Intertwining Automorphisms in Finite Incidence Structures

Alan Camina and Johannes Siemons
School of Mathematics
University of East Anglia
Norwich NR4 7TJ, U.K.

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ABSTRACT

The automorphism group of a finite incidence structure acts as permutation groups on the points and on the blocks of the structure. We view these actions as linear representations and observe that they are intertwined by the incidence relation. Most commonly the intertwining is of maximal linear rank, so that the representation on points appears as a subrepresentation of the action of the blocks. The paper investigates various consequences of this fact.

1. INTRODUCTION

An incidence structure consists of a triple $\mathcal{S} = (\mathcal{P}, \mathcal{B}; \mathcal{I})$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and an incidence relation $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. The elements of $\mathcal{P}$ are called *points*, and the elements of $\mathcal{B}$ are called *blocks*. In this note we are only concerned with finite structures, that is, both $\mathcal{P}$ and $\mathcal{B}$ are finite sets.

We view automorphisms of $\mathcal{S}$ as pairs of permutations of $\mathcal{P}$ and $\mathcal{B}$ which preserve incidence. This note is concerned with the interconnection between these two actions of an automorphism group.

It is very useful to regard both permutation representations as linear representations of the automorphism group. This view has also been taken by Cameron and Liebler [5] and Ott [19]. Incidence preservation can be seen as an intertwining relation between the linear representations of the automorphism group. This important observation was first used in Brauer's paper [2]; its relevance to incidence structures, however, went largely unnoticed. See also Wagner's review [25].

In Theorem 3.2 we show that the linear representation on points is a subrepresentation of the linear representation on blocks if an incidence matrix for $\mathcal{S}$—the intertwining matrix for the two representations—has linear rank equal to the number of points. This result has been mentioned before by Kantor [11] and Lehrer [16] in the context of block transitive designs and flag-transitive incidence structures related to classical groups. There it is also shown that these structures satisfy the assumption of maximality of the incidence rank. It is worth noticing that the implication of Theorem 3.2 plays a role in the number of questions related to the representation theory of the symmetric group. We also mention a result of O'Nan [18] about sharply 1-transitive sets in a permutation group. There it is shown that a containment relation between linear representations prohibits the existence of a sharply 1-transitive sets of permutations on blocks unless the number of points divides the number of blocks.

Theorems 3.1 and 3.2 are obtained in connection with certain standard decompositions of the point and the block modules associated to $\mathcal{S}$. These in turn are due to spectral decompositions, a concept arising from graph theory. Steps towards a combinational interpretation of these decompositions are taken in [23].

The maximality of incidence rank holds for many general classes of incidence structures. At the end of Section 3 we give a short survey on results to this effect. At the same time it becomes apparent that a unified treatment of this question is still missing.

In the fourth section we give applications concerning the number of orbits and the rank for the actions on points and blocks. Parts of these results have been stated before [11, 16], usually under unnecessarily restrictive conditions. At the end of the paper we consider a generalization of these concepts with regard to the induced actions on two subgroups of a group.

2. PRELIMINARIES

Let $p_1, \ldots, p_v$ be the points and $b_1, \ldots, b_w$ the blocks of an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}; \mathcal{I})$. The incidence relation $\mathcal{I}$ can be presented as a 0-1 matrix S, with rows indexed by points and columns indexed by blocks, such that $S_{p,b} = 1$ if and only if p is incident with b. An automorphism of $\mathcal{S}$ is a pair of permutations $g = (g_{\mathcal{P}}, g_{\mathcal{B}})$ such that p is incident with b if and only if $g_{\mathcal{P}}(p)$ is incident with $g_{\mathcal{B}}(b)$, for all p in $\mathcal{P}$ and for all b in $\mathcal{B}$. We represent automorphisms as pairs of permutation matrices (G, H), of size $v \times v$ and $w \times w$ respectively and in correspondence to the original arrangement of the elements of $\mathcal{S}$. In terms of the incidence matrix a pair of permutation matrices represents an automorphism if and only if

$$GS = SH. \tag{2.1}$$

We shall view this as an intertwining relation (see Section 43 in [7]) between the point and block action of automorphisms. A fundamental observation in this respect is Brauer's permutation lemma [2]:

LEMMA 2.1. *If S is a nonsingular square matrix and if G, H are permutation matrices with $GS = SH$, then G and H represent similar permutations.*

Let F be a field. Then an automorphism $g = (G, H)$ of $\mathcal{S}$ acts linearly on F^v and on F^w, the point module and the block module associated to $\mathcal{S}$. The characters of these representations are

$$\begin{aligned} \pi(g) &= \operatorname{trace}(G), \\ \beta(g) &= \operatorname{trace}(H), \end{aligned} \tag{2.2}$$

which count the number of elements fixed by g.

In general, when φ and ψ are characters of a group $\mathcal{G}$, we denote their inner product by $\langle \varphi, \psi \rangle_{\mathcal{G}} = (1/|\mathcal{G}|)\Sigma_{g \in \mathcal{G}} \varphi(g) \cdot \psi(g^{-1})$. The numbers of $\mathcal{G}$-orbits on points and blocks are $n(\mathcal{G}, \mathcal{P}) = \langle \pi, 1 \rangle_{\mathcal{G}}$ and $n(\mathcal{G}, \mathcal{B}) = \langle \beta, 1 \rangle_{\mathcal{G}}$. The *permutation ranks* for the two actions of $\mathcal{G}$ are the numbers $r(\mathcal{G}, \mathcal{P})$ and $r(\mathcal{G}, \mathcal{B})$ of $\mathcal{G}$-orbits on $\mathcal{P} \times \mathcal{P}$ and $\mathcal{B} \times \mathcal{B}$, respectively. Thus $r(\mathcal{G}, \mathcal{P}) = \langle \pi^2, 1 \rangle_{\mathcal{G}} = \langle \pi, \pi \rangle_{\mathcal{G}}$ and $r(\mathcal{G}, \mathcal{B}) = \langle \beta^2, 1 \rangle_{\mathcal{G}} = \langle \beta, \beta \rangle_{\mathcal{G}}$. Note that in the transitive case this agrees with the usual definition of the permutation rank in [26]. Finally, if A is a matrix over a field F, then $\operatorname{rank}_F(A)$ denotes the linear rank of A over F.

3. THE INTERTWINING AND DECOMPOSITION OF THE POINT AND BLOCK MODULES

Let S be the incidence matrix of $\mathcal{S}$, and F some field. Hence we have two incidence maps $S: F^w \to F^v$ and $S^T: F^v \to F^w$ between the point and the block module of $\mathcal{S}$ over F. We are concerned with the following hypothesis:

H1: The characteristic polynomial of SS^T splits into linear factors over F. Furthermore, the algebraic multiplicity of every eigenvalue is the same as its geometric multiplicity.

Under this assumption the collection of eigenvalues of SS^T is the *spectrum* of $\mathscr{S}$ over F, denoted by the spec($\mathscr{S}$). Note that the field of real numbers certainly satisfies H1, as SS^T is self-adjoint. Here spectral values are nonnegative, as SS^T is positive semidefinite.

In the case of graphs our definition of a spectrum differs slightly from the usual one where a spectrum is formed by the eigenvalues of the adjacency matrix. The book of Chetcović, Doob, and Sachs [6] is an excellent reference on graph spectra.

For an eigenvalue λ in spec($\mathscr{S}$), let E_λ be the corresponding eigenspace of $SS^T: F^v \to F^v$. Similarly, let E'_λ denote the eigenspace of $S^TS: F^w \to F^w$ for the same λ.

THEOREM 3.1. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{B}: \mathscr{I})$ be a finite incidence structure, and suppose that F is a field for which* H1 *holds. Let $\mathscr{G}$ be a group of automorphisms of $\mathscr{S}$. Then E_λ and E'_λ are $\mathscr{G}$-invariant modules for every λ in* spec($\mathscr{S}$) *with a $\mathscr{G}$-isomorphism $S^T: E_\lambda \leftrightarrow E'_\lambda$ when $\lambda \neq 0$. Furthermore, $F^v = \bigoplus_{\lambda \in \mathrm{spec}(\mathscr{S})} E_\lambda$ and $\bigoplus_{\lambda \in \mathrm{spec}(\mathscr{S})} E'_\lambda \subseteq F^w$ are $\mathscr{G}$-invariant decompositions.*

Proof. Let x be in E_λ, and let (G, H) be an automorphism. Then $SS^TGx = SHS^Tx = GSS^Tx = \lambda \cdot Gx$ by (2.1), so that E_λ is G-invariant. The same argument applies to E'_λ. Since $S^T(SS^T)x = \lambda S^Tx$, the map S^T takes E_λ to E'_λ, even injectively if $\lambda \neq 0$. For the same reason $S: E'_\lambda \to E_\lambda$ is an injection when $\lambda \neq 0$, so that $E_\lambda \approx E'_\lambda$. Since both maps are $\mathscr{G}$-maps, by (2.1), this is a $\mathscr{G}$-isomorphism. H1 implies that $F^v = \bigoplus_\lambda E_\lambda$. This decomposition is invariant under $\mathscr{G}$ by the first part of the proof. The same applies to $\bigoplus_\lambda E'_\lambda$ in F^w. ∎

THEOREM 3.2. *Under the assumption of Theorem* 3.1, *let π and β be the permutation characters of $\mathscr{G}$ on the elements of $\mathscr{S}$. Then*

$$\pi = \sum_{\lambda \in \mathrm{spec}(\mathscr{S})} \pi_\lambda,$$

where π_λ is the character of $\mathcal{G}$ acting on E_λ. If F is the real field then

$$\beta = \sum_{0 \neq \lambda \in \operatorname{spec}(\mathcal{S})} \pi_\lambda + \beta_0,$$

where β_0 is the character of $\mathcal{G}$ in its action on the kernel of $S: F^w \to F^v$. If in addition all values in spec($\mathcal{S}$) *are distinct, then $\mathcal{G}$ acts on the points of $\mathcal{S}$ as an elementary abelian 2-group.*

We note that the last part of this theorem and variations of it in the case of graphs are the theorems of Babai, Chao, Doob, Mowshowitz, and Sachs; see Theorems 5.1, 5.8–5.11 in [6].

Proof. The first part follows directly from Theorem 3.1. So suppose that $F = \mathbb{R}$. Then $F^w = \oplus_\mu E_\mu^*$, where μ is an eigenvalue of S^TS and E_μ^* the corresponding eigenspace, since S^TS is self-adjoint. But $S^TSx = \mu \cdot x$ implies that $SS^T(Sx) = \mu \cdot (Sx)$, so that μ belongs to spec($\mathcal{S}$) unless $\mu = 0$. Therefore $F^w = \oplus_{\lambda \neq 0} E'_\lambda + E_0^*$ where $E_0^* = \text{kernel}(S^TS)$. The result then follows from Theorem 3.1 once we have shown that $E_0^* = \text{kernel}(S)$. Clearly kernel$(S) \subseteq E_0^*$. Thus suppose $S^TSx = 0$. This implies that $(x^TS^T)(Sx) = 0$, so that $Sx = 0$ and hence $E_0^* \subseteq \text{kernel}(S)$. Finally suppose that every eigenvalue of SS^T has multiplicity one. In this case $\dim E_\lambda = 1$ for $\lambda \in \operatorname{spec}(\mathcal{S})$. Let y be in E_λ and $(G, H) \in \mathcal{G}$. Then Gy belongs to E_λ, so that $Gy = c \cdot y$, where c is a unit in F. Thus $c = \pm 1$ and hence $\mathcal{G}$ acts on the points of $\mathcal{S}$ as an elementary abelian 2-group. ∎

In the next theorem no assumption about the field is made.

THEOREM 3.3. *Let $\mathcal{S}$ be a finite incidence structure, F some field and $\mathcal{G}$ a group of automorphisms of $\mathcal{S}$. Let further π and β be the permutation characters of $\mathcal{G}$ on the points and blocks of $\mathcal{S}$. We assume either that* (i) $\operatorname{rank}_F(S) = v$ *and* $|\mathcal{G}| \neq 0$ *in F, or* (ii) $\operatorname{rank}_F(SS^T) = v$. *Then $\pi = \beta$ if $v = w$ or $\beta = \pi + \psi$ if $v < w$, where ψ is the character of $\mathcal{G}$ on the kernel of $S: F^w \to F^v$.*

Proof. The first hypothesis implies that there is some $w \times v$ matrix S' over F for which $SS' = 1$. Let $\bar{S} := |\mathcal{G}|^{-1} \Sigma HS'G^{-1}$, where the sum extends over all automorphisms (G, H) in $\mathcal{G}$. The intertwining relation (2.1) implies that $S\bar{S} = 1$ and $\bar{S}G = H\bar{S}$ for all (G, H) in $\mathcal{G}$.

The second hypothesis implies that SS^T is invertible, so that we may put $\bar{S} = S^T(SS^T)^{-1}$. Thus $S\bar{S} = 1$. Transposing (2.1), we have $\bar{S}G = S^T(SS^T)^{-1}G = S^T(G^TSS^T)^{-1} = S^T(SS^TG^T)^{-1} = S^TG(SS^T)^{-1} = HS^T(SS^T)^{-1} = H\bar{S}$. Therefore

$S\bar{S}=1$ and $\bar{S}G=H\bar{S}$ in both cases. If $v=w$, then also $\bar{S}S=1$, so that G is conjugate to H and hence $\pi=\beta$. Let therefore $v<w$ and put $\varphi(G,H)=(1-\bar{S}S)H$. It is easy to see that $\varphi(G,H)$ leaves the kernel of S invariant. Also, φ is a linear representation of $\mathcal{G}$ since $\varphi(G'G,H'H)-\varphi(G',H')\cdot\varphi(G,H)=-\bar{S}SH'\bar{S}SH+H'\bar{S}SH=(H'\bar{S}S-H'\bar{S}S\bar{S}S)H=H'(\bar{S}S-\bar{S}S)H=0$. Its character is $\psi(G,H)=\text{trace}((1-\bar{S}S)H)=\text{trace}(H)-\text{trace}(SH\bar{S})=\text{trace}(H)-\text{trace}(G)=\beta(G,H)-\pi(G,H)$. In particular $\psi(1)=w-v$ is the dimension of kernel(S), so that ψ is the representation of $\mathcal{G}$ on this subspace. ■

Remark (The linear rank of incidence). In Theorem 3.3 we have assumed that $\text{rank}_F(S)=v$ or $\text{rank}_F(SS^T)=v$. We note first that the two conditions are equivalent in characteristic zero; see the proof of Theorem 3.2. The assumption $\text{rank}_F(S)=v$ holds for many general classes of incidence structures. Ideally one would hope to have configurational conditions which guarantee this property. Presently only case by case analysis is available. We give a list (not complete) of structures for which $\text{rank}(S)=v$ in characteristic zero: designs (by standard argument); linear spaces [3]; subspace incidence, classical groups [12, 16, 23]; subset incidence, lattices [12, 14, 21, 23]; matroids [14]; nonbipartite graphs [22]. For a discussion of the rank of designs in prime characteristic see [9, 10, 13, 24]. There are instances where $\text{rank}(S)<v$. These include bipartite graphs ($\text{rank}(S)=v-1$; see [22]) and generalized $2n$-gons (see Section 1.9 in [20]).

4. ORBITS AND RANK

In this section let $\mathcal{S}$ be an incidence structure for which

H2: The linear rank of $\mathcal{S}$ in characteristic zero is v.

This is the condition of Theorem 3.2, so that the permutation characters of an automorphism group $\mathcal{G}$ are of the form π and $\beta=\pi+\psi$. Let $f(\mathcal{G})$ be the number of $\mathcal{G}$-orbits on flags (i.e. incident point-block pairs), and let $a(\mathcal{G})$ be the number of $\mathcal{G}$-orbits on anti-flags (i.e. nonincident point-block pairs). We first state an explicit version of a rather well-known result [1]:

Theorem 4.1. *Let $\mathcal{S}$ be a finite incidence structure for which* H2 *holds with an automorphism group $\mathcal{G}$. Then $n(\mathcal{G},\mathcal{B})-n(\mathcal{G},\mathcal{P})=\langle\psi,1\rangle_{\mathcal{G}}\geqslant 0$, where ψ is as in Theorem* 3.3.

The proof is immediate. One may wonder about an equivalent theorem in the case of infinite structures. Many examples show that $n(\mathcal{G}, \mathcal{P}) \leqslant n(\mathcal{G}, \mathcal{B})$ may not hold, even if the incidence map $F\mathcal{P} \to F\mathcal{B}$ is injective; see also Mäurer's paper [17].

THEOREM 4.2. *Let $\mathcal{S}$ be a finite incidence structure for which* H2 *holds with an automorphism group $\mathcal{G}$. Then $r(\mathcal{G}, \mathcal{P}) \leqslant f(\mathcal{G}) + a(\mathcal{G}) \leqslant r(\mathcal{G}, \mathcal{B})$. Furthermore, the following are equivalent:*

(i) $v = w$,
(ii) $f(\mathcal{G}) + a(\mathcal{G}) = r(\mathcal{G}, \mathcal{B})$, *and*
(iii) $r(\mathcal{G}, \mathcal{P}) = r(\mathcal{G}, \mathcal{B})$.

Finally, $f(\mathcal{G}) + a(\mathcal{G}) = r(\mathcal{G}, \mathcal{P})$ if and only if $\langle \pi, \beta - \pi \rangle_{\mathcal{G}} = 0$.

The inequality $r(\mathcal{G}, \mathcal{P}) \leqslant r(\mathcal{G}, \mathcal{B})$ in the case of block transitive designs is given in [11].

Proof. By Theorem 3.3 we can assume that $\beta = \pi + \psi$, where ψ is some character of $\mathcal{G}$. The number of $\mathcal{G}$-orbits on $\mathcal{P} \times \mathcal{B}$ is $\langle \pi \cdot \beta, 1 \rangle_{\mathcal{G}} = \langle \pi, \beta \rangle_{\mathcal{G}} = f(\mathcal{G}) + a(\mathcal{G})$. The theorem then follows from the equation $r(\mathcal{G}, \mathcal{B}) = \langle \beta^2, 1 \rangle_{\mathcal{G}} = \langle \beta, \beta \rangle_{\mathcal{G}} = \langle \pi, \beta \rangle_{\mathcal{G}} + \langle \psi, \beta \rangle_{\mathcal{G}} = \langle \pi, \pi \rangle_{\mathcal{G}} + 2\langle \psi, \pi \rangle_{\mathcal{G}} + \langle \psi, \psi \rangle_{\mathcal{G}}$ and the fact that $r(\mathcal{G}, \mathcal{P}) = \langle \pi^2, 1 \rangle_{\mathcal{G}} = \langle \pi, \pi \rangle_{\mathcal{G}}$. ■

As an illustration of this theorem we derive a generalization of a result due to Dembowski, Theorem 2.3.4 in [8].

THEOREM 4.3. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}; \mathcal{I})$ be an incidence structure with $v \leqslant w$ such that an incidence matrix for $\mathcal{S}$ is not the identity or the matrix of all ones. If $\mathcal{S}$ admits an automorphism group $\mathcal{G}$ acting doubly transitively on $\mathcal{B}$, then $\mathcal{G}$ acts doubly transitively on $\mathcal{P}$, and $\mathcal{S}$ is a symmetric $(v = w)$ 2-design. Furthermore $\mathcal{G}$ is transitive on both flags and antiflags.*

Proof. Since $\mathcal{G}$ acts doubly transitively on $\mathcal{B}$, the character β has the form $1 + \psi$, where ψ is an irreducible character of $\mathcal{G}$. We regard the blocks of $\mathcal{S}$ for the moment as "points" in the structure dual to $\mathcal{S}$ and apply the first part of Theorem 3.2. It follows that $S^T S$ has two eigenvalues μ_0 and μ_1 of respective multiplicities 1 and $w - 1$. The Frobenius-Perron theorem and the nondegeneracy condition in the theorem then imply that $\mu_0 > \mu_1 > 0$, so that $\operatorname{rank}(S^T S) = w$. Since $v \leqslant w$, it follows that in fact $v = w = \operatorname{rank}(SS^T)$. As $2 = r(\mathcal{G}, \mathcal{B}) = f(\mathcal{G}) + a(\mathcal{G}) = r(\mathcal{G}, \mathcal{P})$ by Theorem 4.2, $\mathcal{G}$ acts doubly transitively on the points of $\mathcal{S}$, so that $\mathcal{S}$ is a 2-design and $\mathcal{G}$ acts transitively on both flags and antiflags. ■

We conclude with an application of Theorem 4.2 to the relationship between the orbits of a subgroup in a permutation group and the rank of the group.

THEOREM 4.4. *Let* $(\mathcal{G}, \Omega)$ *be a finite permutation group with a subgroup* $\mathcal{H}$. *If* $\Omega_1, \ldots, \Omega_s$ *are the orbits of* $\mathcal{H}$ *on* Ω, *let* ℓ *be some union of the* Ω_i*'s and put* $\mathcal{B} = \{\ell^g \mid g \in \mathcal{G}\}$. *Assume that* H2 *holds for* $(\Omega, \mathcal{B}; \in)$. *Then* $\mathcal{G}$ *is transitive and* $r(\mathcal{G}, \Omega) \leqslant s$. H2 *holds in particular for* $|\ell| = 2 < |\Omega|$ *and* $(\mathcal{G}, \Omega)$ *primitive.*

Proof. As $\mathcal{G}$ is transitive on $\mathcal{B}$, it follows from Theorem 4.1 that $\mathcal{G}$ is transitive on Ω. This at the same time implies that $a(\mathcal{G}) + f(\mathcal{G})$ is the number of orbits on Ω of the stabilizer of ℓ. This group contains $\mathcal{H}$, and so the required inequality follows from Theorem 4.2. If $|\ell| = 2$, we regard $(\Omega, \mathcal{B}; \in)$ as an undirected graph. The connected components of this graph would form blocks of imprimitivity, so that it must be connected. For the same reason this graph is not bipartite. In [22] it is shown that $(\Omega, \mathcal{B}; \in)$ then satisfies H2. ■

The assumption of primitivity is essential: elementary abelian 2-groups and certain dihedral groups violate the theorem. We conjecture, however, that for primitive groups the condition $|\ell| \leqslant 2$ can be relaxed considerably.

5. A GENERALIZATION

Let $\mathcal{G}$ be a finite group with subgroups $\mathcal{H}$ and $\mathcal{K}$ such that $|\mathcal{G} : \mathcal{H}| \leqslant |\mathcal{G} : \mathcal{K}|$. Let $\mathcal{P}$ be the collection of cosets of $\mathcal{H}$ in $\mathcal{G}$, and $\mathcal{B}$ the collection of cosets of $\mathcal{K}$ in $\mathcal{G}$. We suppose that $\mathcal{G}$ acts faithfully on $\mathcal{P}$ and $\mathcal{B}$ with induced characters $\pi = 1_{\mathcal{H}}^{\mathcal{G}}$ and $\beta = 1_{\mathcal{K}}^{\mathcal{G}}$ as before. Once $\mathcal{P}$ and $\mathcal{B}$ are ordered in some fashion, every $\mathcal{G}$-orbit on $\mathcal{P} \times \mathcal{B}$ in an obvious way can be represented as a 0-1 matrix S_i with $i = 1, \ldots, \langle \pi, \beta \rangle_{\mathcal{G}}$. These matrices are disjoint and hence in particular linearly independent. It follows therefore (see Section 43.11 in [7]) that every intertwining matrix for the actions of $\mathcal{G}$ on $\mathcal{P}$ and $\mathcal{B}$ can be written as a linear combination of the S_i. In particular:

PROPOSITION 5.1. *Let* $\mathcal{G}$ *be a group of automorphisms of some incidence structure* $\mathcal{S} = (\mathcal{P}, \mathcal{B}; \mathcal{I})$ *with* $\mathcal{P}$ *and* $\mathcal{B}$ *given as above. Then an incidence matrix for* $\mathcal{S}$ *is the sum of some of the* S_i.

Let ψ be the largest common constituent character of π and β, i.e. $\langle \pi - \psi, \beta - \psi \rangle_{\mathcal{G}} = 0$. It follows from Theorem 3.2 that the rank of any incidence matrix is bounded by the degree of ψ. It is an open problem to decide whether this bound can always be attained. In particular, if $\pi = \psi$, does there always exist an incidence matrix of rank $\pi(1)$?

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