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Double Coset Decompositions and Intertwining

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Abstract. Let F be a non-archimedean local field of residual characteristic different from 2. This paper is a first step in the description of the smooth representation theory of a unitary group $G \subset GL(N,F)$ via *types*. We intersect certain double cosets in $GL(N,F)$ with G and hence obtain the intertwining of certain characters ψ_β^- of open compact subgroups of G, for $\beta \in \operatorname{Lie} G$. In the case when β is elliptic regular "modulo $\mathfrak{p}_F$", we will then obtain supercuspidal representations of G.

1. Introduction

Let F be a non-archimedean local field of residual characteristic different from 2, with a (possibly trivial) galois involution with fixed field F_0. Let V be an N-dimensional F-vector space equipped with a nondegenerate ϵ-hermitian form h. Put $A = \operatorname{End}_F V$ and let $^-$ be the adjoint involution on A associated to h. We then set $\widetilde{G} = \operatorname{Aut}_F V \simeq GL(N,F)$ and let G be the subgroup of $\widetilde{G}$ of fixed points of the map $\sigma : g \mapsto \overline{g}^{-1}$; thus G is a unitary group (symplectic or orthogonal if $F = F_0$) defined over F_0.

This paper is a step in the attempt to describe the smooth representation theory of G via the theory of *types* (see [5]). Such types, in particular those associated to supercuspidal representations, are intrinsically difficult to construct so we seek to "transfer" types from $\widetilde{G}$ to G. (Note that the types for $\widetilde{G}$ have been constructed in [3], [4].) In particular, we obtain new supercuspidal representations of G, many of which are transferred from types for non-supercuspidal representations of $\widetilde{G}$.

The types for $\widetilde{G}$ are constructed from pairs (J,θ), where J is a compact open subgroup of $\widetilde{G}$ and θ is a character of J. The irreducible representations containing θ are classified by the simple modules over the Hecke algebra $\mathcal{H}(\widetilde{G},\theta)$ and this algebra is "recognizable": that is, it is isomorphic (in a support-preserving way) to a Hecke algebra of standard type in a smaller reductive group. In particular, if the support of the Hecke algebra (which is just the intertwining of θ) is compact modulo centre then any irreducible representation containing θ is supercuspidal.

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This philosophy works well for $\widetilde{G}$ and also for $SL(N)$ (see [6], [7] and the work of Goldberg and Roche [8], in which a complete set of types for $SL(N)$ is obtained) and this paper is a first step in extending this to the classical group G, in a manner which is compatible with the situation for $\widetilde{G}$: that is, we show that the intertwining transfers.

The first result of this paper (2.3) states that, if U is a pro-p subgroup of $\widetilde{G}$, H is a subgroup of $\widetilde{G}$ such that, for all $h \in H$, we have $UhU \cap H = (U \cap H)h(U \cap H)$, and U, H are both fixed by the involution σ, we have

$$UHU \cap G = U \cap G \cdot H \cap G \cdot U \cap G.$$

We may now apply this to computing intertwining of characters for G.

We consider *skew strata*, that is quadruples $[\Lambda, n, r, \beta]$, where Λ is an $\mathfrak{o}_F$-lattice sequence in V fixed by the duality induced by h, $n > r \geq 0$ are integers and $\beta \in \operatorname{Lie} G \cap \mathfrak{a}_{-n}(\Lambda)$, where $\mathfrak{a}_{-n}(\Lambda)$ is the level $-n$ filtration lattice of A induced by Λ (see §3). If $r \geq [\frac{n}{2}]$ then a skew stratum corresponds to a character ψ_β of $U_r = 1 + \mathfrak{a}_r(\Lambda)$ and to the character ψ_β^- of $P_r = U_r \cap G$ obtained by restriction.

Now we consider skew strata which are *simple*, that is where β satisfies certain arithmetical properties relative to Λ (see §4). When the $\mathfrak{o}_F$-lattice sequence Λ is strict, [3] (1.5.8) gives the *formal intertwining* of a (skew) simple stratum in the form UHU and we generalize this (4.5) to arbitrary lattice sequences. If $r \geq [\frac{n}{2}]$, this is precisely the intertwining of ψ_β and the intertwining of ψ_β^- is $UHU \cap G$. The groups U, H satisfy the conditions of (2.3) so we obtain the intertwining of ψ_β^- as $U \cap G \cdot H \cap G \cdot U \cap G$.

We now combine the notion of a simple skew stratum with that of a *split* stratum from [4] (3.6) to define *semisimple* strata. These are orthogonal direct sums of simple skew strata which are "sufficiently different" (see (4.9)) and generalize the characters considered in [10] (2.17). Then we may adapt the intertwining result of [4] (3.7) to calculate the formal intertwining of such a semisimple stratum in the same form UHU. Once again, if $r \geq [\frac{n}{2}]$, the stratum corresponds to a character ψ_β^- of P_r and its intertwining is $UHU \cap G$, which decomposes as in the simple case.

Finally, suppose that $[\Lambda, n, r, \beta]$ is a semisimple stratum with $r \geq [\frac{n}{2}]$ and $F[\beta]$ is maximal in A (of degree N). Then the intertwining of the character ψ_β^- is compact and contained in $P = \mathfrak{a}_0 \cap G$, where $\mathfrak{a}_0 := \mathfrak{a}_0(\Lambda)$ (see § 3). In particular, if ρ is an irreducible representation of P containing the character ψ_β^-, then the (compact) induced representation $\operatorname{Ind}_P^G \rho$ is irreducible and supercuspidal. If $r < n-1$ and $F[\beta]$ is wildly ramified then this is a new supercuspidal representation. The construction of the representation ρ (in the more general setting of *simple characters*) is described in [14] and, again, is more general than that in [10].

We now give a brief summary of the contents of each chapter. In §2 we describe the double coset decomposition, in the generality of an l-group acting on some group G, $l \neq p$. In §3 we introduce the notations for our p-adic groups and recall the definition of a lattice sequence and its associated objects. In §4 we recall the notion of a simple stratum from [3], [4], define semisimple strata and calculate their intertwining, and see that, in the case described above, we obtain supercuspidal representations of G. Finally, in §5, we give the proof of (4.5).

The main results of this paper are based on part of my PhD thesis. I would like to thank my supervisor, Colin Bushnell, for starting me on the project and for his support and encouragement. I would also like to thank Guy Henniart, for suggesting a generalization of the main result and its application to semisimple strata, and Paul Broussous, for his patient explanations during my doctorate and for his corrections to earlier versions of this paper.

2. Double cosets

This section is written in much more generality than the remainder of this paper. Indeed, we begin with a group G and a group Γ of automorphisms of G; we denote the fixed points G^Γ.

Lemma 2.1. *Let G be a group and Γ a group of automorphisms of G. Let U be a subgroup of G fixed by Γ and suppose that, for all $g \in G^\Gamma$, we have*

$$H^1(\Gamma, gUg^{-1} \cap U) = 1.$$

Then, for $g \in G^\Gamma$, we have

$$(UgU)^\Gamma = U^\Gamma g U^\Gamma.$$

Proof. We have a Γ-equivariant exact sequence

$$1 \to gUg^{-1} \cap U \xrightarrow{\delta} U \times U \xrightarrow{\pi} UgU \to 1,$$

where $\delta(u) = (u, g^{-1}ug)$ and $\pi(u,v) = ugv^{-1}$. By [13] Prop. 36, this gives rise to a long exact sequence of pointed sets

$$1 \to (gUg^{-1} \cap U)^\Gamma \to (U \times U)^\Gamma \xrightarrow{\pi} (UgU)^\Gamma \to H^1(\Gamma, gUg^{-1} \cap U) \to H^1(\Gamma, U \times U).$$

We have $(U \times U)^\Gamma = U^\Gamma \times U^\Gamma$ and $H^1(\Gamma, gUg^{-1} \cap U) = 1$. Hence the map π is surjective and $(UgU)^\Gamma = U^\Gamma g U^\Gamma$. □

Lemma 2.2. *Let G be a group and Γ an l-group of automorphisms of G, l prime. Let U be a pro-p subgroup of G fixed by Γ, $p \neq l$ prime. Let $g \in G$. Then UgU is fixed by Γ if and only if $(UgU)^\Gamma \neq \emptyset$.*

Proof. The implication "if" is clear so suppose UgU is fixed by Γ. We can decompose UgU as a disjoint union of single cosets g_iU and Γ permutes the cosets in this decomposition. The number of such single cosets is the index of $gUg^{-1} \cap U$ in U, which is a power of p. In particular, since $l \neq p$, there is a fixed coset for the action of Γ, say kU. For all $\gamma \in \Gamma$, we have $k^\gamma \in kU$ so $\gamma \mapsto k^\gamma k^{-1}$ defines a 1-cocycle of Γ in U. However, $H^1(\Gamma, U) = 1$ so there exists $u \in U$ such that $k^\gamma k^{-1} = u^\gamma u^{-1}$ for all $\gamma \in \Gamma$. Then $ku^{-1} \in (UgU)^\Gamma$. □

Theorem 2.3. *Let G be a group and Γ an l-group of automorphisms of G, l prime. Let U be a pro-p subgroup of G fixed by Γ, $p \neq l$ prime. Let H be a subgroup of G fixed by Γ such that, for all $h \in H$,*

$$UhU \cap H = (U \cap H)h(U \cap H). \tag{2.4}$$

Then we have

$$(UHU)^\Gamma = U^\Gamma H^\Gamma U^\Gamma.$$

We remark that condition (2.4) will be satisfied if H is the fixed point subgroup in G of some group of automorphisms of G satisfying the conditions of Lemma 2.1.

Proof. Suppose that $h \in H$ is such that $(UhU)^\Gamma \neq \emptyset$; hence UhU is fixed by Γ. In particular, for all $\gamma \in \Gamma$, $h^\gamma \in UhU \cap H = (U \cap H)h(U \cap H)$. Thus $(U \cap H)h(U \cap H)$ is fixed by Γ so, by (2.2), there exists $h' \in ((U \cap H)h(U \cap H))^\Gamma$. In particular, we have $UhU = Uh'U$ and $h' \in H^\Gamma$. Hence

$$\begin{aligned}(UHU)^\Gamma &= (UH^\Gamma U)^\Gamma \\ &= U^\Gamma H^\Gamma U^\Gamma \qquad \text{by (2.1)},\end{aligned}$$

since $hUh^{-1} \cap U$ is a pro-p subgroup for all $h \in H^\Gamma$. □

Note that the above results will of course remain valid if Γ is a solvable group of order coprime to p.

3. Preliminaries

Let F be a non-archimedean local field, $\mathfrak{o}_F$ its ring of integers, $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$, $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue class field and $q_F = p^{f_F} = \operatorname{card} k_F$. We assume throughout that the residual characteristic p is not 2.

Suppose that F comes equipped with a galois involution $\bar{\ }$, with fixed field F_0; we allow the possibility that $F = F_0$. Then we denote by $\mathfrak{o}_0$, $\mathfrak{p}_0$, k_0, $q_0 = p^{f_0}$ the objects for F_0 analogous to those above for F. We also fix a uniformizer ϖ_F of F such that $\overline{\varpi_F} = \pm\varpi_F$ (the sign depending on whether F/F_0 is ramified or not).

Let ψ_0 be a character of the additive group of F_0, with conductor $\mathfrak{p}_0$. Then we put $\psi_F = \psi_0 \circ \mathrm{tr}_{F/F_o}$; since $p \neq 2$, F/F_0 is at worst tamely ramified so ψ_F is a character of the additive group of F with conductor $\mathfrak{p}_F$.

Let V be an N-dimensional F-vector space and put $A = \mathrm{End}_F(V)$, $\widetilde{G} = \mathrm{Aut}_F(V)$. Let ψ_A be the character of A given by $\psi_A = \psi_F \circ \mathrm{tr}_{A/F}$. Let h be a nondegenerate ϵ-hermitian form on V and let $^-$ be the adjoint involution on A associated to h so that $h(av, w) = h(v, \overline{a}w)$ for all $v, w \in V$, $a \in A$; this extends the involution on F (for F embedded diagonally in A). We also denote by σ the involution on $\widetilde{G}$ given by $x \mapsto \overline{x}^{-1}$ and by Σ the subgroup of $\mathrm{Aut}\,\widetilde{G}$ consisting of σ and the identity. Note that the action of σ on $\mathrm{Lie}\,\widetilde{G} \simeq A$, via the differential, is given by $x \mapsto -\overline{x}$.

We put $G = \widetilde{G}^{\Sigma} = \{g \in \widetilde{G} : h(gv, gw) = h(v, w) \text{ for all } v, w \in V\}$, a unitary group over F_0 (possibly symplectic or orthogonal). We also put $A_- = \{x \in A : x + \overline{x} = 0\} \simeq \mathrm{Lie}\,G$ and $A_+ = \{x \in A : x = \overline{x}\}$; since F is not of characteristic 2 we have $A = A_- \oplus A_+$ and, moreover, this decomposition is orthogonal with respect to the pairing induced by $\mathrm{tr}_0 = \mathrm{tr}_{F/F_0} \circ \mathrm{tr}_{A/F}$ since, for $x \in A_-$, $y \in A_+$, we have

$$\mathrm{tr}_0(xy) = \overline{\mathrm{tr}_0(xy)} = \mathrm{tr}_0(\overline{xy}) = \mathrm{tr}_0(-yx) = -\mathrm{tr}_0(xy).$$

For S any subset of A, we write S_- (or sometimes S^-) for $S \cap A_-$ and S_+ for $S \cap A_+$. If S is an $\mathfrak{o}_F$-lattice fixed by the involution then we have $S = S_- \oplus S_+$, since the residual characteristic of F is not 2.

Recall from [4] (2.1) that an $\mathfrak{o}_F$-*lattice sequence* in V is a function Λ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V such that

(*i*) $n \geq m$ implies $\Lambda(n) \subset \Lambda(m)$;
(*ii*) there exists a positive integer $e = e(\Lambda)$ (the $\mathfrak{o}_F$-*period* of Λ) such that $\Lambda(n+e) = \mathfrak{p}_F \Lambda(n)$, for all $n \in \mathbb{Z}$.

A lattice sequence Λ gives rise to a filtration on A by

$$\mathfrak{a}_n = \mathfrak{a}_n(\Lambda) = \{x \in A : x\Lambda(m) \subset \Lambda(m+n), m \in \mathbb{Z}\}, \quad n \in \mathbb{Z}.$$

This then gives rise to a “valuation” ν_Λ on A by

$$\nu_\Lambda(x) = \sup\{n \in \mathbb{Z} : x \in \mathfrak{a}_n\}, \quad \text{for } x \in A,$$

with the understanding that $\nu_\Lambda(0) = \infty$.

From a lattice sequence Λ we obtain a compact open subgroup $U = U(\Lambda) = \mathfrak{a}_0(\Lambda)^\times$ of $\widetilde{G}$, equipped with a filtration

$$U_n = U_n(\Lambda) = 1 + \mathfrak{a}_n(\Lambda), \quad n \in \mathbb{Z}, n > 0.$$

This is also the Moy-Prasad filtration associated to a certain rational point in the building of $GL(N, F)$ (see [12], [2]). We define the normalizer of the filtration to be

$$\mathfrak{K}(\Lambda) = \bigcap_{r \geq 0} N_{\widetilde{G}}(U_r),$$

where $N_{\tilde{G}}$ denotes normalizer.

For L an $\mathfrak{o}_F$-lattice in V, we define the dual lattice $L^\#$ by

$$L^\# = \{v \in V : h(v, L) \subset \mathfrak{p}_F\}.$$

Then $L^\#$ can be identified with $\mathrm{Hom}_{\mathfrak{o}_F}(L, \mathfrak{p}_F)$ by the non-degeneracy of h and we have $L^{\#\#} = L$. For Λ an $\mathfrak{o}_F$-lattice sequence, define the dual sequence $\Lambda^\#$ by

$$\Lambda^\#(n) = \Lambda(-n)^\#, \quad n \in \mathbb{Z}.$$

We say that Λ is self-dual if there exists $d \in \mathbb{Z}$ such that $\Lambda^\#(n) = \Lambda(n+d)$, for all $n \in \mathbb{Z}$. In this case, the filtration $\mathfrak{a}_n$ on A induced by Λ satisfies $\overline{\mathfrak{a}}_n = \mathfrak{a}_n$, for $n \in \mathbb{Z}$. In particular, the groups U, U_n, $n \geq 1$, are fixed by Σ and we put $P = U^\Sigma$, a compact open subgroup of G, and $P_n = U_n^\Sigma$, for $n \geq 1$, a filtration on P. Further, by [11] (2.13)(c) we have a bijection $\mathfrak{a}_n^- \to P_n$ given by the Cayley map $x \mapsto (1 + \frac{x}{2})(1 - \frac{x}{2})^{-1}$.

Lemma 3.1. *Let Λ be an $\mathfrak{o}_F$-lattice sequence in V and let $m, n \in \mathbb{Z}$ satisfy $2n \geq m > n \geq 1$.*

(i) The map $x \mapsto 1 + x$ induces an isomorphism of abelian groups $\mathfrak{a}_n/\mathfrak{a}_m \tilde{\to} U_n/U_m$.
(ii) If Λ is self-dual then the map $x \mapsto 1 + x$ induces an isomorphism of abelian groups $\mathfrak{a}_n^-/\mathfrak{a}_m^- \tilde{\to} P_n/P_m$.

Let S be an $\mathfrak{o}_F$-lattice in A, hence an $\mathfrak{o}_0$-lattice in A. We define the $\mathfrak{o}_F$-lattice

$$\begin{aligned} S^* &= \{a \in A : \mathrm{tr}_0(aS) \subset \mathfrak{p}_0\} \\ &= \{a \in A : \mathrm{tr}_{A/F}(aS) \subset \mathfrak{p}_F\}, \end{aligned}$$

since F is at worst tamely ramified over F_0. If S is also stable under the involution, we can define

$$(S_-)^* = \{a \in A_- : \mathrm{tr}_0(aS_-) \subset \mathfrak{p}_0\} = (S^*)_-$$

since the direct sum $S = S_- \oplus S_+$ is orthogonal with respect to tr_0.

We recall from [4] (2.10) that, if Λ is an $\mathfrak{o}_F$-lattice sequence in V with associated filtration $\mathfrak{a}_n$, then we have $\mathfrak{a}_n^* = \mathfrak{a}_{1-n}$.

Let "hat" ^ denote the Pontrjagin dual. Then we have the following:

Lemma 3.2. *Let Λ be an $\mathfrak{o}_F$-lattice sequence in V and let $m, n \in \mathbb{Z}$ satisfy $2n \geq m > n \geq 1$.*

(i) There is a $\mathfrak{K}(\Lambda)$-equivariant isomorphism of abelian groups

$$\begin{array}{rcl} \mathfrak{a}_{1-m}/\mathfrak{a}_{1-n} & \tilde{\to} & (U_n/U_m)^\wedge \\ b + \mathfrak{a}_{1-n} & \mapsto & \psi_b \end{array}$$

where $\psi_b(u) = \psi_F(\mathrm{tr}_{A/F}(b(u-1)))$, for $u \in U_n$.

(*ii*) *If* Λ *is self-dual then there is a* P*-equivariant isomorphism of abelian groups*

$$\begin{array}{rcl}(\mathfrak{a}_{1-m}^{-})/(\mathfrak{a}_{1-n}^{-}) & \tilde{\to} & (P_n/P_m)\hat{} \\ b+(\mathfrak{a}_{1-n}^{-}) & \mapsto & \psi_b^{-}\end{array}$$

where $\psi_b^{-}(p)=\psi_0(\mathrm{tr}_0(b(p-1)))$, *for* $p\in P_n$. *Moreover, for* $b\in(\mathfrak{a}_{1-m}^{-})$, ψ_b^{-} *is the restriction to* P_n *of* ψ_b.

4. Strata and supercuspidal representations

We now give two applications of theorem (2.3) to the intertwining of certain characters involved in representation theory. The language used will be that of [3] and [4], as in the previous section.

4.1. Simple strata

Definition 4.1 (cf. [3] (1.5), [4] (3.1)). A *stratum* in A is a 4-tuple $[\Lambda,n,r,\beta]$ consisting of a lattice sequence Λ in V, $n,r\in\mathbb{Z}$ with $r<n$, and an element $\beta\in\mathfrak{a}_{-n}(\Lambda)$. We say that two strata $[\Lambda,n,r,\beta_i]$, $i=1,2$, are *equivalent* if $\beta_1\equiv\beta_2 \pmod{\mathfrak{a}_{-r}(\Lambda)}$.

Let $[\Lambda,n,r,\beta]$ be a stratum in A and suppose that the integers r,n satisfy

$$n>r\geq[\tfrac{n}{2}]\geq 0, \tag{4.2}$$

where $[x]$ is the greatest integer less than or equal to x. By (3.2)(i), an equivalence class of strata $[\Lambda,n,r,\beta]$ corresponds to the character ψ_β of $U_{r+1}(\Lambda)/U_{n+1}(\Lambda)$.

Definition 4.3 ([3] (1.5.5)). Let $[\Lambda,n,r,\beta]$ be a stratum in A. It is *pure* if

(i) the algebra $E=F[\beta]$ is a field;
(ii) Λ is an $\mathfrak{o}_E$-lattice sequence;
(iii) $\nu_\Lambda(\beta)=-n$.

If $[\Lambda,n,r,\beta]$ is a pure stratum, we put, for $k\in\mathbb{Z}$,

$$\mathfrak{n}_k(\beta,\Lambda)=\{x\in\mathfrak{a}_0(\Lambda):\beta x-x\beta\in\mathfrak{a}_k\},$$

an $\mathfrak{o}_F$-lattice in A. Then we define $k_0(\beta,\Lambda)$ to be the least integer k such that $\mathfrak{n}_{k+1}(\beta,\Lambda)$ is contained in $B\cap\mathfrak{a}_0+\mathfrak{a}_1$, where B is the A-centralizer of β; we understand that if $F[\beta]=F$ then $k_0(\beta,\Lambda)=\infty$. Note that this is the same definition as in [4] (5.1), though this is not clear *a priori* (see §5 for further discussion). If $F[\beta]\neq F$ then $k_0(\beta,\Lambda)$ is an integer greater than or equal to $-n$.

Definition 4.4 ([3] (1.5.5)). A pure stratum $[\Lambda, n, r, \beta]$ is called *simple* if $r < -k_0(\beta, \Lambda)$.

We now define the formal intertwining of a stratum $[\Lambda, n, r, \beta]$ to be

$$\mathcal{I}_{\widetilde{G}}[\Lambda, n, r, \beta] = \left\{ x \in \widetilde{G} : x^{-1}(\beta + \mathfrak{a}_{-r})x \cap (\beta + \mathfrak{a}_{-r}) \neq \emptyset \right\}.$$

If (4.2) is satisfied, then this is nothing other than the intertwining in $\widetilde{G}$ of the character ψ_β:

$$\mathcal{I}_{\widetilde{G}}(\psi_\beta) := \left\{ x \in \widetilde{G} : \psi_\beta(u) = \psi_\beta^x(u) \text{ for any } u \in U_n^x \cap U_n \right\},$$

where $U_n^x = x^{-1}U_n x$ and $\psi_\beta^x(u) = \psi_\beta(xux^{-1})$, for $u \in U_n^x$ (see, e.g. [9] pp.484–485). Then we have the following result, which is a generalization of [3] (1.5.8).

Theorem 4.5. *Let $[\Lambda, n, r, \beta]$ be a simple stratum in A. Let B denote the A-centralizer of β, $\mathfrak{b}_n = \mathfrak{a}_n \cap B$, for $n \in \mathbb{Z}$. Write $k = k_0(\beta, \Lambda)$ and put $\mathfrak{m}_i = \mathfrak{n}_{k+i}(\beta, \Lambda) \cap \mathfrak{a}_i$, for $i \geq 0$. Then*

$$\mathcal{I}_{\widetilde{G}}[\Lambda, n, r, \beta] = (1 + \mathfrak{m}_{-(k+r)})B^\times(1 + \mathfrak{m}_{-(k+r)}).$$

Essentially, the proof is identical to [3] (1.5.8), once we have proved a few preliminary lemmas. We postpone the proof until §5.

We now consider the situation in our group G.

Definition 4.6. A stratum $[\Lambda, n, r, \beta]$ in A is called *skew* if $\beta + \overline{\beta} = 0$ and Λ is self-dual.

Again, if (4.2) is satisfied then, by (3.2)(*ii*), an equivalence class of skew strata $[\Lambda, n, r, \beta]$ corresponds to the character ψ_β^- of $P_{r+1}(\Lambda)/P_{n+1}(\Lambda)$.

We define the formal intertwining in G of a skew stratum $[\Lambda, n, r, \beta]$ to be

$$\mathcal{I}_G[\Lambda, n, r, \beta] = \{x \in G : x^{-1}(\beta + \mathfrak{a}_{-r}^-)x \cap (\beta + \mathfrak{a}_{-r}^-) \neq \emptyset\}.$$

If (4.2) is satisfied, then this is precisely the intertwining in G of the character ψ_β^-.

Theorem 4.7. *Let $[\Lambda, n, r, \beta]$ be a skew simple stratum in A. Let B denote the A-centralizer of β, $\mathfrak{b}_n = \mathfrak{a}_n \cap B$, for $n \in \mathbb{Z}$. Write $k = k_0(\beta, \Lambda)$ and put $\mathfrak{m}_i = \mathfrak{n}_{k+i}(\beta, \Lambda) \cap \mathfrak{a}_i$, for $i \geq 0$. Then $\mathfrak{m}_i$ is fixed by the involution and we define the group $Q_i = (1 + \mathfrak{m}_i) \cap G$, for $i > 0$. Then*

$$\mathcal{I}_G[\Lambda, n, r, \beta] = Q_{-(k+r)}(B \cap G)Q_{-(k+r)}.$$

Proof. We write $\mathcal{I}_G$ for $\mathcal{I}_G[\Lambda,n,r,\beta]$ and $\mathcal{I}_{\widetilde{G}}$ for $\mathcal{I}_{\widetilde{G}}[\Lambda,n,r,\beta]$. We begin with a lemma which will also prove useful later:

Lemma 4.8. *$\mathcal{I}_G = \mathcal{I}_{\widetilde{G}} \cap G$.*

Proof. We clearly have $\mathcal{I}_G \subset \mathcal{I}_{\widetilde{G}} \cap G$. Suppose, on the other hand, $x \in \mathcal{I}_{\widetilde{G}} \cap G$; then there exist $b_i \in \mathfrak{a}_{-r}$, $i=1,2$ such that

$$x(\beta+b_1)x^{-1} = \beta + b_2.$$

We now write $b_i = u_i + v_i$, $u_i \in \mathfrak{a}_{-r}^{+}$, $v_i \in \mathfrak{a}_{-r}^{-}$, for $i=1,2$ so

$$x(\beta+v_1)x^{-1} + xu_1x^{-1} = (\beta+v_2) + u_2.$$

Now $xA_-x^{-1} \subset A_-$ and $xA_+x^{-1} \subset A_+$ so $A = A_- \perp A_+$ implies that

$$x(\beta+v_1)x^{-1} = \beta + v_2,$$

i.e. $x \in \mathcal{I}_G$. □

Returning to the proof of the Theorem, we see that $\mathcal{I}_G = \mathcal{I}_{\widetilde{G}} \cap G = (1+\mathfrak{m}_d)B^{\times}(1+\mathfrak{m}_d) \cap G$ by (4.5), where $d = -r-k > 0$.

Now $1+\mathfrak{m}_d$ is a pro-p subgroup of $\widetilde{G}$ fixed by Σ and, since $\mathfrak{b}_d \subset \mathfrak{m}_d \subset \mathfrak{a}_d$ we have, for $b \in B^{\times}$,

$$\begin{aligned}(1+\mathfrak{b}_d)b(1+\mathfrak{b}_d) &\subset (1+\mathfrak{m}_d)b(1+\mathfrak{m}_d) \cap B \\ &\subset (1+\mathfrak{a}_d)b(1+\mathfrak{a}_d) \cap B = (1+\mathfrak{b}_d)b(1+\mathfrak{b}_d),\end{aligned}$$

by [3] (1.6.1). Hence all the conditions of (2.3) are satisfied and the result follows. □

4.2. Semisimple strata

In this section we use the techniques and results of [4] (in particular §3) to generalize the above result to the case where the element β does not generate a field extension over F. We first recall some notions, in particular that of the characteristic polynomial of a stratum (see e.g. [3] (2.3)).

Let $[\Lambda,n,r,\beta]$ be a stratum in A and put $e=e(\Lambda)$, the $\mathfrak{o}_F$-period of Λ. Set $g=(n,e)$ and consider the element

$$y_\beta = \varpi_F^{n/g}\beta^{e/g}.$$

This is an element of $\mathfrak{a}_0(\Lambda)$ so its characteristic polynomial as an F-endomorphism of V lies in $\mathfrak{o}_F[X]$. We define $\varphi_\beta(X)$ to be the reduction modulo $\mathfrak{p}_F$ of this characteristic polynomial; this depends only on the equivalence class of the stratum.

For $i=1,2$, let V_i be subspaces of V such that $V = V_1 \perp V_2$. In particular, h_i, the restriction of h to $V_i \times V_i$, is a nondegenerate ϵ-hermitian form on V_i, for $i = 1, 2$. We put $A^{ij} = \mathrm{Hom}_F(V^j, V^i)$, for $i, j = 1, 2$, and abbreviate $A^{ii} = A^i$. We use the notation

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}.$$

For $i = 1, 2$, let Λ^i be a self-dual lattice sequence in V_i, $e_i = e(\Lambda^i|\mathfrak{o}_F)$ the $\mathfrak{o}_F$-period of Λ^i and let d_i be the (unique) integer such that $\Lambda^i(k)^{\#} = \Lambda^i(d_i - k)$, for all $k \in \mathbb{Z}$. After renumbering, we may assume that $d_i = 0$ or -1. If $d_i = 0$ then we replace Λ^i by the self-dual lattice sequence Λ'^i given by

$$\Lambda'^i(k) = \Lambda^i([\tfrac{k+1}{2}]) \qquad \text{for } k \in \mathbb{Z};$$

that is, every lattice is taken with double the multiplicity. Note that the filtration on A^i determined by Λ'^i is the same as that determined by Λ^i, upto a factor in the index. We have $\Lambda'^i(k)^{\#} = \Lambda'^i(-1 - k)$, for all $k \in \mathbb{Z}$, so we may assume $d_i = -1$, for $i = 1, 2$.

We set $\Lambda = \Lambda^1 \perp \Lambda^2$, a self-dual lattice sequence in V of $\mathfrak{o}_F$-period $e = \mathrm{lcm}\,(e_1, e_2)$ (see [4] (2.8) for the definition of direct sum of lattice sequences). Let $\beta_i \in A^i$ be skew elements and put $n_i = -\nu_{\Lambda^i}(\beta_i)$. Put $\beta = \beta_1 \oplus \beta_2$ and $n = e \cdot \max\{n_1/e_1, n_2/e_2\}$. Then we obtain a skew stratum $[\Lambda, n, r, \beta]$ in A, with $\nu_\Lambda(\beta) = -n$, for any $0 \le r \le n - 1$.

Definition 4.9 (cf. [4] (3.6)). A skew stratum $[\Lambda, n, r, \beta]$ as above is called *split* if

(*i*) $\beta_1 \in \mathfrak{K}(\Lambda^1)$;
(*ii*) either $n_1/e_1 > n_2/e_2$ or else all the following conditions hold:
  (*a*) $n_1/e_1 = n_2/e_2$,
  (*b*) $\beta_2 \in \mathfrak{K}(\Lambda^2)$,
  (*c*) $\gcd(\varphi_{\beta_1}, \varphi_{\beta_2}) = 1$.

It is called *semisimple* if (inductive definition, on the dimension)

(*iii*) either $V_2 = 0$ and $[\Lambda, n, r, \beta]$ is a simple stratum or it is split and each $[\Lambda^i, n_i, r_i, \beta_i]$ is a semisimple stratum, $i = 1, 2$, where $r_i = [re_i/e]$.

We remark that the characters considered in [10] arise from semisimple strata of the form $[\Lambda, n, n-1, \beta]$ with Λ a strict lattice sequence.

Theorem 4.10 (cf. [4] (3.7)). *Let $[\Lambda, n, r, \beta]$ as above be split, with $r \ge 0$. Put $M = \mathrm{Aut}_F(V^1) \times \mathrm{Aut}_F(V^2)$, considered as a Levi subgroup of $\widetilde{G}$. Then*

$$\mathcal{I}_{\widetilde{G}}[\Lambda, n, r, \beta] \subset U_{n-r}\, M\, U_{n-r},$$

where $U_s = U_s(\Lambda) = 1 + \mathfrak{a}_s(\Lambda)$, for $s > 0$.

Proof. We proceed as in the proof of [4] Theorem 3.7.

Proposition 4.11. *Under the hypotheses of (4.10), let* $c \in \mathfrak{a}_{-r} \cap A^{21}$ *and define a map* $\delta_c : A^{12} \to A^{12}$ *by*

$$\delta_c(x) = \beta_1 x - x\beta_2 + xcx, \qquad x \in A^{12}.$$

Then $\delta_c(\mathfrak{a}_{n-r} \cap A^{12}) = \mathfrak{a}_{-r} \cap A^{12}$.

Proof. We need some lemmas.

Lemma 4.12. *The map* $\delta = \delta_0$ *maps* $\mathfrak{a}_s \cap A^{12}$ *onto* $\mathfrak{a}_{s-n} \cap A^{12}$ *for all* s.

Proof. Suppose first that $n_1/e_1 > n_2/e_2$. The Λ^1-invertibility of β_1 implies that $\mathfrak{a}_{s-n} \cap A^{12} = \beta_1(\mathfrak{a}_s \cap A^{12})$, while $(\mathfrak{a}_s \cap A^{12})\beta_2 \subset \mathfrak{a}_{s-n+1} \cap A^{12}$. Hence

$$\mathfrak{a}_{s-n} \cap A^{12} = \delta_0(\mathfrak{a}_s \cap A^{12}) + \mathfrak{a}_{s-n+1} \cap A^{12}$$

and the lemma follows immediately in this case. For the other case, see [4] (3.7) Lemma 1. □

Lemma 4.13. *We have*

$$\delta_c(\mathfrak{a}_{n-r} \cap A^{12}) + \mathfrak{a}_{k-r} \cap A^{12} = \mathfrak{a}_{-r} \cap A^{12},$$

for all integers $k \geq 0$.

Proof. Since $c \in \mathfrak{a}_{-r}$ we have, for $x \in \mathfrak{a}_{n-r} \cap A^{12}$, $xcx \in \mathfrak{a}_{2n-3r} \subset \mathfrak{a}_{-r}$, as $n - r > 0$, so the lemma holds if $k = 0$. The proof then follows by induction exactly as [4] (3.7) Lemma 2. □

The proposition now follows as in [4] (3.7). □

Write $\mathcal{M}$ for the algebra $A^{11} \oplus A^{22} \subset A$, so that $M = \mathcal{M}^\times$. We first need:

Lemma 4.14. *Let* $x = \beta + y$, $y \in \mathfrak{a}_{-r}$. *Then there exists* $u \in U_{n-r}$ *such that* $uxu^{-1} \in \beta + \mathfrak{a}_{-r} \cap \mathcal{M}$.

Proof. This is essentially identical to [4] (3.7) Lemma 3. □

We now prove the theorem. Let $g \in \mathcal{I}_{\widetilde{G}}[\Lambda, n, r, \beta]$; thus there exist $x, y \in \mathfrak{a}_{-r}$ such that $g^{-1}(\beta + x)g = \beta + y$. By (4.14), we may replace g by $u_1 g u_2$, $u_i \in U_{n-r}$, and assume that $x, y \in \mathfrak{a}_{-r} \cap \mathcal{M}$. We put

$$\beta + x = \begin{pmatrix} \beta_1' & 0 \\ 0 & \beta_2' \end{pmatrix}, \qquad \beta + y = \begin{pmatrix} \beta_1'' & 0 \\ 0 & \beta_2'' \end{pmatrix},$$

and write out the equation $(\beta + x)g = g(\beta + y)$. Comparing $(1,2)$-entries, we have $\beta_1' g_{12} = g_{12}\beta_2''$. But the map $A^{12} \to A^{12}$ given by $z \mapsto \beta_1' z - z\beta_2''$ is injective, as in [4] (3.7) Lemma 4, so $g_{12} = 0$. Likewise $g_{21} = 0$, whence $g \in M$ as required. □

In fact, we even have

$$\mathcal{I} = \mathcal{I}_{\widetilde{G}}[\Lambda, n, r, \beta] = U_{n-r}\,\mathcal{I}_M[\Lambda, n, r, \beta]\,U_{n-r}, \tag{4.15}$$

where $\mathcal{I}_M$ is the intertwining in M. For suppose $x \in \mathcal{I}$, so that

$$\beta x \equiv x\beta \pmod{\mathfrak{a}_{-r}x + x\mathfrak{a}_{-r}},$$

and let $y \in \mathfrak{a}_{n-r}$. Then $a_\beta(y) \in \mathfrak{a}_{-r}$ so $(1+y)\beta(1+y)^{-1} = \beta - a_\beta(y)(1+y)^{-1} \equiv \beta \pmod{\mathfrak{a}_{-r}}$. Then $\beta x \equiv x\beta \equiv x(1+y)\beta(1+y)^{-1} \pmod{\mathfrak{a}_{-r}x + x\mathfrak{a}_{-r}}$ so $x(1+y) \in \mathcal{I}$, since $(1+y) \in \mathfrak{a}_0^\times$. By symmetry, we have $\mathcal{I} = U_{n-r}\,\mathcal{I}\,U_{n-r}$ and (4.15) follows.

Now suppose $[\Lambda, n, r, \beta]$ is a skew stratum in A; by (4.8), we have that the formal intertwining of the stratum in G is $\mathcal{I}_G[\Lambda, n, r, \beta] = \mathcal{I}_{\widetilde{G}}[\Lambda, n, r, \beta] \cap G$. Hence, applying (2.3) with $H = M$ we have:

Corollary 4.16. *Let $[\Lambda, n, r, \beta]$ as above be a skew split stratum, with $r \geq 0$. Then, with notation as above, we have*

$$\mathcal{I}_G[\Lambda, n, r, \beta] \subset P_{n-r}(M \cap G)P_{n-r},$$

where $P_s = U_s \cap G$, for $s > 0$.

Proof. We need only show that M, U_{n-r} satisfy the condition (2.4). But M is the fixed point subgroup in $\widetilde{G}$ of the subgroup $\Gamma = \{1\} \times \{\pm 1\}$ of M acting by inner automorphisms and U_{n-r} is fixed by Γ so (2.4) is satisfied, by (2.1). □

Indeed, as above for (4.15) but using the Cayley map, we have

$$\mathcal{I}_G[\Lambda, n, r, \beta] = P_{n-r}\,\mathcal{I}_{M\cap G}[\Lambda, n, r, \beta]\,P_{n-r}. \tag{4.17}$$

For $[\Lambda, n, r, \beta]$ a skew semisimple stratum as above, with $r \geq 0$, we define a compact open subgroup $K = K(\beta, \Lambda)$ of G inductively as follows: if the stratum is simple then $K = Q_{-(k+r)}$ (see (4.7)); otherwise

$$K = K(\beta_1, \Lambda^1) \times K(\beta_2, \Lambda^2) \cdot P_{n-r}. \tag{4.18}$$

Then, putting together (4.16) with (4.7), we have the following result.

Theorem 4.19. *Let $[\Lambda, n, r, \beta]$ as above be a skew semisimple stratum, with $r \geq 0$. Then we have*

$$\mathcal{I}_G[\Lambda, n, r, \beta] = KZK,$$

where $K = K(\beta, \Lambda)$ and $Z = Z_G(\beta)$ is the centralizer in G of β.

4.3. Supercuspidal representations

Let $[\Lambda, n, r, \beta]$ be a skew semisimple stratum. We will call it *maximal* if the centralizer $B = Z_A(\beta)$ in A of β has dimension N. In this case, we have that $B = \sum_{i=1}^{r} E_i$ is a sum of fields, each of which is stable under the involution but not fixed pointwise. Moreover, the G-centralizer of β is the compact maximal torus $Z = \prod_{i=1}^{r} N_1(E_i)$, where $N_1(E_i) = \{e \in E_i : e\bar{e} = 1\}$ is the group of norm 1 elements of E_i.

In particular, we have in this situation that $Z \subset P$; moreover, the group $K = K(\beta, \Lambda)$ defined in (4.18) satisfies $K \subset P_1$. Hence, (4.19) gives us

$$\mathcal{I}_G[\Lambda, n, r, \beta] \subset P, \tag{4.20}$$

and we deduce the following:

Theorem 4.21. *Let $[\Lambda, n, r, \beta]$ be a maximal skew semisimple stratum such that (4.2) is satisfied. Let ρ be an irreducible representation of P containing the character ψ_β^-. Then the (compactly) induced representation $\pi = \mathrm{Ind}_P^G \rho$ is irreducible and supercuspidal.*

Proof. Since P normalizes P_n, the restriction of ρ to P_n is a sum of P-conjugates of ψ_β^-. Then, if $g \in G$ intertwines ρ, there exist $p, p' \in P$ such that g intertwines ${}^p\psi_\beta^-$ with ${}^{p'}\psi_\beta^-$. Hence $p^{-1}gp' \in I_G(\psi_\beta^-)$ and this is contained in P, by (4.20). We have shown that $I_G(\rho) = P$ and the result now follows. □

5. Proof of Theorem 4.5

Let $[\Lambda, n, r, \beta]$ be a pure stratum in A. Let $\mathfrak{a}_n = \mathfrak{a}_n(\Lambda)$ be the associated filtration on A, $E = F[\beta]$, $B = Z_A(\beta)$, the A-centralizer of β, and $\mathfrak{b}_n = \mathfrak{a}_n \cap B$, for $n \in \mathbb{Z}$. Note that $\mathfrak{a}_1$ is the Jacobson radical of the $\mathfrak{o}_F$-hereditary order $\mathfrak{a}_0$ and, likewise, that $\mathfrak{b}_1$ is the Jacobson radical of the $\mathfrak{o}_E$-hereditary order $\mathfrak{b}_0$.

Let $a_\beta : A \to A$ be the map $x \mapsto \beta x - x\beta$. Recall from §4.1 that, for $k \in \mathbb{Z}$, we put $\mathfrak{n}_k = \mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{a}_0(\Lambda) : a_\beta(x) \in \mathfrak{a}_k\}$, an $\mathfrak{o}_E$-lattice in A. Then, for $i \geq 0$, we define

$$h_i = h_i(\beta, \Lambda) = \inf\{k \in \mathbb{Z} : \mathfrak{n}_{k+e+i}(\beta, \Lambda) \subset \mathfrak{b}_0 + \mathfrak{p}_E \mathfrak{a}_i\}.$$

with the understanding that if $E = F$ then $h_i(\beta, \Lambda) = \infty$,

Suppose first that Λ is a strict lattice sequence (that is $\Lambda(n) \supsetneq \Lambda(n+1)$ for $n \in \mathbb{Z}$). Then the filtration induced by Λ on A is given by

$$\mathfrak{a}_n = (\mathfrak{a}_1)^n,$$

with $(\mathfrak{a}_1)^0 := \mathfrak{a}_0$ and likewise for the filtration $\mathfrak{b}_n$. We now show that $h_i(\beta, \Lambda)$ coincides with $k_0(\beta, \Lambda)$ as defined in [3] (1.4.5).

Lemma 5.1. *Let $[\Lambda, n, r, \beta]$ be a pure stratum in A, with Λ a strict lattice sequence of $\mathfrak{o}_E$-period e and let $i \geq 0$. Then*

$$h_i(\beta,\Lambda) = \inf\{k \in \mathbb{Z} : \mathfrak{n}_{k+1}(\beta,\Lambda) \subset \mathfrak{b}_0 + \mathfrak{a}_1\}.$$

Proof. Put $l = \inf\{k \in \mathbb{Z} : \mathfrak{n}_{k+1}(\beta,\Lambda) \subset \mathfrak{b}_0 + \mathfrak{a}_1\}$ and $h = h_i(\beta,\Lambda)$. By [3] (1.4.8)(i), we have $\mathfrak{n}_{l+e+i} = \mathfrak{b}_0 + (\mathfrak{b}_1)^{e+i}\mathfrak{n}_l \subset \mathfrak{b}_0 + (\mathfrak{b}_1)^{e+i}\mathfrak{a}_0 = \mathfrak{b}_0 + (\mathfrak{a}_1)^{e+i} = \mathfrak{b}_0 + \mathfrak{p}_E\mathfrak{a}_i$. Hence $h \leq l$.

However, by definition $\mathfrak{n}_{h+e+i} \subset \mathfrak{b}_0 + \mathfrak{p}_E\mathfrak{a}_i$ so, by [3] (1.4.8)(i), we have $\mathfrak{n}_{h+1} = (\mathfrak{b}_1)^{1-e-i}\mathfrak{n}_{h+e+i} \cap \mathfrak{a}_0 \subset (\mathfrak{b}_1)^{1-e-i}(\mathfrak{b}_0 + (\mathfrak{a}_1)^{e+i}) \cap \mathfrak{a}_0 = \mathfrak{b}_0 + \mathfrak{a}_1$. Hence $l \leq h$ also, so we have equality. □

The main ingredients in proving the intertwining theorem 4.5 are the exact sequences analogous to those in [3] (1.4) so we proceed by proving these. We start with some preliminary lemmas, where Λ is no longer assumed to be strict and e is the $\mathfrak{o}_E$-period of Λ.

Lemma 5.2 (cf. [3] (1.4.8)).

(i) *We have $\mathfrak{p}_E\mathfrak{n}_k = \mathfrak{n}_{k+e} \cap \mathfrak{p}_E\mathfrak{a}_0$ and $\mathfrak{n}_k = \mathfrak{p}_E^{-1}\mathfrak{n}_{k+e} \cap \mathfrak{a}_0$, for all $k \in \mathbb{Z}$.*
(ii) *For $k \in \mathbb{Z}$, the following are equivalent:*
  (a) *$k \geq h_i(\beta,\Lambda)$;*
  (b) *$\mathfrak{n}_{k+e+i} = \mathfrak{b}_0 + \mathfrak{p}_E(\mathfrak{n}_{k+i} \cap \mathfrak{a}_i)$.*

Proof. This is identical to [3] (1.4.8), since $\mathfrak{p}_E\mathfrak{p}_E^{-1} = \mathfrak{o}_E$. □

In particular, we have that for $k \geq h_i(\beta,\Lambda)$,

$$\mathfrak{n}_{k+e+i} \cap \mathfrak{a}_i = \mathfrak{b}_i + \mathfrak{p}_E(\mathfrak{n}_{k+i} \cap \mathfrak{a}_i).$$

Also observe that the lemma implies that the application $i \mapsto h_i$ is periodic, with period dividing e.

Recall ([3] (1.3.3)) that a *tame corestriction* on A relative to E/F is a (B,B)-bimodule homomorphism $s : A \to B$ such that $s(\mathfrak{A}) = \mathfrak{A} \cap B$ for any hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A which is normalized by $E^\times$. Such a homomorphism exists and is uniquely determined upto multiplication by an element of $\mathfrak{o}_E^\times$. We also have $\ker s = \operatorname{im} a_\beta$ and $\ker a_\beta = \operatorname{im} s$.

Lemma 5.3 (cf. [3] (1.3.4)(ii)). *Let Λ be any $\mathfrak{o}_E$-lattice sequence in V and let s be a tame corestriction on A relative to E/F. Then we have $s(\mathfrak{a}_n) = \mathfrak{b}_n$, for $n \in \mathbb{Z}$.*

Proof. This is identical to [3] (1.3.4)(ii), having observed that, from [4] (2.10), we have $\mathfrak{a}_n^* = \mathfrak{a}_{1-n}$. □

Now we have all the tools necessary to prove (and the proof is *mutatis mutandis* that of [3] (1.4.7)):

Proposition 5.4 (cf. [3] (1.4.7), (1.4.10), [1] (5.2)). *Let $i, k, r \in \mathbb{Z}$ and suppose $i \geq 0$, $k \geq h_i(\beta, \Lambda)$, $r \geq 1$. Then the following sequences are exact:*

$$0 \to \mathfrak{n}_{k+i} \cap \mathfrak{a}_i / \mathfrak{n}_{k+r+i} \cap \mathfrak{a}_i \xrightarrow{a_\beta} \mathfrak{a}_{k+i}/\mathfrak{a}_{k+r+i} \xrightarrow{s} \mathfrak{b}_{k+i}/\mathfrak{b}_{k+r+i} \to 0,$$
$$0 \to \mathfrak{n}_{k+i} \cap \mathfrak{a}_i / \mathfrak{b}_i \xrightarrow{a_\beta} \mathfrak{a}_{k+i} \xrightarrow{s} \mathfrak{b}_{k+i} \to 0.$$

We now give another characterization of the integers h_i (cf. [3] (1.4.11)).

Lemma 5.5. *For $i \geq 0$ we have $h_i(\beta, \Lambda) = \inf\{k \in \mathbb{Z} : a_\beta(\mathfrak{a}_i) \supset \mathfrak{a}_{k+i} \cap \operatorname{im} a_\beta\}$.*

Proof. We have $a_\beta(\mathfrak{a}_i) \supset a_\beta(\mathfrak{n}_{h_i+i} \cap \mathfrak{a}_i) = \ker(s|\mathfrak{a}_{h_i+i})$ by (5.4) and this is precisely $\mathfrak{a}_{h_i+i} \cap \operatorname{im} a_\beta$. Conversely, suppose $k < h_i$ and $a_\beta(\mathfrak{a}_i) \supset \mathfrak{a}_{k+i} \cap \operatorname{im} a_\beta$. Then

$$a_\beta(\mathfrak{p}_E \mathfrak{a}_i) \supset \mathfrak{a}_{k+e+i} \cap \operatorname{im} a_\beta \supset \mathfrak{a}_{h_i+e+i} \cap \operatorname{im} a_\beta = a_\beta(\mathfrak{n}_{h_i-1+e+i} \cap \mathfrak{a}_i)$$

by (5.4) (and since $e - 1 \geq 0$). Hence we have $\mathfrak{n}_{h_i-1+e+i} \cap \mathfrak{a}_i \subset \mathfrak{b}_0 + \mathfrak{p}_E \mathfrak{a}_i$.

Now by (5.2) we have

$$\mathfrak{n}_{h_i-1+2e+i} = \mathfrak{b}_0 + \mathfrak{p}_E(\mathfrak{n}_{h_i-1+e+i} \cap \mathfrak{a}_i) \subset \mathfrak{b}_0 + \mathfrak{p}_E(\mathfrak{b}_0 + \mathfrak{p}_E \mathfrak{a}_i) = \mathfrak{b}_0 + \mathfrak{p}_E^2 \mathfrak{a}_i$$

so, also by (5.2),

$$\mathfrak{n}_{h_i-1+e+i} = \mathfrak{p}_E^{-1} \mathfrak{n}_{h_i-1+2e+i} \cap \mathfrak{a}_0 \subset \mathfrak{p}_E^{-1}(\mathfrak{b}_0 + \mathfrak{p}_E^2 \mathfrak{a}_i) \cap \mathfrak{a}_0 = \mathfrak{b}_0 + \mathfrak{p}_E \mathfrak{a}_i,$$

contradicting the minimality of h_i. □

In order to compare the integers h_i with the value $k_0(\beta, \Lambda)$ as defined in [4] (5.1), we must introduce the notion of a (W, E)-*decomposition* (see [3] (1.2)). Let W be the F-span of an E-basis of V so the canonical map $E \otimes_F W \to V$ is an isomorphism. This induces an algebra isomorphism $A(E) \otimes_F \operatorname{End}_F(W) \simeq A$, where $A(E) := \operatorname{End}_F(E)$, which in turn induces an isomorphism of $(A(E), B)$-bimodules

$$A(E) \otimes_E B \simeq A. \tag{5.6}$$

In the space $A(E)$ there is a unique hereditary $\mathfrak{o}_F$-order normalized by $E^\times$; we denote it $\mathfrak{A}(E)$ and write $\Lambda(E)$ for the associated (strict) lattice sequence in E. Then, for a suitable choice of W, the isomorphism (5.6) restricts to an isomorphism

$$\mathfrak{A}(E) \otimes_{\mathfrak{o}_E} \mathfrak{b}_r \simeq \mathfrak{a}_r,$$

for each $r \in \mathbb{Z}$ (see [4] (5.3)).

We may also consider the lattices $\mathfrak{n}_k(\beta, \Lambda(E))$. Then we also have

$$\mathfrak{n}_{ek}(\beta, \Lambda) = \mathfrak{n}_k(\beta, \Lambda(E)) \otimes_{\mathfrak{o}_E} \mathfrak{b}_0,$$

for $k \in \mathbb{Z}$. This is [3] (1.4.13)(i) in the case of a strict lattice sequence and the general case follows since $\mathfrak{a}_{ek} = \mathfrak{p}_E^k \mathfrak{a}_0$ for all $k \in \mathbb{Z}$.

We put $k_F(\beta) = \inf\{k \in \mathbb{Z} : \mathfrak{n}_{k+1}(\beta, \Lambda(E)) \subset \mathfrak{o}_E + \mathfrak{P}(E)\}$, where $\mathfrak{P}(E) = \mathfrak{p}_E \mathfrak{A}(E)$ is the Jacobson radical of $\mathfrak{A}(E)$. (Note that $k_F(\beta)$ is denoted $k_0(\beta, \mathfrak{A}(E))$ in [3].) We also have $k_F(\beta) = \inf\{k \in \mathbb{Z} : a_\beta(\mathfrak{A}(E)) \supset \mathfrak{P}(E)^k \cap \operatorname{im} a_\beta\}$. Then we define, as in [4] (5.1),

$$k_0(\beta, \Lambda) = e \cdot k_F(\beta).$$

Lemma 5.7 (cf. [3] (1.4.13)(ii)). *With notation as above,* $k_0(\beta, \Lambda) \geq h_i(\beta, \Lambda)$*, for all* $i \geq 0$*, and* $h_1(\beta, \Lambda) = k_0(\beta, \Lambda)$*.*

Proof. Writing $\otimes$ for $\otimes_{\mathfrak{o}_E}$ and $k = k_F(\beta)$, we have

$$\begin{aligned} a_\beta(\mathfrak{a}_i) = a_\beta(\mathfrak{A}(E) \otimes \mathfrak{b}_i) &\supset (\mathfrak{P}(E)^k \cap \operatorname{im} a_\beta) \otimes \mathfrak{b}_i \\ &= \mathfrak{A}(E)\mathfrak{p}_E^k \otimes \mathfrak{b}_i \cap \operatorname{im} a_\beta = \mathfrak{a}_{ek+i} \cap \operatorname{im} a_\beta, \end{aligned}$$

so certainly $h_i(\beta, \Lambda) \leq ek = k_0(\beta, \Lambda)$.

Now suppose $ek > h_1(\beta, \Lambda)$; then $\mathfrak{n}_{ek+e} \subset \mathfrak{b}_0 + \mathfrak{p}_E \mathfrak{a}_1$ so we have $\mathfrak{n}_{ek} = \mathfrak{p}_E^{-1} \mathfrak{n}_{ek+e} \cap \mathfrak{a}_0 \subset \mathfrak{b}_0 + \mathfrak{a}_1$. Then

$$\begin{aligned} \mathfrak{n}_k(\beta, \Lambda(E)) &\subset (\mathfrak{n}_k(\beta, \Lambda(E)) \otimes \mathfrak{b}_0) \cap A(E) \\ &= \mathfrak{n}_{ek} \cap A(E) \subset (\mathfrak{b}_0 + \mathfrak{a}_1) \cap A(E) \subset \mathfrak{o}_E + \mathfrak{P}(E) \end{aligned}$$

contrary to the definition of $k = k_F(\beta)$. □

Observe that the proof also shows that $k_0(\beta, \Lambda) = \inf\{k \in \mathbb{Z} : \mathfrak{n}_{k+1} \subset \mathfrak{b}_0 + \mathfrak{a}_1\}$, which is the definition given in §4.

In order to prove (4.5), we need one more exact sequence (cf. [3] (1.4.16)). Let $i, j, m, n \in \mathbb{N}$, $b_1, b_2 \in B^\times$, $k = k_0(\beta, \Lambda)$ and put

$$\begin{aligned} L &= \mathfrak{a}_{i+k} \cap (b_1 \mathfrak{a}_{j+k} + \mathfrak{a}_{m+k} b_2 + \mathfrak{a}_{n+k}), \\ M &= (\mathfrak{a}_i \cap \mathfrak{n}_{i+k}) \cap (b_1(\mathfrak{a}_j \cap \mathfrak{n}_{j+k}) + (\mathfrak{a}_m \cap \mathfrak{n}_{m+k}) b_2 + (\mathfrak{a}_n \cap \mathfrak{n}_{n+k})). \end{aligned}$$

Then we require that the sequence

$$M \overset{a_\beta}{\to} L \overset{s}{\to} A \tag{5.8}$$

be exact. This will follow exactly as in [3] (1.4.16) if we can show that $s(L) = L \cap B$. (L is E-exact, in the language of [3] (1.3).) But this holds as in [3] (1.3.16) so (5.8) is indeed an exact sequence.

We now have all the tools necessary to prove that

$$\mathcal{I} = \mathcal{I}_{\widetilde{G}}[\Lambda, n, r, \beta] = (1 + \mathfrak{m}_d) B^\times (1 + \mathfrak{m}_d),$$

for $[\Lambda, n, r, \beta]$ a simple stratum and where $\mathfrak{m}_d = \mathfrak{a}_{-r-k_0} \cap \mathfrak{n}_r$. We have $(1 + \mathfrak{m}_d)\mathcal{I}(1 + \mathfrak{m}_d) = \mathcal{I}$ as in [3] (1.5.8) and $B^\times \subset \mathcal{I}$. Furthermore, $x \in \mathcal{I}$ if and only if $\varpi_F x \in \mathcal{I}$ so we only need show that $\mathcal{I} \cap \mathfrak{a}_0 \subset (1+\mathfrak{m}_d) B^\times (1+\mathfrak{m}_d)$. The remainder of the proof is then *mutatis mutandis* that of [3] (1.5.8). □

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