Genericity of supercuspidal representations of p-adic Sp_4

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August 20, 2007

Abstract

We describe the supercuspidal representations of $\mathrm{Sp}_4(F)$, for F a non-archimedean local field of residual characteristic different from 2, and determine which are generic.

Introduction

Let F be a locally compact non-archimedean local field, with ring of integers $\mathfrak{o}_F$, maximal ideal $\mathfrak{p}_F$, and residue field k_F. Whereas every irreducible supercuspidal representation of $\mathrm{GL}_N(F)$ is *generic* – i.e., has a Whittaker model – this is no longer true for classical groups over F. The existence of non-generic supercuspidal representations of classical groups is significant and bears many consequences; let us cite a few of them in local representation theory (global consequences are heavy as well, see for instance [12]): First, of course, the fact that the definition of the L-function attached to a representation of a classical group is available only for generic representations, thanks to the work of Shahidi; in particular, a characterization through local factors of the local Langlands correspondence for a classical group is not fully available. Also, reducibility of parabolic induction is completely understood for $\mathrm{GL}_N(F)$ while in classical groups results are complete only in the case of generic inducing representations.

The most celebrated example of a non-generic supercuspidal representation is the representation Θ_{10} of $\mathrm{Sp}_4(F)$, induced from the inflation to $\mathrm{Sp}_4(\mathfrak{o}_F)$ of the cuspidal unipotent representation θ_{10} of $\mathrm{Sp}_4(k_F)$ constructed by Srinivasan (se [12] again). To our knowledge, until recently, most known non-generic supercuspidal representations were level zero, in particular level zero representations induced from the inflation of a cuspidal unipotent representation of the reductive quotient of a maximal special parahoric subgroup. Our purpose in this paper is to exhaust the non-generic supercuspidal representations of $\mathrm{Sp}_4(F)$ in odd residual characteristic.

It is certainly no surprise, and part of the folklore in the subject, that level zero supercuspidal representations of $\mathrm{Sp}_4(F)$ coming from cuspidal representations of $\mathrm{Sp}_4(k_F)$ are generic if and only if the corresponding cuspidal representation is, and that the only non-generic cuspidal representation of $\mathrm{Sp}_4(k_F)$ is Srinivasan's θ_{10}. For the sake of completeness we include proofs of these results. Actually we prove that generic level zero supercuspidal representations of $\mathrm{Sp}_{2N}(F)$ are obtained by inducing the inflation of a generic cuspidal representation of the reductive quotient of a maximal parahoric subgroup of $\mathrm{Sp}_{2N}(F)$, on the condition that this parahoric subgroup is special.

For positive level supercuspidal representations, the situation is not as simple. Our point of view is to use the exhaustive construction given by the second author in [21]: those representations are

*The second author would like to thank the University of Paris 7 for its hospitality in Spring 2005, during which time most of this research was undertaken.

induced from a set of types, generalizing Bushnell-Kutzko types for $\mathrm{GL}_N(F)$. We give necessary and sufficient conditions on those types for the induced representation to be generic. We obtain surprisingly many (at least with respect to our starting point) non-generic representations.

Let us be more precise. Positive level supercuspidal representations of $\mathrm{Sp}_4(F)$ fall into four categories, according to the nature of the skew semi-simple stratum $[\Lambda, n, 0, \beta]$ at the bottom level of the construction (§2.1):

(I) The first category starts with a skew simple stratum with corresponding field extension $F[\beta]$ of dimension 4 over F. Here non-generic representations are obtained only when $F[\beta]/F$ is the biquadratic extension, and when a binary condition involving β and the symplectic form is fulfilled.

(II) The second category starts with a skew simple stratum with corresponding field extension $F[\beta]$ of dimension 2 over F. Non-generic representations are obtained whenever the $F[\beta]$-skew hermitian form attached to the symplectic form is anisotropic (or equivalently, when the quotient group J/J^1 involved in the construction, which is a reductive group over k_F, is anisotropic).

(III) In the third category, the stratum is the orthogonal sum of two two-dimensional skew simple strata $[\Lambda_i, n_i, 0, \beta_i]$, $i = 1, 2$. Non-genericity occurs only when $F[\beta_1]$ is isomorphic to $F[\beta_2]$, and again when a binary condition involving $\beta_1\beta_2$ and the symplectic form is fulfilled.

(IV) All representations in the fourth category – when the stratum is the orthogonal sum of a skew simple stratum and a null stratum, both two-dimensional – are non-generic.

The main character in the proof is indeed a would-be character: To a stratum as above is attached a function ψ_β on $\mathrm{Sp}_4(F)$ and the crucial question is whether there exists a maximal unipotent subgroup U of $\mathrm{Sp}_4(F)$ on which ψ_β actually defines a character (see [5]); this question is easily solved in §3, where Proposition 3.4 lists the exact conditions alluded to above. Whenever there is no such subgroup U, we prove in §5 that the corresponding representations are not generic, using a criterion of non-genericity given in §1.2 (this says essentially that if a cuspidal representation $c\text{-}\mathrm{Ind}_J^{\mathrm{Sp}_4(F)}\lambda$ is generic, then there is a long root subgroup on which λ is trivial).

Now assume that there is a maximal unipotent subgroup U on which ψ_β is a character. A type attached to our stratum is a representation $\kappa \otimes \sigma$ of a compact open subgroup J, where κ is a suitable β-extension attached to the stratum and σ is a cuspidal representation of some finite reductive group J/J^1 attached to the stratum (see §2.1). The fundamental step is Theorem 4.3, stating that the representation κ contains the character ψ_β of $J \cap U$. This implies genericity in cases (I) and (III), where σ is just a character of an anisotropic group. But for cases (II) and (IV) (in which, we should add, ψ_β defines a *degenerate* character of U) we have to deal with the component σ, with opposite effects. In case (II), the inflation of σ contains the restriction to $J \cap U$ of a character of U and genericity follows. On the contrary, in case (IV), the inflation of σ does not contain the restriction to $J \cap U$ of a character of U and the resulting cuspidal representation is not generic.

We have several remarks to add concerning these results. First, genericity for positive level supercuspidal representations of $\mathrm{Sp}_4(F)$ only depends on the stratum itself, on the symplectic form, and on the representation σ (the "level 0 part"). It does not depend on the choice of a semi-simple character attached to the stratum.

The second remark is that the proofs are quite technical, often on a case-by-case basis. The present work can of course be regarded as a first step towards understanding non-genericity in classical groups, but even the case of $\mathrm{Sp}_{2N}(F)$ might not be just an easy generalisation. In particular, the precise conditions for genericity are surprisingly complicated but it seems to us that they do not admit a simple unified description, as for level zero representations. For example, non-generic positive level supercuspidal representations can be induced from either special or non-special maximal compact subgroups of $\mathrm{Sp}_4(F)$, and likewise for generic representations.

Finally, we have deliberately stuck to the construction of supercuspidal representations of $\mathrm{Sp}_4(F)$ via types. Another very fruitful point of view uses Howe's correspondence. Indeed, in a very recent work ([11]), Gan and Takeda study the Langlands correspondence for $\mathrm{GSp}_4(F)$ and obtain in the process a classification of non-generic supercuspidal representations of $\mathrm{GSp}_4(F)$ in terms of Howe's correspondence; they also announce a sequel dealing with $\mathrm{Sp}_4(F)$. A dictionary between these two points of view would of course be very interesting, especially if it can provide some insight about the way non-generic supercuspidal representations of $\mathrm{Sp}_4(F)$ fit into L-packets. For example, in case (I), the genericity of the supercuspidal representation depends on the embedding of β in the symplectic Lie algebra: up to the adjoint action of $\mathrm{Sp}_4(F)$, there are two such embeddings, precisely one of which gives rise to non-generic representations. This suggests that representations in these two sets might be paired to form L-packets. A closer investigation of such phenomena would certainly deserve some effort.

Notation

Let F be a locally compact non-archimedean local field, with ring of integers $\mathfrak{o}_F$, maximal ideal $\mathfrak{p}_F$, residue field k_F and odd residual characteristic $p = \mathrm{char}\, k_F$. On occasion ϖ_F will denote a uniformizing element of F. Similar notations will be used for field extensions of F. We let ν_F denote the additive valuation of F, normalised so that $\nu_F(F^\times) = \mathbb{Z}$. We fix, once and for all, an additive character ψ_F of F with conductor $\mathfrak{p}_F$.

Let V be a $2N$-dimensional F-vector space, equipped with a nondegenerate alternating form h. By a *symplectic basis* for V, we mean an ordered basis $\{e_{-N}, ..., e_{-1}, e_1, ..., e_N\}$ such that, for $1 \le i, j \le N$,

$$h(e_i, e_j) \;=\; h(e_{-i}, e_{-j}) \;=\; 0, \quad h(e_{-i}, e_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Let $A = \mathrm{End}_F(V)$ and let $^-$ denote the adjoint anti-involution on A associated to h, so

$$h(av, w) \;=\; h(v, \bar{a}w), \qquad \text{for } a \in A,\ v, w \in V.$$

We will let G be the corresponding symplectic group $G = \{g \in \mathrm{GL}_F(V) / \bar{g} = g^{-1}\}$, or $G = \mathrm{Sp}_{2N}(F)$ whenever a symplectic basis is fixed. For most of the paper N will be 2. Similarly $\tilde{G}$ will denote either $\mathrm{GL}_F(V)$ or $\mathrm{GL}_{2N}(F)$.

Skew semi-simple strata in A are the basic objects in what follows. We recall briefly the essential notations attached to those objects and refer to [20] for definitions.

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in A. Then Λ is a self-dual lattice sequence in V and defines a decreasing filtration $\{\mathfrak{a}_i(\Lambda), i \in \mathbb{Z}\}$ of A by $\mathfrak{o}_F$-lattices $\mathfrak{a}_i(\Lambda) = \{x \in A / \forall k \in \mathbb{Z},\ x\Lambda(k) \subseteq \Lambda(k+i)\}$. We put $\mathfrak{a}(\Lambda) = \mathfrak{a}_0(\Lambda)$, a self-dual $\mathfrak{o}_F$-order in A. We will also need $P(\Lambda) = \mathfrak{a}(\Lambda)^\times \cap G$ and $P_i(\Lambda) = (1 + \mathfrak{a}_i(\Lambda)) \cap G$ for $i \ge 1$. Note that the lattice sequence Λ gives rise to a valuation ν_Λ on V (6.1) and the filtration $\{\mathfrak{a}_i(\Lambda), i \in \mathbb{Z}\}$ gives rise to a valuation on A, also denoted by ν_Λ.

Next, β is a skew element in A: $\overline{\beta} = -\beta$, and n is a positive integer with $n = -\nu_\Lambda(\beta)$. Furthermore the algebra $E = F[\beta]$ is a sum of fields $E = \oplus_{i=1}^l E_i$, corresponding to a decomposition $V = \perp_{i=1}^l V^i$ of V as an orthogonal direct sum, and accordingly of Λ: $\Lambda = \Sigma_{i=1}^l \Lambda^i$ with $\Lambda^i(k) = \Lambda(k) \cap V^i$, and of β: $\beta = \Sigma_{i=1}^l \beta_i$. The centralizer B of β in A is $B = \oplus_{i=1}^l B_i$ where B_i is the centralizer of β_i in $\mathrm{End}_F(V^i)$.

Last, for β a skew element of A we define ψ_β as the following function on G: $\psi_\beta(x) = \psi\left(\mathrm{tr}(\beta(x-1))\right)$, $x \in G$.

1 Generalities on genericity

1.1 Genericity

The results of this subsection are valid in a much more general setting than the remainder of this paper so we temporarily suspend our usual notations.

Let $G = \mathbf{G}(F)$ be the group of F-rational points of a connected reductive algebraic group $\mathbf{G}$ defined over F. Let $\mathbf{S}$ be a maximal F-split torus in $\mathbf{G}$ with $\mathbf{G}$-centralizer $\mathbf{T}$, let $\mathbf{B}$ be an F-parabolic subgroup of $\mathbf{G}$ with Levi component $\mathbf{T}$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$.

Let χ be a smooth (unitary) character of $U = \mathbf{U}(F)$. The torus $S = \mathbf{S}(F)$ acts on the set of such characters by conjugation. We say that χ is *nondegenerate* if its stabilizer for this action is just the centre Z of G.

Example 1.1. Let $G = \mathrm{Sp}_4(F)$, which we write as a group of matrices with respect to some symplectic basis, let T be the torus of diagonal matrices, and let U be the subgroup of upper triangular unipotent matrices in G. Any character χ of U is given by

$$\chi \begin{pmatrix} 1 & u & x & y \\ 0 & 1 & v & x \\ 0 & 0 & 1 & -u \\ 0 & 0 & 0 & 1 \end{pmatrix} = \psi_F(au + bv),$$

for some $a, b \in F$, and it is easy to see that χ is nondegenerate if and only if a, b are both non-zero. Moreover, there are four orbits of nondegenerate characters of U, given by the class of b in $F^\times/(F^\times)^2$.

Returning to a general connected reductive group G, let π be a smooth irreducible representation of G. We say that π is *generic* if there exist $U = \mathbf{U}(F)$ as above and a nondegenerate character χ of U such that

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \chi) \neq 0.$$

Note that, since all such subgroups U are conjugate in G we may choose to fix one. Moreover, we need only consider nondegenerate characters χ up to T-conjugacy. A basic result here is:

Theorem 1.2 ([15] Theorem 3). *Assume* $\mathbf{G}$ *is split over* F. *Let* π *be a smooth irreducible representation of* G *and let* χ *be a nondegenerate character of a maximal unipotent subgroup* U *of* G. *Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \chi) \leq 1.$$

On the other hand, and again in a general G, when dealing with supercuspidal representations we may not bother about the nondegeneracy of the character in the definition of genericity, a fact that will be useful in the sequel. Indeed:

Lemma 1.3. *Assume* $\mathbf{G}$ *is split over* F*. Let* π *be a smooth irreducible supercuspidal representation of* G*. Let* $\mathbf{U}$ *be a maximal connected unipotent subgroup of* $\mathbf{G}$ *and let* χ *be a character of* $U = \mathbf{U}(F)$ *such that*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \chi) \neq 0.$$

Then the character χ *is nondegenerate.*

Proof. Assume χ is degenerate and use the definition of nondegeneracy in [7], 1.2, as well as the corresponding notation: there is a simple root α such that the restriction of χ to $U_{(\alpha)}$ is trivial. The character χ is then trivial on the subgroup $\langle U_{(\alpha)}, U_{\mathrm{der}} \rangle$ of U.

We claim that this subgroup contains the unipotent radical N of a proper parabolic subgroup P of G. The assumption $\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \chi) \neq 0$ provides us, by Frobenius reciprocity, with a non-zero linear form λ on the space V of π which satisfies $\lambda \circ \pi(u) = \chi(u)\lambda$ for any $u \in U$. Since λ is in particular N-invariant, we get that the space V_N of N-coinvariants is non-zero, which contradicts cuspidality.

We now prove the claim. Let Δ be the set of simple roots as in [7]. From [2], 21.11, the unipotent radical of the standard F-parabolic subgroup $\mathbf{P}_I$ of $\mathbf{G}$ attached to the subset $I = \Delta - \{\alpha\}$ is $\mathbf{U}_{\Psi(I)}$ where $\Psi(I)$ is the set of positive roots that can be written $\alpha + \beta$ with β either 0 or a positive root. Hence the elements in $\Psi(I)$ other than α are positive roots γ of length at least 2, and $\mathbf{U}_{\Psi(I)}$ is directly spanned by $\mathbf{U}_{(\alpha)}$ and the $\mathbf{U}_{(\gamma)}$ for non-divisible such roots γ. From [6], Theorem 4.1, for any such γ we have $\mathbf{U}_{(\gamma)}(F) \subset U_{\mathrm{der}}$ (recall the characteristic of F is not 2), hence $N = \mathbf{U}_{\Psi(I)}(F)$ is contained in $\langle U_{(\alpha)}, U_{\mathrm{der}} \rangle$ as asserted. ■

Now let π be an irreducible supercuspidal representation of G. We suppose, as is the case for all known supercuspidal representations, that π is irreducibly compact-induced from some open compact mod centre subgroup of G. Then (the proof of) [5] 1.6 Proposition immediately gives us the following:

Proposition 1.4. *Let* K *be an open compact mod centre subgroup of* G *and* ρ *an irreducible representation of* K *such that* $\pi = c\text{-}\mathrm{Ind}_K^G \rho$ *is an irreducible, so supercuspidal, representation of* G*. Then* π *is generic if and only if there exist a maximal connected unipotent subgroup* $\mathbf{U}$ *of* $\mathbf{G}$ *and a character* χ *of* $U = \mathbf{U}(F)$ *such that* $\rho | K \cap U$ *contains* $\chi | K \cap U$*. Moreover, if this is the case then* $\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \chi) \neq 0$ *so* χ *is nondegenerate, and the character* $\chi | K \cap U$ *occurs in* $\rho | K \cap U$ *with multiplicity* 1.

1.2 A criterion for non-genericity

Now we look more closely at the situation for symplectic groups so we return to our usual notation: $G = \mathrm{Sp}_{2N}(F)$. In particular, using Proposition 1.4 and a decomposition of G, we will obtain a criterion to determine when an irreducible representation of G is *not* generic.

Let T denote the standard (diagonal) maximal split torus of G, let U be the subgroup of all upper triangular unipotent matrices in G, and put $\mathcal{B} = TU$, the Borel subgroup of all upper triangular matrices in G. Let $\Phi = \Phi(G, T)$ be the root system and, for $\gamma \in \Phi$, let U_γ denote the corresponding root subgroup. Let W denote the Weyl group $N_G(T)/T$; by abuse of notation, we will also use W for a set of representatives in the compact maximal subgroup $K_0 = \mathrm{Sp}_{2N}(\mathfrak{o}_F)$ of G.

We write K_1 for the pro-unipotent radical of K_0, so that $K_0/K_1 \simeq \mathrm{Sp}_{2N}(k_F)$. We note that $\mathcal{B} \cap K_0 / \mathcal{B} \cap K_1$ is the standard Borel subgroup (of upper triangular matrices) of K_0/K_1, that $T \cap K_0 / T \cap K_1$ is the diagonal torus, and that W is the Weyl group.

Let I_1 denote the inverse image of the maximal unipotent subgroup $U \cap K_0/U \cap K_1$ of K_0/K_1, that is, the pro-unipotent radical of the standard Iwahori subgroup I consisting of matrices which are upper triangular modulo $\mathfrak{p}_F$. Then the Bruhat decomposition for K_0/K_1 gives

$$K_0 \,/\, K_1 = (\mathcal{B} \cap K_0) W I_1 \,/\, K_1.$$

Since $K_1 \subset I_1$, we obtain $K_0 = (\mathcal{B} \cap K_0) W I_1$. Finally, using the Iwasawa decomposition $G = \mathcal{B} K_0$ (since K_0 is a good maximal compact subgroup of G), we obtain

$$G = \mathcal{B} W I_1. \tag{1.5}$$

Now we can use this decomposition to translate Proposition 1.4 into a sufficient condition for non-genericity of compactly-induced supercuspidal representations of G.

Proposition 1.6. *Let J be a compact open subgroup of G and λ an irreducible representation of J such that $\pi = $ c-$\mathrm{Ind}_J^G \lambda$ is an irreducible supercuspidal representation of G. Let U be the subgroup of all upper triangular unipotent matrices in G and we also use the other notations from above. Then π is generic if and only if there exist $w \in W$, $p \in I_1$ and a character χ of U such that ${}^p\lambda$ contains the character χ^w of ${}^pJ \cap U^w$. In particular,*

> *if π is generic then there are $p \in I_1$ and a* long *root $\gamma \in \Phi$ such that ${}^p\lambda$ contains the trivial character of ${}^pJ \cap U_\gamma$.*

We remark that, in our symplectic basis, the long roots correspond to the entries on the anti-diagonal.

Proof. Since all maximal unipotent subgroups of G are conjugate to U, Proposition 1.4 implies that π is generic if and only if there exist $g \in G$ and a character χ of U such that λ contains the character χ^g of $J \cap U^g$. Now we use the decomposition (1.5) to write $g = bwp$, with $b \in B$, $w \in W$ and $p \in I_1$. Since $U^b = U$ and χ^b is another character of U, we can absorb the b and the result follows on conjugating by p.

The final assertion follows since the derived subgroup U^w_{der} contains U_γ for some long root γ. ■

2 The supercuspidal representations of $\mathrm{Sp}_4(F)$

What we seek is a complete list of which supercuspidal representations of $\mathrm{Sp}_4(F)$ are generic, which are not. So we start with a description of positive level supercuspidal representations of $\mathrm{Sp}_4(F)$ – level zero supercuspidal representations are obtained by inducing from a maximal parahoric subgroup $\mathcal{P}$ the inflation of a cuspidal representation of the (finite) quotient of $\mathcal{P}$ by its pro-p-radical. We are then in a position to state our main theorem, identifying non-generic representations from our list. We end the section with a proof of the theorem for level zero representations. The proof for positive level occupies the remaining sections.

2.1 The positive level supercuspidal representations

In this section we describe the construction of the positive level supercuspidal representations of $\mathrm{Sp}_4(F)$. We refer to [21] for more details and for proofs of the results stated here.

The construction begins with a skew semisimple stratum $[\Lambda, n, 0, \beta]$ in A such that $\mathfrak{a}(\Lambda) \cap B$ is a *maximal* self-dual order normalized by $E^\times$ in B. There are essentially four cases here. In the first two, the stratum is actually simple:

(I) "maximal case": $[\Lambda, n, 0, \beta]$ is a skew simple stratum and $E = F[\beta]$ is an extension of F of degree 4.

(II) "2 then 0 case": $[\Lambda, n, 0, \beta]$ is a skew simple stratum, $E = F[\beta]$ is an extension of F of degree 2, and $\mathfrak{a}_0(\Lambda)$ is maximal amongst (self-dual) $\mathfrak{o}_F$-orders in A normalized by $E^\times$.

Otherwise, we have a splitting $V = V^1 \perp V^2$ of $[\Lambda, n, 0, \beta]$ into two 2-dimensional F-vector spaces, and we write: Λ^i for the lattice sequence in V^i given by $\Lambda^i(k) = \Lambda(k) \cap V^i$, for $k \in \mathbb{Z}$; $\beta_i = \mathbf{1}^i\beta\mathbf{1}^i$, where $\mathbf{1}^i$ is the projection onto V^i with kernel V^{3-i}; if $\beta_i \neq 0$ then $n_i = -\nu_{\Lambda^i}(\beta_i)$, otherwise $n_i = 0$. Then $n = \max\{n_1, n_2\}$.

(III) "2+2 case": for $i = 1, 2$, $[\Lambda^i, n_i, 0, \beta_i]$ is a skew simple stratum and $E_i = F[\beta_i]$ is an extension of F of degree 2.

(IV) "2+0 case": $[\Lambda^1, n_1, 0, \beta_1]$ is a skew simple stratum and $E_1 = F[\beta_1]$ is an extension of F of degree 2; $\beta_2 = 0$, so that in V^2 we have the null stratum $[\Lambda^2, 0, 0, 0]$, and $\mathfrak{a}_0(\Lambda^2)$ is maximal amongst (self-dual) $\mathfrak{o}_F$-orders in $A^2 = \mathrm{End}_F(V^2)$.

We often think of (IV) as a degenerate case of (III) by thinking of a null stratum as a degenerate simple stratum.

In each case, we have the subgroups $\bar{H}^1 = H^1(\beta, \Lambda)$, $\bar{J}^1 = J^1(\beta, \Lambda)$ and $\bar{J} = J(\beta, \Lambda)$ of $\bar{G}$. We write $H^1 = \bar{H}^1 \cap G$, and similarly for the other groups. There is a family $\mathcal{C}(\beta, \Lambda)$ of rather special characters of H^1, called *semisimple characters*; one of their properties is the fact that their restriction to $P_i(\Lambda)$ for suitable i is equal to ψ_β. For each $\theta \in \mathcal{C}(\beta, \Lambda)$, there is a unique irreducible representation η of J^1 containing θ. In each case, there is a "suitable" extension κ of η to a representation of J, which we call a *β-extension* – see below for more details of this step.

The extensions E, E_i in each case come equipped with a non-trivial galois involution, which we write $^-$ as usual. We use the same notation for the induced involution on the residue fields k_E, k_i; note that this involution may be trivial. Then the quotient J/J^1 has one of the following forms:

(I) $k_E^1 = \{x \in k_E : x\overline{x} = 1\}$;

(II) $U(1,1)(k_E/k_F)$ or $k_E^1 \times k_E^1$ if E/F is unramified; $SL_2(k_F)$ or $O_2(k_F)$ if E/F is ramified;

(III) $k_1^1 \times k_2^1$;

(IV) $k_1^1 \times SL_2(k_F)$.

Let σ be the inflation to J of an irreducible cuspidal representation of J/J^1. (Note that, in the case of $O_2(k_F)$ in (II), this just means any irreducible representation of the (anisotropic) dihedral group $O_2(k_F)$.)

Now we put $\lambda = \kappa \otimes \sigma$ and $\pi = c\text{-}\mathrm{Ind}_J^G \lambda$ is an irreducible supercuspidal representation of G. All irreducible supercuspidal representations of G can be constructed in this way (though we remark that often we cannot, as yet, tell when two such representations are equivalent).

Finally in this subsection, we recall briefly some properties of the β-extensions which we will need, especially in the cases (II) and (IV) where their construction is somewhat more involved. Indeed, in cases (I) and (III), J/J^1 has no unipotent elements so there is never any problem here.

We define another skew semisimple stratum $[\Lambda_m, n_m, 0, \beta]$ as follows:

- in cases (I) and (III), we have $\Lambda_m = \Lambda$, $n_m = n$;

- in case (II), Λ_m is a self-dual $\mathfrak{o}_E$-lattice sequence in V with $\mathfrak{a}_0(\Lambda_m) \subset \mathfrak{a}_0(\Lambda)$ minimal amongst (self-dual) $\mathfrak{o}_F$-orders normalized by $E^\times$, and $n_m = -\nu_{\Lambda_m}(\beta)$;

- in case (IV), we take Λ_m^2 a self-dual $\mathfrak{o}_F$-lattice sequence in V with $\mathfrak{a}_0(\Lambda_m^2) \subset \mathfrak{a}_0(\Lambda^2)$ minimal amongst (self-dual) $\mathfrak{o}_F$-orders, $\Lambda_m = \Lambda^1 \perp \Lambda_m^2$, and $n_m = -\nu_{\Lambda_m}(\beta)$.

In each case, we have the subgroups $\bar{H}_m^1 = H^1(\beta, \Lambda_m)$, $\bar{J}_m^1 = J^1(\beta, \Lambda_m)$ of $\bar{G}$, and we put $H_m^1 = \bar{H}_m^1 \cap G$ etc. Let $\theta_m \in \mathcal{C}(\beta, \Lambda_m)$ be the *transfer* of θ, that is $\theta_m = \tau_{\Lambda, \Lambda_m, \beta}\theta$ in the notation of [20] §3.6, and let η_m be the unique irreducible representation of J_m^1 containing θ_m. We form the group

$$\tilde{J}^1 = (P_1(\Lambda_m) \cap B)J^1.$$

Then (see [21]) there is a unique irreducible representation $\tilde{\eta}$ of $\tilde{J}^1$ which extends η and such that $\tilde{\eta}$ and η_m induce equivalent irreducible representations of $P_1(\Lambda_m)$. Moreover, if $I_g(\tilde{\eta})$ denotes the g-intertwining space of $\tilde{\eta}$, we have

$$\dim I_g(\tilde{\eta}) = \begin{cases} 1 & \text{if } g \in \tilde{J}^1(B \cap G)\tilde{J}^1; \\ 0 & \text{otherwise.} \end{cases}$$

A β-extension of η is an irreducible representation κ of J such that $\kappa|_{\tilde{J}^1} = \tilde{\eta}$.

2.2 The main theorem

Theorem 2.1. *The non-generic supercuspidal representations of* $\mathrm{Sp}_4(F)$ *are the following.*

(i) *The positive level supercuspidal representations attached to a skew semisimple stratum* $[\Lambda, n, 0, \beta]$ *as above and such that:*

- *either* $[\Lambda, n, 0, \beta]$ *is a sum of non-zero simple strata (cases (I), (II) and (III)) and there is no maximal unipotent subgroup of* G *on which* ψ_β *is a character;*
- *or* $[\Lambda, n, 0, \beta]$ *is the sum of a non-zero simple stratum and a null stratum in dimension 2 (case (IV)).*

(ii) *The level zero supercuspidal representations induced from the inflation to a maximal parahoric subgroup* $\mathcal{P}$ *of a cuspidal representation* σ *of* $\mathcal{P}/\mathcal{P}^1$*, where* $\mathcal{P}^1$ *is the pro-p-radical of* $\mathcal{P}$*, satisfying one of the following:*

(a) $\mathcal{P}$ *is attached to a non-connected subset of the extended Dynkin diagram of* G*, that is,* $\mathcal{P}/\mathcal{P}^1$ *is isomorphic to* $\mathrm{Sp}(2, k_F) \times \mathrm{Sp}(2, k_F)$.

(b) $\mathcal{P}$ *is isomorphic to* $\mathrm{Sp}_4(\mathfrak{o}_F)$ *and* σ *is a non-regular cuspidal representation of* $\mathrm{Sp}_4(k_F)$.

For positive level representations the theorem will follow from Proposition 3.4, which establishes the conditions on β for there to exist a maximal unipotent subgroup of G on which ψ_β is a character, and from Theorem 4.5 and section 5. The proof for level zero representations is given below, with a more detailed list.

2.3 The generic level zero representations of $\mathrm{Sp}_{2N}(F)$

Note that for finite reductive groups, the notion equivalent to genericity is called regularity: a representation of $\mathrm{Sp}_{2N}(k_F)$ is called *regular* if it contains a nondegenerate character of a maximal unipotent subgroup. Part (ii) of the above theorem, i.e. the level zero case, actually holds for $\mathrm{Sp}_{2N}(F)$, as a consequence of Propositions 1.4 and 1.6.

Proposition 2.2. *Let $\mathcal{P}$ be a maximal parahoric subgroup of $\mathrm{Sp}_{2N}(F)$ with pro-p-radical $\mathcal{P}^1$ and let σ be a cuspidal representation of $\mathcal{P}/\mathcal{P}^1$. The representation π of $\mathrm{Sp}_{2N}(F)$ induced from the inflation of σ to $\mathcal{P}$ is irreducible supercuspidal. It is generic if and only if the quotient $\mathcal{P}/\mathcal{P}^1$ is isomorphic to $\mathrm{Sp}_{2N}(k_F)$ and σ identifies to a regular cuspidal representation of $\mathrm{Sp}_{2N}(k_F)$.*

Proof. Up to conjugacy, we may assume that $\mathcal{P}$ is standard; in particular, using the notation in 1.2, $\mathcal{P}$ contains I. Then our standard $\mathcal{P}$ is the group of invertible and symplectic elements in the order

$$\mathfrak{A} = \begin{pmatrix} M_i(\mathfrak{o}_F) & M_{i,\,2N-i}(\mathfrak{o}_F) & M_i(\mathfrak{p}_F^{-1}) \\ M_{2N-\,i,\,i}(\mathfrak{p}_F) & M_{2N-2i}(\mathfrak{o}_F) & M_{2N-\,i,\,i}(\mathfrak{o}_F) \\ M_i(\mathfrak{p}_F) & M_{i,\,2N-i}(\mathfrak{p}_F) & M_i(\mathfrak{o}_F) \end{pmatrix} \qquad \text{for some integer } i, 0 \le i \le N.$$

Assume first that $\mathcal{P}/\mathcal{P}^1$ is isomorphic to $\mathrm{Sp}_{2N}(k_F)$, i.e. $i = 0$ or N, and use Proposition 1.4. If σ is regular, then π is generic. Conversely if π is generic, there is a maximal unipotent subgroup U' and a character χ' of U' such that σ contains $\chi_{|\mathcal{P}\cap U'}$. The subgroup U' is conjugate to the subgroup U of all upper triangular unipotent matrices so, using the Iwasawa decomposition $G = \mathcal{P}\mathcal{B}$ as in 1.2, we may replace U' by U. Since σ is cuspidal, Lemma 1.3, applied to $\mathrm{Sp}_{2N}(k_F)$, tells us that $\chi_{|\mathcal{P}\cap U}$ identifies with a nondegenerate character of $\mathcal{P}\cap U/\mathcal{P}^1\cap U$, a maximal unipotent subgroup of $\mathrm{Sp}_{2N}(k_F)$, hence σ is regular.

Assume now that $1 \le i \le N-1$: then $\mathcal{P}/\mathcal{P}^1$ is isomorphic to $\mathrm{Sp}_{2i}(k_F) \times \mathrm{Sp}_{2N-2i}(k_F)$, the relevant entries being those in $\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$ in the above description of $\mathfrak{A}$. Assume for a contradiction that π is generic: from Proposition 1.6, plus the inclusion $I_1 \subset \mathcal{P}$, there exist $w \in W$ and a character χ of U such that σ contains the character χ^w of $\mathcal{P}\cap U^w$. Let $\overline{U^w} = \mathcal{P}\cap U^w/\mathcal{P}^1\cap U^w$ and let $\overline{\chi^w}$ be the character of $\overline{U^w}$ defined by χ^w. The group $\overline{U^w}$ is a maximal unipotent subgroup of $\mathcal{P}/\mathcal{P}^1$ (a simple combinatoric argument suffices here). We will show that, since χ is trivial on U_{der}, the character $\overline{\chi^w}$ is degenerate, thus contradicting the cuspidality of σ.

To fix ideas, suppose that $w = 1$. The intersection of the image of U with $\mathrm{Sp}_{2i}(k_F)$ is the subgroup $\bar{U}_i$ of upper triangular unipotent matrices while the image of U_{der} contains the simple long root: the restriction of $\bar{\chi}$ to $\bar{U}_i$ is degenerate hence σ cannot be cuspidal (Lemma 1.3).

In general, observe that w must map the N positive long roots (corresponding to the antidiagonal entries in U) onto a set $\mathcal{E}$ of N long roots that correspond to the long root entries in U^w. Those N long roots separate into i long roots in $\bar{U}^w \cap \mathrm{Sp}_{2i}(k_F)$ and $N-i$ long roots in $\bar{U}^w \cap \mathrm{Sp}_{2N-2i}(k_F)$. The $N-1$ positive not simple long roots corresponding to antidiagonal entries in U_{der} are sent onto a subset of $N-1$ long roots in $\mathcal{E}$: only one is missing, so either in $\bar{U}^w \cap \mathrm{Sp}_{2i}(k_F)$ or in $\bar{U}^w \cap \mathrm{Sp}_{2N-2i}(k_F)$, the unique long root entry that does not belong to the derived group does belong to U^w_{der}: on this group, the restriction of $\bar{\chi}^w$ is degenerate, so σ is not cuspidal. ■

2.4 The cuspidal representations of $\mathbf{Sp_4(\mathbb{F}_q)}$

We come back to Sp(4). It has been known for a long time that among the cuspidal representations of $\mathrm{Sp}_4(k_F)$, only one is non-regular, the famous representation θ_{10} of Srinivasan ([17] II.8.3, [18]); it is the unique cuspidal unipotent representation of $\mathrm{Sp}_4(k_F)$. Yet this common knowledge lacks of a reference in the modern setting of Deligne-Lusztig characters, we thus pause here to detail the list of cuspidal representations of $\mathrm{Sp}(4, \mathbb{F}_q)$ as they arise from the Lusztig classification. The necessary background and notations are taken from the book [10], specially chapter 14.

For this subsection only we let $G = \mathrm{Sp}(4, \bar{\mathbb{F}}_q)$ and we let F be the standard Frobenius on G, acting as $x \mapsto x^q$ on each entry, so that $G^F = \mathrm{Sp}(4, \mathbb{F}_q)$. We let G^* be the dual group $SO(5, \bar{\mathbb{F}}_q)$ with standard Frobenius F^*.

Deligne-Lusztig characters of G^F are parameterized by pairs (T^*, s), T^* an F^*-stable maximal torus of G^* and s an element of ${T^*}^{F^*}$, up to ${G^*}^{F^*}$-conjugacy. A rational series of irreducible characters of G^F is made of all irreducible components of Deligne-Lusztig characters $R_{T^*}^G(s)$ where the rational conjugacy class of s (i.e. the ${G^*}^{F^*}$-conjugacy class of s) is fixed. Rational series of characters are disjoint and exhaust irreducible characters of G^F. Cuspidal (irreducible) characters are those characters that appear in some $R_{T^*}^G(s)$ for a minisotropic torus T^* and do not appear in any $R_{T^*}^G(s)$ where the torus T^* is contained in a proper F^*-stable Levi subgroup of G^*.

Let s be a rational semi-simple element contained in an F^*-stable maximal torus T^* of G^*, let $C_{G^*}(s)$ (resp. $C_{G^*}^o(s)$) be its centralizer in G^* (resp. the connected component of its centralizer) and let $W(s)$ (resp. $W^o(s)$) be the Weyl group of $C_{G^*}(s)$ (resp. $C_{G^*}^o(s)$) relative to T^*, contained in the Weyl group $W(T^*)$ of G^* relative to T^*.

For w in $W(T^*)$, there exists an F^*-stable maximal torus T_w^* of G^* of type w with respect to T^* and containing s if and only if w belongs to $W^o(s)$. Letting x be the type of T^* with respect to some split torus, one defines by the formula

$$\chi(s) = (-1)^{l(x)} |W^o(s)|^{-1} \sum_{w \in W^o(s)} (-1)^{l(w)} R_{T_w^*}^G(s)$$

a proper character $\chi(s)$ which is a multiplicity one sum of regular irreducible characters, each appearing with multiplicity ± 1 in Deligne-Lusztig characters underlying the series attached to s. We have

$$\left\langle R_{T_w^*}^G(s), R_{T_w^*}^G(s) \right\rangle_{G^F} = \mathrm{Card}\ W(s)^{wF^*} \quad \text{and} \quad \langle \chi(s), \chi(s) \rangle_{G^F} = |(W(s)/W^o(s))^{F^*}|$$

and the results in *loc. cit.*, §14, imply the following, for the rational series of characters attached to s:

(i) If only minisotropic rational maximal tori contain s, all characters in the series are cuspidal. The number of regular cuspidal characters in the series is the number of components of $\chi(s)$.

(ii) If no minisotropic rational maximal torus contains s, there is no cuspidal character in the series.

(iii) If at least one minisotropic rational maximal torus and at least one non-minisotropic rational maximal torus contain s, no cuspidal character in the series (if any) is regular.

(iv) If exactly one minisotropic rational maximal torus (up to rational conjugacy) and at least one non-minisotropic rational maximal torus contain s, there is no cuspidal character in the series.

The Weyl group of G^* has 8 elements. A rational maximal torus of type w with respect to a split torus is minisotropic if and only if w is either a Coxeter element h (there are two of them, conjugate in the Weyl group) or the element of maximal length w_0. Rational points of such a torus are conjugate to $T_0^{*wF^*}$, isomorphic to

$$\begin{aligned} &\mathbb{K}_4^2 = \ker N_{\mathbb{F}_{q^4}^\times/\mathbb{F}_{q^2}^\times} && \text{for } w = h, \\ &\mathbb{K}_2^1 \times \mathbb{K}_2^1 = \ker N_{\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times} \times \ker N_{\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times} && \text{for } w = w_0. \end{aligned}$$

The table below lists the families of geometric conjugacy classes of rational semi-simple elements of G^* through a representative s_0 (not necessarily rational) in the diagonal torus

$$T_0^* = \{\ t^*(\lambda,\mu) = \begin{pmatrix} \lambda &&&& \\ & \mu &&& \\ && 1 && \\ &&& \mu^{-1} & \\ &&&& \lambda^{-1} \end{pmatrix}\ /\ \lambda,\mu \in \bar{\mathbb{F}}_q^\times\},$$

using the following notation: we fix ζ_4, a primitive (q^4-1)-th root of unity in $\bar{\mathbb{F}}_q^\times$, and we let

$$\zeta_{4,2} = \zeta_4^{q^2-1}, \quad \zeta_2 = \zeta_4^{q^2+1}, \quad \zeta_{2,1} = \zeta_2^{q-1} \quad \text{and} \quad \zeta = \zeta_2^{q+1}.$$

In cases 13 and 14, $\chi(s) = R_{T^*}^G(s)$ is irreducible, cuspidal and regular (it actually contains any nondegenerate character of a maximal unipotent subgroup) (i); we get $\dfrac{(q-1)(q-3)}{8} + \dfrac{(q^2-1)}{4}$ equivalence classes of such representations.

Cases 4, 5, 6, 7, 8 and 9 give no cuspidal representations (ii), neither do cases 10 and 12 (iv). Cases 2 and 3 each determine two rational series, in which again (iv) applies: they contain no cuspidal.

Missing cuspidals ([17] II.8.3, [18]) now must come from cases 1 and 11. Indeed case 11 produces two rational series, one of which satisfying (ii), but the other (i), for a torus of type w_0. Here $\chi(s) = R_{T^*}^G(s)$ is the sum of two irreducible, cuspidal and regular (but for different choices of a nondegenerate character of a maximal unipotent subgroup) representations and we get $2\,\dfrac{(q-1)}{2}$ equivalence classes of such representations.

Last, case 1 gives the so-called unipotent series, which for $\mathrm{Sp}(4,\mathbb{F}_q)$ contains exactly one cuspidal representation ([13], Theorem 3.22), non-regular by (iii).

Case	s_0	condition	number	$W^o(s_0)$	$\lvert W(s_0)/W^o(s_0)\rvert$
1	$t^*(1,1)$		1	W	1
2	$t^*(-1,-1)$		1	$\langle s_{\alpha'}, s_{\alpha'+2\beta'}\rangle$	2
3	$t^*(-1,1)$		1	$\langle s_{\beta'}\rangle$	2
4	$t^*(\zeta^i,1)$	$\zeta^i \neq \pm 1$	$\frac{q-3}{2}$	$\langle s_{\beta'}\rangle$	1
5	$t^*(\zeta^i,-1)$	$\zeta^i \neq \pm 1$	$\frac{q-3}{2}$	1	2
6	$t^*(\zeta^i,\zeta^i)$	$\zeta^i \neq \pm 1$	$\frac{q-3}{2}$	$\langle s_{\alpha'}\rangle$	1
7	$t^*(\zeta^i,\zeta^j)$	$\zeta^i \neq \pm 1$ $\zeta^j \neq \pm 1$ $\zeta^i \neq \zeta^{\pm j}$	$\frac{(q-3)(q-5)}{8}$	1	1
8	$t^*(\zeta_2^i,\zeta_2^{qi})$	$\zeta_2^i \notin \mathbb{K}_2^1$ $\zeta_2^i \notin \mathbb{F}_q^\times$	$\frac{(q-1)^2}{4}$	1	1
9	$t^*(\zeta^i,\zeta_{2,1}^j)$	$\zeta^i \neq \pm 1$ $\zeta_{2,1}^j \neq \pm 1$	$\frac{(q-1)(q-3)}{4}$	1	1
10	$t^*(1,\zeta_{2,1}^i)$	$\zeta_{2,1}^i \neq \pm 1$	$\frac{(q-1)}{2}$	$\langle s_{\alpha'+\beta'}\rangle$	1
11	$t^*(-1,\zeta_{2,1}^i)$	$\zeta_{2,1}^i \neq \pm 1$	$\frac{(q-1)}{2}$	1	2
12	$t^*(\zeta_{2,1}^i,\zeta_{2,1}^{-i})$	$\zeta_{2,1}^i \neq \pm 1$	$\frac{(q-1)}{2}$	$\langle s_{\alpha'+2\beta'}\rangle$	1
13	$t^*(\zeta_{2,1}^i,\zeta_{2,1}^j)$	$\zeta_{2,1}^i \neq \pm 1$ $\zeta_{2,1}^j \neq \pm 1$ $\zeta_{2,1}^i \neq \zeta_{2,1}^{\pm j}$	$\frac{(q-1)(q-3)}{8}$	1	1
14	$t^*(\zeta_{4,2}^{qi},\zeta_{4,2}^i)$	$\zeta_{4,2}^i \neq \pm 1$	$\frac{q^2-1}{4}$	1	1

3 The function ψ_β on maximal unipotent subgroups

A key step in the determination of Whittaker functions in $\mathrm{GL}_N(F)$ in [5] is the construction of a maximal unipotent subgroup U of $\mathrm{GL}_N(F)$ on which ψ_β defines a character (*loc. cit.* propositions 2.1 and 2.2). This will be a key step indeed in the determination of generic supercuspidal representations of $\mathrm{Sp}_4(F)$: the existence of such a subgroup on which ψ_β defines a **non degenerate** character will turn out to be a sufficient condition for genericity (see §4), whereas the non existence will imply non genericity (see §5). In cases where such a U exists but ψ_β is degenerate, we will find both generic and non generic representations.

3.1 The quadratic form $h(v,\beta v)$

Proposition 3.1. *Let β be an element of A such that $\bar\beta = -\beta$ and let ψ_β be the function on G defined by $\psi_\beta(x) = \psi\left(\mathrm{tr}(\beta(x-1))\right)$, $x \in G$. The following are equivalent:*

(i) *There exists a maximal unipotent subgroup U of G such that the restriction of ψ_β to U is a character of U.*

(ii) *There exists a maximal unipotent subgroup U of G such that $\psi_\beta(x) = 1$ for all $x \in U_{\mathrm{der}}$.*

(iii) *There exists a totally isotropic flag of subspaces of V:*

$$\{0\} \subset V_1 \subset V_2 \subset V_3 \subset V$$

such that $\beta V_i \subset V_{i+1}$ for $i = 1, 2$.

(iv) *The quadratic form $v \mapsto h(v, \beta v)$ on V has non trivial isotropic vectors.*

Proof. The equivalence of the first three statements is straightforward and a variant of [5] 2.1; note that a maximal flag of subspaces of V determines a maximal unipotent subgroup of G if and only if it is totally isotropic.

Certainly (iii) implies (iv): a basis vector v for V_1 satisfies $h(v, \beta v) = 0$ since V_2 is its own orthogonal. Assuming (iv), let v be a non-zero vector in V such that $h(v, \beta v) = 0$ and put $V_1 = \mathrm{Span}\ \{v\}$, $V_3 = V_1^\perp$. If βv and v are colinear, let V_2 be any totally isotropic 2-dimensional subspace of V containing V_1, otherwise put $V_2 = \mathrm{Span}\ \{v, \beta v\}$: (iii) is satisfied since, for a totally isotropic flag as in (iii), the conditions $\beta V_1 \subset V_2$ and $\beta V_2 \subset V_3$ are equivalent (recall $\bar\beta = -\beta$). ■

Remark 3.2. *Assume the conditions in Proposition 3.1 hold and let U be a maximal unipotent subgroup of G attached to a totally isotropic flag $\{0\} \subset V_1 \subset V_2 \subset V_3 \subset V$ such that $\beta V_i \subset V_{i+1}$ for $i = 1, 2$. A simple inspection shows that the character ψ_β of U is non degenerate if and only if βV_1 is not contained in V_1 and βV_2 is not contained in V_2.*

We need to investigate those cases where the element β appears in a skew semi-simple stratum $[\Lambda, n, 0, \beta]$ as listed in §2.1. We need an extra piece of notation in cases I or II, where the stratum is simple: the field extension $E = F[\beta]$ has degree 4 or 2; we let E_0 be the field of fixed points of the involution $x \mapsto \bar x$ on E, so that $[E : E_0] = 2$, and we define a skew-hermitian form δ on V relative to E/E_0 by

$$h(av, w) = \mathrm{tr}_{E/F}(a\delta(v, w)) \quad \text{for all } a \in E, v, w \in V. \tag{3.3}$$

(This notation will also be used in case III when E_1 and E_2 are isomorphic, with $E = E_1$.) The determinant of δ belongs to $F^\times$ if $[E : F] = 2$; it is a skew element in $E^\times$ if $[E : F] = 4$.

Proposition 3.4. *Let β be an element of A appearing in a skew semi-simple stratum $[\Lambda, n, 0, \beta]$ as in §2.1. The only cases in which there does not exist a maximal unipotent subgroup U of G on which ψ_β is a character are the following:*

(i) *The element β generates a biquadratic extension $E = F[\beta]$ of F (case I) and the coset $\beta \det(\delta)\, N_{E/E_0}(E^\times)$ in $E_0^\times$ is the $N_{E/E_0}(E^\times)$-coset that does not contain the kernel of $\mathrm{tr}_{E_0/F}$.*

(ii) *The element β generates a quadratic extension $E = F[\beta]$ of F (case II) and the skew-hermitian form δ on V, $\dim_E V = 2$, is anisotropic – that is, $\det(\delta) \notin N_{E/F}(E^\times)$.*

(iii) *The symplectic space V decomposes as $V = V^1 \perp V^2$ and the element β decomposes accordingly as $\beta = \beta_1 + \beta_2$ where for $i = 1, 2$, β_i generates a quadratic extension $E_i = F[\beta_i]$ (case III), E_1 is isomorphic to E_2 and $\beta_1/\beta_2 \notin \det(\delta)\, N_{E/F}(E^\times)$.*

Remark 3.5. *Let β be as above. Assume there exists a maximal unipotent subgroup U of G on which ψ_β is a character. Then:*

- *in cases I and III the character ψ_β of U is non degenerate;*
- *in cases II and IV the character ψ_β of U is degenerate.*

The proof of those statements occupies the next two subsections. We recall that, up to isomorphism, there is exactly one anisotropic quadratic form on V: its determinant is a square and its Hasse-Minkowski symbol is equal to $-(-1,-1)_F$ ([14], §63C).

3.2 The biquadratic extension

Let us examine the case of a maximal simple stratum (case I). The determinant of the quadratic form $v \mapsto h(v, \beta v)$ on V is the determinant of β, i.e. $N_{E/F}(\beta)$.

Lemma 3.6. *The norm $N_{E/F}(\beta)$ of β is a square in $F^\times$ if and only if E is biquadratic. If this holds we have: $N_{E/E_0}(E^\times) = F^\times {E_0^\times}^2$.*

Proof. A four-dimensional extension of F is called *biquadratic* if it is Galois with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Biquadratic extensions of F are all isomorphic, their norm subgroup is $F^{\times 2}$(class field theory). The "if" part is then clear. Now assume that $N_{E/F}(\beta)$ is a square. Since β is skew and generates a degree four field extension of F, its square β^2 generates E_0 over F and is not a square in $E_0^\times$, while $N_{E_0/F}(\beta^2) = N_{E/F}(\beta)$ must be a square in $F^\times$. We proceed according to ramification.
If E_0 is ramified over F: β^2 must have even valuation, its squareroot generates an unramified extension of E_0. So E/E_0 is unramified and $N_{E/E_0}(E^\times)$ is made of even valuation elements, i.e. is equal to $F^\times {E_0^\times}^2$ since $\mathfrak{o}_{E_0}^\times = \mathfrak{o}_F^\times(1 + \mathfrak{p}_{E_0})$. It follows that $N_{E/F}(E^\times) = F^{\times 2}$.
If E_0 is unramified over F, we have in the residual field k_{E_0}:

$u \in k_{E_0}^\times$ *is a square in* $k_{E_0}^\times$ *if and only if* $N_{k_{E_0}/k_F}(u)$ *is a square in* $k_F^\times$.

We write $\beta^2 = \varpi_F^j u$ with $u \in \mathfrak{o}_{E_0}^\times$. Then $N_{E_0/F}(\beta^2) = \varpi_F^{2j} N_{E_0/F}(u)$. It follows that u must be a square and β^2 must have odd valuation: its square root generates a ramified extension of E_0. Then E is the extension $E_0[\alpha]$ where α^2 is a uniformizing element in F, and $N_{E/E_0}(E^\times) = (-\alpha^2)^{\mathbb{Z}} \mathfrak{o}_{E_0}^{\times 2} = F^\times {E_0^\times}^2$ because $k_F^\times \subset k_{E_0}^{\times 2}$. It follows that $N_{E/F}(E^\times) = F^{\times 2}$. ■

If $N_{E/F}(\beta)$ is not a square, we are done. Assume from now on that E is biquadratic and use the form δ defined in 3.3. Since β is skew and δ is skew-hermitian, the element $\beta\delta(v,v)$ belongs to E_0. Since V is one-dimensional over E the form $\delta(v,v)$ is anisotropic and the subset $D(V) = \{\beta\delta(v,v) \ / \ v \in V, v \neq 0\}$ of $E_0^\times$ is one of the two cosets of $N_{E/E_0}(E^\times)$ in $E_0^\times$. On the other hand, we have $N_{E/E_0}(E^\times) = F^\times {E_0^\times}^2$ hence the set of non-zero elements in Ker $\mathrm{tr}_{E_0/F}$ is fully contained in one of those two cosets. Proposition 3.4 follows in this case (and Remark 3.5 directly follows from 3.2 since β generates a degree 4 extension of F).

Remark. We can be more precise about this condition: if E_0 is unramified over F and $|k_F| \equiv 3\,[4]$, then $h(\beta v, v)$ is anisotropic if and only if $\beta\delta(v,v) \notin F^\times {E_0^\times}^2$; otherwise $h(\beta v, v)$ is anisotropic if and only if $\beta\delta(v,v) \in F^\times {E_0^\times}^2$.

3.3 Cases II, III and IV

The case numbered IV in §2.1 is obvious. The set of isotropic vectors for the quadratic form $h(v, \beta v)$ is the subspace V^2. The flags that satisfy 3.1(iii) are the flags that can be written in the form $\{0\} \subset Fe_{-2} \subset Fe_{-2} + Fe_{-1} \subset Fe_{-2} + Fe_{-1} + Fe_1 \subset V$ where $\{e_{-i}, e_i\}$ is a symplectic basis of V^i for $i = 1, 2$.

Case II is also quite clear: as in case I the element $\beta\delta(v,v)$ belongs to $E_0 = F^\times$ hence $h(\beta v, v) = 2\beta\delta(v,v)$ has isotropic vectors if and only if $\delta(v,v)$ does. Furthermore a flag $\{0\} \subset V_1 \subset V_2 \subset V_3 \subset V$ as in 3.1 must have the form $V_1 = Fv$, where v is non zero and isotropic for δ, and $V_2 = < v, \beta v >$. Since β^2 belongs to $F^\times$ we always have $\beta V_2 = V_2$ so, if ψ_β defines a character of the corresponding unipotent subgroup of G, this character is degenerate.

We finish with case III. We have $V = V^1 \perp V^2$ and $\beta = \beta_1 + \beta_2$. For $v \in V$, writing $v = v_1 + v_2$ on $V = V^1 \perp V^2$, we get $h(v, \beta v) = h(v_1, \beta_1 v_1) + h(v_2, \beta_2 v_2)$. The determinant of this form is the product $N_{E_1/F}(\beta_1) N_{E_2/F}(\beta_2)$. For the form to be anisotropic, the determinant must be a square hence

$$N_{E_1/F}(\beta_1) \equiv N_{E_2/F}(\beta_2) \bmod F^{\times 2}.$$

Each β_i is skew with characteristic polynomial $X^2 - (-N_{E_i/F}(\beta_i))$: the class of $-N_{E_i/F}(\beta_i)$ mod the squares determines, up to isomorphism, the extension E_i. So if E_1 and E_2 are not isomorphic we are done.

We pursue assuming they are and let $E = E_1 \simeq E_2$. We may see V as a vector space over E and define δ as in 3.3. The decomposition $V = V^1 \perp V^2$ is orthogonal for δ as well, and for $i = 1, 2$, $v_i \mapsto h(v_i, \beta_i v_i)$ is an anisotropic quadratic form on V^i: a (non zero) isotropic vector for $h(v, \beta v)$ must have the form $v = v^1 + v^2$ with $v^i \in V^i$, $v^i \neq 0$. We then have:

$$h(v, \beta v) = h(v^1, \beta_1 v^1) + h(v^2, \beta_2 v^2) = 2\,\beta_1\,\delta(v^1, v^1) + 2\,\beta_2\,\delta(v^2, v^2),$$

and 3.4 follows.

The remark on the non degeneracy of ψ_β on U whenever it defines a character follows from the fact that V^1 and V^2 are anisotropic for $h(v, \beta v)$: the corresponding totally isotropic flag has the form (with notations as above) $\{0\} \subset V_1 = < v^1 + v^2 > \subset V_2 = < v^1 + v^2, \beta_1 v^1 + \beta_2 v^2 > \subset V_1^\perp \subset V$. We never have $\beta V_2 \subseteq V_2$ unless $\beta_1^2 = \beta_2^2$ – but this belongs to case II, not to case III.

4 Generic representations

In section 3, we have found necessary and sufficient conditions for there to exist a maximal unipotent subgroup U of G on which ψ_β defines a character. When there is such a U, this gives us a candidate for trying to build a Whittaker model, as is the case in $\mathrm{GL}_N(F)$ (see [5]). In this section we consider the case where there is such a U.

4.1 Characters and β-extensions

Proposition 4.1 (cf. [5, Lemma 2.10]). *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in A. Let $\theta \in \mathcal{C}(\Lambda, 0, \beta)$ be a skew semisimple character and let U be a maximal unipotent subgroup of G such that $\psi_\beta|_{U_{der}} = 1$. Then*

$$\theta|_{H^1 \cap U} = \psi_\beta|_{H^1 \cap U}.$$

Unfortunately, we have been unable to find a unified proof of this Proposition; the proof is therefore rather ugly, on a case-by-case basis, and we postpone it to the appendix.

We continue with the notation of Proposition 4.1. Then we can define a character Θ_β of $\tilde{H}^1 = (J \cap U)H^1$ by

$$\Theta_\beta(uh) = \psi_\beta(u)\theta(h), \qquad \text{for } u \in J \cap U,\ h \in H^1.$$

Notice that this is a character, since J normalizes H^1 and intertwines θ with itself.

Corollary 4.2 (cf. [5, Lemma 2.11]). *Let η be the unique irreducible representation of J^1 which contains θ. Then the restriction of η to $J^1 \cap U$ contains the character $\psi_\beta|_{J^1 \cap U}$.*

Proof. The proof is essentially identical to that of [5, Lemma 2.11]. We recall that $\mathbf{k}_\theta(x, y) = \theta[x, y]$ defines a nondegenerate alternating form on the finite group J^1/H^1 ([20, Proposition 3.28]). Notice also that the image of $\tilde{H}^1 \cap J^1$ in J^1/H^1 is a totally isotropic subspace for the form $\mathbf{k}_\theta$, since θ extends to a character Θ_β of $\tilde{H}^1$. Now we can construct η by first extending Θ_β to (the inverse image in J^1 of) a maximal totally isotropic subspace of J^1/H^1 and then inducing to J^1. In particular, η contains Θ_β and hence $\psi_\beta|_{J^1 \cap U}$. ■

Now put $\tilde{J}^1 = (J \cap U)J^1 = (P(\Lambda) \cap B^\times \cap U)J^1$. Note that, if J/J^1 is anisotropic then $J \cap U = J^1 \cap U$ so $\tilde{J}^1 = J^1$.

Theorem 4.3 (cf. [16, Theorem 2.6]). *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in A as listed in §2.1 and let κ be a β-extension of η to J as described there. Assume there is a maximal unipotent subgroup U of G such that $\psi_\beta|_{U_{der}} = 1$ and use the notation above. Then:*

$$\kappa|_{\tilde{J}^1} \simeq \mathrm{Ind}_{\tilde{H}^1}^{\tilde{J}^1} \Theta_\beta.$$

In particular, we deduce that the restriction of κ to $J \cap U$ contains the character $\psi_\beta|_{J \cap U}$ (cf. [5, Lemma 2.12]). Moreover, except in case IV, and case II when J/J^1 is isotropic, this means that the restriction of the simple type λ to $J \cap U$ contains $\psi_\beta|_{J \cap U}$, since $\lambda = \kappa$.

Proof. The proof is in essence the same as that of [16, Theorem 2.6], which it may be useful to read first: because of the similarities, we do not give all the details here.

We begin by proving

$$\mathrm{Ind}_{(J^1 \cap U)H^1}^{J^1} \Theta_\beta \simeq \eta. \tag{4.4}$$

We prove this first in cases I and III. Here $E^1 = \{x \in E / x\bar{x} = 1\}$ is a maximal torus of G. Then $J = E^1 J^1$ and $\pi = c\text{-Ind}_J^G \kappa$. Since $J \cap U = J^1 \cap U$, Corollary 4.2 implies that κ contains $\psi_\beta|_{J\cap U}$. Hence π contains ψ_β and, since ψ_β is then a nondegenerate character, κ contains $\psi_\beta|_{J\cap U}$ with multiplicity one (Proposition 1.4). Hence $\Theta_\beta|_{(J^1\cap U)H^1}$ occurs in η with multiplicity precisely one and (4.4) follows (see [16, Lemma 2.5]). Note that this already gives the Theorem cases I and III, since E^1 is maximal and $\tilde{J}^1 = J^1$ in these cases.

Now we consider the other cases II and IV. Recall that U is given by a flag $\{0\} \subset V_1 \subset V_2 \subset V_3 \subset V$ (see Proposition 3.1), described in §3.3. What we need here is to define a parabolic subgroup P_0 of G, with unipotent radical U_0 contained in U and with a specific Levi factor M_0 conforming to Λ in the sense of [4, §10]. We achieve this according to the case as follows.

(II) From §3.3, the unipotent subgroup U is attached to a flag

$$\{0\} \subset Fw_{-1} \subset Fw_{-1} + F\beta w_{-1} \subset Fw_{-1} + F\beta w_{-1} + F\beta w_1 \subset V$$

where $\{w_{-1}, w_1\}$ is a Witt basis for V over E. We then let M_0 be the stabilizer of the decomposition $V = Ew_{-1} \oplus Ew_1$ and P_0 be the stabilizer of the flag $\{0\} \subset Ew_{-1} \subset V$.

(IV) Here U is attached to a flag $\{0\} \subset Fe_{-2} \subset Fe_{-2} + Fe_{-1} \subset Fe_{-2} + V^1 \subset V$. We can pick e_2 so that $\{e_{-2}, e_2\}$ is a symplectic basis of V^2 adapted to Λ^2. We then let M_0 be the stabilizer of the decomposition $V = Fe_{-2} \oplus V^1 \oplus Fe_2$ and P_0 be the stabilizer of the flag $\{0\} \subset Fe_{-2} \subset Fe_{-2} \oplus V^1 \subset V$.

Let $\boldsymbol{k}_\theta$ be the form defined in the proof of Corollary 4.2. In each case, we have the following properties (see [4, §10] and [21, Lemma 5.6, Corollary 5.10]):

(i) U, H^1 and J^1 have Iwahori decompositions with respect to (M_0, P_0);

(ii) $J^1 \cap U_0 / H^1 \cap U_0$ is a totally isotropic subspace of J^1/H^1 with respect to the form $\boldsymbol{k}_\theta$;

(iii) $J^1 \cap M_0 \cap U / H^1 \cap M_0 \cap U$ is a maximal totally isotropic subspace of $J^1 \cap M_0 / H^1 \cap M_0$;

(iv) there is an orthogonal sum decomposition

$$\frac{J^1}{H^1} = \frac{J^1 \cap M_0}{H^1 \cap M_0} \perp \left(\frac{J^1 \cap U_0}{H^1 \cap U_0} \times \frac{J^1 \cap U_0^-}{H^1 \cap U_0^-} \right),$$

where U_0^- is the unipotent subgroup opposite to U_0 relative to M_0.

Then $(J^1 \cap U)H^1$ has an Iwahori decomposition with respect to (M_0, P_0) and $(J^1 \cap U)H^1/H^1 \simeq J^1 \cap M_0 \cap U / H^1 \cap M_0 \cap U \perp J^1 \cap U_0 / H^1 \cap U_0$ is a maximal totally isotropic subspace of J^1/H^1. In particular, from the construction of Heisenberg extensions, equation (4.4) follows.

For the final stage, as in the construction of β-extensions, there is an $\mathfrak{o}_E$-lattice sequence Λ_m such that $\tilde{J}^1 = (P^1(\Lambda_m) \cap B)J^1$, and $P^1(\Lambda^m)$ still has an Iwahori decomposition with respect to (M_0, P_0). Defining $\tilde{\eta}$ to be $\text{Ind}_{\tilde{H}^1}^{\tilde{J}^1} \Theta_\beta$, we see that $\tilde{\eta}|J^1 = \eta$, and one checks that

$$\text{Ind}_{\tilde{J}^1}^{P^1(\Lambda_m)} \tilde{\eta} \simeq \text{Ind}_{J_m^1}^{P^1(\Lambda_m)} \eta_m.$$

Since this uniquely determines $\tilde{\eta}$, from the definition of β-extension we get that $\kappa|_{\tilde{J}^1} = \tilde{\eta}$, and the result follows. ∎

4.2 The positive level generic supercuspidal representations of $\mathrm{Sp}_4(F)$

Here we show that all the positive level supercuspidal representations which are not in the list of Theorem 2.1 are indeed generic.

Theorem 4.5. *Let $\pi = c\text{-Ind}_J^G\lambda$ be a positive level irreducible supercuspidal representation of $G = \mathrm{Sp}_4(F)$ with underlying skew semisimple stratum $[\Lambda, n, 0, \beta]$. Suppose that this stratum is not in case IV and that there exists a maximal unipotent subgroup U of G such that $\psi_\beta|_{U_{der}} = 1$. Then π is generic.*

Proof. Except in case II when J/J^1 is isotropic, this is immediate from Theorem 4.3, since $\lambda|_{\tilde{J}^1} = \kappa|_{\tilde{J}^1}$ in these cases so $\lambda|_{J\cap U}$ contains the character $\psi_\beta|_{J\cap U}$; hence, by Proposition 1.4, π is generic.

Suppose now we are in case II and choose a Witt basis $\{w_{-1}, w_1\}$ for V over E attached to the unipotent subgroup U as in the proof of Theorem 4.3. The quotient $J/J^1 \simeq P(\Lambda)\cap B^\times/P_1(\Lambda)\cap B^\times$ is then isomorphic to $SL(2, k_F)$ if E/F is ramified, to $U(1,1)(k_E/k_F)$ if E/F is unramified.

We have $U \cap B^\times = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} /x \in F\}$ with respect to the E-basis $\{w_{-1}, w_1\}$ and

$$U \cap B^\times = \{u(x) = \begin{pmatrix} 1 & 0 & 0 & N(\beta)x \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} /x \in F\}$$

in the symplectic basis $\{w_{-1}, \beta w_{-1}, \frac{\beta}{N(\beta)}\frac{w_1}{2}, \frac{w_1}{2}\}$ of V over F.

Now recall that $\lambda = \kappa \otimes \sigma$, for σ some irreducible cuspidal representation of J/J^1. Note that all cuspidal representations σ of $U(1,1)(k_E/k_F)$ or $SL(2,k_F)$ are generic, since the maximal unipotent subgroup is abelian and σ cannot contain the trivial character, by Lemma 1.3. Here we only need the fact that (the inflation of) σ restricted to $J \cap U \cap B^\times$ contains some character of $J \cap U \cap B^\times$, which must have the form $u(x) \mapsto \psi_F(\alpha x)$ for some α in F. Since $\chi : \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & x & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \psi_F(\alpha x)$ is a character of U, we see that $\sigma|J \cap U$ contains $\chi|J \cap U$. Since $\kappa|_{J\cap U}$ contains ψ_β (Theorem 4.3), we deduce that $\lambda|_{J\cap U}$ contains the restriction to $J\cap U$ of the character $\chi\psi_\beta$ of U and, by Proposition 1.4, $\pi = c\text{-Ind}_J^G\lambda$ is generic (and the character $\chi\psi_\beta$ is non-degenerate). ■

4.3 Case IV

In case IV, any maximal unipotent subgroup U of G on which ψ_β defines a character can be written as upper triangular unipotent matrices in a symplectic basis $\{e_{-2}, e_{-1}, e_1, e_2\}$ as in the beginning of §3.3. Here $\{e_{-2}, e_2\}$ is a symplectic basis of V^2 hence $U \cap B^\times$ is made of matrices of the form $\begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Even though σ, as a cuspidal hence generic representation of $SL(2, k_F)$, does contain a non trivial character of $J \cap U \cap B^\times$, this character is by no means the restriction of a character of U. This is an heuristic explanation of the fact that none of the case IV supercuspidal representations are generic, unfortunately not a proof. In this section, we prove a crucial result towards non-genericity. The last step will be taken in section 5.4.

Proposition 4.6. *Let $\pi = c\text{-Ind}_J^G \lambda$ be a positive level supercuspidal representation from case IV and let U be a maximal unipotent subgroup of G such that $\psi_\beta|_{U_{der}} = 1$. Then the restriction of λ to $J \cap U \cap B^\times$ is a sum of non-trivial characters.*

Proof. We retain all the notation of section 4.1 and write U as in the beginning of this section. We also write $U^2 = U \cap B^\times = U \cap \mathrm{Sp}(V^2)$. Since $J = (J \cap P(\Lambda) \cap B^\times) J^1$ we have $J \cap U = (J \cap U^2)(J^1 \cap U)$ hence: $\tilde{J}^1 = (J \cap U^2) J^1$ and $\tilde{H}^1 = (J \cap U^2)(J^1 \cap U) H^1$. We claim that

(i) $J^1 \cap U^2 = H^1 \cap U^2$;

(ii) $\mathrm{Ind}_{\tilde{H}^1}^{\tilde{J}^1} \Theta_\beta \simeq 1_{J \cap U^2} \otimes \eta$.

The first claim comes from the definitions in [19] on p.131. Indeed $J^1 \cap \mathrm{Sp}(V^2) = H^1 \cap \mathrm{Sp}(V^2) = P_1(\Lambda^2)$. For the second, we note first that $1_{J \cap U^2} \otimes \eta$ does define a representation of $\tilde{J}^1$ since $J \cap U^2$ normalizes (J^1, η) and $\eta | J^1 \cap U^2 = \eta | H^1 \cap U^2$ is a multiple of ψ_β, trivial on U^2 ($\beta_2 = 0$). Frobenius reciprocity gives a non-zero intertwining operator between those representations, which are irreducible (recall from §4.1 that $\mathrm{Ind}_{\tilde{H}^1}^{\tilde{J}^1} \Theta_\beta | J^1 = \eta$).
We have $\lambda = \kappa \otimes \sigma$, with σ an irreducible cuspidal representation of $J/J^1 \cong SL_2(k_F)$. In particular, σ is generic and its restriction to $\tilde{J}^1 / J^1$ is a sum of nondegenerate characters χ. On the other hand, the second claim above, plus Theorem 4.3, tell us that κ is trivial on $J \cap U^2$, q.e.d. ■

5 Non-generic representations

Our goal in this section is to show that the positive level supercuspidal representations which are in the list of Theorem 2.1 are indeed non generic. For cases (I), (II) and (III) we will prove that whenever there is no maximal unipotent subgroup on which the function ψ_β is a character (see Proposition 3.4), an irreducible supercuspidal representation of $G = \mathrm{Sp}_4(F)$ with underlying skew semisimple stratum $[\Lambda, n, 0, \beta]$ is not generic. The main tool here is the criterion given in Proposition 1.6, hence we will work out in some detail the restriction of ψ_β to one-parameter subgroups. This technique will also provide us with a last piece of argument to settle non-genericity in case (IV).

5.1 The function ψ_β on some one-parameter subgroups

In a given symplectic basis $(e_{-2}, e_{-1}, e_1, e_2)$ of V, we will denote by U_k, for $k \in \{-2, -1, 1, 2\}$, the following root subgroup :

$$U_k = \{1 + x\, t_k \ ;\ x \in F\} \ \text{ with } \ t_k(e_j) = \begin{cases} e_{-k} & \text{if } j = k; \\ 0 & \text{otherwise.} \end{cases}$$

It is attached to a long root and has a filtration indexed by $s \in \mathbb{Z}$: $U_k(s) = \{1 + x\, t_k \ :\ x \in \mathfrak{p}_F^s\}$.
Let β be an element of A and let ψ_β be the function on G defined by $\psi_\beta(x) = \psi(\mathrm{tr}\beta(x-1))$, $x \in G$. For k in $\{-2, -1, 1, 2\}$, let $\epsilon(k)$ be the sign of k.

Lemma 5.1. *Fix $g \in G$ and k in $\{-2, -1, 1, 2\}$. Let $s \in \mathbb{Z}$.*

(i) *For any x in F we have: $\psi_\beta\left(1 + x\, {}^g t_k\right) = \psi\left(\epsilon(k)\, x\, h(ge_{-k}, \beta g e_{-k})\right)$.*

(ii) *The character ψ_β of ${}^g U_k$ is non trivial on ${}^g U_k(s)$ if and only if $s \leq -\nu_F h(ge_{-k}, \beta g e_{-k})$.*

Proof. We keep the usual notation ${}^g u = gug^{-1}$, $u^g = g^{-1}ug$. We have for $x \in F$:

$$\psi_\beta\left(1 + x\,{}^g t_k\right) = \psi\left(\mathrm{tr}(\beta x\,{}^g t_k)\right) = \psi\left(\mathrm{tr}(x\beta^g\, t_k)\right) = \psi(x(\beta^g)_{k,-k})$$

while the $(k,-k)$ entry of β^g is $(\beta^g)_{k,-k} = \epsilon(k)h(e_{-k}, \beta^g e_{-k}) = \epsilon(k)h(ge_{-k}, \beta ge_{-k})$. ■

5.2 Proof for a maximal simple stratum

We work in this paragraph with a maximal simple stratum $[\Lambda, n, 0, \beta]$ in $\mathrm{End}_F V$, satisfying the assumption in Theorem 2.1 namely, in terms of the form δ (see (3.3) and §3.2):

(i) *$E = F[\beta]$ is a biquadratic extension of F with fixed points E_0 under $x \mapsto \bar{x}$; in particular $N_{E/E_0}(E^\times) = F^\times E_0^{\times 2}$.*

(ii) *The subset $D(V) = \{\beta\delta(v,v) \ :\ v \in V, v \neq 0\}$ of $E_0^\times$ is the $N_{E/E_0}(E^\times)$-coset in $E_0^\times$ that does not contain the kernel of $\mathrm{tr}_{E_0/F}$.*

Note that lattice duality with respect to the form δ, defined by $L^\# = \{v \in V : \delta(v, L) \subseteq \mathfrak{p}_E\}$ for L an $\mathfrak{o}_E$-lattice in V, coincides with lattice duality with respect to h. We may and do assume that the self-dual lattice chain Λ satisfies $\Lambda(i)^\# = \Lambda(d - i)$, $i \in \mathbb{Z}$, with $d = 0$ or 1. We also may and do assume that Λ is strict, i.e. has period 2.

Lemma 5.2. *Pick v_0 in V such that $\Lambda(i) = \mathfrak{p}_E^i v_0$ for $i \in \mathbb{Z}$. We have $\nu_E\delta(v_0, v_0) = 1 - d$ and, for $v \in V$ and s in $\mathbb{Z}$:*

$$\nu_F h(v, \beta v) = \frac{1}{e(E_0/F)}\nu_{E_0}(\beta\delta(v,v));$$
$$\nu_F h(v, \beta v) = s \iff v \in \Lambda(s + \tfrac{1}{2}(n + d - 1)) - \Lambda(s + \tfrac{1}{2}(n + d + 1)).$$

Proof. We have $h(v, \beta v) = -2\,\mathrm{tr}_{E_0/F}(\beta\delta(v,v))$ (§3.2); the first statement is thus a consequence of the following property:

> *Let x be an element of the $F^\times E_0^{\times 2}$-coset in $E_0^\times$ that does not contain the kernel of $\mathrm{tr}_{E_0/F}$. Then $\nu_F \mathrm{tr}_{E_0/F}\, x = \frac{1}{e(E_0/F)}\nu_{E_0} x$.*

Indeed we have $\nu_F \mathrm{tr}_{E_0/F}\, x = \frac{1}{e(E_0/F)}\left[\nu_{E_0} x + \nu_{E_0}\left(1 + \frac{\tilde{x}}{x}\right)\right]$ where $x \mapsto \tilde{x}$ is the Galois conjugation of E_0 over F. Let x and y in $E_0^\times$ such that $\dfrac{\tilde{x}}{x}$ and $\dfrac{\tilde{y}}{y}$ belong to $-1 + \mathfrak{p}_{E_0}$ and let $u = x/y$. Then $\tilde{u}/u$ belongs to $1 + \mathfrak{p}_{E_0}$, which implies that u belongs to $F^\times E_0^{\times 2}$ (easy checking according to the ramification of E_0 over F). Hence x and y are in the same $F^\times E_0^{\times 2}$-coset, that must be the coset containing the kernel of $\mathrm{tr}_{E_0/F}$, q.e.d.

The second statement is now immediate. We can write $v \in V$ as $v = u\, v_0$ with $u \in E$. Then:

$$\nu_F h(v, \beta v) = \frac{1}{e(E_0/F)}\frac{1}{e(E/E_0)}\left[\nu_E(\beta\delta(v_0, v_0)) + \nu_E(u\bar{u})\right]$$

hence $\nu_F h(v, \beta v) = \frac{1}{2}(-n + 1 - d) + \nu_E u$, q.e.d. ■

We now fix a symplectic basis $(e_{-2}, e_{-1}, e_1, e_2)$ adapted to Λ: there are non-decreasing functions $\alpha_s : \mathbb{Z} \to \mathbb{Z}$ such that

$$\Lambda(j) = \bigoplus_{s\in\{-2,-1,1,2\}} \mathfrak{p}_F^{\alpha_s(j)} e_s \quad (j \in \mathbb{Z}). \tag{5.3}$$

Proposition 5.4. *For any $g \in P(\Lambda)$, for any $k \in \{-2,-1,1,2\}$, the character ψ_β of ${}^gU_k$ is non trivial on ${}^gU_k \cap P_{[\frac{n}{2}]+1}(\Lambda)$.*

Proof. Note first that $P(\Lambda)$ normalizes $P_{[\frac{n}{2}]+1}(\Lambda)$, so ${}^gU_k \cap P_{[\frac{n}{2}]+1}(\Lambda)$ is equal to ${}^gU_k(s)$, where $s \in \mathbb{Z}$ is defined by $U_k \cap P_{[\frac{n}{2}]+1}(\Lambda) = U_k(s)$. Using Lemma 6.2 for our lattice sequence Λ of period 2 we get, for $x \in F$:

$$\begin{aligned} 1 + x\,t_k \in P_{[\frac{n}{2}]+1}(\Lambda) &\iff \forall j,\ x\,t_k\Lambda(j) \subset \Lambda(j + \left[\tfrac{n}{2}\right] + 1) \\ &\iff \forall j,\ e_{-k} \in \Lambda(j - 2\alpha_k(j) + \left[\tfrac{n}{2}\right] + 1 - 2\nu_F x) \\ &\iff \nu_\Lambda\left(e_{-k}\right) \geq \max\left\{j - 2\alpha_k(j); j \in \mathbb{Z}\right\} + \left[\tfrac{n}{2}\right] + 1 - 2\nu_F x \\ &\iff \nu_\Lambda\left(e_{-k}\right) \geq \nu_\Lambda\left(e_k\right) + \left[\tfrac{n}{2}\right] + 1 - 2\nu_F x \\ &\iff 2\nu_F x \geq 2\nu_\Lambda\left(e_k\right) + \left[\tfrac{n}{2}\right] + 2 - d. \end{aligned}$$

Hence $s = \nu_\Lambda\left(e_k\right) + 1 + \left[\dfrac{[\frac{n}{2}] - d + 1}{2}\right]$; we have to prove $s \leq -\nu_F h(ge_{-k}, \beta g e_{-k})$ (Lemma 5.1). Since g belongs to $P(\Lambda)$ we have by Lemmas 5.2 and 6.2:

$$\nu_F h(ge_{-k}, \beta g e_{-k}) = \nu_F h(e_{-k}, \beta e_{-k}) = -\nu_\Lambda\left(e_k\right) + d - 1 - \frac{1}{2}(n + d - 1)).$$

The condition we need is thus $\left[\dfrac{[\frac{n}{2}] - d + 1}{2}\right] \leq \frac{1}{2}(n - d - 1)$ which holds for $n \geq 1$ (the right hand side is an integer by Lemma 5.2). ■

We now derive Theorem 2.1 in this case: the representation π is not generic. Indeed if it was, we could find, by Proposition 1.6, some $g \in P(\Lambda)$ and some $k \in \{-2,-1,1,2\}$ such that λ contains the trivial character of ${}^gU_k \cap J$. In particular the restriction of λ to $P_{[\frac{n}{2}]+1}(\Lambda)$ would contain the trivial character of ${}^gU_k \cap P_{[\frac{n}{2}]+1}(\Lambda)$. Since this restriction is a multiple of ψ_β this is impossible.

5.3 Proof for a maximal semi-simple or non-maximal simple stratum

We will in this section treat simultaneously cases (II) and (III) in Theorem 2.1, granted that:

Lemma 5.5. *Under the assumption of case (II) in Theorem 2.1, any splitting $V = V^1 \perp V^2$ of V into two one-dimensional E-vector spaces splits the lattice chain Λ, that is, for any $t \in \mathbb{Z}$:*

$$\Lambda(t) = \Lambda^1(t) \perp \Lambda^2(t) \quad \text{with} \quad \Lambda^i(t) = \Lambda(t) \cap V^i,\ i = 1, 2.$$

Proof. We know from section 3 that the given assumption amounts to the fact that the quadratic form $v \mapsto h(v, \beta v)$ on V has no non trivial isotropic vectors, which implies that the anti-hermitian form δ on V defined by $h(av, w) = \mathrm{tr}_{E/F}(a\delta(v, w))$ for all $a \in E$, $v, w \in V$, is anisotropic. From [1]A.2, the lattice chain underlying Λ is thus the unique self-dual $\mathfrak{o}_E$-lattice chain in V. On the other hand self-dual $\mathfrak{o}_E$-lattice chains Λ^i in V^i, $i = 1, 2$, can be summed into a self-dual $\mathfrak{o}_E$-lattice chain in V. Unicity implies that Λ is obtained in this way. ■

We can now let $[\Lambda, n, 0, \beta]$, with $\Lambda = \Lambda^1 \perp \Lambda^2$, $\beta = \beta_1 + \beta_2$ and $n = \max\{n_1, n_2\}$, be a maximal skew semi-simple stratum or a non-maximal skew simple stratum in $\mathrm{End}_F V$: we place ourselves in the situation of case (II) or case (III) in Proposition 3.4, case (II) being obtained by letting $E_1 = E_2 = E$ and $\beta_1 = \beta_2 = \beta$. In particular $E_1 = F[\beta_1]$ *is always isomorphic to* $E_2 = F[\beta_2]$ and one checks easily that the conditions (ii) or (iii) in Proposition 3.4 are equivalent to saying that *the symplectic form satisfies:*

$$\forall v_1 \in V^1,\ \forall v_2 \in V^2,\ -h(v_1, \beta v_1)\, h(v_2, \beta v_2) \notin N_{E_i/F}(E_i^\times).$$

For an homogeneous treatment regardless of the ramification over F of the quadratic extensions involved, we use the conventions explained in section 6.2, namely the lattices sequences Λ^1, Λ^2 and Λ are all normalized in such a way that they have period 4 over F and duality given by $d = 1$.

Lemma 5.6. *Let* $v_1 \in V^1$, $v_2 \in V^2$ *and* $v = v_1 + v_2 \in V$. *Let* $\mathbf{e} = e(E_i/F)$.

(1) $\nu_F h(v, \beta v) = \min\{\nu_F h(v_1, \beta_1 v_1),\ \nu_F h(v_2, \beta_2 v_2)\}$;

(2) $\nu_\Lambda(v) \geq 2\nu_F h(v, \beta v) - \max\{\dfrac{2\nu_{E_1}\beta_1}{\mathbf{e}}, \dfrac{2\nu_{E_2}\beta_2}{\mathbf{e}}\}$.

Proof. (1) We have $h(v, \beta v) = h(v_1, \beta_1 v_1) + h(v_2, \beta_2 v_2)$. The assertion follows if the valuations of $h(v_i, \beta_i v_i)$, $i = 1, 2$, are distinct. If they are equal and finite, write $h(v, \beta v) = h(v_1, \beta_1 v_1)(1 + \frac{h(v_2,\beta_2 v_2)}{h(v_1,\beta_1 v_1)})$. Since $\frac{h(v_2,\beta_2 v_2)}{h(v_1,\beta_1 v_1)}$ cannot be congruent to -1 mod $\mathfrak{p}_F$ (its opposite would be a square hence a norm) the result follows.
(2) From Lemma 6.4 and (1) we get $\nu_\Lambda(v) = \min_{i=1,2} \nu_{\Lambda_i}(v_i) = \min_{i=1,2}(2\nu_F h(v_i, \beta_i v_i) - \dfrac{2\nu_{E_i}\beta_i}{\mathbf{e}})$ whence the result. ■

We fix symplectic bases $\{e_i, e_{-i}\}$ of V^i, $i = 1, 2$, adapted to Λ_i and use notation 5.3. We denote by $\mathbf{1}^1$ and $\mathbf{1}^2$ the orthogonal projections of V onto V^1 and V^2 respectively.
We need information on the intersection of one-parameter subgroups ${}^gU_k$ with subgroups of G of the following form (with a_1, a_2 positive integers and $a = \max\{a_1, a_2\}$):

$$L = \big(1 + \mathfrak{a}_{a_1}(\Lambda^1) + \mathfrak{a}_{a_2}(\Lambda^2) + \mathfrak{a}_a(\Lambda)\big) \cap G.$$

Lemma 5.7. *Let* $g \in P(\Lambda)$; *let* $g_i = \nu_\Lambda\left(\mathbf{1}^i\,(g e_{-k})\right) \in \mathbb{Z} \cup +\infty$. *Then*

$${}^gU_k \cap L = {}^gU_k(s) \quad \textit{with} \quad s = \left\lceil \frac{\nu_\Lambda\,(e_k) + \max_{i=1,2}\{a_i - g_i\} + 3}{4} \right\rceil.$$

Proof. We have for $x \in F$ and $g \in G$ (see [9] 2.9):

$$1 + x\,{}^g t_k \in L \iff x\,{}^g t_k \Lambda(j) \subseteq \Lambda^1(j + a_1) + \Lambda^2(j + a_2) \text{ for any } j \in \mathbb{Z}.$$

Since $t_k\,\Lambda(j) = \mathfrak{p}_F^{\alpha_k(j)} e_{-k}$ we have, if g belongs to $P(\Lambda)$:

$$\begin{aligned} 1 + x\,{}^g t_k \in L &\iff \forall j,\ x g e_{-k} \in \Lambda^1(j + a_1 - 4\alpha_k(j)) + \Lambda^2(j + a_2 - 4\alpha_k(j)) \\ &\iff \text{for } i = 1, 2,\ \forall j,\ 4\nu_F x + g_i \geq j - 4\alpha_k(j) + a_i. \end{aligned}$$

We conclude with Lemma 6.2. ■

We now apply this lemma to the subgroup L obtained with $a_i = \left[\frac{n_i}{2}\right]+1$, $i=1,2$. Define integers l_1 and l_2 by $l_i = \nu_F h(\mathbf{1}^i\,(ge_{-k})\,, \beta_i \mathbf{1}^i\,(ge_{-k}))$. From Lemmas 5.1 and 5.7, the character ψ_β of gU_kg^{-1} is non trivial on $gU_kg^{-1} \cap L$ if and only if

$$\left[\frac{\nu_\Lambda\,(e_k) + \max_{\,i=1,2}\{\left[\frac{n_i}{2}\right]+1-g_i\}+3}{4}\right] \leq -\min\{l_1, l_2\}.$$

On the other hand we have $g_i = 2l_i - \dfrac{2\nu_{E_i}\beta_i}{\mathbf{e}}$ by Lemma 6.4, and Lemma 6.3 relates $n_i = -\nu_{\Lambda_i}(\beta_i)$ and $\nu_{E_i}\beta_i$; in any case, one checks:

$$\max_{\,i=1,2}\{\left[\frac{n_i}{2}\right]+1-g_i\} = \varepsilon - 2\min\{l_1,l_2\} \quad \text{ with } \varepsilon = 0 \text{ or } 1.$$

It follows that ψ_β is non trivial on $gU_kg^{-1} \cap L$ if and only if $\nu_\Lambda\,(e_k) \leq -2\min\{l_1,l_2\} - \varepsilon$, that is, $\nu_\Lambda\,(e_k) \leq -2\nu_F h(ge_{-k}, \beta ge_{-k}) - \varepsilon$. Now Lemma 5.6 gives us:

$$\nu_\Lambda\,(ge_{-k}) \geq 2\nu_F h(ge_{-k}, \beta ge_{-k}) - \max\{\frac{2\nu_{E_1}\beta_1}{\mathbf{e}}, \frac{2\nu_{E_2}\beta_2}{\mathbf{e}}\}.$$

Since g belongs to $P(\Lambda)$, we have $\nu_\Lambda\,(ge_{-k}) = \nu_\Lambda\,(e_{-k}) = -\nu_\Lambda\,(e_k)$ (Lemma 6.2), which implies the desired inequality. We have just proven:

Proposition 5.8. *For any $g \in P(\Lambda)$, for any $k \in \{-2,-1,1,2\}$, the character ψ_β of gU_kg^{-1} is non trivial on $gU_kg^{-1} \cap L$, where*

$$L = \left(I + \mathfrak{a}_{\left[\frac{n_1}{2}\right]+1}(\Lambda^1) + \mathfrak{a}_{\left[\frac{n_2}{2}\right]+1}(\Lambda^2) + \mathfrak{a}_{\left[\frac{n}{2}\right]+1}(\Lambda)\right) \cap G.$$

At this point we can derive Theorem 2.1 in cases (II) and (III) as in the previous subsection: the representation π is not generic. Indeed if it was, we could find, by Proposition 1.6, some $g \in P(\Lambda)$ and some $k \in \{-2,-1,1,2\}$ such that λ contains the trivial character of ${}^gU_k \cap J$. In particular the restriction of λ to the above subgroup L would contain the trivial character of ${}^gU_k \cap L$. Since this restriction is a multiple of ψ_β this is impossible.

5.4 Non-genericity in the degenerate case (IV)

We let again $[\Lambda, n, 0, \beta]$, with $\Lambda = \Lambda^1 \oplus \Lambda^2$, $\beta = \beta_1 + \beta_2$ and $n = \max\{n_1, n_2\}$, be a skew semi-simple stratum in $\mathrm{End}_F V$, but we assume that one of the two simple strata involved is null (case IV). Although there does exist a maximal unipotent subgroup on which ψ_β is a character, this character is then degenerate (Proposition 3.4 and Remark 3.5). We will show that the corresponding supercuspidal representation is non-generic, using Proposition 4.6 and the criterion 1.6.

Criterion 1.6 involves conjugacy by elements in an Iwahori subgroup. We will find convenient to use the standard Iwahori subgroup, and to use an Iwahori subgroup normalizing the lattice chain Λ. These conditions can both be fulfilled at the possible cost of exchanging Λ^1 and Λ^2, that is, we have to complicate notations and let $\{1,2\} = \{r,s\}$ with $[\Lambda^r, n_r, 0, \beta_r]$ not null and $[\Lambda^s, n_s, 0, \beta_s] = [\Lambda^s, 0, 0, 0]$; in particular $n = n_r$.

Since the proof below is rather technical, we first sketch it, assuming $\beta_2 = 0$. In a symplectic basis as in §3.3, ψ_β does define a character of the upper and lower triangular unipotent subgroups, trivial on the long root subgroups $U_{\pm 2}$ corresponding to the null stratum but non trivial on the other long root subgroups $U_{\pm 1}$. As in the previous case, we have a subgroup L of J on which λ restricts to a multiple of ψ_β. We will show that for any Iwahori conjugate ${}^gU_{\pm 1}$, ψ_β is non trivial on ${}^gU_{\pm 1}\cap L$. Next we will identify subsets X_2 and X_{-2} of I such that, for $g\in X_{\pm 2}$, ψ_β is again non trivial on ${}^gU_{\pm 2}\cap L$. The last step will be to show that for $g\in I-X_{\pm 2}$, if the representation λ contains the trivial character of ${}^gU_{\pm 2}\cap J$, then it contains the trivial character of $U_{\pm 2}\cap J$ – this last possibility being excluded by Proposition 4.6. Hence, by Proposition 1.6, the representation induced from λ cannot be generic.

We pick a symplectic basis (e_{-2},e_{-1},e_1,e_2) of V adapted to Λ and such that $P(\Lambda)$ contains the standard Iwahori subgroup I consisting of matrices with entries in $\mathfrak{o}_F$ which are upper triangular modulo $\mathfrak{p}_F$. We normalize lattice sequences Λ_1 and Λ_2 such that they have period 4 over F and duality invariant 1 and define $\epsilon=0$ if Λ_r contains a self-dual lattice, $\epsilon=1$ otherwise (§6.2). Let:

$$L=\Big(I+\mathfrak{a}_1(\Lambda^s)+\mathfrak{a}_{[\frac{n}{2}]+1}(\Lambda^r)+\mathfrak{a}_{[\frac{n}{2}]+1}(\Lambda)\Big)\cap G.$$

This is a subgroup of H^1 (from the definition [19] p.131) on which θ restricts to ψ_β ([20] Lemma 3.15).

Lemma 5.9. *Let k in $\{-2,-1,1,2\}$. For $y\in P(\Lambda)$, define $y_r=\nu_\Lambda(\mathbf{1}^r(ye_{-k}))$, $y_s=\nu_\Lambda(\mathbf{1}^s(ye_{-k}))$ and $l_r=\nu_F h(\mathbf{1}^r\,(ye_{-k})\,,\beta_r\mathbf{1}^r\,(ye_{-k}))$. The character ψ_β of yU_ky^{-1} is non trivial on $yU_ky^{-1}\cap L$ if and only if either $\mathbf{1}^r(ye_{-k})=0$ or $\mathbf{1}^r(ye_{-k})\neq 0$ and*

$$\begin{cases} y_r\le \nu_\Lambda(e_{-k})+\left[\frac{n}{2}\right]+1-2\epsilon\\ y_s\ge -\nu_\Lambda(e_{-k})+4\,l_r+1\end{cases}$$

Proof. We have $h(v,\beta v)=h(v_r,\beta_r v_r)$ (for $v=v_1+v_2$, $v_i\in V^i$), so with Lemmas 6.4 and 6.3:

$$\nu_F h(v,\beta v)=t\iff \mathbf{1}^r(v)\in\Lambda^r(2t+\left[\tfrac{n}{2}\right]+1-\epsilon)-\Lambda^r(2t+\left[\tfrac{n}{2}\right]+1-\epsilon+1).$$

In particular, if $\mathbf{1}^r(ye_{-k})=0$, the character ψ_β is trivial on yU_ky^{-1} (Lemma 5.1).
We now assume $\mathbf{1}^r(ye_{-k})\neq 0$ and get from Lemma 6.4:

$$y_r=2\,l_r+\left[\frac{n}{2}\right]+1-\epsilon. \tag{5.10}$$

We apply Lemma 5.7 to L:

$$yU_ky^{-1}\cap L=yU_k(t)y^{-1}\text{ with }t=\left[\frac{\nu_\Lambda(e_k)+\max\{1-y_s,\epsilon-2\,l_r\}+3}{4}\right].$$

Using Lemma 5.1 we conclude that the character ψ_β of yU_ky^{-1} is non trivial on $yU_ky^{-1}\cap L$ if and only if $t\le -l_r$ whence the result (note that $\nu_\Lambda(e_k)=-\nu_\Lambda(e_{-k})$ by Lemma 6.2). ■

Lemma 5.11. *Let $y\in I$. If $|k|=r$, or if $|k|=s$ and $\mathbf{1}^r(ye_{-k})\notin\Lambda^r(\nu_\Lambda(e_{-k})+\left[\frac{n+1}{2}\right])$, the character ψ_β of yU_ky^{-1} is non trivial on $yU_ky^{-1}\cap L$.*

Proof. Since y belongs to $P(\Lambda)$ we certainly have $\mathbf{1}^r(ye_{-k})\in\Lambda^r(\nu_\Lambda(e_{-k}))$, $\mathbf{1}^s(ye_{-k})\in\Lambda^s(\nu_\Lambda(e_{-k}))$, and:

$$\begin{aligned}&\text{either }\textbf{(a)}\ \ \mathbf{1}^r(ye_{-k})\notin\Lambda^r(\nu_\Lambda(e_{-k})+1)\\ &\quad\text{or }\textbf{(b)}\ \ \mathbf{1}^r(ye_{-k})\in\Lambda^r(\nu_\Lambda(e_{-k})+1)\text{ and }\mathbf{1}^s(ye_{-k})\notin\Lambda^s(\nu_\Lambda(e_{-k})+1).\end{aligned}\tag{5.12}$$

Assume first **(a)**. Then the first condition in Lemma 5.9 is satisfied and the second will hold if $-\nu_\Lambda(e_{-k}) + 4\,l_r + 1 \le \nu_\Lambda(e_{-k})$. But we have $y_r = 2\,l_r + \left[\frac{n}{2}\right] + 1 - \epsilon$ (5.10) whence

$$4l_r = 2(\nu_\Lambda(e_{-k}) - \left[\frac{n}{2}\right] - 1 + \epsilon) \le 2\nu_\Lambda(e_{-k}) - 1.$$

Hence:

$$\textit{if } y \in P(\Lambda) \textit{ and } \mathbf{1}^r(ye_{-k}) \notin \Lambda^r(\nu_\Lambda(e_{-k}) + 1),\ \psi_\beta \textit{ is non trivial on } yU_ky^{-1} \cap L. \tag{5.13}$$

The discussion now will rely on $|k|$. If $|k| = r$ and $y \in I$, then **(a)** holds and (5.13) gives the result. If now $y \in I$ and $|k| = s$, we are left with case **(b)** in (5.12); in particular $y_s = \nu_\Lambda(e_{-k})$. We only have to check that the assumption $\mathbf{1}^r(ye_{-k}) \notin \Lambda^r(\nu_\Lambda(e_{-k}) + \left[\frac{n+1}{2}\right])$ implies the two inequalities in 5.9, which is straightforward granted that, when $\epsilon = 1$, n is even (6.3). ■

Lemma 5.14. *Let $|k| = s$ and $y \in I$ such that $\mathbf{1}^r(ye_{-k}) \in \Lambda^r(\nu_\Lambda(e_{-k}) + \left[\frac{n+1}{2}\right])$. Define $z = I + Z - \bar{Z}$ with $Z(e_{-k}) = -y_{-k,-k}^{-1}\mathbf{1}^r(ye_{-k})$ and $Z(e_t) = 0$ for $t \neq -k$. Then z belongs to $P_{\left[\frac{n+1}{2}\right]}(\Lambda)$, contained in $J \cap I$, and $\mathbf{1}^r(zye_{-k}) = 0$.*

Proof. One checks easily that z belongs to I and $\mathbf{1}^r(zye_{-k}) = 0$. It remains to show that Z belongs to $\mathfrak{a}_{\left[\frac{n+1}{2}\right]}(\Lambda)$. We have, using notation 6.1 and Lemma 6.2:

$$\begin{aligned}
Z \in \mathfrak{a}_{\left[\frac{n+1}{2}\right]}(\Lambda) &\iff Z\,\mathfrak{p}_F^{\alpha_{-k}(t)} e_{-k} \subseteq \Lambda(t + \left[\tfrac{n+1}{2}\right]) \text{ for any } t \in \mathbb{Z} \\
&\iff Ze_{-k} \in \Lambda^r(t - 4\alpha_{-k}(t) + \left[\tfrac{n+1}{2}\right]) \text{ for any } t \in \mathbb{Z} \\
&\iff \mathbf{1}^r(ye_{-k}) \in \Lambda^r(\max{}_{t\in\mathbb{Z}}\{t - 4\alpha_{-k}(t)\} + \left[\tfrac{n+1}{2}\right]) \\
&\iff \mathbf{1}^r(ye_{-k}) \in \Lambda^r(\nu_\Lambda(e_{-k}) + \left[\tfrac{n+1}{2}\right]).
\end{aligned}$$

■

With this in hand we are ready to conclude:

Proposition 5.15. *If the representation $\mathrm{Ind}_J^G\lambda$ has a Whittaker model, there exists $k \in \{-s, s\}$ such that λ contains the trivial character of $U_k \cap J = U_k \cap P(\Lambda^s)$.*

Proof. Recall Proposition 1.6: *if the representation $\mathrm{Ind}_J^G\lambda$ has a Whittaker model, there exists $k \in \{-2, -1, 1, 2\}$ and $y \in I$ such that λ contains the trivial character of $yU_ky^{-1} \cap J$.* Note that if (k, y) is such a pair, so is (k, zyx) for any $z \in J \cap I$ and $x \in I \cap N_G(U_k)$.
Assume we are in this situation and pick such a pair (k, y). Since the restriction of λ to L is a multiple of ψ_β, Lemma 5.11 tells us that we must have $|k| = s$ and $\mathbf{1}^r(ye_{-k}) \in \Lambda^r(\nu_\Lambda(e_{-k}) + \left[\frac{n+1}{2}\right])$. The Proposition is then an immediate consequence of the following fact:

($*$) *The double class $(J \cap I)\,y\,(I \cap N_G(U_k))$ contains an element of $J \cap I$, indeed of $I \cap P(\Lambda^s)$.*

Let us now prove this fact. Assume first that $k = s = 2$ (then $r = 1$). Using the standard Iwahori decomposition of $y \in I$ and the fact that upper triangular matrices normalize U_2 we may assume that y is a lower triangular unipotent matrix. Now since $\mathbf{1}^1(ye_{-k})$ belongs to $\Lambda^r(\nu_\Lambda(e_{-k}) + \left[\frac{n+1}{2}\right])$

we may change y into zy where z is defined in Lemma 5.14. Note that z is also lower triangular unipotent, hence so is zy. Since $\mathbf{1}^1(zye_{-k}) = 0$, zy has the following shape:

$$zy = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ c & 0 & 0 & 1 \end{pmatrix}.$$

The middle block $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ centralizes U_2 so the double class contains $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{pmatrix}$ which belongs to $I \cap \mathrm{Sp}(V^2)$ hence to $P(\Lambda) \cap \mathrm{Sp}(V^2) = P(\Lambda^2)$, q.e.d. The case $k = -s = -2$ is identical, replacing lower triangular by upper triangular. The cases $k = s = 1$ and $k = -s = -1$ are obtained similarly, using conjugation by $w = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varpi_F & 0 & 0 & 0 \\ 0 & \varpi_F & 0 & 0 \end{pmatrix}$ or w^{-1}, elements of $\mathrm{GSp}_4(F)$ normalizing I, to get a convenient Iwahori decomposition for the initial element y. ■

We are at last ready to prove non-genericity of supercuspidal representations coming from case (IV), which will finish the proof of Theorem 2.1. The group $U_k \cap J$ above is equal to $J \cap U \cap B^\times$ for some unipotent subgroup U chosen as in Proposition 4.6 – indeed we have $k \in \{-s, s\}$ so U_k is a long root subgroup attached to the two-dimensional space in which we have a null stratum. We know from Proposition 4.6 that the restriction of λ to $U_k \cap J$, a sum of non trivial characters, cannot contain the trivial character. Hence $\mathrm{Ind}_J^G \lambda$ doesn't have a Whittaker model.

6 Appendix: normalization of lattice sequences

We gather in the first two subsections the technical information about direct sums of lattice sequences that we need in several parts of the paper. Specifically, in most cases we deal with direct sums of self-dual lattice sequences in two-dimensional symplectic spaces and it will be convenient to homogeneize their F-periods and duality invariants.

Then we proceed with preliminary results in view of the proof, in §6.4, of Proposition 4.1

6.1 Self-dual lattice sequences

Let Λ be an $\mathfrak{o}_F$-lattice sequence in a finite dimensional F-vector space V and let $\mathfrak{a}_k(\Lambda)$ be the corresponding filtration of A. We define:

$$\nu_\Lambda(v) = \max\{i \in \mathbb{Z}; v \in \Lambda(i)\} \quad (v \in V); \quad \nu_\Lambda(g) = \max\{i \in \mathbb{Z}; g \in \mathfrak{a}_i(\Lambda)\} \quad (g \in A).$$

When V is equipped with a symplectic form h, lattice duality with respect to h is defined by $L^\# = \{v \in V : h(v, L) \subseteq \mathfrak{p}_F\}$ for L an $\mathfrak{o}_F$-lattice in V. An $\mathfrak{o}_F$-lattice sequence Λ in V is *self-dual* if there is an integer $d(\Lambda)$, the duality invariant of Λ, such that $\Lambda(t)^\# = \Lambda(d(\Lambda) - t)$.

We now let Λ be a self-dual lattice sequence of period e in V and we pick a symplectic basis $(e_{-2}, e_{-1}, e_1, e_2)$ adapted to Λ: there are non-decreasing functions $\alpha_s : \mathbb{Z} \to \mathbb{Z}$ such that

$$\Lambda(j) = \bigoplus_{s \in \{-2,-1,1,2\}} \mathfrak{p}_F^{\alpha_s(j)} e_s \quad (j \in \mathbb{Z}). \tag{6.1}$$

We will need the following straightforward property:

Lemma 6.2. *For k in $\{-2,-1,1,2\}$, we have*

$$\nu_\Lambda(e_k) = -\nu_\Lambda(e_{-k}) + d(\Lambda) - 1 = \max\{j - e\alpha_k(j); j \in \mathbb{Z}\}.$$

Proof. Since the period of Λ is e, the two sets $\{j - e\alpha_k(j); j \in \mathbb{Z}\}$ and $\{i \in \mathbb{Z}; \alpha_k(i) = 0\}$ coincide: the valuation of e_k is the maximum of either one. Next we use duality to check that

$$e_{-k} \notin \Lambda(x+d) \iff e_{-k} \notin \Lambda(-x)^{\#} \iff h(e_{-k}, \mathfrak{p}_F^{\alpha_k(-x)} e_k) \notin \mathfrak{p}_F \iff \alpha_k(-x) \leq 0.$$

■

6.2 Normalization of our lattice sequences

We start as in §2 case (III) with an orthogonal decomposition $V = V^1 \perp V^2$ and not null skew simple strata $[\Lambda^i, n_i, 0, \beta_i]$ in $\mathrm{End}_F(V^i)$. We let $E_i = F[\beta_i]$. We will find convenient to normalize the lattice sequences Λ^i in such a way that their sum Λ is given by $\Lambda(t) = \Lambda^1(t) \perp \Lambda^2(t)$ for any $t \in \mathbb{Z}$, and is self-dual. This will be the case provided that Λ^1 and Λ^2 have the same period and $d(\Lambda^1) = d(\Lambda^2) = 1$ (see [20] for instance).

The quadratic form $v \mapsto h(v, \beta_i v)$ on V^i has no non trivial isotropic vectors, since the anti-hermitian form δ_i on V^i defined by $h(av, w) = \mathrm{tr}_{E_i/F}(a\delta_i(v,w))$ for all $a \in E_i$, $v, w \in V^i$, is anisotropic. Let v_i be a basis of V^i over E_i. There is some u_i in E_i, satisfying $\bar{u}_i = -u_i$, such that this form reads $\delta_i(xv_i, yv_i) = ux\bar{y}$ for all $x, y \in E_i$. Since E_i is tame over F, lattice duality is the same for h and δ_i, namely $\Lambda^i(t)^{\#} = \{v \in V^i; \delta_i(v, \Lambda^i(t)) \subset \mathfrak{p}_{E_i}\}$. We have here $(\mathfrak{p}_{E_i}^t v_i)^{\#} = \mathfrak{p}_{E_i}^{d_i - t} v_i$ with $d_i = 1 - \nu_E u_i$. Hence, up to a translation in indices, the unique self-dual $\mathfrak{o}_{E_i}$-lattice chain $(L_i)_{i\in\mathbb{Z}}$ in V^i satisfies one of three possibilities:

(i) E_i is ramified over F and $d_i = 0$;

(ii) E_i is unramified over F and $d_i = 0$;

(iii) E_i is unramified over F and $d_i = 1$.

In the two first cases we put $L_i'(t) = L_i([\frac{t}{2}])$ and get a lattice sequence with duality invariant $d_i' = 1$; in the third case we keep $L_i' = L_i$. In the ramified case we put $\Lambda_i = L_i'$: it has period 4. We now need to put $\Lambda_i = 2L_i'$ in the second case, $\Lambda_i = 4L_i'$ in the third case, and we get a normalization of our $\mathfrak{o}_{E_i}$- lattice sequences in V_i such that their period is 4 and duality invariant 1. We will use the following straightforward properties of Λ_i, in each of the three cases above:

Lemma 6.3. *Normalize the lattice sequence Λ^i such that its period over F is 4 and $d(\Lambda^i) = 1$.*

(i) *If E_i is ramified over F then $\nu_{\Lambda_i}(V^i - \{0\}) = 2\mathbb{Z} + 1$ and $\nu_{\Lambda_i}(\beta_i) = 2\nu_{E_i}(\beta_i) + 1$;*

(ii) *If E_i is unramified over F and Λ_i contains a self-dual lattice then $\nu_{\Lambda_i}(V^i - \{0\}) = 4\mathbb{Z} + 2$ and $\nu_{\Lambda_i}(\beta_i) = 2(2\nu_{E_i}(\beta_i) + 1)$;*

(iii) *If E_i is unramified over F and Λ_i does not contain a self-dual lattice then $\nu_{\Lambda_i}(V^i - \{0\}) = 4\mathbb{Z}$ and $\nu_{\Lambda_i}(\beta_i) = 4\nu_{E_i}(\beta_i)$.*

We need to relate, under these conventions, the valuations relative to the lattice chains Λ_i with the valuations over F of the quadratic forms $h(v, \beta_i v)$, $v \in V^i$.

Lemma 6.4. *Let $\mathbf{e}_i = e(E_i/F)$ and normalize the lattice sequence Λ^i such that its period over F is 4 and $d(\Lambda^i) = 1$. For any $v \in V^i$ we have:*

$$\nu_{\Lambda_i}(v) = 2\nu_F h(v, \beta_i v) - \frac{2\nu_{E_i}\beta_i}{\mathbf{e}_i}.$$

Proof. For v in V^i and a in E_i we have $h(av, \beta_i av) = \mathrm{N}_{E_i/F}(a)h(v, \beta_i v)$. The map $v \mapsto h(v, \beta_i v)$ on V_i is thus constant on the sets $\Lambda_i(t) - \Lambda_i(t+1)$ hence factors through the valuation ν_{Λ_i}: there is a map ϕ, defined on the image of ν_{Λ_i} and with values in $\mathbb{Z}$, such that $\nu_F h(v, \beta_i v) = \phi(\nu_{\Lambda_i}(v))$ for any non zero v in V^i. Periodicity over E_i implies $\phi(t + 2\mathbf{e}_i) = \phi(t) + \mathbf{e}_i$, $t \in \mathbb{Z}$: computing one value of ϕ is enough. We certainly have $\nu_F h(v, \beta_i v) = \dfrac{1}{\mathbf{e}_i}(\nu_{E_i}\beta_i + \nu_{E_i}\delta_i(v, v))$ whence:

(i) if E_i is ramified over F, then $\Lambda_i(1) = \Lambda_i(0) = \Lambda_i(1)^{\#}$ so $\nu_{\Lambda_i}(v) = 1$ implies $\nu_{E_i}\delta_i(v, v)) = 1$ and $\nu_F h(v, \beta_i v) = \dfrac{1}{2}(\nu_{E_i}\beta_i + 1)$;

(ii) if E_i is unramified over F and Λ_i contains a self-dual lattice then $\Lambda_i(2)^{\#} = \Lambda_i(-1) = \Lambda_i(2)$ so $\nu_{\Lambda_i}(v) = 2$ implies $\nu_{E_i}\delta_i(v, v)) = 1$ and $\nu_F h(v, \beta_i v) = \nu_{E_i}\beta_i + 1$;

(iii) If E_i is unramified over F and Λ_i does not contain a self-dual lattice then $\Lambda_i(0)^{\#} = \Lambda_i(1) = \varpi_F \Lambda_i(0)$ so $\nu_{\Lambda_i}(v) = 0$ implies $\nu_{E_i}\delta_i(v, v)) = 0$ and $\nu_F h(v, \beta_i v) = \nu_{E_i}\beta_i$.

■

6.3 Some intersections

The following Lemma and Corollary were suggested by Vytautas Paskunas. In the proofs, we may (and often will) ignore the condition that the stratum and character be skew, since the results in the skew case follow immediately by restriction to G – that is, we actually prove the statements for σ-stable semisimple strata in GL_4. To ease notation, this will be implicit so that, in the proofs below, U should be a σ-stable maximal unipotent subgroup of GL_4, etc.

First we need some notation in the semisimple case. For $[\Lambda, n, 0, \beta]$ a semisimple stratum in V with splitting $V = \bigoplus_{i=1}^{2} V^i$, we write $A = \oplus_{i,j=1}^{2} A^{ij}$ in block notation, where $A^{ij} = \mathrm{Hom}_F(V^j, V^i) = \mathbf{1}^j A \mathbf{1}^i$ and $\mathbf{1}^i$ is the projection onto V^i with kernel V^{3-i}. For $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$, with $k_1 \geq k_2 \geq 1$, define

$$\mathfrak{a}_{\mathbf{k}} = \mathfrak{a}_{\mathbf{k}}(\Lambda) = \begin{pmatrix} \mathfrak{a}_{k_1}^{11} & \mathfrak{a}_{k_1}^{12} \\ \mathfrak{a}_{k_1}^{21} & \mathfrak{a}_{k_2}^{22} \end{pmatrix}$$

and $\mathrm{U}^{\mathbf{k}}(\Lambda) = 1 + \mathfrak{a}_{\mathbf{k}}$.

Lemma 6.5. *Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum in A, with splitting $V = \bigoplus_{i=1}^{l} V^i$, $1 \leq l \leq 2$. Suppose also that $F[\beta]$ is of maximal degree over F, and let U be a maximal unipotent subgroup of G. Write B for the centralizer in A of β. For $k \geq m \geq 1$, we have*

$$\Big((\mathrm{U}^m(\Lambda) \cap B)\mathrm{U}^k(\Lambda)\Big) \cap U = (\mathrm{U}^k(\Lambda) \cap U).$$

We remark that the proof below is just for the case which interests us here (so that there are at most 2 pieces in the splitting) but it is straightforward to generalize the Lemma to the case where there are an arbitrary number of pieces.

Proof. We will prove the corresponding additive statement. Writing $U = 1+\mathbb{N}$, $\mathfrak{b}_m = \mathfrak{a}_m(\Lambda)\cap B$, it is:

$$(\mathfrak{b}_m + \mathfrak{a}_k)\cap\mathbb{N} = \mathfrak{a}_k\cap\mathbb{N}.$$

Notice that, since $B = F[\beta]$, the lattice $\mathfrak{b}_m$ contains no non-trivial nilpotent elements. We will show that, for $1\le m<k$,

$$(\mathfrak{b}_m + \mathfrak{a}_k)\cap\mathbb{N} \subset (\mathfrak{b}_{m+1} + \mathfrak{a}_k)\cap\mathbb{N}$$

and the result follows at once by an easy induction. So suppose $\epsilon\in\mathfrak{b}_m$ is such that $(\epsilon+\mathfrak{a}_k)\cap\mathbb{N}\ne\emptyset$. In particular, $(\epsilon+\mathfrak{a}_{m+1})\cap\mathbb{N}\ne\emptyset$ so there exists $s>0$ such that $\epsilon^s\in\mathfrak{a}_{sm+1}$. But then $\epsilon^s\in\mathfrak{b}_{sm+1}$ so (by [3], working block-by-block), the coset $\epsilon+\mathfrak{b}_{m+1}$ contains a nilpotent element, which must be 0. We deduce that $\epsilon\in\mathfrak{b}_{m+1}$, as required. ■

For $[\Lambda,n,0,\beta]$ a semisimple stratum in V with splitting $V=\bigoplus_{i=1}^2 V^i$, we write $M_{sp}=\mathrm{Aut}_F(V^1)\times\mathrm{Aut}_F(V^2)$, $U_{sp}=1+A^{12}$, $P_{sp}=M_{sp}U_{sp}$ and $\bar U_{sp}=1+A^{21}$.

Corollary 6.6. *Let $[\Lambda,n,0,\beta]$ be a semisimple stratum in A, with splitting $V=\bigoplus_{i=1}^l V^i$, $1\le l\le 2$, and let U be a maximal unipotent subgroup of G. Write B for the centralizer in A of β. We suppose that*

(i) *U has an Iwahori decomposition with respect to (M_{sp},P_{sp});*

(ii) *$U\cap B^\times$ is a maximal unipotent subgroup of $B^\times$.*

Then, for $k_1\ge\cdots\ge k_l\ge m\ge 1$ and $\mathbf{k}=(k_1,\dots,k_l)$, we have

$$\Big((\mathrm{U}^m(\Lambda)\cap B)\mathrm{U}^{\mathbf{k}}(\Lambda)\Big)\cap U = (\mathrm{U}^m(\Lambda)\cap B\cap U)(\mathrm{U}^{\mathbf{k}}(\Lambda)\cap U). \tag{6.7}$$

Again the statement is easily generalized to the case when the splitting has more than two pieces. Note also that, in the simple case, condition (i) is empty while condition (ii) is implied, for example, by the condition $\psi_\beta|_{U_{der}}=1$ ([5] Proposition 2.2). This is not true for semisimple strata.

Proof. First we reduce to the simple case. We notice that $\mathrm{U}^{\mathbf{k}}(\Lambda)$ and $B^\times\subset M_{sp}$ have Iwahori decompositions with respect to (M_{sp},P_{sp}). Since U also has such a decomposition, we reduce to proving that we have equality in (6.7) when we intersect both sides with U_{sp}, with M_{sp} and with $\bar U_{sp}$ respectively. Since $B\subset M_{sp}$, this is immediate for the unipotent radicals $\bar U_{sp}$, U_{sp}. Hence we are reduced to the intersection with M_{sp}, which, block-by-block, is just the simple case.

So now suppose $[\Lambda,n,0,\beta]$ is a simple stratum. As in Lemma 6.5, we will prove the corresponding additive statement: writing $U=1+\mathbb{N}$, it is:

$$(\mathfrak{b}_m+\mathfrak{a}_{\mathbf{k}})\cap\mathbb{N} = (\mathfrak{b}_m\cap\mathbb{N}) + (\mathfrak{a}_{\mathbf{k}}\cap\mathbb{N}), \tag{6.8}$$

where $k=k_1$. We will reduce to the case where $E=F[\beta]$ is maximal and invoke Lemma 6.5.

Write $d=[E:F]$; then, in the flag corresponding to U,

$$\{0\}=V_0\subset V_1\subset\cdots\subset V_N=V,$$

the subspace V_{di} is an E-subspace, for $0\le i\le N/d$. Let $U_0=1+\mathbb{N}_0$ be the unipotent subgroup corresponding to the maximal E-flag

$$\{0\}=V_0\subset V_d\subset\cdots\subset V_{di}\subset\cdots\subset V_N=V$$

and let $P_0 = 1 + \mathbb{P}_0$ the corresponding parabolic subgroup. There exists an E-decomposition $V = \oplus_{i=1}^{N/d} W_i$ of V such that for $0 \leq i \leq N/d$, $V_{di} = \oplus_{j=1}^{i} W_j$ and such that for every $t \in \mathbb{Z}$, $\Lambda(t) = \oplus_{i=1}^{N/d} \Lambda(t) \cap W_i$ (as a suitable variant of, e.g., [22] §II.1). Let L_0 be the corresponding Levi component of P_0 and let $\bar{U}_0 = 1 + \bar{\mathbb{N}}_0$ be the unipotent subgroup opposite U_0 with respect to L_0. The lattices $\mathfrak{b}_m$ and $\mathfrak{a}_k$ have (additive) Iwahori decompositions with respect to $\bar{\mathbb{N}}_0, \mathbb{P}_0$ ([4] §10) so

$$(\mathfrak{b}_m + \mathfrak{a}_k) \cap \mathbb{P}_0 = \mathfrak{b}_m \cap \mathbb{L}_0 + \mathfrak{b}_m \cap \mathbb{N}_0 + \mathfrak{a}_k \cap \mathbb{L}_0 + \mathfrak{a}_k \cap \mathbb{N}_0$$

and we have

$$(\mathfrak{b}_m + \mathfrak{a}_k) \cap \mathbb{N} = (\mathfrak{b}_m + \mathfrak{a}_k) \cap \mathbb{P}_0 \cap \mathbb{N} = (\mathfrak{b}_m \cap \mathbb{L}_0 + \mathfrak{a}_k \cap \mathbb{L}_0) \cap \mathbb{N} + \mathfrak{b}_m \cap \mathbb{N}_0 + \mathfrak{a}_k \cap \mathbb{N}_0$$

Hence we are reduced to showing

$$(\mathfrak{b}_m \cap \mathbb{L}_0 + \mathfrak{a}_k \cap \mathbb{L}_0) \cap \mathbb{N} = \mathfrak{b}_m \cap \mathbb{L}_0 \cap \mathbb{N} + \mathfrak{a}_k \cap \mathbb{L}_0 \cap \mathbb{N},$$

which is the same as (6.8) in $\mathbb{L}_0$, where we can work block-by-block. In each block, the field extension E is maximal, so we have indeed reduced to the maximal case, which is Lemma 6.5. ■

Finally, we will need one more similar result, in the case of a minimal semisimple stratum – in this case the extra conditions of the previous Corollary are not satisfied.

Lemma 6.9. *Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum, with splitting $V = V^1 \bigoplus V^2$, $1 \leq l \leq 2$. Suppose that $\beta_i = \mathbf{1}^1 \beta \mathbf{1}^i$ is minimal, for $i = 1, 2$ and that $[\Lambda, n, n-1, \beta]$ is* not *equivalent to a simple stratum. Assume also (without loss of generality) that $n_1 \geq n_2 \geq 1$. Put $k_i = [\frac{n_i}{2}] + 1$, for $i = 1, 2$, and $\mathbf{k} = (k_1, k_2)$. Then $H^1 = (\mathrm{U}^1(\Lambda) \cap B)\mathrm{U}^{\mathbf{k}}(\Lambda)$ and, if U is a maximal unipotent subgroup of G on which ψ_β defines a character:*

$$H^1 \cap U = \mathrm{U}^{\mathbf{k}}(\Lambda) \cap U.$$

Proof. We work under the conventions made in the previous subsection. The subgroup U is necessarily given (see Proposition 3.4) by a flag of the following form:

$$\{0\} \subset \langle v_1 + v_2 \rangle \subset \langle v_1 + v_2, v_{-1} - v_{-2} \rangle \subset \langle v_1, v_2, v_{-1} - v_{-2} \rangle \subset V,$$

where $v_i \in V^i$ are non zero vectors such that $h(v_1, \beta_1 v_1) = -h(v_2, \beta_2 v_2) = \mu$ and $v_{-1} = \mu^{-1}\beta_1 v_1$, $v_{-2} = -\mu^{-1}\beta_2 v_2$. In particular $\{v_i, v_{-i}\}$ is a symplectic basis for V^i, adapted to Λ^i.

That $H^1 = (\mathrm{U}^1(\Lambda) \cap B)\mathrm{U}^{\mathbf{k}}(\Lambda)$ is clear from the definition. For the intersection property, we will need:

Lemma 6.10. *Put $\nu = \nu_{\Lambda_1}(v_1) - \nu_{\Lambda_2}(v_2)$ and let $E_{i,j}$ be the linear map sending v_j to v_i and all other basis elements to* 0*. We have*

$$\nu_\Lambda(E_{1,2}) = \nu, \qquad \nu_\Lambda(E_{-2,-1}) = n_1 - n_2 - \nu. \tag{6.11}$$

Furthermore, if $n_1 \geq n_2$, then $0 \leq \nu \leq n_1 - n_2$.

Proof. Computing the valuations of $E_{1,2}$ and $E_{-2,-1}$ is simple checking. Lemma 6.4 combined with our asssumption on v_1, v_2 implies $\nu = -\dfrac{2\nu_{E_1}\beta_1}{\mathbf{e}_1} + \dfrac{2\nu_{E_2}\beta_2}{\mathbf{e}_2}$. The inequality $0 \leq \nu \leq n_1 - n_2$ then follows from Lemma 6.3. ■

Write $U = 1 + \mathbb{N}$. Then the elements of $\mathbb{N}$ can be written (as matrices with respect to the basis $\{v_1, v_{-1}, v_2, v_{-2}\}$)

$$x = \begin{pmatrix} a & e & -a & c \\ -d & -b & d & -b \\ a & e+f & -a & c+f \\ d & b & -d & b \end{pmatrix},$$

for $a, b, c, d, e, f \in F$.

Now suppose that x as above also lies in the lattice (in block matrix form, each block 2×2)

$$\begin{pmatrix} \mathfrak{a}_m & \mathfrak{a}_{k_1} \\ \mathfrak{a}_{k_1} & \mathfrak{a}_{k_2} \end{pmatrix},$$

for some $m < k_1$. Then, using (6.11), we get:

- $aE_{2,1} \in \mathfrak{a}_{k_1}$ so $aE_{1,1} = aE_{1,2}E_{2,1} \in \mathfrak{a}_{k_1+\nu}$;
- $bE_{-1,-2} \in \mathfrak{a}_{k_1}$ so $bE_{-1,-1} = bE_{-1,-2}E_{-2,-1} \in \mathfrak{a}_{k_1+n_1-n_2-\nu}$;
- $(c+f)E_{2,-2} \in \mathfrak{a}_{k_2}$ so $(c+f)E_{1,-2} = (c+f)E_{1,2}E_{2,-2} \in \mathfrak{a}_{k_2+\nu}$;
  $cE_{1,-2} \in \mathfrak{a}_{k_1}$ so $fE_{1,-2} \in \mathfrak{a}_{\min\{k_1,k_2+\nu\}}$
  and $fE_{1,-1} = fE_{1,-2}E_{-2,-1} \in \mathfrak{a}_{\min\{k_1+n_1-n_2-\nu,k_2+n_1-n_2\}}$.
  $(e+f)E_{2,-1} \in \mathfrak{a}_{k_1}$ so $(e+f)E_{1,-1} = (e+f)E_{1,2}E_{2,-1} \in \mathfrak{a}_{k_1+\nu}$.

Writing $t = \min\{\nu, n_1-n_2-\nu, k_2+n_1-n_2-k_1\} \geq 0$, we have $aE_{1,1}, bE_{-1,-1}, eE_{1,-1} \in \mathfrak{a}_{k_1+t} \subset \mathfrak{a}_{m+1}$. In particular, looking at the "top-left" block of x, we have

$$\begin{pmatrix} a & e \\ -d & -b \end{pmatrix} \in \begin{pmatrix} 0 & 0 \\ -d & 0 \end{pmatrix} + \mathfrak{a}_{m+1}.$$

Now we prove that, for $m \leq k_2$ and $\mathbf{k} = (k_1, k_2)$,

$$\Big((\mathrm{U}^m(\Lambda) \cap B)\mathrm{U}^{\mathbf{k}}(\Lambda)\Big) \cap U = \mathrm{U}^{\mathbf{k}}(\Lambda) \cap U.$$

By Lemma 6.5, we have

$$\Big((\mathrm{U}^m(\Lambda) \cap B)\mathrm{U}^{\mathbf{k}}(\Lambda)\Big) \cap U \subset \Big((\mathrm{U}^m(\Lambda) \cap B)\mathrm{U}^{k_2}(\Lambda)\Big) \cap U = \mathrm{U}^{k_2}(\Lambda) \cap U.$$

Then we need only prove, for $k_2 \leq m < k_1$ (the additive statement)

$$\begin{pmatrix} \mathfrak{b}_m + \mathfrak{a}_{k_1} & \mathfrak{a}_{k_1} \\ \mathfrak{a}_{k_1} & \mathfrak{a}_{k_2} \end{pmatrix} \cap \mathbb{N} \subset \begin{pmatrix} \mathfrak{b}_{m+1} + \mathfrak{a}_{k_1} & \mathfrak{a}_{k_1} \\ \mathfrak{a}_{k_1} & \mathfrak{a}_{k_2} \end{pmatrix}.$$

But we have seen above that, if $\epsilon \in \begin{pmatrix} \mathfrak{b}_m & 0 \\ 0 & 0 \end{pmatrix}$ is such that $\epsilon + \mathfrak{a}_{\mathbf{k}}$ contains an element of $\mathbb{N}$, then $\epsilon + \begin{pmatrix} \mathfrak{a}_{m+1} & 0 \\ 0 & 0 \end{pmatrix}$ also contains a nilpotent element. But then (as in the proof of Lemma 6.5) $\epsilon + \begin{pmatrix} \mathfrak{b}_{m+1} & 0 \\ 0 & 0 \end{pmatrix}$ also contains a nilpotent element, which must be 0. Hence $\epsilon \in \mathfrak{b}_{m+1}$, as required. ∎

6.4 Proof of Proposition 4.1

It remains to prove Proposition 4.1: If $\theta \in \mathcal{C}(\Lambda, 0, \beta)$ is a skew semisimple character and U is a maximal unipotent subgroup of G such that $\psi_\beta|_{U_{der}} = 1$, then

$$\theta|_{H^1 \cap U} = \psi_\beta|_{H^1 \cap U}.$$

As in the previous section, we actually prove the same statement for σ-stable semisimple strata in GL_4, and the result follows by restriction to G. We remark that a skew element β generating a field such that $[F[\beta] : F] = 2$ is necessarily minimal. We proceed on a case-by-case basis:

Simple case (Cases (I) and (II)) We proceed by induction on $r = -k_0(\beta, \Lambda)$. We make the following additional hypothesis:

($*$) there exists a simple stratum $[\Lambda, n, r, \gamma]$ equivalent to $[\Lambda, n, r, \beta]$ such that $\psi_\gamma|_{U_{der}} = 1$.

In particular, we can then use the inductive hypothesis for the stratum $[\Lambda, n, 0, \gamma]$ with the same unipotent subgroup U. Note that this hypothesis is certainly satisfied when β is minimal, since we can take $\gamma = 0$. We will show later that it is also satisfied in the other cases that are relevant to us here.

We have $H^1(\beta, \Lambda) = (\mathrm{U}^1(\Lambda) \cap B) H^{[\frac{r}{2}]+1}(\beta, \Lambda)$ so, by Corollary 6.6, $H^1 \cap U = H^{[\frac{r}{2}]+1} \cap U$. Moreover, $H^{[\frac{r}{2}]+1}(\beta, \Lambda) = H^{[\frac{r}{2}]+1}(\gamma, \Lambda)$ and $\theta|_{H^{[\frac{r}{2}]+1}} = \theta_0|_{H^{[\frac{r}{2}]+1}} \psi_{\beta-\gamma}$, for some $\theta_0 \in \mathcal{C}(\Lambda, 0, \gamma)$. But, by the inductive hypothesis, θ_0 agrees with ψ_γ on $H^{[\frac{r}{2}]+1} \cap U$ and the result follows.

To finish, we must show that ($*$) is satisfied. The only case to consider is when $[\Lambda, n, 0, \beta]$ is a skew simple stratum with $F[\beta]$ maximal (of degree 4) and $r = -k_0(\beta, \Lambda) < n$. Then $[\Lambda, n, r, \beta]$ is equivalent to some skew simple stratum $[\Lambda, n, r, \gamma_0]$, with γ_0 *minimal* and $F[\gamma_0]$ of degree 2 over F. Note that, since $p \neq 2$, all extensions are tame here. Note also that the flag corresponding to the unipotent subgroup U is given by $V_i = \langle v, \beta v, \cdots, \beta^{i-1} v \rangle$, for some $v \in V$ with $h(v, \beta v) = 0$. Also put $\zeta = h(\beta v, \beta^2 v) \neq 0$, and $d = -v_F(\zeta)$.

Let $P(X) = X^2 + \lambda \in F[X]$ be the minimal polynomial of γ_0 and put $e_{-2} = v$, $e_{-1} = \beta v$, $e_1 = \zeta^{-1} P(\beta) v$, $e_2 = -\beta \zeta^{-1} P(\beta) v - k \beta v$, where $k = \zeta^{-2} h(\beta v, \beta^2 v) - 2\lambda\zeta^{-1}$ (this is a symplectic basis). With respect to this basis, β has matrix

$$\begin{pmatrix} 0 & -\lambda & 0 & \mu \\ 1 & 0 & k & 0 \\ 0 & \zeta & 0 & \lambda \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

for some $\mu \in F$. For $i, j \in \{\pm 1, \pm 2\}$, write $E_{i,j}$ for the linear map sending e_j to e_i and all other basis vectors to 0. We then have

$$E_{-2,-1} \in \mathfrak{a}_n, \qquad E_{1,-1} \in \mathfrak{a}_{de-r}.$$

Write $\beta = \gamma_0 + c_0$, with $c_0 \in \mathfrak{a}_{-r}$. Then $\beta^2 + \lambda = c_0\gamma_0 + \gamma_0 c_0 + c_0^2 \in \mathfrak{a}_{-n-r}$. From the matrix description of β, we get that

$$\beta^2 + \lambda = \begin{pmatrix} 0 & 0 & -(\mu + k\lambda) & 0 \\ 0 & k\zeta & 0 & (\mu + k\lambda) \\ \zeta & 0 & k\zeta & 0 \\ 0 & -\zeta & 0 & 0 \end{pmatrix}$$

Then:

- $(\mu+k\lambda)E_{-1,2} \in \mathfrak{a}_{-n-r}$ and $(\mu+k\lambda)E_{-2,2} = (\mu+k\lambda)E_{-2,-1}E_{-1,2} \in \mathfrak{a}_{-r}$;
- $\zeta E_{1,-1} \in \mathfrak{a}_{-r}$.

Hence

$$c = \begin{pmatrix} 0 & 0 & 0 & \mu+k\lambda \\ 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{a}_{-r}.$$

Put $\gamma = \beta - c$. Then $[\Lambda, n, r, \gamma]$ is a stratum as required by hypothesis $(*)$.

Minimal semisimple case (Case (III)) Now suppose $[\Lambda, n, 0, \beta]$ is a skew semisimple stratum, with splitting $V = V^1 \bigoplus V^2$, $1 \le l \le 2$, that $\beta_i = \mathbf{1}^1 \beta \mathbf{1}^i$ is minimal, for $i = 1,2$ and that $[\Lambda, n, n-1, \beta]$ is *not* equivalent to a simple stratum. Assume also (without loss of generality) that $n_1 \ge n_2 \ge 1$. Put $k_i = [\frac{n_i}{2}]+1$, for $i=1,2$, and $\mathbf{k} = (k_1, k_2)$. Then, by Lemma 6.9, $H^1 = (\mathrm{U}^1(\Lambda) \cap B)\mathrm{U}^{\mathbf{k}}(\Lambda)$ and $H^1 \cap U = \mathrm{U}^{\mathbf{k}}(\Lambda) \cap U$. But, by definition of θ, it agrees with ψ_β on $\mathrm{U}^{\mathbf{k}}(\Lambda)$ so the result follows.

Degenerate semisimple case (case (IV)) Now suppose $[\Lambda, n, 0, \beta]$ is again a skew semisimple stratum, but with $\beta_2 = 0$. In this case it is straightforward to see that the flag defining U must be given by

$$\{0\} \subset V_1 = \langle v_2 \rangle \subset V_2 = \langle v_2, v_1 \rangle \subset V_3 = V_1 \oplus V^1 \subset V_4 = V,$$

where $v_2 \in V^2$ and $v_1 \in V^1$. In particular, U satisfies the extra conditions in Corollary 6.6. Putting $k_1 = [\frac{n_1}{2}]+1$ and $k_2 = 1$, we have $H^1 = (\mathrm{U}^1(\Lambda) \cap B)\mathrm{U}^{\mathbf{k}}(\Lambda)$ so, by Corollary 6.6,

$$H^1 \cap U = (\mathrm{U}^1(\Lambda) \cap B \cap U)(\mathrm{U}^{\mathbf{k}}(\Lambda) \cap U) = \mathrm{U}^{\mathbf{k}}(\Lambda) \cap U.$$

As in semisimple case (i), the result now follows by the definition of semisimple characters.

Non-minimal semisimple case (case (III)) Finally, suppose $[\Lambda, n, 0, \beta]$ is a skew semisimple stratum, with splitting $V = V^1 \bigoplus V^2$, $1 \le l \le 2$, that $\beta_i = \mathbf{1}^1 \beta \mathbf{1}^i$ is minimal, for $i=1,2$ but that $[\Lambda, n, n-1, \beta]$ *is* equivalent to a simple stratum. Let $r = -k_0(\beta, \Lambda)$. As in the simple case, we will show that

$(*)$ there exists a *simple* stratum $[\Lambda, n, r, \gamma]$ equivalent to $[\Lambda, n, r, \beta]$ such that $\psi_\gamma|_{U_{der}} = 1$.

(Indeed, γ will be minimal.) The proof is then the same as that in the simple case, since we can use the simple case for γ. In this case we invoke Lemma 6.5 to show that $H^1 \cap U = H^{[\frac{n}{2}]+1} \cap U$.

As in the simple case, the flag corresponding to the unipotent subgroup U is given by $V_i = \langle v, \beta v, \cdots, \beta^{i-1} v \rangle$, for some $v \in V$ with $h(v, \beta v) = 0$. So $v = v_1 + v_2$, with $v_i \in V^i$ such that $h(v_1, \beta_1 v_1) = -h(v_2, \beta_2 v_2)$. Note that, for $i = 1,2$, $\{v_i, \beta_i v_i\}$ is a basis for V^i, with respect to which β_i has matrix

$$\begin{pmatrix} 0 & \lambda_i \\ 1 & 0 \end{pmatrix},$$

for some $\lambda_i \in F$. Also, let E_i denote the linear map in A^{ii} which send $\beta_i v_i$ to v_i and v_i to 0; then $E_i \in \mathfrak{a}_n^{ii}$.

Let $[\Lambda, n, r, \gamma_0]$ be a skew simple stratum equivalent to $[\Lambda, n, r, \beta]$, with $\gamma_0 \in M_{sp}$, and let $X^2 - \lambda$ be the minimal polynomial of γ_0 over F. For $i = 1,2$, let $\gamma_i \in A^{ii}$ have matrix

$$\begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix},$$

with respect to the basis $\{v_i, \beta_i v_i\}$ of V^i. Since $\beta - \gamma_0 \in \mathfrak{a}_{-r}$, we get that $\lambda - \lambda_i \in \mathfrak{a}^{ii}_{-n-r}$, for $i = 1, 2$, so $(\lambda - \lambda_i)E_i \in \mathfrak{a}^{ii}_{-r}$. Hence $\beta_i - \gamma_i \in \mathfrak{a}^{ii}_{-r}$ and $\gamma = \gamma_1 + \gamma_2$ is as required, since $\gamma v = \beta v$.

This completes the proof of Proposition 4.1. ■

Remarks It surely will not have escaped the reader's notice that the methods in each case are rather similar. It may well be possible to unify the cases into a single proof but we have not been able to do this. We also note that we could not have used [5] Lemma 2.10 here, since the proof given there unfortunately does not work. It seems likely that the result there is true (at least in the tame case, as here) but we have not (yet) been able to find a proof.

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