

Covers for self-dual supercuspidal representations of the Siegel Levi subgroup of classical p -adic groups

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Abstract

We study components of the Bernstein decomposition of a p -adic classical group (with p odd) with inertial support a self-dual positive level supercuspidal representation of a Siegel Levi subgroup. More precisely, we use the method of covers to construct a Bushnell-Kutzko type for such a component. A detailed knowledge of the Hecke algebra of the type should have number-theoretic implications.

Introduction

Let F_0 be a locally compact nonarchimedean local field, and G the F_0 -points of a connected reductive quasi-split algebraic group \mathbf{G} defined over F_0 . The smooth representation theory of G plays a key role in the modern theory of automorphic forms, and is therefore of deep interest to number theorists. A theorem of Bernstein shows that, in the terminology of [7], it is enough to understand the subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$ of smooth representations with supercuspidal support in a given inertial class \mathfrak{s} . The class $\mathfrak{s} = [L, \pi]_G$ is determined by a Levi subgroup $L \subset G$ and an irreducible supercuspidal representation π of L . The theory of types and covers [7] gives one a method for trying to understand the subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$. In particular, if one knows supercuspidal types for all Levi subgroups of G , and how to construct G -covers of all such types, then one has a complete set of types for G . Here, we study the case where F_0 has odd residual characteristic, G is a classical group, and $\mathfrak{s} = [L, \pi]_G$ with L a Siegel Levi subgroup and π an irreducible supercuspidal representation fixed by the non-trivial Weyl group element. While this work gives a complete set of G -covers (and hence types) in this case, the theory cannot be considered complete without knowledge of the associated Hecke algebras. As shown in [16], a precise understanding of such Hecke algebras gives information on reducibility of induced representations and therefore determines poles of certain Langlands L -functions. These considerations will be the subject of future work.

Let \mathbf{G} be any of the groups \mathbf{Sp}_{2M} , \mathbf{SO}_{2M} , \mathbf{SO}_{2M+1} , \mathbf{U}_n or \mathbf{SO}_n^* , i.e., a symplectic, quasi-split special orthogonal group, or quasi-split unitary group. (Here \mathbf{SO}_n^* stands for a quasi-split but non-split special orthogonal group.) Let $\mathbf{B} = \mathbf{TN}$ be a Borel subgroup with unipotent radical \mathbf{N} and

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maximal torus T . There is a choice of standard maximal parabolic subgroup $P = LU$ containing B whose Levi component L is isomorphic to either $GL_M \times G_0$, or $\mathbf{Res}_{F/F_0}(GL_M) \times G_0$; here G_0 is an anisotropic group and F/F_0 is the quadratic extension defining the unitary group. We study the theory of types for the inertial class $\mathfrak{s} = [L, \pi]_G$, where π is fixed by the non-trivial element w_0 of the Weyl group $N_G(L)/L$. The work of Bushnell and Kutzko [6] gives one a type (J, λ) for the inertial class $\mathfrak{s}_L = [L, \pi]_L$. The purpose of this paper is to construct a cover (J, λ) of (J_L, λ_L) for each group G . (This generalizes the work of Blondel [3] when $G = Sp_{2M}$, though her proofs are somewhat different.) Therefore, by [7], the pair (J, λ) is a type for \mathfrak{s} . One then knows that the category $\mathfrak{A}^{\mathfrak{s}}(G)$ is isomorphic to the category $\mathcal{H}(G, \lambda)$ -Mod of unital (left) modules over the λ -spherical Hecke algebra, and we also give the structure of this Hecke algebra.

This isomorphism of categories has implications both for the location of poles of certain L -functions, and for the classification of local galois representations. One example of this may be seen in recent work of Kutzko and Morris [16], where G is one of the groups Sp_{2M} , SO_{2M} or SO_{2M+1} , and π is a level zero self-dual supercuspidal representation. Specific information about the Hecke algebra $\mathcal{H}(G, \lambda)$ is employed to give a purely local proof of Shahidi's theorem on the reducibility of parabolic induction [21], which can also be described in terms of the poles of L -functions. On the other hand, recent work of Henniart [15], building on work of Harris and Taylor [12] and Henniart [14], relates this to the classification of local galois representations by their images (e.g. symplectic, orthogonal). In the case of the level zero representations in [16], the classification of galois representations obtained via the Hecke algebra isomorphism was known already. However, in the general situation of this paper, the corresponding classification of galois representations is not fully understood and has been sought for some time. We hope to address this problem in a sequel.

We now give a summary of the contents of the paper. In §1.1, we review the theory of types for self-dual representations, via results of Blondel, which is crucial to the construction (see [3, Proposition 2.2]). (Note that our use of the word *self-dual* is somewhat non-standard: it does *not* mean self-contragredient.) In §1.3, we fix a self-dual representation π of L , and use these results to define a well-adapted inner product on an F -vector space V , which defines the group G . In §1.4, we show that there is a choice of *simple stratum* (see [6]), defining the type in π , which is particularly well suited. In particular, it gives rise to a *skew semisimple* stratum (see [24]) in $\text{End}_F(V)$, which underpins the construction of G -cover.

We construct the G -cover (J, λ) in §2, following the recipe of [6, §7]. The first stages, when we are working with pro- p groups, are performed first in $\text{Aut}_F(V)$ (see §2.1), and then transferred to G using the Glauberman correspondence (see §2.2). Indeed, the principle of the construction is the same as that of Blondel for symplectic groups [3], though the details are complicated by the presence of an anisotropic part. We diverge from [3] in the verification that (J, λ) is indeed a G -cover, where we construct an invertible element f in the Hecke algebra $\mathcal{H}(G, \lambda)$, which is supported on a strongly (P, J) -positive element in the centre of L (see [7]). To do this, we use an argument which harks back to the work of Borel [4]: we find two invertible elements of the Hecke algebra $\mathcal{H}(G, \lambda)$, which each have support in a compact subgroup of G , and whose convolution is supported on a strongly positive element of L (see Lemmas 2.10, 2.11); a suitable power of this is the required element f .

Finally, in §2.3, we prove a result on the Hecke algebra $\mathcal{H}(G, \lambda)$, following the techniques and philosophy of [6, §5] (see also [19]). We show that the subalgebra of elements with support in a fixed maximal compact subgroup is isomorphic to an algebra of the form $\mathcal{H}(\mathcal{G}', \rho')$, where \mathcal{G}' is a (possibly disconnected) classical group over a finite field, and ρ' is an irreducible cuspidal

representation of the Siegel Levi subgroup of \mathcal{G}' . In many cases, the calculations in [16] will give the parameters of these Hecke algebras; in general, the calculations will be similar. This will simplify the computation of the structure of the Hecke algebra $\mathcal{H}(G, \lambda)$; these specific calculations, and their implications to the classification of galois representations, are left to future work.

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1 Preliminaries

Let F be a locally compact nonarchimedean local field. Let μ be an automorphism of F with $\mu^2 = 1$; we allow the possibility that μ is trivial. We set $F_0 = F^\mu$ to be the fixed points of μ . Let \mathfrak{o}_F be the ring of integers in F , and \mathfrak{p}_F the maximal ideal in \mathfrak{o}_F . Denote by k_F the residual field $\mathfrak{o}_F/\mathfrak{p}_F$, and let $q_F = |k_F|$. We adopt similar notation for F_0 and for any extension of F_0 .

Let p denote the characteristic of k_F . We assume that p is not 2 throughout.

For r a real number, we write $\lfloor r \rfloor$ for the greatest integer less than or equal to r , and $\lceil r \rceil$ for the least integer greater than or equal to r .

1.1 Self-dual representations of $\mathrm{GL}_M(\mathbf{F})$

We begin by looking at the self-dual representations of $\mathrm{GL}_M(F)$, following Blondel [3, §2.2]. Let W be an M -dimensional F -vector space with basis $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_M\}$, equipped with the non-degenerate hermitian form $\langle \cdot, \cdot \rangle_W$ given by $\langle \mathbf{w}_i, \mathbf{w}_j \rangle_W = \delta_{i+j, M+1}$. Denote by $\tilde{\varepsilon}_W$ the induced adjoint involution on $A_W = \mathrm{End}_F(W)$ and by ε_W the involution $g \mapsto \tilde{\varepsilon}_W(g^{-1})$ of $G_W = \mathrm{Aut}_F(W) \cong \mathrm{GL}_M(F)$. Then, writing $a \in A_W$ with respect to the basis \mathcal{B} , we have

$$\tilde{\varepsilon}_W(a) = {}^\dagger a^\mu,$$

where $a^\mu \in A_W$ is obtained by applying μ to the coefficients of a , and † denotes transpose with respect to the off-diagonal. For ρ a representation of a subgroup J of G_W , we write ρ^{ε_W} for the representation of $\varepsilon_W(J)$ given by $\rho^{\varepsilon_W}(\varepsilon_W(j)) = \rho(j)$, for $j \in J$.

Let π be an irreducible supercuspidal representation of G_W such that $\pi \simeq \pi^{\varepsilon_W}$. Note that, if $F = F_0$, then, by Gel'fand and Kazhdan [11], this is equivalent to π being self-contragredient. Let (J_W, λ_W) be a maximal simple type in G_W corresponding to the inertial equivalence class $[G_W, \pi]_{G_W}$. We will need to use the construction of λ_W quite explicitly so we recall it briefly here (see [6] for more details).

It begins with a *simple stratum* $[\mathfrak{A}_W, n_W, 0, \beta]$, where \mathfrak{A}_W is a principal hereditary \mathfrak{o}_F -order in A_W , with Jacobson radical \mathfrak{P}_W , and $n_W \in \mathbb{N}$ is such that $\beta \in \mathfrak{P}_W^{-n_W} \setminus \mathfrak{P}_W^{1-n_W}$. Further, $E = F[\beta]$ is a field extension of F and E^\times normalizes \mathfrak{A}_W . We set B_W to be the A_W -centralizer of E and put $\mathfrak{B}_W = \mathfrak{A}_W \cap B_W$, a maximal hereditary \mathfrak{o}_E -order in B_W . We also fix a uniformizer ϖ_E of E .

From the stratum are defined certain subgroups $H_W^k = H^k(\beta, \mathfrak{A}_W)$ and $J_W^k = J^k(\beta, \mathfrak{A}_W)$, for $k \geq 0$, along with some sets $\mathcal{C}(\mathfrak{A}_W, k, \beta)$ of characters of H_W^{k+1} called *simple characters* (see [6, §3]). The construction of the type continues with a simple character $\theta_W \in \mathcal{C}(\mathfrak{A}_W, 0, \beta)$. There is then a unique irreducible representation η_W of J_W^1 which contains θ_W .

Now we take κ_W to be a β -extension of η_W , that is, one of a certain family of representations of $J_W = J_W^0$ which restrict to η_W . We also recall the construction of κ_W here, from [6, §5.1–2]. Let \mathfrak{B}_W^m be a minimal hereditary \mathfrak{o}_E -order in B_W contained in \mathfrak{B}_W and let $\mathfrak{A}_W^m \subset \mathfrak{A}_W$ be the unique hereditary \mathfrak{o}_F -order in A_W which is normalized by E^\times and such that $\mathfrak{A}_W^m \cap B_W = \mathfrak{B}_W^m$. Then $[\mathfrak{A}_W^m, n_W^m, 0, \beta]$ is also a simple stratum, for some integer n_W^m , and we can define simple characters associated to this stratum also. Moreover, there is a canonical bijection $\tau_{\mathfrak{A}_W, \mathfrak{A}_W^m, \beta} : \mathcal{C}(\mathfrak{A}_W, 0, \beta) \rightarrow \mathcal{C}(\mathfrak{A}_W^m, 0, \beta)$ (see [6, §3.6]). Let θ_W^m be the transfer of θ_W under this bijection. There is a unique irreducible representation η_W^m of $J^1(\beta, \mathfrak{A}_W^m)$ which contains θ_W^m .

Now we form the group $\tilde{J}_W^1 = U^1(\mathfrak{B}_W^m)J_W^1$. There is a unique representation $\tilde{\eta}_W$ of \tilde{J}_W^1 such that $\tilde{\eta}_W|_{J_W^1} = \eta_W$ and $\tilde{\eta}_W, \eta_W^m$ induce equivalent irreducible representations of $U^1(\mathfrak{A}_W^m)$. Then κ_W is any representation of J_W such that $\kappa_W|_{\tilde{J}_W^1} = \tilde{\eta}_W$. Finally, $J_W/J_W^1 \cong U(\mathfrak{B}_W)/U^1(\mathfrak{B}_W)$ is isomorphic to $GL_r(k_E)$, where $r = M/[E : F]$. Then there is a cuspidal representation ρ_W of J_W/J_W^1 such that

$$\lambda_W = \kappa_W \otimes \rho_W.$$

By conjugating (J_W, λ_W) if necessary, we may and do assume \mathfrak{A}_W is *standard*; that is, with respect to our chosen basis \mathcal{B} , it consists of matrices with entries in \mathfrak{o}_F which are upper block triangular modulo \mathfrak{p}_F . Note that this means that $\tilde{\varepsilon}_W(\mathfrak{A}_W) = \mathfrak{A}_W$.

Proposition 1.1 (cf. [3, 2.2 Proposition]). (i) *There exists $\sigma \in U(\mathfrak{A}_W)$ such that J_W is stable under $\tilde{\sigma} : g \rightarrow \sigma \varepsilon_W(g) \sigma^{-1}$ and λ_W is equivalent to $\lambda_W \circ \tilde{\sigma}$.*

(ii) *Such an element σ is unique up to left multiplication by J_W . It satisfies:*

- (a) $\sigma \varepsilon_W(\sigma) \in J_W$ and $\varpi_E^{-1} \sigma \varepsilon_W(\varpi_E^{-1} \sigma) \in J_W$.
- (b) *The map $\tilde{\sigma}$ stabilizes H_W^1 and J_W^1 and we have $\theta_W = \theta_W \circ \tilde{\sigma}$.*
- (c) *The lattices $\mathfrak{J}_W, \mathfrak{H}_W^1$ are stable under $X \mapsto \sigma \tilde{\varepsilon}_W(X) \sigma^{-1}$.*

The proof is identical to that of [3, 2.2 Proposition]. Note that we will actually have to be somewhat more careful in our choices of β and σ , as in [3, 2.3 Corollary] – see Remark 1.4 for more details.

1.2 Simple characters

We record here the following useful lemma from [3, 4.3 Lemma]. Note that this uses very strongly the condition that $p \neq 2$.

Lemma 1.2 ([3, 4.3 Lemma 1]). *Suppose V'' is any F -vector space and $[\mathfrak{A}'', n'', 0, \beta'']$ is a simple stratum in $A'' = \text{End}_F(V'')$, where \mathfrak{A}'' is a hereditary \mathfrak{o}_F -order in V'' .*

- (i) *$[\mathfrak{A}'', n'', 0, \frac{1}{2}\beta'']$ is a simple stratum in A'' , with $H^k(\frac{1}{2}\beta'', \mathfrak{A}'') = H^k(\beta'', \mathfrak{A}'')$, for each $k \geq 0$, and similarly for J^k .*
- (ii) *For each $m \geq 0$, the map $\theta \mapsto \theta^2$ is a bijection from $\mathcal{C}(\mathfrak{A}'', m, \frac{1}{2}\beta'')$ onto $\mathcal{C}(\mathfrak{A}'', m, \beta'')$ which is compatible with the canonical bijections τ of [6, §3.6]. We denote the inverse bijection by $\theta \mapsto \theta^{1/2}$.*

We will write $\theta_L = \theta_W^{1/2} \in \mathcal{C}(\mathfrak{A}_W, 0, \frac{1}{2}\beta)$ and $\theta_L^m = (\theta_W^m)^{1/2} \in \mathcal{C}(\mathfrak{A}_W^m, 0, \frac{1}{2}\beta)$. We also let η_L be the unique irreducible representation of $J^1(\beta, \mathfrak{A}_W)$ which contains θ_L , and likewise η_L^m .

Note also that we have $(\theta_L \circ \tilde{\sigma})^2 = \theta_L^2 \circ \tilde{\sigma} = \theta_W$. In particular, since the squaring map is a bijection, we see that $\theta_L \circ \tilde{\sigma} = \theta_L$.

1.3 Classical groups

Let V_0 be an F -vector space equipped with a nondegenerate anisotropic ν -hermitian form $\langle \cdot, \cdot \rangle_0$, with $\nu = \pm 1$. Thus $\langle \mathbf{v}_0, \mathbf{w}_0 \rangle_0 = \nu \langle \mathbf{w}_0, \mathbf{v}_0 \rangle_0^\mu$, for all $\mathbf{v}_0, \mathbf{w}_0 \in V_0$. We write $G_0 = \text{Aut}_F(V_0)$ and denote by \overline{G}_0^+ the (anisotropic) group corresponding to the form $\langle \cdot, \cdot \rangle_0$:

$$\overline{G}_0^+ := \{g_0 \in G_0 : \langle g_0 \mathbf{v}_0, g_0 \mathbf{w}_0 \rangle_0 = \langle \mathbf{v}_0, \mathbf{w}_0 \rangle_0 \text{ for all } \mathbf{v}_0, \mathbf{w}_0 \in V_0\}.$$

We put $\overline{G}_0 := \{g_0 \in \overline{G}_0^+ : N_{F/F_0} \circ \det_{G_0/F}(g_0) = 1\}$. We allow the possibility that $V_0 = \{0\}$.

Set $V = W \oplus V_0 \oplus W$ and define a form $\langle \cdot, \cdot \rangle$ on V by

$$\langle (\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2), (\mathbf{w}_1, \mathbf{w}_0, \mathbf{w}_2) \rangle = \langle \mathbf{v}_1, \sigma^{-1} \mathbf{w}_2 \rangle_W + \langle \mathbf{v}_0, \mathbf{w}_0 \rangle_0 + \nu \langle \sigma^{-1} \mathbf{v}_2, \mathbf{w}_1 \rangle_W,$$

for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2 \in W$ and $\mathbf{v}_0, \mathbf{w}_0 \in V_0$, where σ is the element given by Proposition 1.1. Note that $\langle \cdot, \cdot \rangle$ is now a nondegenerate ν -hermitian form in which the two copies of W are (dual) maximal isotropic subspaces, and V_0 is the maximal anisotropic subspace. Let $N = \dim_F V$; then $N = 2M + D$, where $D = \dim_F V_0$.

We put $A = \text{End}_F(V) \cong \mathbb{M}(N, F)$. We set $G = A^\times = \text{Aut}_F(V) \cong GL_N(F)$ and

$$\overline{G}^+ = \{g \in G : \langle g\mathbf{v}, g\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle \text{ for all } \mathbf{v}, \mathbf{w} \in V\},$$

a unitary, symplectic or orthogonal group over F_0 . We also put

$$\overline{G} := \{g \in \overline{G}^+ : N_{F/F_0} \circ \det_{G/F}(g) = 1\}.$$

More generally, for H a subgroup of G , we will write \overline{H}^+ for the intersection $\overline{H}^+ = H \cap \overline{G}^+$, and \overline{H} for the intersection $\overline{H} = H \cap \overline{G}$.

We denote by $\tilde{\varepsilon}$ the adjoint involution on A determined by the form; that is, for $a \in A$, $\tilde{\varepsilon}(a)$ is the unique element of A such that

$$\langle av, w \rangle = \langle v, \tilde{\varepsilon}(a)w \rangle, \quad \text{for all } v, w \in V,$$

We have an involution ε on G given by $\varepsilon(g) = \tilde{\varepsilon}(g^{-1})$, for $g \in G$, so that $\overline{G}^+ = G^\varepsilon$.

Denote by ϕ_1 (respectively ϕ_2) the embedding of A_W into A via the embedding of W into the first (respectively last) factor of $V = W \oplus V_0 \oplus W$, so that $\phi_1(a)(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2) = (a\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2)$. The involution $\tilde{\varepsilon}$ interchanges the images of these two embeddings, and a straightforward calculation shows that $\tilde{\varepsilon}(\phi_1(a)) = \phi_2(\sigma \varepsilon_W(a) \sigma^{-1}) = \phi_2(\tilde{\sigma}(a))$, using the notation of Proposition 1.1.

Given a representation ρ of a subgroup J of G , we denote by ρ^ε the representation of $\varepsilon(J)$ given by $\rho^\varepsilon(\varepsilon(j)) = \rho(j)$, for $j \in J$.

For L an \mathfrak{o}_F -lattice in V , we define its dual lattice to be

$$L^\# = \{v \in V : \langle v, L \rangle \subset \mathfrak{p}_F\}.$$

We note that, since \overline{G}_0^+ stabilizes lattices $L_0 \supset L_0^\# \supset \mathfrak{p}_F L_0$ in V_0 (see [17, §1.8]), it is contained in a maximal ε -stable compact open subgroup of G_0 (namely, the G_0 -stabilizer of these lattices).

Let P be the parabolic subgroup of G stabilizing the self-dual flag

$$\{0\} \subsetneq W \oplus \{0\} \oplus \{0\} \subseteq (W \oplus \{0\} \oplus \{0\})^\perp = W \oplus V_0 \oplus \{0\} \subsetneq V,$$

with unipotent radical U . Let L denote the Levi component of P which stabilizes each copy of W along with V_0 . So $L \cong GL_M(F) \times GL_D(F) \times GL_M(F)$. Let P_- denote the opposite parabolic subgroup, $P_- = LU_-$. We also put $A_L = A_W \oplus A_0 \oplus A_W = \text{Lie}(L)$.

We set $\overline{P} = P \cap \overline{G}$, the Siegel parabolic subgroup of \overline{G} , with unipotent radical $\overline{U} = U \cap \overline{G}$, and $\overline{L} = L \cap \overline{G}$, a Levi component of \overline{P} . Then $\overline{L} \cong GL_M(F) \times \overline{G}_0$; in block matrix form, we identify $G_W \times \overline{G}_0$ with \overline{L} via the isomorphism

$$i(g, g_0) = \begin{pmatrix} g & 0 & 0 \\ 0 & g_0 & 0 \\ 0 & 0 & \tilde{\sigma}(g) \end{pmatrix}, \quad \text{for } g \in G_W, g_0 \in \overline{G}_0.$$

Note here that we are writing matrices with respect to the non-Witt basis $\mathcal{B} \cup \mathcal{B}_0 \cup \mathcal{B}$ of $V = W \oplus V_0 \oplus W$, for \mathcal{B}_0 any basis of V_0 . We choose to write all matrices with respect to this basis, rather than with respect to the Witt basis $\mathcal{B} \cup \mathcal{B}_0 \cup \sigma\mathcal{B}$.

If ρ is a representation of a subgroup J_W of G_W , and \overline{J}_0 is a subgroup of \overline{G}_0 , then we denote by $i(\rho)$ the representation of $i(J_W \times \overline{J}_0)$ given by $i(\rho)(i(j, j_0)) = \rho(j)$, for $j \in J_W, j_0 \in \overline{J}_0$. If the group \overline{J}_0 is not specified then we take it to be the whole of \overline{G}_0 .

More generally, if \overline{J} is a subgroup of \overline{G} such that $\overline{J} \cap \overline{L} = i(J_W \times \overline{J}_0)$ and \overline{J} has an Iwahori decomposition with respect to $(\overline{L}, \overline{P})$, then we denote by $\widetilde{i(\rho)}$ the representation of \overline{J} given by

$$\widetilde{i(\rho)}(u_- l u) = i(\rho)(l), \quad \text{for } u_- \in \overline{J} \cap \overline{U}_-, l \in \overline{J} \cap \overline{L}, u \in \overline{J} \cap \overline{U},$$

whenever this defines a representation.

It will also be useful, later, to put $V' = W \oplus \{0\} \oplus W \subset V$, equipped with the restriction of the form \langle, \rangle . We put $A' = \text{End}_F(V')$, $G' = \text{Aut}_F(V')$, ε the involution of G' associated to the form, and $\overline{G}'^+ = (G')^\varepsilon$. We also let P' be the maximal parabolic subgroup of G which stabilizes the flag

$$\{0\} \subsetneq V' \subsetneq V,$$

with unipotent radical U' . Let L' denote the Levi component of P' which stabilizes the decomposition $V = V' \oplus V_0$, so that $L' \cong G' \times G_0$, and let P'^- denote the opposite parabolic subgroup, with unipotent radical U'^- . We note that, while L' is stable under the involution ε , \overline{L}' is *not* a Levi subgroup of \overline{G} .

We consider the inertial class $\mathfrak{s}_{\overline{L}} = [\overline{L}, i(\pi)]_{\overline{L}}$ and a type $(J_{\overline{L}}, \lambda_{\overline{L}})$ for it, where

$$J_{\overline{L}} := i(J_W) = i(J_W \times \overline{G}_0) \quad \text{and} \quad \lambda_{\overline{L}} := i(\lambda_W).$$

We are going to construct a \overline{G} -cover of this type, which will give a \overline{G} -type for the inertial class $\mathfrak{s} = [\overline{L}, i(\pi)]_{\overline{G}}$.

1.4 Semisimple characters

We continue with the notation of the previous sections. Let $\mathcal{L}_W = \{L_k^W : k \in \mathbb{Z}\}$ be the \mathfrak{o}_E -lattice chain in W corresponding to \mathfrak{A}_W , so that

$$\mathfrak{A}_W = \{a \in A_W : aL_k^W \subset L_k^W \text{ for all } k \in \mathbb{Z}\},$$

normalized so that $L_0^W = \mathfrak{o}_F \mathfrak{w}_1 \oplus \cdots \oplus \mathfrak{o}_F \mathfrak{w}_M$. Let e_W denote the \mathfrak{o}_F -period of \mathfrak{A}_W . We define \mathcal{L}' to be the \mathfrak{o}_E -lattice chain of \mathfrak{o}_F -period $2e_W$ in V' given by

$$\cdots \supset L_k^W \oplus L_k^W \supset L_k^W \oplus L_{k+1}^W \supset L_{k+1}^W \oplus L_{k+1}^W \supset \cdots$$

It is straightforward to check, since $\sigma \in U(\mathfrak{A}_W)$, we have

$$(L_k^W \oplus L_k^W)^\# = L_{e_W - k}^W \oplus L_{e_W - k}^W \quad \text{and} \quad (L_k^W \oplus L_{k+1}^W)^\# = L_{e_W - k - 1}^W \oplus L_{e_W - k}^W$$

so that \mathcal{L}' is a self-dual lattice chain. We write $\mathcal{L}' = \{L'_k : k \in \mathbb{Z}\}$, where we number the lattices so that $L'_0 = L_{\lfloor \frac{e_W}{2} \rfloor}^W \oplus L_{\lceil \frac{e_W}{2} \rceil}^W = (L'_0)^\#$. Let $\mathfrak{A}' = \mathfrak{a}_0(\mathcal{L}')$ be the \mathfrak{o}_F -order associated with \mathcal{L}' .

Now let Λ' be the \mathfrak{o}_E -lattice *sequence* of period $4e_W$ in $V' = W \oplus W$ given by

$$\Lambda'(k) = L'_{\lfloor \frac{k}{2} \rfloor}$$

so that every lattice of \mathcal{L}' occurs twice in the sequence and $\Lambda'(k)^\# = \Lambda'(1 - k)$, for $k \in \mathbb{Z}$.

We consider the element $\frac{1}{2}\beta \oplus \frac{1}{2}\beta \in A_W \oplus A_W$, which we will also call $\frac{1}{2}\beta$. Then $[\mathcal{L}', 2n_W, 0, \frac{1}{2}\beta]$ and $[\Lambda', 4n_W, 0, \frac{1}{2}\beta]$ are both simple strata in A' so we can define the orders $\mathfrak{H}, \mathfrak{J}$ and the groups H^k, J^k for them, and also simple characters (see [8]). These are in fact the same, up to a scaling of the index (*loc. cit.*); so, for example,

$$H^k(\frac{1}{2}\beta, \Lambda') = H^{\lceil \frac{k}{2} \rceil}(\frac{1}{2}\beta, \mathcal{L}').$$

In particular, $H^1(\frac{1}{2}\beta, \Lambda') = H^1(\frac{1}{2}\beta, \mathcal{L}')$ and we shall denote this group H'^1 . Similarly, we put $J'^1 = J^1(\frac{1}{2}\beta, \Lambda') = J^1(\frac{1}{2}\beta, \mathcal{L}')$ and $J' = J(\frac{1}{2}\beta, \Lambda') = J(\frac{1}{2}\beta, \mathcal{L}')$. Likewise, the simple characters of H'^1 are the same: $\mathcal{C}(\Lambda', 0, \frac{1}{2}\beta) = \mathcal{C}(\mathcal{L}', 0, \frac{1}{2}\beta)$.

Moreover, the groups H'^1, J'^1 and J' are described in [3, 2.2 Lemma]. So, for example, with respect to the basis $\mathcal{B} \cup \mathcal{B}$ of V' (which, we recall, is *not* a Witt basis),

$$H'^1 = \begin{pmatrix} H_W^1 & \mathfrak{J}_W \\ \varpi_E \mathfrak{J}_W & H_W^1 \end{pmatrix} \quad \text{and} \quad J'^1 = \begin{pmatrix} J_W^1 & \varpi_E^{-1} \mathfrak{H}_W \\ \mathfrak{H}_W & J_W^1 \end{pmatrix}.$$

Finally, we have the bijection $\tau_{\mathfrak{A}_W, \mathfrak{A}', \frac{1}{2}\beta} : \mathcal{C}(\mathfrak{A}_W, 0, \frac{1}{2}\beta) \rightarrow \mathcal{C}(\mathfrak{A}', 0, \frac{1}{2}\beta)$ from [6, §3.6]. Let θ' be the image of θ_L under this map. Note that, by [6, §7.1–2], θ' is trivial on $H'^1 \cap U$ and $H'^1 \cap U_-$, while

$$\theta'|_{H'^1 \cap L} = \theta_L \otimes \theta_L.$$

It is straightforward to check that Proposition 1.1, together with our definition of the form $\langle \cdot, \cdot \rangle$, implies that θ' is fixed by the involution ε (cf. [3, 2.3 Corollary]).

Now let \mathcal{L}_0 be the unique self-dual \mathfrak{o}_F -lattice chain in V_0

$$\cdots \supseteq L_0 \supseteq L_0^\# \supseteq \mathfrak{p}_F L_0 \supseteq \mathfrak{p}_F L_0^\# \supseteq \cdots.$$

Note that we may have $L_0 = L_0^\#$ or $L_0^\# = \mathfrak{p}_F L_0$ so this lattice chain has \mathfrak{o}_F -period $e_0 = 1$ or 2 . Let Λ_0 be the self-dual \mathfrak{o}_F -lattice sequence of period $4e_W$ given by

$$\Lambda_0(k) = \begin{cases} \mathfrak{p}_F^j L_0 & \text{if } \lceil \frac{k}{2e_W} \rceil = 2j, \\ \mathfrak{p}_F^j L_0^\# & \text{if } \lceil \frac{k}{2e_W} \rceil = 2j + 1, \end{cases}$$

so that every lattice of \mathcal{L}_0 appears with equal multiplicity $4e_W/e_0$. Note also that $\Lambda_0(k)^\# = \Lambda_0(1-k)$, for $k \in \mathbb{Z}$. Moreover, the filtration of A_0 determined by Λ_0 is the same as that determined by \mathcal{L}_0 up to a scaling of the index. In particular, $\mathfrak{a}_0(\Lambda_0) = \mathfrak{A}(\mathcal{L}_0)$ and $\mathfrak{a}_1(\Lambda_0) = \mathfrak{B}(\mathcal{L}_0)$.

Finally, we define Λ to be the \mathfrak{o}_F -lattice sequence in V given by

$$\Lambda(k) = \Lambda'(k) \oplus \Lambda_0(k), \quad \text{for } k \in \mathbb{Z}.$$

Then, by construction, Λ is self-dual and of \mathfrak{o}_F -period $e = 4e_W$. We consider the element $\frac{1}{2}\beta \oplus 0 \in A' \oplus A_0$ (in fact, in A_L); by abuse of notation, we will still call this element $\frac{1}{2}\beta$. Then $[\Lambda, n, 0, \frac{1}{2}\beta]$ is a *semisimple stratum* in A , where $n = 4n_W$. (See [24, §3.1] for the definition of semisimple stratum, which is more general than the definition in [22, §3.3]; in particular, null strata are thought of as simple strata so, alternatively, the definition in [22] could be used with “simple” replaced by “simple or null” everywhere. The results of [22] all remain valid in this situation – the proofs are the same and they are also proved in [24].)

We put $J = J(\frac{1}{2}\beta, \Lambda)$ (see [22] or [24]) and similarly for $J^1, H^1, \mathfrak{J}, \mathfrak{H}$ etc. In matrix form we have

$$\mathfrak{J} = \begin{pmatrix} \mathfrak{J}_W & \mathfrak{a}_{\lceil n/2 \rceil}(\Lambda) & \varpi_E^{-1} \mathfrak{H}_W^1 \\ \mathfrak{a}_{\lceil n/2 \rceil}(\Lambda) & \mathfrak{A}(\mathcal{L}_0) & \mathfrak{a}_{\lceil n/2 \rceil}(\Lambda) \\ \mathfrak{H}_W^1 & \mathfrak{a}_{\lceil n/2 \rceil}(\Lambda) & \mathfrak{J}_W \end{pmatrix} \quad \text{and} \quad \mathfrak{H} = \begin{pmatrix} \mathfrak{H}_W & \mathfrak{a}_{\lfloor n/2 \rfloor}(\Lambda) & \mathfrak{J}_W^1 \\ \mathfrak{a}_{\lfloor n/2 \rfloor}(\Lambda) & \mathfrak{A}(\mathcal{L}_0) & \mathfrak{a}_{\lfloor n/2 \rfloor}(\Lambda) \\ \varpi_E \mathfrak{J}_W^1 & \mathfrak{a}_{\lfloor n/2 \rfloor}(\Lambda) & \mathfrak{H}_W \end{pmatrix},$$

and there are similar decompositions for \mathfrak{J}^k and \mathfrak{H}^k , $k \geq 1$. Since J' is stable under the involution ε and Λ is self-dual, we see that J is stable under the involution ε , and likewise for J^1 and H^1 .

Let θ be the unique semisimple character (see [22, §3.3] or [24, §3.2]) of H^1 such that

$$\theta|_{H'^1} = \theta'.$$

Now θ is trivial on $H^1 \cap U'$ and $H^1 \cap U'^-$ by definition so, since θ' is trivial on $H'^1 \cap U$ and $H'^1 \cap U_-$, we see that θ is in fact trivial on $H^1 \cap U$ and $H^1 \cap U_-$. Since $\theta|_{H'^1}$ is fixed by ε and $\theta|_{U^1(\mathcal{L}_0)}$ is trivial (by definition), we see that θ is fixed by ε . Moreover, since $\theta_L^2 = \theta_W$, we have

$$\theta|_{\overline{H}^1} = i(\widetilde{\theta}_W).$$

Proposition 1.3 (cf. [3, 2.3 Theorem]). *There exists a semisimple stratum $[\Lambda, n, 0, \alpha]$ in A with $\alpha \in A_L$ and $\alpha = -\widetilde{\varepsilon}(\alpha)$ such that $\theta \in \mathcal{C}(\Lambda, 0, \alpha)$.*

Proof Let $\varphi : A' \rightarrow A'$ be the involution given by conjugation by

$$h = \begin{pmatrix} I_M & 0 \\ 0 & -I_M \end{pmatrix}.$$

Then $(\theta')^\varphi = \theta'$, since it is trivial on the unipotent parts, and $(\theta')^\varepsilon = \theta'$. Thus, θ' is invariant under the subgroup Ω of $\text{Aut}(G')$ generated by ε and φ . As $\varepsilon\varphi = \varphi\varepsilon$, we see $\Omega \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and thus by [22, Theorem 6.3] (see the Remarks at the bottom of page 139 there also), there is a choice of α , fixed by φ and with $\alpha = -\widetilde{\varepsilon}(\alpha)$, such that $\theta' \in \mathcal{C}(\Lambda', 0, \alpha)$. Then $[\Lambda, n, 0, \alpha]$ is as required. \blacksquare

Remark 1.4. Writing $\alpha = \text{diag}(\alpha_W, 0, -\tilde{\sigma}(\alpha_W))$ for the element of Proposition 1.3, we see that our original type (J_W, λ_W) could also have been defined using $2\alpha_W$ in place of β . Replacing β by $2\alpha_W$, the discussion following [3, 2.3 Theorem] applies in this situation (see especially [3, 2.3 Corollary] and the Remarks following). In particular, we may assume that σ is such that $\tilde{\sigma}$ induces a non-trivial galois involution on $E = F[\beta]$ which extends the involution μ on F and sends β to $-\beta$. From now on, we assume that σ is chosen in this more careful way; this implies that $\sigma_{\varepsilon_W}(\sigma) \in B_W$. Moreover, when μ is trivial, the degree $[E : F]$ is even so that $M = \dim_F(W)$ is even also.

As in the construction of the type (J_W, λ_W) , we will also need the transfer of θ to a minimal order, which we give now. Let B' denote the centralizer in A' of E and put $\mathfrak{B}' = \mathfrak{A}' \cap B'$, an \mathfrak{o}_E -order of period 2 with radical \mathfrak{Q}' . Recall that we had, in §1.1, a minimal \mathfrak{o}_E -order $\mathfrak{B}_W^m \subset \mathfrak{B}_W$. Let $\mathfrak{B}^m \subset \mathfrak{B}'$ be the $\tilde{\varepsilon}$ -stable minimal \mathfrak{o}_E -order in B' given by

$$\mathfrak{B}^m = (\mathfrak{B}_W^m \oplus \tilde{\varepsilon}(\mathfrak{B}_W^m)) + \mathfrak{Q}',$$

(cf. [3, 4.3]). Let \mathcal{L}^m be the corresponding self-dual \mathfrak{o}_E -lattice chain in V' , of \mathfrak{o}_E -period $2r$, where $r = M/[E : F]$. Let Λ^m be the self-dual \mathfrak{o}_E -lattice sequence in V' in which every lattice of \mathcal{L}^m occurs twice and with the indexing chosen such that

$$\Lambda^m(k)^\# = \Lambda^m(1 - k), \quad \text{for all } k \in \mathbb{Z}.$$

Note that $[\mathcal{L}^m, 2n_W^m, 0, \frac{1}{2}\beta]$ and $[\Lambda^m, 4n_W^m, 0, \frac{1}{2}\beta]$ are simple strata in V' whose associated groups and characters are the same up to a scaling of index.

Now let Λ^m be the \mathfrak{o}_F -lattice sequence in V defined by

$$\Lambda^m(k) = \Lambda^m(k) \oplus \Lambda_0 \left(\left\lceil \frac{k}{r} \right\rceil \right).$$

It is a self-dual lattice sequence of period re such that $\mathfrak{a}_0(\Lambda^m) \subset \mathfrak{a}_0(\Lambda)$. Then $[\Lambda^m, n^m, 0, \frac{1}{2}\beta]$ is a semisimple stratum in A , where $n^m = 4n_W^m = nr$.

We put $\theta'^m = \tau_{\Lambda', \Lambda'^m, \frac{1}{2}\beta}(\theta')$, a simple character of $H^1(\frac{1}{2}\beta, \Lambda'^m)$; then also $\theta'^m = \tau_{\mathfrak{A}_W^m, \Lambda'^m, \frac{1}{2}\beta}(\theta_L^m)$. Finally, let θ^m be the unique semisimple character of $H_m^1 = H^1(\frac{1}{2}\beta, \Lambda^m)$ such that

$$\theta^m|_{H^1(\frac{1}{2}\beta, \Lambda^m)} = \theta'^m.$$

In the language of [24, §3.5], we have $\theta^m = \tau_{\Lambda, \Lambda^m, \frac{1}{2}\beta}(\theta)$.

2 Covers and Hecke Algebras

In this chapter, we construct our cover and find the structure of its Hecke algebra. The method of construction of the cover is (apart from the added complication of the anisotropic part) the same as that in [3] (the reader may find it helpful to consult the diagram on page 552 of [3], especially the left side). However, here we obtain full information on the intertwining of the cover (rather than just a bound), and it is this which allows us, in §2.3, to identify the Hecke algebra. In order to obtain this information on the intertwining, we need to review carefully the construction and keep track of the intertwining at each stage.

We continue with the notation above, so we have $[\Lambda, n, 0, \frac{1}{2}\beta]$ a skew semisimple stratum in A , with $\beta \in A_L$, and $\theta \in \mathcal{C}(\Lambda, 0, \frac{1}{2}\beta)$ such that

$$\theta|_{\overline{H^1}} = \widetilde{i(\theta_W)}.$$

We also have $E = F[\beta]$ and B the A -centralizer of E ; note here that, with notation as above, $B^\times = (B')^\times \times G_0$. From Remark 1.4, the galois involution μ extends to E (as $\tilde{\sigma}$ when we consider E embedded in A_W , or as $\tilde{\varepsilon}$ when we consider E embedded in A), and we write E_0 for the fixed subfield, which has index 2. We fix a uniformizer ϖ_E of E such that $\varpi_E^\mu = \pm\varpi_E$. Note that $\beta \in A_W$ embeds in A as $\text{diag}(\beta, 0, -\tilde{\sigma}(\beta)) = \text{diag}(\beta, 0, \beta)$.

2.1 G-Covers

We begin by doing some work in G , before using Glauberman's correspondence to transfer this information to \overline{G} .

Recall that, given ρ a representation of a subgroup H of G and $g \in G$, the *intertwining space* $I_g(\rho|H)$ is

$$I_g(\rho|H) = \text{Hom}_{H \cap {}^g H}(\rho, {}^g \rho),$$

where ${}^g H = gHg^{-1}$ and ${}^g \rho$ is the representation of ${}^g H$ given by ${}^g \rho(ghg^{-1}) = \rho(h)$, for $h \in H$. The G -intertwining $I_G(\rho|H)$ of ρ is then defined to be

$$I_G(\rho) = I_G(\rho|H) = \{g \in G : I_g(\rho|H) \neq 0\}.$$

In our situation, we recall the following, from [22, Theorem 3.14, Corollary 4.2 and Proposition 4.3]:

Lemma 2.1. (i) $I_G(\theta) = J^1 B^\times J^1$.

(ii) *There exists a unique irreducible representation η of J^1 which contains θ . Moreover, $\dim \eta = (J^1 : H^1)^{\frac{1}{2}}$ and, for $g \in G$,*

$$\dim I_g(\eta|J^1) = \begin{cases} 1 & \text{if } g \in J^1 B^\times J^1; \\ 0 & \text{otherwise.} \end{cases}$$

From [6, §7.1] (together with [22, §3.3]), H^1 has an Iwahori decomposition with respect to (L, P) and

$$H^1 \cap L = H^1(\frac{1}{2}\beta, \mathfrak{A}_W) \times U^1(\Lambda_0) \times H^1(\frac{1}{2}\beta, \mathfrak{A}_W).$$

There are also similar decompositions for J^1 and for J . Moreover, θ is trivial on $H^1 \cap U$ and $H^1 \cap U_-$, and the restriction of θ to $H^1 \cap L$ takes the form

$$\theta|_{H^1 \cap L} = \theta_L \otimes \mathbf{1} \otimes \theta_L.$$

We recall ([22, Proposition 4.1] – see also [6, §3.4]) that the pairing

$$(j, j') \mapsto \theta[j, j'], \quad \text{for } j, j' \in J^1$$

induces a nondegenerate alternating bilinear form \mathbf{k}_θ on J^1/H^1 ; likewise, we have a nondegenerate alternating bilinear form \mathbf{k}_{θ_L} on $J^1(\frac{1}{2}\beta, \mathfrak{A}_W)/H^1(\frac{1}{2}\beta, \mathfrak{A}_W)$. Then, exactly as in [6, Proposition 7.2.3], we get:

Lemma 2.2 (cf. [6, Proposition 7.2.3]). (i) *The subspaces $J^1 \cap U_- / H^1 \cap U_-$ and $J^1 \cap U / H^1 \cap U$ of J^1 / H^1 are both totally isotropic for the form \mathbf{k}_θ and orthogonal to the subspace $J^1 \cap L / H^1 \cap L$.*

(ii) *The restriction of \mathbf{k}_θ to the group*

$$J^1 \cap L / H^1 \cap L = J^1(\frac{1}{2}\beta, \mathfrak{A}_W) / H^1(\frac{1}{2}\beta, \mathfrak{A}_W) \times J^1(\frac{1}{2}\beta, \mathfrak{A}_W) / H^1(\frac{1}{2}\beta, \mathfrak{A}_W)$$

is the orthogonal sum of the pairings $\mathbf{k}_{\theta_L}, \mathbf{k}_{\theta_U}$.

(iii) *We have an orthogonal sum decomposition*

$$\frac{J^1}{H^1} = \frac{J^1 \cap L}{H^1 \cap L} \perp \left(\frac{J^1 \cap U_-}{H^1 \cap U_-} \times \frac{J^1 \cap U}{H^1 \cap U} \right).$$

We define the groups

$$H_P^1 = H^1(J^1 \cap U), \quad J_P^1 = H^1(J^1 \cap P), \quad J_P = H^1(J \cap P).$$

Since J^1 normalizes θ and $\theta|_{H^1 \cap U}$ is trivial, we can define the character θ_P of H_P^1 by

$$\theta_P(hu) = \theta(h), \quad \text{for } h \in H^1, u \in J^1 \cap U.$$

As in [6, §7.2], we immediately get:

Corollary 2.3 (cf. [6, Propositions 7.2.4, 7.2.9]). *There is a unique irreducible representation η_P of J_P^1 such that $\eta_P|_{H_P^1}$ contains θ_P . Moreover, $\eta \simeq \text{Ind}_{J_P^1}^{J^1} \eta_P$ and, for each $b \in B^\times$, there is a unique (J_P^1, J_P^1) -double coset in $J^1 b J^1$ which intertwines η_P .*

We note also that we certainly have $\eta_P|_{J_P^1 \cap L} \simeq \eta_L \otimes \mathbf{1} \otimes \eta_L$.

Proposition 2.4. *We have $I_G(\theta_P) = J_P^1 B^\times J_P^1$ and hence $I_G(\eta_P) = J_P^1 B^\times J_P^1$.*

Proof This is a special case of [25, Lemma 5.8]. ■

We will also need similar results for our character θ^m of $H_m^1 = H^1(\frac{1}{2}\beta, \Lambda^m)$. Let η^m be the unique irreducible representation of $J_m^1 = J^1(\frac{1}{2}\beta, \Lambda^m)$ which contains θ^m . As above, H_m^1 and J_m^1 have Iwahori decompositions with respect to (L, P) and we can define the character θ_P^m of $H_{m,P}^1 = H_m^1(J_m^1 \cap U)$ by trivial extension of θ^m . The same proofs (indeed, they are somewhat simpler) as those of Lemma 2.2, Corollary 2.3 and Proposition 2.4 show that

$$I_G(\theta_P^m) = J_{m,P}^1 B^\times J_{m,P}^1,$$

where $J_{m,P}^1 = H_m^1(J_m^1 \cap P)$, that there is a unique irreducible representation η_P^m of $J_{m,P}^1$ which contains θ_P^m , and that $\eta^m = \text{Ind}_{J_{m,P}^1}^{J_m^1} \eta_P^m$.

Let θ_L^m denote the transfer to $H^1(\frac{1}{2}\beta, \mathfrak{A}_W^m)$ of θ_L (this is just the restriction of θ^m to one of the copies of $G_W \subset L$). Then, if η_L^m denotes the unique irreducible representation of $J^1(\frac{1}{2}\beta, \mathfrak{A}_W^m)$ which contains θ_L^m , we have

$$\eta_P^m|_{H_m^1 \cap L} \simeq \eta_L^m \otimes \mathbf{1} \otimes \eta_L^m.$$

2.2 \overline{G} -Covers

Now we will transfer the information obtained in the last section to \overline{G} , using Glauberman's correspondence (see [10], or [22] for the situation here). Let Ω denote a 2-group of automorphisms of G . Recall that if H is a pro- p subgroup of G and H^Ω is the group of Ω -fixed points, then there is a bijection, denoted $\rho \leftrightarrow \mathbf{g}_\Omega(\rho)$ between (equivalence classes of) irreducible representations of H with $\rho^\omega \simeq \rho$, for all $\omega \in \Omega$, and (equivalence classes of) irreducible representations of H^Ω . Further, this correspondence commutes with irreducible restriction and irreducible induction. Recall also that the representation $\mathbf{g}_\Omega(\rho)$ is characterized as the unique component of $\rho|_{H^\Omega}$ appearing with odd multiplicity.

We will usually apply this correspondence with Ω the group of automorphisms of G consisting of ε and the identity, in which case we will just write \mathbf{g} for the correspondence. Note also that, for ε -stable pro- p subgroups H of G , we have $H^\varepsilon = \overline{H}^+ = \overline{H}$.

We write $\overline{\theta} = \mathbf{g}(\theta) = \theta|_{\overline{H}^1}$, a skew semisimple character (see [22, §3.4]). We also set

$$\overline{G}_E^+ = B \cap \overline{G}^+ \text{ and } \overline{G}_E = B \cap \overline{G}$$

so that \overline{G}_E^+ (respectively \overline{G}_E) is the direct product of a unitary group, for the quadratic extension E/E_0 , and the anisotropic group \overline{G}_0^+ (respectively \overline{G}_0).

We recall first the following, from [22, Theorem 3.16] (see also the Remarks following *op. cit.* Corollary 4.2), and [24, Proposition 3.3.1]:

Lemma 2.5. (i) $I_{\overline{G}^+}(\overline{\theta}) = \overline{J}^1 \overline{G}_E^+ \overline{J}^1$ and $I_{\overline{G}}(\overline{\theta}) = \overline{J}^1 \overline{G}_E \overline{J}^1$.

(ii) *There exists a unique irreducible representation $\overline{\eta}$ of \overline{J}^1 which contains $\overline{\theta}$. Moreover, $\overline{\eta} = \mathbf{g}(\eta)$, $\dim \overline{\eta} = (\overline{J}^1 : \overline{H}^1)^{\frac{1}{2}}$ and, for $g \in \overline{G}^+$,*

$$\dim I_g(\overline{\eta}|\overline{J}^1) = \begin{cases} 1 & \text{if } g \in \overline{J}^1 \overline{G}_E^+ \overline{J}^1; \\ 0 & \text{otherwise.} \end{cases}$$

We also put $\overline{\theta}_P = \mathbf{g}(\theta_P) = \theta_P|_{\overline{H}_P^1}$. Note that we have

$$\overline{\theta}_P = \widetilde{i(\theta_W)}.$$

Remark 2.6. (i) Note that, in the language of [25, Definition 5.1], the decomposition $V = W \oplus V_0 \oplus W$ (of which the Levi subgroup L is the stabilizer) is *properly subordinate* to the stratum $[\Lambda, n, 0, \beta]$. Then [25, Lemma 5.3] implies that $U(\Lambda) \cap B \cap U = U^1(\Lambda) \cap B \cap U$. In particular, $\overline{J} \cap \overline{U} = \overline{J}^1 \cap \overline{U}$.

(ii) From (i), we have $\overline{J}_P \cap \overline{U} = \overline{H}_P^1 \cap \overline{U}$, and the same is true for \overline{U}_- . Then the property $\overline{\theta}_P = \widetilde{i(\theta_W)}$ implies (see [2, Lemma 1(ii)]) that

$$(\overline{J}_P \cap \overline{U})(\overline{J}_P \cap \overline{U}_-) \subset (\overline{J}_P \cap \overline{U}_-) \text{Ker}(i(\theta_W))(\overline{J}_P \cap \overline{U}).$$

In particular (by *loc. cit.*), if ρ is a representation of a subgroup K of $J_W \times \overline{G}_0$ which restricts to a multiple of θ_W , then $\widetilde{i(\rho)}$ is a well-defined representation of $(\overline{J}_P \cap \overline{U}_-)(i(K))(\overline{J}_P \cap \overline{U})$. We will use this several times, for example, in the definitions of $\widetilde{\eta}_P$ and $\overline{\kappa}_P$ below.

From Proposition 2.4, together with [22, Corollary 2.5] and [23, Theorem 2.3], we get:

Proposition 2.7. $I_{\overline{G}^+}(\overline{\theta}_P) = \overline{J}_P^1 \overline{G}_E^+ \overline{J}_P^1$ and $I_{\overline{G}}(\overline{\theta}_P) = \overline{J}_P^1 \overline{G}_E \overline{J}_P^1$.

Let $\overline{\theta}^m = \mathbf{g}(\theta^m) = \theta^m|_{\overline{H}_m^1}$. Let $\overline{\eta}$ be the unique irreducible representation of \overline{J}^1 which contains $\overline{\theta}$, as in Lemma 2.5, and let $\overline{\eta}^m$ be the unique irreducible representation of \overline{J}_m^1 which contains $\overline{\theta}^m$. Note that $\overline{\theta}_P = \mathbf{g}(\theta_P)$ is the trivial extension of $\overline{\theta}$ to \overline{H}_P^1 and, similarly, $\overline{\theta}_P^m = \mathbf{g}(\theta_P^m)$ the trivial extension of $\overline{\theta}^m$ to $\overline{H}_{m,P}^1$.

We also put $\overline{\eta}_P = \mathbf{g}(\eta_P)$ and $\overline{\eta}_P^m = \mathbf{g}(\eta_P^m)$. By taking ε -fixed points in J^1/H^1 , we can imitate Lemma 2.2 and Corollary 2.3 to show that there is a unique irreducible representation of \overline{J}_P^1 which contains $\overline{\theta}_P$ and, since $\eta_P|_{\overline{H}_P^1}$ is a multiple of $\overline{\theta}_P$, we see that this must be $\overline{\eta}_P$. Likewise, $\overline{\eta}_P^m$ is the unique irreducible representation of $\overline{J}_{m,P}^1$ containing $\overline{\theta}_P^m$. We also have

$$\overline{\eta}_P = \widetilde{i(\eta_W)} \quad \text{and} \quad \overline{\eta}_P^m = \widetilde{i(\eta_W^m)},$$

since the representations on the right restrict to multiples of $\widetilde{i(\theta_W)} = \overline{\theta}_P$ and $\widetilde{i(\theta_W^m)} = \overline{\theta}_P^m$ respectively.

The group $U(\Lambda) \cap B$ has Iwahori decomposition with respect to (L, P) , and normalizes $J^1(\frac{1}{2}\beta, \Lambda)$ and $H^1(\frac{1}{2}\beta, \Lambda)$. Hence $U(\Lambda) \cap B$ normalizes J_P^1 and, since $U(\Lambda^m)$ is contained in $U(\Lambda)$, we can form the group $\widetilde{J}_P^1 = (\overline{U}^1(\Lambda^m) \cap \overline{G}_E) \overline{J}_P^1$. We also recall that we have the representation $\widetilde{\eta}_W$ of $\widetilde{J}_W^1 = U^1(\mathfrak{B}_W^m) J_W^1$ (see §1.1) and observe that $\widetilde{J}_P^1 \cap \overline{L} = i(\widetilde{J}_W^1 \times \overline{U}^1(\Lambda_0))$.

Proposition 2.8 (cf. [6, Propositions 5.1.15, 5.1.19]). *There is a unique representation $\widetilde{\eta}_P$ of \widetilde{J}_P^1 such that*

(i) $\widetilde{\eta}_P|_{\overline{J}_P^1} = \overline{\eta}_P$;

(ii) $\widetilde{\eta}_P$ and $\overline{\eta}_P^m$ induce equivalent irreducible representations of $\overline{U}^1(\Lambda^m)$.

Moreover, $\widetilde{\eta}_P = \widetilde{i(\widetilde{\eta}_W)}$ and

$$\dim I_g(\widetilde{\eta}_P) = \begin{cases} 1 & \text{if } g \in \widetilde{J}_P^1 \overline{G}_E^+ \widetilde{J}_P^1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let Ω_1 be the group of automorphisms of \overline{G}^+ generated by conjugation by

$$h_1 = \begin{pmatrix} -I_M & & \\ & I_D & \\ & & -I_M \end{pmatrix}.$$

Then $(\overline{G}^+)^{\Omega_1} = (\overline{G}')^+ \times \overline{G}_0^+$. We will denote the Glauberman correspondence \mathbf{g}_{Ω_1} on representations of pro- p subgroups of \overline{G} by \mathbf{g}_1 . We note that, since all our representations are trivial on U' and U'^- , they are fixed by Ω_1 and, moreover, \mathbf{g}_1 is just restriction for these representations (since their restrictions are irreducible).

Let Ω_2 be the group of automorphisms of $(\overline{G}^+)^{\Omega_1}$ generated by conjugation by

$$h_2 = \begin{pmatrix} I_M & & \\ & I_D & \\ & & -I_M \end{pmatrix}.$$

Then $((\overline{G}^+)^{\Omega_1})^{\Omega_2} = \overline{L}$. We will denote the Glauberman correspondence \mathbf{g}_{Ω_2} on representations of pro- p subgroups of $(\overline{G}^+)^{\Omega_1}$ by \mathbf{g}_2 . As above, since all our representations are trivial on U and U^- , their transfers to $(\overline{G}^+)^{\Omega_1}$ are fixed by Ω_2 and, moreover, \mathbf{g}_2 is again just restriction for these representations (since their restrictions are still irreducible).

We write \mathbf{g}_{Ω} for the composition $\mathbf{g}_2 \circ \mathbf{g}_1$. Then we have

$$\mathbf{g}_{\Omega}(\overline{\eta}_P) = i(\eta_W) \quad \text{and} \quad \mathbf{g}_{\Omega}(\overline{\eta}_P^m) = i(\eta_W^m).$$

We define $\tilde{\eta}_P$ to be $i(\widetilde{\eta}_W)$. Then we clearly have $\mathbf{g}_{\Omega}(\tilde{\eta}_P) = i(\tilde{\eta}_W)$ and $\tilde{\eta}_P|_{\overline{J}_P^1} = \overline{\eta}_P$. Also, since the intertwining of $\tilde{\eta}_P$ is contained in that of $\overline{\eta}_P$,

$$I_{\overline{G}^+}(\tilde{\eta}_P) \cap \overline{U}^1(\Lambda^m) \subset \overline{J}_P^1 \overline{G}_E^+ \overline{J}_P^1 \cap \overline{U}^1(\Lambda^m) = (\overline{U}^1(\Lambda^m) \cap \overline{G}_E) \overline{J}_P^1 = \tilde{J}_P^1,$$

so the induced representation $\text{Ind}_{\tilde{J}_P^1}^{\overline{U}^1(\Lambda^m)} \tilde{\eta}_P$ is irreducible. Likewise, $\text{Ind}_{\overline{J}_{m,P}^1}^{\overline{U}^1(\Lambda^m)} \overline{\eta}_P^m$ is irreducible. Then, since Glauberman's correspondence commutes with irreducible induction, we have

$$\begin{aligned} \mathbf{g}_{\Omega} \left(\text{Ind}_{\tilde{J}_P^1}^{\overline{U}^1(\Lambda^m)} \tilde{\eta}_P \right) &= \text{Ind}_{i(\tilde{J}_W^1 \times \overline{U}^1(\Lambda_0))}^{i(U^1(\Lambda_W^m) \times \overline{U}^1(\Lambda_0))} i(\tilde{\eta}_W) \\ &\simeq \text{Ind}_{i(J^1(\beta, \mathfrak{A}_W^m) \times \overline{U}^1(\Lambda_0))}^{i(U^1(\Lambda_W^m) \times \overline{U}^1(\Lambda_0))} i(\eta_W^m) = \mathbf{g}_{\Omega} \left(\text{Ind}_{\overline{J}_{m,P}^1}^{\overline{U}^1(\Lambda^m)} \overline{\eta}_P^m \right). \end{aligned}$$

Condition (ii) now follows as \mathbf{g}_{Ω} is injective.

Now we show that these two conditions determine $\tilde{\eta}_P$ uniquely. For this, we need only show that $\overline{\eta}_P$ occurs in $\text{Ind}_{\overline{J}_{m,P}^1}^{\overline{U}^1(\Lambda^m)} \overline{\eta}_P^m$ with multiplicity one. We use the Mackey formula to compute the restriction

$$\text{Res}_{\overline{J}_P^1}^{\overline{U}^1(\Lambda^m)} \text{Ind}_{\tilde{J}_P^1}^{\overline{U}^1(\Lambda^m)} \tilde{\eta}_P.$$

If $x \in \overline{U}^1(\Lambda^m)$ intertwines $\tilde{\eta}_P$ with $\overline{\eta}_P$, then it intertwines $\overline{\eta}_P$ with itself so lies in \tilde{J}_P^1 , as above. Thus the multiplicity of $\overline{\eta}_P$ in $\text{Ind}_{\overline{J}_{m,P}^1}^{\overline{U}^1(\Lambda^m)} \overline{\eta}_P^m$ is equal to its multiplicity in $\tilde{\eta}_P$, which we know to be one.

Finally, we must show that all of \overline{G}_E^+ intertwines $\tilde{\eta}_P$. So suppose $b \in \overline{G}_E^+$. Since b intertwines $\overline{\eta}_P^m$, it certainly intertwines

$$\text{Ind}_{\tilde{J}_P^1}^{\overline{U}^1(\Lambda^m)} \tilde{\eta}_P \simeq \text{Ind}_{\overline{J}_{m,P}^1}^{\overline{U}^1(\Lambda^m)} \overline{\eta}_P^m.$$

We deduce that there exist $u, v \in \overline{U}^1(\Lambda^m)$ such that ubv intertwines $\tilde{\eta}_P$. In particular, ubv intertwines $\overline{\eta}_P$ so there exist $j_1, j_2 \in \overline{J}_P^1$ such that $j_1 ubv j_2 \in \overline{G}_E^+$. Note that this element also still intertwines $\tilde{\eta}_P$.

Now

$$\begin{aligned} U^1(\Lambda^m) b U^1(\Lambda^m) \cap B^\times &= U^1(\Lambda^m) b U^1(\Lambda^m) \cap L' \cap B^\times \\ &= (U^1(\Lambda^m) \cap L') b (U^1(\Lambda^m) \cap L') \cap B^\times, \end{aligned}$$

by [23, Lemma 2.1] applied with $\Gamma = \Omega_1$. Further,

$$(U^1(\Lambda^{\mathfrak{m}}) \cap L')b(U^1(\Lambda^{\mathfrak{m}}) \cap L') \cap B^\times = (U^1(\Lambda^{\mathfrak{m}}) \cap B^\times)b(U^1(\Lambda^{\mathfrak{m}}) \cap B^\times),$$

by [6, Theorem 1.6.1] (which we may apply since, in A' we have a simple stratum). Then, applying [23, Lemma 2.1] again, with $\Gamma = \{1, \varepsilon\}$, we get

$$\overline{U}^1(\Lambda^{\mathfrak{m}})b\overline{U}^1(\Lambda^{\mathfrak{m}}) \cap \overline{G}_E^+ = (\overline{U}^1(\Lambda^{\mathfrak{m}}) \cap \overline{G}_E)b(\overline{U}^1(\Lambda^{\mathfrak{m}}) \cap \overline{G}_E).$$

So there exist $u', v' \in \overline{U}^1(\Lambda^{\mathfrak{m}}) \cap \overline{G}_E$ such that $u'bv' = j_1ubvj_2$ and, since $u', v' \in \widetilde{J}_P^1$, we see that b intertwines $\widetilde{\eta}_P$.

Finally, the assertion that the intertwining spaces are all 1-dimensional follows immediately from Lemma 2.5(ii), since $\widetilde{\eta}_P$ extends $\bar{\eta}$. \blacksquare

Now we define $\bar{\kappa}_P = \widetilde{i(\kappa_W)}$, a representation of \overline{J}_P , so that $\bar{\kappa}_P|_{\widetilde{J}_P^1} \simeq \widetilde{\eta}_P$. We also put $\bar{\rho}_P = \widetilde{i(\rho_W)}$, a representation of \overline{J}_P trivial on \overline{J}_P^1 , and

$$\bar{\lambda}_P = \widetilde{i(\lambda_W)} = \bar{\kappa}_P \otimes \bar{\rho}_P.$$

Writing matrices with respect to the (non-Witt) basis $\mathcal{B} \cup \mathcal{B}_0 \cup \mathcal{B}$ of $V = W \oplus V_0 \oplus W$, we set

$$w_1 = \begin{pmatrix} & & I_M \\ & I_D & \\ \nu\sigma\varepsilon_W(\sigma) & & \end{pmatrix},$$

where I_k denotes the $k \times k$ identity matrix. We also put

$$\overline{W} = \{i(\varpi_E^a, \mathbf{1}_{V_0}) : a \in \mathbb{Z}\} \cup \{i(\varpi_E^a, \mathbf{1}_{V_0})w_1 : a \in \mathbb{Z}\},$$

where ϖ_E is our fixed uniformizer of $E \subset A_W$ and $\mathbf{1}_{V_0}$ is the identity map on V_0 .

Proposition 2.9. $I_{\overline{G}}(\bar{\lambda}_P) = \overline{J}_P \overline{W} \overline{J}_P$.

Proof Since $\overline{J}_P \cap \overline{L} = i(J_W)$, while ϖ_E and $\sigma\varepsilon_W(\sigma)$ normalize J_W , the elements of \overline{W} normalize $\overline{J}_P \cap \overline{L}$. Since ϖ_E normalizes λ_W and $\lambda_W \simeq \lambda_W \circ \tilde{\sigma}$ (see Proposition 1.1), we see that the elements of \overline{W} normalize $i(\lambda_W)$. On the other hand, they either preserve \overline{U} and \overline{U}_- or interchange them so we see that every element of \overline{W} intertwines $\bar{\lambda}_P = \widetilde{i(\lambda_W)}$. Hence we have $I_{\overline{G}}(\bar{\lambda}_P) \supset \overline{J}_P \overline{W} \overline{J}_P$.

The proof of the opposite containment, which is a variant of the proof of [6, Proposition 5.3.2], is inspired by [3, page 551]; in place of [3, 4.2 Lemma], we use [25, Proposition 1.1], which is a slight generalization of [18, Proposition 4.13]. It is almost identical to the proof of [25, Proposition 6.14], except that the definition of $\bar{\kappa}_P$ there is *a priori* slightly different.

Suppose $g \in \overline{G}$ intertwines $\bar{\lambda}_P = \bar{\kappa}_P \otimes \bar{\rho}_P$, so that $g \in I_{\overline{G}}(\bar{\eta}_P|_{\overline{J}_P^1}) = \overline{J}_P \overline{G}_E \overline{J}_P$, as $\bar{\rho}_P$ is trivial on \overline{J}_P^1 . Thus, we may assume g lies in \overline{G}_E . Moreover $\overline{J}_P \cap \overline{G}_E = \overline{U}(\Lambda) \cap \overline{G}_E$ is a parahoric subgroup of \overline{G}_E containing the Iwahori subgroup $\overline{U}(\Lambda^{\mathfrak{m}}) \cap \overline{G}_E$. Therefore, we may further assume g is a *distinguished double coset representative* for $\overline{U}(\Lambda) \cap \overline{G}_E \backslash \overline{G}_E / \overline{U}(\Lambda) \cap \overline{G}_E$ (see [18, §3] or [25, §1] for this notion).

Since $\dim I_g(\bar{\eta}_P, \bar{J}_P^1) = 1$, we can imitate the proof of [6, Proposition 5.3.2] to get that any non-zero intertwining operator in $I_g(\bar{\lambda}_P, \bar{J}_P)$ has the form $S \otimes T$, with $S \in I_g(\bar{\eta}_P, \bar{J}_P^1)$ and T an endomorphism of the space of $\bar{\rho}_P$. Now the operator S also intertwines the restriction $\kappa_P|_{\tilde{J}_P^1} = \tilde{\eta}_P$ so, again as in [6, Proposition 5.3.2], it follows that T belongs to $I_g(\bar{\rho}_P|_{\tilde{J}_P^1})$. In particular, g intertwines $\bar{\rho}_P|_{\tilde{J}_P^1 \cap \bar{G}_E}$. But $\tilde{J}_P^1 \cap \bar{G}_E = \bar{U}^1(\Lambda^m) \cap \bar{G}_E$ is the radical of the Iwahori subgroup $\bar{U}(\Lambda^m) \cap \bar{G}_E$ of \bar{G}_E contained in $\bar{U}(\Lambda) \cap \bar{G}_E$. By [25, Proposition 1.1] and the Remarks that follow it, we conclude that we can assume that g normalizes $\bar{U}(\Lambda) \cap \bar{G}_E \cap \bar{L}$.

Finally, we notice (cf. [3, 4.1]) that $N_{\bar{G}_E}(\bar{U}(\Lambda) \cap \bar{G}_E \cap \bar{L}) = (\bar{U}(\Lambda) \cap \bar{G}_E \cap \bar{L})\bar{W}(\bar{U}(\Lambda) \cap \bar{G}_E \cap \bar{L})$. Since $\bar{U}(\Lambda) \cap \bar{G}_E \cap \bar{L} \subset \bar{J}_P$, we are done. \blacksquare

Recall that we write $J_{\bar{L}} = i(J_W)$ and $\lambda_{\bar{L}} = i(\lambda_W)$. In order to prove that $(\bar{J}_P, \bar{\lambda}_P)$ is a cover of $(J_{\bar{L}}, \lambda_{\bar{L}})$, the only thing remaining is to find a strongly (\bar{P}, \bar{J}_P) -positive element in the centre of \bar{L} which supports an invertible element of the spherical Hecke algebra $\mathcal{H}(\bar{G}, \bar{\lambda}_P)$. To achieve this, we look at the $\bar{\lambda}_P$ -spherical Hecke algebras of two parahoric subgroups whose intersections with \bar{G}_E are non-conjugate maximal compact open subgroups of \bar{G}_E . Note that we could here have used instead the methods of [3] – indeed, the crucial ingredients (for example, the elements w_1, w_2 below) are the same.

Let $\mathcal{L}'^{(1)}$ be the self-dual \mathfrak{o}_E -lattice chain of \mathfrak{o}_F -period e_W in $V' = W \oplus W$ given by

$$\cdots \supset L_k^W \oplus L_k^W \supset L_{k+1}^W \oplus L_{k+1}^W \supset \cdots,$$

so that $\mathcal{L}'^{(1)}$ consists of every second lattice of \mathcal{L}' . Let $\Lambda'^{(1)}$ be the self-dual \mathfrak{o}_E -lattice sequence in V' in which every lattice of $\mathcal{L}'^{(1)}$ occurs twice and with the indexing chosen such that

$$\Lambda'^{(1)}(k)^\# = \Lambda'^{(1)}(1-k), \quad \text{for all } k \in \mathbb{Z}.$$

Let $\Lambda^{(1)}$ be the \mathfrak{o}_F -lattice sequence in V defined by

$$\Lambda^{(1)}(k) = \Lambda'^{(1)}(k) \oplus \Lambda_0(2k).$$

It is a self-dual lattice sequence of period $2e_W = e/2$ such that $\mathfrak{a}_0(\Lambda^{(1)}) \supset \mathfrak{a}_0(\Lambda)$. Put $\bar{K}_1 = \bar{U}(\Lambda^{(1)})$.

We define $\bar{K}_2 = \bar{U}(\Lambda^{(2)})$ by the same process, starting from the self-dual \mathfrak{o}_E -lattice chain $\mathcal{L}'^{(2)}$ in V' given by

$$\cdots \supset L_k^W \oplus L_{k+1}^W \supset L_{k+1}^W \oplus L_{k+2}^W \supset \cdots$$

Then $\bar{U}(\Lambda) \subset \bar{K}_1 \cap \bar{K}_2$ so, in particular, $\bar{J}_P \subset \bar{K}_1 \cap \bar{K}_2$.

Note that the element w_1 lies in $\bar{W} \cap \bar{K}_1$. We also set $w_2 = i(\varpi_E^{-1}, \mathbf{1}_{V_0})w_1 \in \bar{W} \cap \bar{K}_2$.

Lemma 2.10. (i) $\mathcal{H}(\bar{K}_1, \bar{\lambda}_P) = \langle f_1, f_{w_1} \rangle$ where f_1 is supported on \bar{J}_P and f_{w_1} is supported on $\bar{J}_P w_1 \bar{J}_P$.

(ii) $\mathcal{H}(\bar{K}_2, \bar{\lambda}_P) = \langle f_1, f_{w_2} \rangle$, with f_1 as in (i) and f_{w_2} supported on $\bar{J}_P w_2 \bar{J}_P$.

Proof Both parts follow from the following consideration. For $i = 1, 2$ the \bar{K}_i -intertwining of $\bar{\lambda}_P$ is given by

$$I_{\bar{K}_i}(\bar{\lambda}_P) = \bar{J}_P \bar{W} \bar{J}_P \cap \bar{K}_i = \bar{J}_P (\bar{W} \cap \bar{K}_i) \bar{J}_P.$$

But $\bar{W} \cap \bar{K}_i = \{1, w_i\}$. Moreover, the restriction of $\bar{\lambda}_P$ to $\bar{J}_P \cap \bar{L}$ is irreducible and w_i normalizes $\bar{J}_P \cap \bar{L}$ so the intertwining space $I_{w_i}(\bar{\lambda}_P)$ is 1-dimensional. \blacksquare

Lemma 2.11. *Consider f_{w_1}, f_{w_2} as elements of $\mathcal{H}(\overline{G}, \overline{\lambda}_P)$. Then the convolution $f_{w_1} * f_{w_2}$ is supported on $\overline{J}_P w_1 w_2 \overline{J}_P$.*

Proof We know $f_{w_1} * f_{w_2}$ is supported on

$$\begin{aligned} & \overline{J}_P w_1 \overline{J}_P w_2 \overline{J}_P \\ &= \overline{J}_P (w_1 (\overline{J}_P \cap \overline{U}_-) w_1^{-1}) (w_1 (\overline{J}_P \cap \overline{L}) w_1^{-1}) w_1 w_2 (w_2^{-1} (\overline{J}_P \cap \overline{U}) w_2) \overline{J}_P. \end{aligned}$$

From the explicit matrix descriptions in §1.4, \overline{J}_P contains $w_1 (\overline{J}_P \cap \overline{U}_-) w_1^{-1} = w_1 (\overline{H}^1 \cap \overline{U}_-) w_1^{-1}$, $w_1 (\overline{J}_P \cap \overline{L}) w_1^{-1}$ and $w_2^{-1} (\overline{J}_P \cap \overline{U}) w_2 = w_2^{-1} (\overline{J}^1 \cap \overline{U}) w_2$, and the lemma follows. \blacksquare

We will prove that $f_\zeta := f_{w_1} * f_{w_2}$ is invertible. To accomplish this we prove that f_{w_1} and f_{w_2} are each invertible. In each case we know

$$f_{w_i} * f_{w_i} = c_{1i} f_1 + d_i f_{w_i} \tag{2.12}$$

by Lemma 2.10. Thus, we only need to show $c_{1i} \neq 0$ for each i .

Lemma 2.13. *In equation (2.12) $c_{1i} \neq 0$ for $i = 1, 2$.*

Proof We treat only the case $i = 1$, since the other case is identical. We need to check that $f_{w_1} * f_{w_1}(1) \neq 0$. Since

$$f_{w_1}(x) = \begin{cases} 0 & \text{if } x \notin \overline{J}_P w_1 \overline{J}_P \\ \overline{\lambda}_P^\vee(j_1) f_{w_1}(w_1) \overline{\lambda}_P^\vee(j_2) & \text{if } x = j_1 w_1 j_2, \text{ with } j_1, j_2 \in \overline{J}_P, \end{cases}$$

we can write

$$\begin{aligned} f_{w_1} * f_{w_1}(1) &= \int_{\overline{K}_1} f_{w_1}(y) f_{w_1}(y^{-1}) dy \\ &= \frac{1}{|\overline{J}_P \cap w_1 \overline{J}_P|} \int_{\overline{J}_P \times \overline{J}_P} f_{w_1}(j_1 w_1 j_2) f_{w_1}(j_2^{-1} w_1^{-1} j_1^{-1}) dj_1 dj_2 \\ &= \frac{|\overline{J}_P|}{|\overline{J}_P \cap w_1 \overline{J}_P|} \int_{\overline{J}_P} \overline{\lambda}_P^\vee(j_1) f_{w_1}(w_1) f_{w_1}(w_1^{-1}) \overline{\lambda}_P^\vee(j_1^{-1}) dj_1. \end{aligned}$$

Now w_1 intertwines $\overline{\lambda}_P^\vee = \widetilde{i(\lambda_W^\vee)}$ and normalizes $\overline{J}_P \cap \overline{L} = i(J_W)$ so $f_{w_1}(w_1)$ is an equivalence $w_1 i(\lambda_W^\vee) \simeq i(\lambda_W^\vee)$. But $w_1^{-1} = h_1^{-1} w_1$, where $h_1 = i(\nu \sigma \varepsilon_W(\sigma), \mathbf{1}_{V_0})$ and, by Proposition 1.1, $\nu \sigma \varepsilon_W(\sigma) \in J_W$; hence $w_1^{-1} \in \overline{J}_P w_1$ and $f_{w_1}(w_1^{-1})$ is an equivalence $i(\lambda_W^\vee) \simeq w_1 i(\lambda_W^\vee)$. Thus $f_{w_1}(w_1) f_{w_1}(w_1^{-1})$ is an equivalence of $i(\lambda_W^\vee)$ and hence a scalar $c \neq 0$. Thus

$$f_{w_1} * f_{w_1}(1) = c \frac{|\overline{J}_P|^2}{|\overline{J}_P \cap w_1 \overline{J}_P|} \neq 0.$$

Therefore, f_{w_1} is invertible, as required. \blacksquare

Put $\zeta = i(\varpi_E, \mathbf{1}_{V_0})$; since $\sigma \varepsilon_W(\sigma) \in J_W$ the function f_ζ is supported on $\overline{J}_P \zeta \overline{J}_P$.

Lemma 2.14. *For each $k \in \mathbf{N}$, the k -fold convolution f_ζ^k is supported on $\overline{J}_P \zeta^k \overline{J}_P$.*

Proof This is by simple induction on k , since

$$\begin{aligned} & \bar{J}_P \zeta \bar{J}_P \zeta^k \bar{J}_P \\ &= \bar{J}_P (\zeta (\bar{J}_P \cap \bar{U}) \zeta^{-1}) (\zeta (\bar{J}_P \cap \bar{L}) \zeta^{-1}) \zeta^{k+1} (\zeta^{-k} (\bar{J}_P \cap \bar{U}_-) \zeta^k) \bar{J}_P, \end{aligned}$$

while \bar{J}_P contains $\zeta (\bar{J}_P \cap \bar{U}) \zeta^{-1}$, $\zeta (\bar{J}_P \cap \bar{L}) \zeta^{-1}$ and $\zeta^{-k} (\bar{J}_P \cap \bar{U}_-) \zeta^k$ (cf. Lemma 2.11). \blacksquare

In particular, since $\varpi_E^{e(E/F)}$ and ϖ_F differ only by a unit of E , the function $f_\zeta^{e(E/F)}$ is an invertible element of $\mathcal{H}(\bar{G}, \bar{\lambda}_P)$ which is supported on the double coset $\bar{J}_P \zeta^{e(E/F)} \bar{J}_P = \bar{J}_P \zeta_F \bar{J}_P$, where

$$\zeta_F = i(\varpi_F \mathbf{1}_W, \mathbf{1}_{V_0})$$

is a strongly (\bar{P}, \bar{J}_P) -positive element of the centre of \bar{L} . We conclude:

Theorem 2.15. *Let π_W be an irreducible supercuspidal representation of $G_W \cong GL_M(F)$, with $\pi_W^{\varepsilon_W} \simeq \pi_W$. Using the notation above, the pair $(\bar{J}_P, \bar{\lambda}_P)$ is a \bar{G} -cover of $(J_{\bar{L}}, \lambda_{\bar{L}})$. In particular, it is an \mathfrak{s} -type, with $\mathfrak{s} = [\bar{L}, i(\pi_W)]_{\bar{G}}$.*

Proof By construction, \bar{J}_P is decomposed with respect to \bar{P} so (i) and (ii) of [7, Definition 8.1] are satisfied for \bar{P} , and likewise for \bar{P}_- . By Lemma 2.13 the strongly (\bar{P}, \bar{J}_P) -positive element ζ_F supports an invertible element of $\mathcal{H}(\bar{G}, \bar{\lambda}_P)$. Thus $(\bar{J}_P, \bar{\lambda}_P)$ also satisfies (iii) of [7, Definition 8.1] for \bar{P} , and is therefore a \bar{G} -cover of $(J_{\bar{L}}, \lambda_{\bar{L}})$. Then [7, Theorem 8.3] implies $(\bar{J}_P, \bar{\lambda}_P)$ is an \mathfrak{s} -type. \blacksquare

2.3 Hecke Algebras

In this section we derive results analogous to those of Chapter 5 of [6]. In particular, we show that the Hecke algebra of our type, $\mathcal{H}(\bar{G}, \bar{\lambda}_P)$ can be computed by using analogous computations in a case where the representation π of \bar{L} is of level zero. For many of these situations, [16] will give us the parameters of the Hecke algebra.

We fix $i = 1$ or 2 and, in the notation of the previous section, we put $\Lambda^M = \Lambda^{(i)}$. Then $[\Lambda^M, n_M, 0, \frac{1}{2}\beta]$ is a skew semisimple stratum and we set $\bar{J}_M = \bar{J}(\frac{1}{2}\beta, \Lambda^M)$, and similarly \bar{J}_M^1 and \bar{H}_M^1 . Let $\bar{\theta}_M$ denote the transfer of $\bar{\theta}$ to \bar{H}_M^1 . By [22, Corollary 4.2], there is a unique irreducible representation $\bar{\eta}_M$ of \bar{J}_M^1 which extends $\bar{\theta}_M$, and by [25, Theorem 4.1], we may choose a β -extension, $\bar{\kappa}_M$ of $\bar{\eta}_M$ to \bar{J}_M – we recall here what we mean by β -extension:

Recall the lattice sequence Λ^m from §1.4 such that $\bar{U}(\Lambda^m) \cap \bar{G}_E$ is an Iwahori subgroup of $\bar{U}(\Lambda) \cap \bar{G}_E$; then we have $\bar{\theta}^m$ the transfer of $\bar{\theta}$ and $\bar{\eta}^m$ the unique irreducible representation of $\bar{J}^1(\frac{1}{2}\beta, \Lambda^m)$ containing $\bar{\theta}^m$. We abbreviate $\bar{U}(\Lambda) \cap \bar{G}_E = \bar{U}(\Lambda_{\mathfrak{o}_E})$ (and similarly for other lattice sequences) and define $\hat{J}_M = \bar{U}(\Lambda_{\mathfrak{o}_E}) \bar{J}_M^1$ and $\hat{J}_M^1 = \bar{U}^1(\Lambda_{\mathfrak{o}_E}^m) \bar{J}_M^1$. Thus, $\bar{J}_M \supset \hat{J}_M \supset \hat{J}_M^1 \supset \bar{J}_M^1$ and \hat{J}_M^1 is a pro- p Sylow subgroup of \bar{J}_M . By [25, Proposition 3.7], there is a unique irreducible representation, $\hat{\eta}_M$ of \hat{J}_M^1 which extends $\bar{\eta}_M$ and such that $\hat{\eta}_M$ and $\bar{\eta}^m$ induce equivalent irreducible representations of $\bar{U}^1(\Lambda_m)$. Then a β -extension $\bar{\kappa}_M$ is an extension to \bar{J}_M of $\hat{\eta}_M$.

Similarly (as in [6, Proposition 5.2.5] – see [25, Lemma 4.2]), there is a unique irreducible representation, $\hat{\mu}_M$ of \hat{J}_M which extends $\bar{\eta}_M$ and such that

$$\text{Ind}_{\hat{J}_M}^{\bar{U}(\Lambda_{\mathfrak{o}_E}) \bar{U}^1(\Lambda)} \hat{\mu}_M \simeq \text{Ind}_{\bar{J}_P}^{\bar{U}(\Lambda_{\mathfrak{o}_E}) \bar{U}^1(\Lambda)} \bar{\kappa}_P. \quad (2.16)$$

Moreover, as in [6, Proposition 5.2.6], we have $\hat{\mu}_M|_{\hat{J}_M^1} = \hat{\eta}_M$. The following Lemma will be crucial:

Lemma 2.17. *Let $\bar{\tau}_1, \bar{\tau}_2$ be irreducible representations of $\bar{U}(\Lambda_{\mathfrak{o}_E})/\bar{U}^1(\Lambda_{\mathfrak{o}_E})$, which we regard as representations of \bar{J}_P or \hat{J}_M . Then there is a Hecke algebra isomorphism*

$$\begin{aligned} \mathcal{H}(\bar{G}, \bar{\kappa}_P \otimes \bar{\tau}_1, \bar{\kappa}_P \otimes \bar{\tau}_2) &\longrightarrow \mathcal{H}(\bar{G}, \hat{\mu}_M \otimes \bar{\tau}_1, \hat{\mu}_M \otimes \bar{\tau}_2) \\ \bar{f} &\longmapsto \hat{f}, \end{aligned}$$

which preserves support: if \bar{f} has support $\bar{J}_P g \bar{J}_P$, for some $g \in \bar{G}_E$, then \hat{f} has support $\hat{J}_M g \hat{J}_M$.

Proof Note that we can also think of $\bar{\tau}_j$ as a representation of $\bar{U}(\Lambda_{\mathfrak{o}_E})\bar{U}^1(\Lambda)$, and we get

$$\text{Ind}_{\bar{J}_P}^{\bar{U}(\Lambda_{\mathfrak{o}_E})\bar{U}^1(\Lambda)}(\bar{\kappa}_P \otimes \bar{\tau}_j) = \left(\text{Ind}_{\bar{J}_P}^{\bar{U}(\Lambda_{\mathfrak{o}_E})\bar{U}^1(\Lambda)} \bar{\kappa}_P \right) \otimes \bar{\tau}_j,$$

and similarly for $\hat{\mu}_M$. The first assertion is then immediate from (2.16), by Lemma A.1 in the Appendix. The assertion concerning supports is a mild generalization of, for example, [25, Lemma 6.1 and Proposition 7.1] (see also [6, Proposition 5.5.13]), using Lemma A.1 in place of [6, Corollary 4.1.5]. More precisely, put $\bar{\kappa} = \text{Ind}_{\bar{J}_P}^{\bar{U}(\Lambda_{\mathfrak{o}_E})\bar{U}^1(\Lambda)} \bar{\kappa}_P$ (which is irreducible, since $I_{\bar{G}}(\bar{\kappa}_P) \subset I_{\bar{G}}(\bar{\eta}_P) = \bar{J}_P \bar{G}_E \bar{J}_P$), and consider $\bar{\tau}_j$ also as representations of \bar{J}_M ; then the results cited are analogous to showing that there are support-preserving isomorphisms from each Hecke algebra to $\mathcal{H}(\bar{G}, \bar{\kappa} \otimes \bar{\tau}_1, \bar{\kappa} \otimes \bar{\tau}_2)$, and our isomorphism is the composition of these. \blacksquare

Using this, we obtain:

Lemma 2.18. *There is a choice of κ_W for which $\kappa_W \circ \tilde{\sigma} \simeq \kappa_W$.*

Proof Recall that we have fixed $i = 1$ or 2 ; in the notation of the previous section, we put $w = w_i$. Note that, by Proposition 2.8, w intertwines the representation $\bar{\eta}_P = \widetilde{i(\eta_W)}$ of \bar{J}_P .

We fix some choice of κ_W . Now $\kappa_W \circ \tilde{\sigma}$ is also a β -extension of η_W so $\kappa_W \circ \tilde{\sigma} \simeq \kappa_W \otimes \chi_W$, for some character χ_W of $U(\mathfrak{B}_W)/U^1(\mathfrak{B}_W) \cong J_W/J_W^1$ which factors through the determinant $\det_{B_W/E}$. Then

$${}^w i(\kappa_W) \simeq i(\kappa_W \circ \tilde{\sigma}) = i(\kappa_W) \otimes i(\chi_W)$$

and w intertwines $\bar{\kappa}_P$ with $\bar{\kappa}_P \otimes \bar{\chi}_P$ (where $\bar{\chi}_P = \widetilde{i(\chi_W)}$ on \bar{J}_P), since both are trivial on the unipotent parts of \bar{J}_P . We also put $\hat{\chi}_M = \widetilde{i(\chi_W)}$ on \hat{J}_M . Then Lemma 2.17 (applied with $\bar{\tau}_1$ the trivial representation and $\bar{\tau}_2 = \widetilde{i(\chi_W)}$) implies that w intertwines $\hat{\mu}_M$ with $\hat{\mu}_M \otimes \hat{\chi}_M$.

Recall that we have $\bar{\kappa}_M|_{\hat{J}_M^1} = \hat{\eta}_M = \hat{\mu}_M|_{\hat{J}_M^1}$, by construction of β -extensions in [25, Theorem 4.1]. Thus, if we let $\hat{\kappa}_M = \bar{\kappa}_M|_{\hat{J}_M}$, we must have $\hat{\kappa}_M \simeq \hat{\mu}_M \otimes \bar{\psi}_M$ for a character $\bar{\psi}_M$ of $\bar{U}(\Lambda_{\mathfrak{o}_E})/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^m)$, that is, a character of the Siegel Levi subgroup of $\bar{U}(\Lambda_{\mathfrak{o}_E}^M)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M)$ which is trivial on the maximal unipotent subgroup $\bar{U}^1(\Lambda_{\mathfrak{o}_E}^m)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M)$. Then $\bar{\psi}_M$ factorizes through the determinant on the Levi subgroup and we can write $\bar{\psi}_M = i(\psi_W)$, for some character ψ_W of J_W/J_W^1 .

Now $w \in \bar{J}_M$ so w certainly intertwines $\hat{\kappa}_M$ with itself. Hence w intertwines $\hat{\mu}_M \otimes \bar{\psi}_M$ with $\hat{\mu}_M \otimes \hat{\chi}_M \otimes {}^w \bar{\psi}_M$. Chasing back through the isomorphism of Lemma 2.17, we see that w intertwines $\bar{\kappa}_P \otimes \bar{\psi}_M$ with $\bar{\kappa}_P \otimes \hat{\chi}_M \otimes {}^w \bar{\psi}_M$. Since w normalizes \bar{L} , this implies that conjugation by w gives an equivalence $\kappa_W \otimes \psi_W \simeq \kappa_W \otimes \chi_W(\psi_W \circ \tilde{\sigma})$. By [6, Theorem 5.2.2], we deduce that $\psi_W = \chi_W(\psi_W \circ \tilde{\sigma})$ and, in particular, $\kappa_W \otimes \psi_W$ is a β -extension with the required property. \blacksquare

We now choose κ_W as in Lemma 2.18 and make the same constructions as before: $\bar{\kappa}_P$, $\bar{\kappa}$ and $\hat{\mu}_M$. Comparing $\hat{\mu}_M$ with $\hat{\kappa}_M$, we again get a character $\bar{\psi}_M$ but now with the property that ${}^w\bar{\psi}_M = \bar{\psi}_M$. If E/E_0 is unramified then this implies that $\bar{\psi}_M = {}^w\bar{\psi}_0\bar{\psi}_0$, for some character $\bar{\psi}_0$. Then $\bar{\psi}_0$ extends to a character of J_M and, replacing $\bar{\kappa}_M$ by $\bar{\kappa}_M \otimes \bar{\psi}_0^{-1}$, we may assume $\bar{\psi}_M = 1$. In the ramified case, the condition on $\bar{\psi}_M$ is that $\bar{\psi}_M^2 = 1$ but it is (at least in principle) possible that $\bar{\psi}_M \neq 1$.

We write $\bar{\psi}_M = i(\widetilde{\psi}_W)$; then we may assume $\psi_W^2 = 1$.

Proposition 2.19. *With the notation as above, there is an algebra isomorphism*

$$\mathcal{H}(\bar{J}_M, \bar{\lambda}_P) \xrightarrow[\iota]{\sim} \mathcal{H}(\bar{U}(\Lambda_{\mathfrak{o}_E}^M)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M), i(\widetilde{\rho_W \otimes \psi_W}))$$

which preserves support: if $\bar{f} \in \mathcal{H}(\bar{J}_M, \bar{\lambda}_P)$ has support $\bar{J}_P b \bar{J}_P$, for some $b \in \bar{J}_M \cap \bar{G}_E = \bar{U}(\Lambda_{\mathfrak{o}_E}^M)$, then $\iota(\bar{f})$ has support $(\bar{U}(\Lambda_{\mathfrak{o}_E}^M)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M)) b (\bar{U}(\Lambda_{\mathfrak{o}_E}^M)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M))$.

Proof From Lemma 2.17, we get a support-preserving isomorphism

$$\mathcal{H}(\bar{G}, \bar{\lambda}_P) \simeq \mathcal{H}(\bar{G}, \hat{\mu}_M \otimes i(\widetilde{\rho_W})) = \mathcal{H}(\bar{G}, (\hat{\mu}_M \otimes \bar{\psi}_M) \otimes (i(\widetilde{\rho_W \otimes \psi_W}))),$$

where we are using the fact that $\psi_W^2 = 1$. On the other hand, we have support-preserving isomorphisms

$$\begin{aligned} \mathcal{H}(\bar{J}_M, (\hat{\mu}_M \otimes \bar{\psi}_M) \otimes (i(\widetilde{\rho_W \otimes \psi_W}))) &\simeq \mathcal{H}(\bar{J}_M, i(\widetilde{\rho_W \otimes \psi_W})) \\ &\simeq \mathcal{H}(\bar{U}(\Lambda_{\mathfrak{o}_E}^M)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M), i(\widetilde{\rho_W \otimes \psi_W})), \end{aligned}$$

where the first isomorphism follows from the fact that $\hat{\mu}_M \otimes \bar{\psi}_M = \hat{\kappa}_M$ extends to a representation $\bar{\kappa}_M$ of \bar{J}_M (cf. [6, Lemma 5.6.3]), and the second by reduction modulo \bar{J}_M^1 , since $\bar{J}_M/\bar{J}_M^1 \cong \bar{U}(\Lambda_{\mathfrak{o}_E}^M)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M)$. Putting these isomorphisms together gives the isomorphism of the Proposition. \blacksquare

Remarks 2.20. (i) Note that, writing \mathfrak{B}'_M for the self-dual \mathfrak{o}_E -order $\mathfrak{a}_0(\Lambda^M) \cap B'$, we have

$$\begin{aligned} \bar{U}(\Lambda_{\mathfrak{o}_E}^M)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M) &\cong \bar{U}(\mathfrak{B}'_M)/\bar{U}^1(\mathfrak{B}'_M) \times \bar{G}_0/\bar{G}_0^1 \quad \text{and} \\ \bar{U}(\Lambda_{\mathfrak{o}_E}^M)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M) &\cong \bar{U}(\mathfrak{B}')/\bar{U}^1(\mathfrak{B}') \times \bar{G}_0/\bar{G}_0^1, \end{aligned}$$

where \bar{G}_0^1 is the pro- p radical of the anisotropic group \bar{G}_0 . Then we have an isomorphism

$$\mathcal{H}(\bar{U}(\Lambda_{\mathfrak{o}_E}^M)/\bar{U}^1(\Lambda_{\mathfrak{o}_E}^M), i(\widetilde{\rho_W \otimes \psi_W})) \simeq \mathcal{H}(\bar{U}(\mathfrak{B}'_M)/\bar{U}^1(\mathfrak{B}'_M), i(\widetilde{\rho_W \otimes \psi_W})).$$

The quotient $\bar{U}(\mathfrak{B}'_M)/\bar{U}^1(\mathfrak{B}'_M)$ is a unitary (if E/E_0 is unramified), symplectic or orthogonal group over k_{E_0} and the Hecke algebra on the right is described in [13]. Alternatively, reduction modulo $\bar{U}^1(\mathfrak{B}'_M)$ gives a support-preserving isomorphism

$$\mathcal{H}(\bar{U}(\mathfrak{B}'_M)/\bar{U}^1(\mathfrak{B}'_M), i(\widetilde{\rho_W \otimes \psi_W})) \simeq \mathcal{H}(\bar{U}(\mathfrak{B}'_M), i(\widetilde{\rho_W \otimes \psi_W})),$$

and the latter is described in [18]. This will allow us, in a future work, to compute explicitly the parameters for the Hecke algebra.

- (ii) Since (in the case where E/E_0 is ramified) we have $\psi_W^2 = 1$, we may replace our choice of κ_W by $\kappa_W \otimes \psi_W$ (which has the same property of being fixed by $\tilde{\sigma}$); this replaces ρ_W by $\rho_W \otimes \psi_W$, another self-dual cuspidal representation of $U(\mathfrak{B}_W)/U^1(\mathfrak{B}_W)$, and we lose the character ψ_W from the RHS of the isomorphism in Proposition 2.19. *However*, we cannot do this independently for the two choices $\Lambda^{(1)}, \Lambda^{(2)}$ for Λ^M . In particular, if we choose (as we always can) to dispose with the character ψ_W in one case, then it may *a priori* still be non-trivial in the other.
- (iii) Since $(\bar{J}_P, \bar{\lambda}_P)$ is a cover of $(J_{\bar{L}}, \lambda_{\bar{L}})$, by [7, Corollary 7.12] we have a canonical embedding of Hecke algebras $t_P: \mathcal{H}(\bar{L}, \lambda_{\bar{L}}) \hookrightarrow \mathcal{H}(\bar{G}, \bar{\lambda}_P)$ and we identify $\mathcal{H}(\bar{L}, \lambda_{\bar{L}})$ (which is just the algebra of Laurent polynomials in a single variable) with its image \mathcal{B}_P . We also put $\mathcal{K} = \mathcal{H}(\bar{J}_M, \bar{\lambda}_P)$. Then [9, Theorem 1.5] implies that the map

$$\begin{aligned} \mathcal{B}_P \otimes_{\mathbb{C}} \mathcal{K} &\rightarrow \mathcal{H}(\bar{G}, \bar{\lambda}_P) \\ f \otimes \phi &\mapsto f * \phi \end{aligned}$$

is an isomorphism of $(\mathcal{B}_P, \mathcal{K})$ -bimodules.

We conclude with a description of the Hecke algebra as a convolution algebra. We put $S = \{w_1, w_2\}$ and recall that we have elements $f_{w_i} \in \mathcal{H}(\bar{G}, \bar{\lambda}_P)$ with support $\bar{J}_P w_i \bar{J}_P$.

Theorem 2.21 (cf. [1, (1.11) Théorème]). *The algebra $\mathcal{H}(\bar{G}, \bar{\lambda}_P)$ is a convolution algebra on (\bar{W}, S) . As a vector space, it has basis $f_w = f_{s_1} * \cdots * f_{s_r}$, where $w = s_1 \cdots s_r$ is a reduced decomposition of w in \bar{W} . It is generated as an algebra by f_{w_1} and f_{w_2} , with relations*

$$f_{w_i} * f_{w_i} = c_{1i} f_{w_i} + d_i f_{w_i}, \quad \text{for } i = 1, 2.$$

Proof The proof is the same as that of [1, (1.11) Théorème]. ■

A Intertwining and Hecke algebras

In this short appendix we give the statement of a result, which is a mild generalization of [6, Proposition 4.1.3, Corollary 4.1.5]. The notation here is independent of that in the rest of this paper.

Let G be a unimodular locally compact totally disconnected topological group with a countable base of open sets, and fix a Haar measure on G . Let K be a compact open subgroup of G and (ρ_i, W_i) a continuous finite-dimensional representation of K , for $i = 1, 2$.

Let $\mathcal{H}(G, \rho_1, \rho_2)$ be the complex vector space of compactly supported functions $\phi: G \rightarrow \text{Hom}_{\mathbb{C}}(W_2^{\vee}, W_1^{\vee})$ satisfying

$$\phi(k_1 g k_2) = \rho_1^{\vee}(k_1) \circ \phi(g) \circ \rho_2^{\vee}(k_2), \quad \text{for all } k_1, k_2 \in K, g \in G.$$

Then $\mathcal{H}(G, \rho_1, \rho_2)$ is an $(\mathcal{H}(G, \rho_1), \mathcal{H}(G, \rho_2))$ -bimodule, under convolution of functions. If $g \in G$ then we write $\mathcal{H}(G, \rho_1, \rho_2)_g$ for the subspace of functions which are null outside KgK .

Recall that the G -intertwining of ρ_1 with ρ_2 is given by

$$I_G(\rho_1, \rho_2) = \{g \in G : \text{Hom}_{K \cap {}^g K}({}^g \rho_1, \rho_2) \neq 0\}.$$

The result we need is the following:

Lemma A.1 (cf. [6, 4.1.1–5]). *In the above situation, let $g \in G$.*

- (i) *There is a function $\phi \in \mathcal{H}(G, \rho_1, \rho_2)$ with $\phi(g) \neq 0$ if and only if $g^{-1} \in I_G(\rho_1, \rho_2)$. Moreover, if g satisfies these conditions then there is a canonical isomorphism of vector spaces between $\text{Hom}_{K \cap K^g}(W_1, W_2^g)$ and $\mathcal{H}(G, \rho_1, \rho_2)_g$.*
- (ii) *Let H be a compact open subgroup containing K . Then there is a canonical isomorphism*

$$\mathcal{H}(G, \rho_1, \rho_2) \simeq \mathcal{H}(G, \text{Ind}_K^H \rho_1, \text{Ind}_K^H \rho_2)$$

of $(\mathcal{H}(G, \rho_1), \mathcal{H}(G, \rho_2))$ -bimodules. This restricts to an isomorphism

$$\mathcal{H}(G, \text{Ind}_K^H \rho_1, \text{Ind}_K^H \rho_2)_g \simeq \coprod_{\substack{g' \in K \backslash G / K \\ Hg'H = HgH}} \mathcal{H}(G, \rho_1, \rho_2)_{g'}.$$

The proof is very similar to those of [6, 4.1.1–5]; one point to note is that we are using the canonical isomorphisms of [6, Proposition 4.1.3] to identify $\mathcal{H}(G, \rho_i)$ with $\mathcal{H}(G, \text{Ind}_K^H \rho_i)$, for $i = 1, 2$, so that $\mathcal{H}(G, \text{Ind}_K^H \rho_1, \text{Ind}_K^H \rho_2)$ is indeed an $(\mathcal{H}(G, \rho_1), \mathcal{H}(G, \rho_2))$ -bimodule.

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