29th MIDWEST SYMPOSIUM ON CIRCUITS AND SYSTEMS

Proceedings of the 29th Midwest Symposium
on Circuits and Systems, held August 10–12, 1986,
in Lincoln, Nebraska.

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SPECTRA OF INCIDENCE STRUCTURES

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§1 INTRODUCTION

Let P be a set of v *points* and B a set of b *blocks* disjoint from P, both finite. When I is some non-empty subset of $P \times B$ we say that a point p is *incident* with a block b if (p,b) belongs to I. The triple $S = (P,B,I)$ is called an *incidence structure*.

Once the points and blocks of S are arranged in some way we define the *incidence matrix* S as follows : its rows are indexed by P, its columns by B, so that the (p,b)-entry is 1 if p is incident with b and is 0 otherwise. The *spectrum* of S, denoted by spec(S), now is defined as the collection of all eigenvalues with multiplicities of the matrix $S \cdot S^T$. Note that the spectrum does not depend upon the particular way in which points and blocks are arranged as this corresponds to conjugation of $S \cdot S^T$ by a permutation matrix. The v eigenvalues in the spectrum are all non-negative and real as $S \cdot S^T$ is a symmetric and positive semi-definite matrix. By the Frobenius-Perron Theorem its maximal eigenvalue appears with multiplicity 1 if S is connected (in the usual sense). To avoid ambiguity let always $v \leq b$.

In an earlier paper [2] we have investigated the spectrum in terms of tactical decompositions of the structure. In the present note we are concerned with spectral properties of $S \cdot S^T$ and related matrices in terms of the automorphisms of the structure. In section 3 we shall see that certain theorems about graph spectra hold for incidence structures in general.

§2 SPECTRA AND AUTOMORPHISMS

An *automorphism* of S is a pair (G,H) of permutation matrices such that $G \cdot S = S \cdot H$. Here G, a $v \times v$ matrix , is called the *point action* and H, a $b \times b$ matrix , the *block action* of the automorphism. These matrices represent permutations of the point and block sets which preserve incidence. The set of all such automorphisms of S , with the obvious multiplication, is the *automorphism group* $A(S)$. The group of all point actions is denoted by $\mathcal{G}(S)$, the group of all block actions by $\mathcal{H}(S)$. Note that $\mathcal{G}(S)$ and $\mathcal{H}(S)$ are homomorphic images of $A(S)$. The *point character* $\pi(G)$ = trace(G) counts the number of points fixed by an automorphism while the *block character* $\beta(H)$ = trace(H) counts the number of blocks fixed. Both characters can be expressed in terms of the distinct, over $\mathbf{C}$ irreducible characters $\chi_1,..,\chi_s$ of $A(S)$ as

$$(1)\quad \begin{aligned} \pi &= \Sigma\, m_i \cdot \chi_i \quad , 0 \leq m_i \in \mathbb{Z} \quad , \text{and} \\ \beta &= \Sigma\, n_i \cdot \chi_i \quad , 0 \leq n_i \in \mathbb{Z} \quad . \end{aligned}$$

We define $d_i = \chi_i(1)$ to be the degree and m_i (resp. n_i) the multiplicity of this character. Clearly the sum of the products $m_i \cdot d_i$ (resp. $n_i \cdot d_i$) is the number of points (blocks) of S . The sum over the squares† is the *permutation rank* $r(\mathcal{G})$ and $r(\mathcal{H})$ of $\mathcal{G}$ and $\mathcal{H}$ respectively. If these groups act transitively, then this rank is the number of orbits of the stabilizer of a point or a block.

We now introduce some further notation. For a given pair (m_i,d_i) of integers and a collection Λ_i of $m_i \cdot d_i$ real numbers we say that Λ_i is (m_i,d_i)-*partitioned* if this collection contains at most m_i values each appearing with multiplicity a multiple of d_i . For a given set $\{(m_i,d_i) \mid 1 \leq i \leq s\}$ a collection Λ is $\{(m_i,d_i)\}$-*partitioned* if Λ can be arranged in subcollections $\Lambda_1,..,\Lambda_i,..,\Lambda_s$ such that each Λ_i is (m_i,d_i)-partitioned.

For a square matrix A in general spec(A) denotes the collection of its eigenvalues, with multiplicities. The *point degree matrix* of a structure is the $v \times v$ diagonal matrix D whose (p,p)-entry is the number of blocks incident with p . The *block degree matrix* is the $b \times b$ matrix D' defined correspondingly. The matrix of all 1's and appropriate size is denoted by J .

† i.e. $(m_i)^2$ and $(n_i)^2$ respectively.

29th Midwest Symposium on Circuits and Systems
M. Ismail, Editor

THEOREM: Let $\mathcal{S}$ be an incidence structure with incidence matrix S and automorphism group $A(\mathcal{S})$. Suppose that the point and the block characters are decomposed as in (1) in terms of irreducible characters of $A(\mathcal{S})$ over $\mathbb{C}$. Then the spectrum of any matrix A in the algebra generated by $S \cdot S^T$, D (the point degree matrix), the identity matrix and the J-matrix is $\{(m_i,d_i)\}$-partitioned.

Similarly, when A' is a matrix in the algebra generated by $S^T \cdot S$, D' (the block degree matrix), the identity matrix and the J-matrix, then spec(A') is $\{(n_i,d_i)\}$-partitioned.

PROOF: 1. Let (G,H) be an automorphism of $\mathcal{S}$. As $G \cdot S = S \cdot H$ also $S^T \cdot G^T = H^T \cdot S^T$ so that $(G \cdot S) \cdot (S^T \cdot G^T) = S \cdot (H \cdot H^T) \cdot S^T$. Since G and H are permutation matrices we have that $S \cdot S^T$ commutes with all point actions of the automorphism group. The same is true for the point degree matrix and, trivially for the identity and J-matrices. Therefore a matrix A as in the theorem commutes with all G in $\mathcal{G}(\mathcal{S})$.

2. Corresponding to the decomposition (1) of the point character there is a unitary matrix U (see for instance §29 in Wielandt [3]) such that $U \cdot G \cdot U^T$ for all G in $\mathcal{G}(\mathcal{S})$ has the following diagonal form

$$(2)\qquad \begin{pmatrix} G_1 & & & & & \\ & \ddots & & & 0 & \\ & & G_1 & & & \\ & & & \ddots & & \\ & 0 & & & G_s & \\ & & & & & \ddots \\ & & & & & & G_s \end{pmatrix} \begin{matrix} \Big\} \; m_1 \text{ times} \\ \vdots \\ \Big\} \; m_s \text{ times} \end{matrix}$$

where $G \longrightarrow G_i$ (a $d_i \times d_i$ matrix) is the irreducible representation corresponding to the character χ_i.

3. As A commutes with all G, $U \cdot A \cdot U^T$ commutes with all matrices of the form (2). Hence, by Schur's Lemma $U \cdot A \cdot U^T$ has the form

$$(3)\qquad \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_s \end{pmatrix}$$

where A_i (a square matrix of size $m_i \cdot d_i$) is the tensor product of some $(m_i \times m_i)$ matrix M_i with the identity matrix of size d_i. Thus spec(A_i) consists of the eigenvalues of M_i, each repeated d_i times, i. e. spec(A_i) is (m_i,d_i)-partitioned. Thus spec(A) = spec($U \cdot A \cdot U^T$) has the required property.

4. The result for A' follows in precisely the same way. QED .

§3 CONSEQUENCES OF THE THEOREM

We retain the notation of the previous section. The *dual* of $\mathcal{S}$, denoted by $\mathcal{S}' = (\mathcal{P}';\mathcal{B}';I')$ is the structure obtained from $\mathcal{S}$ by interchanging the rôle of points and blocks : $\mathcal{P}' = \mathcal{B}$, $\mathcal{B}' = \mathcal{P}$ with incidence as in $\mathcal{S}$. In [2] , theorem 4.2 , it is shown that spec($\mathcal{S}'$) is obtained by appending b-v zeros to the spectrum of $\mathcal{S}$. This fact together with the theorem , applied to $A = S \cdot S^T$ and $A' = S^T \cdot S$, gives the first part of the following

COROLLARY 1: (a) The spectrum of $\mathcal{S}$ is $\{(m_i,d_i)\}$-partitioned and when b-v zeros are appended to it also $\{(n_i,d_i)\}$-partitioned.
(b) The linear rank of S over $\mathbb{Q}$ is an integer of the form $a_1 \cdot d_1 + \ldots + a_s \cdot d_s$ where $a_i \leq m_i$ or where $a_i \leq n_i$ for all $i = 1,..,s$.

PROOF: (b) Clearly if $x \cdot S = 0$ for some real v-vector x then $x \cdot (S \cdot S^T) = 0$. Conversely let $y \cdot (S \cdot S^T) = 0$. Then $(y \cdot S) \cdot (y \cdot S)^T = o$ so that $y \cdot S = 0$. Therefore S and $S \cdot S^T$ have the same kernel and hence the same rank r over $\mathbb{Q}$. As r is the number of non-zero values in spec($\mathcal{S}$) or spec($\mathcal{S}'$) the result follows from (a). QED .

We say that $\mathcal{S}$ is *point-connected* if for any pair p_1, p_r of points there are $p_1, \ell_1, \ldots, p_i, \ell_i, \ldots, p_r, \ell_r$ such that p_{i-1} and p_i are incident with ℓ_i for $i = 1,..,r$.

COROLLARY 2: For a point-connected structure the maximal eigenvalue in its spectrum (or in the spectrum of its dual) has multiplicity 1 .

PROOF: Let $A = S \cdot S^T$. As $\mathcal{S}$ is connected some power of A has strictly positive entries. Hence the Frobenius-Perron Theorem (see for instance theorem 0.3 in [1]) implies that the maximal eigenvalue of that matrix , and hence of A, has multiplicity one. QED .

The condition on the connectedness is essential as there are many structures with repeated maximal eigenvalue that are not connected.

COROLLARY 3: Suppose that the multiplicities m_i in (1) are all at most 1. Then a matrix A as in the theorem can be written as a linear combination of the class matrices of $\mathcal{G}(\mathcal{S})$.

A class matrix is defined as the sum over all matrices in a conjugacy class of $\mathcal{G}(\mathcal{S})$. For a proof of the corollary see theorem 29.8 in Wielandt's book.

COROLLARY 4: Let a matrix A as in the theorem have (at least) t distinct eigenvalues. Then

(a) $t \leq \Sigma m_i$,

(b) $t \leq r(\mathcal{G})$, the permutation rank of $\mathcal{G}$,

(c) If $t = r(\mathcal{G})$ then $m_i \leq 1$ for all i,

(d) If $t = v$ then $\mathcal{G}(S)$ is an elementary abelian 2-group.

PROOF:(a).This follows from the theorem.

(b)-(c).As $r(\mathcal{G}) = \Sigma (m_i)^2$ we have $t \leq r(\mathcal{G})$ with equality only if $m_i \leq 1$ for all $i = 1,\ldots,s$, by part (a).

(d).Let x be a real eigenvector for A . As A commutes with all point actions and as the eigenvalues of A are all distinct , it follows that x also is an eigenvector for every G in $\mathcal{G}(S)$ with real eigenvalue μ_G . As this eigenvalue also is a unit it follows that $\mu_G = \pm 1$ so that G^2 is the identity. This proves that $\mathcal{G}(S)$ is an elementary abelian 2-group. QED .

For a graph theorist the spectrum of a graph usually means the sequence of eigenvalues of its adjacency matrix, i.e. of $A = S \cdot S^T - D$ where D is the degree matrix of the graph. With this specialization the above corollaries are standard theorems about graph spectra , see for instance theorems 5.1, 5.8, 5.9 etc. in chapter 5 of [1]. Some of the results there about simple (non-maximal) eigenvalues - giving structural information about the graph - can be adapted for incidence structures in general.

We remark here also that the spectrum does not identify the isomorphism type nor the automorphism group of a structure. A strong form of this statement is known as Babai's theorem. (See theorem 5.13 in [1].)

We conclude with some remarks about structures whose automorphism group has small permutation rank on points. It is known in general that if $r(\mathcal{G})$ does not exceed 5 then the multiplicities of the permutation character are all at most 1 (with the only exception that $\mathcal{G}$ fixes a single point and is doubly transitive on the remaining points). Thus - apart from the exception- $\mathcal{G}$ in particular is transitive on points. For the remaining two corollaries let A be the matrix $S \cdot S^T$.

COROLLARY 5: Let $A(S)$ have rank 2 on points. Then $A = x \cdot J + y \cdot I$ where $x + y$ is the degree of a point and x is the number of blocks joining distinct points.

PROOF: The algebra of all matrices commuting with all G in $\mathcal{G}(S)$ has dimension $r(\mathcal{G}) = 2$, see proposition 29.2 in [3] . Hence A is a linear combination of the identity and the J-matrix . The remainder follows easily. QED .

Note that $y = 0$ in the last corollary implies that $\mathcal{B}$ consists of a "single" block repeated x times. Otherwise rank(S)=v.

COROLLARY 6: Let $A(S)$ have rank 3 on points, $\pi = 1 + \psi + \chi$. Then A^2 is a linear combination of A , the identity and the J-matrix . Furthermore the rank of S over $\mathbf{Q}$ is of the form $1 + \psi(1) + \chi(1)$ or $1 + \psi(1)$.

PROOF: Here the dimension of the algebra of all matrices commuting with the point action is three so that A^2 is linearly dependent on A , J and the identity. If the rank of S is one then the block set consists of a single repeated block. This would allow the full symmetric group to act on the point set. Thus , by corollary 1 , the rank has the given form. QED .

Examples with rank(S) < v are provided by bipartite graphs with point-transitive automorphism group acting doubly transitively on the points of each part.

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