Journal of Algebra **226**, 236–249 (2000)
doi:10.1006/jabr.1999.8169, available online at http://www.idealibrary.com on IDEAL®

On a Conjecture of Foulkes

Suzie C. Dent and Johannes Siemons

School of Mathematics, University of East Anglia, Norwich NR4 7TJ, United Kingdom

Communicated by Walter Feit

Received September 10, 1998

Suppose that $\Omega = \{1, 2, \ldots, ab\}$ for some non-negative integers a and b. Denote by $P(a, b)$ the set of unordered partitions of Ω into a parts of cardinality b. In this paper we study the decomposition of the permutation module $\mathbf{C}P(a, b)$ where $\mathbf{C}$ is the field of complex numbers. In particular, we show that $\mathbf{C}P(3, b)$ is isomorphic to a submodule of $\mathbf{C}P(b, 3)$ for $b \geq 3$. This settles the next unproven case of a conjecture of Foulkes. © 2000 Academic Press

1. INTRODUCTION

For positive integers a and b let $\Omega = \{1, 2, \ldots, ab\}$ and denote by

$$P(a, b) = \big\{\{\Delta_1, \Delta_2, \ldots, \Delta_a\} \mid \Delta_i \subset \Omega,\ \Delta_i \cap \Delta_j = \varnothing \text{ if } i \neq j,\ |\Delta_i| = b\big\}$$

the *unordered* (a, b)*-partitions* of Ω. Throughout G denotes the symmetric group $\mathrm{Sym}(ab)$. If F is any field let $FP(a, b)$ be the permutation module arising from the natural action of G on $P(a, b)$. Equivalently, one can view $FP(a, b)$ as the permutation representation of G on the cosets of the wreath product $\mathrm{Sym}(b) \wr \mathrm{Sym}(a)$.

In this paper we study the decomposition of $\mathbf{C}P(a, b)$ where $\mathbf{C}$ denotes the field of complex numbers and $(a, b) = (3, k)$ or $(k, 3)$ for arbitrary k. The decomposition of $\mathbf{C}P(3, k)$ is already given in [16]. In Section 4 we give a new proof of this result which allows us to prove the main theorem of this paper:

THEOREM. *If* $3 \leq k$ *then* $\mathbf{C}P(3, k)$ *is isomorphic to a submodule of* $\mathbf{C}P(k, 3)$.

0021-8693/00 $35.00

This settles a special case of a conjecture of Foulkes [6] which says that $\mathbf{C}P(a,b)$ is isomorphic to a submodule of $\mathbf{C}P(b,a)$ for any $a \leq b$. The literature on this question includes [1, 2, 4, 6, 7, 9, 10, 11, 12, 13, 16]. The results in Thrall's 1942 paper [16] give the decompositions of the permutation representations $\mathbf{C}P(2,b)$ and $\mathbf{C}P(b,2)$; from these one can verify Foulkes' conjecture for $a = 2$ and b arbitrary. In 1944 Littlewood [11] went on to give the complete decompositions of the representations $\mathbf{C}P(b,3)$ for $b \leq 6$ and $\mathbf{C}P(b,4)$ for $b \leq 5$. In 1950 Foulkes [6] decomposed $\mathbf{C}P(b,5)$ for $b \leq 4$ and $\mathbf{C}P(b,6)$ for $b \leq 4$. Using Littlewood's decomposition of $\mathbf{C}P(5,4)$ Foulkes noticed that every term in the decomposition of $\mathbf{C}P(4,5)$ occurs in $\mathbf{C}P(5,4)$. On the basis of these results Foulkes made the aforementioned conjecture. In [9] a proof of the conjecture for $a = 2$ and $b \geq 2$ is given in terms of Gaussian coefficients and plethysms. In [4] eigenvalue arguments already suggested in [1, 17] provide yet another proof. In [1] it is shown that the truth of Foulkes' conjecture for the pair of integers (b,b) implies the result for (a,b) with $a \leq b$.

2. NOTATION AND GENERAL RESULTS

We adopt the basic notation about the representation theory of the symmetric groups introduced by James in [8]. We briefly recap some of the definitions and introduce a join operation for tableaux.

If $\mu = (\mu_1, \mu_2, \ldots, \mu_\tau)$ is a partition of n and if t is a μ-tableau then the *row-stabilizer* of the tableau t is the subgroup R_t of $\mathrm{Sym}(n)$ which fixes all rows of t set-wise. Similarly, the *column-stabilizer* is the subgroup C_t of $\mathrm{Sym}(n)$ which fixes all columns of t set-wise. The *signed column sum* κ_t is the element of $\mathbf{C}\,\mathrm{Sym}(n)$ obtained by summing the elements of the column stabilizer of t with signs attached. To each μ-tableau t we associate the *polytabloid* $e_t := \{t\}\kappa_t$ where $\{t\}$ is the tabloid obtained from t. The module spanned by this polytabloid is known as the Specht module for μ and is denoted by S^μ. It is a well known fact that S^μ is irreducible and that all irreducible $\mathbf{C}\,\mathrm{Sym}(n)$-modules arise in this fashion. We will denote by M^μ and $\mathbf{C}\,\mathrm{Sym}(n)$-module generated by μ-tabloids.

We will also need the concept of a tableau with repeated entries. To distinguish these from tableaux with distinct entries we use capital letters and say that a μ-tableau T has *type* $\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_l^*)$ if i occurs μ_i^* times in T for $i \in \{1, 2, \ldots, l\}$. Denote by $\mathscr{T}(\mu, \mu^*)$ the set of μ-tableaux of type μ^*. We can define an action of $\mathrm{Sym}(n)$ on $\mathscr{T}(\mu, \mu^*)$ in the following way. We label the place numbers of $T \in \mathscr{T}(\mu, \mu^*)$ according to the positions of the numbers $1, 2, \ldots, n$ in t. That is, for $i = 1, \ldots, n$ let $T(i)$ be the entry in T which occurs in the same position as i occurs in t. Then for $g \in \mathrm{Sym}(n)$ we have $Tg(i) = T(ig^{-1})$. We will say that T and T'

are *row equivalent* if $T' = Tg$ for some permutation g in the row stabilizer of the given tableau t.

We use the standard diagrams for tableaux as in [8] and distinguish the associated tabloids by drawing lines between rows. Given a μ-tableau t and a μ-tableau T of type μ^*, we can construct a μ^*-tabloid by putting the ij-entry of t into the row of the tabloid given by the ij-entry of T. It is easy to see that given a μ-tableau t there is a one-to-one correspondence between μ-tableaux T of type μ^* and μ^*-tabloids. Therefore we will not distinguish between the module M^{μ^*} and the isomorphic module of μ-tableaux of type μ^*.

For $T \in \mathscr{T}(\mu, \mu^*)$ define a $\mathbf{C}\,\mathrm{Sym}(n)$-homomorphism from M^{μ} to M^{μ^*} given by

$$\Theta_T : \{t\} \mapsto \sum \{T' \mid T' \text{ is row equivalent to } T\}$$

and extend linearly to the whole of M^{μ}. We will express elements in the image of this homomorphism in terms of tableaux of type μ^*. We denote the restriction of Θ_T to S^{μ} by $\hat{\Theta}_T$.

Remark. For $g_1 \in R_t$ it is easy to see that $Tg_1\kappa_t = 0$ if and only if some column of Tg_1 contains two identical numbers. Thus we only need to consider those Tg_1 which have distinct entries in each column.

A tableau T in $\mathscr{T}(\mu, \mu^*)$ will be called *semistandard* if its entries are non-decreasing along the rows and strictly increasing down its columns. Let $\mathscr{T}_o(\mu, \mu^*)$ be the set of semistandard tableaux in $\mathscr{T}(\mu, \mu^*)$. The homomorphisms $\hat{\Theta}_T$ with T in $\mathscr{T}_o(\mu, \mu^*)$ are called *semistandard homomorphisms*. We need the following standard results.

THEOREM 2.1. *If F is a field of characteristic zero then* $\{\hat{\Theta}_T \mid T \in \mathscr{T}_o(\mu, \mu^*)\}$ *is a basis for* $\mathrm{Hom}_{F\,\mathrm{Sym}(n)}(S^{\mu}, M^{\mu^*})$.

COROLLARY 2.2. *If F is a field of characteristic zero then the dimension of the space* $\mathrm{Hom}_{F\,\mathrm{Sym}(n)}(S^{\mu}, M^{\mu^*})$ *is the number of semistandard μ-tableaux of type μ^*.*

The proof of our main result is based on the fact that the multiplicity of S^{μ} in $\mathbf{C}P(a,b)$ is the dimension of $\mathrm{Hom}_{\mathbf{C}\,\mathrm{Sym}(n)}(S^{\mu}, \mathbf{C}P(a,b))$. Thus it is sufficient to show for $k \geq 3$ that the number of linearly independent homomorphisms from S^{μ} to $\mathbf{C}P(k,3)$ is greater than or equal to the multiplicity of S^{μ} in $\mathbf{C}P(3,k)$. We will therefore construct a suitably sized set of non-zero homomorphisms $\overline{\Theta}_{T_i}$ from S^{μ} to $\mathbf{C}P(3,k)$ and $\mathbf{C}P(k,3)$, respectively, with the property that $\overline{\Theta}_{T_i}(e_t)$ involves an element which is not involved in $\overline{\Theta}_{T_j}(e_t)$ for all j with $j < i$. This ensures that our chosen homomorphisms are linearly independent.

Essential will be the following *join operation* for tableaux. Let T^1 and T^2 be tableaux (with or without repeated entries) and let l be the larger of the number of rows of T^1 and T^2. Write down the columns of T^1 of length l followed by the columns of T^2 of length l, in the same order. Continue this process for columns of length $l-i$ with $i = 1, 2, \ldots, l-1$ or until all columns of T^1 and T^2 have been used up. The resulting tableau is the *join* of T^1 and T^2, denoted by $T^1 \vee T^2$.

EXAMPLE 2.3. If T^1 and T^2 are the tableaux

$$T^1 = \begin{matrix} 11123 \\ 22 \\ 33 \end{matrix} \qquad \text{and} \qquad T^2 = \begin{matrix} 44455568 \\ 66777 \\ 88 \end{matrix},$$

then we have

$$T^1 \vee T^2 = \begin{matrix} 1144455123568 \\ 2266777 \\ 3388 \end{matrix}.$$

3. DECOMPOSING $\mathbf{C}P(a,b)$

Semistandard homomorphisms give rise to $\mathbf{C}G$-homomorphisms from S^μ to a module isomorphic to $\mathbf{C}P(a,b)$, in the following way.

DEFINITION 3.1. Two μ-tableaux T and T^* have the *same pattern* if the entries of T can be relabeled to give T^*.

For example,

$$T = \begin{matrix} 44423 \\ 223 \\ 3 \end{matrix} \qquad \text{and} \qquad T^* = \begin{matrix} 22213 \\ 113 \\ 3 \end{matrix}$$

have the same pattern.

If T is a tableau of type (b^a) let $\overline{T}$ denote the corresponding unordered partition in $P(a,b)$. In particular, if T and T^* have the same pattern then $\overline{T} = \overline{T}^*$. This yields a map $^-$ from the space of tableaux to $\mathbf{C}P(a,b)$. The composition of $\hat{\Theta}_T$ and $^-$ is denoted by $\overline{\Theta}_T$. Thus any element involved in $\overline{\Theta}_T\{t\}\kappa_t$ can be written in the form $\overline{T}g_1g_2$ with $g_1 \in R_t$ and $g_2 \in C_t$.

Given a partition μ of ab we can test whether S^μ appears in $\mathbf{C}P(a,b)$. If S^μ is a submodule of $\mathbf{C}P(a,b)$ then the image under $\overline{\Theta}_T$ of S^μ is non-zero for some T. So, in fact, it is sufficient to show that $\overline{\Theta}_T(e_t)$ is non-zero for some T.

The following example and lemma illustrate these definitions.

EXAMPLE 3.2. Let T and t be given by

$$T = \begin{matrix}1112\\22\end{matrix} \qquad \text{and} \qquad t = \begin{matrix}1356\\24\end{matrix}.$$

Then

$$\begin{aligned}\hat{\Theta}_T\{t\} &= \begin{matrix}1112\\22\end{matrix} + \begin{matrix}1121\\22\end{matrix} + \begin{matrix}1211\\22\end{matrix} + \begin{matrix}2111\\22\end{matrix}\\ &= \frac{\overline{135}}{\underline{246}} + \frac{\overline{136}}{\underline{245}} + \frac{\overline{156}}{\underline{243}} + \frac{\overline{356}}{\underline{241}}\end{aligned}$$

and so

$$\begin{aligned}\hat{\Theta}_T\{t\}\kappa_t = {} & \frac{\overline{135}}{\underline{246}} + \frac{\overline{136}}{\underline{245}} + \frac{\overline{156}}{\underline{243}} + \frac{\overline{356}}{\underline{241}} - \frac{\overline{235}}{\underline{146}} - \frac{\overline{236}}{\underline{145}}\\ & - \frac{\overline{256}}{\underline{143}} - \frac{\overline{356}}{\underline{142}} + \frac{\overline{245}}{\underline{136}} + \frac{\overline{246}}{\underline{135}} + \frac{\overline{256}}{\underline{134}} + \frac{\overline{456}}{\underline{132}}\\ & - \frac{\overline{145}}{\underline{236}} - \frac{\overline{146}}{\underline{235}} - \frac{\overline{156}}{\underline{234}} - \frac{\overline{456}}{\underline{231}}\\ = {} & \frac{\overline{135}}{\underline{246}} + \frac{\overline{136}}{\underline{245}} - \frac{\overline{235}}{\underline{146}} - \frac{\overline{236}}{\underline{145}} + \frac{\overline{245}}{\underline{136}} + \frac{\overline{246}}{\underline{135}}\\ & - \frac{\overline{145}}{\underline{236}} - \frac{\overline{146}}{\underline{235}}.\end{aligned}$$

Upon applying $^{-}$ we get the following element of $\mathbf{C}P(a,b)$:

$$\overline{\Theta}_T\{t\}\kappa_t = 2\begin{matrix}(135)\\(246)\end{matrix} + 2\begin{matrix}(136)\\(245)\end{matrix} - 2\begin{matrix}(235)\\(146)\end{matrix} - 2\begin{matrix}(236)\\(145)\end{matrix}.$$

LEMMA 3.3. *If $0 \leq c \leq b$ the Specht module $S^{(ab-c(a-1),\, c^{(a-1)})}$ appears in $\mathbf{C}P(a,b)$ if and only if c is even. Moreover, when c is even then $S^{(ab-c(a-1),\, c^{(a-1)})}$ appears in $\mathbf{C}P(a,b)$ with multiplicity one.*

Proof. There is only one semistandard $(ab - c(a-1), c^{(a-1)})$-tableau of type (b^a). Denote this tableau by T. By calculating the coefficients of (a, b)-partitions involved in $\overline{\Theta}_T(\{t\}\kappa_t)$, where t is any $(ab -$

$c(a-1), c^{(a-1)})$-tableau, we see that the coefficient is $a!$ if c is even and zero otherwise. ∎

We remark that the modules in the decomposition of $\mathbf{C}P(2,k)$ come straight from Lemma 3.3 and it can be checked by calculating dimensions that these are all the modules in its decomposition.

4. THE DECOMPOSITION OF $\mathbf{C}P(3,k)$

In this section we give the decomposition of $\mathbf{C}P(3,k)$ for k arbitrary. This decomposition is less straightforward and was done by Thrall [16] in 1942. We will outline a somewhat more modern proof of it.

THEOREM 4.1. *The irreducible constituents of* $\mathbf{C}P(3,k)$ *are*:

(i) $S^{(3k-4u-s,\,2u+s,\,2u)}$, *where* $0 \le u, s$ *and* $3(k-2u) \ge 2s$, *with multiplicity* $m_{k-2u,s}$ *and*

(ii) $S^{(3k-4u-s-5,\,2u+s+4,\,2u+1)}$, *where* $0 \le u, s$ *and* $3(k-2u-3) \ge 2s$, *with multiplicity* $m_{k-2u-3,s}$.

The multiplicities $m_{v,s}$ *are given by*

$$m_{v,s} = \begin{cases} 0 & \text{if } s = 1 \\ c+1 & \text{if } s \neq 1 \text{ and } s \le v \\ c - \left[\dfrac{s-v+1}{2}\right] + 1 & \text{if } s \neq 1 \text{ and } s > v, \end{cases}$$

where $s = 6c + r$ *with* $r \in \{0,2,3,4,5,7\}$ *and* $[\,]$ *denotes the integer part function.*

Proof (*outline*). *Part 1*. For each μ in the theorem we construct an integer set I and a collection $\{T_w : w \in I\}$ of semistandard tableaux of type (k^3) with the property that $\{\overline{\Theta}_{T_w} : w \in I\}$ are linearly independent homomorphisms from S^μ to $\mathbf{C}P(3,k)$. To show that the multiplicity of S^μ in $\mathbf{C}P(3,k)$ is at least m_μ we require $|I| \ge m_\mu$. We choose the following semistandard tableaux of type (k^3):

(i) If $\mu = (3k-4u-s, 2u+s, 2u)$ and $k - 2u \ge s$ then for w ranging over the even numbers between 0 and $2c$ let T_w be the semistandard tableaux

$$\begin{array}{llll} \underbrace{\begin{array}{c}1\cdots1\\2\cdots2\\3\cdots3\end{array}}_{2u} & \begin{array}{c}1\cdots1\\ \underbrace{2\cdots2}_{s-w}\\ \end{array} & \begin{array}{c}1\cdots1\\ \underbrace{3\cdots3}_{w}\\ \end{array} & \begin{array}{c}1\cdots12\cdots23\cdots3\\ \\ \end{array} \end{array}.$$

(ii) If $\mu = (3k - 4u - s, 2u + s, 2u)$ and $k - 2u < s$ then for w ranging over the even numbers between $s - (k - 2u)$ and $2c$ let T_w be the semistandard tableaux

$$\begin{array}{llll}
1\cdots1 & 1\cdots1 & 1\cdots1 \quad 2 \quad\;\; 2 & 2\cdots2\,3\cdots3 \\
2\cdots2 & \underbrace{2\cdots2}_{s-w} & \underbrace{3\cdots3 \quad 3\cdots3}_{w} & \\
\underbrace{3\cdots3}_{2u} & & &
\end{array}.$$

(iii) If $\mu = (3k - 4u - s - 5, 2u + s + 4, 2u + 1)$ and $k - 2u - 3 \geq s$ then for w ranging over the even numbers between 0 and $2c$ let T_w be the semistandard tableaux

$$\begin{array}{lllll}
1\cdots1 & 1111 & 1\cdots1 & 1\cdots1 & 1\cdots1\,2\cdots2\,3\cdots3 \\
2\cdots2 & 2222 & \underbrace{2\cdots2}_{s-w} & \underbrace{3\cdots3}_{w} & \\
\underbrace{3\cdots3}_{2u} & 3 & & &
\end{array}.$$

(iv) If $\mu = (3k - 4u - s - 5, 2u + s + 4, 2u + 1)$ and $k - 2u - 3 < s$ then for w ranging over the even numbers between $s - (k - 2u - 3)$ and $2c$ let T_w be the semistandard tableaux

$$\begin{array}{lllll}
1\cdots1 & 1111 & 1\cdots1 & 1\cdots1\,2\cdots2 & 2\cdots2\,3\cdots3 \\
2\cdots2 & 2222 & \underbrace{2\cdots2}_{s-w} & \underbrace{3\cdots3\,3\cdots3}_{w} & \\
\underbrace{3\cdots3}_{2u} & 3 & & &
\end{array}.$$

For linear independence, we show that for a fixed tableau t there is a $(3, k)$-partition involved in $\overline{\Theta}_{T_w}(e_t)$ which is not involved in $\overline{\Theta}_{T_i}(e_t)$ for $i \in I$ with $i < w$. This then implies that each S^μ has multiplicity at least m_μ.

Part 2. Determine the permutation rank of Sym$(3k)$ on $P(3, k)$. This involves counting 3×3 integer matrices with constant row and column sum k, up to a certain equivalence. This can be done using Chapter 1 of [15] and it turns out that this permutation rank is

$$\begin{array}{ll}
\frac{1}{288}\{k^4 + 6k^3 + 64k^2 + 192k + 160 + 128c\} & \text{if } k \text{ is even} \\
\frac{1}{288}\{k^4 + 6k^3 + 64k^2 + 138k + 79 + 128c\} & \text{if } k \text{ is odd,}
\end{array}$$

where

$$c = \begin{cases} 1 & \text{if 3 divides } k \\ 0 & \text{otherwise.} \end{cases}$$

These numbers are sequence 973 in Sloane's book [14] from where one can see that they relate to certain seventhics of Cayley [3].

Part 3. Complete the proof by showing that the expression for the permutation rank is equal to Σm_μ^2 where m_μ are the integers from Part 1. Full details can be found in [5]. ∎

Remark. Thrall gives the following method of calculating m_μ when $\mu = (\mu_1, \mu_2, \mu_3)$: "To the minimum of $1 + \mu_1 - \mu_2$ and $1 + \mu_2 - \mu_3$ we add whichever one of $-2, 0, +2$ will give a result divisible by 3. If this result is even divide by 6 to get m_μ. If this result is odd add or subtract 3 according as μ_2 is even or odd and then divide by 6 to get m_μ."

5. MODULES IN THE DECOMPOSITION OF $\mathbf{C}P(k,3)$

We aim to show that all modules of $\mathbf{C}P(3,k)$ appear in $\mathbf{C}P(k,3)$ with no lesser multiplicity. We begin by describing a general method for constructing μ-tableaux of type (3^k) from smaller tableaux and explain how we can use these to obtain linearly independent homomorphisms from S^μ to $\mathbf{C}P(k,3)$. This method will be the basis of the proof of our main theorem.

Consider the tableaux $P^1, P^2, \ldots, P^{10}$ given by

$$P^1 = \begin{matrix} 111222 \\ 433344 \\ 665556 \end{matrix} \qquad P^2 = \begin{matrix} 1112 \\ 2233 \\ 3444 \end{matrix}$$

$$P^3 = \begin{matrix} 11123 \\ 22 \\ 33 \end{matrix} \qquad P^4 = \begin{matrix} 1112 \\ 2233 \\ 3 \end{matrix}$$

$$P^5 = \begin{matrix} 112314 \\ 2244 \\ 33 \end{matrix} \qquad P^6 = \begin{matrix} 11122235 \\ 33444 \\ 55 \end{matrix}$$

$$P^7 = \begin{matrix} 112221 \\ 3333444 \end{matrix} \qquad P^8 = \begin{matrix} 112122 \\ 333 \end{matrix}$$

$$P^9 = \begin{matrix} 1112 \\ 22 \end{matrix} \qquad \text{and} \qquad P^{10} = 111.$$

Note that any three identical digits of the P^i are arranged in one of the following ways.

1. They appear together in a single row.
2. Two of the them appear together in the second or the third row with the remaining digit in the top row.
3. Two of them appear together in the second row with the remaining digit in the third row.

We will use these properties to show that certain homomorphisms are non-zero and linearly independent.

Step 1 (*construction of μ-tableaux*). We construct two sets of μ-tableaux of type (3^k) and a μ-tableau of type $(1)^k$ as follows. The first of these, of type (3^k), will be of the form

$$T^* = \bigvee_{i=1}^{10} a_i T^{*i},$$

where $a_i T^{*i}$ is the join of T^{*i} with itself a_i times in the normal sense. Here T^{*i} is a tableau with the same pattern as P^i with the property that the labeling set for the first tableau in the $\vee$-expression is $1, 2, \dots$ and so that the remaining tableaux are labeled by consecutive numbers. Let t be the μ-tableau, of type (1^k), given by

$$t = \bigvee_{i=1}^{10} a_i t^i$$

where t^i has the same shape as T^{*i}. Without loss of generality we will always label the first tableau in the expression with consecutive numbers $1, 2, \dots$ and in turn label each of the remaining tableaux increasingly. With each T^* we associate a tableau T formed by permuting the digits in its rows until the numbers increase along rows. By construction the digits will automatically be strictly increasing down the columns, so this tableau will be semistandard.

Step 2 ($\overline{\Theta}_T(\{t\}\kappa_t)$ *is non-zero*). We use the tableaux constructed in Step 1 to show that the coefficient of $\overline{T}^*$ in $\overline{\Theta}_T\{t\}\kappa_t$ is non-zero. This coefficient is equal to the sum of the coefficients in $\hat{\Theta}_T\{t\}\kappa_t$ will be the form Tgh with $g \in R_t$ and $h \in C_t$. Moreover, a tableau with the same pattern as T^* can be written correspondingly in the form

$$\bigvee_{i=1}^{10} a_i P'^i$$

such that P'^i has the same pattern as P^i and such that for $i \neq j$ the labelling set for P'^i is disjoint from that of P'^j. Thus tableaux involved in $\overline{\Theta}_T\{t\}\kappa_t$ with the same pattern as T^* can be written in the form

$$\bigvee_{i=1}^{10} a_i T'^i g_i$$

where $g_i \in C_{t^i}$, the tableau $\bigvee_{i=1}^{10} a_i T'^i$ is row equivalent to T and $T'^i g_i$ has the same pattern as P^i. Since any three identical digits can only appear in the third rows of P^1 and P^2, they can appear only in the third rows of T'^1 and T'^2. Therefore T'^3, T'^5, and T'^6 must have pairs of digits in their third rows. Similarly, the second rows of T'^1, T'^6, T'^7, and T'^8 must contain three identical digits, leaving pairs of digits in the second rows of T'^2, T'^3, T'^4, T'^5, T'^6, and T'^9. By case by case analysis it can be checked that T'^i must have the same pattern as a tableau row equivalent to T^{*i}. Hence the coefficient of $\overline{T}^*$ in $\overline{\Theta}_T\{t\}\kappa_t$ is given by the product of the coefficients of $\overline{T}^{*i}$ in $\overline{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$ multiplied by the number of possible choices for the entries of the T^i, where T^i is the semistandard tableau row equivalent to T^{*i}. Trivially we can choose $T'^i = T^{*i}$ so it is sufficient to prove that the coefficient of $\overline{T}^{*i}$ in $\overline{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$ is non-zero for all i. The following lemma therefore concludes Step 2.

LEMMA 5.1. *For $i = 1, 2, \ldots, 10$ the coefficient of $\overline{T}^{*i}$ in $\overline{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$ is non-zero.*

Proof. Without loss of generality we can choose the labelling set for each of the tableaux T^{*i} to be $\{1, 2, \ldots\}$ and t^i to be the tableaux with the digits $1, 2, 3, \ldots$ placed in increasing order down its columns. It is a straightforward procedure to calculate the coefficient of $\overline{T}^{*i}$ in $\overline{\Theta}_{T^i}\{t^i\}\kappa_{t^i}$. These coefficients can be seen to be 48, 24, 6, 6, 8, 4, 8, 2, 2, and 1, respectively. Since these coefficients are all non-zero the proof is complete. ∎

Remark. As a direct consequence of the lemma, we have that $S^{(6,6,6)}$ appears in $\mathbf{C}P(6,3)$; the module $S^{(8,5,2)}$ appears in $\mathbf{C}P(5,3)$; the modules $S^{(4,4,4)}$, $S^{(6,4,2)}$, and $S^{(6,6)}$ appear in $\mathbf{C}P(4,3)$; the modules $S^{(5,2,2)}$, $S^{(4,4,1)}$, and $S^{(6,3)}$ appear in $\mathbf{C}P(3,3)$; the module $S^{(4,2)}$ appears in $\mathbf{C}P(2,3)$; and the module $S^{(3)}$ appears in $\mathbf{C}P(1,3)$.

Step 3 (*a lower bound for the multiplicity of S^μ in $\mathbf{C}P(k,3)$*). The final step is to use the semistandard tableaux constructed in Step 1 to show that the corresponding homomorphisms from S^μ to $FP(k,3)$ are linearly independent, hence giving a lower bound for the multiplicity of S^μ in $FP(k,3)$. Let $\{T_i^* : 0 \leq i \leq q\}$ and $\{T_i : 0 \leq i \leq q\}$ be the tableaux and semistandard tableaux, respectively, constructed in Step 1. We show that if

$j \in \{0, 1, \ldots, q\}$ with $j < i$ then $\overline{T}_i^*$ is involved in $\overline{\Theta}_{T_i}\{t\}\kappa_t$ but not in $\overline{\Theta}_{T_j}\{t\}\kappa_t$. This condition ensures that $\overline{\Theta}_{T_0}, \overline{\Theta}_{T_1}, \ldots, \overline{\Theta}_{T_q}$ are linearly independent and gives the lower bound $q + 1$ for the multiplicity of S^{μ} in $\mathbf{C}P(k, 3)$.

From Theorem 4.1 we see that if S^{μ} appears in $FP(3, k)$ then μ has one of two shapes:

Shape 1.

$$\begin{array}{l}
* \; * \cdots * \; * \;\; * \cdots * \;\; * \cdots * \\
* \; * \cdots * \; * \;\; \underbrace{* \cdots *}_{s} \\
\underbrace{* \; * \cdots * \; *}_{2u}
\end{array},$$

with u and s non-negative integers satisfying $3k \geq 6u + 2s$. Here S^{μ} has multiplicity $m_{k-2u, s}$:

Shape 2.

$$\begin{array}{l}
* \; * \cdots * \; * \; * \; * \; * \; * \;\; * \cdots * \;\; * \cdots * \\
* \; * \cdots * \; * \; * \; * \; * \; * \;\; \underbrace{* \cdots *}_{s} \\
\underbrace{* \; * \cdots * \; *}_{2u} \; *
\end{array},$$

with u and s non-negative integers satisfying $3k \geq 6u + 2s + 9$. Here S^{μ} has multiplicity $m_{k-2u-3, s}$.

We first deal with partitions of Shape 1:

THEOREM 5.2. *For* $3(k - 2u) \geq 2s$ *the multiplicity of* $S^{(3k-4u-s, s+2u, 2u)}$ *in the decomposition of* $\mathbf{C}P(k, 3)$ *is at least* $m_{k-2u, s}$.

Proof. The idea is to carry out the three steps described at the beginning of this section.

Step 1. We begin by considering the case when $s = 1$. Since $m_{k-2u,1} = 0$ for all values of k, there is nothing to prove. Therefore assume that $s \neq 1$. Then s can be written uniquely in the form $s = 6c + r$ for a non-negative integer c and $r \in \{0, 2, 3, 4, 5, 7\}$. When $k - 2u < s - 2c$ we have $m_{k-2u, s} = c - [(s - k + 2u + 1)/2] + 1 = 0$ and so we only need to consider the case when $k - 2u \geq s - 2c$. Write $c = e + f$ and $r = 2r_1 + 3r_2$ for $0 \leq e, f \leq c$, $r_1 \in \{0, 1, 2\}$ and $r_2 \in \{0, 1\}$. Note that there is a unique way of writing r in this form. To construct suitably sized sets of tableaux of Shape 1, we split this step into four cases to accommodate all possible values of u and s.

Case 1. If $u \neq 1$ then we write $2u$ (in a unique way) in the form $2u = 6u_1 + 4u_2$ where u_1 and u_2 are non-negative integers with $u_1 \in \{0, 1\}$. Denote by T_f^* the $(3k - 4u - s, s + 2u, 2u)$-tableau of type (3^k) given by

$$u_1 T^{*1} \vee u_2 T^{*2} \vee eT^{*7} \vee 3fT^{*9} \vee r_2 T^{*8} \vee r_1 T^{*9}$$
$$\vee (k - s - 2u + 2e) T^{*10}.$$

Since $k - 2u \geq s - 2c$ and $e \leq c$ we can always construct a tableau of this kind by taking e large enough so that $k - 2u \geq s - 2e$. Let t be the $(3k - s, s)$-tableau constructed from the tableaux t^1, t^2, t^7, t^8, t^9, and t^{10} as described at the beginning of this section and let T_f be the semistandard tableau row equivalent to T_f^*. The number of times T^{*9} appears in the expression for T_f^* depends on the size of $k - 2u$. If $k - 2u \geq s$ then f can take any value between 0 and c since $k - s - 2u + 2e$ will always be non-negative. If $k - 2u < s$ then $k - s - 2u + 2e = k - s - 2u + 2c - 2f$ will be non-negative for $0 \leq f \leq c - (s - k + 2u)/2$. Thus f can take $c + 1$ different values if $s \leq k - 2u$ and $c - [(s - k + 2u + 1)/2] + 1$ different values if $s > k - 2u$.

Case 2. If $u = 1$ and $k - 2 > s$ then let T_f^* be given by

$$T^{*3} \vee eT^{*7} \vee 3fT^{*9} \vee r_2 T^{*8} \vee r_1 T^{*9} \vee (k - s - 3 + 2e) T^{*10}.$$

Let t be the usual $(3k - 4u - s, s + 2u, 2u)$-tableau and let T_f be the semistandard tableau row equivalent to T_f^*. It is clear that f can take any value between 0 and c since $k - s - 3 + 2e = k - s - 3 + 2(c - f)$ is always non-negative for f in this range.

Case 3. $u = 1$ and $k - 2 \leq s$ with $c \geq 1$. For $f < c$ (that is, $e \neq 0$), let T_f^* be given by

$$T^{*4} \vee T^{*4} \vee (e - 1) T^{*7} \vee 3fT^{*9} \vee r_2 T^{*8} \vee r_1 T^{*9}$$
$$\vee (k - s - 2 + 2e) T^{*10}$$

and when $f = c$ (so $e = 0$), let T_f^* be given by

$$T^{*5} \vee T^{*9} \vee T^{*9} \vee 3(c - 1) T^{*9} \vee r_2 T^{*8} \vee r_1 T^{*9} \vee (k - s - 2) T^{*10}.$$

We require $k - s - 2 + 2e$ to be non-negative so f can take any value in the range $0 \leq f \leq c - (s - k + 2)/2$. Let t be the usual $(3(k - 2u) - s + 2u, s + 2u, 2u)$-tableau and let T_f be the semistandard tableau row equivalent to T_f^*.

Case 4. If $u = 1$ and $k - 2 \leq s$ with $c = 0$ then $r \neq 0$ (since $k \geq 3$). If $r_2 = 1$ then let

$$T_f^* = T^{*6} \vee r_1 T^{*9} \vee (k - r - 2) T^{*10}.$$

Otherwise, if $r_2 = 0$ then let

$$T_f^* = T^{*5} \vee (r_1 - 1) T^{*9} \vee (k - r - 2) T^{*10}.$$

By assumption $k - r - 2 = k - s - 2$ is non-zero and it is clear that f can only be zero. Let t be the usual $(3(k-2u) - s + 2u, s + 2u, 2u)$-tableau and let T_f be the semistandard tableau row equivalent to T_f^*.

Step 2. Since the tableaux given in Step 1 were constructed in the way described at the beginning of this section, Lemma 5.1 tells us that the homomorphisms $\overline{\Theta}_{T_f}$ are non-zero.

Step 3. For each T_f^* where f runs over the values in the appropriate range we need to show that the $\overline{\Theta}_{T_f}$ are linearly independent. To do this we show that if $i < f$ then $\overline{T}_f^*$ is not involved in $\overline{\Theta}_{T_i}\{t\}\kappa_t$. In the first two cases it can be seen, by counting pairs in the second row of T_i, that the coefficient of $\overline{T}_f^*$ in $\overline{\Theta}_{T_i}\{t\}\kappa_t$ is zero. In Case 3, when $f < c$ we can again count pairs in the second row of T_i and see that $\overline{T}_f^*$ is not involved in $\overline{\Theta}_{T_i}\{t\}\kappa_t$. To complete Case 3 we show that $\overline{T}_o^*, \overline{T}_1^*, \ldots, \overline{T}_{c-1}^*$ are not involved in $\overline{\Theta}_{T_c}\{t\}\kappa_t$. Since f can only be equal to c when $k = s + 2$, this is the only case which needs to be considered. We know that T_c has a pair of digits in its third row and so this pair will always be in the first two columns of an element involved in $\hat{\Theta}_{T_c}\{t\}\kappa_t$. However, the first two columns of $\overline{T}_i^*$ with $i < c$ have distinct entries. As $\overline{\Theta}_{T_c}\{t\}\kappa_t \neq 0$, this case is now complete. In Case 4 we know from Theorem 4.1 that $m_{k-2,r} \leq 1$ and since we have constructed one non-zero homomorphism from $S^{(3(k-2)-r+2, s+2, 2)}$ to $\mathbf{C}P(k, 3)$ there is nothing more to prove.

Comparing case by case the number of values f can take with the multiplicity $m_{k-2u,s}$ given in Theorem 4.1 shows that we have constructed the required number of linearly independent homomorphisms and the proof is complete. ∎

THEOREM 5.3. *For* $3(k - 2u - 3) \geq 2s$ *the multiplicity of* $S^{(3k-4u-s-5, s+2u+4, 2u+1)}$ *in the decomposition of* $\mathbf{C}P(k, 3)$ *is at least* $m_{k-2u-3,s}$.

Proof. This proof also follows the three steps outlined at the beginning of the section and is completely analogous to the proof of Theorem 5.2. For each T_f^* which we constructed in the last proof, we "replace" three of the T^{*10} by T^{*4}. Therefore f can take $c + 1$ different values if $k - 2u -$

$3 > s$ and $c - [(s - (k - 2u + 3) + 1)/2] + 1$ different values if $k - 3 - 2u \le s$. This is the number of linearly independent homomorphisms from $S^{(3(k-2u-3)-s+2u+4,\, s+2u+4,\, 2u+1)}$ to $\mathbf{C}P(k,3)$. ∎

The main theorem therefore follows from Theorems 4.1, 5.2, and 5.3.

REFERENCES

1. S. C. Black and R. J. List, A note on plethysm, *European J. Combin.* **10** (1989), 111–112.
2. M. Brion, Stable properties of plethysm: On two conjectures of Foulkes, *Manuscripta Math.* **80** (1993), 347–371.
3. A. Cayley, Calculation of the minimum N.G.F. of the binary seventhic, *Amer. J. Math.* **2** (1879), 71–84.
4. C. Coker, A problem related to Foulke's conjecture, *Graphs and Combin.* **9** (1993), 117–134.
5. S. C. Dent, "Incidence Structures of Partitions," Ph.D. thesis, School of Mathematics, UEA, Norwich, 1997.
6. H. O. Foulkes, Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form, *J. London Math. Soc.* **25** (1950), 205–209.
7. R. Howe, (GLN, GLM)-duality and symmetric plethysym, *Proc. Indian Acad. Sci. Math. Sci.* **97**, No. 1–3 (1987), 85–109.
8. G. D. James, "The Representation Theory of the Symmetric Groups," Springer-Verlag, Berlin, 1978.
9. G. D. James and A. Kerber, "The Representation Theory of the Symmetric Group," Encyclopedia of Mathematics and its Applications, Vol. 16, Addison-Wesley, Reading, MA, 1981.
10. A. Kerber, "Algebraic Combinatorics via Finite Group Actions," Wissenschaftsverlag, Mannheim, 1991.
11. D. E. Littlewood, Invariant theory, tensors and group characters, *Philos. Trans. Roy. Soc. London Ser. A* **293** (1944), 305–365.
12. L. Manivel, Gauss Maps and Plethysm, *Ann. Inst. Fourier* **47**, No. 3 (1997), 715.
13. J. B. Remmel, Combinatorial algorithms for the expansion of various products of Schur-functions, *Acta Appl. Math.* **21**, No. 1–2 (1990), 105–135.
14. N. J. A. Sloane, "A Handbook of Integer Sequences," Academic Press, New York, 1973.
15. R. P. Stanley, "Combinatorics and Commutative Algebra," Birkhäuser, Boston, 1983.
16. R. M. Thrall, On symmetrized Kronecker powers and the structure of the free Lie ring, *Amer. J. Math.* **64** (1942), 371–388.
17. A. Wagner, private communication with J. Siemons, 1987.