q) .$$

If $d'' \leqslant d$ then $K_{d''}$ will denote a restriction of K_d to $PG(d'' - 1, q)$ such that $K_{d''}$ extended to $PG(d - 1, q)$ is K_d, and

$$PGL(d'', q) < K_{d''} \leqslant P\Gamma L(d'', q) .$$

The definition of a K-base set is given in § 2 (Definition 2.1).

When these lemmas are applied to the exceptional fields of order $q = 4$, 8 or 9 we take $K_d = P\Gamma L(d, q)$; for $q = 16$ we take $K_d = PGL(d, 16) \cdot \langle \sigma^2 \rangle$ where σ is a generator for $\mathrm{Aut}(GF(16))$.

The notation is analogous for the affine case, which is dealt with in parallel in this section.

LEMMA 5.1. – *Let Λ be a subset of the points in $PG(d'-1, q)$ (in $AG(d', q)$) such that Λ is a $K_{d'}$-base set for $PGL(d', q)$ (for $AGL(d', q)$).*

Suppose that Λ is contained in a subspace S and let $PGL(d, q)$, (or $AGL(d, q)$) be the restriction of $PGL(d', q)$ (or $AGL(d', q)$) to S. Then Λ is a K_d-base set for $PGL(d, q)$ or $AGL(d, q)$ respectively.

PROOF. – Suppose $\Lambda^h = \Lambda^g$ for some $h \in P\Gamma L(d, q)$, and some $g \in PGL(d, q)$. Then clearly $h \in P\Gamma L(d', q)$ and as Λ is a $K_{d'}$-base set, h belongs to $PGL(d', q)$. But this implies $h \in PGL(d, q)$ and so Λ is a K_d-base set for $PGL(d, q)$. For the affine groups the same arguments hold.

LEMMA 5.2. – *Let Λ be a set of points in $PG(d-1, q)$ (in $AG(d, q)$) so that Λ is a K_d-base set for $PGL(d, q)$ (for $AGL(d, q)$).*

Suppose $PG(d-1, q)$ is contained as a subspace in $PG(d'-1, q)$ ($AG(d, q)$ as a subspace of $AG(d', q)$). Then Λ is a $K_{d'}$-base set for $PGL(d', q)$ (for $AGL(d', q)$).

PROOF. – We have to show the following: if $\Lambda^h = \Lambda^g$ for $h \in P\Gamma L(d', q)$ and $g \in PGL(d', q)$ then h belongs to $PGL(d', q)$. Let h and g have these properties. By induction and lemma 5.1 we may assume that Λ is not contained in any proper subspace of $PG(d-1, q)$. Then gh^{-1} fixes setwise Λ and thus also the set S of points in $PG(d-1, q)$. Let h^* be the action of gh^{-1} on S. As Λ is a K_d-base set for $PGL(d, q)$, $h^* \in PGL(d, q)$. This implies that there is an element a in $PGL(d', q)$ such that h^* is the action of a restricted to S. Therefore $a^{-1} \cdot g \cdot h^{-1} = h_1$ fixes every point of S. It can be shown in general that an element in $P\Gamma L$ fixing a proper subspace pointwise belongs to PGL. For this reason h_1 and consequently h belong to $PGL(d', q)$. This completes the proof of the lemma in case of the projective groups. For affine groups the arguments are analogous.

LEMMA 5.3. – (i) *For $d' > d \geqslant 3$ let Λ be a $(d+2)$-set in $PG(d'-1, q)$. If Λ is a $K_{d'}$-base set for $PGL(d', q)$, then Λ is inside some $PG(d-1, q)$ and is a K_d-base set for $PGL(d, q)$.*

(ii) *For $d' > d \geqslant 2$ let Λ be a $(d+2)$-set in $AG(d', q)$. If Λ is a $K_{d'}$-base set for $AGL(d', q)$, then Λ is inside some $AG(d, q)$ and is a K_d-base set for $AGL(d, q)$.*

PROOF. – Let Λ be a $K_{d'}$-base set of $d+2$ points for $PGL(d', q)$. If the projective space spanned by Λ has dimension d, i.e. if Λ is a frame for $PG(d, q)$, then as $PGL(d+1, q)$ and $P\Gamma L(d+1, q)$ have only one orbit on frames (by [7] p. 32), Λ cannot be a $K_{d'}$-

base set for $PGL(d', q)$. Thus Λ is inside a $PG(d-1, q)$, so that Lemma 5.1 may be applied.

If Λ is a $K_{d'}$-base set for $AGL(d', q)$, then if the affine space spanned by Λ has dimension $d+1$, then Λ is a frame for $AG(d+1, q)$ and we have a contradiction as above. Thus Λ is inside a $AG(d, q)$, so that again Lemma 5.1 may be applied.

We now deal with the exceptional cases, i.e. when $q = 4, 8, 9, 16$. Here we have already shown in Theorems 3.1 and 4.1 that $PGL(d, q)$ is not 4-closed and $AGL(d, q)$ is not 3-closed. We examine these groups for k-closure for $k \geqslant 5$ in the projective case and $k \geqslant 4$ in the affine case.

The general computational procedure is to establish k-closure for as low dimension d as possible. Lemmas 5.1 to 5.3 may then be used to obtain k-closure for higher values of d. We examine each value of $q = 4, 8, 9, 16$ separately and give the arguments specific to each case to establish, where feasible, the minimum value of k for k-closure. The general method was to construct the groups $P\Gamma L(d, q) = \Gamma$ and $PGL(d, q) = G$ for some small d (i.e. 2, 3 or 4) and to find a k-set Δ such that

$$|G_{\{\Delta\}}| = |\Gamma_{\{\Delta\}}| .$$

If this holds, then since $|G| = |\Delta^G||G_{\{\Delta\}}| \neq |\Gamma| = |\Delta^\Gamma||\Gamma_{\{\Delta\}}|$, we have $\Delta^G \neq \Delta^\Gamma$ i.e. G and Γ have distinct orbits on k-sets. In the cases $q = 4, 8$ or 9, $[\Gamma{:}G]$ is a prime, so that Δ is a Γ-base set for G. By lemma 3.1, k-closure of Γ ensures k-closure of G. In the case $q = 16$, we have $H = PGL(d, q)\cdot\langle\sigma^2\rangle$ properly containing G, and so we apply the same method to H and G. The method for the affine groups was the same.

We remark that k-closure of $P\Gamma L(2, q)$ (or $A\Gamma L(1, q)$) is not assumed, even in the cases when it is known to be established. Thus our results on k-closure for $d \geqslant 3$ for the projective case (or $d \geqslant 2$ for the affine case) are independent of the classification theorem (C.T.). However, we use lemma 3.1 constantly.

Most of the computations were done with the aid of the Cayley package of J. Cannon [4] on the Birmingham University computer.

(1) *The field* $GF(4)$. Here $K_d = P\Gamma L(d, 4)$ or $A\Gamma L(d, 4)$ respectively. The group $PGL(2, 4)$ is Alt(5) with 2-closure equal to Sym(5). We constructed $P\Gamma L(3, 4)$ and $PGL(3, 4)$ and found a K_3-base set Λ of 10 points for $PGL(3, 4)$ thus establishing 10-closure of $PGL(3, 4)$. To show that 10 is in fact a minimum for k-closure, we computed the lengths of all the 17 orbits of $PGL(3, 4)$ on 9-sets and found these to be the same as those of $P\Gamma L(3, 4)$ on 9-sets. Thus $PGL(3, 4)$ is not 9-closed.

Lemma 5.2 assures the 10-closure of $PGL(d, 4)$ for all $d \geqslant 3$, but since the 21 point plane $PG(2, 4)$ is well known to have unusual properties, we constructed $P\Gamma L(4, 4)$ and $PGL(4, 4)$ and applied the method described. A K_4-base set Λ of 8 points was found. A complete determination of the number of orbits of $PGL(4, 4)$ and $P\Gamma L(4, 4)$ on 6-sets, showed this number to be 18, and thus established that $PGL(4, 4)$ is not 6-closed. If k is the minimum for k-closure of $PGL(4, 4)$ then $7 \leqslant k \leqslant 8$.

Lemma 5.2 then can be used to show that $PGL(d, 4)$ for $d \geqslant 4$ is 8-closed and Lemma 5.3 then shows that $PGL(d, 4)$ is not 6-closed for $d \geqslant 4$. Thus $PGL(d, 4)$ is k-closed for $d \geqslant 4$ where the minimum value of k is at most 8, greater than 6, but might possibly be 7.

The geometrical configurations of the K_d-base sets of 10 and 8 points for $PGL(3, 4)$ and $PGL(d, 4)$ (for $d \geqslant 4$), respectively are shown below. Here we have shown a line through two points if and only if the base set contains at least three points of the line through the two points.

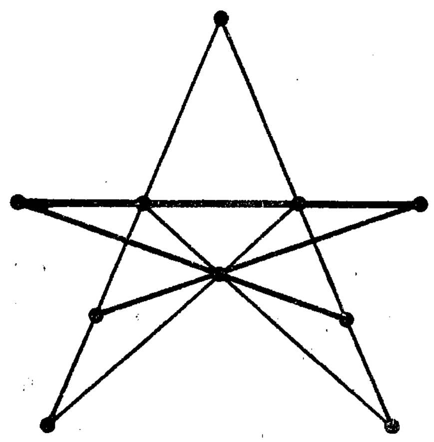

Fig. 5.1. – Geometrical configuration of K_3-base set Λ for $PGL(3, 4)$

$|PGL(3, 4)_{\{\Lambda\}}| = 2.$

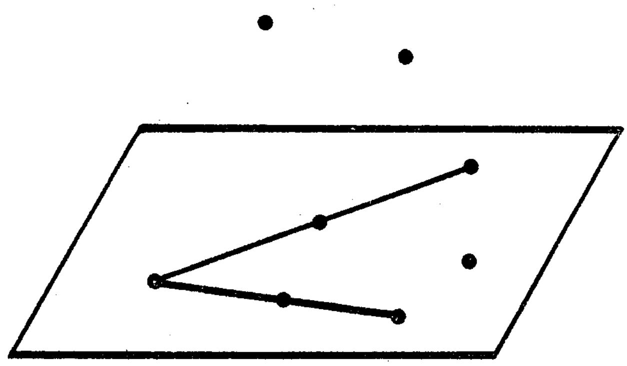

Fig. 5.2. – Geometrical configuration of non-coplanar K_d-base set Λ for

$PGL(d, 4)$, $d \geqslant 4$, $|PGL(4, 4)_{\{\Lambda\}}| = 2.$

In the affine case, $A\Gamma L(2,4)$ and $AGL(2,4)$ were constructed and a K_2-base set of 6 points for $AGL(2,4)$ was found. A complete enumeration of the orbits of $AGL(2,4)$ on 5-sets of $AG(2,4)$ showed that 6 is the minimum value of k for k-closure.

$AGL(d,4)$ is 6-closed for all $d \geq 2$ by lemma 5.2, but that 6 is the minimum for k-closure does not follow from lemma 5.3. In order to prove that $AGL(3,4)$ is not 5-closed we constructed $A\Gamma L(3,4)$ and determined the lengths of the orbits of $AGL(3,4)$ and $A\Gamma L(3,4)$ on 5-sets. There are 9 orbits of 5-sets in both cases, so $AGL(3,4)$ is not 5-closed. Now lemma 5.3 can be used to show that 6 is the minimum k for k-closure for all $AGL(d,4)$ with $d \geq 2$.

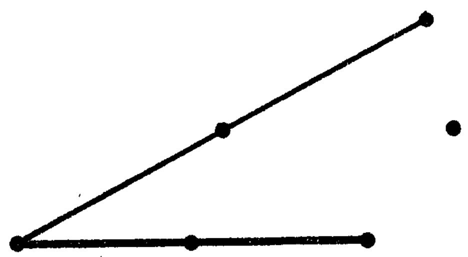

Fig. 5.3. – Geometrical configuration of coplanar K_d-base set Λ for $AGL(d,4)$, $d \geq 2$, $|AGL(2,4)_{\{\Lambda\}}| = 2$.

(2) *The field* $GF(8)$. Here $K_d = P\Gamma L(d,8)$ or $A\Gamma L(d,8)$ respectively. $P\Gamma L(2,8)$ and $PGL(2,8)$ are 4-homogeneous (by Result 2.2) and thus not 4-closed, or k-closed for any k.

$P\Gamma L(3,8)$ and $PGL(3,8)$ were constructed and a K_3-base set Λ for $PGL(3,8)$ of 6 points was found by the computational method described. A complete determination of the lengths of the orbits of $P\Gamma L(3,8)$ and $PGL(3,8)$ on 5-sets showed that there are 5 such orbits in both cases. Thus $PGL(3,8)$ is not 5-closed. By Lemma 5.2 $PGL(d,8)$ is 6-closed for all $d \geq 3$, and by Lemmas 5.1 and 5.3, $PGL(d,8)$ is not 5-closed for any $d \geq 3$. Thus 6 is the minimum for k-closure for $PGL(d,8)$ $d \geq 3$.

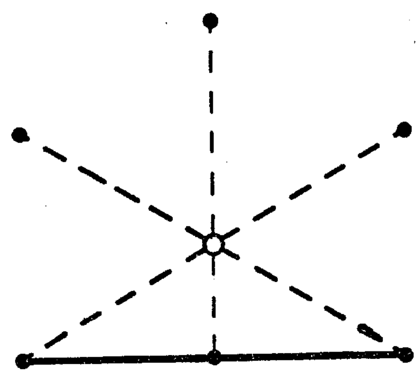

Fig. 5.4. – Geometrical configuration of coplanar K_d-base set Λ for $PGL(d,8)$, $d \geq 3$, $|PGL(3,8)_{\{\Lambda\}}| = 1$.

In the affine case, $AGL(2,8)$ was constructed and a K_2-base set of 4 points was found. Since $AGL(d,8)$ is not 3-closed for any $d \geqslant 2$, by Theorem 4.1, 4 is the minimum for k-closure of $AGL(2,8)$. By Lemma 5.2 $AGL(d,8)$ is 4-closed for all $d \geqslant 2$, and by Theorem 4.1, 4 is the minimum. The geometrical configuration for a K_d-set Λ for $AGL(d,8)$ is a quadrangle (i.e. 4 coplanar points, no 3 of which are collinear) and $|AGL(2,8)_{\{\Lambda\}}| = 4$.

(3) *The field* $GF(9)$. Here $K_d = P\Gamma L(d,9)$ or $A\Gamma L(d,9)$ respectively. By computation it was shown that $PGL(2,9)$ has the same orbits on 5-sets as $P\Gamma L(2,9)$ and thus cannot be 5-closed (irrespective of the closure properties of $P\Gamma L(2,9)$). $P\Gamma L(3,9)$ and $PGL(3,9)$ were constructed and a K_3-base set of 6 points was found for $PGL(3,9)$. A complete determination of the lengths of the orbits on 5-sets showed that $PGL(3,9)$ and $P\Gamma L(3,9)$ have the same number (i.e. 9) of orbits on 5-sets. Thus 6 is the minimum for k-closure of $PGL(3,9)$. By Lemma 5.2 $PGL(d,q)$ is 6-closed for all $d \geqslant 3$, and by Lemmas 5.1 and 5.3, 6 is the minimum value for k-closure of $PGL(d,9)$ $d \geqslant 3$. The geometrical configuration of K_d-base set Λ for $PGL(d,9)$ of 6 points is the same as that shown in Fig. 5.4. Here $|PGL(3,9)_{\{\Lambda\}}| = 1$.

In the affine case, $AGL(2,9)$ was constructed and a K_2-base set of 4 points was found. As in the case of $GF(8)$, we obtain the 4-closure of $AGL(d,9)$ for $d \geqslant 3$, where 4 is the minimum. The geometrical configuration of a K_d-base set Λ of 4 points is again a quadrangle and $|AGL(2,9)_{\{\Lambda\}}| = 2$.

(4) *The field* $GF(16)$. Here $K_d = PGL(d,16) \cdot \langle \sigma^2 \rangle$ or $AGL(d,16) \cdot \langle \sigma^1 \rangle$ respectively. $P\Gamma L(2,16)$ was constructed. Its orbits on 4-sets have already been shown to be distinct from those of $K_2 = PGL(2,16) \cdot \langle \sigma^2 \rangle$ on 4-sets (see Lemma 3.4), which are the same as those of $PGL(2,16)$ on 4-sets. A K_2-base set Λ of 6 points was found for $PGL(2,16)$, and $PGL(2,16)$ was shown to have the same orbits on 5-sets as K_2. By Theorem 3.2, $P\Gamma L(2,16)$ is 4-closed, so that K_2 is also 4-closed and $PGL(2,16)$ is 6-closed, by Lemma 3.1. Here 6 is the minimum since the 5-closure of $PGL(2,16)$ is K_2.

Independently of Theorem 3.2, for $d \geqslant 3$ and $K_d = PGL(d,16) \cdot \langle \sigma^2 \rangle$ it follows from Lemma 5.2 that Λ will be a K_d-base set of 6 points for $PGL(d,16)$. By Lemma 3.2, $P\Gamma L(d,16)$ is 3-closed for $d \geqslant 3$, and by Lemma 3.1, K_d is 4-closed. Thus $PGL(d,16)$ is 6-closed. The minimum for k-closure of $PGL(d,16)$ for $d \geqslant 3$ thus satisfies $5 \leqslant k \leqslant 6$. A K_d-base set Λ of 6 points for $PGL(d,16)$ is the set of 6 collinear points with parametric coordinates $\{\omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6\}$, where ω is a primitive element of $GF(16)$.

$A\Gamma L(1,16)$ has distinct orbits from $H = AGL(1,16)\cdot\langle\sigma^2\rangle$ on 3-sets, and the latter has the same orbits on 3-sets as $AGL(1,16)$, by Lemma 4.4. A K_1-base set Λ of 4 points was found. With reasoning as previously, we prove that $AGL(d,16)$ is 4-closed for all $d\geq 1$.

Thus we have the following theorem:

THEOREM 5.1. – *For the projective and affine groups in the exceptional cases, the minimum value of k for which the group is k-closed is given in the table below.*

(d, q)	$(2, 4)$	$(3, 4)$	$(d, 4)$ $d\geq 4$	$(2, 8)$	$(d, 8)$ $d\geq 3$	$(2, 9)$	$(d, 9)$ $d\geq 3$	$(2, 16)$	$(d, 16)$ $d\geq 3$
$PGL(d, q)$	—	10	$7\leq k\leq 8$	—	6	—	6	6	$5\leq k\leq 6$
$AGL(d-1, q)$	—	6	6	—	4	—	4	4	4

Here an entry in the table of the form $7\leq k\leq 8$ indicates that the group is 8-closed, not 6-closed, and that the question of 7-closure is still open. An entry « — » indicates that the group is not closed for any value of k.

NOTE. – The K_d-base sets are not, of course, unique and we have given only one example in a single orbit on k-sets in each case. In most cases other orbits containing K_d-base sets were found. Computer print-outs of any of the computational results of this section are available on request from the authors.

We show below that the K_d-base sets Λ obtained to prove Theorem 5.1 are in fact base sets.

LEMMA 5.4. – *For all values of $d\geq 2$ and q as in Theorem 5.1, the K_d-base sets Λ are base sets. Further, $PGL(d, q)$ and $AGL(d, q)$ are the full automorphism groups of the designs $\mathfrak{D}(G,\Lambda)$ where G is $PGL(d, q)$ or $AGL(d, q)$ respectively.*

PROOF. – The designs $\mathfrak{D}(G,\Lambda)$ are defined in § 2. In this lemma we will use the notation $\mathfrak{D}_{\mathcal{P}}(d,\Lambda)$ for $\mathfrak{D}(G,\Lambda)$ where $G = PGL(d, q)$ and Λ is a K_d-base set, and $\mathfrak{D}_{\mathcal{A}}(d,\Lambda)$ for $\mathfrak{D}(G,\Lambda)$ where $G = AGL(d, q)$ and Λ is a K_d-base set. This is not the notation of Definitions 3.1 and 4.1, which do not apply for these values of q.

For the projective case when $q = 4, 8$ or 9, that the K_d-base sets Λ are base sets follows from Result 2.4 since $d\geq 3$. For the case $q = 16$ and $d = 2$ of case (4), Result 2.4 is not applicable, but we may use either Result 2.5 or argue as in Theorem 3.1, since in this case the K-base set is on a line. Similarly for $q = 16$ and $d \geqslant 3$. In all these cases then $PGL(d, q)$ is the full automorphism group of the 2-design $\mathfrak{D}_{\mathcal{P}}(d, \Lambda)$.

For the affine case the situation is not quite so simple as the analogous result, Result 2.3, indicates the possibility of other groups that could contain $AGL(d, q)$ and fix the base set. We can however deal with the case $q = 16$ as for the projective case and argue as in Theorems 3.1 and 4.1, since the K-base set is on a line. Here we need $d \geqslant 2$.

In the cases $q = 4, 8$ or 9 where $d \geqslant 2$, we have $AGL(d, q) \leqslant \operatorname{Aut}(\mathfrak{D}_{\mathcal{A}}(d, \Lambda))$ where $\mathfrak{D}_{\mathcal{A}}(d, \Lambda)$ is a $2-(q^d, k, \lambda)$ design with $AGL(d, q)$ acting 2-transitively on points and transitively on blocks.

If $AGL(d, q) < K \leqslant \operatorname{Sym}(q^d)$ and $\Lambda^K = \Lambda^{AGL(d,q)}$, then $K \leqslant \operatorname{Aut}(\mathfrak{D}_{\mathcal{A}}(d, \Lambda))$. The possibilities for K are given in Result 2.3. We consider the fields of order $q = 4, 8$ and 9 in turn. In all cases we rule out $K = \operatorname{Alt}(q^d)$ or $\operatorname{Sym}(q^d)$ since $AGL(d, q)$ is not k-homogeneous for these values of k.

The field $GF(4)$: $q = 4 = 2^2$ and so from Result 2.3 we have

(i) $ASL(2d, 2) = AGL(2d, 2) = K$ for $d \geqslant 2$, or

(ii) $AGL(2, 4) < K < AGL(4, 2)$ with $K_\alpha \simeq \operatorname{Alt}(7)$ for $d = 2$.

In case (ii), with $d = 2$, we have $K \leqslant \operatorname{Aut}(\mathfrak{D}_{\mathcal{A}}(2, \Lambda))$ where $\mathfrak{D}_{\mathcal{A}}(2, \Lambda)$ is a 2-(16, 6, λ) design with b blocks where

$$b = \frac{|AGL(2, 4)|}{|AGL(2, 4)_{\{\Lambda\}}|} = \frac{|AGL(2, 4)|}{2} = 1440 .$$

Since $K > AGL(2, 4)$, it is also transitive on blocks, so that

$$|K_{\{\Lambda\}}| = \frac{|K|}{1440} = 28 .$$

Now $|\Lambda| = 6$, so that $K_{\{\Lambda\}}/K_{(\Lambda)} \leqslant \operatorname{Sym}(6)$, so that $|K_{\{\Lambda\}}|/K_{(\Lambda)}|$ divides $6!$

Thus 7 divides $|K_{(\Lambda)}|$, so that K contains an element of order 7 fixing 6 points at least, and thus fixing 9 points. Now by a theorem of Jordan quoted in Wielandt [21] p. 39, $K \geqslant \operatorname{Alt}(16)$, which is a contradiction.

Similarly for $d = 2$ (i) cannot hold, since as above we can deduce that $AGL(4, 2) = K$ must contain a 7-cycle fixing 9 points.

For $d \geqslant 3$ we have only the case (i) to consider, i.e. $K = AGL(2d, 2)$ acting as an automorphism group on the 2-$(4^d, 6, \lambda)$ design. Since $AGL(2d, 2)$ is 3-transitive on points, the 2-design must in fact be a 3-design. We count blocks through 3 points of the design and show that this number depends on whether we choose 3 collinear points or a triangle, and hence that the design is not a 3-design.

A triangle of points of $AG(d, 4)$ must be in a plane of $AG(d, 4)$, and by the construction of the design $\mathfrak{D}_{\mathcal{A}}(d, \Lambda)$, all the blocks containing the triangle will be in the plane. This number is less than the number through 2 points in a plane, i.e. less than 180 which is the value of λ for the 2-$(16, 6, \lambda)$ design in $AG(2, 4)$.

For 3 collinear points P, Q, R: the number of blocks in any plane containing P, Q, R is 36 and the number of planes in $AG(d, 4)$ containing the line is $(4^d - 4)/12$. Thus the number of blocks is $12(4^{d-1} - 1)$ and for $d \geqslant 3$ this number is $\geqslant 180$. This contradicts the number of blocks through 3 points being less than 180, so that $\mathfrak{D}_{\mathcal{A}}(d, \Lambda)$ (for $d \geqslant 3$) cannot be a 3-design.

Thus Λ is a base set for $AGL(d, 4)$ for $d \geqslant 2$, and $AGL(d, 4)$ is the full collineation group of the design.

The field $GF(8)$: $q = 8 = 2^3$ and so from Result 2.3 we have the possibility

$$ASL(3d, 2) = AGL(3d, 2) = K .$$

Again, K is 3-transitive on points of the design, so that the design is a 3-design. But in this case the geometrical configuration for Λ inside $AG(d, 8)$ is a quadrangle, so that if we choose 3 collinear points of $AG(d, 8)$ there is no block containing them. Thus K cannot act on the design and Λ is a base set for $AG(d, 8)$, $d \geqslant 2$.

The field $GF(9)$: $q = 9 = 3^2$ and from Result 2.3 we have the possibility

$$AGL(d, 9) < ASL(2d, 3) \leqslant K \leqslant AGL(2d, 3) .$$

We show that $K = ASL(2d, 3)$ cannot act in the way required. For $d = 2$, $\mathfrak{D}_{\mathcal{A}}(2, \Lambda)$ is a 2-$(81, 4, \lambda)$ design.

Using the fact that K is also transitive on blocks, we can compute, as in the case $q = 4$, that 13 divides $|K_{\{\Lambda\}}|$. Since $K_{\{\Lambda\}}/K_{(\Lambda)} \leqslant \operatorname{Sym}(4)$ we find that 13 divides $|K_{(\Lambda)}|$ and thus K has an element of order 13 fixing 4 points, and thus at least 16 points. This element moves at most 65 points. By a theorem of Marggraf in Wielandt [21] p. 38, we obtain $K \geqslant \operatorname{Alt}(81)$ which contradicts our conditions. Thus Λ is a base set for $d = 2$.

For $d \geqslant 3$, consider the action of $K = ASL(2d, 3)$ on $AG(2d, 3)$ and on $AG(d, 9)$. Since the block Λ of $\mathfrak{D}_{\mathcal{A}}(d, \Lambda)$ has just 4 points, it is inside a subspace $\mathcal{A}$ of $AG(2d, 3)$ where $\mathcal{A} = AG(4, 3)$. As a set of points in $AG(d, 9)$, $\mathcal{A} = AG(2, 9)$. K will induce the group $AGL(4, 3)$ on $\mathcal{A}$ and this will act on the design $\mathfrak{D}_{\mathcal{A}}(2, \Lambda)$ on points of $\mathcal{A}$. Since $AGL(4, 3) > AGL(2,9)$, and the latter is transitive on blocks of $\mathfrak{D}_{\mathcal{A}}(2, \Lambda)$, $AGL(4, 3)$ is also transitive on blocks of $\mathfrak{D}_{\mathcal{A}}(2, \Lambda)$. Now we can argue as in the case $d = 2$ to obtain a contradiction.

Thus in all cases we have shown that the K_a-base sets Λ are base sets, and the lemma is proved. The lemma then also shows that the groups are geometric.

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