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On Modular Homology in the Boolean Algebra, III

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Let F be a field of characteristic p, and if Ω is an n-set let M^n be the vector space over F with basis 2^{Ω}. We continue our investigation of modular homological S_n-representations which arise from the r-step inclusion map. This is the FS_n-homomorphism ∂_r: $M^n \to M^n$ which sends a k-element subset $\Delta \subseteq \Omega$ onto the sum of all $(k-r)$-element subsets of Δ. Using homological methods one can give explicit character and dimension formulae. © 2001 Academic Press

1. INTRODUCTION

If F is a field, Ω is a set of size n, and $0 \leq k \leq n$ is an integer, let M_k^n be the vector space over F with the k-element subsets of Ω as basis. Then M_k^n is the natural FS_n permutation module for the symmetric group $S_n := \text{Sym}(\Omega)$ acting on the collection of all k-element subsets of Ω. For the integer $r > 0$ define the FS_n-homomorphism ∂_r: $M_k^n \to M_{k-r}^n$ on a basis as follows. If $\Delta \subseteq \Omega$ then

$$\partial_r(\Delta) := \sum \Gamma,$$

where the summation runs over all $\Gamma \subset \Delta$ of size $|\Delta| - r$. We refer to ∂_r as the *r-step inclusion map*. A simple computation shows that if F has characteristic $p > 0$ then $\partial_r^p \equiv 0$. For any $0 < i < p$ therefore the sequence

$$M_{k-ir}^n \xleftarrow{\partial_r^i} M_k^n \xleftarrow{\partial_r^{p-i}} M_{k+(p-i)r}^n$$

is homological. The corresponding homology module is denoted by

$$H_{k,i}^n := (\ker \partial_r^i \cap M_k^n)/\partial_r^{p-i}(M_{k+(p-i)r}^n).$$

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(To avoid confusion later note that in this notation r must be determined from the context.)

The first case to consider is $r = 1$ or, more generally, when r is a power of p. In [3] and [1] it was shown that for $r = p^j$ with j arbitrary one has

$$(*): \quad H^n_{k,i} = 0 \qquad \text{unless } n < 2k + (p - i)r < n + pr.$$

For fixed k and $0 < i < p$ consider therefore the sequence

$$\mathcal{M}: 0 \leftarrow \cdots \leftarrow M^n_{k-2pr} \leftarrow M^n_{k-(p+i)r} \leftarrow M^n_{k-pr} \leftarrow M^n_{k-ir} \leftarrow M^n_k \leftarrow \cdots \leftarrow 0,$$

in which each arrow is the appropriate power of ∂_r. As ∂_r^p is zero $\mathcal{M}$ is homological, and from $(*)$ it follows that there is at most one position in which $\mathcal{M}$ can fail to be exact. Any such sequence will be called *almost exact*.

For such almost exact sequences standard results from algebraic topology can be used to express the S_n-character on the nontrivial homology module in terms of the natural S_n-characters on the modules appearing in $\mathcal{M}$. In other words, the character on the nontrivial homology is a Lefschetz character.

In [1] this situation has been analyzed completely when $r = 1$: various irreducible S_n-representations can be realized in this fashion, and indeed whole inductive systems for symmetric groups arise in this way, for arbitrary p. In two recent papers [12, 13] it is shown that these modules play a fundamental role for the modular homology of simplicial complexes in general and for shellable complexes in particular. Our interest here is partly guided by the fact, that in the geometrical setting rank selected posets are important, and this leads to the consideration of r-step maps for $r > 1$. From the viewpoint of representation theory homological representations are interesting because in many situations the Hopf–Lefschetz trace formula provides explicit character and dimension formulae. Identifying representation as homological therefore is of general use. This is explained in more detail in Section 2.

The purpose of this paper is to make some progress toward determining the homology modules when $r > 1$ is a power of p. In Section 3 we consider the homology modules arising from the 2-step map in characteristic 2. It is shown that these are either irreducible, when n is odd (see Theorem 3.4), or otherwise have a unique factor of multiplicity two (see Theorem 3.10). Here we also have explicit matrix representations.

In Section 4 we deal with the r-step inclusion map when r is a p-power in general. Theorem 4.1 shows that $H^n_{k,i}$ is irreducible for $2k - ir + 1 = n$, generalizing Theorem 6.4 in[1].

In Section 5 we return to $p = 2$ with r a power of 2. In Theorem 5.1 all composition factors of $H^n_{k,1}$ with $2k - r + 2 = n$ are determined and in Corollary 5.2 we make some comments about branching rules.

We attempt to keep this paper as self-contained as possible. However, for some details it may be necessary to consult [1] directly.

2. CHARACTER FORMULAE

We begin by recollecting some standard facts from algebraic topology which are relevant for this paper. Such ideas can also be found in the work of S. Sundaram, such as [18]. We feel that these homological techniques are quite fundamental and deserve attention.

Let F be a field, let A_k for $0 \le k \le k_0 < \infty$ be finite-dimensional vector spaces over F, and suppose that $\delta\colon \oplus_k A_k \to \oplus_k A_k$ is a linear map with $\delta(A_{k+1}) \subseteq \ker\delta \cap A_k$ for all $0 \le k \le k_0$. Thus

$$\begin{aligned}\mathscr{A}\colon 0 \xleftarrow{\delta} A_0 \xleftarrow{\delta} \cdots \xleftarrow{\delta} A_{k-1} \xleftarrow{\delta} A_k \xleftarrow{\delta} A_{k+1}\\ \xleftarrow{\delta} \cdots \xleftarrow{\delta} A_{k_0} \xleftarrow{\delta} 0\end{aligned}$$

is a homological sequence, and we denote its homology modules by

$$H_k := (\ker\delta \cap A_k)/\delta(A_{k+1}).$$

For convenience we shall put $A_k = 0 = H_k$ if $k < 0$ or $k > k_0$.

If a group G acts linearly on each A_k and commutes with δ, then G also acts on each H_k. Furthermore, if trace$(g, *)$ denotes the trace of $g \in G$ on the module indicated by $*$ then the Hopf–Lefschetz trace formula says the following (see, for instance, [14]).

THEOREM 2.1. $\sum_{k\in\mathbb{Z}}(-1)^k \mathrm{trace}(g, H_k) = \sum_{k\in\mathbb{Z}}(-1)^k \mathrm{trace}(g, A_k)$.

Of interest to us is the situation where all but at most one of the homology modules of $\mathscr{A}$ are zero. If this happens then $\mathscr{A}$ is said to be *almost exact* and the nontrivial homology module is its *Lefschetz module*. Therefore,

COROLLARY 2.2. *If $\mathscr{A}$ is almost exact with Lefschetz module H_k then*

$$\mathrm{trace}(g, H_k) = \sum_{j\in\mathbb{Z}}(-1)^{k+j}\mathrm{trace}(g, A_{k+j}),$$

and

$$\dim H_k = \sum_{j\in\mathbb{Z}}(-1)^{k+j}\dim A_{k+j}$$

is the Euler characteristic of $\mathscr{A}$.

Furthermore, in our case F has characteristic $p > 0$ and each A_k is a permutation module. So here trace(g, A_k) is the number fix(g, A_k) of elements in the permutation set underlying A_k which are fixed by g, when evaluated in the field F. However, it is well known that the lift of a permutation character is unique, and therefore

$$\chi(g, A_k) := \mathrm{fix}(g, A_k) \qquad \text{for any } p'\text{-element } g \in G$$

is the Brauer character associated to trace(g, A_k).

We return more specifically to the sequences discussed in the Introduction. So we fix some $n > 0$, let $A_k = M_k^n$, $G = S_n$, and assume that r is some fixed power of p. Fix also some $0 < k^*$ and $0 < i^* < p$. For ease of reading write M_k instead of M_k^n and consider the sequence

$$\mathcal{M}\colon 0 \xleftarrow{\partial_r^*} M_0 \xleftarrow{\partial_r^*} \cdots \xleftarrow{\partial_r^*} M_{k^*-i^*r} \xleftarrow{\partial_r^*} M_{k^*}$$
$$\xleftarrow{\partial_r^*} M_{k^*+pr-i^*r} \xleftarrow{\partial_r^*} \cdots \xleftarrow{\partial_r^*} 0,$$

where ∂_r^* is $\partial_r^{i^*}$ or $\partial_r^{p-i^*}$ as appropriate. For any

$$k \in \{k^* + zpr - i^*r, k^* + zpr\colon z \in \mathbb{Z}\}$$
$$\text{and appropriate } i \in \{i^*, p - i^*\}$$

we may define the homology module $H_{k,i}^n$ as in the introduction. By Theorem 5.3 in [1] $\mathcal{M}$ is almost exact. To define the character on the Lefschetz module let

$$\mathrm{fix}(g, M_k^n) := |\{\, \Delta \subseteq \Omega : g\Delta = \Delta \text{ and } |\Delta| = k\}|$$

be the number of k-element subsets from Ω fixed by g and put

$$\beta(g, n, k, i) := \sum_{j\in\mathbb{Z}} \Big\{ \mathrm{fix}(g, M_{k+prj}^n) - \mathrm{fix}(g, M_{k+prj-ir}^n) \Big\}.$$

The main prerequisite of this paper is the following restatement of Theorem 5.3 in [1] and Corollary 2.2.

THEOREM 2.3. *$\mathcal{M}$ is almost exact. In particular,* $H_{k,i}^n = 0$ *unless* $n < 2k + (p - i)r < n + pr$. *In the latter case*

$$\chi(g, H_{k,i}^n) := \beta(g, n, k, i)$$

is the Brauer character of $H_{k,i}^n$. *In particular,* $H_{k,i}^n$ *has dimension* $\beta(\mathrm{id}, n, k, i) = \sum_{j\in\mathbb{Z}} \{\binom{n}{k+prj} - \binom{n}{k+prj-ir}\}$, *and this is the Euler characteristic of* $\mathcal{M}$.

We now list some elementary properties of β used later. The next proposition uses only the definition and the fact that $\mathrm{fix}(g, M_k^n) = \mathrm{fix}(g, M_{n-k}^n)$.

PROPOSITION 2.4. (a) $\beta(g, n, k, i) = -\beta(g, n, k - ir, p - i)$.

(b) *If* $k \equiv k^*$ (mod pr) *then* $\beta(g, n, k, i) = \beta(g, n, k^*, i)$.

(c) $\beta(g, n, k, i) = \beta(g, n, n - k, p - i)$.

(d) *If* $2k - ir \equiv n \bmod pr$ *then* $\beta(g, n, k, i) = 0$.

We note several nice inductive properties of β which help to evaluate characters and often yield dimension formulae in closed form; see Theorem 2.6 and Corollaries 4.4 and 5.3 later on.

PROPOSITION 2.5. (a) *If g is an n-cycle then* $\beta(g, n, k, i) = a + e$, *where*

$$a = \begin{cases} 1 & \text{if } k \equiv 0 \bmod pr \\ -1 & \text{if } k - ir \equiv 0 \bmod pr \\ 0 & \text{otherwise} \end{cases}$$

and

$$e = \begin{cases} 1 & \text{if } k \equiv n \bmod pr, \\ -1 & \text{if } k - ir \equiv n \bmod pr \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If* $g = g_1 g_2$, *where g_1 is a cycle of length b disjoint from g_2, then* $\beta(g, n, k, i) = \beta(g_2, n - b, k, i) + \beta(g_2, n - b, k - b, i)$.

Proof. An n-cycle fixes only Ω and $\varnothing$, and so (a) follows from the definition. For (b) observe that $\mathrm{fix}(g, M_k^n) = \mathrm{fix}(g_2, M_k^{n-b}) + \mathrm{fix}(g_2, M_{k-b}^{n-b})$. ∎

To identify homology modules in terms of the standard representations of S_n let λ be a partition of n. Then the Specht module corresponding to λ is denoted by S^λ, and as usual $D^\lambda := S^\lambda/(S^\lambda \cap S^{\lambda\perp})$. In Theorem 5.3 of [1] we have identified certain D^λ's as homology modules arising from the 1-step map:

THEOREM 2.6. *Let* $p > 2$, $r = 1$, *and* $0 < i < p$. *If* $n < 2k + p - i < n + p$ *then* $H_{k,i}^n$ *is irreducible if and only if* $2k + p - i = n + p - 1$. *If* $2k + p - i = n + p - 1$ *then* $H_{k,i}^n \cong D^\lambda$ *with* $\lambda = (k,\ k - i + 1)$ *and* $\dim H_{k,i}^n = \sum_{j\in\mathbb{Z}}\{\binom{n}{k+pj} - \binom{n}{k+pj-ir}\}$.

The expression for the dimension is a linear recurrence of degree at most $(p-1)/2$, and so one may attempt to produce a nice closed expression for it. For instance, if $p = 5$ this results in Fibonacci numbers, and the modules are the ones described in Ryba's paper [15].

Also for $r > 1$ the composition factors of $H_{k,i}^n$ are indexed by two-part partitions of n. In fact, we shall represent several new irreducible modules as homology modules (see Theorems 3.4 and 4.1).

In all cases where D^λ is identified as a Lefschetz module the Theorem 2.3 gives the dimension as well as its Brauer character. It is often also possible to construct the representation explicitly (see Section 3), and in this respect the homological methods turn out to be very efficient.

In Erdmann's paper [5] dimensions of representations labeled by two-part partitions are given in terms of generating functions and certain Chebyshev polynomials. As these can be evaluated for all two-part partitions, her treatment in this respect is more general. Another method for computing dimensions for two-part partitions would be to derive these from the decomposition numbers of James' papers [8, 9].

To keep this paper as self-contained as possible we shall go through some facts from [1] which are to be used here without further mention. For more detail be advised to consult [1] directly.

Throughout let $M^n := \oplus_k M_k^n$, where M_k^n is as in the Introduction. If $r > 0$ is an integer let $\partial_r \colon M \to M$ be as before. If, $i, s > 0$ are integers, then

$$\partial_r \partial_s = \binom{r+s}{r} \partial_{r+s} \quad \text{and in particular} \quad \partial_r^i = \binom{ir}{r} \cdots \binom{2r}{r} \binom{r}{r} \partial_{ir}.$$

If $f = \sum_\Delta f_\Delta \Delta$ and $h = \sum_\Gamma h_\Gamma \Gamma$, with coefficients f_Δ, h_Γ in F, belong to M, define

$$f \cup h := \sum_{\Delta, \Gamma} f_\Delta h_\Gamma (\Delta \cup \Gamma).$$

This turns M into an FS_n-algebra with the empty set as identity element. We say that f and h are *disjoint* if $f_\Delta \neq 0 \neq h_\Gamma$ implies $\Delta \cap \Gamma = \emptyset$. Fundamental is the formula

$$\partial_r (f \cup h) = \sum_{j=0}^{r} \partial_j (f) \cup \partial_{r-j} (h)$$

which holds for disjoint f, $g \in M$. Also, if $f \in M$ and $\alpha \in \Omega$ are arbitrary then f can be written uniquely as

$$f = \{\alpha\} \cup f_1 + f_2$$

with f_1 and f_2 disjoint from α.

3. THE 2-STEP INCLUSION MAP

Throughout this section let F have characteristic $p = 2$ and let $r = 2$. Hence $i = 1$ and for convenience we abbreviate $H_k^n := H_{k,1}^n$ and $K_k^n := \ker \partial_2 \cap M_k^n$. Our aim is to analyze H_k^n. Some of the results that follow are not new and are contained in [3] or may be deduced from Gow's paper [6] by considering the restriction to S_n of an appropriate symplectic representation. Here we follow a different approach which leads us to determine a basis for H_k^n and hence to an explicit matrix representation.

THEOREM 3.1. *For $2 < n$ and $k \leq n$ there is an FS_{n-2}-isomorphism $H_k^n \cong H_{k-1}^{n-2} \oplus H_{k-1}^{n-2}$.*

Proof. Let $\Omega = \{\alpha_1, \ldots, \alpha_{n-1}, \alpha_n\}$ and take $f \in M^n$. Then there are unique elements $f_0, f_1, f_2, f_3 \in M^{n-2}$ disjoint from $\{\alpha_1, \alpha_2\}$ with

$$f = \{\alpha_n, \alpha_{n-1}\} \cup f_0 + \{\alpha_n\} \cup f_1 + \{\alpha_{n-1}\} \cup f_2 + f_3,$$

and by Lemma 2.1 of [1] we have

$$\begin{aligned} \partial_2(f)_0 &= \partial_2(f_0) \\ \partial_2(f)_1 &= \partial_1(f_0) + \partial_2(f_1) \\ \partial_2(f)_2 &= \partial_1(f_0) + \partial_2(f_2) \\ \partial_2(f)_3 &= f_0 + \partial_1(f_1 + f_2) + \partial_2(f_3). \end{aligned} \tag{1}$$

If $f \in K_k^n$ therefore,

$$\begin{aligned} 0 &= \partial_1(f_0 + \partial_1(f_1 + f_2) + \partial_2(f_3)) \\ &= \partial_1(f_0) + \partial_2\partial_1(f_3) \\ &= \partial_2(f_1 + \partial_1(f_3)) \\ &= \partial_2(f_2 + \partial_1(f_3)), \end{aligned}$$

and defining $\varphi(f) := ([f_1 + \partial_1(f_3)], [f_2 + \partial_1(f_3)])$ yields a map $\varphi\colon K_k^n \to H_{k-1}^{n-2} \oplus H_{k-1}^{n-2}$. Clearly, this is an FS_{n-2}-homomorphism. Furthermore, if $h \in M_{k+2}^n$ then

$$\begin{aligned} \varphi(\partial_2(h)) &= ([\partial_2(h)_1 + \partial_1(\partial_2(h)_3)], [\partial_2(h)_2 + \partial_1(\partial_2(h)_3)]) \\ &= ([\partial_2(h_1 + \partial_1(h_3))], [\partial_2(h_2 + \partial_1(h_3))]) \end{aligned}$$

shows that $\varphi\colon H_k^n \to H_{k-1}^{n-2} \oplus H_{k-1}^{n-2}$ induces an FS_{n-2}-homomorphism between homologies.

To show that φ is injective suppose that $\varphi([f]) = ([0], [0])$. Then there are $x_i \in M_{k+1}^{n-2}$ for $i = 1, 2$ with $\partial_2(x_i) = f_i + \partial_1(f_3)$. If we set

$$x := \{\alpha_n, \alpha_{n-1}\} \cup (f_3 + \partial_1(x_1 + x_2)) + \{\alpha_n\} \cup x_1 + \{\alpha_{n-1}\} \cup x_2$$

then $x \in M_{k+2}^n$ and

$$\begin{aligned} \partial_2(x)_0 &= \partial_2(f_3 + \partial_1(x_1 + x_2)) \\ \partial_2(x)_1 &= \partial_1(f_3) + \partial_2(x_1) \\ \partial_2(x)_2 &= \partial_1(f_3) + \partial_2(x_2) \\ \partial_2(x)_3 &= f_3. \end{aligned}$$

Substituting for $\partial_2(x_i)$ and applying (1) we have $\partial_2(x) = f$. To show that φ is surjective let $j_1, j_2 \in K_{k-1}^{n-2}$. Setting $h := \{\alpha_n, \alpha_{n-1}\} \cup \partial_1(j_1 + j_2) + \{\alpha_n\} \cup j_1 + \{\alpha_{n-1}\} \cup j_2$ we observe that $h \in K_k^n$ with $\varphi([h]) = ([j_1], [j_2])$. ∎

By Theorem 3.2 of [1] we know that the only nontrivial homology modules are

$$H_k^n \qquad \text{when } n = 2k \text{ is even, and}$$

$$H_k^n \quad \text{and} \quad H_{k-1}^n \qquad \text{when } n = 2k-1 \text{ is odd.}$$

This may also be seen by induction directly from the preceding theorem. The characters are given in Section 2, and we see from Proposition 2.4(c) that the characters for H_k^n and H_{k-1}^n coincide if n is odd.

An alternative description of these characters is the following. Let $\lfloor * \rfloor$: {odd integers} $\to \{\pm 1\}$ be the function

$$\lfloor z \rfloor := \begin{cases} 1 & \text{if } z \equiv 1, 7 \pmod 8 \\ -1 & \text{if } z \equiv 3, 5 \pmod 8 \end{cases}.$$

THEOREM 3.2. (a) *If* $n = 2k-1$ *let* $g \in S_n$ *have odd order and cycle type* $(b_1, \ldots, b_{2l-1})$. *Then*

$$\chi(g, H_k^n) = 2^{l-1} \cdot \lfloor b_1 \rfloor \cdot \lfloor b_2 \rfloor \cdots \lfloor b_{2l-1} \rfloor.$$

In particular, $\dim H_k^n = 2^{k-1}$.

(b) *If* $n = 2k$ *let* $g \in S_n$ *have odd order and cycle type* $(b_1, \ldots, b_{2l})$. *Then*

$$\chi(g, H_k^n) = 2^{l} \cdot \lfloor b_1 \rfloor \cdot \lfloor b_2 \rfloor \cdots \lfloor b_{2l} \rfloor.$$

In particular, $\dim H_k^n = 2^{k}$.

Proof. The result holds when $n = 1, 2$. If $n > 2$ write $g = g_1 g_2$, where g_1 is a b-cycle disjoint from $g_2 \in S_{n-b}$ with b odd.

(a) If $n = 2k-1$ then $\beta(g, n, k, 1) = \beta(g_2, n-b, k, 1) + \beta(g_2, n-b, k-b, 1)$ by Proposition 2.5 with $n - b = 2(k - (b+1)/2)$. Furthermore,

$$b \equiv 1 \pmod 8 \iff k - b \equiv k - (b+1)/2 \pmod 4 \tag{2}$$

$$b \equiv 3 \pmod 8 \iff k - 2 \equiv k - (b+1)/2 \pmod 4 \tag{3}$$

$$b \equiv 5 \pmod 8 \iff k - b - 2 \equiv k - (b+1)/2 \pmod 4 \tag{4}$$

$$b \equiv 7 \pmod 8 \iff k \equiv k - (b+1)/2 \pmod 4 \tag{5}$$

and

$$b \equiv 1 \pmod 4 \iff 2k - 2 \equiv n - b \pmod 4$$

$$b \equiv 3 \pmod 4 \iff 2(k-b) - 2 \equiv n - b \pmod 4.$$

By Theorem 2.3, therefore, $\chi(g, H_k^n) = \lfloor b \rfloor \cdot \chi(g_2, H_{k-(b+1)/2}^{n-b})$.

(b) If $n = 2k$ then $\beta(g, n, k, 1) = \beta(g_2, n-b, k, 1) + \beta(g_2, n-b, k-b, 1)$ with $n - b = 2(k-(b-1)/2) - 1 = 2(k-(b+1)/2)+1$. In addition to the congruences (2)–(5) we have

$$b \equiv 1 \pmod 8 \iff k \equiv k-(b-1)/2 \pmod 4$$
$$b \equiv 3 \pmod 8 \iff k-b-2 \equiv k-(b-1)/2 \pmod 4$$
$$b \equiv 5 \pmod 8 \iff k-2 \equiv k-(b-1)/2 \pmod 4$$
$$b \equiv 7 \pmod 8 \iff k-b \equiv k-(b-1)/2 \pmod 4).$$

By Theorem 2.3, therefore, $\chi(g, H^n_k) = \lfloor b \rfloor \cdot \{\chi(g_2, H^{n-b}_{k-(b+1)/2}) + \chi(g_2, H^{n-b}_{k-(b-1)/2})\} = \lfloor b \rfloor \cdot 2 \cdot \chi(g_2, H^{n-b}_{k-(b-1)/2})$. ∎

To examine H^n_k in detail we need to distinguish between n odd and n even. First let $n = 2k-1$ and $\Omega = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Set

$$v^n_0 := (\{\alpha_1\} + \{\alpha_2\}) \cup (\{\alpha_3\} + \{\alpha_4\}) \cup \cdots \cup (\{\alpha_{n-2}\} + \{\alpha_{n-1}\}) \cup \{\alpha_n\}$$

and verify that $v^n_0 \in K^n_k$. For $0 \le l < 2^{k-1}$ with the 2-adic expansion $l = \sum_{j=0}^{k-2} l_j 2^j$ we define

$$v^n_l := (23)^{l_0}(45)^{l_1} \cdots (n-3, n-2)^{l_{k-3}}(n-1, n)^{l_{k-2}}(v^n_0).$$

THEOREM 3.3. *If $n = 2k-1$ then $\{[v^n_l] : 0 \le l < 2^{k-1}\}$ is a basis of H^n_k.*

Proof. As the result holds for $n = k = 1$ we suppose $n > 1$ and that

$$\{\varphi^{-1}([v^{n-2}_l], [0]) : 0 \le l < 2^{k-2}\} \cup \{\varphi^{-1}([0], [v^{n-2}_l]) : 0 \le l < 2^{k-2}\}$$

is a basis for H^n_k. Here φ is the isomorphism of Theorem 3.1. Then we have

$$\begin{aligned}\varphi^{-1}([v^{n-2}_0], [0]) &= [\{\alpha_n, \alpha_{n-1}\} \cup \partial_1(v^{n-2}_0) + \{\alpha_n\} \cup v^{n-2}_0]\\ &= [\partial_1(v^{n-2}_0) \cup (\{\alpha_{n-2}\} + \{\alpha_{n-1}\}) \cup \{\alpha_n\}]\\ &= [v^n_0]\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}([0], [v^{n-2}_0]) &= [\{\alpha_n, \alpha_{n-1}\} \cup \partial_1(v^{n-2}_0) + \{\alpha_{n-1}\} \cup v^{n-2}_0]\\ &= [\partial_1(v^{n-2}_0) \cup (\{\alpha_{n-2}\} + \{\alpha_n\}) \cup \{\alpha_{n-1}\}]\\ &= (n-1, n)[v^n_0].\end{aligned}$$

Therefore

$$[v^n_l] = \begin{cases} \varphi^{-1}([v^{n-2}_l], [0]) & \text{for } 0 \le l < 2^{k-2} \\ \varphi^{-1}([0], [v^{n-2}_l]) & \text{for } 2^{k-2} \le l < 2^{k-1}, \end{cases}$$

and this completes the proof. ∎

We recall that the characters for H_k^n and H_{k-1}^n coincide for $n = 2k - 1$. The next result therefore shows that for odd n both homology modules are irreducible.

THEOREM 3.4. *If $n = 2k - 1$ then H_k^n is irreducible and $H_k^n \cong D^{(k,k-1)}$.*

Proof. The result holds when $n = 1$, and for $n > 1$ let $U \neq 0$ be a submodule of H_k^n. Let $0 \neq ([f], [h])$ be in $\varphi(U)$. Then we may assume $[f] = [h]$, for otherwise

$$\begin{aligned}([f],[h]) + \varphi((n-1,n)\varphi^{-1}([f],[h])) &= ([f],[h]) + ([h],[f]) \\ &= ([f+h],[f+h])\end{aligned}$$

is a nonzero element of $\varphi(U)$. Thus $([f],[f]) \neq 0$ implies that $\langle [f] \rangle$ is a nonzero FS_{n-2}-submodule of H_{k-1}^{n-2}, and by induction we may assume $\langle [f] \rangle = H_{k-1}^{n-2}$. In particular, $([v_0^{n-2}], [v_0^{n-2}]) \in \langle ([f],[f]) \rangle$, and so

$$\begin{aligned}\varphi^{-1}([v_0^{n-2}],[v_0^{n-2}]) &= [v_0^n] + (n-1,n)[v_0^n] \\ &= (n-2,n)[v_0^n]\end{aligned}$$

belongs to U. By Theorem 3.3 the FS_n-span of this coset is H_k^n, and hence $U = H_n^k$.

To identify this module in terms of the standard representations of S_n note that the FS_n-span of v_0^n is the Specht module $S^{(k,k-1)}$. By Theorem 3.3

$$H_k^n = (\langle v_0^n \rangle + \partial_2(M_{k+2}^n))/\partial_2(M_{k+2}^n) \cong \langle v_0^n \rangle / (\langle v_0^n \rangle \cap \partial_2(M_{k+2}^n)),$$

and as $D^{(k,k-1)}$ is the unique top composition factor of $S^{(k,k-1)}$ we see that H_k^n must be isomorphic to this module. ∎

From this identification and Theorems 3.2 and 3.1 we obtain the following two corollaries.

COROLLARY 3.5. *If $g \in S_{2k-1}$ has odd order and cycle type $(b_1, \ldots, b_{2l-1})$ then*

$$\chi(g, D^{(k,k-1)}) = 2^{l-1} \cdot \lfloor b_1 \rfloor \cdot \lfloor b_2 \rfloor \cdots \lfloor b_{2l-1} \rfloor,$$

and in particular $\dim D^{(k,k-1)} = 2^{k-1}$.

Remark. In Theorem 5.1 of [2] it is shown that the restriction (mod 2) of the basic spin module for the double cover of S_n is $D^{(k,k-1)}$. Hence it is also possible (yet much less direct) to obtain these character formulae using results from [17] together with Theorem 2 of [11].

COROLLARY 3.6. *For all integers k there exists an FS_{2k-1}-isomorphism*

$$D^{(k+1,k)} \cong D^{(k,k-1)} \oplus D^{(k,k-1)}.$$

Let ρ_n now denote the matrix representation of S_n corresponding to the basis of H_k^n given in Theorem 3.3. To provide an explicit description of ρ_n it suffices to compute the images of all transpositions of the form $(j, j+1)$ since these generate S_n.

EXAMPLE. Let $\Omega = \{\alpha_1, \alpha_2, \alpha_3\}$. Then H_2^3 has (ordered) basis $[(\{\alpha_1\} + \{\alpha_2\}) \cup \{\alpha_3\}]$ and $[(\{\alpha_1\} + \{\alpha_3\}) \cup \{\alpha_2\}]$. Therefore

$$\rho_3(12) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \rho_3(23) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

More generally, we prove

LEMMA 3.7. *If $n = 2k-1$ and $0 < j < n-2$ then*

$$\rho_n(j, j+1) = \begin{pmatrix} \rho_{n-2}(j, j+1) & 0 \\ 0 & \rho_{n-2}(j, j+1) \end{pmatrix}.$$

Proof. In the proof of Theorem 3.3 we showed that

$$[v_l^n] = \begin{cases} \varphi^{-1}([v_l^{n-2}], [0]) & \text{for } 0 \le l < 2^{k-2} \\ \varphi^{-1}([0], [v_l^{n-2}]) & \text{for } 2^{k-2} \le l < 2^{k-1}, \end{cases}$$

which implies this result. ∎

LEMMA 3.8. *If $n = 2k-1$ then*

$$\rho_n(n-1, n) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where 1 *and* 0 *denote the* $2^{k-2} \times 2^{k-2}$ *identity and zero matrices*, *respectively.*

Proof. For $0 \le l < 2^{k-2}$ we have $v_{l+2^{k-2}}^n = (n-1, n)(v_l^n)$. ∎

LEMMA 3.9. *If $n = 2k-1 \ge 5$ then*

$$\rho_n(n-2, n-1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

where 1 *and* 0 *denote the* $2^{k-3} \times 2^{k-3}$ *identity and zero matrices*, *respectively.*

Proof. (1) For $0 \le l < 2^{k-3}$ we have $v_l^n = \partial_1(v_l^{n-2}) \cup (\{\alpha_{n-2}\} + \{\alpha_{n-1}\}) \cup \{\alpha_n\}$, and such vectors are clearly fixed by the transposition $(n-2, n-1)$.

(2) For $2^{k-3} \leq l < 2^{k-2}$ we have

$$v^n_{l-2^{k-3}} = \partial_1(v^{n-4}_{l-2^{k-3}}) \cup (\{\alpha\} + \{\alpha_{n-3}\}) \cup (\{\alpha_{n-2}\} + \{\alpha_{n-1}\}) \cup \{\alpha_n\}$$
$$v^n_l = \partial_1(v^{n-4}_{l-2^{k-3}}) \cup (\{\alpha\} + \{\alpha_{n-2}\}) \cup (\{\alpha_{n-3}\} + \{\alpha_{n-1}\}) \cup \{\alpha_n\},$$

where α is given by $v^{n-4}_{l-2^{k-3}} = \partial_1(v^{n-4}_{l-2^{k-3}}) \cup \{\alpha\}$. Therefore

$$(n-2, n-1)(v^n_l) = v^n_{l-2^{k-3}} + v^n_l.$$

(3) For $2^{k-2} \leq l < 3 \cdot 2^{k-3}$ we have

$$v^n_{l-2^{k-2}} = \partial_1(v^{n-2}_{l-2^{k-2}}) \cup (\{\alpha_{n-2}\} + \{\alpha_{n-1}\}) \cup \{\alpha_n\}$$
$$v^n_l = \partial_1(v^{n-2}_{l-2^{k-2}}) \cup (\{\alpha_{n-2}\} + \{\alpha_n\}) \cup \{\alpha_{n-1}\}.$$

Therefore

$$(n-2, n-1)(v^n_l) = v^n_{l-2^{k-2}} + v^n_l.$$

(4) For $3 \cdot 2^{k-3} \leq l < 2^{k-1}$ we have

$$v^n_{l-2^{k-2}} = \partial_1(v^{n-4}_{l-3\cdot 2^{k-3}}) \cup (\{\alpha\} + \{\alpha_{n-2}\}) \cup (\{\alpha_{n-3}\} + \{\alpha_{n-1}\}) \cup \{\alpha_n\}$$
$$v^n_{l-2^{k-3}} = \partial_1(v^{n-4}_{l-3\cdot 2^{k-3}}) \cup (\{\alpha\} + \{\alpha_{n-3}\}) \cup (\{\alpha_{n-2}\} + \{\alpha_n\}) \cup \{\alpha_{n-1}\}$$
$$v^n_l = \partial_1(v^{n-4}_{l-3\cdot 2^{k-3}}) \cup (\{\alpha\} + \{\alpha_{n-2}\}) \cup (\{\alpha_{n-3}\} + \{\alpha_n\}) \cup \{\alpha_{n-1}\},$$

where α is given by $v^{n-4}_{l-3\cdot 2^{k-3}} = \partial_1(v^{n-4}_{l-3\cdot 2^{k-3}}) \cup \{\alpha\}$. Therefore

$$(n-2, n-1)(v^n_l) = v^n_{l-2^{k-2}} + v^n_{l-2^{k-3}} + v^n_l$$
$$+ \partial_2(v^{n-4}_{l-3\cdot 2^{k-3}} \cup \{\alpha_{n-3}, \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}).$$

This completes the proof. ∎

We shall now analyze H^n_k when n is even.

THEOREM 3.10. *If $n = 2k$ then H^n_k has a unique composition factor of multiplicity two, and this factor is $D^{(k+1,\, k-1)}$.*

Proof. By Theorem 3.2 the restriction to FS_n of H^{n+1}_{k+1} has composition factors coinciding with those of H^n_k. By Lemma 3.7 and Lemma 3.9 this restriction has matrix representation ρ_{n+1} given as follows. For $0 < j < n-1$ we have

$$\rho_{n+1}(j, j+1) = \begin{pmatrix} \rho_{n-1}(j, j+1) & 0 \\ 0 & \rho_{n-1}(j, j+1) \end{pmatrix} \quad \text{and}$$

$$\rho_{n+1}(n-1, n) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

where 1 and 0 are the $2^{k-1} \times 2^{k-1}$ identity and zero matrices, respectively. Observe that the vectors whose first 2^{k-1} entries are zero form an FS_n-submodule $U \subseteq H_{k+1}^{n+1}$ for which $H_{k+1}^{n+1}/U \cong U$. By Theorems 3.1 and 3.4 the restriction to FS_{n-1} of H_{k+1}^{n+1} has a unique composition factor of multiplicity two. So U is irreducible.

To identify the composition factors of H_k^n in terms of standard representations note that the span of

$$u^n := (\{\alpha_1\} + \{\alpha_2\}) \cup (\{\alpha_3\} + \{\alpha_4\}) \cup \cdots \cup (\{\alpha_{n-3}\} + \{\alpha_{n-2}\}) \cup \{\alpha_{n-1}, \alpha_n\}$$

is the Specht module $S^{(k+1,\, k-1)}$. Observe that $\partial_1(u^n) \in K_k^n$ satisfies

$$[\partial_1(u^n)] = \varphi^{-1}([\partial_1(u^{n-2})], [\partial_1(u^{n-2})]),$$

and by induction this coset is nonzero in H_k^n. Therefore H_k^n contains a submodule isomorphic to a quotient of $S^{(k+1,\, k-1)}$. Since $D^{(k+1,\, k-1)}$ is the unique top composition factor of this Specht module it must also be the repeated factor of H_k^n. ∎

COROLLARY 3.11. *If $g \in S_{2k}$ has odd order and cycle type $(b_1, b_2, \ldots, b_{2l})$ then*

$$\chi(g, D^{(k+1,\, k-1)}) = 2^{l-1} \cdot \lfloor b_1 \rfloor \cdot \lfloor b_2 \rfloor \cdots \lfloor b_{2l} \rfloor,$$

and in particular $\dim D^{(k+1,\, k-1)} = 2^{l-1}$.

Remark. This character formula can also be obtained from [17] together with Theorem 2 of [11].

COROLLARY 3.12. *The matrix representation of S_{2k} on $D^{(k+1,\, k-1)}$ is given by*

$$\rho_{2k}(j, j+1) = \rho_{2k-1}(j, j+1)$$

for $0 < j < 2k - 1$ *and*

$$\rho_{2k}(2k-1, 2k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(*Here* 1 *and* 0 *are* $2^{k-2} \times 2^{k-2}$ *identity and zero matrices*, *respectively.*)

Proof. These matrices represent the action of S_{2k} on the module U defined in the proof of Theorem 3.10. ∎

4. THE r-STEP INCLUSION MAP

In this section let F be a field of characteristic $p > 0$. The first result generalizes Theorem 6.4 of [1] and Theorem 3.4 of Section 3.

THEOREM 4.1. *If r is a power of p, $0 < i < p$, and $0 \leq k \leq n$ satisfies $2k - ir + 1 = n$ then $H^n_{k,i} \cong D^{(k, k-ir+1)}$.*

Proof. In this proof we abbreviate $S^{(k)} := S^{(k, n-k)}$ and $D^{(k)} := D^{(k, n-k)}$ for convenience. The result holds when $n = k = ir - 1$. For $n > k$ we use Theorem 2.3 to write

$$\begin{aligned}\chi(g, H^n_{k,i}) &= \sum_{z \in Z} \{\chi(g, M^n_{k+pzr}) - \chi(g, M^n_{k+(pz-i)r})\} \\ &= \sum_{z \geq 0} \{\chi(g, M^n_{k+pzr}) - \chi(g, M^n_{k-(pz+i)r})\} \\ &\quad - \sum_{z > 0} \{\chi(g, M^n_{k+(pz-i)r}) - \chi(g, M^n_{k-pzr})\}.\end{aligned}$$

As $M^n_{k-(pz+i)r} \cong M^n_{k+pzr+1}$ for all $z \geq 0$ Example 17.17 of [7] shows that

$$\chi(g, M^n_{k+pzr}) - \chi(g, M^n_{k-(pz+i)r}) = \chi(g, S^{(k+pzr)}),$$

and similarly

$$\chi(g, M^n_{k+(pz-i)r}) - \chi(g, M^n_{k-pzr}) = \chi(g, S^{(k+(pz-i)r)})$$

for all $z > 0$. Therefore

$$\chi(g, H^n_{k,i}) = \sum_{z \geq 0} \{\chi(g, S^{(k+pzr)}) - \chi(g, S^{(k+(p(z+1)-i)r)})\}. \tag{6}$$

In Corollary 5.4 and Lemma 5.5 of [1] we have shown that $H^n_{k,i}$ is isomorphic to a quotient of $S^{(k)}$. Suppose for a contradiction that $D^{(n)}$ is a factor of $H^n_{k,i}$. Then by Theorem 24.15 of [7]

$$f_p(2k - ir + 1, k - ir + 1) = 1,$$

where f_p is the function defined in Chapter 24 of James' book. More generally, suppose that $D^{(n)}$ is a factor of some $S^{(k+pzr)}$. Then consider the two p-adic expansions

$$\begin{aligned}2k - ir + 2 &= a_0 + a_1 p + \cdots + a_s p^s + \cdots + a_t p^t \\ k - (pz + i)r + 1 &= b_0 + b_1 p + \cdots + b_s p^s,\end{aligned}$$

from which, setting $c_j := a_j - b_j$, we obtain

$$k + pzr + 1 = c_0 + c_1 p + \cdots + c_s p^s + \cdots + c_t p^t.$$

If $f_p(2k-ir+1,\, k-pzr-ir+1)=1$ one can check easily that $f_p(2k-ir+1, k-(a_tp^t-pzr)+1)=1$, and from Theorem 24.15 of [7] we see that $D^{(n)}$ is a composition factor of $S^{(k+(a_tp^t-pzr)-ir)}$. However, since $pzr \le k-ir+1 < a_tp^t$, the latter appears as a summand of $\chi(g, H^n_{k,i})$ with negative coefficient, unless $z=0$ and $p^t \le r$. In that case we compare the p-adic expansions of $k+1$ and $k-ir+1$ and see that $b_j=c_j=0$ for $0\le j<t$ with $a_tp^t=ir$. This forces $n=ir-1$, a contradiction.

To complete the proof, let j be some positive integer. By Theorem 24.15 of [7] and (6) the multiplicity of $D^{(j,n-j)}$ as a factor of $H^n_{k,i}$ is that of $D^{(j-1,(n-2)-(j-1))}$ as a factor of $H^{n-2}_{k-1,i}$. By induction this multiplicity is given by 1 if $(j-1)=(k-1)$ and by 0 otherwise. ∎

COROLLARY 4.2. *If r is a power of p, $0<i<p$, and if $0\le k\le n$ satisfies $2k-ir+1=n$ then*

$$\chi(g, D^{(k,k-ir+1)}) = \sum_{z\in Z}\{\mathrm{fix}(g, M^n_{k+pzr}) - \mathrm{fix}(g, M^n_{k+(pz-i)r})\}$$

for all p'-elements g in S_n. In particular,

$$\dim D^{(k,k-ir+1)}) = \sum_{z\in Z}\left\{\binom{n}{k+pzr} - \binom{n}{k+(pz-i)r}\right\}.$$

Proof. The result follows from Theorem 2.3 and Theorem 4.1. ∎

The character of an n-cycle is particularly simple to evaluate, as can be seen from Proposition 2.5.

COROLLARY 4.3. *Let r be a power of p and let $0<i<p$. If $0\le k\le n$ satisfies $2k-ir+1=n$ and if n is coprime to p then*

$$\chi((12\cdots n), D^{(k,k-ir+1)}) = \begin{cases} 1 & \text{if } n\equiv \pm(ir-1) \pmod{pr} \\ -1 & \text{if } n\equiv \pm(ir+1) \pmod{pr} \\ 0 & \text{otherwise.}\end{cases}$$

We also have the following closed-dimension formula:

COROLLARY 4.4. *For $p=2$ and arbitrary k we have*

$$\dim D^{(k,k-3)} = \frac{1}{2\sqrt{2}}\left\{\left(2+\sqrt{2}\right)^{k-2} - \left(2-\sqrt{2}\right)^{k-2}\right\}.$$

Proof. Let $f(k) := \dim D^{(k,k-3)}$. From Proposition 2.5 and Corollary 4.2 we see that

$$f(k)-4f(k-1)+2f(k-2)=0.$$

The result follows by solving this difference equation, subject to the boundary conditions $f(2)=0$ and $f(3)=1$. ∎

Remark. As mentioned in Section 2, this result can also be derived from Erdmann's paper [5] or from the decomposition numbers of James' papers [8, 9]. The same applies for Corollary 5.3.

5. SOME BRANCHING RULES

In this section F has characteristic 2; the notation is the same as in Section 3.

THEOREM 5.1. *If* $0 \le k \le n$ *and* $r = 2^d > 2$ *satisfy* $2k - r + 2 = n$ *then* H_k^n *has composition factors*

$$D^{(k,\,k-r+2)} \qquad \text{with multiplicity one and}$$

$$D^{(k+2^l,\,k-r+2-2^l)} : 0 \le l < d \qquad \text{each with multiplicity two.}$$

Proof. As in the proof of Theorem 4.1 we put $S^{(k)} := S^{(k,\,n-k)}$ and $D^{(k)} := D^{(k,\,n-k)}$. By Theorem 2.3 we have

$$\begin{aligned}\chi(g, H_k^n) &= \sum_{z \in Z} \{\chi(g, M_{k+2zr}^n) - \chi(g, M_{k+(2z-1)r}^n)\} \\ &= \sum_{z \ge 0} \{\chi(g, M_{k+2zr}^n) - \chi(g, M_{k-(2z+1)r}^n)\} \\ &\quad - \sum_{z > 0} \{\chi(g, M_{k+(2z-1)r}^n) - \chi(g, M_{k+2zr}^n)\}.\end{aligned}$$

By arguments similar to those in the proof of Theorem 4.1 we therefore obtain

$$\chi(g, H_k^n) = \sum_{z \ge 0} (-1)^z \{\chi(g, S^{(k+zr)}) + \chi(g, S^{(k+zr+1)})\}.$$

It is easy to see that all composition factors of H_k^n are of the form $D^{(k+j)}$ with $j \ge 0$, and by Theorem 24.15 of [7] the multiplicity of this module is

$$\sum_{z \ge 0} (-1)^z \{f_2(2j + r - 2, j - zr) + f_2(2j + r - 2, j - (zr + 1))\}.$$

In particular, $D^{(k)}$ has multiplicity one.

First suppose that $j \ge r$. Then we have the 2-adic expansion

$$2j + r - 1 = 1 + a_1 2 + a_2 4 + \cdots + 2^N$$

with $N > d$, and since $2j + r - 1 = (j - l) + (j + r + l - 1)$ we have $f_2(2j + r - 2, j - l) = 1$ if and only if $f_2(2j + r - 2, j - ((2^{N-d} - 1)r - l + 1)) = 1$. However, we have

$$\begin{aligned} l = 2zr &\iff (2^{N-d} - 1)r - l + 1 = (2^{N-d} - 2z - 1)r + 1 \\ l = (2z + 1)r &\iff (2^{N-d} - 1)r - l + 1 = (2^{N-d} - 2(z + 1))r + 1 \\ l = 2zr + 1 &\iff (2^{N-d} - 1)r - l + 1 = (2^{N-d} - 2z - 1)r \\ l = (2z + 1)r + 1 &\iff (2^{N-d} - 1)r - l + 1 = (2^{N-d} - 2(z + 1))r, \end{aligned}$$

and so $D^{(k+j)}$ is not a factor of H_k^n.

Next suppose that $0 < j < r$ is not a power of 2 and write

$$j = 2^{l_1} + 2^{l_2} + \cdots$$

with $0 < l_i < l_{i+1} < d$. Here we have the 2-adic expansion

$$2j + r - 1 = 1 + 2 + \cdots + 2^{l_1} + 2^{l_2+1} + \cdots$$

with $l_1 < l_2$, and in particular $f_2(2j+r-2, j) = f_2(2j+r-2, j-1) = 0$. So $D^{(k+j)}$ again is not a factor of H_k^n.

Finally, if $j = 2^l$ with $0 \le l < d$ then

$$2j + r - 1 = 1 + 2 + \cdots + 2^l + 2^d,$$

and so

$$f_2(2j+r-2, j) = f_2(2j+r-2, j-1) = 1.$$

This completes the proof. ∎

COROLLARY 5.2. *If $0 \le k \le n$ and $2 < r = 2^d$ satisfy $2k - r + 1 = n$ then the restriction to FS_{n-1} of $D^{(k,\, k-r+1)}$ has composition factors*

$$\begin{cases} D^{(k-1,\, k-r+1)} & \text{with multiplicity one, and} \\ D^{(k-1+2^l, k-r+1-2^l)} : 0 \le l < d & \text{each with multiplicity two.} \end{cases}$$

Proof. If $g \in S_{n-1}$ has odd order then $\chi(g, H_k^n) = \beta(g, n, k, 1) = \beta(g, n-1, k, 1) + \beta(g, n-1, k-1, 1) = \chi(g, H_{k-1}^{n-1})$ by Theorem 2.3 and Proposition 2.5. Since $2(k-1) - r + 2 = n - 1$ the result follows from Theorem 4.1 and Theorem 5.1. ∎

Remarks. (1) While we have assumed here that F has characteristic 2 it appears that the proof of Theorem 5.1 could be adapted to any nonzero characteristic and more general conditions on k and r. This would provide branching rules for H_k^n and arbitrary prime power r in general, similar to Theorem 6.1 in [1], where the same was done for $r = 1$.

(2) More general results on branching rules for representations labeled by two-part partitions in arbitrary characteristic are contained in Sheth's paper [16].

COROLLARY 5.3. *For all k we have*

$$\dim D^{(k,\, k-5)} = \tfrac{1}{4}\left\{\left(2+\sqrt{2}\right)^{k-3} + \left(2-\sqrt{2}\right)^{k-3}\right\} - 2^{k-4}.$$

Proof. We see from Corollary 5.2 and Conjecture 1 of Benson (proved in [10]) that

$$\begin{aligned} \dim D^{(k,\, k-3)} &= \dim D^{(k-1,\, k-3)} + 2\dim D^{(k,\, k-4)} + 2\dim D^{(k+1,\, k-5)} \\ &= \dim D^{(k-2,\, k-3)} + 2\dim D^{(k-1,\, k-4)} + 2\dim D^{(k,\, k-5)}. \end{aligned}$$

The result follows by evaluating the first three dimensions, using Corollary 4.4 and Theorem 3.5 as appropriate. ∎

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