On the realisation of maximal simple types and epsilon factors of pairs

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Abstract

Let G be the group of rational points of a general linear group over a non-archimedean local field F. We show that certain representations of open, compact-mod-centre subgroups of G, (the maximal simple types of Bushnell and Kutzko) can be realized as concrete spaces. In the level zero case our result is essentially due to Gel′fand. This allows us, for a supercuspidal representation π of G, to compute a distinguished matrix coefficient of π. By integrating, we obtain an explicit Whittaker function for π. We use this to compute the ε-factor of pairs, for supercuspidal representations π_1, π_2 of G, when π_1 and the contragredient of π_2 differ only at the 'tame level' (more precisely, π_1 and $\check{\pi}_2$ contain the same simple character). We do this by computing both sides of the functional equation defining the epsilon factor, using the definition of Jacquet, Piatetskii-Shapiro, Shalika. We also investigate the behaviour of the ε-factor under twisting of π_1 by tamely ramified quasi-characters. Our results generalise the special case $\pi_1 = \pi_2$ totally wildly ramified, due to Bushnell and Henniart.

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Introduction

Let F be a non-archimedean local field and fix an additive character ψ_F of F, with conductor $\mathfrak{p}_F$ (the maximal ideal of the ring of integers $\mathfrak{o}_F$ of F). Let V be an N-dimensional F-vector space and let $G = \mathrm{Aut}_F(V)$. This paper concerns the supercuspidal representations of G, so we adopt the notation of [6], where these are classified in terms of *maximal simple types* ([6]§6).

A maximal simple type is a pair (J, λ), consisting of a (rather special) compact open subgroup J of G and an irreducible representation λ of J. It is constructed from a *simple stratum* $[\mathfrak{A}, n, 0, \beta]$, where $\mathfrak{A}$ is a principal hereditary $\mathfrak{o}_F$-order in A and $\beta \in A$ satisfy various properties (see [6] (1.5.5) for the full definition). The algebra $E = F[\beta]$ is a field, and $E^\times$ normalises J. If we set $\mathbf{J} = E^\times J$ and let Λ be a representation of $\mathbf{J}$ such that $\Lambda|_J \cong \lambda$, then $\text{c-Ind}_{\mathbf{J}}^G \Lambda$ is an irreducible supercuspidal representation of G. Conversely, any irreducible supercuspidal representation of G arises this way.

It follows from the fact that $\text{c-Ind}_{\mathbf{J}}^G \Lambda$ has a Whittaker model, that there exist a maximal unipotent subgroup U of G and a non-degenerate character ψ_α of U such that $\mathrm{Hom}_{U\cap\mathbf{J}}(\psi_\alpha, \Lambda) \neq 0$. Moreover, the uniqueness of the Whittaker model implies that the pair (U, ψ_α) is determined up to conjugation by $\mathbf{J}$ (see [3]).

Associated to the maximal simple type, we have two other groups $H^1 \subset J^1 \subset J$ (see [6]§3.1). They are normal subgroups of J and $\lambda|_{H^1}$ is a multiple of a *simple character* θ. Hence, $\psi_\alpha|_{U\cap H^1} = \theta|_{U\cap H^1}$ and we may define a character Ψ of $(J \cap U)H^1$, by

$$\Psi(uh) = \psi_\alpha(u)\theta(h), \quad \forall u \in J \cap U, \quad \forall h \in H^1.$$

Let $0 \subset V_1 \subset \cdots \subset V_N = V$ be the maximal flag corresponding to U, and set $\mathcal{M} = \{g \in G : (g-1)V \subseteq V_{N-1}\}$, so that $\mathcal{M}$ is a mirabolic subgroup of G. Our first main result, in §4, is

Theorem A. *Let U be a maximal unipotent subgroup of G and let ψ_α be a non-degenerate character of U such that $\mathrm{Hom}_{U\cap\mathbf{J}}(\psi_\alpha, \Lambda) \neq 0$. Then $\Lambda|_{(\mathcal{M}\cap\mathbf{J})J^1}$ is irreducible and*

$$\Lambda|_{(\mathcal{M}\cap\mathbf{J})J^1} \cong \mathrm{Ind}_{(U\cap\mathbf{J})H^1}^{(\mathcal{M}\cap\mathbf{J})J^1} \Psi.$$

Moreover, the same result holds if we replace: $\mathbf{J}$ by $\mathfrak{K}(\mathfrak{A})$, the G-normaliser of $\mathfrak{A}$; Λ by $\rho = \mathrm{Ind}_{\mathbf{J}}^{\mathfrak{K}(\mathfrak{A})} \Lambda$; and $(\mathcal{M}\cap\mathbf{J})J^1$ by $(\mathcal{M}\cap\mathfrak{K}(\mathfrak{A}))\mathbf{U}^1(\mathfrak{A})$, where $\mathbf{U}^1(\mathfrak{A})$ is the group of principal units of $\mathfrak{A}$.

We show in §4.4 that this property in fact characterises the representations of the form $\mathrm{Ind}_{\mathbf{J}}^{\mathfrak{K}(\mathfrak{A})}\Lambda$. More precisely, writing $\mathcal{M}_{\mathfrak{A}} = (\mathcal{M} \cap \mathfrak{K}(\mathfrak{A}))\mathbf{U}^1(\mathfrak{A})$ we have

Proposition B. *Let τ be a representation of $\mathfrak{K}(\mathfrak{A})$ such that*

$$\tau|_{\mathcal{M}_{\mathfrak{A}}} \cong \mathrm{Ind}_{(J\cap U)H^1}^{\mathcal{M}_{\mathfrak{A}}} \Psi.$$

Then

$$\tau \cong \mathrm{Ind}_{\mathbf{J}}^{\mathfrak{K}(\mathfrak{A})} \Lambda',$$

for some representation Λ' of $\mathbf{J}$, such that $(J, \Lambda'|_J)$ is a maximal simple type containing θ.

This is an analogue of Gel′fand's characterisation of the cuspidal representations of $\mathrm{GL}_N(\mathbf{F}_q)$.

Before continuing with the applications of Theorem A, we will say a few words about its proof. The strategy is as follows: We first construct a special pair (U, ψ_α), such that $\mathrm{Hom}_{U\cap\mathbf{J}}(\psi_\alpha, \Lambda) \neq 0$, by carefully picking a basis of V, and then letting U be the group of upper-triangular matrices with respect to this basis, and ψ_α be the 'standard' nondegenerate character. The choice of basis is made so that we can control the restriction of ψ_α to $U \cap B^\times$, where B is the centraliser of β. We then prove Theorem A for this particular pair (U, ψ_α). The general case follows from the fact that any other such pair $(U', \psi_{\alpha'})$ is conjugate to our particular choice by some $g \in \mathbf{J}$.

We remark also that Theorem A should follow easily from [3] Theorem 2.9, but there are problems with the proof of that result: the unipotent group U used in the proof is not (in general) the group required by the statement of the Theorem; moreover, there is a gap in the proof of [3] Lemma 2.10 which, so far as we know, nobody has been able to fix. In the course of the proof of Theorem A, we end up proving an analogue of [3] Theorem 2.9, see Theorem 3.3. We get around the problem of [3] Lemma 2.10 by using the case when E is maximal in A as a 'black box'. If [3] Lemma 2.10 were true then the basis of V referred to above could be written explicitly in terms of β and some of our proofs would simplify.

The interest in Theorem A is that, since $F^\times \operatorname{Ker}\Lambda$ is of finite index in $\mathbf{J}$, it allows us to apply a very general Theorem of Alperin and James [1] for finite groups. Following [1], we define the Bessel function $\mathcal{J}_\Lambda$ of Λ by

$$\mathcal{J}_\Lambda(g) = Q^{-1} \sum_{u \in (U\cap\mathbf{J})H^1/\mathbf{U}^{n+1}(\mathfrak{A})} \Psi(u)\,\mathrm{tr}_\Lambda(gu^{-1}).$$

where $Q = ((U \cap \mathbf{J})H^1 : \mathbf{U}^{n+1}(\mathfrak{A}))$. Now, [1] implies:

Theorem C. *Let $\mathcal{S}$ be the space of functions $f : (\mathcal{M} \cap \mathbf{J})J^1 \to \mathbb{C}$, such that*

$$f(ug) = \Psi(u)f(g), \quad \forall u \in (U \cap \mathbf{J})H^1, \quad \forall g \in (\mathcal{M} \cap \mathbf{J})J^1,$$

and, for all $g \in \mathbf{J}$, let $L(g) \in \mathrm{End}_{\mathbb{C}}(\mathcal{S})$ be the operator:

$$[L(g)f](m) = \sum_{m_1 \in (\mathcal{M}\cap\mathbf{J})J^1/(U\cap\mathbf{J})H^1} \mathcal{J}_\Lambda(mgm_1)f(m_1^{-1}),$$

Then L defines a representation of $\mathbf{J}$ on $\mathcal{S}$, which is isomorphic to Λ.

The analogous result for $\mathrm{GL}_N(\mathbf{F}_q)$ (or, equivalently, for level zero supercuspidal representations of G) was first observed by Gel′fand [7].

Now we proceed as in [3] §3 and construct a Whittaker function for $\pi = \text{c-Ind}_{\mathbf{J}}^G \Lambda$. However, the observation that Λ can be realised via Bessel functions results in explicit formulae and extra information about this Whittaker function. Our main result concerning Whittaker functions (see §5.2) is as follows:

Theorem D. *Define $\mathcal{W} \in \mathrm{Ind}_U^G \psi_\alpha$ by $\mathrm{Supp}\,\mathcal{W} \subseteq U\mathbf{J}$ and*

$$\mathcal{W}(ug) = \psi_\alpha(u)\mathcal{J}_\Lambda(g), \quad \forall u \in U, \quad \forall g \in \mathbf{J},$$

then $\mathcal{W}$ is a Whittaker function for $\pi = \text{c-Ind}_{\mathbf{J}}^G \Lambda$. Moreover, $(\mathrm{Supp}\,\mathcal{W}) \cap \mathcal{M} = U(H^1 \cap \mathcal{M})$ and

$$\mathcal{W}(uh) = \psi_\alpha(u)\theta(h), \quad \forall u \in U, \quad \forall h \in H^1 \cap \mathcal{M}.$$

It is the second part of Theorem D that really requires Theorem A and the explicit realization in terms of Bessel functions. The first part of Theorem D has also been obtained by Roberto Johnson [9], in the special case when π is a Carayol representation.

Finally, in §7 we use our Whittaker functions to compute ε-factors of pairs in the following situation. We fix a simple stratum as above and consider two supercuspidal representations $\pi_1 = \text{c-Ind}_{\mathbf{J}}^G \Lambda_1$ and $\pi_2 = \text{c-Ind}_{\mathbf{J}}^G \Lambda_2$, such that $\Lambda_1|_{H^1}$ and $\Lambda_2|_{H^1}$ are multiples of *the same simple character θ*. Then, for $i = 1, 2$, we can write $\Lambda_i|_J \cong \kappa \otimes \sigma_i$, where κ is a β-extension (see [6] (5.2.1)) and σ_i is the lift of a cuspidal representation of $J/J^1 \cong \mathrm{GL}_r(\mathfrak{k}_E)$, where $\mathfrak{k}_E$ is the residue field of E and $r = \dim_E(V)$.

One may show that, for $i = 1, 2$, we have $\Lambda_i \cong \tilde{\kappa} \otimes \Sigma_i$, where $\tilde{\kappa}$ and Σ_i are representations of $\mathbf{J}$ which restrict to κ and σ_i respectively. Moreover, we may think of Σ_i as a representation of $\mathfrak{K}(\mathfrak{A}) \cap B^\times \cong E^\times \operatorname{GL}_r(\mathfrak{o}_E)$ (where, we recall, B is the centraliser of E). We set $\tau_i = \text{c-Ind}_{\mathfrak{K}(\mathfrak{B})}^{B^\times} \Sigma_i$; then τ_i is a supercuspidal level zero representation of $B^\times \cong \operatorname{GL}_r(E)$.

In Theorem 7.1 we relate $\varepsilon(\pi \times \check{\pi}_2, s, \psi_F)$ and $\varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E)$, where ψ_E is an additive character of E with conductor $\mathfrak{p}_E$ which extends ψ_F. We obtain:

Theorem E.

$$\varepsilon(\pi_1 \times \check{\pi}_2, s, \psi_F) = \zeta \omega_{\tau_1}(\nu^{-r}) \omega_{\tau_2}(\nu^r) q^{(s-1/2)r v_E(\nu) N/e} \varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E),$$

where: $\zeta = \omega_{\tau_2}(-1)^{r-1} \omega_{\pi_2}(-1)^{N-1}$; $q = q_F$ *is the cardinality of* $\mathfrak{k}_F$; v_E *is the additive valuation on* E *with image* $\mathbb{Z}$; *and* $\nu = \nu(\theta, \psi_F, \psi_E) \in E^\times/(1+\mathfrak{p}_E)$ *is an invariant which we define in §6.*

We prove this by computing both sides of the functional equation for the epsilon factor, using the definition of Jacquet, Piatetskii-Shapiro, Shalika [8], with the Whittaker functions of Theorem D. We are able to do the calculation because the fact that the operator L in Theorem C defines a group action imposes various identities on the Bessel function $\mathcal{J}_\Lambda$.

Theorem E implies:

Corollary F. *Let* $\chi : F^\times \to \mathbb{C}^\times$ *be a tamely ramified quasi-character and put* $\chi_E = \chi \circ \mathrm{N}_{E/F}$; *then*

$$\frac{\varepsilon(\pi_1 \chi \times \check{\pi}_2, s, \psi_F)}{\varepsilon(\pi_1 \times \check{\pi}_2, s, \psi_F)} = \chi(\mathrm{N}_{E/F}(\nu^{-r^2})) \frac{\varepsilon(\tau_1 \chi_E \times \check{\tau}_2, s, \psi_E)}{\varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E)},$$

where $\nu = \nu(\theta_F, \psi_F, \psi_E)$.

We recover a result of Bushnell and Henniart [5], concerning the effect on $\varepsilon(\pi \times \check{\pi}, s, \psi_F)$ of twisting π by tamely ramified quasi-characters, when π is totally wildly ramified, as a special case of the above Corollary. Moreover, we show in §6.1 that the invariant ν behaves well under the tame lifting operation for simple characters of Bushnell and Henniart [2], which implies [5] Theorem 7.1.

We end the introduction with a brief summary of the contents of each section. We begin in §1 with notation and some elementary results about nondegenerate characters and induced supercuspidal representations. In §2 we begin the groundwork for the proof of Theorem A, proving a similar result for a β-extension κ. In §3 we define the particular unipotent subgroup, and the basis, used in the proof of Theorem A; this proof appears in §4, along with the proof of Proposition B. In §5, we apply the Theorem of Alperin and James to define Bessel functions, and construct our explicit Whittaker function from Theorem D. In §6, we define the numerical invariant ν which appears in Theorem E. Finally, the proof of Theorem E, and its application to twisting by tamely ramified quasi-characters, appears in §7.

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1 Notation and Preliminaries

Let F be a locally compact non-archimedean local field, with ring of integers $\mathfrak{o}_F$, maximal ideal $\mathfrak{p}_F$, and residue field $\mathfrak{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$ with $q_F = p^f$ elements, p prime. We fix ϖ_F a uniformizing element of F and let v_F denote the additive valuation of F, normalised so that $v_F(\varpi_F) = 1$. We use similar notation for any field extension of F.

Let V be an N-dimensional F-vector space, $A = \mathrm{End}_F(V)$ and $G = \mathrm{Aut}_F(V)$ so, after choosing a basis for V, we have

$$A \cong \mathbb{M}_N(F), \qquad G \cong \mathrm{GL}_N(F).$$

1.1 Unipotent subgroups and characters

The results of this section are stated without proof, since these proofs are straightforward. One way to prove them would be to choose a suitable basis for V with respect to which the unipotent subgroups considered consist of matrices which are upper triangular.

We fix, once and for all, an additive character $\psi_F : F \to \mathbb{C}$ which is trivial on $\mathfrak{p}_F$, non-trivial on $\mathfrak{o}_F$. For any $a \in A$, we define a function $\psi_a : A \to \mathbb{C}$ by

$$\psi_a(x) \;=\; (\psi_F \circ \mathrm{tr}_{A/F})(a(x-1)), \qquad \text{for } x \in A,$$

where $\mathrm{tr}_{A/F}$ denotes the matrix trace. We use the same notation for the restriction of ψ_a to various subsets of A.

Let $\mathcal{F}$ be an F-flag in V,

$$\mathcal{F}: \qquad 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_s = V,$$

Let $P = P_{\mathcal{F}}$ be the G-stabiliser of $\mathcal{F}$, a parabolic subgroup of G, and let $U = U_{\mathcal{F}}$ be its unipotent radical. We also put

$$\begin{aligned} X_{\mathcal{F}} &= \{x \in A : xV_i \subseteq V_{i+1},\ 0 \le i \le s-1\}, \\ X_{\mathcal{F}}^{+} &= \{x \in X_{\mathcal{F}} : xV_i \not\subseteq V_i,\ 0 \le i \le s-1\}, \\ X_{\mathcal{F}}^{-} &= \{x \in X_{\mathcal{F}} : xV_i \subseteq V_i,\ 0 \le i \le s-1\}. \end{aligned}$$

Lemma 1.1. *Let $a \in A$. The function ψ_a defines a linear character of U if and only if $a \in X_{\mathcal{F}}$. Moreover, ψ_a is trivial on U if and only if $a \in X_{\mathcal{F}}^{-}$.*

Now suppose that $\mathcal{F}$ is a maximal F-flag so that $s = N$ and $\dim_F V_i = i$, for $0 \le i \le N$. A smooth linear character χ of U is said to be *nondegenerate* if its G-normaliser is $F^{\times}U$. We can describe this more concretely by choosing a basis $v_1, ..., v_N$ for V such that $V_i = \bigoplus_{j=1}^{i} F v_j$, for $1 \le i \le N$. Then U is identified with the upper triangular unipotent matrices in $\mathrm{GL}_N(F)$ and the smooth characters χ of U are given by

$$\chi(u) = \psi_F\left(\sum_{i=1}^{N-1} \mu_i u_{i,i+1}\right), \qquad \text{for } u = (u_{ij}) \in U,$$

where $\mu_i \in F$, $1 \le i \le N-1$, are fixed scalars. It is easy to see that χ is nondegenerate if and only if all $\mu_i \ne 0$; or, equivalently, if and only if, for all $1 \le j \le N-1$, there exists $u^{(j)} \in U$ with $u^{(j)}_{i,i+1} = 0$, for $i \ne j$, such that $\chi(u^{(j)}) \ne 1$.

Lemma 1.2. *Let $\mathcal{F}$ be a maximal F-flag and $a \in X_{\mathcal{F}}$. Then ψ_a is nondegenerate if and only if $a \in X_{\mathcal{F}}^{+}$.*

1.2 Induced supercuspidals

Since the supercuspidal representations of G are all obtained by irreducible induction from compact-mod-centre subgroups, the following Proposition (which is mostly taken from [3] §1) will be useful.

Proposition 1.3. *Let $\mathfrak{K}$ be an open, compact-mod-centre subgroup of G, and suppose that ρ is a representation of $\mathfrak{K}$, such that $\pi = \text{c-Ind}_{\mathfrak{K}}^G \rho$ is an irreducible supercuspidal representation of G. Let U be a maximal unipotent subgroup of G, and let χ be a smooth character of U.*

(i) *If $\mathrm{Hom}_{U\cap\mathfrak{K}}(\rho, \chi) \neq 0$ then χ is non-degenerate.*

(ii) *If χ is non-degenerate then there exists $g \in G$ such that $\mathrm{Hom}_{U\cap\mathfrak{K}^g}(\chi, \rho^g) \neq 0$.*

(iii) *If $\mathrm{Hom}_{U\cap\mathfrak{K}}(\chi, \rho) \neq 0$ and $\mathrm{Hom}_{U\cap\mathfrak{K}^g}(\chi, \rho^g) \neq 0$, for some $g \in G$, then there exists $u \in U$ such that $\mathfrak{K}^u = \mathfrak{K}^g$ and $\rho^u \cong \rho^g$.*

Proof. (i) Let $\phi \in \mathrm{Hom}_{U\cap\mathfrak{K}}(\rho, \chi) \neq 0$ be such that $\phi \neq 0$, and fix a Haar measure du on U. Then this gives a non-zero $\Phi \in \mathrm{Hom}_U(\pi, \chi)$, by

$$\Phi(f) = \int_U \chi(u^{-1})\phi(f(u))du, \quad \forall f \in \text{c-Ind}_{\mathfrak{K}}^G \rho.$$

If χ is degenerate then there exists a unipotent radical U' of some proper parabolic subgroup of G, such that the restriction of χ to U' is trivial, but this implies $\mathrm{Hom}_{U'}(\pi, \mathbf{1}) \neq 0$. However this may not happen as π is supercuspidal.

Parts (ii) and (iii) follow from [3] Proposition 1.6 and (1.8). □

2 A note on β-extensions

The main result of this section is Theorem 2.6, which asserts that the restriction of a β-extension κ to a certain subgroup of J is isomorphic to a representation induced from a linear character. This result will be used in §4. In section 2.1 we recall some results on Iwahori decompositions. We will use the definitions and notations of [6] with little introduction.

Let $[\mathfrak{A}, n, 0, \beta]$ be a principal simple stratum in A (see [6] (1.5.5)). In particular, $\mathfrak{A}$ is a hereditary, principal $\mathfrak{o}_F$-order in A, with Jacobson radical $\mathfrak{P}$, and $\beta \in \mathfrak{P}^{-n} \setminus \mathfrak{P}^{1-n}$ is such that $E = F[\beta]$ is a field with $E^\times$ normalising $\mathfrak{A}$. We denote by B the A-centraliser of E and put $\mathfrak{B} = \mathfrak{A} \cap B$.

Let $\mathcal{L} = \{L_k : k \in \mathbb{Z}\}$ be the $\mathfrak{o}_F$-lattice chain in V associated to $\mathfrak{A}$, see [6] (1.1.2). Since $E^\times$ normalises $\mathfrak{A}$ we may consider $\mathcal{L}$ also as an $\mathfrak{o}_E$-lattice chain. Let $e = e(\mathfrak{B}|\mathfrak{o}_E)$ be the $\mathfrak{o}_E$-period of $\mathcal{L}$. We fix an E-basis $\{w_1, \ldots, w_r\}$ of V, where $r[E : F] = N$, such that

$$L_0 = \mathfrak{o}_E w_1 + \mathfrak{o}_E w_2 + \cdots + \mathfrak{o}_E w_r$$

$$L_i = \mathfrak{o}_E w_1 + \cdots + \mathfrak{o}_E w_{\frac{r}{e}(e-i)} + \mathfrak{p}_E w_{\frac{r}{e}(e-i)+1} + \cdots + \mathfrak{p}_E w_r$$

for $0 < i < e$. The choice of this basis identifies B with $\mathbb{M}_r(E)$ and $\mathfrak{B}$ with a subring of $\mathbb{M}_r(\mathfrak{o}_E)$, which consists of block upper-triangular matrices modulo $\mathfrak{p}_E$, such that each block on the diagonal is of the size $\frac{r}{e} \times \frac{r}{e}$.

Let $\mathcal{F}_0$ be the F-flag in V, given by

$$\mathcal{F}_0 : 0 \subset Ew_1 \subset \cdots \subset \oplus_{i=1}^{j} Ew_i \subset \cdots \subset \oplus_{i=1}^{r} Ew_i = V.$$

Let P_0 be the G-stabiliser of $\mathcal{F}_0$. Moreover, put $G_i = \mathrm{Aut}_F(Ew_i)$, for $1 \leq i \leq r$, and

$$M_0 = \prod_{i=1}^{r} G_i,$$

a Levi component of the parabolic subgroup P_0 of G. Let U_0 be the unipotent radical of P_0, so that $P_0 = M_0 U_0$. We also denote by U_0^- the unipotent radical of the parabolic subgroup opposite to P_0 relative to M_0.

We note that $\mathcal{F}_0$ is also a maximal E-flag in V. This yields:

Lemma 2.1. *Let $\mathcal{F}$ be an F-flag refining $\mathcal{F}_0$ and let U be the unipotent radical of the G-stabiliser of $\mathcal{F}$, then*

$$U \cap B^\times = U_0 \cap B^\times.$$

2.1 Iwahori decompositions

Put $J = J(\beta, \mathfrak{A})$, $J^1 = J^1(\beta, \mathfrak{A})$ and $H^1 = H^1(\beta, \mathfrak{A})$ (see [6] §3 for the definitions of these groups). By [2] Example 10.9, J^1 and H^1 have Iwahori decompositions with respect to (M_0, P_0):

$$J^1 = (J^1 \cap U_0^-)(J^1 \cap M_0)(J^1 \cap U_0)$$
$$H^1 = (H^1 \cap U_0^-)(H^1 \cap M_0)(H^1 \cap U_0)$$

It will also be useful for us to form the group

$$(J^1 \cap P_0)H^1 = (H^1 \cap U_0^-)(J^1 \cap M_0)(J^1 \cap U_0).$$

Now let $\mathcal{F}$ be a maximal F-flag refining $\mathcal{F}_0$ and U the corresponding unipotent subgroup. We want to understand the group $(J \cap U)H^1$. For $1 \leq i \leq r$, let $\mathcal{F}_i$ be the maximal F-flag in Ew_i given by intersection of $\mathcal{F}$ with Ew_i. Let U_i be the unipotent radical of the G_i-stabiliser of $\mathcal{F}_i$; then $U_i = U \cap G_i$ and

$$U \cap M_0 = \prod_{i=1}^{r} U_i.$$

For $1 \leq i \leq r$, we denote by $\mathfrak{A}_i$ the hereditary $\mathfrak{o}_E$-order in $A_i = \mathrm{End}_F(Ew_i)$ given by the lattice chain $\mathcal{L}_i = \{L_k \cap Ew_i : k \in \mathbb{Z}\}$.

Lemma 2.2. *$(J^1 \cap U)H^1$ has an Iwahori decomposition with respect (M_0, P_0) and*

$$((J^1 \cap U)H^1) \cap M_0 = \prod_{i=1}^{r} (J^1(\beta, \mathfrak{A}_i) \cap U_i) H^1(\beta, \mathfrak{A}_i).$$

Proof. Since J^1 and U have Iwahori decompositions with respect to (M_0, P_0), so does $J^1 \cap U$. Since this group normalises H^1, which also has an Iwahori decomposition, we see that $(J^1 \cap U)H^1$ has an Iwahori decomposition with respect (M_0, P_0) and, in particular,

$$((J^1 \cap U)H^1) \cap M_0 = (J^1 \cap U \cap M_0)(H^1 \cap M_0).$$

The lemma now follows from the decompositions of $J^1 \cap M_0$ and $H^1 \cap M_0$ (see [2] §10) and the decompositions above. □

Since $\dim_E(Ew_i) = 1$, the algebra $E = F[\beta]$ is a maximal subfield of A_i. We will use the lemma above as a reduction step to the case when E is a maximal subfield of A.

Lemma 2.3. *Let U' be a unipotent radical of some parabolic subgroup of G, then the image of*

$$U' \cap J \to J/J^1 \cong \mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$$

is contained in the unipotent radical of some Borel subgroup of $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$.

Corollary 2.4. *We have*

$$J \cap U = (\mathbf{U}(\mathfrak{B}) \cap U_0)(J^1 \cap U).$$

Moreover, $(J \cap U)H^1$ has an Iwahori decomposition with respect to (M_0, P_0).

Proof. The E-basis $\{w_1, \ldots, w_r\}$ of V identifies $B^\times$ with $\mathrm{GL}_r(E)$ and $\mathbf{U}(\mathfrak{B}) \cap U_0$ with a subgroup of unipotent upper-triangular matrices with entries in $\mathfrak{o}_E$. This implies that the image of $\mathbf{U}(\mathfrak{B}) \cap U_0$ in $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$ is the unipotent radical of some Borel subgroup of $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$. Lemma 2.3 implies that

$$(J \cap U)J^1 = (\mathbf{U}(\mathfrak{B}) \cap U_0)J^1.$$

Intersecting both sides with U gives the first part of the lemma. The second follows immediately from Lemma 2.2 □

2.2 β-extensions

Let $\mathcal{C}(\mathfrak{A}, 0, \beta)$ be the set of simple characters of H^1, in the sense of [6] (3.2.3). Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$; then θ is a linear character and, there exists a unique irreducible representation η of J^1 containing θ, [6] (5.1.1).

Lemma 2.5. *Let $\mathbf{k}_\theta$ be the nondegenerate alternating form on J^1/H^1 given by*

$$\mathbf{k}_\theta(x, y) = \theta([x, y]), \qquad \textit{for } x, y \in J^1,$$

introduced in [6] §3.4. Let $\mathcal{U}$ be a subgroup of J^1 containing H^1 and let $\overline{\mathcal{U}}$ be the image of $\mathcal{U}$ in J^1/H^1. Suppose there exists a linear character χ of $\mathcal{U}$ such that $\chi|_{H^1} = \theta$. Then the following are equivalent:

(i) χ *occurs in* η *with multiplicity one;*

(ii) $\overline{\mathcal{U}}$ *is a maximal totally isotropic subspace of* J^1/H^1 *for the form* $\mathbf{k}_\theta$*;*

(iii) $\eta \cong \operatorname{Ind}_{\mathcal{U}}^{J^1} \chi$.

Proof. Since H^1 is normal in J^1 and J^1/H^1 is abelian (by [6] (3.1.15)), $\mathcal{U}$ is a normal subgroup of J^1. Moreover, since $\chi|_{H^1} = \theta$ and J^1/H^1 is abelian, the commutator subgroup of $\mathcal{U}$ will lie in the kernel of θ and hence $\overline{\mathcal{U}}$ is a totally isotropic subspace of J^1/H^1 for the form $\mathbf{k}_\theta$.

(i)$\Rightarrow$(ii) Let $\overline{\mathcal{U}}_{max}$ be a maximal totally isotropic subspace of J^1/H^1 containing $\overline{\mathcal{U}}$ and let $\mathcal{U}_{max}$ be its inverse image in J^1, so that $\mathcal{U}_{max}/\operatorname{Ker}(\theta)$ is a maximal abelian subgroup of $J^1/\operatorname{Ker}(\theta)$. The character χ admits extension to a linear character of $\mathcal{U}_{max}$ in exactly $(\mathcal{U}_{max} : \mathcal{U})$ ways and every one of these extensions occurs in η. Then χ occurs in η with multiplicity at least this index so that $\mathcal{U}_{max} = \mathcal{U}$.

(ii)$\Rightarrow$(iii) Suppose that $j \in J^1$ intertwines χ with itself. Let $\mathcal{U}'$ be the subgroup of J^1 generated by j and $\mathcal{U}$ and let $\overline{\mathcal{U}'}$ be the image of $\mathcal{U}'$ in J^1/H^1. A typical element of $\overline{\mathcal{U}'}$ is a coset $j^a x H^1$, where a is an integer and $x \in \mathcal{U}$. Since

$$\chi([j^a x, j^b y]) = \chi(j^a x j^{-a})\chi(j^{a+b} y x^{-1} j^{-a-b})\chi(j^b y^{-1} j^{-b}) = \chi([x, y]) = 1$$

for all $x, y \in \mathcal{U}$, the subspace $\overline{\mathcal{U}'}$ is totally isotropic for the form $\mathbf{k}_\theta$. Hence $\overline{\mathcal{U}'} = \overline{\mathcal{U}}$ and $j \in \mathcal{U}$. In particular, $\operatorname{Ind}_{\mathcal{U}}^{J^1} \chi$ is irreducible. Since η is the unique irreducible representation of J^1 containing θ and χ contains θ, we have $\eta \cong \operatorname{Ind}_{\mathcal{U}}^{J^1} \chi$.

(iii)$\Rightarrow$(i) Since η is irreducible, this is just Frobenius reciprocity. $\square$

Now let κ be a representation of J, such that $\kappa|_{J^1} \cong \eta$ and κ is intertwined by the whole of $B_\beta^\times$, that is, a β-extension of η in the sense of [6] (5.2.1).

Theorem 2.6. *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum. Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ and let κ be a β-extension of η as above. Let $\mathcal{F}$ be a maximal F-flag in V, let U be the unipotent radical of the G-stabiliser of $\mathcal{F}$ and let χ be a smooth character of U, such that*

(i) $\mathcal{F}_0 \subseteq \mathcal{F}$,

(ii) $\theta|_{U\cap H^1} = \chi|_{U\cap H^1}$,

(iii) χ *is trivial on* U_0.

Let Θ be the linear character of $(J\cap U)H^1$ defined by

$$\Theta(uh) = \chi(u)\theta(h), \quad \forall u \in U\cap J, \quad \forall h \in H^1.$$

Then

$$\kappa|_{(J\cap U)J^1} \cong \operatorname{Ind}_{(J\cap U)H^1}^{(J\cap U)J^1} \Theta.$$

Before proving Theorem 2.6, we remark that it is not clear that there exist a flag $\mathcal{F}$ and a character χ satisfying the hypotheses. This would follow from [3] Lemma 2.10, if we could fix the proof of that result. Instead, we will have to wait for Theorem 3.3 to see that there are indeed such a flag and character.

Proof. We begin by proving

$$\operatorname{Ind}_{(J^1\cap U)H^1}^{J^1} \Psi \cong \eta. \tag{$*$}$$

Step 1. We will prove $(*)$ in the special case when $E = F[\beta]$ is a maximal subfield of A.

Suppose that E is a maximal subfield of A so that $B = E$ and $J = \mathfrak{o}_E^\times J^1$. The pair (J, κ) is a simple type in the sense of [6] (5.5.10) and, since $\mathfrak{B} = \mathfrak{o}_E$, we have $e(\mathfrak{B}|\mathfrak{o}_E) = 1$. Now [6] (6.2.2) and (6.2.3) imply that there exists a representation Λ of $E^\times J$ such that $\Lambda|_J \cong \kappa$ and $\pi = \text{c-Ind}_{E^\times J}^G \Lambda$ is an irreducible supercuspidal representation of G. If $u \in U$, then $\det_A(u) = 1$ and this implies that

$$(E^\times J) \cap U = J \cap U.$$

Since $\dim_E V = 1$ the unipotent radical U_0 is trivial and hence Corollary 2.4 implies that

$$J \cap U = J^1 \cap U.$$

Hence $\Lambda|_{(E^\times J)\cap U} \cong \eta|_{J^1\cap U}$. Since η is the unique irreducible representation containing θ and $\Theta|_{H^1} = \theta$, we obtain that Θ occurs in $\eta|_{(J^1\cap U)H^1}$. Since $\Theta|_{U\cap J} = \chi|_{U\cap J}$, we obtain

$$\begin{aligned} 1 \ \leq\ \dim \mathrm{Hom}_{(J^1\cap U)H^1}(\Theta,\eta) &\ \leq\ \dim \mathrm{Hom}_{J^1\cap U}(\chi,\eta) \\ &\ =\ \dim \mathrm{Hom}_{(E^\times J^1)\cap U}(\chi,\Lambda) \ \leq\ 1, \end{aligned}$$

where the last inequality follows from [3] Proposition 1.6(iii). Hence

$$\dim \mathrm{Hom}_{(J^1\cap U)H^1}(\Theta,\eta) = 1.$$

The equivalence $(*)$ now follows immediately from Lemma 2.5 applied to $\mathcal{U} = (J^1\cap U)H^1$ and $\chi = \Theta$.

Step 2. We will prove $(*)$ in the general case by reducing to Step 1.

For $1 \leq j \leq r$ we have an equality of sets:

$$\{L_i \cap Ew_j : L_i \in \mathcal{L}\} = \{\mathfrak{p}_E^k w_j : k \in \mathbb{Z}\}.$$

This follows from the explicit description of lattices in $\mathcal{L}$, in terms of the E-basis $\{w_1, \dots w_r\}$ of V. Hence [2] Example 10.9 implies that the character θ is trivial on its restrictions to $H^1 \cap U_0$ and $H^1 \cap U_0^-$. Moreover, by [2] page 167, the restriction of θ to $H^1 \cap G_i = H^1(\beta, \mathfrak{A}_i)$ is the simple character θ_i in $\mathcal{C}(\mathfrak{A}_i, 0, \beta)$ corresponding to θ under the canonical bijection $\tau_{\mathfrak{A},\mathfrak{A}_i,\beta}$ of [6] §3.6, where $\mathfrak{A}_i$ is the hereditary $\mathfrak{o}_F$-order corresponding to the lattice chain $\{\mathfrak{p}_E^k w_i : k \in \mathbb{Z}\}$ in Ew_i. In particular, [2] Example 10.9 implies that the analogue of [6] (7.2.3) holds in our situation:

(i) the subspaces $(J^1 \cap U_0)/(H^1 \cap U_0)$ and $(J^1 \cap U_0^-)/(H^1 \cap U_0^-)$ of J^1/H^1 are both totally isotropic for the form $\mathbf{k}_\theta$, and orthogonal to the subspace $(J^1 \cap M_0)/(H^1 \cap M_0)$;

(ii) the restriction of $\mathbf{k}_\theta$ to the group

$$(J^1 \cap M_0)/(H^1 \cap M_0) = \prod_{i=1}^{r} J^1(\beta, \mathfrak{A}_i)/H^1(\beta, \mathfrak{A}_i)$$

is the orthogonal sum of the pairings $\mathbf{k}_{\theta_i}$;

(iii) we have an orthogonal sum decomposition

$$\frac{J^1}{H^1} = \frac{J^1\cap M_0}{H^1\cap M_0} \perp \left(\frac{J^1\cap U_0^-}{H^1\cap U_0^-} \times \frac{J^1\cap U_0}{H^1\cap U_0} \right).$$

In particular, the restriction of $\mathbf{k}_\theta$ to the group $(J^1 \cap U_0^-)/(H^1 \cap U_0^-) \times (J^1 \cap U_0)/(H^1 \cap U_0)$ is non-degenerate.

Let $\overline{((J^1 \cap U)H^1) \cap M_0}$ be the image of the natural homomorphism

$$((J^1 \cap U)H^1) \cap M_0 \to (J^1 \cap M_0)/(H^1 \cap M_0).$$

Lemma 2.2 and (ii) above imply that

$$\overline{((J^1 \cap U)H^1) \cap M_0} = \prod_{i=1}^{r} \overline{J^1(\beta, \mathfrak{A}_i) \cap U_i}$$

where $\overline{J^1(\beta, \mathfrak{A}_i) \cap U_i}$ is the image of the natural homomorphism

$$J^1(\beta, \mathfrak{A}_i) \cap U_i \to J^1(\beta, \mathfrak{A}_i)/H^1(\beta, \mathfrak{A}_i).$$

Since E is a maximal subfield of A_i and we have proved $(*)$ when E is maximal, $\overline{J^1(\beta, \mathfrak{A}_i) \cap U_i}$ is a maximal isotropic subspace in $J^1(\beta, \mathfrak{A}_i)/H^1(\beta, \mathfrak{A}_i)$ for the form $\mathbf{k}_{\theta_i}$.

Now (ii) implies that $\overline{((J^1 \cap U)H^1) \cap M_0}$ is a maximal isotropic subspace of $(J^1 \cap M_0)/(H^1 \cap M_0)$ for $\mathbf{k}_\theta$. Moreover, (i) and (iii) imply that $(J^1 \cap U_0)/(H^1 \cap U_0)$ is a maximal isotropic subspace of $(J^1 \cap U_0^-)/(H^1 \cap U_0^-) \times (J^1 \cap U_0)/(H^1 \cap U_0)$.

It follows from the orthogonal sum decomposition in (iii) that

$$\overline{((J^1 \cap U)H^1) \cap M_0} \times (J^1 \cap U_0)/(H^1 \cap U_0)$$

is a maximal isotropic subspace in J^1/H^1 for the form $\mathbf{k}_\theta$.

Since $(J^1 \cap U)H^1$ contains $J^1 \cap U_0$, the image of $(J^1 \cap U)H^1$ in J^1/H^1 contains a maximal totally isotropic subspace of J^1/H^1 described above. Since there exists a linear character Θ of $(J^1 \cap U)H^1$ extending θ, we see that this image is itself isotropic and hence must be a maximal totally isotropic subspace. The equivalence $(*)$ now follows from Lemma 2.5.

Step 3. Finally, we will deduce Theorem 2.6 from $(*)$ by examining the construction of κ in [6] §5.

Let $\mathcal{L}_m$ be the $\mathfrak{o}_E$-lattice chain in V given by

$$\mathcal{L}_m = \{\varpi_E^k(\mathfrak{o}_E w_1 + \cdots + \mathfrak{o}_E w_j + \mathfrak{p}_E w_{j+1} + \cdots + \mathfrak{p}_E w_r) : k \in \mathbb{Z}, 1 \le j \le r\}$$

Let $\mathfrak{B}_m = \mathrm{End}^0_{\mathfrak{o}_E}(\mathcal{L}_m)$ so that $\mathfrak{B}_m$ is a minimal $\mathfrak{o}_E$-order in B. Similarly, let $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L}_m)$ so that $\mathfrak{A}_m$ is the unique hereditary $\mathfrak{o}_F$-order in A normalised by $E^\times$ such that $\mathfrak{A}_m \cap B = \mathfrak{B}_m$. Moreover, $[\mathfrak{A}_m, nr/e, 0, \beta]$, where $e = e(\mathfrak{B}|\mathfrak{o}_E)$, is a simple stratum in A and the groups

$$H^1_m = H^1(\beta, \mathfrak{A}_m), \qquad J^1_m = J^1(\beta, \mathfrak{A}_m)$$

have Iwahori decompositions with respect to (M_0, P_0).

We denote by θ_m the simple character in $\mathcal{C}(\mathfrak{A}_m, 0, \beta)$ corresponding to θ via the canonical bijection $\tau_{\mathfrak{A},\mathfrak{A}_m,\beta}$ of [6] §3.6. Then θ_m is trivial on $H^1_m \cap U_0$ and $H^1_m \cap U_0^-$ (see [2] (10.9)). We also denote by η_m the unique irreducible representation of J^1_m which contains θ_m.

Since $\mathcal{L} \subseteq \mathcal{L}_m$ we have $\mathfrak{A}_m \subseteq \mathfrak{A}$ and $\mathfrak{B}_m \subseteq \mathfrak{B}$. Moreover,

$$\mathbf{U}(\mathfrak{B}) \cap U_0 = \mathbf{U}(\mathfrak{B}_m) \cap U_0$$

since with respect to the E-basis $\{w_1, \ldots, w_r\}$ of V, both groups are identified with unipotent upper-triangular matrices, with entries in $\mathfrak{o}_E$.

We define

$$\tilde{\eta} \ := \ \mathrm{Ind}_{(J\cap U)H^1}^{(J\cap U)J^1} \Theta.$$

Note first that $\tilde{\eta}|_{J^1} = \mathrm{Ind}_{(J^1\cap U)H^1}^{J^1} \Theta \cong \eta$ so $\tilde{\eta}$ is certainly irreducible. Moreover, [6] (5.1.1) implies that the $\mathbf{U}^1(\mathfrak{A}_m)$-intertwining of $\tilde{\eta}$ is contained in

$$I_{\mathbf{U}^1(\mathfrak{A}_m)}(\eta) = (J^1 B^\times J^1) \cap \mathbf{U}^1(\mathfrak{A}_m) = \mathbf{U}^1(\mathfrak{B}_m) J^1 = (J \cap U) J^1$$

where the last equality follows from Corollary 2.4. Hence $\mathrm{Ind}_{(J\cap U)J^1}^{\mathbf{U}^1(\mathfrak{A}_m)} \tilde{\eta}$ is irreducible.

Now we claim that

$$\mathrm{Ind}_{(J\cap U)J^1}^{\mathbf{U}^1(\mathfrak{A}_m)} \tilde{\eta} \cong \mathrm{Ind}_{J^1_m}^{\mathbf{U}^1(\mathfrak{A}_m)} \eta_m.$$

It is enough to show that

$$\mathrm{Hom}_{\mathbf{U}^1(\mathfrak{A}_m)} \left(\mathrm{Ind}_{(J\cap U)J^1}^{\mathbf{U}^1(\mathfrak{A}_m)} \tilde{\eta}, \mathrm{Ind}_{H^1_m}^{\mathbf{U}^1(\mathfrak{A}_m)} \theta_m \right) \neq 0,$$

since the latter is a multiple of $\mathrm{Ind}_{J^1_m}^{\mathbf{U}^1(\mathfrak{A}_m)} \eta_m$, which is irreducible by the same argument as above. By Mackey Theory, it is enough to show that

$$\mathrm{Hom}_{((J\cap U)H^1)\cap H^1_m}(\Theta, \theta_m) \neq 0.$$

But $((J\cap U)H^1)\cap H^1_m$ has an Iwahori decomposition with respect to (M_0, P_0), since both $(J\cap U)H^1$ and H^1_m do. Moreover, Θ is trivial on $((J\cap U)H^1)\cap U_0^-$, since $((J\cap U)H^1)\cap U_0^- = H^1 \cap U_0^-$ and θ is trivial on $H^1\cap U_0^-$. Since χ is trivial on U_0 we get that Θ is trivial on $((J\cap U)H^1)\cap U_0$. Hence both Θ and θ_m are trivial on the subgroups $((J\cap U)H^1)\cap H^1_m \cap U_0^-$ and $((J\cap U)H^1)\cap H^1_m \cap U_0$. Finally, by [2] §10,

$$H^1_m \cap M_0 \ = \ \prod_{i=1}^r H^1(\beta, \mathfrak{A}_i) \ = \ H^1 \cap M_0$$

so

$$(J \cap U)H^1 \cap H_m^1 \cap M_0 \;=\; H^1 \cap M_0 \;\subset\; H^1 \cap H_m^1,$$

where $\Theta = \theta$ and θ_m agree by [6] (3.6.1).

In particular, we have shown that, upto equivalence, $\tilde{\eta}$ satisfies the conditions of the representation (also denoted $\tilde{\eta}$) in [6] (5.1.15). Since these conditions uniquely determine $\tilde{\eta}$, we conclude that our $\tilde{\eta}$ is, up to equivalence, the same one used in the construction of κ in [6] (5.2.4) and hence

$$\kappa|_{(J\cap U)J^1} \cong \tilde{\eta} \cong \mathrm{Ind}_{(J\cap U)H^1}^{(J\cap U)J^1} \Theta$$

as required. □

3 A particular unipotent subgroup

The main result of this section says that we may choose a maximal F-flag $\mathcal{F}$ in V and a character χ of the G-stabiliser U of $\mathcal{F}$ which satisfy the hypotheses of Theorem 2.6 and, moreover, such that we can control the restriction of certain characters of U to $U \cap \mathbf{U}(\mathfrak{B})$ (see Theorem 3.3 for details). We will continue with the notation of the previous section but, since we will consider only supercuspidal representations in the applications, we will assume that $e(\mathfrak{B}|\mathfrak{o}_E) = 1$.

3.1 An F-basis of E

Although we would like to work in a basis-free way, we are forced to choose one, since the zeta function in §7, whose functional equation defines ε-factors of pairs, is defined on matrices. In this subsection we consider the special case when E is maximal, so we may identify $V = E$. We show that there exists an F-basis $\mathcal{B}$ of E with 'nice' properties. The basis is chosen to ease the pain of calculations in §7, see also Corollary 3.4. The conditions imposed on $\mathcal{B}$ imply certain uniqueness result, which will be used to define a numerical invariant in §6.

Let $\mathcal{F} = \{V_i : 1 \le i \le d\}$, where $d = [E : F]$, be a maximal F-flag in E, let U be the unipotent radical of the $\mathrm{Aut}_F(E)$-stabiliser of $\mathcal{F}$, and let χ be a smooth, non-degenerate character of U. Let ψ_E be an additive character of E, trivial on $\mathfrak{p}_E$, and such that

$$\psi_E(x) = \psi_F(x), \quad \forall x \in F.$$

Definition 3.1. *An F-basis $\mathcal{B} = \{x_1, \dots, x_d\}$ of E is (U, χ, ψ_E)-*balanced *if the following hold:*

(i) $V_i = Fx_1 + \dots + Fx_i$, $1 \leq i \leq d$;

(ii) *there exists functions $a_i : \mathbb{Z} \to \mathbb{Z}$, for $1 \leq i \leq d$ such that*

$$\mathfrak{p}_E^k = \sum_{i=1}^{d} \mathfrak{p}_F^{a_i(k)} x_i, \quad \forall k \in \mathbb{Z};$$

(iii) *if $u \in U$, and (u_{ij}) is the matrix of u with respect to $\mathcal{B}$, then*

$$\chi(u) = \psi_F(\sum_{i=1}^{d-1} u_{i,i+1});$$

(iv) *if we embed naturally $E \hookrightarrow \mathrm{End}_F(E)$ and, for $\xi \in E$, we let (ξ_{ij}) be the matrix of ξ with respect to $\mathcal{B}$, then*

$$\psi_E(\xi) = \psi_F(\xi_{dd}), \quad \forall \xi \in E.$$

Proposition 3.2. *There exists an (U, χ, ψ_E)-balanced basis. Moreover, if $\mathcal{B} = \{x_1, \dots x_d\}$ is (U, χ, ψ_E)-balanced and $\mathcal{B}' = \{y_1, \dots, y_d\}$ is a basis of E over F which satisfies Definition 3.1 (i),(iii) and (iv), then $x_d x_1^{-1} = y_d y_1^{-1}$.*

Proof. Proposition II-3 in [13] applied to $\mathcal{F}$ and $\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$ implies that there exists an F-basis $\mathcal{B} = \{x_1, \dots, x_d\}$ of E which satisfies (i) and (ii). Since χ is non-degenerate, after replacing x_i by some $\lambda_i x_i$, where $\lambda_i \in F^\times$, we may ensure that $\mathcal{B}$ satisfies (iii).

For $\xi \in E$, let (ξ_{ij}) be a matrix of ξ with respect to $\mathcal{B}$. Consider the function $\phi_{\mathcal{B}} : E \to \mathbb{C}^\times$ given by $\phi_{\mathcal{B}}(\xi) = \psi_F(\xi_{dd})$. It is clear that $\phi_{\mathcal{B}}$ is an additive character. If $x \in F$, then $\phi_{\mathcal{B}}(x) = \psi_F(x)$, hence $\phi_{\mathcal{B}}$ is non-trivial on $\mathfrak{o}_E$. Since $x_d \in \mathfrak{p}_E^{v_E(x_d)}$ and $x_d \notin \mathfrak{p}_E^{v_E(x_d)+1}$ we have $a_d(v_E(x_d)) = 0$ and $a_d(v_E(x_d)+1) = 1$, hence if $\xi \in \mathfrak{p}_E$, then $\xi_{dd} \in \mathfrak{p}_F$, and so $\phi_{\mathcal{B}}$ is trivial on $\mathfrak{p}_E$. Since ψ_E and $\phi_{\mathcal{B}}$ have the same conductor, there exists $\alpha \in \mathfrak{o}_E^\times$, such that $\psi_E(\xi) = \phi_{\mathcal{B}}(\alpha\xi)$, for all $\xi \in E$.

Set $A(E) = \mathrm{End}_F(E)$, let $\mathfrak{A}(E)$ be the hereditary order corresponding to the lattice chain $\mathcal{L}(E) = \{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$, and, for $1 \leq i, j \leq d$, let $\mathbf{1}_{ij} \in A(E)$ be given by $\mathbf{1}_{ij} x_k = \delta_{ik} x_j$, where δ_{ik} is the Kronecker delta. Since $\mathcal{B}$ satisfies (ii) we have $\mathbf{1}_{ij} \in \mathfrak{A}$.

Set $u = 1 + (\alpha - 1)\mathbf{1}_{dd} \in \mathfrak{A}(E)$. We have

$$\psi_F(\lambda) = \psi_E(\lambda) = \phi_{\mathcal{B}}(\alpha\lambda) = \psi_F(\alpha_{dd}\lambda), \quad \forall \lambda \in F.$$

Hence $\alpha_{dd} = 1$ and so $u \in U \cap \mathfrak{A}(E)$. So the basis $u\mathcal{B} = \{ux_1, \ldots, ux_d\}$ also satisfies (i),(ii) and (iii). Now

$$\phi_{u\mathcal{B}}(\xi) = \phi_{\mathcal{B}}(u^{-1}\xi u) = \psi_F\left(\sum_{i=1}^{d} \xi_{di}\alpha_{id}\right) = \psi_F((\xi\alpha)_{dd}) = \phi_{\mathcal{B}}(\xi\alpha) = \psi_E(\xi).$$

Hence $u\mathcal{B}$ is (U, χ, ψ_E)-balanced.

Suppose that $\mathcal{B} = \{x_1, \ldots x_d\}$ is (U, χ, ψ_E)-balanced and that $\mathcal{B}' = \{y_1, \ldots, y_d\}$ satisfies Definition 3.1(i),(iii) and (iv). After translating by some $\lambda \in F^\times$, we may assume that $x_1 = y_1$. Parts (i),(iii) imply that there exists $u \in U$ such that $\mathcal{B}' = u\mathcal{B}$. Let (u_{ij}) be the matrix of u with respect to $\mathcal{B}$. Then, for all $\xi \in E$, we have

$$\psi_E(\xi) = \phi_{u\mathcal{B}}(\xi) = \psi_F\left((u^{-1}\xi u)_{dd}\right) = \psi_F\left(\sum_{i=1}^{d} \xi_{di}u_{id}\right) = \psi_A\left(\xi\left(\sum_{i=1}^{d} u_{id}\mathbf{1}_{di}\right)\right).$$

where $\psi_A = \psi_F \circ \mathrm{tr}_{A(E)/F}$. Moreover, $\mathbf{1}_{id} = x_i x_d^{-1}\mathbf{1}_{dd}$ so

$$\psi_E(\xi) = \psi_A\left(\xi\left(\sum_{i=1}^{d} u_{id}x_i x_d^{-1}\right)\mathbf{1}_{dd}\right).$$

By [6] (1.3.4) there exists an (E, E)-bimodule homomorphism $s : A \to E$ (a *tame corestriction*), such that

$$\psi_A(\xi a) = \phi_{\mathcal{B}}(\xi s(a)), \quad \forall \xi \in E, \quad \forall a \in A.$$

Since $\phi_{\mathcal{B}}(\xi) = \psi_F(\xi_{dd}) = \psi_A(\xi\mathbf{1}_{dd}) = \phi_{\mathcal{B}}(\xi s(\mathbf{1}_{dd}))$, for all $\xi \in E$, we obtain that $s(\mathbf{1}_{dd}) = 1$. Hence we get

$$\psi_E(\xi) = \phi_{\mathcal{B}}\left(\xi\left(\sum_{i=1}^{d} u_{id}x_i x_d^{-1}\right)\right) = \psi_E\left(\xi\left(\sum_{i=1}^{d} u_{id}x_i x_d^{-1}\right)\right).$$

Thus $x_d = \sum_{i=1}^{d} u_{id}x_i$, which implies that $u_{id} = 0$, if $i \neq d$. In particular, we get $y_d = ux_d = x_d$. □

3.2 A choice of maximal flag

Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, such that $e(\mathfrak{B}|\mathfrak{o}_E) = 1$. Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ and let $\mathcal{F}_0$ be a maximal E-stable flag in V, U_0 the unipotent radical of the G-stabiliser of $\mathcal{F}_0$.

Theorem 3.3. *There exist a maximal F-flag $\mathcal{F}$ in V, a smooth character χ of the unipotent radical $U = U_{\mathcal{F}}$ and an element $b \in X_{\mathcal{F}} \cap \mathfrak{A}$, such that the following hold:*

(i) $\mathcal{F}_0 \subseteq \mathcal{F}$,

(ii) $\theta|_{U \cap H^1} = \chi|_{U \cap H^1}$,

(iii) *χ is trivial on U_0,*

(iv) *the character*

$$\overline{\psi}_b : (\mathbf{U}(\mathfrak{B}) \cap U)/(\mathbf{U}^1(\mathfrak{B}) \cap U) \to \mathbb{C}^\times, \quad u(\mathbf{U}^1(\mathfrak{B}) \cap U) \mapsto \psi_b(u)$$

defines a non-degenerate character of a maximal unipotent subgroup of $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$.

Proof. Let us consider first, as in Step 1 of the proof of Theorem 2.6 the case when E is a maximal subfield of A. We identify $V = E$ and $\mathfrak{A} = \mathfrak{A}(E)$ is the hereditary order associated to the lattice chain $\mathcal{L}(E) = \{\mathfrak{p}_E^k : k \in \mathbb{Z}\}$. The parts (i),(iii) and (iv) are empty in this case since $\mathcal{F}_0 = \{0 \subseteq V\}$, $U_0 = \{1\}$, and $\mathbf{U}(\mathfrak{B}) = \mathfrak{o}_E^\times$ does not contain non-trivial unipotent elements, and so $\mathbf{U}(\mathfrak{B}) \cap U = \{1\}$. Consider the supercuspidal representation $\pi = \text{c-Ind}_{E^\times J}^G \Lambda$, as in Step 1 of the proof of Theorem 2.6. According to [3] Proposition 1.6(i), there exist a maximal F-flag $\mathcal{F}$ in E, and a smooth character χ of the unipotent radical U of the G-stabiliser of $\mathcal{F}$ such that

$$\mathrm{Hom}_{U \cap (E^\times J)}(\chi, \Lambda) \neq 0.$$

Since $\Lambda|_{H^1} = (\dim \Lambda)\theta$, we obtain

$$\theta|_{U \cap H^1} = \chi|_{U \cap H^1}.$$

Hence the theorem holds for E maximal. Note that, since $E^\times$ normalises H^1 and $\theta = \theta^x$, for all $x \in E^\times$, we may replace the pair (U, χ) by a conjugate (U^x, χ^x), where $x \in E^\times$.

Now let us consider the general case. Put $d = [E : F]$.

Construction of $\mathcal{F}$ and χ. According to [13] Proposition II-3 we may choose an E-basis $\mathcal{B}_E = \{w_1, \ldots, w_r\}$ of V such that

$$\mathcal{F}_0 = \{\sum_{i=1}^{j} Ew_i : 1 \leq j \leq r\}, \quad \text{and} \quad L_k = \mathfrak{p}_E^k w_1 + \cdots + \mathfrak{p}_E^k w_r, \quad \forall k \in \mathbb{Z}.$$

Let P_0 be the G-stabiliser of $\mathcal{F}_0$. Moreover, put $G_i = \mathrm{Aut}_F(Ew_i)$, for $1 \leq i \leq r$, and

$$M_0 \;=\; \prod_{i=1}^{r} G_i,$$

a Levi component of the parabolic subgroup P_0 of G. Then U_0 is the unipotent radical of P_0, so that $P_0 = M_0 U_0$. We are in the situation considered in Step 2 of the proof of Theorem 2.6. In particular,

$$H^1(\beta, \mathfrak{A}) \cap M_0 = \prod_{i=1}^{r} H^1(\beta, \mathfrak{A}_i) \cong \prod_{i=1}^{r} H^1(\beta, \mathfrak{A}(E)),$$

where the last isomorphism is induced by identifying Ew_i with E. Moreover, according to [2] Corollary 10.16, there exists $\theta_F \in \mathcal{C}(\mathfrak{A}(E), 0, \beta)$ such that

$$\theta|_{H^1(\beta,\mathfrak{A})\cap M_0} = \theta_F \otimes \cdots \otimes \theta_F$$

via the above identification. Now, by the case when E is maximal considered above, we know that there exist a maximal F-flag $\mathcal{F}_1$ in E, and a smooth character $\chi_1 : U_{\mathcal{F}_1} \to \mathbb{C}^\times$ such that

$$\chi_1(u) = \theta_F(u), \quad \forall u \in U_{\mathcal{F}_1} \cap H^1(\beta, \mathfrak{A}(E)),$$

where $U_{\mathcal{F}_1}$ is the unipotent radical of $\mathrm{Aut}_F(E)$-stabiliser of $\mathcal{F}_1$. Proposition 1.3 (i) implies that χ_1 is non-degenerate. Choose an additive character ψ_E of E, such that $\psi_E(x) = \psi_F(x)$, for all $x \in F$, and ψ_E trivial on $\mathfrak{p}_E$. Proposition 3.2 gives a $(U_{\mathcal{F}_1}, \chi_1, \psi_E)$-balanced F-basis $\mathcal{B}_1 = \{x_{11}, \ldots x_{1d}\}$ of E. Set $y = x_{1d}x_{11}^{-1}$ and for $2 \leq j \leq r$, let $\mathcal{B}_j = \{x_{j1}, \ldots, x_{jd}\}$ be the basis of E over F given by

$$x_{ji} = y^{j-1} x_{1i}, \quad 1 \leq i \leq d.$$

Note that, in particular, $x_{j1} = x_{j-1,d}$, for $2 \leq j \leq r$.

Let $\mathcal{F}_j = \{\sum_{i=1}^{k} F x_{ji} : 1 \leq k \leq d\}$; then $\mathcal{F}_j = y\mathcal{F}_{j-1}$ and hence $U_{\mathcal{F}_j} = U_{\mathcal{F}_{j-1}}^y$ so we may define a character $\chi_j : U_{\mathcal{F}_j} \to \mathbb{C}^\times$ by $\chi_j = \chi_{j-1}^y$. Since $y \in E^\times$ normalises θ_F, we obtain that

$$\chi_j(u) = \theta_F(u), \quad \forall u \in U_{\mathcal{F}_j} \cap H^1(\beta, \mathfrak{A}(E)), \quad 1 \leq j \leq r.$$

Let $\mathcal{F} = \{V_k : 1 \le k \le N\}$ be the maximal F-flag in V given by

$$V_{(i-1)d+j} = Fx_{11}w_1 + \cdots + Fx_{ji}w_i, \quad 1 \le i \le r, \quad 1 \le j \le d,$$

and let $U_\mathcal{F}$ be the unipotent radical of the G-stabiliser of $\mathcal{F}$. Since $U_\mathcal{F}/U_0 \cong U_\mathcal{F} \cap M_0 \cong \prod_{i=1}^r U_i$ we may define $\chi : U_\mathcal{F} \to \mathbb{C}^\times$ by

$$\chi|_{U_\mathcal{F} \cap M_0} = \chi_1 \otimes \cdots \otimes \chi_r, \quad \chi|_{U_0} = \mathbf{1}.$$

The Iwahori decomposition implies that

$$\begin{aligned} U_\mathcal{F} \cap H^1(\beta, \mathfrak{A}) &= \big(U_\mathcal{F} \cap M_0 \cap H^1(\beta, \mathfrak{A})\big)\big(H^1(\beta, \mathfrak{A}) \cap U_0\big) \\ &\cong \Big(\prod_{i=1}^r U_{\mathcal{F}_i} \cap H^1(\beta, \mathfrak{A}(E))\Big)\big(H^1(\beta, \mathfrak{A}) \cap U_0\big) \end{aligned}$$

Since θ is trivial on $U_0 \cap H^1(\beta, \mathfrak{A})$, it follows that

$$\theta(u) = \chi(u), \quad \forall u \in U \cap H^1(\beta, \mathfrak{A}).$$

Construction of b. For $\boldsymbol{\mu} = (\mu_1, ..., \mu_{r-1}) \in \mathfrak{o}_F^{r-1}$, we define $b = b(\boldsymbol{\mu}) \in X_\mathcal{F}$ by

$$b(x_{ji}w_j) = \begin{cases} \mu_j x_{j+1,1} w_{j+1}, & \text{if } i = d,\ 1 \le j \le r-1; \\ 0 & \text{otherwise.} \end{cases}$$

We claim that such b also lies in $\mathfrak{A}$. For $1 \le j \le r$ we have constructed a $(U_{\mathcal{F}_j}, \chi_j, \psi_E)$-balanced basis $\mathcal{B}_j = \{x_{j1}, \ldots, x_{jd}\}$, of E over F, so there exist functions $a_{ji} : \mathbb{Z} \to \mathbb{Z}$, for $1 \le i \le d$, such that

$$\mathfrak{p}_E^k = \sum_{i=1}^d \mathfrak{p}_F^{a_{ji}(k)} x_{ji}, \quad \forall k \in \mathbb{Z}.$$

Hence,

$$L_k = \sum_{j=1}^r \mathfrak{p}_E^k w_j = \sum_{j=1}^r \sum_{i=1}^d \mathfrak{p}_F^{a_{ji}(k)} x_{ji} w_j, \quad \forall k \in \mathbb{Z}.$$

Since, by construction, $x_{jd} = x_{j+1,1}$, for $1 \le j \le r-1$, we have $a_{jd}(k) = a_{j+1,1}(k)$ for all $k \in \mathbb{Z}$. Hence

$$bL_k = \sum_{j=1}^{r-1} \mu_j \mathfrak{p}_F^{a_{jd}(k)} x_{j+1,1} w_{j+1} \subseteq L_k, \quad \forall k \in \mathbb{Z}$$

which implies that $b \in \mathfrak{A}$.

Since $b \in X_{\mathcal{F}}$, Lemma 1.1 implies that ψ_b defines a linear character of U. Since $b \in \mathfrak{A}$ we have $\psi_b(u) = 1$ for all $u \in \mathbf{U}^1(\mathfrak{A})$. Hence the character $\overline{\psi}_b$ is well defined. Moreover, $(\mathbf{U}(\mathfrak{B}) \cap U)/(\mathbf{U}^1(\mathfrak{B}) \cap U)$ is the unipotent radical of the $\mathrm{Aut}_{\mathfrak{k}_E}(L_0/L_1)$-stabiliser of the flag $\{\sum_{j=1}^{i} \mathfrak{k}_E(w_j + L_1) : 0 \le i \le r\}$. This follows from Lemma 2.1.

Let $\boldsymbol{\mu} = (1, ..., 1)$ and let $b = b(\boldsymbol{\mu})$ as above. (Indeed, any $\boldsymbol{\mu} \in (\mathfrak{o}_F^{\times})^{r-1}$ would do.) We claim that $\overline{\psi}_b$ is non-degenerate. For $2 \le k \le r$ and $\xi \in E$ we define $u_{\xi,k} \in U \cap B^{\times} = U_0 \cap B^{\times}$ by

$$u_{\xi,k}(w_j) = \begin{cases} w_k + \xi w_{k-1}, & \text{if } k = j; \\ w_j, & \text{otherwise.} \end{cases}$$

We will compute $(\chi\psi_b)(u_{\xi,k}) = \psi_b(u_{\xi,k})$. Since $x_{ji} \in E$, it commutes with $u_{\xi,k}$ and hence

$$(u_{\xi,k} - 1)(x_{ji} w_j) = \begin{cases} \xi x_{ki} w_{k-1} & \text{if } j = k\ , \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{B}_{k-1}$ is an F-basis of E we may write uniquely

$$\xi x_{ki} = \sum_{j=1}^{d} \lambda_{ji}^{(k)} x_{k-1,j}$$

where $\lambda_{ji}^{(k)} \in F$. Then

$$(b(u_{\xi,k} - 1))(x_{ji} w_j) = \begin{cases} \lambda_{di}^{(k)} x_{k1} w_k & \text{if } j = k\ , \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\mathrm{tr}_{A/F}(b(u_k - 1)) = \lambda_{d1}^{(k)}.$$

Since, by construction, $x_{k1} = x_{k-1,d}$, we have $\lambda_{d1}^{(k)} = \xi_{dd}^{(k)}$, where $(\xi_{ij}^{(k)})$ is the matrix of $\xi \in \mathrm{End}_F(E)$ with respect to $\mathcal{B}_{k-1}$. However, since $\mathcal{B}_k$ is $(U_{\mathcal{F}_k}, \chi_k, \psi_E)$-balanced

$$\psi_b(u_{\xi,k}) = \psi_F(\xi_{dd}^{(k)}) = \psi_E(\xi).$$

Now, ψ_E has conductor $\mathfrak{p}_E$, and so $\overline{\psi}_b$ is non-degenerate. □

We record the following corollary to the proof of the theorem.

Corollary 3.4. *Fix an additive character $\psi_E : E \to \mathbb{C}^\times$ such that $\psi_E(x) = \psi_F(x)$, for all $x \in F$ and ψ_E is trivial on $\mathfrak{p}_E$. Choose an E-basis $\mathcal{B}_E = \{w_1, \dots, w_r\}$ of V such that*

$$\mathcal{F}_0 = \{\sum_{i=1}^{j} Ew_i : 1 \leq j \leq r\}, \quad \textit{and} \quad L_k = \mathfrak{p}_E^k w_1 + \cdots + \mathfrak{p}_E^k w_r, \quad \forall k \in \mathbb{Z}.$$

Let $\mathcal{F}$, χ and ψ_b be as in Theorem 3.3. There exists a basis $\{x_1, \dots, x_d\}$ of E over F, satisfying Definition 3.1(iv) with respect to ψ_F and ψ_E, such that $x_1 = 1$ and such that, if we putt

$$v_{d(i-1)+j} = x_d^{i-1} x_j w_i, \quad 1 \leq i \leq r, \quad 1 \leq j \leq d,$$

then the set $\mathcal{B}_F = \{v_1, \dots, v_N\}$ is an F-basis of V with the following properties:

(i) $\mathcal{F} = \{\sum_{i=1}^{j} Fv_i : 1 \leq j \leq N\}$;

(ii) *there exist functions $a_i : \mathbb{Z} \to \mathbb{Z}$, for $1 \leq i \leq N$ such that*

$$L_k = \sum_{i=1}^{N} \mathfrak{p}_F^{a_i(k)} v_i, \quad \forall k \in \mathbb{Z};$$

(iii) *if $u \in U$ has matrix (u_{ij}) with respect to $\mathcal{B}_F$, then*

$$(\chi\psi_b)(u) = \psi_F(\sum_{i=1}^{N-1} u_{i,i+1});$$

(iv) *if $u \in U \cap B^\times$ has matrix (u_{ij}) with respect to $\mathcal{B}_E$, then*

$$(\chi\psi_b)(u) = \psi_E(\sum_{i=1}^{r-1} u_{i,i+1}).$$

If we put $\beta = 0$, so that $E = F$, $\mathfrak{o}_E = \mathfrak{o}_F$ and $r = N$ then the proof of Theorem 3.3 formally goes through with $\mathcal{F} = \mathcal{F}_0$, $\mathcal{B}_j = \{1\}$, for $1 \leq j \leq N$, and $b = b(\boldsymbol{\mu})$, for any $\boldsymbol{\mu} \in (\mathfrak{o}_F^\times)^{N-1}$. We obtain a 'level zero version' of Theorem 3.3:

Corollary 3.5. *Let $\mathfrak{A}$ be a maximal $\mathfrak{o}_F$-order in A. Then there exist a maximal F-flag $\mathcal{F}$ in V and an element $b \in X_{\mathcal{F}} \cap \mathfrak{A}$ such that*

$$\overline{\psi}_b : (\mathbf{U}(\mathfrak{A}) \cap U)/(\mathbf{U}^1(\mathfrak{A}) \cap U) \to \mathbb{C}^\times, \quad u(\mathbf{U}^1(\mathfrak{A}) \cap U) \mapsto \psi_b(u)$$

defines a non-degenerate character of a maximal unipotent subgroup of the group $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A})$.

4 Supercuspidal representations

Let π be an irreducible supercuspidal representation of G. By [6] §6, π is compactly induced:

$$\pi \cong \text{c-Ind}_{\mathbf{J}}^{G} \Lambda,$$

from a (rather special) open compact-mod-centre subgroup $\mathbf{J}$ of G. More precisely, $\mathbf{J}$ has a unique maximal compact open subgroup J and if we put $\lambda := \Lambda|_J$ then (J, λ) is a maximal simple type in the sense of [6] §6.

Definition 4.1. *The pair* (J, λ) *is a* maximal simple type *if one of the following holds:*

(a) $J = J(\beta, \mathfrak{A})$ *is a subgroup associated to a simple stratum* $[\mathfrak{A}, n, 0, \beta]$, *such that if we write* $E = F[\beta]$ *and* $B = \mathrm{End}_E(V)$ *then* $\mathfrak{B} = \mathfrak{A} \cap B$ *a maximal* $\mathfrak{o}_E$*-order in* B. *Moreover, there exists a simple character* $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta, \psi_F)$ *such that*

$$\lambda \cong \kappa \otimes \sigma,$$

where κ *is a* β*-extension of the unique irreducible representation* η *of* $J^1 = J^1(\beta, \mathfrak{A})$, *which contains* θ, *and* σ *is the inflation to* J *of a cuspidal representation of* $J/J^1 \cong \mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B}) \cong \mathrm{GL}_r(\mathfrak{k}_E)$.

(b) $(J, \lambda) = (\mathbf{U}(\mathfrak{A}), \sigma)$, *where* $\mathfrak{A}$ *is a maximal hereditary* $\mathfrak{o}_F$*-order in* A *and* σ *is an inflation of a cuspidal representation of* $\mathbf{U}(\mathfrak{A})/\mathbf{U}^1(\mathfrak{A}) \cong \mathrm{GL}_N(\mathfrak{k}_F)$.

In case (a), $\mathbf{J} = E^{\times} J$, and in case (b), $\mathbf{J} = F^{\times}\mathbf{U}(\mathfrak{A})$. In practice we will treat (b) as a special case of (a), with $\beta = 0$, $E = F$, $\mathfrak{B} = \mathfrak{A}$, $J^1 = H^1 = \mathbf{U}^1(\mathfrak{A})$ and θ, η, κ all trivial. We will refer to (b) as the level zero case.

We are going to describe the main result of this section. Let $\mathcal{F} = \{V_i : 1 \leq i \leq N\}$ be **any** maximal F-flag in V, let U be the unipotent radical of the G-stabiliser of $\mathcal{F}$, and let $\psi_\alpha : U \to \mathbb{C}^\times$ be **any** non-degenerate character of U. Let π be a supercuspidal representation of G. We know by [3] Proposition 1.6 that there exists $(\mathbf{J}, \Lambda)$ as above, such that

$$\pi \cong \text{c-Ind}_{\mathbf{J}}^{G} \Lambda, \quad \mathrm{Hom}_{U \cap \mathbf{J}}(\psi_\alpha, \Lambda) \neq 0.$$

Moreover we know by Proposition 1.3 that the above properties determine such a pair $(\mathbf{J}, \Lambda)$ up to conjugation by $u \in U$. Since $\Lambda|_{H^1} = (\dim \Lambda)\theta$, we obtain that

$$\theta(u) = \psi_\alpha(u), \quad \forall u \in U \cap H^1.$$

Since J normalises H^1 and intertwines θ, we may define:

Definition 4.2. *Let* $\Psi_\alpha : (J \cap U)H^1 \to \mathbb{C}^\times$ *be the character given by*

$$\Psi_\alpha(uh) = \psi_\alpha(u)\theta(h), \quad \forall u \in J \cap U, \quad \forall h \in H^1.$$

We also define the following subgroups:

Definition 4.3. *Set*

$$\mathcal{M}_F = \{g \in G : (g-1)V \subseteq V_{N-1}\}, \quad \mathcal{M}_{\mathfrak{A}} = (\mathcal{M}_F \cap \mathbf{U}(\mathfrak{A}))\mathbf{U}^1(\mathfrak{A}),$$

and

$$\mathcal{G}_F = \{g \in G : gv_1 = v_1, \ \forall v_1 \in V_1\}, \quad \mathcal{G}_{\mathfrak{A}} = (\mathcal{G}_F \cap \mathbf{U}(\mathfrak{A}))\mathbf{U}^1(\mathfrak{A}).$$

Let $\mathfrak{K}(\mathfrak{A}) = \{g \in G : g^{-1}\mathfrak{A}g = \mathfrak{A}\}$ be the G-normaliser of $\mathfrak{A}$ and let ρ be the representation of $\mathfrak{K}(\mathfrak{A})$ given by

$$\rho = \operatorname{Ind}_{\mathbf{J}}^{\mathfrak{K}(\mathfrak{A})} \Lambda.$$

The main result of this Section is the following Theorem.

Theorem 4.4. (i) *The restriction* $\Lambda|_{\mathcal{M}_{\mathfrak{A}} \cap J}$ *is an irreducible representation of* $\mathcal{M}_{\mathfrak{A}} \cap J$*. Moreover,*

$$\Lambda|_{\mathcal{M}_{\mathfrak{A}} \cap J} \cong \operatorname{Ind}_{(J \cap U)H^1}^{\mathcal{M}_{\mathfrak{A}} \cap J} \Psi_\alpha.$$

(ii) *The restriction* $\rho|_{\mathcal{M}_{\mathfrak{A}}}$ *is an irreducible representation of* $\mathcal{M}_{\mathfrak{A}}$*. Moreover,*

$$\rho|_{\mathcal{M}_{\mathfrak{A}}} \cong \operatorname{Ind}_{(J \cap U)H^1}^{\mathcal{M}_{\mathfrak{A}}} \Psi_\alpha.$$

Further, both (i) *and* (ii) *hold if we replace* $\mathcal{M}_{\mathfrak{A}}$ *with* $\mathcal{G}_{\mathfrak{A}}$*.*

The strategy is to show that Theorem 4.4 holds for a particular choice of U and ψ_α, constructed from Theorem 3.3, and then show that the general result may be obtained by conjugating by some $g \in \mathbf{J}$. Before proceeding with the proof we note that Ψ_α occurs in Λ with multiplicity 1, since ψ_α occurs in Λ with multiplicity 1 and the proof of [3] Lemma 3.1 implies:

Corollary 4.5. *The character* Ψ_α *occurs in* π *with multiplicity* 1*.*

We note that the level 0 case can be formally recovered from the general case with $\beta = 0$ and θ the trivial character, and is a well known result of Gel′fand [7].

4.1 Some decompositions

We will need some decompositions of G, and also of other general linear groups, so we state the following Theorem for a general field F.

Theorem 4.6. *Let* F *be any field, let* V *be an* N*-dimensional* F*-vector space, set* $\mathrm{G} = \mathrm{Aut}_{\mathrm{F}}(\mathrm{V})$*. Let* K *be an extension of* F *of degree* N *and suppose that we are given an embedding of algebras* $\iota : \mathrm{K} \hookrightarrow \mathrm{End}_{\mathrm{F}}(\mathrm{V})$*. Then the following hold:*

(i) *Let* $v \in \mathrm{V}$*,* $v \neq 0$*, and put* $\mathcal{G}_v = \{g \in \mathrm{G} : gv = v\}$*; then every* $g \in \mathrm{G}$ *can be uniquely decomposed as*

$$g = xh, \quad x \in \iota(\mathrm{K}^\times), \quad h \in \mathcal{G}_v;$$

(ii) *Let* $\mathrm{V}' \subset \mathrm{V}$ *be an* F*-subspace of* V *of dimension* $N-1$ *and set* $\mathcal{M} = \{g \in \mathrm{G} : (g-1)\mathrm{V} \subseteq \mathrm{V}'\}$*; then every* $g \in \mathrm{G}$ *can be uniquely decomposed as*

$$g = xh, \quad x \in \iota(\mathrm{K}^\times), \quad h \in \mathcal{M}.$$

Proof. We can view V as a K-vector space, via ι. Since, $[\mathrm{K} : \mathrm{F}] = \dim_{\mathrm{F}} \mathrm{V}$, we obtain $\dim_{\mathrm{K}} \mathrm{V} = 1$. Hence $\mathrm{K}^\times$ acts transitively on the set of non-zero vectors in V and that $\mathrm{G} = \iota(\mathrm{K}^\times)\mathcal{G}_v$. If $x \in \mathcal{G}_v$ then it has eigenvalue 1, and so $\iota(\mathrm{K}^\times) \cap \mathcal{G}_v = \{1\}$. This establishes Part (i).

Choose a basis $\mathcal{B} = \{v_1, \ldots, v_N\}$ of V such that $\mathrm{V}' = \mathrm{F}v_1 + \cdots + \mathrm{F}v_{N-1}$. Let $\delta : \mathrm{G} \to \mathrm{G}$ be the map $g \mapsto \mathrm{w}(g^\top)^{-1}\mathrm{w}$, where $\mathrm{w} \in \mathrm{G}$ is defined on $\mathcal{B}$ by $\mathrm{w}(v_i) = v_{N-i+1}$, and $g^\top$ denotes the transpose of g with respect to the basis $\mathcal{B}$. We have $\delta^2 = \mathrm{id}$ and $\delta(\mathcal{M}) = \mathcal{G}_{v_1}$. Part (ii) follows from Part (i) with ι replaced with $\delta \circ \iota$ and $v = v_1$. □

We apply our decomposition theorem to prove several results on the intersection with the groups $\mathcal{M}_F$ and $\mathcal{G}_F$. Analogously, we set

$$\mathcal{M}_E = \{g \in B^\times : (g-1)V \subseteq V_{N-d}\}, \quad \mathcal{M}_{\mathfrak{B}} = (\mathcal{M}_E \cap \mathbf{U}(\mathfrak{B}))\mathbf{U}^1(\mathfrak{B})$$

and

$$\mathcal{G}_E = \{g \in B^\times : gw_1 = w_1\}, \quad \mathcal{G}_{\mathfrak{B}} = (\mathcal{G}_E \cap \mathbf{U}(\mathfrak{B}))\mathbf{U}^1(\mathfrak{B}).$$

Let K be a maximal unramified extension of E, which normalises $\mathfrak{A}$, so that $[K : E] = N/d$. Theorem 4.6 implies that $B^\times = K^\times \mathcal{M}_E$. Since $V_{N-d} \subseteq V_{N-1}$, we obtain that

$$\mathcal{M}_F \cap B^\times = \mathcal{M}_E.$$

Corollary 4.7. *The following decompositions hold:*

(i) $\mathfrak{K}(\mathfrak{A}) = K^{\times}(\mathbf{U}(\mathfrak{A}) \cap \mathcal{M}_F), \quad \mathbf{U}^1(\mathfrak{A}) = (\mathbf{U}^1(\mathfrak{A}) \cap \mathcal{M}_F)(1 + \mathfrak{p}_K)$;

(ii) $\mathbf{J} = K^{\times}(J \cap \mathcal{M}_F), \quad J^1 = (J^1 \cap \mathcal{M}_F)(1 + \mathfrak{p}_K)$;

(iii) $\mathfrak{K}(\mathfrak{B}) = K^{\times}(\mathbf{U}(\mathfrak{B}) \cap \mathcal{M}_E), \quad \mathbf{U}^1(\mathfrak{B}) = (\mathbf{U}^1(\mathfrak{B}) \cap \mathcal{M}_E)(1 + \mathfrak{p}_K)$.

Moreover, the statements remain true if we replace $\mathcal{M}_F$ with $\mathcal{G}_F$ and $\mathcal{M}_E$ with $\mathcal{G}_E$.

Proof. Note, that $K^{\times} \subset \mathfrak{K}(\mathfrak{B}) \subset \mathbf{J} \subset \mathfrak{K}(\mathfrak{A})$. It is enough to prove (i), since (ii) and (iii) are obtained by intersecting with $\mathbf{J}$ and $\mathfrak{K}(\mathfrak{B})$, respectively. According to Theorem 4.6 we may write $\mathfrak{K}(\mathfrak{A}) = K^{\times}(\mathfrak{K}(\mathfrak{A}) \cap \mathcal{M}_F)$.

Let $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ be the $\mathfrak{o}_E$-lattice chain in V associated to $\mathfrak{A}$. According to [13] Theorem II-1 there exists a decomposition of $V = \sum_{i=1}^{N} V^i$ into one dimensional subspaces such that $V_{N-1} = \sum_{i=1}^{N-1} V^i$ and $L_j = \sum_{i=1}^{N} L_j \cap V^i$, for all $j \in \mathbb{Z}$. If $g \in \mathfrak{K}(\mathfrak{A}) \cap \mathcal{M}_F$ then by projecting to the V^N subspace we obtain $L_{j+v_{\mathfrak{A}}(g)} \cap V^N = L_j \cap V^N$, for all $j \in \mathbb{Z}$. This implies that $v_{\mathfrak{A}}(g) = 0$, hence $g \in \mathbf{U}(\mathfrak{A}) \cap \mathcal{M}_F$. Hence $\mathfrak{K}(\mathfrak{A}) \cap \mathcal{M}_F = \mathbf{U}(\mathfrak{A}) \cap \mathcal{M}_F$.

Let $g \in \mathbf{U}^1(\mathfrak{A})$. We may write by above $g = hx$, where $h \in \mathbf{U}(\mathfrak{A}) \cap \mathcal{M}_F$ and $x \in \mathfrak{o}_K^{\times}$. Let $\bar{g}$, $\bar{h}$ and $\bar{x}$ be the images of g, h, x in $\mathrm{Aut}_{\mathfrak{k}_F}(L_m/L_{m+1})$, where $L_m \in \mathcal{L}$, such that $L_{m+1} + L_m \cap V_{N-1} \neq L_m$. Since $g \in \mathbf{U}^1(\mathfrak{A})$, we have $\bar{g} = 1$. Now

$$(\bar{h} - 1)(L_m/L_{m+1}) \subseteq (L_m \cap V_{N-1} + L_{m+1})/L_{m+1}.$$

Our assumption on L_m, implies that $\bar{h}$ has eigenvalue 1. Since $\bar{h}\bar{x} = 1$, $\bar{x}$ also has eigenvalue 1, hence $\bar{x} = 1$, which implies that $x \in 1 + \mathfrak{p}_K$, and so $h \in \mathbf{U}^1(\mathfrak{A}) \cap \mathcal{M}_F$. □

Corollary 4.8. *We have*

$$\mathfrak{K}(\mathfrak{A}) = \mathcal{M}_{\mathfrak{A}}\mathbf{J}, \quad \mathcal{M}_{\mathfrak{A}} \cap \mathbf{J} = (\mathcal{M}_F \cap \mathbf{J})J^1 = \mathcal{M}_{\mathfrak{B}}J^1.$$

Moreover, the statement is true if we replace $\mathcal{M}_{\mathfrak{A}}$ with $\mathcal{G}_{\mathfrak{A}}$; $\mathcal{M}_F$ with $\mathcal{G}_F$ and $\mathcal{M}_{\mathcal{B}}$ with $\mathcal{G}_{\mathcal{B}}$.

We end this section with an observation which will prove useful later.

Lemma 4.9. *Suppose that $g \in \mathfrak{K}(\mathfrak{A})$ then*

$$v_F(\det g) = \frac{N v_{\mathfrak{A}}(g)}{e(\mathfrak{A}|\mathfrak{o}_F)}.$$

In particular, $\mathbf{J} \cap U = J \cap U$.

Proof. Let $e = e(\mathfrak{A}|\mathfrak{o}_F)$ and set $h = g^e \varpi_F^{-v_{\mathfrak{A}}(g)}$, then $h \in \mathfrak{K}(\mathfrak{A})$ and $v_{\mathfrak{A}}(h) = 0$. Hence, $h \in \mathbf{U}(\mathfrak{A})$. Since $\mathbf{U}(\mathfrak{A})$ is compact, this implies that $\det h \in \mathfrak{o}_F^\times$ and this yields the Lemma. □

4.2 The proof of Theorem 4.4 in a special case

Let us fix a maximal simple type (J, λ). Let $[\mathfrak{A}, n, 0, \beta]$ be an associated simple stratum, and let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ be such that $\lambda|_{H^1} = (\dim \lambda)\theta$. Let $(\mathcal{F}, \chi, b)$ be a triple given by Theorem 3.3 and let U be the unipotent radical of the G-stabiliser of $\mathcal{F}$. By construction, a subflag $\mathcal{F}_0 := \{V_{di} : 1 \le i \le r\}$ of $\mathcal{F}$ is a maximal stable E-flag in V, and let $\mathcal{B}_E = \{w_1, \dots w_r\}$ denote an E-basis of V chosen as in Corollary 3.4. We choose some $a \in X_{\mathcal{F}}$ such that $\chi = \psi_a$, and we set

$$\alpha = a + b.$$

Then we have

$$\psi_\alpha(u) = \psi_a(u)\psi_b(u) = \psi_a(u) = \theta(u), \quad \forall u \in U \cap H^1.$$

The first equality is trivial; the second holds, since $b \in \mathfrak{A}$ and $u \in \mathbf{U}^1(\mathfrak{A})$; the third is Theorem 3.3(ii). Hence we may define $\Psi_\alpha : (J \cap U)H^1 \to \mathbb{C}^\times$ by $\Psi_\alpha(uh) = \psi_\alpha(u)\theta(h)$, for $u \in J \cap U$ and $h \in H^1$, as above.

Lemma 4.10. *The function ψ_b defines a linear character on $(J \cap U)J^1$ and*

$$\Psi_\alpha(j) = \Theta(j)\psi_b(j), \quad \forall j \in (J \cap U)H^1$$

where the character Θ is given by $\Theta(uh) = \psi_a(u)\theta(h)$, for $u \in (J \cap U)$ and $h \in H$, as in Theorem 2.6.

Proof. Since $b \in X_{\mathcal{F}}$, by Lemma 1.1, ψ_b defines a linear character of U. Since $b \in \mathfrak{A}$, we have $\psi_b(u) = 1$, for all $u \in \mathbf{U}^1(\mathfrak{A})$. This implies that

$$\psi_b(uj) = \psi_b(u), \quad \forall u \in U \cap J, \quad j \in J^1.$$

Now J normalises J^1 and hence ψ_b is a character on $(J \cap U)J^1$. Since $\alpha = a+b$, we have an equality of functions $\psi_\alpha = \psi_a\psi_b$ and hence for every $u \in J \cap U$ and $h \in H^1$ we have

$$\Psi_\alpha(uh) = \psi_b(u)\psi_a(u)\theta(h) = \psi_b(uh)\Theta(uh).$$

□

Let $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ be the $\mathfrak{o}_E$-lattice chain in V associated to $\mathfrak{A}$, and put

$$\mathcal{M}_{\mathfrak{k}_E} = \left\{ b \in \mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B}) : (b-1)(L_0/L_1) \subseteq \sum_{j=1}^{r-1} \mathfrak{k}_E(w_j + L_1) \right\},$$

the image of $\mathcal{M}_{\mathfrak{B}}$ in $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$. Similarly, let

$$\mathcal{G}_{\mathfrak{k}_E} = \{b \in \mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B}) : b(w_1 + L_1) = w_1 + L_1\}$$

be the image of $\mathcal{G}_{\mathfrak{B}}$ in $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$.

The group $\mathcal{M}_{\mathfrak{k}_E}$ is known as a mirabolic subgroup. We note that it contains the image of $\mathbf{U}(\mathfrak{B}) \cap U$ in $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$. Moreover, with respect to the basis $w_1 + L_1, ..., w_r + L_1$ of L_0/L_1, the group $\mathcal{M}_{\mathfrak{k}_E}$ is identified with the subgroup of matrices of the form

$$\begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

and the image of $\mathbf{U}(\mathfrak{B}) \cap U$ in $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$ is identified with the subgroup of unipotent upper-triangular matrices.

Lemma 4.11. *Let σ be a cuspidal representation of $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$. Let $U_{\mathfrak{k}_E}$ be the image of $\mathbf{U}(\mathfrak{B}) \cap U$ in $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$, so that*

$$U_{\mathfrak{k}_E} \cong (\mathbf{U}(\mathfrak{B}) \cap U)/(\mathbf{U}^1(\mathfrak{B}) \cap U),$$

and let ψ be any non-degenerate character of $U_{\mathfrak{k}_E}$. Then

$$\sigma|_{\mathcal{M}_{\mathfrak{k}_E}} \cong \mathrm{Ind}_{U_{\mathfrak{k}_E}}^{\mathcal{M}_{\mathfrak{k}_E}} \psi, \quad \sigma|_{\mathcal{G}_{\mathfrak{k}_E}} \cong \mathrm{Ind}_{U_{\mathfrak{k}_E}}^{\mathcal{G}_{\mathfrak{k}_E}} \psi.$$

Moreover, the representations $\sigma|_{\mathcal{M}_{\mathfrak{k}_E}}$ and $\sigma|_{\mathcal{G}_{\mathfrak{k}_E}}$ are irreducible representations of $\mathcal{M}_{\mathfrak{k}_E}$ and $\mathcal{G}_{\mathfrak{k}_E}$, respectively.

Proof. The statement for $\mathcal{M}_{\mathfrak{k}_E}$ is [7] Theorem 8. Set $\bar{w}_i = w_i + L_1$, for $1 \le i \le r$, and $\mathcal{B}_{\mathfrak{k}_E} = \{\bar{w}_1, \dots, \bar{w}_r\}$, a $\mathfrak{k}_E$-basis of L_0/L_1. We identify $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$ with $\mathrm{GL}_r(\mathfrak{k}_E)$, via $\mathcal{B}_{\mathfrak{k}_E}$. Let $\delta : \mathrm{GL}_r(\mathfrak{k}_E) \to \mathrm{GL}_r(\mathfrak{k}_E)$ be the automorphism given by $\delta(g) = \mathrm{w}(g^\top)^{-1}\mathrm{w}$, where $g^\top$ denotes the transpose of g and w is given by $\mathrm{w}(\bar{w}_i) = \bar{w}_{r-i+1}$, for $1 \le i \le r$. Then $\delta(U_{\mathfrak{k}_E}) = U_{\mathfrak{k}_E}$, and ψ^δ, given by $\psi^\delta(u) = \psi(\delta(u))$, for all $u \in U_{\mathfrak{k}_E}$, is a non-degenerate character. The representation of $\mathrm{GL}_r(\mathfrak{k}_E)$, given by $\sigma^\delta(g) = \sigma(\delta(g))$ is cuspidal. Hence, by the statement for $\mathcal{M}_{\mathfrak{k}_E}$,

$$\sigma^\delta|_{\mathcal{M}_{\mathfrak{k}_E}} \cong \mathrm{Ind}_{U_{\mathfrak{k}_E}}^{\mathcal{M}_{\mathfrak{k}_E}} \psi^\delta.$$

Since $\delta(\mathcal{G}_{\mathfrak{k}_E}) = \mathcal{M}_{\mathfrak{k}_E}$ and $\delta^2 = \mathrm{id}$, by twisting by δ we obtain

$$\sigma|_{\mathcal{G}_{\mathfrak{k}_E}} \cong \mathrm{Ind}_{U_{\mathfrak{k}_E}}^{\mathcal{G}_{\mathfrak{k}_E}} \psi.$$

The irreducibility follows from the irreducibility of $\sigma^\delta|_{\mathcal{M}_{\mathfrak{k}_E}}$. □

Theorem 4.12. *The restriction $\lambda|_{\mathcal{M}_{\mathfrak{B}}J^1}$ is an irreducible representation of $\mathcal{M}_{\mathfrak{B}}J^1$ and*

$$\lambda|_{\mathcal{M}_{\mathfrak{B}}J^1} \cong \mathrm{Ind}_{(J\cap U)H^1}^{\mathcal{M}_{\mathfrak{B}}J^1} \Psi_\alpha.$$

The statement remains true if we replace $\mathcal{M}_{\mathfrak{B}}$ with $\mathcal{G}_{\mathfrak{B}}$.

Proof. Set $\mathcal{M} = \mathcal{M}_{\mathfrak{B}}$. In the level 0 case, $\mathcal{M} = \mathcal{M}_{\mathfrak{A}}$ and the Theorem asserts that $\sigma|_{\mathcal{M}_{\mathfrak{A}}}$ is irreducible and

$$\sigma|_{\mathcal{M}_{\mathfrak{A}}} \cong \mathrm{Ind}_{(U\cap \mathbf{U}(\mathfrak{A}))\mathbf{U}^1(\mathfrak{A})}^{\mathcal{M}_{\mathfrak{A}}} \psi_b.$$

Since σ is an inflation of a cuspidal representation and $\mathcal{F}$ and b were chosen in Corollary 3.5 so that $\overline{\psi}_b$ is non-degenerate, the assertion is Lemma 4.11.

Let us consider the general case. Since σ is an inflation of a cuspidal representation of $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$ and $\mathcal{F}$ and b were chosen in Theorem 3.3, so that the character $\overline{\psi}_b$ is non-degenerate, we may again apply Lemma 4.11 to obtain

$$\lambda|_{\mathcal{M}J^1} \cong \kappa|_{\mathcal{M}J^1} \otimes \sigma|_{\mathcal{M}J^1} \cong \kappa|_{\mathcal{M}J^1} \otimes \mathrm{Ind}_{(J\cap U)J^1}^{\mathcal{M}J^1} \psi_b \cong \mathrm{Ind}_{(J\cap U)J^1}^{\mathcal{M}J^1} \kappa \otimes \psi_b.$$

Theorem 2.6 implies that $\kappa|_{(J\cap U)J^1} \cong \mathrm{Ind}_{(J\cap U)H^1}^{(J\cap U)J^1} \Theta$ and hence

$$\lambda|_{\mathcal{M}J^1} \cong \mathrm{Ind}_{(J\cap U)J^1}^{\mathcal{M}J^1} \mathrm{Ind}_{(J\cap U)H^1}^{(J\cap U)J^1} \Theta \otimes \psi_b \cong \mathrm{Ind}_{(J\cap U)H^1}^{\mathcal{M}J^1} \Psi_\alpha$$

where the last isomorphism is given by Lemma 4.10 and the transitivity of induction. Moreover, since by Lemma 4.11 the restriction $\sigma|_{\mathcal{M}J^1}$ is irreducible, a straightforward modification of [6] (5.3.2) implies that $\lambda|_{\mathcal{M}J^1}$ is an irreducible representation of $\mathcal{M}J^1$. The proof for $\mathcal{G}_{\mathfrak{B}}$ is analogous. □

Since, by Corollary 4.8, we have $\mathcal{M}_{\mathfrak{A}} \cap \mathbf{J} = \mathcal{M}_{\mathfrak{B}}J^1$, we have now proved Theorem 4.4(i) in our special case. We also record the following: Since $\mathbf{J} \cap U = J \cap U$, we immediately get

Corollary 4.13. $\mathrm{Hom}_{U\cap \mathbf{J}}(\psi_\alpha, \Lambda) \neq 0$.

Finally, part (ii) of Theorem 4.4 is given by:

Proposition 4.14. *The restrictions of ρ to $\mathcal{M}_{\mathfrak{A}}$ and to $\mathcal{G}_{\mathfrak{A}}$ are irreducible representations of $\mathcal{M}_{\mathfrak{A}}$ and $\mathcal{G}_{\mathfrak{A}}$, respectively. Moreover,*

$$\rho|_{\mathcal{M}_{\mathfrak{A}}} \cong \mathrm{Ind}_{(J\cap U)H^1}^{\mathcal{M}_{\mathfrak{A}}} \Psi_\alpha, \quad \rho|_{\mathcal{G}_{\mathfrak{A}}} \cong \mathrm{Ind}_{(J\cap U)H^1}^{\mathcal{G}_{\mathfrak{A}}} \Psi_\alpha.$$

Proof. We will prove statement for $\mathcal{M}_{\mathfrak{A}}$, the proof for $\mathcal{G}_{\mathfrak{A}}$ is analogous. We have

$$\rho|_{\mathcal{M}_{\mathfrak{A}}} \cong \mathrm{Ind}_{\mathcal{M}_{\mathfrak{A}}\cap \mathbf{J}}^{\mathcal{M}_{\mathfrak{A}}} \Lambda|_{\mathcal{M}_{\mathfrak{A}}\cap \mathbf{J}} \cong \mathrm{Ind}_{\mathcal{M}_{\mathfrak{B}} J^1}^{\mathcal{M}_{\mathfrak{A}}} \lambda|_{\mathcal{M}_{\mathfrak{B}} J^1} \cong \mathrm{Ind}_{(J\cap U)H^1}^{\mathcal{M}_{\mathfrak{A}}} \Psi_\alpha$$

where the last two isomorphisms follow from Corollary 4.8 and Theorem 4.12. Since $\Psi_\alpha|_{H^1} = \theta$ we have

$$I_G(\Psi_\alpha) \subseteq I_G(\theta) = J^1 B^\times J^1$$

by [6] (3.3.2). Since $\lambda|_{\mathcal{M}_{\mathfrak{B}} J^1} \cong \mathrm{Ind}_{(J\cap U)H^1}^{\mathcal{M}_{\mathfrak{B}} J^1} \Psi_\alpha$, [6] (4.1.1) and (4.1.5) imply that

$$I_G(\lambda|_{\mathcal{M}_{\mathfrak{B}} J^1}) = \mathcal{M}_{\mathfrak{B}} J^1 I_G(\Psi_\alpha) \mathcal{M}_{\mathfrak{B}} J^1 \subseteq J^1 B^\times J^1.$$

Hence,

$$I_{\mathcal{M}_{\mathfrak{A}}}(\lambda|_{\mathcal{M}_{\mathfrak{B}} J^1}) \subseteq \mathcal{M}_{\mathfrak{A}} \cap (J^1 B^\times J^1) = (\mathbf{U}(\mathfrak{B}) \cap \mathcal{M}_{\mathfrak{A}}) J^1 = \mathcal{M}_{\mathfrak{B}} J^1$$

and hence $\rho|_{\mathcal{M}_{\mathfrak{A}}}$ is irreducible. We note that $\mathcal{M}_{\mathfrak{A}}$ contains $\mathbf{U}^1(\mathfrak{A})$, and hence J^1; $\mathbf{U}(\mathfrak{A}) \cap B^\times = \mathbf{U}(\mathfrak{B})$ and the last equality above follows from Corollary 4.7. □

This completes the proof of Theorem 4.4 for our special choice of $\mathcal{F}$ and ψ_α.

4.3 The proof of Theorem 4.4 in the general case

We will prove Theorem 4.4 in the general case, by showing that after conjugation by some $g \in \mathbf{J}$ we end up in the special case, considered above.

Proof. Let $\mathcal{F}' = \{V_i' : 1 \leq i \leq N\}$ be any maximal F-flag in V, and let U' be the unipotent radical of the G-stabiliser of $\mathcal{F}'$, and let $\psi_{\alpha'}$ be any smooth non-degenerate character of U'. Let π be a supercuspidal representation of G, then there exists a pair $(\mathbf{J}, \Lambda)$, such that $\pi \cong \text{c-}\mathrm{Ind}_{\mathbf{J}}^G \Lambda$ and $\mathrm{Hom}_{U'\cap \mathbf{J}}(\psi_{\alpha'}, \Lambda) \neq 0$, and $\Lambda|_J \cong \lambda$, where J is the maximal compact open subgroup of $\mathbf{J}$, and (J, λ) is a maximal simple type, with the stratum $[\mathfrak{A}, n, 0, \beta]$. Moreover, $\lambda|_{H^1} = (\dim \lambda)\theta$, where $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. We define $\mathcal{M}'_{\mathfrak{A}}$ as in Definition 4.3 and $\Psi_{\alpha'}$ as in Definition 4.2. Let $\mathcal{F} = \{V_i : 1 \leq i \leq N\}$, U, ψ_α and $\mathcal{M}_{\mathfrak{A}}$ be as in §4.2. By Corollary 3.4 the character ψ_α is non-degenerate, and we know that

$$\operatorname{Hom}_{U' \cap \mathbf{J}}(\psi_{\alpha'}, \Lambda) \neq 0, \quad \text{and} \quad \operatorname{Hom}_{U \cap \mathbf{J}}(\psi_\alpha, \Lambda) \neq 0.$$

Hence by [3] Proposition 1.6 (ii), there exists $g \in \mathbf{J}$, such that $U' = U^g$ and $\psi_{\alpha'} = \psi_\alpha^g$. In particular, $\mathcal{F}' = g\mathcal{F}$, and hence $V'_{N-1} = gV_{N-1}$, which implies that $\mathcal{M}'_{\mathfrak{A}} = \mathcal{M}_{\mathfrak{A}}^g$. Since $g \in \mathbf{J}$, we have $J = J^g$, $H^1 = (H^1)^g$, $\theta = \theta^g$. Hence $(J \cap U')H^1 = \big((J \cap U)H^1\big)^g$ and $\Psi_{\alpha'} = \Psi_\alpha^g$. We have proved the Theorem for $\mathcal{F}$ and ψ_α, now twisting by g, we obtain the result for $\mathcal{F}'$ and $\psi_{\alpha'}$. □

Remark 4.15. *It follows from the proof that any $\mathcal{F}'$ and $\psi_{\alpha'}$, with the property that $\operatorname{Hom}_{U' \cap \mathbf{J}}(\psi_{\alpha'}, \Lambda) \neq 0$, arise from the construction in Theorem 3.3, once we replace β by $g\beta g^{-1}$, for some $g \in \mathbf{J}$, and so the construction in Theorem 3.3 is a natural one.*

4.4 A characterisation of $\operatorname{Ind}_{\mathbf{J}}^{\mathfrak{K}(\mathfrak{A})} \Lambda$

We observe that a result of Gel′fand characterising cuspidal representations of $\mathrm{GL}_N(\mathbf{F}_q)$ implies a very similar result, for the representations of $\mathfrak{K}(\mathfrak{A})$ of the form $\operatorname{Ind}_{\mathbf{J}}^{\mathfrak{K}(\mathfrak{A})} \Lambda$.

Proposition 4.16. *Let τ be a representation of $\mathfrak{K}(\mathfrak{A})$ such that*

$$\tau|_{\mathcal{M}_{\mathfrak{A}}} \cong \operatorname{Ind}_{(J \cap U)H^1}^{\mathcal{M}_{\mathfrak{A}}} \Psi_\alpha,$$

then

$$\tau \cong \operatorname{Ind}_{\mathbf{J}}^{\mathfrak{K}(\mathfrak{A})} \Lambda,$$

for some representation Λ of $\mathbf{J}$, such that $(J, \Lambda|_J)$ is a maximal simple type, as in Definition 4.1.

Proof. Since $\operatorname{Ind}_{(J \cap U)H^1}^{\mathcal{M}_{\mathfrak{A}}} \Psi_\alpha$ is irreducible, so is τ. Now,

$$\operatorname{Ind}_{(J \cap U)H^1}^{J} \Psi_\alpha \cong \kappa \otimes \operatorname{Ind}_{(J \cap U)J^1}^{J} \psi_b \cong \prod_\sigma \kappa \otimes \sigma,$$

where the product runs over all the generic representations σ of $J/J^1 \cong \mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B}) \cong \mathrm{GL}_r(\mathfrak{k}_E)$. Hence $\tau|_J$ will contain a summand of the form $\kappa \otimes$

σ. It follows from [6] (5.3.2) that the $I_{\mathbf{U}(\mathfrak{A})}(\kappa \otimes \sigma) \subseteq (JB^{\times}J) \cap \mathbf{U}(\mathfrak{A}) = J$. Hence $\operatorname{Ind}_J^{\mathbf{U}(\mathfrak{A})} \kappa \otimes \sigma$ is irreducible, and so is isomorphic to τ. Restricting to $\mathcal{M}_{\mathfrak{A}}$, we obtain that $\sigma|_{\mathcal{M}_{\mathfrak{B}} J^1}$ is irreducible, and [7] implies that σ is cuspidal. Hence, $\tau|_J$ will contain some simple type λ, it follows from [6] §6.2 that $\tau|_{\mathbf{J}}$ will contain some Λ, and hence $\tau \cong \operatorname{Ind}_{\mathbf{J}}^{\mathfrak{K}(\mathfrak{A})} \Lambda$. □

5 Realisation of maximal simple types

We continue with the situation described in the beginning of §4. Let U be a maximal unipotent subgroup of G, and ψ_α non-degenerate character of U. Let $\pi \cong \text{c-Ind}_{\mathbf{J}}^G \Lambda$ be a supercuspidal representation, and $\operatorname{Hom}_{U\cap \mathbf{J}}(\psi_\alpha, \Lambda) \neq 0$. Theorem 4.4 allows us to use a rather general result of Alperin and James [1], and realize the representation Λ as a concrete space, and describe the action of $\mathbf{J}$ on this space in terms of the character of Λ and Ψ_α. This concrete realization enables us to compute a certain matrix coefficient of π, and by integrating it we obtain an explicit Whittaker function for π.

5.1 Bessel functions

We will adapt the result of Alperin and James [1] to our setting. Let $\mathcal{K}$ be an open, compact-modulo-centre subgroup of G and let τ be an irreducible smooth representation of $\mathcal{K}$.

Assumption 5.1. *Suppose that there exists compact open subgroups $\mathcal{U} \subseteq \mathcal{M} \subseteq \mathcal{K}$, and a linear character Ψ of $\mathcal{U}$, such that the following hold:*

(i) *$\tau|_{\mathcal{M}}$ is irreducible representation of $\mathcal{M}$;*

(ii) *$\tau|_{\mathcal{M}} \cong \operatorname{Ind}_{\mathcal{U}}^{\mathcal{M}} \Psi$.*

Let $\mathcal{N}$ be an open, normal subgroup of $\mathcal{K}$ contained in the $\operatorname{Ker} \tau$. Set

$$e_\Psi = (\mathcal{U} : \mathcal{N})^{-1} \sum_{h \in \mathcal{U}/\mathcal{N}} \Psi(h) h^{-1}.$$

Let $\chi = \chi_\tau$ be the (trace) character of τ and let $\omega = \omega_\tau$ be the central character of τ, so that

$$\chi(xg) = \omega(x)\chi(g), \quad \forall x \in F^{\times}, \quad \forall g \in \mathcal{K}.$$

Definition 5.2. *The* Bessel function $\mathcal{J} : \mathcal{K} \to \mathbb{C}$ *of* τ *is defined by:*

$$\mathcal{J}(g) = \operatorname{tr}_\tau(e_\Psi g) = (\mathcal{U} : \mathcal{N})^{-1} \sum_{h \in \mathcal{U}/\mathcal{N}} \Psi(h^{-1})\chi(gh).$$

Proposition 5.3. *The Bessel function* $\mathcal{J}$ *has the following properties:*

(i) $\mathcal{J}(1) = 1$*;*

(ii) $\mathcal{J}(xg) = \mathcal{J}(gx) = \omega(x)\mathcal{J}(g), \quad \forall x \in F^\times, \quad \forall g \in \mathcal{K}$*;*

(iii) $\mathcal{J}(hg) = \mathcal{J}(gh) = \Psi(h)\mathcal{J}(g), \quad \forall h \in \mathcal{U}, \quad \forall g \in \mathcal{K}$*;*

(iv) *if* $\mathcal{J}(g) \neq 0$ *then* g *intertwines* Ψ*; in particular, if* $m \in \mathcal{M}$ *then* $\mathcal{J}(m) \neq 0$ *if and only if* $m \in \mathcal{U}$*;*

(v) *for all* g_1, $g_2 \in \mathcal{K}$ *we have*

$$\sum_{m \in \mathcal{M}/\mathcal{U}} \mathcal{J}(g_1 m)\mathcal{J}(m^{-1}g_2) = \mathcal{J}(g_1 g_2).$$

Proof. We observe that it is enough to prove the Proposition for a twist of τ by an unramified character. Twisting by $[g \mapsto (\omega(\varpi_F))^{-v_F(\det(g))/N}]$ we ensure that $\varpi_F^{\mathbb{Z}}\mathcal{N}$ lies in the kernel of τ. Hence, $\operatorname{Ker}\tau$ is of finite index in $\mathcal{K}$ and we may consider τ as a representation of a finite group.

Part (i) is a reformulation of the fact that $\operatorname{Ind}_{\mathcal{U}}^{\mathcal{M}} \Psi$ is irreducible. Since χ is defined by matrix trace, we have $\chi(gg_1) = \chi(g_1 g)$, for all g, $g_1 \in \mathcal{K}$, and Parts (ii) and (iii) are straightforward consequences of the definition of $\mathcal{J}$.

Part (iv): Part (iii) implies that $\mathcal{J}$ is a $\check{\Psi}$-spherical function on $\mathcal{K}$, in the sense of [6] (4.1), where $\check{\Psi}$ is the dual of Ψ. Hence if $\mathcal{J}(g) \neq 0$ then according to [6] (4.1.1), g intertwines Ψ.

Since $\operatorname{Ind}_{\mathcal{U}}^{\mathcal{M}} \Psi$ is irreducible, the $\mathcal{M}$-intertwining of Ψ is equal to $\mathcal{U}$. Now Parts (i),(iii) and the argument above finish the proof of Part (iv).

Part (v) is [1] Lemma 2, or [7] Theorem 9. □

Theorem 5.4 (cf. [1]). *Let* $\mathcal{S}$ *be the space of functions from* $\mathcal{M}$ *to* $\mathbb{C}$ *satisfying the condition*

$$f(hm) = \Psi(h)f(m), \quad \forall h \in \mathcal{U}, \quad \forall m \in \mathcal{M}.$$

For each $g \in \mathcal{K}$ *we define an operator* $L(g)$ *on* $\mathcal{S}$ *by the formula*

$$[L(g)f](m) = \sum_{m_1 \in \mathcal{M}/\mathcal{U}} \mathcal{J}(mgm_1)f(m_1^{-1}).$$

Then L *defines a representation of* $\mathcal{K}$*, which is isomorphic to* τ*.*

Proof. Again, it is enough to prove the statement after twisting by unramified character, and this way we may ensure that $\mathcal{K}/\operatorname{Ker}\tau$ is finite. The assertion now follows from the main Theorem in [1]. The level 0 case is [7] Theorem 10. □

If $\check{\tau}$ is the dual of τ, then $\check{\tau}$, $\mathcal{M}$, $\mathcal{U}$ and $\check{\Psi}$ satisfy Assumption 5.1. Hence Theorem 5.4 holds for $\check{\tau}$, with $\check{\Psi}$ instead of Ψ and with $\check{\mathcal{S}}$ the space of functions from $\mathcal{M}$ to $\mathbb{C}$ satisfying the condition

$$f(hm) = \check{\Psi}(h)f(m), \quad \forall h \in \mathcal{U}, \quad \forall m \in \mathcal{M}.$$

Moreover, the Bessel function $\check{\mathcal{J}} = \mathcal{J}_{\check{\tau}}$ satisfies

$$\check{\mathcal{J}}(g) = \mathcal{J}(g^{-1}), \quad \forall g \in \mathcal{K}.$$

Let $(\ ,\ )$ be a non-degenerate $\mathcal{K}$-invariant pairing on $\mathcal{S} \times \check{\mathcal{S}}$. Since, τ is irreducible, the pairing is determined up to a scalar multiple. Let $\varphi \in \mathcal{S}$ and $\check{\varphi} \in \check{\mathcal{S}}$ be such that $\operatorname{Supp}\varphi = \operatorname{Supp}\check{\varphi} = \mathcal{U}$ and $\varphi(1) = \check{\varphi}(1) = 1$. Since φ and $\check{\varphi}$ span the Ψ- and $\check{\Psi}$-isotypical subspaces in $\mathcal{S}$ and $\check{\mathcal{S}}$ respectively, we may normalise $(\ ,\ )$, so that

$$(\varphi, \check{\varphi}) = 1.$$

This determines the pairing uniquely.

Lemma 5.5. *We have*

$$(L(g)\varphi, \check{\varphi}) = \mathcal{J}(g), \quad \forall g \in \mathcal{K}.$$

Proof. It follows from Theorem 5.4 that

$$L(g)\varphi = \sum_{m \in \mathcal{M}/\mathcal{U}} \mathcal{J}(m^{-1}g)L(m)\varphi.$$

Since the $\mathcal{M}$-intertwining of Ψ is just $\mathcal{U}$, for $m \in \mathcal{M}$ we have $(L(m)\varphi, \check{\varphi}) \neq 0$ if and only if $m \in \mathcal{U}$. This implies the Lemma. □

We will apply the preceding results in several situations but, for now, we observe that Theorem 4.4 implies (in the notation of §4):

Theorem 5.6. *Assumption 5.1 (and hence Proposition 5.3 and Theorem 5.4) holds in the following contexts:*

(i) $\mathcal{K} = \mathfrak{K}(\mathfrak{A})$, $\tau = \rho$, $\mathcal{M} = \mathcal{M}_{\mathfrak{A}}$ *or* $\mathcal{M} = \mathcal{G}_{\mathfrak{A}}$, $\mathcal{U} = (J \cap U)H^1$, $\Psi = \Psi_\alpha$;

(ii) $\mathcal{K} = \mathbf{J}$, $\tau = \Lambda$, $\mathcal{M} = \mathcal{M}_{\mathfrak{A}} \cap J$ *or* $\mathcal{M} = \mathcal{G}_{\mathfrak{A}} \cap J$, $\mathcal{U} = (J \cap U)H^1$, $\Psi = \Psi_\alpha$.

If π has level 0 then we recover the result of Gel′fand [7].

5.2 Explicit Whittaker functions

Now we argue along the lines of [3] §3. However, not only do we get a uniqueness statement as in [3], but we also obtain explicit formulae in terms of the character of Λ and Ψ_α.

Let $\mathcal{V}$ be the underlying vector space of π and let $(\check{\pi}, \check{\mathcal{V}})$ be the smooth dual of $(\pi, \mathcal{V})$. We denote by $\langle , \rangle$ the pairing on $\mathcal{V} \times \check{\mathcal{V}}$ given by the evaluation. Let $v_\alpha \in \mathcal{V}$ and $\check{v}_\alpha \in \check{\mathcal{V}}$ be non-zero vectors such that $\pi(h)v_\alpha = \Psi_\alpha(h)v_\alpha$ and $\check{\pi}(h)v_\alpha = \check{\Psi}_\alpha(h)\check{v}_\alpha$, for all $h \in (J \cap U)H^1$. Corollary 4.5 implies that such vectors exist and they are unique up to scalar multiple. We rescale so that $\langle v_\alpha, \check{v}_\alpha \rangle = 1$.

Proposition 5.7. *The representation π admits a unique coefficient function $f = f_{\alpha,U}$ with the following properties:*

(i) $f(1) = 1$, *and*

(ii) $f(h_1 g h_2) = \Psi_\alpha(h_1 h_2) f(g), \quad \forall h_1, h_2 \in (J \cap U)H^1, \quad \forall g \in G.$

Moreover, $\operatorname{Supp} f \subseteq \mathbf{J}$ *and*

$$f(g) = \langle \pi(g)v_\alpha, \check{v}_\alpha \rangle = \mathcal{J}(g), \quad \forall g \in \mathbf{J}$$

where $\mathcal{J} = \mathcal{J}_\Lambda$ is the Bessel function.

Proof. For Parts (i) and (ii) we argue as in the proof of [3] Proposition 3.2. If we set $f(g) = \langle \pi(g)v_\alpha, \check{v}_\alpha \rangle$, then f satisfies (i) and (ii); the uniqueness is implied by Corollary 4.5.

If $\langle g v_\alpha, \check{v}_\alpha \rangle \neq 0$ then $e_\Psi \pi(g) v_\alpha \neq 0$ and, since $v_\alpha \in \mathcal{V}^\Lambda$, Corollary 4.5 implies $e_\Lambda(\pi(g)\mathcal{V}^\Lambda) \neq 0$. Hence g intertwines Λ and so $g \in \mathbf{J}$.

If $g \in \mathbf{J}$ then Theorem 5.4 and Lemma 5.5 imply that

$$\langle \pi(g)v_\alpha, \check{v}_\alpha \rangle = (L(g)\varphi, \check{\varphi}) = \mathcal{J}(g).$$

□

Theorem 5.8. *Let du be an invariant Haar measure on U, normalised so that $\int_{U \cap \mathbf{J}} du = 1$. Let $\Upsilon : \pi \to \operatorname{Ind}_U^G \psi_\alpha$ be a linear map given by*

$$v \mapsto [g \mapsto \int_U \psi_\alpha(u) \langle \pi(u^{-1}g)v, \check{v}_\alpha \rangle du].$$

Then Υ *is non-zero and* G*-equivariant. Moreover,* $\operatorname{Supp}\Upsilon(v_\alpha) \subseteq U\mathbf{J}$ *and*

$$[\Upsilon(v_\alpha)](ug) = \psi_\alpha(u)\mathcal{J}(g), \quad \forall u \in U, \quad \forall g \in \mathbf{J}.$$

Further, $\operatorname{Supp}\Upsilon(v_\alpha) \cap \mathcal{M}_F = U(H^1 \cap \mathcal{M}_F)$ *and*

$$[\Upsilon(v_\alpha)](uh) = \psi_\alpha(u)\theta(h), \quad \forall u \in U, \quad \forall h \in H^1 \cap \mathcal{M}_F.$$

Proof. The first assertion follow directly from Proposition 5.7. Now

$$\operatorname{Supp}\Upsilon(v_\alpha) \cap \mathcal{M}_F \subseteq (U\mathbf{J}) \cap \mathcal{M}_F = U(\mathbf{J} \cap \mathcal{M}_F) = U(J \cap \mathcal{M}_{\mathfrak{A}}) \cap \mathcal{M}_F,$$

where the last equality follows from Corollary 4.8. The second assertion now follows from Proposition 5.3(iv). □

Remark 5.9. *The Whittaker function* $\Upsilon(v_\alpha)$ *above, and the bound on the support, can be obtained by integrating the matrix coefficient appearing in [2] (see also [10]) and this is sufficient for the purposes of [5]. However, the fact that we can realize the representation* Λ *via Bessel functions gives us the precise knowledge of* $\operatorname{Supp}\Upsilon(v_\alpha) \cap \mathcal{M}_F$*. We use this in §7 to compute, in some cases, epsilon factors of pairs.*

Corollary 5.10. *Let* $\mathcal{J}_\rho$ *be the Bessel function of* ρ*. For* $g \in \mathfrak{K}(\mathfrak{A})$*,*

$$\mathcal{J}_\rho(g) = \begin{cases} \mathcal{J}(g) & \text{if } g \in \mathbf{J}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is implied by the uniqueness of the matrix coefficient in Proposition 5.7. □

5.3 Multiplicative property

Our maximal simple types are of the form (J, λ), where $\lambda = \kappa \otimes \sigma$. In this section, we show that the Bessel function associated to the extension Λ of λ to $\mathbf{J}$ can be split as a product of two Bessel function (see Proposition 5.13).

Lemma 5.11. *There exists a representation* $\tilde{\kappa}$ *of* $\mathbf{J}$ *such that* $\tilde{\kappa}|_J \cong \kappa$*. Moreover, given such a representation* $\tilde{\kappa}$*, there exists a unique representation* Σ *of* $\mathbf{J}$*, such that* $\Sigma|_J \cong \sigma$ *and* $\Lambda \cong \tilde{\kappa} \otimes \Sigma$*.*

Proof. We may extend the action of J on κ to the action of $F^\times J$, by making some uniformiser ϖ_F act trivially. By definition of κ, [6] (5.2.1), $E^\times$, (and hence $\mathbf{J}$) intertwines κ. Since, $F^\times J$ is normal in $\mathbf{J}$ and the quotient $\mathbf{J}/(F^\times J) \cong E^\times/F^\times \mathfrak{o}_E^\times$ is a cyclic group, we may extend the action to $\mathbf{J}$.

Now suppose that we are given a representation $\tilde{\kappa}$ of $\mathbf{J}$ such that $\tilde{\kappa}|J \cong \kappa$. By the same argument as above, there exists a representation Σ of $\mathbf{J}$ such that $\Sigma|_J \cong \sigma$. Moreover, we may ensure that $\omega_{\tilde{\kappa}}(\varpi_F)\omega_\Sigma(\varpi_F) = \omega_\Lambda(\varpi_F)$, where ω denotes the central character. Hence

$$\Lambda|_{F^\times J} \cong (\tilde{\kappa} \otimes \Sigma)|_{F^\times J}.$$

Thus Λ is a direct summand of

$$\mathrm{Ind}_{F^\times J}^{\mathbf{J}} \tilde{\kappa} \otimes \Sigma \cong \tilde{\kappa} \otimes \Sigma \otimes \mathrm{Ind}_{F^\times J}^{\mathbf{J}} \mathbf{1} \cong \bigoplus_\chi \tilde{\kappa} \otimes \Sigma \otimes \chi,$$

where χ runs over characters of $\mathbf{J}/F^\times J \cong E^\times/F^\times \mathfrak{o}_E^\times$. Hence, after replacing Σ by some $\Sigma \otimes \chi$, we may ensure that $\Lambda \cong \tilde{\kappa} \otimes \Sigma$. Moreover, by [6] §6 we know that $\Lambda \cong \Lambda \otimes \chi$ implies that χ is the trivial character. Hence, such Σ is unique. □

Let us now fix some $\tilde{\kappa}$ as above and let Σ be the unique representation of $\mathbf{J}$, given by Lemma 5.11. Let (U, ψ_α) be as in §4.2. In particular, we require that $U \cap B^\times$ is a maximal unipotent subgroup of $B^\times$ and we may write $\psi_\alpha = \psi_a \psi_b$, such that ψ_a is trivial on $U \cap B^\times$; ψ_b is a non-degenerate character on $U \cap B^\times$, which descends to a non-degenerate character of a maximal unipotent subgroup of $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$ (see Theorem 3.3(iv)).

Lemma 5.12. *Assumption 5.1 (and hence Proposition 5.3 and Theorem 5.4) holds in the following contexts:*

(i) $\mathcal{K} = \mathbf{J}$, $\tau = \tilde{\kappa}$, $\mathcal{M} = J^1$, $\mathcal{U} = (J^1 \cap U)H^1$, $\Psi = \Theta$, *where* $\Theta(uh) = \psi_\alpha(u)\theta(h)$, *for* $u \in J^1 \cap U$ *and* $h \in H^1$.

(ii) $\mathcal{K} = \mathbf{J}$, $\tau = \Sigma$, $\mathcal{M} = \mathcal{M}_\mathfrak{A} \cap J$ *or* $\mathcal{G}_\mathfrak{A} \cap J$, $\mathcal{U} = (J \cap U)J^1$, $\Psi = \psi_b$.

Proof. Part (i) is just Theorem 2.6. Since $\mathbf{J}/J^1 \cong \mathfrak{K}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$ and J^1 acts trivially on Σ, part (ii) is given by Theorem 5.6(i) (together with Corollary 4.8) applied in the level zero case. □

Theorem 5.6 and Lemma 5.12 imply that we may associate Bessel functions to Λ, $\tilde{\kappa}$ and Σ, via Definition 5.2.

Proposition 5.13. *We have*

$$\mathcal{J}_\Lambda(g) = \mathcal{J}_{\tilde{\kappa}}(g)\mathcal{J}_\Sigma(g), \quad \forall g \in \mathbf{J}.$$

Proof. We have $\Lambda \cong \tilde{\kappa} \otimes \Sigma$, and $\Psi_\alpha = \Theta\psi_b$, see Lemma 4.10. Let $\mathcal{V}_{\tilde{\kappa}}$ and $\mathcal{V}_\Sigma$ be the underlying vector spaces of $\tilde{\kappa}$ and Σ, respectively. We claim that

$$e_{\Psi_\alpha}(v \otimes w) = (e_\Theta v) \otimes (e_{\psi_b} w), \quad \forall v \in \mathcal{V}_{\tilde{\kappa}}, \quad \forall w \in \mathcal{V}_\Sigma. \tag{$\dagger$}$$

We choose non-zero vectors $v_\Theta \in e_\Theta \mathcal{V}_{\tilde{\kappa}}$ and $w_{\psi_b} \in e_{\psi_b}\mathcal{V}_\Sigma$. It follows from Theorem 2.6 that $\dim e_\Theta \mathcal{V}_{\tilde{\kappa}} = 1$ and from Lemma 4.11 that $\dim e_{\psi_b}\mathcal{V}_\Sigma = 1$. Now $(J \cap U)H^1$ acts on $v_\Theta \otimes w_{\psi_b}$ via Ψ_α and hence Theorem 4.12 implies that the set

$$\{c(v_\Theta \otimes w_{\psi_b}) : \bar{c} \in \mathcal{M}_{\mathfrak{B}} J^1/(J \cap U)H^1\},$$

is a basis of $\mathcal{V}_{\tilde{\kappa}} \otimes \mathcal{V}_\Sigma$, where c denotes a coset representative of a coset $\bar{c}$. It is enough to show the claim ($\dagger$) holds for the elements of this basis.

If $g \in \mathcal{M}_{\mathfrak{B}} J^1$ and $e_{\Psi_\alpha} g(v_\Theta \otimes w_{\psi_b})$ is not equal to zero then g intertwines Ψ_α and hence, by Theorem 4.12, we obtain that $g \in (J \cap U)H^1$. Conversely, if $g \in (J \cap U)H^1$ then $e_{\Psi_\alpha} g(v_\Theta \otimes w_{\psi_b}) = \Psi_\alpha(g) v_\Theta \otimes w_{\psi_b}$.

If $g \in \mathcal{M}_{\mathfrak{B}} J^1$ and $(e_\Theta g v_\Theta) \otimes (e_{\psi_b} g w_{\psi_b}) \neq 0$ then g intertwines ψ_b and hence, by Lemma 4.11, $g \in (J \cap U)J^1$. Moreover, g intertwines Θ and so by Theorem 2.6, $g \in (J \cap U)H^1$. If $g \in (J \cap U)H^1$ then

$$(e_\Theta g v_\Theta) \otimes (e_{\psi_b} g w_{\psi_b}) = \Theta(g)\psi_b(g) v_\Theta \otimes w_{\psi_b} = \Psi_\alpha(g) v_\Theta \otimes w_{\psi_b}.$$

Hence we obtain the claim ($\dagger$).

Now let Θ' be the restriction of Θ to $(J^1 \cap U)H^1$. According to Theorem 2.6, Θ' also occurs in κ with multiplicity one and hence $e_{\Theta'} v = e_\Theta v$, for all $v \in \mathcal{V}_{\tilde{\kappa}}$. Hence,

$$\mathcal{J}_\Lambda(g) = \operatorname{tr}_\Lambda(e_{\Psi_\alpha} g) = \operatorname{tr}_{\tilde{\kappa}}(e_{\Theta'} g) \operatorname{tr}_\Sigma(e_{\psi_b} g) = \mathcal{J}_{\tilde{\kappa}}(g)\mathcal{J}_\Sigma(g).$$

□

6 A numerical invariant

In this section, we will define a certain numerical invariant which appears in our formula for epsilon factors in §7. We continue with the notation of previous sections.

We suppose that $E = F[\beta]$ is maximal in A and we identify V with E. Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. Let $\mathcal{F} = \{V_i : 1 \leq i \leq d\}$ be a maximal F-flag in E; let U be a unipotent radical of $\mathrm{Aut}_F(E)$-stabiliser of $\mathcal{F}$; let χ be a smooth, non-degenerate character of U, such that

$$\chi|_{U \cap H^1} = \theta|_{U \cap H^1};$$

and let ψ_E be an additive character of E, trivial on $\mathfrak{p}_E$, and such that

$$\psi_E(x) = \psi_F(x), \quad \forall x \in F.$$

Definition 6.1. *Choose a basis $\mathcal{B} = \{x_1, \ldots, x_d\}$ of E over F, which satisfies Definition 3.1(i),(iii) and (iv) with (U, χ, ψ_E). We define $\nu \in E^\times/(1 + \mathfrak{p}_E)$, by*

$$\nu = \nu(\theta, \psi_F, \psi_E) = x_d x_1^{-1} \pmod{\mathfrak{p}_E}.$$

Proposition 6.2. *$\nu(\theta, \psi_F, \psi_E)$ depends only on θ, ψ_F and ψ_E.*

Proof. Suppose that we have another triple U', χ', $\mathcal{B}'$ which satisfy the conditions above. Then Proposition 1.3 and the first part of the proof of Theorem 3.3 imply that there exists $g \in \mathbf{J}$ such that $U' = U^g$ and $\chi' = \chi^g$. Since E is maximal, we may write $g = xh$, where $x \in E^\times$ and $h \in J^1$.

Let $\xi \in E$ be the unique element such that $\xi h x_1 = h x_d$. Then $(x_1 x_d^{-1} h^{-1} \xi h) \in \mathcal{G} \cap \mathbf{J}$, where $\mathcal{G} = \{g \in \mathrm{Aut}_F(E); g x_1 = x_1\}$. Now, according to Corollary 4.7, $\mathcal{G} \cap \mathbf{J} = \mathcal{G} \cap J^1$ and hence the image of $(x_1 x_d^{-1} h^{-1} \xi h)$ in $\mathbf{J}/J^1 \cong E^\times/1 + \mathfrak{p}_E$, is equal to 1. This implies that $\xi \equiv x_d x_1^{-1} \pmod{\mathfrak{p}_E}$ and hence we may assume that $U = U'$ and $\chi = \chi'$. The second part of Proposition 3.2 implies that ν does not depend on the choice of basis $\mathcal{B}$. □

Remark 6.3. *Suppose E is not necessarily maximal. Let $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$ and let $\theta_F \in \mathcal{C}(\mathfrak{A}(E), \beta, \psi_F)$ be the simple character corresponding to θ via the correspondence of [6] §3.6, where $\mathfrak{A}(E)$ is the hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(E)$, corresponding to the lattice chain $\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$. Let $\{x_1, ..., x_d\}$ be the F-basis of E given by Corollary 3.4, with $x_1 = 1$. Then it follows from the construction in the proof of Theorem 3.3, that $\nu(\theta_F, \psi_F, \psi_E) \equiv x_d \pmod{\mathfrak{p}_E}$.*

6.1 Behaviour under tame lifting

We continue with the assumption that E is maximal in A and let us further assume that E is totally wildly ramified. Let K be a tame extension of F. The algebra $L = K \otimes_F E$ is a field, which is the compositum of E and K.

Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$; then Bushnell and Henniart in [2] define the *tame lift* of θ, which is a simple character θ^K of $H^1_K = H^1(\beta, \mathfrak{A}_K)$, where $\mathfrak{A}_K$ is the hereditary $\mathfrak{o}_K$-order in $\mathrm{End}_K(L)$ corresponding to the $\mathfrak{o}_K$-lattice chain $\{\mathfrak{p}_L^j : j \in \mathbb{Z}\}$. We will investigate how the invariant ν varies with the tame lifting.

Let ψ_L and ψ_K be the additive characters of L and K, respectively, given by

$$\psi_L(x) = \psi_E(\mathrm{tr}_{L/E}\, x), \quad \forall x \in L, \quad \psi_K(x) = \psi_F(\mathrm{tr}_{K/F}\, x), \quad \forall x \in K.$$

Since K is tame over F, ψ_K has conductor $\mathfrak{p}_K$; likewise, ψ_L has conductor $\mathfrak{p}_L$.

Let U be a maximal unipotent subgroup of G and let $\mathcal{F} = \{V_i : 1 \le i \le d\}$ be a maximal flag corresponding to U. Let χ be a smooth non-degenerate character of U, and let $\mathcal{B} = \{x_1, \ldots, x_d\}$ be an F-basis of E, with respect to which U is the group of unipotent upper-triangular matrices and, if $u \in U$ and $(u_{ij}) \in \mathbb{M}_d(F)$ is a matrix of u with respect to $\mathcal{B}$, then

$$\chi(u) = \psi_F(\sum_{i=1}^{d-1} u_{i,i+1}).$$

Set $\mathcal{F}_K = \{V_i \otimes_F K : 1 \le i \le d\}$ and let U_K be the unipotent radical of the $\mathrm{Aut}_K(L)$-stabiliser of $\mathcal{F}_K$. For $u \in U_K$, write $(u_{ij}) \in \mathbb{M}_d(K)$ for the matrix of u with respect to $\{x_1, \ldots, x_d\}$, and let $\chi^K : U_K \to \mathbb{C}^\times$ be the character given by

$$\chi^K(u) = \psi_K(\sum_{i=1}^{d-1} u_{i,i+1}).$$

Proposition 6.4. *We have* $\theta|_{U \cap H^1} = \chi|_{U \cap H^1}$ *if and only if*

$$\theta^K|_{U_K \cap H^1_K} = \chi^K|_{U_K \cap H^1_K}.$$

Proof. By [2] Corollary 9.13(iii), tame lifting is transitive in the field extension: if K' is a subfield of K containing F, then

$$(\theta^{K'})^K = \theta^K.$$

So it is enough to prove the Proposition when K/F is Galois, cyclic, and either unramified or totally tamely ramified, as in [2] (12.2). Let Γ be the Galois group of K/F and fix a generator σ of Γ. For $g \in \mathrm{Aut}_K(L)$, let N_σ be the cyclic norm map, given by $N_\sigma g = g\sigma(g)\cdots\sigma^{l-1}(g)$, where $l = [K : F]$. Define $\mathfrak{H}^1_F$, $\mathfrak{H}^1_K$, $\mathfrak{U}_F$, $\mathfrak{U}_K$ by

$$H^1 = 1 + \mathfrak{H}^1_F, \quad H^1_K = 1 + \mathfrak{H}^1_K, \quad U = 1 + \mathfrak{U}_F, \quad U_K = 1 + \mathfrak{U}_K.$$

We observe that the proof of [2] (12.3) Proposition (including the results required from [2]§11) goes through if we replace $\mathfrak{H}^1_F$ with $\mathfrak{H}^1_F \cap \mathfrak{U}_F$ and $\mathfrak{H}^1_K$ with $\mathfrak{H}^1_K \cap \mathfrak{U}_K$. We obtain the following:

(i) For $x \in H^1_K \cap U_K$, there exists $u \in H^1_K \cap U_K$ such that $y_x = ux\sigma(u)^{-1}$ satisfies $N_\sigma y_x \in H^1 \cap U$.

(ii) The map $x \mapsto N_\sigma y_x$ induces a bijection between σ-conjugacy classes in $H^1_K \cap U_K$ and conjugacy classes in $H^1 \cap U$.

Now, [2] (12.6), and the fact that both θ^K and χ^K are stable under Γ, imply that

$$\theta^K(x) = \theta^K(y_x) = \theta(N_\sigma y_x), \quad \chi^K(x) = \chi^K(y_x) = \chi(N_\sigma y_x).$$

The above coupled with (ii) gives the Proposition. □

Remark 6.5. *The above Proposition would follow easily from [3] Lemma 2.10, if the gap in its proof were fixed.*

Corollary 6.6. *We have* $\nu(\theta^K, \psi_K, \psi_L) \equiv \nu(\theta, \psi_F, \psi_E) \pmod{\mathfrak{p}_L}$.

Proof. Let (U, χ) be such that $\theta|_{H^1 \cap U} = \chi|_{H^1 \cap U}$, let $\mathcal{B} = \{x_1, \ldots, x_d\}$ be an F-basis of E, which satisfies Definition 3.1(i),(iii) and (iv), with respect to U,χ,ψ_F and ψ_E. Propositions 6.2 and 6.4 imply that it is enough to show the following: If $a \in L$ and (a_{ij}) is the matrix of $a \in \mathrm{End}_K(L)$ with respect to $\mathcal{B}$ then $\psi_L(a) = \psi_K(a_{dd})$.

Since ψ_L and ψ_K are additive, and $L = K \otimes_F E$, it is enough to prove this for $a = c \otimes b$, where $c \in K$ and $b \in E$. Let (b_{ij}) be a matrix of b with respect to $\mathcal{B}$ then $b_{ij} \in F$ and $a_{ij} = cb_{ij}$. Hence,

$$\psi_K(a_{dd}) = \psi_F(b_{dd} \operatorname{tr}_{K/F} c) = \psi_E(b \operatorname{tr}_{K/F} c) = \psi_E(\operatorname{tr}_{L/E}(bc)) = \psi_L(a).$$

□

7 Application to ε-factors of pairs

We will use the Whittaker function constructed in Theorem 5.8 to compute ε-factor of pairs in the following situation:

As before, let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A, such that $e(\mathfrak{B}|\mathfrak{o}_E) = 1$. Let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$be a simple character and let κ and η be representations of $J = J(\beta, \mathfrak{A})$ and $J^1 = J^1(\beta, \mathfrak{A})$, as in Definition 4.1. Let σ_1 and σ_2 be lifts of cuspidal representations of $J/J^1 \cong \mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B}) \cong \mathrm{GL}_r(\mathfrak{k}_E)$ to J. We allow the case $\sigma_1 \cong \sigma_2$. For $i = 1, 2$, set $\lambda_i = \kappa \otimes \sigma_i$ and let π_i be a supercuspidal representation of G such that $\pi_i|_J$ contains λ_i. According to [6] §6, there exists an irreducible representation Λ_i of $\mathbf{J} = E^\times J$, such that $\Lambda_i|_J \cong \lambda_i$ and

$$\pi_i \cong \text{c-Ind}_{\mathbf{J}}^G \Lambda_i.$$

We fix some extension $\tilde{\kappa}$ of κ to $\mathbf{J}$, as in Lemma 5.11. For $i = 1, 2$, let Σ_i be the unique representation of $\mathbf{J}$, also given by Lemma 5.11, such that $\Lambda_i \cong \tilde{\kappa} \otimes \Sigma_i$ and $\Sigma_i|_J \cong \sigma_i$; we view Σ_i as a representation of $\mathfrak{K}(\mathfrak{B}) = E^\times \mathbf{U}(\mathfrak{B})$ and set

$$\tau_i = \text{c-Ind}_{\mathfrak{K}(\mathfrak{B})}^{B^\times} \Sigma_i.$$

Then τ_1 and τ_2 are level zero supercuspidal representations of $B^\times \cong \mathrm{GL}_r(E)$. Let $\mathfrak{A}(E)$ be the hereditary order in $\mathrm{End}_F(E)$, corresponding to the lattice chain $\{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$. Let $\theta_F \in \mathcal{C}(\mathfrak{A}(E), n, 0, \beta)$be the simple character corresponding to θ via the correspondence of [6] §3.6.

Theorem 7.1. *Choose an additive, unitary character $\psi_E : E \to \mathbb{C}^\times$, such that ψ_E is trivial on $\mathfrak{p}_E$ and $\psi_E(x) = \psi_F(x)$, for all $x \in F$. Then*

$$\varepsilon(\pi_1 \times \check{\pi}_2, s, \psi_F) = \zeta \omega_{\tau_1}(\nu^{-r}) \omega_{\tau_2}(\nu^r) q^{(s-1/2) r v_E(\nu) N/e} \varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E),$$

where: $\nu = \nu(\theta_F, \psi_F, \psi_E) \in E^\times/(1 + \mathfrak{p}_E)$ is the invariant defined in Definition 6.1; $r = \dim_E(V)$; $\check{\pi}$ denotes the contragredient of π; $q = q_F$ is the cardinality of $\mathfrak{k}_F$; and $\zeta = \omega_{\tau_2}(-1)^{r-1} \omega_{\pi_2}(-1)^{N-1}$.

We remark that, although the representation τ_1 and τ_2 depend on the choice of β-extension κ, and the choice of $\tilde{\kappa}$, the ε-factor in Theorem 7.1 does not. For a different choice of $\tilde{\kappa}$ would twist τ_1 and τ_2 by the same tamely ramified character χ and we have

$$\varepsilon(\tau_1 \chi \times \check{\tau}_2 \chi^{-1}, s, \psi_E) = \varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E).$$

In §7.5, we use our invariant ν to recover (and generalise) certain results in [5] on the behaviour of ε-factors of pairs under twists by tamely ramified characters.

7.1 Preparation

Let U be a maximal unipotent subgroup of G, which is the G-stabiliser of the maximal flag $\mathcal{F} = \{V_i : 1 \leq i \leq N\}$, and let $\psi_\alpha = \chi \psi_b$ be a smooth non-degenerate character of U, as constructed in Theorem 3.3. Let $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ be the lattice chain in V corresponding to $\mathfrak{A}$. Recall that, in Corollary 3.4, we constructed an F-basis $\mathcal{B}_F = \{v_1, \ldots, v_N\}$ of V, with respect to which U

is the group of upper triangular matrices and ψ_α is the 'standard' character. Moreover, the vectors v_i are of the form

$$v_{d(i-1)+j} = x_d^{i-1} x_j w_i, \quad 1 \leq i \leq r, \quad 1 \leq j \leq d,$$

where $\mathcal{B}_E = \{w_1, \ldots, w_r\}$ is an E-basis of V such that $L_0 = \sum_{i=1}^r \mathfrak{o}_E w_i$, and $\{x_1, \ldots, x_d\}$ is an F-basis of E, which depends on ψ_E (see Corollary 3.4). Further, $x_1 = 1$. Whenever it is required of us we will identify G with $\mathrm{GL}_N(F)$ via $\mathcal{B}_F$. Let $\mathrm{w} \in G$ be the element defined on the basis $\mathcal{B}_F$ by

$$\mathrm{w}(v_i) = v_{N-i+1}, \quad 1 \leq i \leq N.$$

We also define an involution $\delta : G \to G$, by

$$\delta(g) = \mathrm{w} g^{\top -1} \mathrm{w},$$

where $g^\top$ is the transpose of g with respect to $\mathcal{B}_F$ and $g^{\top -1} = (g^\top)^{-1} = (g^{-1})^\top$.

We will briefly recall the definition of ε-factors of pairs, using the the formulation of Jacquet, Piatetskii-Shapiro and Shalika [8], rather than Shahidi [11].

Let $\mathcal{W}_1 = \mathcal{W}(\pi, \psi_\alpha)$ and $\mathcal{W}_2 = \mathcal{W}(\check{\pi}_2, \overline{\psi}_\alpha)$ be the Whittaker models of π_1 and $\check{\pi}_2$ respectively. Let $\mathcal{S}(F^N)$ be the set of compactly-supported, locally constant functions $\phi : F^N \to \mathbb{C}$. We denote by $\mathrm{e}_1 = (1, 0, \ldots, 0), \ldots, \mathrm{e}_N = (0, \ldots, 0, 1)$ the standard basis of F^N. Given $W_1 \in \mathcal{W}_1$, $W_2 \in \mathcal{W}_2$ and $\Phi \in \mathcal{S}(F^N)$, we define the zeta function

$$Z(W_1, W_2, \Phi, s) = \int_{U \backslash G} W_1(g) W_2(g) \Phi(\mathrm{e}_N g) |\det g|^s dg,$$

where dg is a G-equivariant measure on $U \backslash G$. Note that, under our identification of G with $\mathrm{GL}_N(F)$, via $\mathcal{B}_F$, the term $\mathrm{e}_N g$ is the N^{th} row of the matrix of g with respect to $\mathcal{B}_F$. The integral converges absolutely for $\mathrm{Re}(s)$ sufficiently large, and is a rational function of q^{-s}. This zeta function satisfies a functional equation, [8](2.7):

$$\frac{Z(\widetilde{W_1}, \widetilde{W_2}, \hat{\Phi}, 1-s)}{L(\check{\pi}_1 \times \pi_2, 1-s)} = \omega_{\pi_2}(-1)^{N-1} \varepsilon(\pi_1 \times \check{\pi}_2, s, \psi_F) \frac{Z(W_1, W_2, \Phi, s)}{L(\pi_1 \times \check{\pi}_2, s)},$$

where, for $i = 1, 2$, $\widetilde{W_i}(g) = W_i(\mathrm{w} g^{\top -1})$; $\hat{\Phi}$ is the Fourier transform of Φ, given by

$$\hat{\Phi}(\mathrm{y}) = \int_{F^N} \Phi(\mathrm{x}) \psi_F(\mathrm{x}\mathrm{y}^\top) d\mathrm{x}, \quad \forall \mathrm{y} \in F^N,$$

where $d\mathrm{x}$ is normalised so that $\hat{\hat{\Phi}}(\mathrm{x}) = \Phi(-\mathrm{x})$; and $L(\pi_1 \times \check{\pi}_2, s)$ is the L-function. Since π_1 and π_2 are supercuspidal, it is enough for our purposes to know that

$$L(\pi_1 \times \check{\pi}_2, s) = \prod_{\chi} L(\chi, s),$$

where the product is taken over all the unramified characters $\chi : F^\times \to \mathbb{C}^\times$ such that $\pi_1 \cong \pi_2 \otimes \chi \circ \det$, and $L(\chi, s)$ is as in Tate's thesis [12]. In particular, if $\sigma_1 \not\cong \sigma_2$ then the product is taken over an empty set and so $L(\pi_1 \times \check{\pi}_2, s) = 1$. If $\sigma_1 \cong \sigma_2$ then there exists χ as above such that $\pi_1 \cong \pi_2 \otimes \chi \circ \det$. In this case, it follows from [6] (6.2.5) that

$$L(\pi_1 \times \check{\pi}_2, s) = (1 - \chi(\varpi_F)^{-N/e} q^{-sN/e})^{-1},$$

where $e = e(\mathfrak{A}|\mathfrak{o}_F) = e(E|F)$.

7.2 Computation

Let $W_1 \in \mathcal{W}(\pi_1, \psi_\alpha)$ and $W_2 \in \mathcal{W}(\check{\pi}_2, \overline{\psi}_\alpha)$ be the Whittaker functions constructed in Theorem 5.8. Then $\operatorname{Supp} W_1 \subseteq U\mathbf{J}$, $\operatorname{Supp} W_2 \subseteq U\mathbf{J}$ and

$$W_1(ug) = \psi_\alpha(u)\mathcal{J}_{\Lambda_1}(g), \quad W_2(ug) = \overline{\psi}_\alpha(u)\mathcal{J}_{\check{\Lambda}_2}(g), \quad \forall u \in U, \quad \forall g \in \mathbf{J}.$$

Set $\mathcal{J}_1 = \mathcal{J}_{\Lambda_1}$ and $\mathcal{J}_2 = \mathcal{J}_{\check{\Lambda}_2}$, and let $\Phi \in \mathcal{S}(F^N)$ be the indicator function on the set $\mathrm{e}_N J^1$. We are going to compute the zeta functions on both sides of the functional equation for this particular choice of W_1, W_2 and Φ. This will give us Theorem 7.1.

For X a subset of G which is a union of right U-cosets, we write $\mathrm{vol}_U(X)$ for the volume of $U\backslash X$ with respect to the measure du on $U\backslash G$.

Proposition 7.2. *Let* $\mathrm{F} : G \to \mathbb{C}$ *be the function given by*

$$\mathrm{F}(g) = W_1(g)W_2(g)\Phi(\mathrm{e}_N g).$$

Then F *is an indicator function on the set* UH^1*. In particular,*

$$Z(W_1, W_2, \Phi, s) = \mathrm{vol}_U(UH^1).$$

Proof. We have

$$\operatorname{Supp} W_1 \subseteq U\mathbf{J}, \quad \operatorname{Supp} W_2 \subseteq U\mathbf{J}, \quad \operatorname{Supp}[g \mapsto \Phi(\mathrm{e}_N g)] = \mathcal{M}_F J^1,$$

where $\mathcal{M}_F = \{g \in G : (g-1)V \subseteq V_{N-1}\}$ is the mirabolic subgroup, as in §4. Hence,

$$\operatorname{Supp} \mathrm{F} \subseteq U\mathbf{J} \cap \mathcal{M}_F J^1 = U(\mathbf{J} \cap \mathcal{M}_F)J^1 = U\mathcal{M}_{\mathfrak{B}} J^1,$$

where the last equality is given by Corollary 4.8. We have

$$\mathrm{F}(ug) = W_1(g)W_2(g)\Phi(\mathrm{e}_N g) = \mathcal{J}_1(g)\mathcal{J}_2(g), \quad \forall u \in U, \forall g \in \mathcal{M}_{\mathfrak{B}} J^1.$$

It follows from Proposition 5.3(ii) and (iv) that $\mathcal{J}_1(g)\mathcal{J}_2(g) = \Psi_\alpha(g)\overline{\Psi_\alpha}(g) = 1$, if $g \in (J \cap U)H^1$, and $\mathcal{J}_1(g)\mathcal{J}_2(g) = 0$, otherwise. Hence F is an indicator function on the set UH^1. Since H^1 is compact and U is unipotent, we obtain that $|\det g| = 1$, for all $g \in UH^1$. Hence $Z(W_1, W_2, s, \Phi) = \mathrm{vol}_U(UH^1)$. □

We write $\mathcal{G}_F = \{g \in G : gv_1 = v_1\}$, as in §4.

Lemma 7.3. *For all $g_1 \in \mathcal{G}$, $h \in G$ and $g_2 \in (J \cap \mathcal{M}_F)J^1$ we have:*

$$\hat{\Phi}(\mathrm{e}_1(g_1hg_2)^{\top-1}) = \hat{\Phi}(\mathrm{e}_1 h^{\top-1}).$$

Proof. Since $\mathrm{e}_1 g_1^{\top-1} = \mathrm{e}_1$, we obtain $\hat{\Phi}(\mathrm{e}_1(g_1h)^{\top-1}) = \hat{\Phi}(\mathrm{e}_1 h^{\top-1})$. Since Φ is an indicator function on the set $\mathrm{e}_N J^1 = \mathrm{e}_N(J \cap \mathcal{M}_F)J^1$, we have $g_2\Phi = \Phi$, hence

$$\begin{aligned}\hat{\Phi}(\mathrm{e}_1(hg_2)^{\top-1}) &= \int_{F^N} \Phi(\mathrm{x})\psi_F(\mathrm{x}(g_2^{-1}h^{-1}\mathrm{e}_1^\top))d\mathrm{x} \\ &= \int_{F^N} [g_2\Phi](\mathrm{x})\psi_F(\mathrm{x}(h^{-1}\mathrm{e}_1^\top))d\mathrm{x} = \hat{\Phi}(\mathrm{e}_1 h^{\top-1}).\end{aligned}$$

□

For L a lattice in F^N, we write $\mathrm{vol}_F(L)$ for the volume of L with respect to the measure $d\mathrm{x}$ on F^N.

Lemma 7.4. *Let $i, j \in \mathbb{Z}$; then* $\mathrm{vol}_F(\mathrm{e}_N\mathfrak{P}^i) = q^{(j-i)N/e}\,\mathrm{vol}_F(\mathrm{e}_N\mathfrak{P}^j)$.

Proof. Since there exists $\gamma \in \mathfrak{K}(\mathfrak{A})$ such that $\mathrm{e}_N = \mathrm{e}_1\gamma$, and then $\mathrm{e}_N\mathfrak{P}^i = \mathrm{e}_1\gamma\mathfrak{P}^i = \mathrm{e}_1\mathfrak{P}^{i+v_{\mathfrak{A}}(\gamma)}$, it is enough to prove that

$$\mathrm{vol}_F(\mathrm{e}_1\mathfrak{P}^i) = q^{(j-i)N/e}\,\mathrm{vol}_F(\mathrm{e}_1\mathfrak{P}^j).$$

Since our basis $\mathcal{B}_F$ splits the lattice chain we may write

$$\mathfrak{P}^k = \bigoplus_{1 \le i,j \le N} \mathfrak{p}_F^{c_{ij}(k)}\mathbf{1}_{ij},$$

where $\mathbf{1}_{ij} \in A$ are the projections given by $\mathbf{1}_{ij}(v_k) = \delta_{ik} v_j$, for $1 \leq k \leq N$, and δ_{ik} is the Kronecker delta. Hence, $\mathrm{e}_1 \mathfrak{P}^k = \sum_{j=1}^N \mathfrak{p}_F^{c_{1j}(k)} \mathrm{e}_j$. According to [6] (1.1.4) we have

$$\mathfrak{P}^{1-k} = \{a \in A : \psi_F(\mathrm{tr}_{A/F}(xa)) = 1, \forall x \in \mathfrak{P}^k\}, \quad \forall k \in \mathbb{Z},$$

which, since ψ_F has conductor $\mathfrak{p}_F$, implies that $1 - c_{ij}(k) = c_{ji}(1-k)$, for all $k \in \mathbb{Z}$. Since we have chosen v_1, so that $v_1 \in L_0$, $v_1 \notin L_1$ and the lattice chain is principal, we have $\mathfrak{P}^k v_1 = L_k$, for all $k \in \mathbb{Z}$. Now, $\mathfrak{P}^k v_1 = \sum_{j=1}^N \mathfrak{p}_F^{c_{j1}(k)} v_j = \sum_{j=1}^N \mathfrak{p}_F^{1-c_{1j}(1-k)} v_j$. Hence

$$(\mathrm{e}_1 \mathfrak{P}^i : \mathrm{e}_1 \mathfrak{P}^j) = (L_{1-j} : L_{1-i}),$$

where the brackets denote the generalised index. Since $(L_i : L_{i+e}) = q^N$ and $\mathcal{L}$ is principal, we have $(L_i : L_{i+1}) = q^{N/e}$, for all $i \in \mathbb{Z}$, and the result follows. □

We write $q_{\mathfrak{A}} = q^{N/e}$, so Lemma 7.4 says that $\mathrm{vol}_F(\mathrm{e}_N \mathfrak{P}^i) = q_{\mathfrak{A}}^{j-i} \mathrm{vol}_F(\mathrm{e}_N \mathfrak{P}^j)$. Let $\mathrm{w}_E \in \mathbf{U}(\mathfrak{B})$ be the element defined by its action on the basis $\mathcal{B}_E$ by

$$\mathrm{w}_E(w_i) = w_{r-i+1}, \quad 1 \leq i \leq r.$$

From our construction of the bases $\mathcal{B}_E$ and $\mathcal{B}_F$, we have $x_d^r \mathrm{w}_E v_1 = x_d^r w_r = v_N$. In terms of matrices with respect to $\mathcal{B}_F$ we can rephrase this as

$$(x_d^r \mathrm{w}_E) \mathrm{e}_1^\top = \mathrm{e}_N^\top.$$

Lemma 7.5. *Let $\phi : A \to \mathbb{C}^\times$ be the function*

$$\phi : a \mapsto \psi_F((\mathrm{e}_N a)\mathrm{e}_1^\top) = \psi_F(a_{N1}),$$

where (a_{ij}) is the matrix of $a \in A$ with respect to $\mathcal{B}_F$. Then

$$\phi(a x_d^r \mathrm{w}_E) = \psi_F(a_{NN}), \quad \forall a \in A.$$

Hence ϕ defines an additive character on A, which is trivial on $\mathfrak{P}^{1+rv_E(x_d)}$, and non-trivial on $\mathfrak{P}^{rv_E(x_d)}$.

Lemma 7.6. *Let $b \in B$, let (b_{ij}) be the matrix of b with respect to $\mathcal{B}_E$, and define $\phi : A \to \mathbb{C}^\times$ as in Lemma 7.5; then*

$$\phi(b) = \psi_E(x_d^{-r} b_{r1}).$$

Proof. Set $a = bx_d^{-r}\mathrm{w}_E$, let (a_{ij}^F) be the matrix of a with respect to $\mathcal{B}_F$ and let (a_{ij}^E) be the matrix of a with respect to $\mathcal{B}_E$. According to Lemma 7.5, we have $\phi(b) = \psi_F(a_{NN}^F)$. We have $aw_r + V_{N-r} = a_{rr}^E w_r + V_{N-r}$ and, since $x_d \in E$, we obtain $a(x_d^r w_r) + V_{N-r} = a_{rr}^E(x_d^r w_r) + V_{N-r}$. Since $a_{rr}^E \in E$, we may consider it as $a_{rr}^E \in \mathrm{End}_F(E)$. Let (α_{ij}) be the matrix of a_{rr}^E with respect to the basis $\{x_1, \ldots, x_d\}$. Since $v_{d(i-1)+j} = x_d^{i-1}x_j w_i$, for $1 \le i \le r$ and $1 \le j \le d$, and in particular $v_N = x_d^r w_r$, we obtain that

$$av_N + V_{N-1} = \alpha_{dd}v_N + V_{N-1}.$$

In particular, $\alpha_{dd} = a_{NN}$. Recall that the basis $\{x_1, \ldots x_d\}$ was chosen so that $\psi_E(a_{rr}^E) = \psi_F(\alpha_{dd})$, see Definition 3.1(iv). Hence, $\phi(b) = \psi_E(a_{rr}^E)$. Since $x_d \in E$, we obtain that $a_{rr}^E = x_d^{-r}(b\mathrm{w}_E)_{rr} = x_d^{-r}b_{r1}$. □

To ease the notation, we set

$$c = \mathrm{vol}_F(\mathrm{e}_N\mathfrak{P}^{1+rv_E(x_d)}).$$

As in §4.1 let K be a maximal unramified extension of E, such that $K^\times$ normalises $\mathfrak{A}$.

Lemma 7.7. *Let $h \in \mathfrak{K}(\mathfrak{A})$ and set $j = v_{\mathfrak{A}}(h) - v_E(x_d^{-r})$; then*

$$\hat{\Phi}(\mathrm{e}_1 h^{\top -1}) = \begin{cases} 0 & \textit{if } j > 0; \\ c|\det h|\phi(h^{-1}) & \textit{if } j = 0; \\ c|\det h|q_{\mathfrak{A}}^{j} & \textit{if } j < 0. \end{cases}$$

Proof. Corollary 4.7 implies that $\mathrm{e}_N\mathbf{U}^1(\mathfrak{A}) = \mathrm{e}_N J^1 = \mathrm{e}_N(1 + \mathfrak{p}_K)$. Hence,

$$\begin{aligned}\hat{\Phi}(\mathrm{e}_1 h^{\top -1}) &= \int_{\mathrm{e}_N\mathbf{U}^1(\mathfrak{A})} \psi_F((\mathrm{x}h^{-1})\mathrm{e}_1^\top)d\mathrm{x} \\ &= \psi_F((\mathrm{e}_N h^{-1})\mathrm{e}_1^\top)\int_{\mathrm{e}_N\mathfrak{P}} \psi_F((\mathrm{x}h^{-1})\mathrm{e}_1^\top)d\mathrm{x} \\ &= |\det h|\psi_F((\mathrm{e}_N h^{-1})\mathrm{e}_1^\top)\int_{\mathrm{e}_N\mathfrak{P}^{1-v_{\mathfrak{A}}(h)}} \psi_F(\mathrm{x}\mathrm{e}_1^\top)d\mathrm{x}.\end{aligned}$$

Lemmas 7.4, 7.5 and the orthogonality of characters imply the Lemma. □

Lemma 7.8. $\mathrm{vol}_U(U\delta(H^1)) = \mathrm{vol}_U(UH^1)$.

Proof. Let $\mathcal{K}_m = \{g \in \mathrm{GL}_N(\mathfrak{o}_F) : g \equiv 1 \pmod{\mathfrak{p}_F^m}\}$. Since $\mathcal{K}_m$, for $m \ge 1$, form a basis of neighbourhoods of 1 in G, there exists m, such that $\mathcal{K}_m \subseteq H^1 \cap \delta(H^1)$. Since $\delta(U) = U$, $\delta(\mathcal{K}_m) = \mathcal{K}_m$ and the measure on $U\backslash G$ is G-invariant, we obtain that $\mathrm{vol}_U(U\delta(H^1)) = \mathrm{vol}_U(UH^1)$. □

Let $\widetilde{\mathrm{F}} : G \to \mathbb{C}$ be the function given by

$$\widetilde{\mathrm{F}}(g) = \widetilde{W_1}(g)\widetilde{W_2}(g)\hat{\Phi}(\mathrm{e}_N g).$$

We have $\operatorname{Supp}\widetilde{\mathrm{F}} \subseteq \operatorname{Supp}\widetilde{W_1} = \delta(\operatorname{Supp} W_1)\mathrm{w} \subseteq U\delta(\mathbf{J})\mathrm{w}$, and

$$\widetilde{\mathrm{F}}(u\delta(g)\mathrm{w}) = \mathcal{J}_1(g)\mathcal{J}_2(g)\hat{\Phi}(\mathrm{e}_1 g^{\top -1}), \quad \forall g \in \mathbf{J}, \forall u \in U.$$

For $x \in E^\times$ we define $S(x)$, by

$$S(x) = \int_{U\delta(Jx)\mathrm{w}} \widetilde{\mathrm{F}}(g)|\det g|^{1-s} dg.$$

Then $S(x)$ depends only on $v_E(x)$. For $\mathrm{Re}(-s)$ sufficiently large,

$$Z(\widetilde{W_1}, \widetilde{W_2}, \hat{\Phi}, 1-s) = \sum_{x \in \mathfrak{o}_E^\times \backslash E^\times} S(x).$$

Since H^1 is normal in $\mathbf{J}$, Proposition 5.3(iii), and Lemma 7.3 imply that

$$\widetilde{\mathrm{F}}(\delta(h)g) = \widetilde{\mathrm{F}}(g), \quad \forall h \in H^1, \forall g \in G.$$

Hence,

$$S(x) = \mathrm{vol}_U(U\delta(H^1))|\det x|^{s-1} \sum_{h \in (J\cap U)H^1 \backslash J} \mathcal{J}_1(hx)\mathcal{J}_2(hx)\hat{\Phi}(\mathrm{e}_1 (hx)^{\top -1}).$$

We forget the volume term, by using Lemma 7.8 and normalising the measure on $U\backslash G$, so that $\mathrm{vol}_U(U\delta(H^1)) = \mathrm{vol}_U(UH^1) = 1$.

As in §4.1, we put

$$\mathcal{G}_E = \{g \in B^\times : gw_1 = w_1\}, \quad \mathcal{G}_{\mathfrak{B}} = (\mathcal{G}_E \cap \mathbf{U}(\mathfrak{B}))\mathbf{U}^1(\mathfrak{B}),$$

where $\mathcal{B}_E = \{w_1, ..., w_r\}$ is our E-basis of V. Corollary 4.7 implies that we have $J = (\mathcal{G}_{\mathfrak{B}} J^1)\mathfrak{o}_K^\times$. Then, using Lemma 7.3, we obtain:

Lemma 7.9.

$$S(x) = |\det x|^{s-1} \sum_{y \in (1+\mathfrak{p}_K)\backslash \mathfrak{o}_K^\times} \hat{\Phi}(\mathrm{e}_1 (yx)^{\top -1}) \sum_{h \in (J\cap U)H^1 \backslash \mathcal{G}_{\mathfrak{B}} J^1} \mathcal{J}_1(hyx)\mathcal{J}_2(hyx).$$

7.3 The case $\sigma_1 = \sigma_2$

Suppose that $\sigma_1 \cong \sigma_2$; then it follows from [6] (6.2.3) that there exists an unramified quasi-character $\chi : F^\times \to \mathbb{C}^\times$, such that $\Lambda_1 \cong \Lambda_2 \otimes \chi \circ \det$, and hence $\pi_1 \cong \pi_2 \otimes \chi \circ \det$. Then $\mathcal{J}_2(g) = \mathcal{J}_1(g^{-1})\chi(\det g)$, for all $g \in \mathbf{J}$. Hence, for all $g \in \mathbf{J}$, we have

$$\sum_h \mathcal{J}_1(hg)\mathcal{J}_2(hg) = \chi(\det g) \sum_{h \in (J\cap U)H^1 \backslash \mathcal{G}_{\mathfrak{B}} J^1} \mathcal{J}_1(hg)\mathcal{J}_1(g^{-1}h^{-1}) = \chi(\det g),$$

where the last equalities follow from Proposition 5.3(v),(i). It follows from Corollary 4.7 that $(J : \mathcal{G}_{\mathfrak{B}} J^1) = (\mathfrak{o}_K^\times : 1 + \mathfrak{p}_K) = q^{N/e} - 1 = q_{\mathfrak{A}} - 1$. There exists $a \in \mathbb{C}$ such that $\chi(x) = |x|_F^a$, for all $x \in F$. If $x \in E$ then Lemma 4.9 implies that $\chi(\det x) = q_{\mathfrak{A}}^{-a v_E(x)}$. Let $x \in E^\times$ and set $j = v_E(x) - v_E(x_d^{-r})$, Lemmas 4.9 and 7.7 imply that

$$S(x) = \begin{cases} 0 & \text{if } j > 0; \\ c q_{\mathfrak{A}}^{r v_E(x_d)(s-a)} \sum_y \phi(y x_d^r) & \text{if } j = 0; \\ c q_{\mathfrak{A}}^{-(s-1-a)j + (s-a) r v_E(x_d)} (q_{\mathfrak{A}} - 1) & \text{if } j < 0. \end{cases}$$

where, in the sum, y runs over the cosets $\mathfrak{o}_K^\times / 1 + \mathfrak{p}_K$ and ϕ is defined in Lemma 7.5. It follows from 7.5 that ϕ restricted to K defines an additive character, which is trivial on $\mathfrak{p}_K^{r v_E(x_d)+1}$ and non-trivial on $\mathfrak{p}_K^{r v_E(x_d)}$. Hence

$$\sum_{y \in \mathfrak{o}_K^\times / 1 + \mathfrak{p}_K} \phi(y x_d^r) = -\phi(0) = -1.$$

Set $\widetilde{Z} = Z(\widetilde{W}_1, \widetilde{W}_2, \hat{\Phi}, 1 - s)$; then, for $\mathrm{Re}(-s)$ sufficiently large, we obtain that

$$\begin{aligned} \widetilde{Z} &= c q_{\mathfrak{A}}^{r v_E(x_d)(s-a)} (-1 + (q_{\mathfrak{A}} - 1) \sum_{k \geq 1} q_{\mathfrak{A}}^{(s-1-a)k}) \\ &= c q_{\mathfrak{A}}^{r v_E(x_d)(s-a)} \frac{q_{\mathfrak{A}}^{s-a} - 1}{1 - q_{\mathfrak{A}}^{s-1-a}} = c q_{\mathfrak{A}}^{(r v_E(x_d)+1)(s-a)} \frac{L(\check{\pi}_1 \times \pi_2, 1-s)}{L(\pi_1 \times \check{\pi}_2, s)}. \end{aligned}$$

It follows from the functional equation and Proposition 7.2 that

$$\varepsilon(\pi_1 \times \check{\pi}_2, \psi_F, s) = \omega_{\pi_2}(-1)^{N-1} c q_{\mathfrak{A}}^{(s-a)(r v_E(x_d)+1)}.$$

Following [4], we observe that the symmetry in the functional equation implies that $\varepsilon(\pi_1 \times \check{\pi}_1, \psi_F, 1/2)^2 = 1$. Hence,

$$c = \mathrm{vol}_F(\mathrm{e}_N \mathfrak{P}^{1 + r v_E(x_d)}) = q_{\mathfrak{A}}^{-(r v_E(x_d)+1)/2}.$$

We will now prove Theorem 7.1 in the case $\sigma_1 \cong \sigma_2$.

Proof. Note that $\Lambda_1 \cong \Lambda_2 \otimes \chi \circ \det_A$ implies $\Sigma_1 \cong \Sigma_2 \otimes \chi_E \circ \det_B$, where $\chi_E = \chi \circ \mathrm{N}_{E/F}$ and $\mathrm{N}_{E/F}$ denotes the field norm. Moreover, Remark 6.3 implies that $v_E(x_d) = v_E(\nu)$, where $\nu = \nu(\theta_F, \psi_F, \psi_E)$. Hence,

$$q^{av_E(x_d^{-r})N/e} = \chi(\det \nu^{-r}) = \omega_{\tau_1}(\nu^{-r})\omega_{\tau_2}(\nu^r).$$

If we compute $\varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E)$ by the same recipe we obtain that

$$\varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E) = \omega_{\tau_2}(-1)^{r-1} q_{\mathfrak{B}}^{-1/2} q_{\mathfrak{B}}^{s-a},$$

where $q_{\mathfrak{B}} = q_E^r = q_F^{N/e} = q_{\mathfrak{A}}$. Hence,

$$\varepsilon(\pi_1 \times \check{\pi}_2, s, \psi_F) = \zeta \omega_{\tau_1}(\nu^{-r})\omega_{\tau_2}(\nu^r) q^{(s-1/2)rv_E(\nu)N/e} \varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E),$$

where $\zeta = \omega_{\pi_2}(-1)^{N-1}\omega_{\tau_2}(-1)^{r-1}$. □

7.4 The case $\sigma_1 \neq \sigma_2$

Now let us suppose that $\sigma_1 \not\cong \sigma_2$.

Lemma 7.10. *Let $x \in E^\times$. If $v_E(x) \neq v_E(x_d^{-r})$ then $S(x) = 0$.*

Proof. Set $j = v_E(x) - v_E(x_d^{-r})$. If $j > 0$ then Lemma 7.7 gives us $S(x) = 0$. If $j < 0$ then Lemma 7.7 implies that

$$S(x) = c|\det x|^s q_{\mathfrak{A}}^j \sum_{h \in (J\cap U)H^1 \backslash J} \mathcal{J}_1(hx)\mathcal{J}_2(hx).$$

Recall from §5.1 that we have the idempotent e_{Ψ_α} given by

$$e_{\Psi_\alpha} = Q^{-1} \sum_{h \in \mathbf{U}^{n+1}(\mathfrak{A}) \backslash (J\cap U)H^1} \Psi_\alpha(h)h^{-1},$$

where $Q = ((J \cap U)H^1 : \mathbf{U}^{n+1}(\mathfrak{A}))$, and similarly $e_{\overline{\Psi}_\alpha}$. Now,

$$\begin{aligned}\sum_{h \in (J\cap U)H^1 \backslash J} \mathcal{J}_1(hx)\mathcal{J}_2(hx) &= \sum_{h \in (J\cap U)H^1 \backslash J} \operatorname{tr}_{\Lambda_1}(xe_{\Psi_\alpha}h) \operatorname{tr}_{\check{\Lambda}_2}(xe_{\overline{\Psi}_\alpha}h) \\ &= Q^{-1} \operatorname{tr}_{\Lambda_1 \otimes \check{\Lambda}_2}\left(x(e_{\Psi_\alpha} \otimes e_{\overline{\Psi}_\alpha}) \sum_{h \in \mathbf{U}^{n+1}(\mathfrak{A}) \backslash J} h \right),\end{aligned}$$

Since $\sigma_1 \not\cong \sigma_2$ we have $\Lambda_1 \not\cong \Lambda_2$, hence $\operatorname{Hom}_J(\mathbf{1}, \Lambda_1 \otimes \check{\Lambda}_2) = 0$. This implies that

$$\sum_{h \in \mathbf{U}^{n+1}(\mathfrak{A}) \backslash J} \Lambda_1(h) \otimes \check{\Lambda}_2(h) = 0,$$

and so $S(x) = 0$. □

Lemma 7.11. *Let $b \in \mathfrak{K}(\mathfrak{B})$; then*

$$\sum_{h\in(J^1\cap U)H^1\backslash J^1} \mathcal{J}_1(hb)\mathcal{J}_2(hb) = \mathcal{J}_{\Sigma_1}(b)\mathcal{J}_{\check{\Sigma}_2}(b).$$

Proof. According to Proposition 5.13, we have $\mathcal{J}_{\Lambda_i}(g) = \mathcal{J}_{\tilde{\kappa}}(g)\mathcal{J}_{\Sigma_i}(g)$, for $i = 1,2$ and for all $g \in \mathbf{J}$. The assertion follows from the fact that J^1 acts trivially on Σ_i and Proposition 5.3(v),(i) applied to $\tilde{\kappa}$, via Lemma 5.12. □

We now prove Theorem 7.1 in the case when $\sigma_1 \not\cong \sigma_2$.

Proof. Set $\widetilde{Z} = Z(\widetilde{W}_1, \widetilde{W}_2, \hat{\Phi}, 1-s)$. It follows from Lemma 7.10 that, for $\mathrm{Re}(-s)$ sufficiently large, $\widetilde{Z} = S(x_d^{-r})$. Lemmas 7.9, 4.9 and 7.7 imply that

$$\widetilde{Z} = cq_{\mathfrak{A}}^{-rv_E(x_d)s} \sum_{y\in(1+\mathfrak{p}_K)\backslash\mathfrak{o}_K^\times} \phi(y^{-1}x_d^r) \sum_{h\in(J\cap U)H^1\backslash\mathcal{G}_{\mathfrak{B}}J^1} \mathcal{J}_1(hyx_d^{-r})\mathcal{J}_2(hyx_d^{-r}),$$

where $c = q_{\mathfrak{A}}^{(-rv_E(x_d)-1)/2}$. Lemma 7.11 implies that

$$\widetilde{Z} = cq_{\mathfrak{A}}^{rv_E(x_d)s} \sum_{y\in(1+\mathfrak{p}_K)\backslash\mathfrak{o}_K^\times} \phi(y^{-1}x_d^r) \sum_{h\in\mathbf{U}^1(\mathfrak{B}_m)\backslash\mathcal{G}_{\mathfrak{B}}} \mathcal{J}_{\Sigma_1}(hyx_d^{-r})\mathcal{J}_{\check{\Sigma}_2}(hyx_d^{-r}),$$

where $\mathbf{U}^1(\mathfrak{B}_m) = (U \cap \mathbf{U}(\mathfrak{B}))\mathbf{U}^1(\mathfrak{B})$. Now $x_d \in E$ so we can use Proposition 5.3(ii) and Lemma 7.3 to obtain

$$\widetilde{Z} = cq_{\mathfrak{A}}^{rv_E(x_d)s}\omega_{\Sigma_1}(x_d^{-r})\omega_{\Sigma_2}(x_d^r) \sum_{h\in\mathbf{U}^1(\mathfrak{B}_m)\backslash\mathbf{U}(\mathfrak{B})} \phi(hx_d^r)\mathcal{J}_{\Sigma_1}(h^{-1})\mathcal{J}_{\check{\Sigma}_2}(h^{-1}).$$

Lemma 7.6 implies that

$$\widetilde{Z} = cq_{\mathfrak{A}}^{rv_E(x_d)s}\omega_{\Sigma_1}(x_d^{-r})\omega_{\Sigma_2}(x_d^r) \sum_{h\in\mathbf{U}^1(\mathfrak{B}_m)\backslash\mathbf{U}(\mathfrak{B})} \psi_E(h_{r1})\mathcal{J}_{\Sigma_1}(h^{-1})\mathcal{J}_{\check{\Sigma}_2}(h^{-1}),$$

where h_{r1} is the $r1$-coefficient of the matrix of h with respect to the basis $\mathcal{B}_E$. It now follows from the functional equation and Proposition 7.2 that $\varepsilon(\pi_1 \times \check{\pi}_2, s, \psi_F) = \omega_{\pi_2}(-1)^{N-1}\widetilde{Z}$.
If we compute $\varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E)$ by the same recipe, we obtain that

$$\varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E) = q_{\mathfrak{B}}^{-1/2}\omega_{\tau_2}(-1)^{r-1} \sum_{h\in\mathbf{U}^1(\mathfrak{B}_m)\backslash\mathbf{U}(\mathfrak{B})} \psi_E(h_{r1})\mathcal{J}_{\Sigma_1}(h^{-1})\mathcal{J}_{\check{\Sigma}_2}(h^{-1}),$$

where $q_{\mathfrak{B}} = q_E^r = q_F^{N/e} = q_{\mathfrak{A}}$. Hence,

$$\varepsilon(\pi_1 \times \check{\pi}_2, s, \psi_F) = \zeta\omega_{\tau_1}(x_d^{-r})\omega_{\tau_2}(x_d^r)q^{rv_E(x_d)(s-1/2)N/e}\varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E),$$

where $\zeta = \omega_{\tau_2}(-1)^{r-1}\omega_{\pi_2}(-1)^{N-1}$. Now ω_{τ_1} and ω_{τ_2} are trivial on $1+\mathfrak{p}_E$ and Remark 6.3 finishes the proof, as in the case $\sigma_1 \cong \sigma_2$. □

7.5 Twists by tamely ramified quasi-characters

We continue in the same situation as above. Theorem 7.1 immediately implies the following:

Corollary 7.12. *Let $\chi : F^\times \to \mathbb{C}^\times$ be a tamely ramified quasi-character and put $\chi_E = \chi \circ \mathrm{N}_{E/F}$; then*

$$\frac{\varepsilon(\pi_1\chi \times \check{\pi}_2, s, \psi_F)}{\varepsilon(\pi_1 \times \check{\pi}_2, s, \psi_F)} = \chi(\mathrm{N}_{E/F}(\nu^{-r^2}))\frac{\varepsilon(\tau_1\chi_E \times \check{\tau}_2, s, \psi_E)}{\varepsilon(\tau_1 \times \check{\tau}_2, s, \psi_E)},$$

where $\nu = \nu(\theta_F, \psi_F, \psi_E)$.

In the case E is maximal, totally ramified over F and $\pi_1 = \pi_2$ we recover [5]§6.1 Corollaire 2, with (in the notation of [5]) $c(\pi_1, \check{\pi}_1, \psi_F) = \mathrm{N}_{E/F}(\nu)$. Moreover, Corollary 6.6 implies [5]§7.1 Théorème, which describes how the constant $c(\pi_1, \check{\pi}_1, \psi_F)$ changes under the tame lifting operation.

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