A NEW LOOK AT COUNTEREXAMPLES IN TOPOLOGY

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Contents

Abstract

The book Counterexamples in Topology is a useful catalogue of topological spaces and properties. This thesis extends that catalogue to the properties of sobriety and packedness, and describes some related theory.

A purely topological account of sobriety and sober reflections is given, together with an account of the connection with point-free topology which motivates it. Concrete constructions of the sober, T_0 and T_1 reflections of a topological space are given, and these are calculated for each space in Counterexamples in Topology. These are used to study the relationship between sobriety and the T_1 separation property.

The notion of a specialization topology is introduced as a means of constructing topological spaces from quasiordered sets. The Alexandrov, Scott and $\mathcal{W}$-topologies are described and shown to be examples of this notion. The sobriety and sober reflections of specialization topologies are considered, and these motivate a suggestion for a generalization of the notion of a topological space.

The calculations in this thesis are summarized in two reference tables.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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This thesis was typeset using LaTeX. The arrows and diagrams were drawn using Paul Taylor's commutative diagrams package.

Chapter 1

Introduction

1.1 Background

The book Counterexamples in Topology, [15], by Steen and Seebach, is a collection of 143 topological spaces together with discussions of some 61 properties of spaces. For most pairs of properties A,B, either it is shown that A implies B or there is an example with A but not B. Thus the book provides a useful catalogue of the connections between these properties. The book contains several reference tables showing these various connections.

Counterexamples in Topology was first published in 1970, and so the properties considered are those that were of interest at the time. In particular, the majority of the examples are Hausdorff, since in the branches of mathematics which used topology then, particularly analysis and algebraic topology, almost every space considered was Hausdorff. More recently, non-Hausdorff spaces have been found to be useful, particularly in computer science. *Domain theory* (see [11]) is an approach to the denotational semantics of programming languages which makes great use of the Scott and Alexandrov topologies, which are described in this thesis. Alexandrov topologies are also used in *digital topology*, which is a method of studying digital images (see [7]). Some of the properties useful in the study of non-Hausdorff spaces have been discovered since 1970, and others were not well known then. They were therefore not included in Counterexamples in Topology. Two such properties are *sobriety* and *packedness.* Although the notion of sobriety originated in algebraic geometry (in France in the late 1950s and early 1960s), the recent uses of point-free topology in logic, computer science and topos theory have brought the notion to the fore.

Point-free topology is concerned with the relationship between topological spaces and their frames of open sets. This relationship is described in chapter 5. The accounts of point free topology in the literature (for example, in [2], [5] and [9]) are generally based on locales rather than frames. The difference is that the arrows in the category **Loc** of locales point in the opposite direction. The arrows in the category **Frm** of frames are functions, so the arrows in **Loc** are cofunctions. Locales are considered mainly for historical reasons, but in order to do calculations with locales, the cofunctions have to be turned round and considered as functions. This creates a lot of extra notation and impairs the clarity of the presentation, so I have chosen to work with **Frm** instead.

A space is packed if each of its compact subsets is closed. Although this notion arises quite naturally, it seems not to have been analysed as a property in its own right. I begin such an analysis here.

1.2 Summary

This thesis makes five contributions to the study of non-Hausdorff spaces.

Firstly, it extends the reference tables in Counterexamples in Topology to include the properties of sobriety and packedness. The new reference tables can be found in appendix B.

Secondly, it contains calculations of the sober reflection of each space, in chapter 3. It is possible to describe sobriety and sober reflections purely in topological terms, and this is done in chapter 2. A point-free account is given in chapter 5.

Thirdly, the thesis contains a study of the relationship between sobriety and other properties, particularly the T_1 separation property. A concrete construction of the T_1 reflector is given in chapter 6, and the T_1 reflection of each space in Counterexamples in Topology is calculated in chapter 8.

Fourthly, in carrying out these calculations, it became clear that the specialization order of a space plays a large part in the structure of non-T_1 spaces. Indeed, for many examples of topological spaces, the information in the topology is essentially all contained in its specialization order. I have coined the term *specialization topologies* for such topologies, and they are investigated in chapters 7, 8 and 9. Some of the spaces dealt with are not T_0, so the specialization order will be a quasiorder but not, in general, a partial order. Some definitions which are standard for partial orders are extended to quasiorders in chapter 4.

Fifthly, the packedness property is considered in chapter 10. As mentioned above, there is no general theory of this notion to draw upon. Thus this chapter sets down a few basic facts, and then analyses the relevant examples from Counterexamples in Topology.

Most of this thesis is concerned with obtaining information about topological spaces via ordered sets, and constructing spaces from the ordered sets. This is done in two different ways – via the frame of open subsets and via the specialization order. Chapter 9 is concerned with the relationship between these ideas.

1.3 Prerequisites and notation

This thesis presumes a basic knowledge of general topology. The separation properties T_0, T_1 and T_2 (Hausdorff) are occasionally needed in the first part of the thesis. Definitions of them can be found at the start of chapter 6, where separation properties are studied specifically. For a topological space X, we write $\mathcal{O}X$ to denote the set of open subsets of X. We use the notation $A - B$ for set-theoretic subtraction, and for any subset $A \subseteq X$ we write A^C for the complement $X - A$. The symbol $\sqcup$ is used to mean disjoint union.

In many chapters from 4 onwards, a basic knowledge of category theory is assumed. Appendix A covers all the material on adjunctions which is used. For a category $\mathbb{C}$, we write $\mathrm{ob}\,\mathbb{C}$ for the collection of objects of $\mathbb{C}$. We write $A \in \mathbb{C}$ to mean $A \in \mathrm{ob}\,\mathbb{C}$. If $A, B \in \mathbb{C}$, we write $\mathbb{C}(A, B)$ for the collection of arrows of $\mathbb{C}$ with domain A and codomain B. We write $A \xrightarrow{f} B$ in $\mathbb{C}$ to mean $A, B \in \mathbb{C}$ and $f \in \mathbb{C}(A, B)$, and f in $\mathbb{C}$ to mean that f is an arrow of $\mathbb{C}$. The opposite of a category $\mathbb{C}$ is denoted $\mathbb{C}^{\mathrm{op}}$. We write a contravariant functor F with domain $\mathbb{C}$ and codomain $\mathbb{D}$ as either $\mathbb{C} \xrightarrow{F} \mathbb{D}^{\mathrm{op}}$ or $\mathbb{C}^{\mathrm{op}} \xrightarrow{F} \mathbb{D}$. This is just a notational convenience similar to writing $b \geqslant a$ for $a \leqslant b$. In particular, although we write $\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Frm}^{\mathrm{op}}$ for the open sets functor, we are considering the category of frames and not that of locales.

We use juxtaposition both for function application and for composition of arrows (including functions). Where necessary for clarity, or to avoid ambiguity, we write e.g. $f(a)$ for function application and $g \circ f$ for composition. Where possible and practicable I have kept to a type system for constants and variables which removes the possibility of ambiguity. For example, topological spaces are generally denoted X or Y, and their elements x or y. Continuous maps are denoted θ or φ. Frames are denoted A or B, their elements a, b or c, and frame morphisms are denoted f or g.

In the text of this thesis, each of the definitions and examples ends with a black square ■. Each proof ends with a white square □. The white square is also used at the end of a result where no proof is given, either because the result follows from the preceding text or because it is omitted.

Chapter 2

Sobriety

In this chapter we introduce the concept of a sober topological space and then go on to describe how given any topological space, we can produce a sober space from it in a canonical way – the sober reflection. The main motivation for this will be explained in the later chapter on point spaces of frames, which can be seen as a generalization of this material, but it is worth explaining the concepts purely in terms of spaces first, since some details which are clear in this formulation are less clear in the more general setting.

2.1 Sober spaces

2.1 Definition. Let X be a topological space, and $F \subseteq X$ a non-empty closed subset. We say that F is *reducible* (or $\cup$-reducible) if there are non-empty, proper, closed subsets F_1, F_2 of F, such that $F = F_1 \cup F_2$. Otherwise F is *irreducible* (or $\cup$-irreducible). By convention, $\emptyset$ is neither reducible nor irreducible. ■

2.2 Example. Let X be any space and $x \in X$. Write $\overline{x}$ for the smallest closed set containing x. That is,

$$\overline{x} = \bigcap \{F \text{ closed} \mid x \in F\}$$

Such a set is called a *point closure*. Every point closure is irreducible. ■

If F is an irreducible closed set then it may be the case that it is the point closure of some point x. If so, we call x a *generic point* for F. With this terminology we can now make the key definition of this section.

2.3 Definition. A topological space is *sober* iff every irreducible closed subset has a unique generic point. ■

Sobriety is thus a combination of two properties: the existence of generic points and their uniqueness. It is sometimes useful to consider these properties individually. Indeed, it is straightforward to see that in a space X, the generic points are unique iff X satisfies the T_0 separation property. We define a space to be *presober* iff each irreducible closed set has at least one generic point. So a space is sober precisely when it is T_0 and presober.

2.4 Example. Let X be the two point indiscrete space. Then X itself is irreducible, and has two generic points. So X is presober but not T_0. ■

2.5 Example. Now let X be an infinite set with the cofinite topology. The closed subsets of X are the finite subsets and X itself. If F is finite and $|F| \geqslant 2$, then for any $x \in F$, $F = \{x\} \cup (F - \{x\})$, so F is reducible. The singletons are closed, so must be irreducible. The set X itself is irreducible, since if $X = A \cup B$, at least one of A and B must be infinite, so not closed. Also, X is not a point closure, since every point closure is a singleton, so it has no generic point. This shows that X is not presober, and thus not sober. ■

We can generalise part of this calculation to any T_1 space. Since a space is T_1 iff every finite set is closed, we have the following lemma.

2.6 Lemma. *In a T_1 space, the finite irreducible closed sets are exactly the singletons.* □

The cofinite topology is an example of a space which is T_1 but not sober, so is also an example which is T_0 and not presober. The following example is sober but not T_1.

2.7 Example. Sierpinski space is the two point space $\{0,1\}$ with closed sets $\emptyset, \{0\}$, and $\{0,1\}$. The two non-empty closed sets are both irreducible and both point closures. However, the space is plainly not T_1. ■

2.8 Lemma. *If X is a T_2 space then every irreducible closed set is a singleton, and X is sober.*

Proof. Suppose $F \subseteq X$ is closed, $x, y \in F$, $x \neq y$. Since X is T_2, there are disjoint open sets U, V with $x \in U$, $y \in V$. Let $F_1 = F \cap U^C$, $F_2 = F \cap V^C$. Then F_1, F_2 are closed, non-empty proper subsets of F, and $F = F_1 \cup F_2$, so F is reducible. So an irreducible closed set must be a singleton, and since every T_2 space is T_1, the singletons are closed. □

There are spaces which are both T_1 and sober, but not T_2. Some examples will be given in the next chapter.

2.2 Sober reflections

Suppose we have a space X which is not sober. We would like to produce a new space which is sober, but which is otherwise as similar to X as possible, in some sense to be made precise later. This new space, SX, will be called the *sober reflection* of X. If X is not sober then it has some irreducible closed sets with more or less than one generic point. We would like to throw away any duplicate points, and add in extra points where required to produce SX from X. Essentially this is what we shall do, although we need to be careful that we define the topology on our our new space correctly.

Since we want each irreducible closed set of X to have a unique generic point, we take the points of SX to be the irreducible closed sets of X. Each point closure in X is irreducible, so the assignment

$$\begin{array}{ccc} X & \xrightarrow{\psi} & SX \\ x & \longmapsto & \overline{x} \end{array}$$

produces a well defined function. When we have defined the topology on SX, this will become a continuous map.

From our earlier remarks, we immediately have the following.

2.9 Lemma. *The function ψ is injective iff X is T_0, surjective iff X is presober, and bijective iff X is sober.* $\square$

We need SX to be a space, not just a set, so we must define a topology on it. For $U \in \mathcal{O}X$, define

$$\Psi U = \{F \in SX \mid U \text{ meets } F\}$$

where U meets F means $U \cap F \neq \emptyset$. This defines a function $\mathcal{O}X \xrightarrow{\Psi} \mathcal{P}(SX)$, where $\mathcal{P}$ is the power set, and we will use this function to transfer the topology on X to SX. We need a lemma to do this.

2.10 Lemma. *The equalities*

$$\begin{array}{rclcrcl} \Psi X & = & SX & \quad & \Psi\emptyset & = & \emptyset \\ \Psi(U \cap V) & = & \Psi U \cap \Psi V & \quad & \Psi(\bigcup \mathcal{U}) & = & \bigcup\{\Psi U \mid U \in \mathcal{U}\} \end{array}$$

hold for all open sets U, V and families $\mathcal{U}$ of open sets of X.

Proof. The first two equalities are immediate. For the third,

$$\begin{aligned} F \in \Psi(U \cap V) &\iff F \text{ meets } U \cap V \\ &\implies F \text{ meets } U \ \& \ F \text{ meets } V \\ &\implies F \in \Psi U \cap \Psi V \end{aligned}$$

and hence $\Psi(U \cap V) \subseteq \Psi U \cap \Psi V$. Suppose $F \in \Psi U \cap \Psi V$ but $F \notin \Psi(U \cap V)$. Then both $F \cap U^C$ and $F \cap V^C$ are non-empty, proper, closed subsets of F, and $F = (F \cap U^C) \cup (F \cap V^C)$, so F is reducible. But $F \in SX$, so F is irreducible. This gives a contradiction, and we deduce that $\Psi(U \cap V) = \Psi U \cap \Psi V$.

For the fourth equality,

$$\begin{aligned} F \in \Psi(\bigcup \mathcal{U}) &\iff F \text{ meets } \bigcup \mathcal{U} \\ &\iff (\exists U \in \mathcal{U})(F \text{ meets } U) \\ &\iff F \in \bigcup \{\Psi U \mid U \in \mathcal{U}\} \end{aligned}$$

which completes the proof. □

2.11 Corollary. *The image of* Ψ *is the collection of open sets for a topology on* SX. □

2.12 Example. Consider again an infinite set X with the cofinite topology. This was described in example 2.5. We have seen that the finite irreducible subsets are the singletons, and that X itself is irreducible. No other infinite subsets of X are closed, so $SX = \{\overline{x} \mid x \in X\} \cup \{X\}$, where $\overline{x} = \{x\}$ in this case. For a nonempty open $U \subseteq X$, $\Psi U = \{\overline{x} \mid x \in U\} \cup \{X\}$. ■

We can generalize this example to the following definition and result, which are used a number of times in the next chapter.

2.13 Definition. For any space X, let X^+ be the space obtained from X by adding one point, which is added to each of the non-empty open subsets. That is, $X^+ = X \sqcup \{\infty\}$ with the open subsets of X^+ being $\emptyset$ and $U \cup \{\infty\}$ for each non-empty open subset U of X. The closed subsets of X^+ are then X^+ and the proper closed subsets of X. ■

2.14 Lemma. *Suppose the irreducible closed sets of X are the point closures and X itself. If X is not a point closure, and all the point closures are distinct, then $SX = X^+$.*

Proof. The proof is essentially the calculation done in example 2.12 above. □

Having produced our space SX, we must check that it has the properties we want. We need to check is that it is sober, and also that the map ψ is continuous. One lemma gives us most of what we need for this.

2.15 Lemma. *As a function $\mathcal{O}X \xrightarrow{\Psi} \mathcal{O}SX$, Ψ is invertible. Its inverse is $\psi^{\leftarrow}$, the inverse image map of ψ.*

Proof. Let $U \in \mathcal{O}X$, $x \in X$. Then

$$x \in \psi^{\leftarrow}(\Psi U) \iff \psi x \in \Psi U \iff \overline{x} \text{ meets } U \iff x \in U$$

so $\psi^{\leftarrow} \circ \Psi = 1_{\mathcal{O}X}$. Since Ψ is defined to be surjective, an arbitrary member of $\mathcal{O}SX$ has the form ΨU, where $U \in \mathcal{O}X$. Then

$$(\Psi\psi^{\leftarrow})(\Psi U) = \Psi((\psi^{\leftarrow}\Psi)U) = \Psi U$$

by the above, so $\Psi \circ \psi^{\leftarrow} = 1_{\mathcal{O}SX}$. □

2.16 Corollary. *The map $X \xrightarrow{\psi} SX$ is continuous.*

Proof. Let $\mathcal{U} \in \mathcal{O}SX$. Then $\mathcal{U} = \Psi U$ for some $U \in \mathcal{O}X$, and $\psi^{\leftarrow}\mathcal{U} = U$. □

We have to do slightly more work to show that SX is sober, but the basis is again lemma 2.15.

2.17 Proposition. *For any topological space X, the sober reflection SX is sober.*

Proof. The bijection $\mathcal{O}X \xrightarrow{\Psi} \mathcal{O}SX$ extends to closed sets by taking complements. Specifically, if $F \subseteq X$ is closed, define $\mathcal{F} = (\Psi(F^C))^C$. Then we have that $\mathcal{F} = \{H \in SX \mid H \subseteq F\}$.

If F is irreducible then $F \in SX$, so $\mathcal{F} = \downarrow(F)$ and is irreducible. Suppose that F is reducible, say $F = F_1 \cup F_2$ with F_1, F_2 closed, proper subsets of F. Suppose also that H is closed in X, H meets F_1 and H meets F_2. Then $H = (H \cap F_1) \cup (H \cap F_2)$, so is reducible, so $H \notin \mathcal{F}$. Thus, with obvious notation, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, so $\mathcal{F}$ is reducible. Since Ψ is surjective, the irreducible closed sets of SX are precisely the point closures and SX is presober.

For T_0, let $F, H \in SX$ with $F \neq H$. Without loss of generality, $F \not\subseteq H$, and then F meets H^C. So $F \in \Psi(H^C)$ and $H \notin \Psi(H^C)$, so SX is T_0, and hence sober. □

We can also get one more important result from this lemma. Suppose X is already sober. What does SX look like? We would hope that it is homeomorphic to X, and in fact this is so.

2.18 Proposition. *If X is sober then $X \xrightarrow{\psi} SX$ is a homeomorphism.*

Proof. From lemma 2.9 we know that ψ is bijective. Since it is surjective, the direct image of $U \in \mathcal{O}X$ is ΨU, so ψ is an open map. It is also continuous by corollary 2.16, so is a homeomorphism. □

We have now constructed, for any topological space X, a sober space SX and a continuous map $X \xrightarrow{\psi} SX$. We claimed earlier that SX would be as similar as possible to X. Certainly it is homeomorphic to X when it can be, but we could just have defined SX to be X when X was sober and the one point space otherwise, and that would have all the properties we have mentioned, except being similar to X. The way in which SX is similar to X is that the collections of open sets are in bijective correspondence. Furthermore, it is easy to show that $\psi^{\leftarrow}$ and Ψ are inclusion preserving, and hence $\mathcal{O}X$ and $\mathcal{O}SX$ are actually isomorphic as partially ordered sets. We will see in chapter 5 that this is the key reason why the construction of SX works. This is one reason why SX is the canonical sober space related to X, but there is also a categorical reason. The map ψ has the following universal property.

2.19 Theorem. *For each continuous map $X \xrightarrow{\theta} Y$ from an arbitrary space X to a sober space Y, there is a unique map $SX \xrightarrow{\theta^\sharp} Y$ such that*

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[dr, "\theta"'] & SX \arrow[d, dotted, "\exists! \, \theta^\sharp"] \\ & Y \end{tikzcd}$$

commutes.

This universal property is, in category theory, the defining property of a reflection. This theorem can be proved directly, but we will not give a proof here, and will instead deduce it in chapter 5 from the more general theory developed there.

Chapter 3

Calculations of sober reflections

This chapter contains calculations of the sober reflections of most of those examples from Steen and Seebach's book, Counterexamples in Topology, which are not T_2. We know that every T_2 space is sober, so there is nothing to do for the other spaces. For ease of reference, the description of each space is also included here. The numbering follows that in Steen and Seebach. Counterexamples 52, the nested interval topology, 54, the interlocking interval topology and 121, the integer broom, are more easily described using their specialization orders. For this reason, the calculations of their sober reflections are deferred to chapter 8, after the discussion in chapter 6 of specialization orders. In the following calculations, the symbol $\sqcup$ is used to mean disjoint union.

1-7 Partition topology. Let X be a set with a partition on it, and basic open sets the blocks of the partition. Then SX has as points the blocks, and has the discrete topology.

8-11 Particular point topology. Let X be a space of cardinality $\geqslant 2$ and with the particular point topology. That is, for a given point $* \in X$ the open sets are $\emptyset$ and the sets containing $*$. So the closed sets are X and any set not containing $*$. If F is non-empty, closed and $* \notin F$ then F is irreducible iff it is a singleton, since otherwise $F = \{x\} \cup (F - \{x\})$ for any $x \in F$, and both $\{x\}$ and $F - \{x\}$ are closed and non-empty. Every singleton except $\{*\}$ is closed. Also X itself is irreducible, since it is a point closure, $X = \overline{*}$. Thus the irreducible closed sets are exactly the point closures, and so X is sober.

13-15 Excluded point topology. Let X have cardinality $\geqslant 3$ with the excluded point topology. So for a given $* \in X$, a subset $A \subseteq X$ is open iff $A = X$ or $* \notin A$. Thus F is closed iff $F = \emptyset$ or $* \in F$. Then the irreducible closed sets are $\{*, x\}$, for $x \in X$, including the special case $x = *$, and so are exactly the point closures. Hence X is sober.

17 Either-or topology. Let $X = [-1, 1]$, and say a subset $A \subseteq X$ is open iff $0 \notin A$ or $(-1, 1) \subseteq A$. Then the closed sets are $\emptyset, X, \{-1\}, \{1\}, \{-1, 1\}$ and any A such that $0 \in A$. Thus the irreducible closed sets are $\{-1\}$, $\{1\}$, and $\{0, x\}$ for $x \in (-1, 1)$, again exactly the point closures. So X is sober.

18-19 Cofinite topology. This was described in example 2.5 in chapter 2. It is not sober, and its sober reflection was calculated in example 2.12 to be X^+.

20 Cocountable topology. Let X be uncountable, with the cocountable topology. Essentially the same calculation as for the cofinite case shows that X is not sober, and that its sober reflection is X^+.

21 Double pointed cocountable topology. This evidently has the same sober reflection as the cocountable topology, the duplicate points being removed in the process.

22 Cocompact topology on $\mathbb{R}$. Let $\langle \mathbb{R}, \tau \rangle$ be the real numbers with Euclidean topology, and define a topology τ^* on $\mathbb{R}$ by declaring the τ^*-closed sets to be $\mathbb{R}$ and the τ-compact sets. This is a topology, since the τ-compact sets are the τ-closed and bounded sets and, with $\mathbb{R}$, this collection is closed under finite unions and arbitrary intersections. Suppose K is τ-compact, and $x, y \in K$, $x \neq y$. Let

$$K_1 = K \cap (-\infty, \tfrac{x+y}{2}], \quad K_2 = K \cap [\tfrac{x+y}{2}, \infty)$$

so that K_1, K_2 are τ-compact and non-empty. Then $K = K_1 \cup K_2$, so K is reducible. However, if $\mathbb{R} = A \cup B$, then at least one of A, B is unbounded, so $\mathbb{R}$ is τ^*-irreducible. Again, since the irreducible closed sets are the singletons and the whole space, the sober reflection of $\langle \mathbb{R}, \tau^* \rangle$ is given by adding one point to produce $\langle \mathbb{R}^+, \tau^* \rangle$.

27 Modified Fort space. Let N be an infinite set, and X the disjoint union $X = N \sqcup \{x_1, x_2\}$, with $x_1 \neq x_2$. Then declare $A \subseteq X$ to be open iff $A \subseteq N$ or $N - A$ is finite. So $F \subseteq X$ is closed iff $x_1, x_2 \in F$ or F is finite. Then X is T_1, so we know that the finite irreducibles are just the singletons. Suppose F is infinite and closed. Then $\{x_1, x_2\} \subseteq F$, and for $x \in F - \{x_1, x_2\}$ we have the decomposition $F = \{x\} \cup F - \{x\}$, so F is reducible. Thus X is sober.

35 One point compactification of $\mathbb{Q}$. Let $\langle \mathbb{Q}, \tau \rangle$ be the rationals with Euclidean (metric) topology. Let $X = \mathbb{Q} \sqcup \{*\}$, and τ^* be the topology on X for which $A \subseteq X$ is open if A is τ-open or $X - A$ is τ-closed and τ-compact. Then τ^* is T_1, so we just need to look at infinite closed sets. Suppose $F \subseteq X$ is infinite and closed. Set $F' = F - \{*\}$, and choose $x, y \in F'$ with $x < y$. Set

$$F_1' = F' \cap (-\infty, \tfrac{x+y}{2}], \quad F_2' = F' \cap [\tfrac{x+y}{2}, \infty), \quad F_i = F_i' \cup \{*\}$$

for $i = 1, 2$. Then F', F_1', F_2' are closed in $\mathbb{Q}$, so if $* \in F$ then $F = F_1 \cup F_2$. But F_1, F_2 are τ^*-closed, $x \in F_1 - F_2$, and $y \in F_2 - F_1$, so F is reducible.

Now if $* \notin F$ then $F = F'$ and F is also compact in $\mathbb{Q}$. So F_1', F_2' are closed subsets of the compact set F in a Hausdorff space $\mathbb{Q}$, so are compact. Thus they are τ^*-closed, and again F is reducible. So X is sober.

50 Right order topology on $\mathbb{R}$. Consider $\mathbb{R}$ with its usual order, and let τ be the topology generated by open sets of the form $\{x \mid a < x\}$ for $a \in \mathbb{R}$. Then the closed sets are $\mathbb{R}$ and $\downarrow(a)$ for $a \in \mathbb{R}$, where $\downarrow(a) = \{x \mid x \leqslant a\}$. The sets $\downarrow(a)$ are point closures, and $\mathbb{R}$ itself is also irreducible but not a point closure. So every closed set is irreducible, and $\langle \mathbb{R}, \tau \rangle$ is not sober. Its irreducible sets are the point closures and the whole space, so its sober reflection is obtained by adding a point $+\infty$ to produce $\mathbb{R}^+$.

53 Overlapping interval topology. We take $X = [-1, 1]$, with open sets being the Euclidean open intervals containing 0. Then the closed sets are $\emptyset$, X, $[-1, a]$, $[b, 1]$ and $[-1, a] \cup [b, 1]$ for $a < 0$ and $0 < b$. The point closure of x is $[-1, x]$, X, or $[x, 1]$ as $x < 0$, $x = 0$, or $0 < x$ respectively. These are all the irreducible closed sets, so X is sober.

55 Hjalmar-Ekdal topology. Let $X = \mathbb{N}^+$, with a subset $A \subseteq X$ open iff for each odd number $x \in A$, $x + 1$ is also in A. So $A \subseteq X$ is closed iff for each even number $x \in A$, $x - 1$ is also in A. Then the point closure of x is $\{x\}$ or $\{x - 1, x\}$ as x is odd or even respectively. These are all the irreducible closed sets, so X is sober.

In fact, X is just the countable coproduct (or disjoint union, or sum) of copies of Sierpinski space. Sierpinski space is the particular point topology on a two point set, and we have seen above that it is sober. Now suppose X is any coproduct of sober spaces. Then each summand is clopen, so the irreducible closed sets of X are just those of each summand. So X is also sober.

56 Prime ideal topology on $\mathbb{Z}$. Let X be the set of all prime ideals of $\mathbb{Z}$. Take as a basis of open sets the sets $V_x = \{P \in X \mid x \notin P\}$ for $x \in \mathbb{N}$. Writing 0 for the zero ideal, we have that $A \subseteq X$ is open iff $0 \in A$ and A is cofinite, or $A = \emptyset$. Thus X is, up to homeomorphism, the sober reflection of a cofinite space, so is sober.

This space is called the spectrum of prime ideals of $\mathbb{Z}$. This spectrum can be defined for any commutative ring and is always sober. A similar result holds for the spectrum of a distributive lattice.

57 Divisor topology. Let $X = \{n \in \mathbb{N} \mid n \geqslant 2\}$, and give as a basis of open sets $U_n = \{x \in X \mid x \text{ divides } n\}$ for $n \in X$. Then the closed sets of X are those closed under multiplication from $\mathbb{N}$. The point closure of n is $\{an \mid a \in \mathbb{N}^+\}$. Now suppose F is nonempty, closed and not a point closure. Let $n = \min(F)$ and set $F_1 = \{an \mid a \in \mathbb{N}^+\}$. Since F is not a point closure, $F \neq F_1$, so let $m = \min(F - F_1)$ and set $F_2 = \{x \in F \mid m \leqslant x\}$. Then F_1 and F_2 are closed, $F = F_1 \cup F_2$, $n \in F_1 - F_2$ and $m \in F_2 - F_1$, so F is reducible. Hence X is sober.

73 Telophase topology. Form X from the closed unit interval $[0, 1]$ by "splitting" the point 1 in two, to give $X = [0, 1] \sqcup \{1^*\}$. The topology is given by saying that the subspace topology of $[0, 1]$ is the normal Euclidean topology, and 1^* has as a local neighbourhood basis $(a, 1) \cup \{1^*\}$ for $a \in [0, 1)$. X is T_1, so suppose F is closed and infinite. If $1^* \notin F$ then $F \subseteq [0, 1]$, and $[0, 1]$ is Hausdorff, so F is reducible. Similarly, if $1 \notin F$ then $F \subseteq [0, 1) \cup \{1^*\}$, homeomorphic to $[0, 1]$, so F is reducible. But if $1, 1^* \in F$ then $F - \{1\}$ and $F - \{1^*\}$ are both closed, and their union is F, so again F is reducible. Thus X is sober.

99 Maximal compact topology. Let $X = (\mathbb{N} \times \mathbb{N}) \sqcup \{x, y\}$ with $x \neq y$. A *row* of X is a subset of X of the form $\{(i, j) \mid i \in \mathbb{N}\}$ for fixed $j \in \mathbb{N}$. Define a topology τ by saying that the singletons $\{(i,j)\}$ are open, that U is an open neighbourhood of x iff $x \in U$ and $(\mathbb{N}\times\mathbb{N})-U$ contains at most finitely points from each row, and that U is an open neighbourhood of y iff $y \in U$ and $(\mathbb{N} \times \mathbb{N}) - U$ contains points from at most finitely many rows. Then a set $F \subseteq X$ is closed iff $x, y \in F$, or $x \in F, y \notin F$ and F contains finitely many points on each row, or $x \notin F, y \in F$ and F contains points from finitely many rows, or $x, y \notin F$ and F is finite. In each case, F is irreducible iff it is a singleton, so X is sober.

Chapter 4

Ordered sets

Much of the rest of the dissertation is concerned with giving information about topological spaces in the form of ordered sets. There are two different examples of this. One is the frame of open sets, and the other is the specialization quasiorder. In this chapter we give the definitions and terminology we need to explain these.

4.1 Quasiordered sets

We consider structures $\langle X, R\rangle$ consisting of a set X and a binary relation R on X. There is a category **Rel** whose objects are these structures, and whose morphisms are functions $X \xrightarrow{f} Y$ such that

$$(\forall x, y \in X)(xRy \implies (fx)R(fy))$$

holds. This is the usual category of models of the empty theory on this signature. We consider the following properties of the binary relation.

4.1 Definition. We say that a binary relation R on X is

- *reflexive* iff $(\forall x \in X)(xRx)$

- *irreflexive* iff $(\forall x \in X)(\neg xRx)$
- *transitive* iff $(\forall x, y, z \in X)((xRy \ \& \ yRz) \implies xRz)$
- *antisymmetric* iff $(\forall x, y \in X)((xRy \ \& \ yRx) \implies x = y)$
- *symmetric* iff $(\forall x, y \in X)(xRy \implies yRx)$
- *connected* iff $(\forall x, y \in X)(xRy$ or $yRx)$

A binary relation which is reflexive and transitive is called a *quasiorder*. Quasiorders are also known as *preorders*, especially by category theorists. A quasiorder which is antisymmetric is called a *partial order*. If it is also connected it is a *linear order*, also known as a *total order*. A quasiorder which is symmetric is an *equivalence relation*. ■

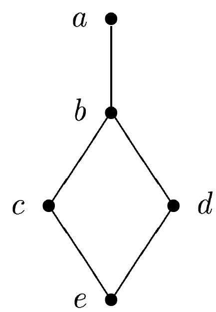
If the relation is a quasiorder then we usually denote it by $\leqslant$, and likewise we usually use $\sim$ to denote an equivalence relation. If $\leqslant$ is a quasiorder on X then we say that $\langle X, \leqslant \rangle$, or just X, is a quasiordered set, or *quoset*. If $\leqslant$ is a partial order, then X is a partially ordered set, or *poset*. The full subcategory (see definition A.4 in the appendix) of **Rel** consisting of all quosets is denoted **Quoset**, and the full subcategory consisting of all posets is denoted **Poset**.

Given any quoset X, we can form a poset as follows. Define an equivalence relation on X by $x \sim y$ iff $(x \leqslant y \ \& \ y \leqslant x)$. Write $\tilde{x}$ for the equivalence class of x, and let DX be the set of equivalence classes. Define an order $\leq$ on DX by $\tilde{x} \leq \tilde{y}$ iff $x \leqslant y$. This is well defined, and makes DX into a poset – the derived poset of X. This construction is actually the reflection of the category **Poset** in **Quoset**, and we shall see in chapter 6 that the same idea gives the reflector for T_0 spaces in **Top**.

4.2 Definition. If $\leqslant$ is a quasiorder, we define another relation $<$, the *strict part of* $\leqslant$, by $x < y$ iff $(x \leqslant y \ \& \ y \nleqslant x)$. ■

The resulting *strict partial orders* are characterised by being irreflexive and transitive. If $\leqslant$ is a partial order then we can recover it from its strict part, but not otherwise.

One way of describing posets is by Hasse diagrams. Only suitably simple posets can be described in this way, but despite their limitations, Hasse diagrams are surprisingly useful. The easiest way to explain them is to give an example. The diagram



represents a poset with underlying set $\{a, b, c, d, e\}$. If two points are joined by an edge this means that the lower endpoint is less than the upper endpoint in the partial order. For example, the edge

indicates that $b \leqslant a$. The diagram above has five edges, saying that $b \leqslant a$, $c \leqslant b$, $d \leqslant b$, $e \leqslant c$ and $e \leqslant d$. This is itself a binary relation, but it is neither transitive nor reflexive. The partial order it represents is the reflexive, transitive closure of this relation, also called the *ancestral* of the relation.

4.3 Definition. If R is a binary relation on X then its *opposite* is the binary relation R^{op} given by $xR^{\mathrm{op}}y \iff yRx$. As usual, we write $\geqslant$ and $>$ for the opposites of $\leqslant$ and $<$. ■

4.4 Definition. Let X be a quoset and $S \subseteq X$. Then an element $x \in X$ is an *upper bound* for S iff $(\forall s \in S)(s \leqslant x)$. It is a *supremum* for S iff it is an upper bound for S and if y is any other upper bound then $x \leqslant y$.

An element $t \in S$ is *maximal* in S iff $(\forall s \in S)(t \leqslant s \implies s \leqslant t)$. It is a *maximum* of S iff $(\forall s \in S)(s \leqslant t)$. Thus every maximum is maximal, but maximal elements are not in general maxima.

We define *lower bounds*, *infima*, *minimal* elements and *minima* dually, using the opposite relation $\geqslant$. ■

If X is a poset, then suprema and maxima are unique, when they exist, but this is not the case for an arbitrary quoset.

4.2 Frames

So far we have seen types of ordered sets which are defined by specifying properties of the binary relation. We now consider types of ordered sets which are defined by specifying *structure*. In categorical terms, the difference between these is that the former are full subcategories of **Rel**, and the latter are not. In other words, the morphisms have to preserve the extra structure we impose.

Our main purpose here is to define frames, but we will first define the more general distributive lattices, since these exhibit many of the features of frames, and every frame is a distributive lattice.

4.5 Definition. Let $\langle X, \leqslant \rangle$ be a poset, and $S \subseteq X$. If S has a supremum it is necessarily unique, and we denote it $\bigvee S$. If it has an infimum, it is denoted $\bigwedge S$. If $S = \{x, y\}$ then its supremum (if it exists) is denoted $x \vee y$ and is called the *join* of x and y. If $\{x, y\}$ has an infimum it is denoted $x \wedge y$ and is called the *meet* of x and y. ■

With this terminology, we can now define distributive lattices.

4.6 Definition. A *distributive lattice* is a structure $\langle A, \leqslant, \vee, \wedge, \bot, \top \rangle$, usually abbreviated to A, where $\langle A, \leqslant \rangle$ is a poset in which every finite subset has a supremum and an infimum, $\vee$ and $\wedge$ are binary functions picking out the join and meet respectively of pairs of elements of A, and $\bot$ and $\top$ are constants picking out the bottom and top elements of A. The top is the maximum of A, which exists because it is the infimum of the empty subset. The bottom is the supremum of the empty subset. In addition, A must satisfy the following two distributive laws.

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \end{aligned}$$

If A, B are distributive lattices then a function $A \xrightarrow{f} B$ is a distributive lattice morphism iff for each $a, b \in A$, the following hold.

$$\begin{aligned} f(a \vee b) &= fa \vee fb & \qquad f(a \wedge b) &= fa \wedge fb \\ f\bot &= \bot & f\top &= \top \end{aligned}$$

In other words, the morphisms between distributive lattices are those functions which preserve $\vee$,$\wedge$,$\bot$ and $\top$. ■

Note that $a \leqslant b$ iff $(a \wedge b = a)$, so these morphisms necessarily preserve $\leqslant$ as well. Note also that the distributive lattices themselves are defined as partially ordered sets with certain properties. (It is convenient but not necessary to have function symbols for the join and meet when giving the distributive laws). However, we wish to state that the meet, join, top and bottom are integral parts of the structure of a distributive lattice, which we do by specifying that the morphisms must preserve them. In categorical terms, we specify the whole category of distributive lattices, not just its objects.

In fact, each of the distributive laws implies the other (see [5, p3]), so it is only necessary to specify one of them. This symmetry means that if $\langle A, \leqslant \rangle$ is a distributive lattice then its opposite $\langle A, \geqslant \rangle$ is too. This symmetry disappears when we consider frames.

Suppose that $\langle X, \leqslant \rangle$ is a poset in which every subset has a supremum. Then X has a top and a bottom, since they are the suprema of X and $\emptyset$ respectively. Every subset S must also have an infimum – the supremum of the set of lower bounds of S. Such a poset is called *complete*, and we use this in the definition of a frame.

4.7 Definition. A *frame* is a structure $\langle A, \leqslant, \bigvee, \wedge, \bot, \top \rangle$, usually abbreviated to A, where $\langle A, \leqslant \rangle$ is a complete poset, $\bigvee$ is a function sending each subset to its supremum, and $\wedge$, $\bot$ and $\top$ are binary meet, bottom and top, as for distributive lattices. It must also satisfy the following *frame distributive law.*

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

A frame morphism is a function between frames preserving $\bigvee$, $\wedge$, $\bot$ and $\top$. ■

If we take $S = \{b, c\}$ in the frame distributive law, we get the first distributive law, so every frame is a distributive lattice. Since each frame is a complete poset, every subset has an infimum and so the dual to the frame distributive law makes sense. However, unlike in the case of the standard distributive laws, this dual law does not follow from the frame distributive law.

4.8 Examples. Any complete Boolean algebra is a frame, in particular the two point Boolean algebra, **2**. The smallest frame is the one point frame in which $\top = \bot$. The canonical example of a frame is the collection of open subsets $\mathcal{O}X$ of a topological space X, where $\leqslant$ is set inclusion, $\bigvee$ is union and $\wedge$ is intersection. Any subset of $\mathcal{O}X$ has an infimum, the interior of its intersection, but this is not in general the same as the intersection. Thus the distinguished structure of the frame is precisely that which is the set theoretical structure of the collection of open sets. ■

Boolean algebras are the algebraic structures naturally occurring in classical logic, and their counterparts in intuitionistic logic are the more general Heyting algebras. Each complete Heyting algebra is a frame, and in fact, every frame is also a complete Heyting algebra. This means that frames also have the property of having an implication operation, which gives much of their power. However, we shall not require this in the dissertation, so will not explain it further.

Chapter 5

The point space of a frame

As remarked upon earlier, if X is any topological space, the collection of open subsets $\mathcal{O}X$ is a frame. The defining property of a continuous function $X \xrightarrow{\theta} Y$ is that if $B \subseteq Y$ is open then $\theta^{\leftarrow}B = \{x \in X \mid \theta x \in B\}$ is open. This gives a function $\mathcal{O}Y \xrightarrow{\mathcal{O}\theta=\theta^{\leftarrow}} \mathcal{O}X$ which is easily seen to be a frame morphism. Furthermore, $\mathcal{O}(\theta\varphi) = (\mathcal{O}\varphi)(\mathcal{O}\theta)$, and $\mathcal{O}1 = 1$, so $\mathcal{O}$ is a functor $\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Frm}^{\mathrm{op}}$. We will now describe the construction of the point space $\mathrm{pt}\,A$ of a frame A. This is a canonical way of producing a topological space from a frame, and we will show that pt is a functor $\mathbf{Frm}^{\mathrm{op}} \xrightarrow{\mathrm{pt}} \mathbf{Top}$, and in fact the right adjoint to $\mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Frm}^{\mathrm{op}}$. The assignment to a space of its sober reflection will turn out to be equivalent to applying the composite functor $\mathrm{pt}\,\mathcal{O}$.

5.1 Definition. Let A be a distributive lattice, $F, I \subseteq A$. Then F is a *filter* on A iff it satisfies

- $\top \in F$
- F is an upper section, i.e. $(a \in F \;\&\; a \leqslant b) \implies b \in F$
- F is closed under meets, i.e. $a, b \in F \implies a \wedge b \in F$

for all $a, b, \in A$. Dually, I is an *ideal* of A iff it satisfies

- $\bot \in I$
- I is a lower section, i.e. $(a \in I \ \& \ b \leqslant a) \implies b \in I$
- I is closed under joins, i.e. $a, b \in I \implies a \vee b \in I$

for all $a, b \in A$. The whole lattice A is both a filter and an ideal. Any other filter is called a *proper* filter, and any other ideal is a *proper* ideal. We say that a filter F is *prime* iff it is proper and satisfies

- $a \vee b \in F \implies (a \in F \text{ or } b \in F)$

and dually that an ideal I is prime iff it is proper and satisfies

- $a \wedge b \in I \implies (a \in I \text{ or } b \in I)$

for all $a, b \in A$. ■

5.2 Lemma. *Let A be a distributive lattice and F a filter on A. Then F is prime iff F^C is an ideal. In that case, F^C is a prime ideal.*

Proof. Suppose F is a prime filter. Then $\top \notin F^C$, $\bot \in F^C$, and F^C is a lower section. If $a, b \in F^C$ then $a \vee b \in F^C$ since F is prime, and if $a \wedge b \in F^C$ then $a \in F^C$ or $b \in F^C$ since F is a filter. So F^C is a prime ideal. Now suppose F^C is an ideal. Then if $a \vee b \in F$ then at least one of $a, b \in F$, i.e. F is prime. □

5.3 Example. For each $a \in A$, $a \neq \top$, the subset $\downarrow(a) = \{b \in A \mid b \leqslant a\}$ is an ideal of A. An ideal of this form is called a *principal* ideal. ■

5.4 Definition. Let A be a distributive lattice and $a \in A$. We say that a is $\wedge$-*irreducible* iff

$$b \wedge c \leqslant a \implies (b \leqslant a \text{ or } c \leqslant a)$$

for all $a, b \in A$, i.e. iff $\downarrow(a)$ is a prime ideal. ■

5.5 Lemma. *Let X be a topological space, and $U \in \mathcal{O}X$. Then U is $\wedge$-irreducible iff U^C is an irreducible closed set.*

Proof. Suppose U is $\wedge$-irreducible and $U^C = F_1 \cup F_2$, some closed F_1, F_2. Then, taking complements, $U = {F_1}^C \cap {F_2}^C$ so, without loss of generality, ${F_1}^C \subseteq U$. But then $U^C = F_1$, so U^C is an irreducible closed set. Conversely, suppose that U^C is irreducible, and $V_1 \cap V_2 \subseteq U$, V_1, V_2 open. Then ${V_1}^C \cup {V_2}^C \supseteq U^C$, so $U = ({V_1}^C \cap U^C) \cup ({V_2}^C \cap U^C)$. Then, without loss of generality, $U^C = V_1 \cap U^C$, so $V_1 \subseteq U$ and U is $\wedge$-irreducible. $\square$

5.6 Definition. Let A be a frame, and F a filter on A. We say that F is *completely prime* iff

$$\bigvee B \in F \implies B \text{ meets } F$$

holds for all subsets $B \subseteq A$. $\blacksquare$

We denote the two point frame by $\mathbf{2}$. Note that for any frame A, a frame morphism $A \xrightarrow{f} \mathbf{2}$ is determined by either of $f^{\leftarrow}\bot$ or $f^{\leftarrow}\top$.

5.7 Lemma. *Let A be a frame.*

1. *A filter F on A is completely prime iff F^C is a principal prime ideal.*

2. *Let $A \xrightarrow{f} \mathbf{2}$ be a frame morphism. Then $f^{\leftarrow}\top$ is a completely prime filter on A and $f^{\leftarrow}\bot$ is a principal prime ideal on A.*

3. *If F is any completely prime filter then the map $A \xrightarrow{f} \mathbf{2}$ given by*

$$fa = \begin{cases} \top & \text{if } a \in F \\ \bot & \text{if } a \notin F \end{cases}$$

is a frame morphism.

Proof. For 1, suppose F is completely prime. Then it is prime, so F^C is a prime ideal. Also $\bigvee F^C \in F^C$, so F^C is principal. Conversely, if $F^C = \downarrow(p)$ then for every $B \subseteq F^C$, $\bigvee B \leqslant p$, so $\bigvee B \in F^C$, so F is completely prime.

For 2, let $I = f^{\leftarrow}\perp$. Then $\perp \in I$, $\top \notin I$, since f preserves $\top$ and $\perp$, and I is a lower section since f is order preserving. $f(a \wedge b) = fa \wedge fb$, so $f(a \wedge b) = \perp$ iff $fa = \perp$ or $fb = \perp$, so I is a prime ideal. Also, $f \bigvee I = \perp$, so I is a principal prime ideal. It follows from part 1 of the lemma that $f^{\leftarrow}\top$ is a completely prime filter.

For 3, the conditions that F is a proper filter say that f preserves $\top$, $\perp$ and $\wedge$. The condition that F is completely prime says that f preserves $\bigvee$. $\square$

5.8 Definition. Let A be a frame. A *point* of A is a frame morphism $A \xrightarrow{p} \mathbf{2}$. We define $\mathrm{pt}\, A$ to be the set of all points of A. $\blacksquare$

It is sometimes useful to consider the points instead as $\wedge$-irreducible elements of A, or completely prime filters or principal prime ideals on A. Lemma 5.7 shows that these are in bijective correspondence with the points, so we can do this. Lemma 5.5 shows that if $A = \mathcal{O}X$ for some space X then the points also correspond to the irreducible closed sets of X – the points of SX.

We wish $\mathrm{pt}\, A$ to be a topological space, so we give it a topology as follows. Define a function

$$\begin{array}{ccc} A & \xrightarrow{\Phi_A} & \mathcal{P}(\mathrm{pt}\, A) \\ a & \longmapsto & \Phi_A a \end{array}$$

where $\Phi_A a = \{p \in \mathrm{pt}\, A \mid pa = \top\}$. We often drop the subscript A and write just Φ for Φ_A.

5.9 Lemma. *The function Φ_A is a frame morphism, i.e. the equalities*

$$\begin{aligned} \Phi(\top) &= \mathrm{pt}\, A & \Phi(\perp) &= \emptyset \\ \Phi(a \wedge b) &= \Phi a \cap \Phi b & \Phi(\bigvee B) &= \bigcup\{\Phi b \mid b \in B\} \end{aligned}$$

hold for all $a, b \in A$ *and subsets* $B \subseteq A$.

This function Φ is a generalization of the function Ψ used in the construction of the sober reflection of a space. Indeed, using the correspondence between points and irreducible closed sets, Ψ is just the special case of Φ when the frame $A = \mathcal{O}X$ for some topological space X.

Proof of lemma 5.9. Every $p \in \operatorname{pt} A$ is a frame morphism, so preserves $\top$ and $\bot$. Thus $p\top = \top$ and $p\bot \neq \top$ for every $p \in \operatorname{pt} A$, and so $\Phi\top = \operatorname{pt} A$ and $\Phi\bot = \emptyset$.

Let $p \in \operatorname{pt} A$. Then

$$p(a \wedge b) = \top \iff pa \wedge pb = \top \iff pa, pb = \top$$

so $\Phi(a \wedge b) = \Phi a \cap \Phi b$. Also

$$p(\bigvee B = \top) \iff \bigvee\{pb \mid b \in B\} = \top \iff (\exists b \in B)(pb = \top)$$

so $\Phi(\bigvee B) = \bigcup\{\Phi b \mid b \in B\}$. □

5.10 Corollary. *The image of* Φ *is the frame of open sets* $\mathcal{O} \operatorname{pt} A$ *for a topology on* $\operatorname{pt} A$. □

Now we have a function $\operatorname{ob}\mathbf{Frm} \xrightarrow{\operatorname{pt}} \operatorname{ob}\mathbf{Top}$, and we wish to define an action on arrows to produce a functor. Let $B \xrightarrow{f} A$ be a frame morphism. We define a continuous map $\operatorname{pt} A \xrightarrow{\operatorname{pt} f} \operatorname{pt} B$ by sending a point $p \in \operatorname{pt} A$ to the composite $B \xrightarrow{p \circ f} \mathbf{2}$. It remains to show the following.

5.11 Lemma. *The function* $\operatorname{pt} f$ *defined above is continuous.*

Proof. Let Φb be open in $\operatorname{pt} B$. Then

$$\begin{aligned} p \in (\operatorname{pt} f)^{\leftarrow}(\Phi b) &\iff p \circ f \in \Phi b \iff (p \circ f)b = \top \\ &\iff p(fb) = \top \iff p \in \Phi(fb) \end{aligned}$$

and $\Phi(fb)$ is open in $\operatorname{pt} A$. □

It follows immediately from the definition that pt sends identities to identities, and that $\mathrm{pt}(fg) = (\mathrm{pt}\, g)(\mathrm{pt}\, f)$, so pt is a contravariant functor from **Frm** to **Top** or, equivalently, a (covariant) functor $\mathbf{Frm}^{\mathrm{op}} \longrightarrow \mathbf{Top}$. We have shown that the composite operation $\mathrm{pt}\,\mathcal{O}$ is the same as the operation S which takes a space to its sober reflection, so it follows that S is a functor. (We only have a correspondence between irreducible closed subsets of a space and points of its frame of open sets rather than equality between them. This means that, the way we have made our definitions, S is naturally isomorphic to $\mathrm{pt}\,\mathcal{O}$, but not actually equal to it. This is not important.)

Now having defined the functor pt, we can explain its importance. It is the canonical way of producing a topological space from a frame, in the same way that the functor $\mathcal{O}$ is the canonical way of producing a frame from a space.

5.12 Proposition. *There is an adjunction*

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift left, "\mathcal{O}"] \arrow[r, phantom, "\perp"] & \mathbf{Frm}^{\mathrm{op}} \arrow[l, shift left, "\mathrm{pt}"] \end{tikzcd}$$

with unit and counit

$$X \xrightarrow{\psi_X} \mathrm{pt}\,\mathcal{O}X \quad \textit{and} \qquad A \xrightarrow{\Phi_A} \mathcal{O}\,\mathrm{pt}\,A$$

respectively. (The counit is written in the direction of arrows in **Frm** *rather than in* $\mathbf{Frm}^{\mathrm{op}}$*.)*

Proof. As for any adjunction, there are at least three ways to proceed (see appendix A). All three formulations give us useful information in this case, and we choose to prove the existence of the adjunction by proving the triangle identities, since we already have the unit and counit.

Let X be a topological space and A a frame. The unit ψ is given by $x \longmapsto \psi x$, where ψx is a frame morphism $\mathcal{O}X \longrightarrow \mathbf{2}$ given by

$$(\psi x)U = \top \iff \overline{x} \text{ meets } U$$

where $\overline{x}$ is the point closure. We also have

$$\Phi a = \{p \in \mathrm{pt}\, A \mid pa = \top\}$$

and the counit is the assignment $a \longmapsto \Phi a$.

The triangle identities to prove are the following.

$$\begin{tikzcd}
\mathcal{O}X \arrow[r, "\Phi_{\mathcal{O}X}"] \arrow[dr, "1_{\mathcal{O}X}"'] & \mathcal{O}\,\mathrm{pt}\,\mathcal{O}X \arrow[d, "\mathcal{O}\psi_X"] \\
& \mathcal{O}X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{pt}\, A \arrow[r, "\psi_{\mathrm{pt}\, A}"] \arrow[dr, "1_{\mathrm{pt}\, A}"'] & \mathrm{pt}\,\mathcal{O}\,\mathrm{pt}\, A \arrow[d, "\mathrm{pt}\, \Phi_A"] \\
& \mathrm{pt}\, A
\end{tikzcd}$$

For the first of these, let $U \in \mathcal{O}X$. Then

$$\begin{aligned}
x \in ((\mathcal{O}\psi_X)\Phi_{\mathcal{O}X})U &\iff x \in (\mathcal{O}\psi_X)(\Phi_{\mathcal{O}X}U) \\
&\iff \psi_X x \in \Phi_{\mathcal{O}X}U \\
&\iff (\psi_X x)U = \top \\
&\iff \overline{x} \text{ meets } U \\
&\iff x \in U
\end{aligned}$$

so $((\mathcal{O}\psi_X)\Phi_{\mathcal{O}X})U = U$ and $(\mathcal{O}\psi_X) \circ (\Phi_{\mathcal{O}X}) = 1_{\mathcal{O}X}$.

For the second identity, let $p \in \mathrm{pt}\, A$. Then

$$((\mathrm{pt}\, \Phi_A)\psi_{\mathrm{pt}\, A})p = (\mathrm{pt}\, \Phi_A)(\psi p) = (\psi p)\Phi$$

and so we must show $(\psi p)\Phi = p$ as a frame morphism $A \longrightarrow \mathbf{2}$. Let $a \in A$. Then

$$((\psi p)\Phi)a = (\psi p)(\Phi a) = \begin{cases} \top & \text{if } \overline{p} \text{ meets } \Phi a \\ \bot & \text{otherwise.} \end{cases}$$

Now $\Phi a = \{q \in \mathrm{pt}\, A \mid qa = \top\}$, and $\overline{p} = \{q \in \mathrm{pt}\, A \mid (\forall b \in A)(qb \leqslant pb)\}$. So

$$(\psi p)(\Phi a) = \top \iff \exists q \in \overline{p} \cap \Phi a \iff (\exists q \in \overline{p})(qa = \top) \iff pa = \top$$

so $(\psi p)\Phi = p$. Thus we have $(\mathrm{pt}\, \Phi_A)\psi_{\mathrm{pt}\, A} = 1_{\mathrm{pt}\, A}$ as we wanted. $\square$

The other two forms of the adjunction also give us useful information, so it is worth stating them as well.

5.13 Corollary. *Let X be a topological space and A a frame. Then there is an isomorphism*

$$\begin{array}{ccc} \mathbf{Top}(X, \mathrm{pt}\, A) & \cong & \mathbf{Frm}(A, \mathcal{O}X) \\ \theta & \longmapsto & \bar{\theta} \\ \bar{f} & \longleftarrow\!\shortmid & f \end{array}$$

*natural in $X \in$ **Top** and $A \in$ **Frm**, given by*

$$(\bar{f}x)a = \begin{cases} \top & \text{if } x \in fa \\ \bot & \text{otherwise} \end{cases} \qquad \text{and} \qquad \bar{\theta}a = \{x \in X \mid (\theta x)a = 1\}$$

for $x \in X$ and $a \in A$. □

5.14 Corollary. *Let X be a topological space, A a frame, and $X \xrightarrow{\theta} \mathrm{pt}\, A$ a continuous map. Then there is a unique frame morphism $A \xrightarrow{\bar{\theta}} \mathcal{O}X$ such that θ factors as*

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[dr, "\theta"'] & \mathrm{pt}\, \mathcal{O}X \arrow[d, "\mathrm{pt}\, \bar{\theta}"] & & \mathcal{O}X \\ & \mathrm{pt}\, A & & A \arrow[u, dotted, "\bar{\theta}"'] \end{tikzcd}$$

Dually, given a frame morphism $A \xrightarrow{f} \mathcal{O}X$, there is a unique continuous map $X \xrightarrow{\bar{f}} \mathrm{pt}\, A$ such that f factors as

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[dr, "f"'] & \mathcal{O}\, \mathrm{pt}\, A \arrow[d, "\mathcal{O}\bar{f}"] & & \mathrm{pt}\, A \\ & \mathcal{O}X & & X \arrow[u, dotted, "\bar{f}"'] \end{tikzcd}$$

□

5.15 Proposition. *The adjunction $\mathcal{O} \dashv \mathrm{pt}$ is idempotent.*

Proof. Lemma 2.15 shows that $\mathcal{O}\psi_X$ is an isomorphism for each space X, and we know that it is natural, so this follows immediately from definition A.7 in appendix A. □

The essential image $\mathcal{I}(\mathrm{pt})$ is the full subcategory **Sob** of **Top** consisting of the sober spaces and all continuous maps between them. A frame in $\mathcal{I}(\mathcal{O})$ is called *spatial*, and $\mathcal{I}(\mathcal{O})$ is called **SpFrm** - the category of spatial frames. A frame will be spatial precisely when it has enough points to distinguish all its elements. In purely frame theoretic terms, this is when for each $a, b \in A$, if $a \nleqslant b$ then there is a $\wedge$-irreducible element $c \in A$ such that $b \leqslant c$ and $a \nleqslant c$.

5.16 Corollary. *The categories* **Sob** *and* **SpFrm** *are dual. That is, each is equivalent to the opposite of the other.*

Proof. Immediate from theorem A.8 in appendix A. □

There are many non-spatial frames, but they are not entirely straightforward to construct. We sketch one such construction here.

5.17 Example. For any Boolean algebra A, an *atom* is a minimal element of $A - \{\bot\}$ and the $\wedge$-irreducible elements of A are precisely the complements of the atoms. Let $\mathbb{R}$ be the space of real numbers with Euclidean topology. The *negation* of an open set U is given by $\neg U = U^{Co}$, the interior of the complement. We say that an open set U is *regular* iff $\neg\neg U = U$. The collection of regular open sets of $\mathbb{R}$ forms a complete Boolean algebra, $\neg\neg\mathcal{O}\mathbb{R}$. Note that, for example, $\neg\neg(0,1) = (0,1)$, but $\neg\neg\big((0,1) \cup (1,2)\big) = (0,2)$, so many open sets are regular, but many are not. Now suppose that U is a non-empty regular open set. Pick $x \in U$. Then there is an open interval V such that $x \in V$ and $V \subset U$ (strict subset). But V is a regular open set, so U is not an atom. Thus $\neg\neg\mathcal{O}\mathbb{R}$ has no atoms, and so $\mathrm{pt}(\neg\neg\mathcal{O}\mathbb{R}) = \emptyset$. But $\mathcal{O}\emptyset = \mathbf{1}$, and $\neg\neg\mathcal{O}\mathbb{R}$ has more than one element, so is not spatial. ■

This construction works on any T_1 space with no isolated points, not just on $\mathbb{R}$.

We also have the universal property promised at the end of chapter 2, and its dual.

5.18 Corollary. *Considered as a functor* $\mathbf{Top} \xrightarrow{S} \mathbf{Sob}$, *$S$ is the left adjoint to the inclusion* $\mathbf{Sob} \hookrightarrow \mathbf{Top}$. *Also,* $\mathbf{Frm} \xrightarrow{\mathcal{O}\,\mathrm{pt}} \mathbf{SpFrm}$ *is the left adjoint to the inclusion* $\mathbf{SpFrm} \hookrightarrow \mathbf{Frm}$.

Proof. This also follows from theorem A.8. $\square$

Chapter 6

Separation Properties

In this chapter we extend the discussion of separation properties given in [15], in particular by discussing reflectors. Firstly, a reminder of the definitions of the properties in question.

6.1 The separation properties

6.1 Definition. Let X be a topological space. We say that X is T_0, T_1 or T_2 iff it satisfies the following axioms respectively.

- $(\forall x, y \in X)\big(x \neq y \implies (\exists U \in \mathcal{O}X)[(x \in U \;\; \& \;\; y \notin U) \text{ or } (x \notin U \;\; \& \;\; y \in U)]\big)$
- $(\forall x, y \in X)\big(x \neq y \implies (\exists U \in \mathcal{O}X)(x \in U \;\; \& \;\; y \notin U)\big)$
- $(\forall x, y \in X)\big(x \neq y \implies (\exists U, V \in \mathcal{O}X)(U, V \text{ disjoint } \;\& \;\; x \in U \;\; \& \;\; y \in V)\big)$

A space satisfying the T_2 axiom is also called *Hausdorff*. ■

6.2 Definition. Let X be a topological space, $A, B \subseteq X$, disjoint. A *Urysohn function* for A and B is a continuous map $X \xrightarrow{\theta} [0,1]$ such that $\theta x = 0$ for all $x \in A$ and $\theta x = 1$ for all $x \in B$. ■

6.3 Definition. A topological space X is said to be *regular* or *completely regular* iff it satisfies respectively

- $(\forall F \text{ closed in } X)(\forall x \in F^C)(\exists U, V \in \mathcal{O}X)(U, V \text{ disjoint } \;\&\; F \subseteq U \;\&\; x \in V)$
- $(\forall F \text{ closed in } X)(\forall x \in F^C)\big(\text{there is a Urysohn function for } F \text{ and } \{x\}\big)$

We say X is T_3 iff it is regular and T_0, and $T_{3\frac{1}{2}}$ iff it is completely regular and T_0.

■

6.4 Definition. The full subcategory of **Top** consisting of the T_n spaces is denoted $\mathbf{Top}_n$ for $n \in \{0, 1, 2, 3, 3\frac{1}{2}\}$. The full subcategories of regular and completely regular spaces are denoted $\mathbf{Top}_R$ and $\mathbf{Top}_{CR}$ respectively. ■

The terminology we have adopted here is probably the most standard, but it is not universal, and in [15] the meanings of T_3 and regular, and $T_{3\frac{1}{2}}$ and completely regular, are reversed. The advantage of our terminology is that we have inclusions

$$\mathbf{Top}_{3\frac{1}{2}} \hookrightarrow \mathbf{Top}_3 \hookrightarrow \mathbf{Top}_2 \hookrightarrow \mathbf{Top}_1 \hookrightarrow \mathbf{Top}_0 \hookrightarrow \mathbf{Top}$$

There are other separation axioms, including completely Hausdorff, normal, completely normal, perfectly normal, Urysohn, T_4 and T_5, but we will not consider them here.

6.2 The specialization order

In the construction of sober reflections, we have seen the notion of a point closure – the smallest closed set containing a given point. These point closures can intersect only if they are nested, so they give rise to a quasiorder on the set.

6.5 Definition. Let X be a topological space. Define a quasiorder $\leqslant$ on X, the *specialization order*, by

$$x \leqslant y \iff \overline{x} \subseteq \overline{y}$$

where $\overline{x}$ is the closure of the singleton $\{x\}$. ■

We have the following result.

6.6 Lemma. *Let X be a topological space. Then every open subset is an upper section of the specialization topology, and every closed subset is a lower section.*

Proof. It is immediate from the definition of the specialization order that $x \leqslant y$ iff every closed subset containing y also contains x, iff every open subset containing x also contains y. □

An important property of the specialization order is that continuous maps respect it. This next lemma will be used in the construction of the T_1 reflector, and its corollary is central to the later chapter on specialization topologies.

6.7 Lemma. *Suppose $X \xrightarrow{\theta} Y$ is a continuous map of topological spaces. Then it is order-preserving on the specialization orders.*

Proof. Suppose $x_1 \leqslant x_2$ in X, and let $F = \overline{\theta x_2}$. Then $\theta^{\leftarrow}F$ is closed in X and $x_2 \in \theta^{\leftarrow}F$, so $x_1 \in \theta^{\leftarrow}F$, i.e. $\theta x_1 \in F$, so $\theta x_1 \leqslant \theta x_2$. □

6.8 Corollary. *The assignment to a space of its specialization order is a functor*

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow{\mathcal{Z}} & \mathbf{Quoset} \\ \langle X, \tau \rangle & \longmapsto & \langle X, \leqslant \rangle \\ f & \longmapsto & f \end{array}$$

where $\leqslant$ is the specialization order of τ. □

The specialization order of a space is important to us here because it encodes all of the information relating to the T_0 and T_1 properties. The following two lemmas are straightforward.

6.9 Lemma. *A space is T_0 iff its specialization order is a partial order.* □

6.10 Lemma. *For any space, X, the following conditions are equivalent.*

1. *X is T_1.*

2. $(\forall x, y \in X)\big(x \neq y \implies (\exists F \subseteq X,\ \textit{closed})(x \in F\ \&\ y \notin F)\big)$

3. *Every singleton of X is closed.*

4. *Every finite subset of X is closed.*

5. *The specialization order of X is the equality relation.* □

If we know the specialization order of a space, we can tell immediately if it is T_0 or T_1. If a space is not T_0 or T_1, the specialization order also tells us how far away from those conditions it is, and we shall see in the next section that it also tells us how to move to a space which is T_0 or T_1.

6.3 Reflectors

We now describe reflectors for some of these inclusions of subcategories. A reflector is a left adjoint to the inclusion functor of a full subcategory. This is explained in appendix A, together with the universal property that a reflector has. We can also think of a reflector simply as a canonical way of turning an arbitrary space into a space with a certain property, e.g. sobriety or one of the separation properties.

We have already seen the construction of the sober reflection in chapter 2. This has the universal property that every map from a space X to a sober space must factor through the map $X \xrightarrow{\psi} SX$. Now every sober space is T_0, so if there is a T_0 reflector, it must be the case that every map to a sober space must also factor through that. In particular, $X \xrightarrow{\psi} SX$ must factor through the T_0 reflection, if it exists. It is easy to see that the T_0 reflection of a space does exist, and indeed that it is obtained simply by removing duplicate points. That is, if two or more points lie in exactly the same open sets, then throw away all but one of them (or quotient out by the appropriate equivalence relation). The sober reflector does exactly this, and may also add some extra points. It is very easy to show the following.

6.11 Proposition. *Let X be a topological space and $X \xrightarrow{\psi} SX$ the sober reflection. Then the T_0 reflection of X is $X \xrightarrow{\psi} \operatorname{Im}\psi$, considering ψ as a map to its image.* $\square$

Note that this is essentially the same construction as that which gives the reflector for **Poset** $\hookrightarrow$ **Quoset**. We factor out the obvious equivalence relation.

It is an immediate consequence that for presober spaces, the T_0 and sober reflections coincide. How about the other way round? Is there a presober reflector, which coincides with the sober reflector for T_0 spaces? No, there is no presober reflector at all. We can take a space and add in the "extra" points given by the sober reflector without removing the duplicate points, and thus produce a presober space in a canonical way, but it lacks the required universal property.

6.12 Theorem. *The full subcageory of* **Top** *consisting of the presober spaces is not reflective.*

Proof. Suppose it were reflective, with reflector P. Let X be an infinite space with the cofinite topology. Since PX is presober, its sober and T_0 reflections coincide. But SPX must be homeomorphic to $SX = X \sqcup \{\infty\}$ since adjunctions, and thus reflections, compose. So PX is SX, possibly with duplicates of some of the points. Let Y be a two point indiscrete space, and $X \xrightarrow{\theta} Y$ a continuous map. Y is presober, so θ factors through PX. However, this factorization is not unique, since we can map the point ∞ to either point in Y and still produce a factorization. Hence PX does not have the universal property required of a reflection, so the reflector does not exist. $\square$

Having seen that the T_0 reflector is very easy to construct, we might hope that each separation property has a reflector, and that they are all straightforward to construct. On the other hand, perhaps some of them are like presobriety, and have no reflector. In fact, we can show that each separation property does have a reflector, but giving a concrete construction of them is, in general, much more difficult.

6.13 Theorem. *Each of the full subcategories* $\mathbf{Top}_n$ *for* $n \in \{0, 1, 2, 3, 3\frac{1}{2}\}$, $\mathbf{Top}_R$ *and* $\mathbf{Top}_{CR}$ *is reflective in* $\mathbf{Top}$.

Proof. We use the version of the general adjoint functor theorem given in appendix A. The proof is very similar for each case, so we give the details just for the case $\mathbf{Top}_2$.

Firstly, $\mathbf{Top}$ is locally small, so each subcategory of it is. Let $\mathcal{X}$ be a set of Hausdorff spaces, and let $Y = \prod \mathcal{X}$, with projection maps $Y \xrightarrow{\pi_X} X$ for $X \in \mathcal{X}$. Let $x, y \in Y$ with $x \neq y$. Then for some $X \in \mathcal{X}$ we have $\pi_X x \neq \pi_X y$. Since X is Hausdorff, there are disjoint $U, V \in \mathcal{O}X$ with $\pi_X x \in U$ and $\pi_X y \in V$. Then $\pi_X^{\leftarrow} U$ and $\pi_X^{\leftarrow} V$ are disjoint open neighbourhoods in Y of x and y respectively. So Y is Hausdorff.

To show that $\mathbf{Top}_2$ has equalizers, let X, Y be Hausdorff spaces and

$$X \underset{\varphi}{\overset{\theta}{\rightrightarrows}} Y$$

be a parallel pair of continuous maps. Then the equalizer

$$E \xrightarrow{\iota} X \underset{\varphi}{\overset{\theta}{\rightrightarrows}} Y$$

is given by $E = \{x \in X \mid \theta x = \varphi x\}$, with the subspace topology and ι being the subspace inclusion. Since any subspace of a Hausdorff space is Hausdorff, E is Hausdorff.

Finally, let $X \in \mathbf{Top}$ and take $\mathbb{S}'$ to be set of all Hausdorff spaces with cardinality less than or equal to that of X. Take $\mathbb{S}$ to be a subset of $\mathbb{S}'$ consisting of one space in each homeomorphism class. Then $\mathbb{S}$ is a small subset of ob $\mathbf{Top}_2$. Every continuous map from X to a Hausdorff space factors through its image, and hence through some $S \in \mathbb{S}$. Thus, by theorem A.9, $\mathbf{Top}_2$ is reflective in $\mathbf{Top}$. $\square$

We shall call the T_n reflector R_n for each $n \in \{0, 1, 2, 3, 3\frac{1}{2}\}$, the regular reflector R_R and the completely regular reflector R_{CR}. The adjoint functor theorem proves that these reflectors exist, but gives no information about them at all. It turns out that we can give a reasonably straightforward description of the T_1 reflector, and this is done in the next section.

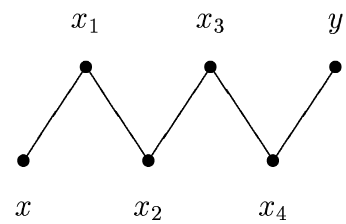
6.4 Construction of the T_1 reflector

Since R_1X must have trivial specialization order, we would expect that the reflection map $X \xrightarrow{\eta} R_1X$ must involve quotienting out the specialization order in some way. Indeed, since any continuous map preserves the specialization order, if $x \leqslant y$ in X then we must have $\eta x = \eta y$.

For any quoset $\langle X, \leqslant \rangle$, define $\leftrightsquigarrow$ to be the smallest equivalence relation containing $\leqslant$, in other words the symmetric, transitive closure of $\leqslant$. The $\leftrightsquigarrow$-blocks are the *components* of X. The symbol used is supposed to represent the existence of a zigzag path through the quasiorder $\leqslant$ from x to y, for example

$$x \leqslant x_1 \geqslant x_2 \leqslant x_3 \geqslant x_4 \leqslant y$$

which looks like a zigzag in a Hasse diagram.



An immediate consequence of the fact that continuous maps preserve the specialization order is the following.

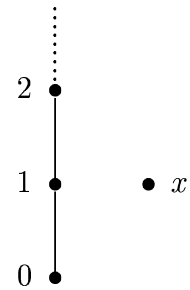
6.14 Lemma. *Let Y be a T_1 space and $X \xrightarrow{\theta} Y$ continuous. Then for $x, y \in X$, if $x \leftrightsquigarrow y$ then $\theta x = \theta y$.* □

Let $X \xrightarrow{\theta} Y$ be any function, and define an equivalence relation $\sim_\theta$ on X by $x \sim_\theta y$ iff $\theta x = \theta y$. Suppose as before that θ is a continuous map and Y a T_1 space. Then the above lemma says that $\leftrightsquigarrow$ refines $\sim_\theta$, that is, that $\leftrightsquigarrow \subseteq \sim_\theta$ as subsets of X^2. But we also know that the singletons in Y are closed, so their inverse images must be closed in X. The inverse images of singletons are precisely the $\sim_\theta$-blocks. Motivated by this, we make the following definition.

6.15 Definition. Let X be a topological space. A *T_1-relation* on X is an equivalence relation $\sim$ refined by $\leftrightsquigarrow$ such that each $\sim$-block is closed. ■

The above remarks show that $\sim_\theta$ is a T_1 relation for every continuous map $X \xrightarrow{\theta} Y$ where Y is a T_1 space. It may be that the $\leftrightsquigarrow$-blocks are closed, but it is not always the case as the following example shows.

6.16 Example. Let X be a poset consisting of the natural numbers $\mathbb{N}$ with their usual ordering, and one other point, x, not related to any other point.



Define a proper subset of X to be closed iff it is a finite lower section. This gives a topology on X, whose specialization order is the order we started with. (In fact, this is the $\mathcal{W}$-topology described in chapter 7.) The $\leftrightsquigarrow$-blocks are $\mathbb{N}$ and $\{x\}$, but $\mathbb{N}$ is not closed. The only T_1-relation on X is the largest equivalence relation on X, that with only one block. ■

6.17 Lemma. *Let X be a topological space. Then the set of all T_1-relations on X is closed under intersections. In particular, taking the empty intersection, there is at least one T_1-relation on X.*

Proof. Let $(\sim_i)_{i\in I}$ be a set of T_1-relations on X, with intersection $\sim$. Then $\sim$ is transitive, reflexive, symmetric, and is refined by $\leftrightsquigarrow$ since these axioms have the form of Horn clauses, and it is immediate that properties defined by Horn clauses are intersection-closed. Now fix $x \in X$. Then $y \sim x \iff y \sim_i x \quad \forall i \in I$, so the block $\tilde{x} = \bigcap_{i\in I} \overset{\sim_i}{x}$. Thus each $\sim$-block is an intersection of closed sets, so closed. Hence $\sim$ is a T_1-relation. □

The empty intersection gives $\sim$ as the equivalence relation on X with only one block. It may be that this is the only T_1 relation on X, as in example 6.16 above. It is also possible that $\leftrightsquigarrow$ is already this relation.

6.18 Corollary. *Let X be a topological space. Then there is a least T_1-relation, $\sim_0$, on X.*

Proof. Take $\sim_0$ to be the intersection of all T_1-relations on X. $\square$

Note that even if a $\leftrightsquigarrow$-block is closed, it may not be a $\sim_0$-block. In example 6.16 above, $\{x\}$ is a closed $\leftrightsquigarrow$-block, but is not $\overset{\sim_0}{x}$.

Define R_1X to be the set of $\sim_0$-blocks of X, and let η be the projection function

$$\begin{array}{ccc} X & \xrightarrow{\eta} & R_1X \\ x & \longmapsto & \overset{\sim_0}{x} \end{array}$$

Topologize R_1X with the weakest topology such that η is continuous, i.e. a subset $U \subseteq R_1X$ is open iff $\eta^{\leftarrow}U$ is open in X.

6.19 Theorem. *This construction $X \xrightarrow{\eta} R_1X$ is the T_1 reflection of X. That is, the following hold.*

1. *R_1X is T_1*
2. *For any T_1 space Y, any map $X \xrightarrow{\theta} Y$ factors uniquely through η as*

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\theta"'] & R_1X \arrow[d, dotted, "\exists!\ \theta^\sharp"] \\ & Y \end{tikzcd}$$

Proof. Let $c \in R_1X$. Then $\eta^{\leftarrow}\{c\}$ is a $\sim_0$-block, so is closed in X, so $\{c\}$ is closed. So R_1X is T_1.

Now suppose that Y is a T_1 space and $X \xrightarrow{\theta} Y$ is continuous. Define $R_1X \xrightarrow{\theta^\sharp} Y$ by $\theta^\sharp \overset{\sim_0}{x} = \theta x$. This is well defined since $\sim_0$ refines $\sim_\theta$, and certainly we have $\theta = \theta^\sharp \circ \eta$. It remains to show that $\theta^\sharp$ is continuous. Suppose $U \subseteq Y$ is open. Then $\eta^{\leftarrow}(\theta^{\sharp\leftarrow}U) = \theta^{\leftarrow}U$, so is open, so $\theta^{\sharp\leftarrow}U$ is open by definition of the topology on R_1X. Thus $\theta^\sharp$ is continuous. $\square$

It follows from theorem A.3 in the appendix that R_1 is a functor and the left adjoint to the inclusion $\mathbf{Top}_1 \hookrightarrow \mathbf{Top}$.

6.5 Sobriety and T_1 separation

We know that both sobriety and the T_1 separation property lie strictly between the separation properties T_0 and T_2, and that neither implies the other. We also know that there are spaces such as the telophase topology which are both sober and T_1 but not T_2, and spaces such as the right order topology on $\mathbb{R}$ which are T_0 but neither T_1 nor sober. We would like to know how sobriety and T_1 interact, and one way to explore this is to study the relationship between their reflectors.

We have seen that the sober reflector factors through the T_0 reflector, and the T_1 reflector also factors it in a straightforward manner, so on the whole we will restrict our attention to T_0 spaces in this section.

The sober reflection SX of a T_0 space X is produced by adding in extra "missing" points. We will shortly make an observation about these points which allows us to make a connection with the T_1 property. Firstly, we must calculate the specialization order of SX.

6.20 Lemma. *The specialization order on SX is just inclusion between the irreducible closed subsets of X.*

Proof. Let F, H be irreducible closed sets of a topological space X and $\leqslant$ be the specialization order on SX. Then

$$\begin{aligned} F \leqslant H \quad &\iff \quad (\forall U \in \mathcal{O}X)(F \in \Psi U \implies H \in \Psi U) \\ &\iff \quad (\forall U \in \mathcal{O}X)(F \text{ meets } U \implies H \text{ meets } U) \\ &\iff \quad (\forall U \in \mathcal{O}X)(H \cap U = \emptyset \implies F \cap U = \emptyset) \\ &\iff \quad (\forall U \in \mathcal{O}X)(H \subseteq U^C \implies F \subseteq U^C) \\ &\iff \quad F \subseteq H \end{aligned}$$

by considering $U^C = H$ for the last step. $\square$

Now for the observation about the extra points in SX.

6.21 Lemma. *Let p be a point of SX, not in the T_0 reflection R_0X of X. Then there is an infinite set of points y of R_0X such that $y \leqslant p$ in the specialization order on SX.*

Proof. The point p corresponds to an irreducible closed set $F \subseteq X$. For each $x \in F$ we have $\overline{x} \subseteq F$, and in fact $F = \bigcup\{\overline{x} \mid x \in F\}$. These point closures are points of R_0X, and $F \neq \emptyset$, so there is at least one point of R_0X below p. Suppose there were just one distinct point closure $\overline{x} \subseteq F$. Then $\overline{x} = F$, and $p \in R_0X$. So there are at least two. Suppose now that there were only finitely many distinct point closures $\overline{x} \subseteq F$, say $F = \bigcup\limits_{i=1}^{n} \overline{x_i}$ for some $n \geqslant 2$. Then $F = \overline{x_1} \cup \bigcup\limits_{i=2}^{n} \overline{x_i}$, so is the union of two proper, nonempty closed sets, so is reducible. So there must be infinitely many distinct point closures which are subsets of F. $\square$

As an immediate corollary of this, we get the following result.

6.22 Theorem. *Let X be a topological space and suppose its sober reflection SX is T_1. Then X is presober. In particular, if X is T_0 and SX is T_1, then $SX \cong X$.*

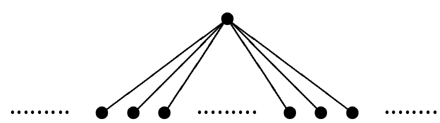
Proof. By lemma 6.21, if X is not presober then SX has non-trivial specialization order. $\square$

For every space in the book Counterexamples in Topology, the T_1 reflection either is equal to the T_0 reflection or is a discrete space. This is therefore not a good source of examples for examining the T_1 reflection. For every sober space X that I have found, R_1X has also been sober. I therefore make the following conjecture.

6.23 Conjecture. *If X is sober then R_1X is sober.*

I have not been able to prove this, nor find a counterexample. If it were true, it would follow that the category of T_1 and sober spaces, $\mathbf{Top}_1 \cap \mathbf{Sob}$, was reflective in $\mathbf{Top}$, with reflector $R_1 \circ S$.

We have been using the cofinite topology and its sober reflection as our canonical example of the sober reflection process. Lemma 6.21 shows us that this is not an accident. It is *the* canonical example, in the sense that for any sober reflection SX, for each added point, the specialization order of SX must contain a subposet the same shape as that of the sober reflection of the cofinite topology, with Hasse diagram



Here subposet is meant in the same sense as subcategory, not necessarily full subcategory. In other words, the points of a cofinite space are unrelated by the specialization order, but the corresponding points in SX may be related.

6.6 Other reflectors

We have proved that the Hausdorff reflector R_2 exists, but we have not given a description of it. Since $\mathbf{Top}_2$ is closed under taking subspaces, it follows from the universal property that the T_2 reflection map $X \xrightarrow{\eta} R_2X$ is surjective. Since any refinement of a Hausdorff topology is also Hausdorff, it follows that the topology on R_2X is the strongest such that η is continuous, i.e. the quotient topology. Thus we should be able to construct R_2 in a similar way to R_1. To do this, we would need to describe the equivalence relation to be factored out. We know that R_2 factors through S and R_1, so through $R_1 \circ S$. If conjecture 6.23 is false, it must also factor through all iterates of it. It must also contain the relation $x \bowtie y$, defined by

$$x \bowtie y \iff \text{there do not exist disjoint neighbourhoods of } x \text{ and } y$$

and may possibly be this relation. However, I have not had time during the course of this MSc to look into this.

The regular and completely regular reflectors have constructions which are quite different from those of R_0, R_1 and R_2. We sketch them now, based on the accounts in [14] and [1] respectively. Further details of both constructions are given in [5].

6.24 Lemma. *A space X is regular iff for each $U, V \in \mathcal{O}X$ with $U \not\subseteq V$, there are $W, Y \in \mathcal{O}X$ such that $U \cup W = X$, $W \cap Y = \emptyset$ and $W \not\subseteq V$.* □

Note that this only refers to the frame of open sets, so is a point-free condition. Thus X is regular iff SX is regular. This also gives a definition of regularity for an arbitrary frame.

For any frame A, we define a binary relation $\lessdot$ on A by

$$a \lessdot b \iff (\exists c \in A)(a \vee c = \top \ \& \ b \wedge c = \bot)$$

for $a, b \in A$. If $a \lessdot b$ we say that a is *well inside* b. Note that this relation is not in general a quasiorder, since it is not in general reflexive. Define

$$A_r = \{a \in A \mid a = \bigvee\{b \in A \mid b \lessdot a\}\}$$

Then A_r is always a subframe of A and $A = A_r$ precisely when A is regular. In general, A_r is not regular. However if we iterate the construction to produce a sequence $A_r, (A_r)_r, \ldots$ then this sequence terminates at some ordinal. Write A_R for the subframe produced by this iterated construction. This A_R is the largest regular subframe of A, and the regular coreflection of A in **Frm**. The regular reflection $R_R X$ of a topological space X has X as its underlying set and $(\mathcal{O}X)_R$

as its collection of open sets. The mediating map is the identity function. Thus the regular reflection of X is obtained by weakening the topology on X.

The completely regular reflection $R_{CR}X$ of a space X is also obtained by weakening the topology.

6.25 Definition. Let $\langle X, \tau \rangle$ be a topological space. A subset $S \subseteq X$ is a *cozero* subset iff there is a continuous map $X \xrightarrow{\theta} [0,1]$ such that $S = \theta^{\leftarrow}(0,1]$. ■

Every cozero subset is open, and the collection of cozero subsets is closed under finite intersections so is the base of a topology, say τ_c on X. The completely regular reflection of $\langle X, \tau \rangle$ is given by $R_{CR}X = \langle X, \tau_c \rangle$, with the identity function as the mediating map. No iteration is required for this construction.

Note that R_0X is regular or completely regular iff X is. We deduce that $\mathbf{Top}_3$ and $\mathbf{Top}_{3\frac{1}{2}}$ are reflective subcategories of $\mathbf{Top}$, with reflectors $R_3 = R_0 \circ R_R$ and $R_{3\frac{1}{2}} = R_0 \circ R_{CR}$. However, R_RX and $R_{CR}X$ can fail to be T_0 even if X is. For example, if X is Sierpinski space, then both R_RX and $R_{CR}X$ are the two point indiscrete space.

Chapter 7

Specialization Topologies

In section 6.2 we saw the notion of the specialization order on a topological space, defined by

$$x \leqslant y \iff \overline{x} \subseteq \overline{y}$$

where $\overline{x}$ is the closure of the singleton $\{x\}$. We also saw how it gives rise to a functor, $\mathbf{Top} \xrightarrow{\mathcal{Z}} \mathbf{Quoset}$.

In this chapter, we are interested in the reverse process. Given a quoset, $\langle X, \leqslant \rangle$, can we construct a topology on X whose specialization order is $\leqslant$? In fact, we shall show that there is always a topology with specialization order $\leqslant$ and there will in general be many of them, for example there are many different T_1 topologies on any infinite set. However, we are interested in those topologies which we can construct directly from the quasiorder structure. We call such topologies *specialization topologies.* This is a new notion, which is very useful in the context of this thesis. It may also be of wider interest. Since these topologies are defined from the quasiorder, they must have the following property, which we turn into a definition.

7.1 Definition. Let $\langle X, \leqslant \rangle$ be a quoset. A *specialization topology* on X is a topology with specialization order $\leqslant$, such that every **Quoset**-automorphism of $\langle X, \leqslant\rangle$ is a homeomorphism. ■

Such topologies are necessarily degenerate in that the information giving the topology is encoded in the first order structure of the quasiorder. They can nonetheless be useful examples. Indeed, we shall see that many of the examples in Counterexamples in Topology are of this form.

Firstly, we will construct the canonical example of a specialization topology – the *Alexandrov* topology.

7.2 Proposition. *The collection of upper sections of a quoset $\langle X, \leqslant\rangle$ is the frame of open sets for a topology on X, whose specialization order is $\leqslant$.*

Proof. Let $\mathcal{U}$ be a set of upper sections of X, $x \in \bigcup\mathcal{U}$ and $x \leqslant y$. Then for some $U \in \mathcal{U}$ we have $x \in U$, and U is an upper section so $y \in U$. So $y \in \bigcup\mathcal{U}$. Similarly, $\bigcap\mathcal{U}$ is an upper section, so certainly a finite intersection of upper sections is an upper section. Hence this is a topology on X. The complement of an upper section is a lower section, and conversely, so the point closure $\overline{x} = {\downarrow}(x)$. But ${\downarrow}(x) \subseteq {\downarrow}(y)$ iff $x \leqslant y$, so $\leqslant$ is the specialization order for this topology. □

7.3 Definition. Let X be a set. A topology on X is an *Alexandrov topology* iff its open sets are all the upper sections of its specialization order. An *Alexandrov space* is a topological space X whose topology is Alexandrov. ■

7.4 Examples. Sierpinski space is Alexandrov. Any partition topology is Alexandrov. In particular, the discrete topology is the only Alexandrov topology with the T_1 separation property. More interestingly, example 57 of Counterexamples in Topology, the divisor topology, is an Alexandrov space. ■

There is also a characterization of these spaces which does not refer to the specialization order at all.

7.5 Lemma. *A topological space X is an Alexandrov space iff its frame of open sets $\mathcal{O}X$ is closed under arbitrary intersections.*

Proof. One direction of this is contained in proposition 7.2. For the other direction, suppose $\mathcal{O}X$ is closed under arbitrary intersections and let $x \in X$. Then $\uparrow(x) = \bigcap\{U \in \mathcal{O}X \mid x \in U\}$, so is open. Now if U is any upper section of the specialization order then $U = \bigcup\{\uparrow(x) \mid x \in U\}$, so U is open. □

This alternative description allows us to characterize the topologies on finite sets.

7.6 Proposition. *Any topological space with only finitely many open sets is an Alexandrov space.*

Proof. Since there are only finitely many open sets, any intersection of them is a finite intersection, so is open. □

7.7 Corollary. *If X is finite, every topology on X is Alexandrov, and there is a bijective correspondence between topologies on X and quasiorders on X.* □

If $\langle X, \leqslant\rangle$ is a quoset, we write ΥX (or $\Upsilon\langle X, \leqslant\rangle$ if necessary) for the Alexandrov space with specialization order $\leqslant$.

7.8 Lemma. *Let $X \xrightarrow{f} Y$ be an order preserving map between quosets. Then f is continuous as a map $\Upsilon X \longrightarrow \Upsilon Y$.*

Proof. Let $B \subseteq Y$ be Υ-open, i.e. an upper section. Suppose $x_1 \in f^{\leftarrow}B$ and $x_1 \leqslant x_2$. Then $fx_1 \leqslant fx_2$ since f is order-preserving and so, since B is an upper section, $fx_2 \in B$. So $x_2 \in f^{\leftarrow}B$ and thus $f^{\leftarrow}B$ is Υ-open. So f is continuous. □

Thus we define Υ to be the identity on morphisms and it becomes a functor **Quoset** $\xrightarrow{\Upsilon}$ **Top**.

7.9 Theorem. *The functor Υ is left adjoint to the specialization functor $\mathcal{Z}$.*

Proof. It is enough to show a natural isomorphism

$$\mathbf{Top}(\Upsilon A, X) \cong \mathbf{Quoset}(A, \mathcal{Z}X)$$

but in fact the isomorphism is the identity map on functions. We therefore just have to show that for a quoset A and a topological space X, a function $A \xrightarrow{f} X$ is order-preserving as a map $A \longrightarrow \mathcal{Z}X$ iff it is continuous as a map $\Upsilon A \longrightarrow X$.

Suppose that f is order preserving and let $U \subseteq X$ be open. We must show that $f^{\leftarrow}U$ is an upper section of A. Let $a \in f^{\leftarrow}U$, and suppose $a \leqslant b$. Then $fa \leqslant fb$, so fb is in every open set containing fa. But $fa \in U$, so $fb \in U$. Thus $b \in f^{\leftarrow}U$, and so $f^{\leftarrow}U$ is an upper section of A.

Conversely, suppose f is continuous, and $a \leqslant b$ in A. Then $f^{\leftarrow}(\overline{fb})$ is closed in ΥA, so a lower section of A. $b \in f^{\leftarrow}(\overline{fb})$, so $a \in f^{\leftarrow}(\overline{fb})$. So $fa \in \overline{fb}$, and thus $fa \leqslant fb$. So f is order preserving. □

The counit of the adjunction

$$\mathbf{Quoset} \underset{\mathcal{Z}}{\overset{\Upsilon}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Top}$$

is the identity function $\Upsilon\mathcal{Z}X \xrightarrow{\epsilon} X$ to a topological space X from its *Alexandrov cocompletion*, the space obtained from X either by moving to the specialization order and back, or purely topologically by refining the topology to insist that every intersection of open sets is open. The composite $\mathcal{Z}\Upsilon$ is the identity functor on **Quoset**, and the unit of the adjunction is the identity natural transformation on this identity functor. We have an immediate consequence.

7.10 Proposition. *The adjunction* **Quoset** $\underset{\mathcal{Z}}{\overset{\Upsilon}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}}$ **Top** *is idempotent.* □

We denote the category $\mathcal{I}(\Upsilon)$ of all Alexandrov spaces as **Alex**. Using theorem A.8 from the appendix, we get two further corollaries of proposition 7.10. The first states that the categories **Alex** and **Quoset** are equivalent, but in this case we can strengthen that to the following.

7.11 Corollary. *The categories* **Alex** *and* **Quoset** *are isomorphic.* □

The second is the universal property of the Alexandrov cocompletion of a space.

7.12 Corollary. *Every continuous map from an Alexandrov space to X is also continuous as a map to the Alexandrov cocompletion of X, and furthermore that this is the strongest topology on X with this property.*

Proof. By theorem A.8, the identity function $\Upsilon\mathcal{Z}X \xrightarrow{\epsilon} X$ is the coreflector of the inclusion $\mathbf{Alex} \hookrightarrow \mathbf{Top}$. It has the universal property that any continuous map $Y \xrightarrow{\theta} X$ from an Alexandrov space Y to X factors uniquely as

$$\begin{tikzcd} Y \arrow[d, dotted, "\theta_\flat"'] \arrow[dr, "\theta"] & \\ \Upsilon\mathcal{Z}X \arrow[r, "\epsilon"'] & X \end{tikzcd}$$

Since ϵ is the identity function, $\theta_\flat$ is the same function as θ. □

We call $\Upsilon\mathcal{Z}X$ the Alexandrov cocompletion rather than completion because it is a coreflection of X rather than a reflection.

If we identify **Alex** and **Quoset** using the isomorphism of corollary 7.11 we can think of **Quoset** as a full subcategory of **Top**. In that case, the adjunction between **Quoset** and **Top** is just the inclusion of a subcategory and its coreflection. Such adjunctions are always idempotent.

The Alexandrov topology is the largest specialization topology in the following sense.

7.13 Proposition. *Let $\langle X, \leqslant \rangle$ be a quoset and τ a topology on X with specialization order $\leqslant$. Then every τ-open set is open in the Alexandrov topology $\Upsilon\langle X, \leqslant \rangle$.*

Proof. The identity function $\Upsilon\mathcal{Z}X \xrightarrow{\epsilon} X$ is continuous, so if U is τ-open then $\epsilon^{\leftarrow}U$ is Υ-open. But $\epsilon^{\leftarrow}U = U$. □

We can also construct the smallest topology with a given specialization order.

7.14 Definition. Let $\langle X, \leqslant \rangle$ be a quoset. The *weak specialization topology*, $\mathcal{W}(X)$, is defined by taking $\{\downarrow(x) \mid x \in X\}$ as a sub-basis of closed sets. So the closed sets are intersections of sets of the form $\downarrow(x_1) \cup \cdots \cup \downarrow(x_n)$. This topology is also known as the *upper-interval topology* or *right order topology*, both names arising from the special case of $\leqslant$ being a linear order. ■

Thus $\mathcal{W}(X)$ is the weakest topology making each principal lower section closed, so is the minimum topology with a given specialization order, since principal lower sections are point closures, so must be closed in any such topology.

7.15 Examples. In any finite space, the $\mathcal{W}$- and Υ-topologies coincide, so every finite space also has the $\mathcal{W}$-topology. The $\mathcal{W}$-topology on the discrete order is the cofinite topology. There are other examples of $\mathcal{W}$-topologies in Counterexamples in Topology. See appendix B for a full list. ■

Since the maximal specialization topology, Υ, is the left adjoint to the specialization order functor $\mathcal{Z}$, we might tentatively hope that the minimal specialization topology, $\mathcal{W}$, is the right adjoint. However, $\mathcal{W}$ is not even a functor. Indeed, we have a stronger result.

7.16 Proposition. *The functor Υ is the only functor* **Quoset** $\xrightarrow{F}$ **Top** *which is the identity on morphisms and such that $\mathcal{Z}F = 1_{\mathbf{Quoset}}$.*

Proof. Suppose F is any such functor, let X be any quoset and $\mathbf{2}$ the two point quoset, $\mathbf{2} = \{\bot, \top\}$, with $\bot < \top$. Let U be any upper section of X. Then the characteristic function of U is an order preserving map $X \xrightarrow{\chi_U} \mathbf{2}$, and $\{\top\}$ is open in $F\mathbf{2}$, so $\chi_U{}^{\leftarrow}\{\top\} = U$ must be open in FX. So every upper section of X must be open, and so FX must be ΥX. □

This lack of functoriality does not cause any problems, but it is worth checking since if $\mathcal{W}$ were a functor we could immediately deduce a collection of categorical results about it.

Apart from $\mathcal{W}$ and Υ, we have to do more work to find general constructions of specialization topologies. We will see one more example in chapter 9, the Scott topology. However, these two constructions are already enough to cover all but one of the examples in Counterexamples in Topology which are specialization topologies. They also cover the majority of examples of spaces which are not T_1. This is unlikely to have been apparent to the authors when the wrote the book, as they probably constructed their examples in a more ad hoc fashion. However, it does mean that the examples in the book are not very representative of non-T_1 spaces. It also fits with the claim that other specialization topologies are more difficult to describe.

The one other example of a specialization topology in Counterexamples in Topology is the cocountable topology on an uncountable set. This has the discrete specialization order, and the topology cannot be defined in terms of the quasiorder. However, the quoset structure includes both the quasiorder and the structure of the underlying set. This structure is the cardinality of the subsets, and it is this which is used to define the topology.

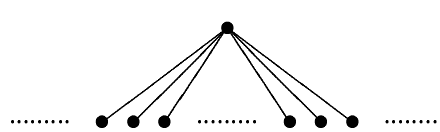
Chapter 8

Specialization calculations

This chapter contains calculations of the specialization order and of the T_0 and T_1 reflections of those examples from Counterexamples in Topology which are not T_2. We also note which spaces have specialization topologies and, if so, if they are Alexandrov or weak topologies (Υ- or $\mathcal{W}$-topologies). For the spaces which have already been considered in chapter 3, the description of the space is not repeated. For those spaces which were not considered in chapter 3, the sober reflection is also computed.

1-7 Partition topology. Any partition topology is an Alexandrov space. It is a $\mathcal{W}$-topology iff there are only finitely many blocks. The T_0 reflection has one point for each block, and the discrete topology. It is therefore Hausdorff, and in particular equal to the sober and T_1 reflections.

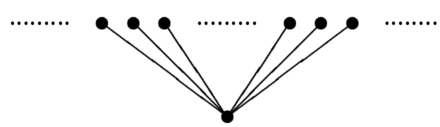
8-11 Particular point topology. This has specialization order



and the Alexandrov topology. It has only one $\leftrightsquigarrow$-block, so its T_1 reflection is the one point space $\mathbf{1}$.

12 Closed extension topology. The closed extension X' of any space X is given by adjoining an extra point, $*$, and extending every open set to include $*$. So the closed sets of X' are X' and the closed sets of X. The specialization order of X' is that of X with the point $*$ added above all the other points. The closed extension operation commutes with the T_0 reflection, so X' is T_0 iff X is. X' has just one $\leftrightsquigarrow$-block, so its T_1 reflection is $\mathbf{1}$. Since the closed sets of X' are those of X, and $\downarrow(*)$, X' is sober iff X is. In fact, the sober reflection also commutes with the closed extension.

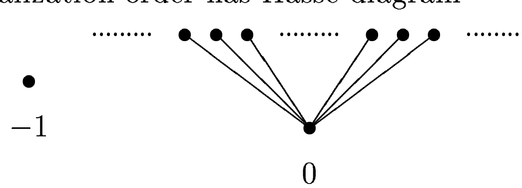
13-15 Excluded point topology. This is the Alexandrov space with specialization order



which is an upside down version of the particular point topology.

16 Open extension topology. This behaves like the closed extension topology, except that the extra point is below rather than above every other point in the specialization order. It also always has T_1 reflection $\mathbf{1}$.

17 Either-or topology. This is just like the excluded point topology, with two extra isolated points, i.e. points whose singletons are clopen. It is an Alexandrov space whose specialization order has Hasse diagram



Its T_1 reflection is the three point discrete space.

18-19 Cofinite topology. The cofinite topology is the $\mathcal{W}$-topology on the discrete order.

20 Cocountable topology. This is a specialization topology on the discrete order, but is neither Υ nor $\mathcal{W}$ if the space is uncountable.

21 Double pointed cocountable topology. The T_0 reflection of this space is the cocountable topology.

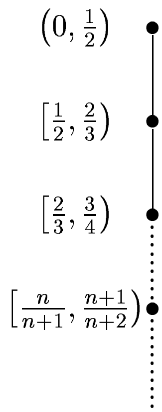
22 Cocompact topology on $\mathbb{R}$. This space is T_1, so it has the discrete specialization order. The subset $\mathbb{Q}$ is not closed, but the subset $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ is. Both are countably infinite subsets of X, so there is a bijection from X to X which maps one to the other. This preserves the order trivially but is not a homeomorphism, i.e. a **Top**-automorphism, so X is not a specialization topology.

27 Modified Fort space. This is a T_1 space, but all of its **Top**-automorphisms must fix or swap the two distinguished points, so it does not have a specialization topology.

35 One point compactification of $\mathbb{Q}$. Again, this is T_1, but its **Top**-automorphisms must fix the distinguished compactifying point, so it is not a specialization topology.

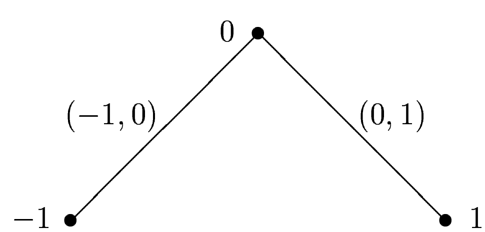
50 Right order topology on $\mathbb{R}$. This is defined to be the $\mathcal{W}$-topology on $\mathbb{R}$ with its usual order. Its T_1 reflection is $\mathbf{1}$.

52 Nested Interval topology. Let X be the real interval $(0,1)$, with open sets $\emptyset$, X and $\left(0, \frac{n+1}{n+2}\right)$ for $n \in \mathbb{N}$. Then X is not T_0, but its T_0 reflection has specialization order



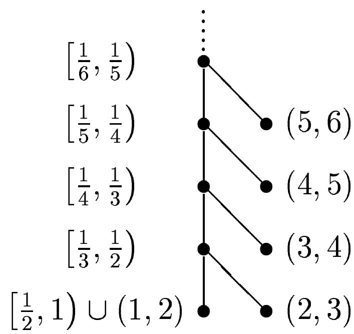
where the interval given as a label is the set of points which are equivalent in the quasiorder. The Υ-topology and the $\mathcal{W}$-topology coincide for this order, so this is, up to homeomorphism, the only space with this specialization order. The Υ- and $\mathcal{W}$-topologies coincide for all finite spaces, but we see here that they can also coincide for infinite spaces. The T_0 reflection, R_0X, is homeomorphic to the set of natural numbers $\mathbb{N}$, with the opposite of their usual order, and the unique topology with this as specialization order. R_0X is sober, since the closed sets are precisely the point closures. The T_1 reflection of X is $\mathbf{1}$.

53 Overlapping interval topology. The specialization order for the overlapping interval topology looks like



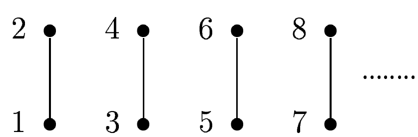
where the lines represent all the points on the real intervals and their order relation, unlike in a standard Hasse diagram where lines and points are distinguished. This is a $\mathcal{W}$-space, and has T_1 reflection $\mathbf{1}$.

54 Interlocking interval topology. Let $X = \mathbb{R}^+ - \mathbb{N}$, the positive reals except for the integers. Take as a basis of open sets $\left(0, \frac{1}{n}\right) \cup (n, n+1)$, for $n \in \mathbb{N}^+$. This space is best described by its specialization order, which is given below. The space is not T_0, so the points of the diagram are those of the T_0 reflection, labelled by the subsets of X they correspond to.



The topology is the $\mathcal{W}$-topology. Since it has only one $\leftrightsquigarrow$-block, its T_1 reflection is the one point space $\mathbf{1}$. Its sober reflection is X^+.

55 Hjalmar-Ekdal topology. The Hjalmar-Ekdal topology is the $\varUpsilon$-topology on the quasiorder

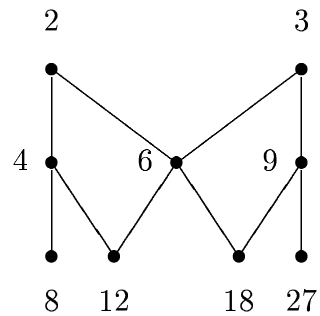


Its T_1-reflection is the countably infinite discrete topology.

56 Prime ideal topology on $\mathbb{Z}$. This has the same specialization order as the particular point topology on a countable set, with particular point the zero ideal, but with the $\mathcal{W}$-topology. Its T_1 reflection is $\mathbf{1}$.

57 Divisor topology. The specialization order is given by $x \leqslant y$ iff y divides x. The graph of this order is not planar, so cannot be drawn in a Hasse diagram.

However, we can draw a small part of it which gives the general idea.



This is also an $\varUpsilon$-space, and has just one $\leftrightsquigarrow$-block, so has T_1 reflection $\mathbf{1}$.

62 Double pointed reals. This is the product of $\mathbb{R}$ with Euclidean topology and a two point indiscrete space. Evidently it has T_0 reflection $\mathbb{R}$.

73 Telophase topology. The telophase space $[0,1] \cup \{1^*\}$ is T_1, but any **Top**-automorphism must fix or swap the points 1 and 1^*, so it is not a specialization topology.

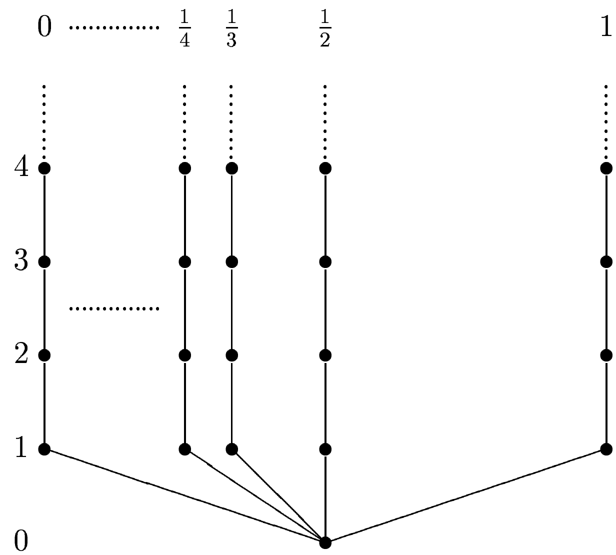
99 Maximal compact topology. This is T_1, but **Top**-automorphisms must fix both the distinguished points x and y, so it is not a specialization topology.

121 The integer broom. Consider polar coordinates (x, θ) on $\mathbb{R}^2$. We take

$$X = \{(n, \theta) \mid n \in \mathbb{N}, \theta \in \{0\} \cup \{1/n \mid n \in \mathbb{N}^+\}\}$$

where all the pairs $(0, \theta)$ refer to the same point, the origin. Take as a basis of open sets all subsets of X of the form $U \times V$ where $U = \{n \mid n \geqslant a\}$, some $a \in \mathbb{N}$ and V is open in the Euclidean subspace topology of $\{0\} \cup \{1/n \mid n \in \mathbb{N}^+\} \subseteq \mathbb{R}$. However, we insist that the only neighbourhood of the origin is X.

The specialization order is a good way of describing this space. It is constructed by taking a copy of the natural numbers $\mathbb{N}$ with their usual order for each element of $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}^+\}$, and identifying the zeros.



An open subset is an upper section for which each row of the diagram is open in the metric topology on the row.

Each copy of $\mathbb{N}$ has the Υ-topology (which is also the $\mathcal{W}$-topology) as a subspace, but the topology of the space as a whole is not a specialization topology, because the leftmost copy of $\mathbb{N}^+$ in the diagram, that indexed by 0, is not open, whereas all the other copies are. There is only one $\leftrightsquigarrow$-block, since the zeros have been identified, so the T_1 reflection is $\mathbf{1}$.

The irreducible closed sets in the integer broom are the point closures and the copies of $\mathbb{N}$. The sober reflection is constructed by adding a point $+\infty$ to each of the copies.

Chapter 9

Sobriety of specialization topologies

We now investigate the question of when a specialization topology is sober. In particular, can we give a condition on the specialization order for the space to be sober? Since a space is T_0 iff its specialization order is a partial order, we will generally be concerned with presobriety rather than sobriety.

9.1 Directed sets

The main property of quasiorders we will need is that of directedness. As we will see, this is closely related to the irreducibility of closed sets.

9.1 Definition. Let $\langle X, \leqslant \rangle$ be a quoset. A subset $S \subseteq X$ is *directed* (or, in full, *upwards directed*) iff $S \neq \emptyset$ and $(\forall x, y \in S)(\exists z \in S)(x, y \leqslant z)$. ■

The following result applies to any space, not just those with a specialization topology.

9.2 Proposition. *Let X be a topological space with specialization order $\leqslant$, and let $S \subseteq X$ be directed in the specialization order. Then $\overline{S}$ is irreducible. In particular, closed directed sets are irreducible.*

Proof. Suppose $\overline{S} = F_1 \cup F_2$ with F_1, F_2 closed, proper subsets of $\overline{S}$. Then $\exists x_1 \in S - F_2, x_2 \in S - F_1$. Since S is directed, $(\exists y \in S)(x_1, x_2 \leqslant y)$. We must have $y \in F_i, i = 1$ or 2, but then also $x_1, x_2 \in F_i$, which is a contradiction. So $\overline{S}$ is irreducible. □

9.3 Definition. We say a quoset $\langle X, \leqslant \rangle$ *has directed suprema* iff every directed subset $S \subseteq X$ has a supremum. ■

9.4 Proposition. *Let X be a presober topological space. Then its specialization order has directed suprema. Also for each directed subset S, and supremum x of S we have $\overline{S} = \downarrow(x)$.*

Proof. Let $S \subseteq X$ be directed. Then $\overline{S}$ is irreducible, so must have a generic point, say x. Then $S \subseteq \downarrow(x)$, so x is an upper bound for S. Now suppose y is another upper bound. Then $S \subseteq \downarrow(y)$, but principal lower sections are closed, so $\overline{S} \subseteq \downarrow(y)$, and $\overline{S} = \downarrow(x)$, so $x \leqslant y$. So x is a supremum of S. □

In the case of Alexandrov spaces there is a converse to proposition 9.2.

9.5 Proposition. *Let X be an Υ-space. Then every irreducible closed set is directed.*

Proof. Suppose $F \subseteq X$ is closed and not directed. Then there are $x, y \in F$ with no upper bound in F. Then $F = (F - \uparrow(x_1)) \cup (F - \uparrow(x_2))$, and $F - \uparrow(x_1)$ and $F - \uparrow(x_2)$ are Υ-closed, so F is reducible. □

This is not true for arbitrary spaces. For example, take (again!) X to be an infinite space with the cofinite topology. Then X itself is irreducible, but not directed. However, we now have enough to give a classification of the sober, and indeed presober, Alexandrov spaces in terms of the specialization order.

9.6 Theorem. *Let X be an Υ-space. Then X is presober iff every directed subset has a maximal element.*

Proof. Suppose X is presober, and $S \subseteq X$ is directed. Then $\overline{S}$ is irreducible, so has a maximal element, say x. Then by the definition of $\overline{S}$, there is some $y \in S$ such that $x \leqslant y$, so y is a maximal element of S. Conversely, suppose every directed subset has a maximal element, and let $F \subseteq X$ be an irreducible closed set. Then F is directed, by proposition 9.5, and so has a maximal element, say x. For any $y \in F$, the pair $\{x, y\}$ has an upper bound $z \in F$, since F is directed. But then $x \leqslant z$, and so $z \leqslant x$ since x is maximal in F. Thus $y \leqslant x$, and this holds for all $y \in F$, i.e. x is a generic point for F. □

This condition is equivalent to a more familiar one, using a weak form of the axiom of choice.

9.7 Definition. Let X be a quoset. We say X satisfies the *ascending chain condition* (or ACC) iff there are no infinite strictly ascending chains

$$x_1 < x_2 < x_3 < \cdots$$

in X. ■

9.8 Proposition. *A quoset X satisfies ACC iff every directed subset of X has a maximum.*

Proof. Suppose X has an infinite strictly ascending chain, $\mathcal{C}$. Then $\mathcal{C}$ is a directed set with no maximum. Conversely, suppose S is a directed subset of X with no maximum. Choose $x_1 \in S$. Then x_1 is not a maximum of S, so there is some $y \in S$ with $y \nleqslant x_1$. Since S is directed, we may choose $x_2 \in S$ with $x_1, y \leqslant x_2$. If $x_2 \leqslant x_1$ then also $y \leqslant x_1$ which is a contradiction. So $x_1 < x_2$. Repeating this, using the axiom of dependent choice, we get a strictly ascending chain

$$x_1 < x_2 < x_3 < \cdots$$

in S, so X does not satisfy ACC. □

The notion of a directed subset of a quasiorder turns out to allow us to construct another topology on any quasiorder, called the Scott topology.

9.9 Definition. Let X be a quoset and U an upper section of X. We say that U is *inaccessible by directed suprema* iff for every directed subset S of X, if U contains some supremum of S then U meets S. ■

9.10 Proposition. *Let $\langle X, \leqslant \rangle$ be a quoset. Then the collection of upper sections which are inaccessible by directed suprema is closed under finite intersections and arbitrary unions.*

Proof. Let $\mathcal{U}$ be a family of upper sections inaccessible by directed suprema. Suppose S is directed and has a supremum s with $s \in \bigcup \mathcal{U}$. Then $s \in U$ for some $U \in \mathcal{U}$, and so S meets U. Thus S meets $\bigcup \mathcal{U}$.

Now suppose U_1, U_2 are upper sections inaccessible by directed suprema, and S is directed and has a supremum s with $s \in U_1 \cap U_2$. Then $s \in U_1$ and $s \in U_2$, so there is some $x_1 \in S \cap U_1$ and some $x_2 \in S \cap U_2$. Since S is directed, there is a $y \in S$ with $x_1, x_2 \leqslant y$. Then $y \in U_1 \cap U_2$ since U_1, U_2 are open, so upper sections. Thus S meets $U_1 \cap U_2$. □

Thus the upper sections inaccessible by directed suprema form the frame of open sets of a topology on X, called the *Scott* topology, $\Sigma(X)$. This is the maximum sober topology in the following sense.

9.11 Proposition. *Let $\langle X, \leqslant\rangle$ be a quoset, and τ a sober topology on X with specialization order $\leqslant$. Then every τ-open subset of X is Σ-open.*

Proof. Suppose that U is τ-open, S is directed and U contains a supremum, s of S. By proposition 9.4, the τ-closure, $\overline{S}$ is $\downarrow(s)$, so U meets $\overline{S}$. But U is open so U meets S. So U is Σ-open. □

9.2 Linear orders

We look at the case when the specialization order is a linear order, where we can give a complete characterization of the sober spaces for each of our topologies. This is a very special case, but more general examples such as number 121 in Counterexamples in Topology, the integer broom, can be constructed by combining linear orders.

If the specialization order is a linear order then the set of lower sections forms a chain, so every non-empty closed set is irreducible. From this we can deduce at once what the sober spaces are.

9.12 Theorem. *Let X be a topological space whose specialization order $\leqslant$ is a linear order on X. Then X is sober iff it has the $\mathcal{W}$-topology and a top element.*

Proof. The $\mathcal{W}$-closed sets are precisely the principal lower sections and X itself. X is a principal lower section iff it has a top, so the $\mathcal{W}$-topology is sober iff X has a top. Now suppose X has some other topology. Then some non-principal lower section F must be closed, but F is necessarily irreducible and has no generic point, so X is not sober. □

In order to see when the Alexandrov and Scott topologies are sober we can just see when they coincide with the $\mathcal{W}$-topology. We already know when the Alexandrov topology is sober, but this calculation is also interesting in its own right, since we know that there is only one topology with a given specialization order if the Alexandrov and $\mathcal{W}$-topologies coincide, and that there is at most one sober topology when the Scott and $\mathcal{W}$-topologies coincide.

9.13 Lemma. *Let $\langle X, \leqslant \rangle$ be a linearly ordered set. Then the Υ- and $\mathcal{W}$-topologies on X coincide iff every proper lower section of X has a maximum.* □

Proof. By a proper lower section we mean a lower section which is a proper subset. The result follows immediately from the definitions. □

By an *order type*, we mean an isomorphism class of linearly ordered sets. Any ordinal is an order type; in fact the ordinals are defined to be precisely those order types in which every subset has a minimum. For any order type β, we write β^* for the order type obtained by reversing the order on β. So the order types α^* where α is an ordinal are precisely those in which every subset has a maximum. For order types β and γ, write $\beta + \gamma$ for the order type obtained by putting γ on top of β. For example, $\omega^* + \omega$ is the order type of the integers. With this notation, we can give a complete description of those linear orders which are the specialization order of only one topology.

9.14 Theorem. *Let X be a linearly ordered set. Then the Υ- and $\mathcal{W}$-topologies on X coincide iff the order type of X is either α^* for some ordinal α or a stack*

$$\alpha_1{}^* + \alpha_2{}^* + \alpha_3{}^* + \cdots$$

of length ω, of opposites of non-zero ordinals.

Proof. Firstly, note that if X has one of these order types then every proper lower section does have a maximum. It remains to show that these are the only order types with this property. So suppose that X is a linearly ordered set for which every proper lower section has a maximum. If X has a maximum then every subset has a maximum, so X has order type α^* for some ordinal α. If X has no maximum then by proposition 9.8 it has an infinite strictly ascending chain, say $x_1 < x_2 < \cdots$. Then $\bigcup\{\downarrow(x_n) \mid n \in \mathbb{N}^+\}$ is a lower section with no maximum, so must be X. Let $\alpha_1 = \downarrow(x_1)^*$ and $\alpha_{n+1} = (\downarrow(x_{n+1}) - \downarrow(x_n))^*$ for $n \in \mathbb{N}^+$. Then each α_n is the order type of an ordinal, since it is an initial segment of $\downarrow(x_n)^*$, which is an ordinal by the first part of the proof. Each α_n is non-zero since the sequence (x_n) is strictly increasing. In this case, X has order type

$$\alpha_1{}^* + \alpha_2{}^* + \alpha_3{}^* + \cdots$$

as required. □

9.15 Corollary. *Let X be an Alexandrov space with linear specialization order. Then X is sober iff it has order type α^* for some ordinal α.* □

This agrees with our general theorem, 9.6, on when an arbitrary Alexandrov space is sober.

The condition for when the Scott and $\mathcal{W}$-topologies coincide for linear orders looks at first sight much like that for when the Alexandrov and $\mathcal{W}$-topologies coincide. However, the conditions behave very differently in practice, as the list of examples shows.

9.16 Theorem. *Let X be a linearly ordered set. Then the Σ- and $\mathcal{W}$-topologies on X coincide iff every proper lower section of X has a supremum.*

Proof. A Σ-closed subset of X is a lower section which contains its supremum if it has one. It is $\mathcal{W}$-closed iff it does have a supremum. □

Elementary analysis shows that this condition is equivalent to every bounded subset of X having a supremum and an infimum. This immediately gives us our first example.

9.17 Examples. The Scott and $\mathcal{W}$-topologies coincide for the real line $\mathbb{R}$ with its usual ordering. They also coincide for any ordinal, and for the opposite of any ordinal. They do not agree for the order type $\omega + \omega^*$. ■

9.18 Corollary. *Suppose X is a linearly ordered set with the Σ-topology. Then X is sober iff every subset has a supremum.* □

A linearly ordered set with the Alexandrov topology is sober iff every directed subset has a maximum, and we have shown that we can drop the assumption that the order is linear – the result also holds for any partial order. A natural question to ask is if the corresponding generalization holds for the Scott topology. Is it the case that a partially ordered set with the Scott topology is sober iff every directed subset has a supremum? Proposition 9.4 shows that one direction holds. Peter Johnstone showed that the other does not in [4], by giving the following example.

9.19 Example. Let $X = \mathbb{N} \times (\mathbb{N} \cup \{+\infty\})$, and define a partial order $\leq$ on X by

$$(m,n) \leq (m',n') \iff \begin{cases} \text{either} & m = m' \;\&\; n \leqslant n' \\ \text{or} & n' = +\infty \;\&\; n \leqslant m' \end{cases}$$

where $\leqslant$ is the usual ordering on $\mathbb{N} \cup \{+\infty\}$. This poset has directed joins, but its Scott topology is not sober. ■

Having seen when these spaces are sober, we would like to compute the sober reflections of spaces with linear specialization order. The order type of the rational numbers gives a good example.

9.20 Example. Consider $\mathbb{Q}$ as a linearly ordered set.

- Firstly, consider the $\mathcal{W}$-topology on $\mathbb{Q}$. Its closed sets are $\mathbb{Q} \cap (-\infty, q]$ for each $q \in \mathbb{Q}$, and $\mathbb{Q}$ itself, so its sober reflection is $\mathbb{Q} \cup \{+\infty\}$.

- The Υ-topology on $\mathbb{Q}$ has all lower sections as its closed sets. Since every bounded set in $\mathbb{R}$ has a supremum, each is one of four types:

$$\mathbb{Q}, \quad \mathbb{Q}\cap(-\infty,q), \quad \mathbb{Q}\cap(-\infty,q], \quad \text{or} \quad \mathbb{Q}\cap(-\infty,x]$$

  where $q\in\mathbb{Q}$ and $x\in\mathbb{R}-\mathbb{Q}$.

  The sober reflection of $\Upsilon(\mathbb{Q})$ is thus $\mathbb{R}\cup\{+\infty\} \sqcup \mathbb{Q}'$, where $\mathbb{R}\cup\{+\infty\}$ has the usual ordering of the real line, but we have an extra copy of $\mathbb{Q}$, namely $\mathbb{Q}'=\{q^-\mid q\in\mathbb{Q}\}$ with $q^-<q$ for each $q\in\mathbb{Q}$ but $x<q^-$ iff $x<q$ for all other x. Since this space is sober we know it must have the $\mathcal{W}$-topology.

- The Σ-topology on $\mathbb{Q}$ has as open sets the upper sections which are inaccessible by suprema. Hence the Σ-closed sets are the Υ-closed sets with the exception of those of the form $\mathbb{Q}\cap(-\infty,q)$. Thus the sober reflection of $\Sigma(\mathbb{Q})$ is $\mathbb{R}\cup\{+\infty\}$ with the $\mathcal{W}$-topology. ■

Using this example, we can see what the sober reflections of these specialization topologies on linearly ordered sets look like in general.

9.21 Theorem. *Let X be a linearly ordered set. Then the sober reflections $S\mathcal{W}X$, $S\Upsilon X$ and $S\Sigma X$ are $\mathcal{W}$-topologies on the linearly ordered sets given below.*

- *$S\mathcal{W}X$ is X if X has a top element, and X^+, with order type $X+1$, otherwise.*

- *$S\Upsilon X$ is the set of lower sections of X, ordered by inclusion.*

- *$S\Sigma X$ consists of those lower sections of X which either contain their supremum or have no supremum, ordered by inclusion.*

In the latter two cases, the topology is also the Scott topology.

Proof. The descriptions of the spaces follow from the descriptions of the topologies, since every closed set is irreducible. It remains to show that $S\Upsilon X$ and $S\Sigma X$ have Scott topologies. By theorem 9.16 we just have to show that each lower section has a supremum. Suppose that $\mathcal{L}$ is a lower section of $S\Upsilon X$. Then $\bigcup \mathcal{L}$ is a lower section of X, and the supremum of $\mathcal{L}$. Now suppose that $\mathcal{L}$ is a lower section of $S\Sigma X$. If $\bigcup \mathcal{L} \in S\Sigma X$ then then it is the supremum of $\mathcal{L}$. If not, it has a supremum, s, in X but doesn't contain it. Then $\bigcup \mathcal{L} \cup \{s\} \in S\Sigma X$ and is the supremum of $\mathcal{L}$. □

9.3 Generalized topologies

The sober reflection of $\mathbb{Q}$ with the Scott topology is almost $\mathbb{R}$, but not quite. We also have the extra point $+\infty$, which is there since the whole space $\mathbb{Q}$ is closed and irreducible. Apart from this one point, this is the construction of the real numbers from the rationals as an ordered set by Dedekind cuts. (We are not considering the construction of the field operations here.) This begs the question – can we tweak the construction in some way so as not to get this extra point? The answer is yes, but at a price. We want the closed subsets of $\mathbb{Q}$ to be just the proper lower sections which are closed under suprema, and not $\mathbb{Q}$ itself. Nor of course do we want $\emptyset$ to be open. I propose the following definition, which would permit this.

9.22 Definition. A *generalized topology* on a set X is given by a collection of subsets of X called *open* subsets, satisfying the following two conditions.

- All finite intersections of open subsets are open.
- All non-empty unions of open subsets are open.

A subset of X is called *closed* iff its complement is open. ■

There are three obvious questions arising from this definition. Firstly, what are the continuous maps? Secondly, the collection of open sets will not in general be a frame, so what does it look like? Thirdly, is there a point-free notion of this generalized topology, and a contravariant adjunction generalizing that between $\mathcal{O}$ and pt? This second question has a bearing on the first. If we want a point free notion then we must keep the definition that a map is continuous iff the inverse image of an open set is open. This produces the following oddity.

9.23 Proposition. *If X and Y are generalized topological spaces, $X \xrightarrow{\theta} Y$ is a continuous map, and $\emptyset$ is not open in X, then the image of θ is dense in Y.*

Proof. Let U be open in Y. Then $\theta^{\leftarrow}U$ is open in X, so non-empty. □

This may be acceptable, since $\emptyset$ will be open in most spaces – indeed any space with two disjoint open sets.

Our motivation for making this definition is to make the Dedekind cut construction a sober reflection. It may be that there is no adjunction between this generalized topology and a point-free notion. In this case, there will be no composite functor pt $\mathcal{O}$ to produce the sober reflection. However, in chapter 2 we showed that the concept of sobriety and the construction of the sober reflection can be considered as entirely topological in nature. We did not make any use of point-free notions in that chapter. The construction of the sober reflection described there carries over to this generalized topology, so we have succeeded in describing the Dedekind cut construction as a purely topological process.

Chapter 10

Closed and compact subsets

10.1 Motivating results

In many applications of topology, the property of compactness of a subset is important. In geometric topology, the spaces considered are usually Hausdorff, and often compact, and there is a close relationship between closed subsets and compact subsets. Indeed, in a compact Hausdorff space, the closed and compact subsets coincide. We would like to know more generally when every closed subset is compact and when every compact subset is closed. The first of these is easy – every closed subset is compact iff the space itself is compact. The second is more difficult, and we investigate this here.

Firstly, we shall prove the assertions we have just made.

10.1 Proposition. *Let X be a topological space, $F \subseteq K \subseteq X$ with F closed and K compact. Then F is also compact.*

Proof. Let $\mathcal{A}$ be an open cover of F. Then $\mathcal{A} \cup \{F^C\}$ is an open cover of X, and thus also of K. Since K is compact, there is a finite subcover of it, say $\mathcal{B}$. Then $\mathcal{B} - \{F^C\}$ is a finite subcover of the original cover of F. □

10.2 Corollary. *Let X be a topological space. Then every closed subset of X is compact precisely when X itself is compact.* □

10.3 Proposition. *Let X be a Hausdorff space and $K \subseteq X$, compact. Then K is closed.*

Proof. Let $x \in K^C$. Since X is Hausdorff, for each $k \in K$ there are disjoint open sets A_k, B_k, such that $x \in A_k, k \in B_k$. Now $(B_k)_{k \in K}$ forms an open cover of K so, since K is compact, there is a finite subcover, say $B_{k_1}, \ldots, B_{k_n}$. Then we have $K \subseteq \bigcup_{i=1}^{n} B_{k_i}$, so $\bigcap_{i=1}^{n} A_{k_i} \subseteq K^C$. Now $\bigcap_{i=1}^{n} A_{k_i}$ is an open neighbourhood of x, and x is arbitrary in K^C, so K^C is open. Thus K is closed. □

10.4 Corollary. *If X is a compact, Hausdorff space and $S \subseteq X$ then S is compact iff it is closed.* □

10.5 Proposition. *Let X be a topological space in which every compact subset is closed. Then X has the T_1 separation property.*

Proof. The finite subsets of X are trivially compact, so are closed by assumption. But a space is T_1 iff its finite subsets are closed. □

With these results, we see that we can restrict our search to spaces which are T_1 but not T_2. It is also useful to have a name for the property we are investigating, so we make the following definition.

10.6 Definition. Let X be a T_1 space. We say that X is *packed* iff all of its compact subsets are closed. ■

The reason for the restriction to T_1 spaces is that there is a more general notion which coincides with this one for T_1 spaces, and that is also called packedness.

10.2 Calculations

There are seven spaces in Counterexamples in Topology which are T_1 and not T_2. We calculate which of them are packed. See chapter 3 for the definitions of the spaces.

18-19 Cofinite topology Every subset of X is compact. To see this, let $S \subseteq X$, and let $\mathcal{U}$ be an open cover of S. Choose any U_0 from the cover. Then $S - U_0$ is finite, say $S - U_0 = \{x_1, \ldots, x_n\}$. For each $i = 1, \ldots, n$, pick U_i from the cover such that $x_i \in U_i$. Then $U_0, \ldots, U_n$ is a finite subcover of S.

The infinite subsets of X are thus compact but not closed, so X is not packed.

20 Cocountable topology The only compact subsets of X are the finite subsets. For suppose that $S \subseteq X$ is infinite, and let $T \subseteq S$ be countably infinite. For each $t \in T$, let $U_t = (X - T) \cup \{t\}$. Then $(U_t)_{t \in T}$ forms an open cover of X with no finite subcover of S. So S is not compact. So every compact subset is closed, and thus X is packed.

In most respects, the cofinite and cocountable topologies are very similar. However, in this respect they are very different.

22 Cocompact topology on $\mathbb{R}$ Let τ be the Euclidean topology on $\mathbb{R}$ and τ^* be the cocompact topology, so the τ^*-closed subsets are $\mathbb{R}$ and the τ-compact subsets. Consider $\mathbb{Z} \subseteq \mathbb{R}$. It is not τ-compact, so is not τ^*-closed. Now let $\mathcal{U}$ be any τ^*-open cover of $\mathbb{Z}$, and $U \in \mathcal{U}$. Then U^C is τ-compact, hence bounded, so $U^C \cap \mathbb{Z}$ is finite. So U contains all but finitely many members of $\mathbb{Z}$ so, as for the cofinite topology, $\mathbb{Z}$ is τ^*-compact. So $\langle \mathbb{R}, \tau^* \rangle$ is not packed.

27 Modified Fort space Let $S = X - \{x_2\}$. Then S is not closed, since it contains x_1 and is infinite. But any open set containing x_1 is cofinite, so S must be compact by the same argument as for the cofinite topology. Thus X is not packed.

35 One point compactification of $\mathbb{Q}$ We have the Euclidean topology τ on $\mathbb{Q}$ and the one point compactification $X = \mathbb{Q} \sqcup \{*\}$ with topology τ^*. Suppose K is τ^* compact in X, and $* \notin K$. Let $\mathcal{U}$ be a τ-open cover of K in $\mathbb{Q}$. Then, since every τ-open set is also τ^*-open, and K is τ^*-compact, this has a finite subcover. So K is τ-compact. Now τ is a Hausdorff topology, so K is also τ-closed, and hence it is τ^*-closed by the definition of τ^*.

Now suppose that $K \subseteq X$ and $* \in K$. Let $K' = K - \{*\}$ and suppose that K is not τ^*-closed. Then K' is not τ-closed, so there is some sequence of distinct points (x_n) in K' with a limit point $x \in \mathbb{Q} - K'$. Now $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is τ-closed and compact, so its complement, say U is τ^*-open. Also, for each $n \in \mathbb{N}$ there is a τ-open subset of $\mathbb{Q}$, U_n, such that for each $m \in \mathbb{N}$, $x_m \in U_n$ iff $m = n$. Then $\{U\} \cup \{U_n \mid n \in \mathbb{N}\}$ is a τ^*-open cover of K' with no proper subcover, since we need U to cover $*$ and U_n to cover x_n. So K is not compact. Hence X is packed.

73 Telophase topology The subspace $[0, 1]$ has the Euclidean subspace topology, so is compact by the Heine-Borel theorem. However, it is not closed, since its complement $\{1^*\}$ is not open. So the space is not packed.

99 Maximal compact topology This is packed, as explained in paragraph three of the discussion in Counterexamples in Topology.

Appendix A

Adjunctions

In this appendix we give a few definitions and results about adjunctions.

A.1 Definition. Let $\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D}$ be functors between categories. An *adjunction* between F and G is an isomorphism

$$\mathbb{C}(A, GB) \xrightarrow[\cong]{\alpha} \mathbb{D}(FA, B)$$

natural in $A \in \mathbb{C}$ and $B \in \mathbb{D}$. If there is an adjunction between F and G we write $F \dashv G$ and say that F is *left adjoint* to G and G is *right adjoint* to F.

Putting $B = FA$ in the above isomorphism, we get a family of arrows $A \xrightarrow{\eta_A} GFA$ of $\mathbb{C}$, given by $\eta_A = \alpha^{-1}(1_{FA})$. These form a natural transformation $1_{\mathbb{C}} \xrightarrow{\eta} GF$ called the *unit* of the adjunction. Dually, we have a natural transformation $FG \xrightarrow{\epsilon} 1_{\mathbb{D}}$ called the *counit* of the adjunction, with components $\epsilon_B = \alpha(1_{GB})$. ■

A.2 Theorem. *The unit and counit of any adjunction satisfy the* triangle identities. *That is, the following two diagrams commute in the appropriate functor* *categories. Note that, for example, ϵF is the natural transformation with components ϵ_{FA}.*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

Furthermore, if $\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D}$ are functors and $1_{\mathbb{C}} \xrightarrow{\eta} GF$, $FG \xrightarrow{\epsilon} 1_{\mathbb{D}}$ are natural transformations satisfying the triangle identities then there exists a (unique) adjunction $F \dashv G$ for which η is the unit and ϵ the counit.

Proof. See [8, p83]. □

A.3 Theorem. *Let $\mathbb{D} \xrightarrow{G} \mathbb{C}$ be a functor and suppose that for each $A \in \mathbb{C}$ we have an object FA of $\mathbb{D}$ and an arrow $A \xrightarrow{\eta_A} GFA$ with the universal property that for each arrow $A \xrightarrow{f} GB$ there is a unique $FA \xrightarrow{\bar{f}} B$ such that*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & GFA \arrow[d, "G\bar{f}"] \\ & GB \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[d, dotted, "\exists!" ', "\bar{f}"] \\ B \end{tikzcd}$$

commutes. Then F is a functor, and there exists a unique adjunction $F \dashv G$ with unit η. Conversely, if $F \dashv G$ is an adjunction with unit η, then this universal property holds.

Proof. See [8, p83]. □

A.4 Definition. A subcategory $\mathbb{D} \hookrightarrow \mathbb{C}$ is a *full* subcategory iff for each pair of objects $A, B \in \mathbb{D}$, every arrow $A \longrightarrow B$ in $\mathbb{C}$ is also in $\mathbb{D}$. A full subcategory is said to be *reflective* iff the inclusion functor has a left adjoint, R. The left adjoint is called a *reflector*, and has the universal property that for any $A \in \mathbb{C}$

and $B \in \mathbb{D}$, for each arrow $A \xrightarrow{f} B$ in $\mathbb{C}$, there is a unique $RA \xrightarrow{f^\sharp} B$ such that f factors as

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & RA \arrow[d, dotted, "\exists!\, f^\sharp"] \\ & B \end{tikzcd}$$

We often refer to the universal arrow $A \xrightarrow{\eta_A} RA$ as the reflection of A, and not just the object RA. A right adjoint to the inclusion functor is called a *coreflector*, and has the dual universal property. ■

A.5 Definition. Let $\mathbb{C} \xrightarrow{F} \mathbb{D}$ be a functor. The *essential full image of F* is the full subcategory $\mathcal{I}(F)$ of $\mathbb{D}$ whose objects are those B such that $B \cong FA$ for some $A \in \mathbb{C}$. ■

A.6 Proposition (Idempotent adjunctions). *Let* $\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D}$ ($\perp$) *be an adjunction with unit η and counit ϵ. Then the following are equivalent.*

1. *$F\eta$ is a natural isomorphism*
2. *ϵF is a natural isomorphism*
3. *$G\epsilon F$ is a natural isomorphism*
4. $GF\eta = \eta GF$
5. $GF\eta G = \eta GFG$
6. *$G\epsilon$ is a natural isomorphism*
7. *ηG is a natural isomorphism*
8. *$F\eta G$ is a natural isomorphism*
9. $FG\epsilon = \epsilon FG$

10. $FG\epsilon F = \epsilon FGF$

A.7 Definition. An adjunction for which these conditions hold is called *idempotent*. ■

Compare this with the notion of an endomorphism f being called idempotent if $f^2 = f$. For any adjunction $F \dashv G$, given an object $A \in \mathbb{C}$ we can form a sequence of objects $FA, FGFA, FGFGFA, \ldots \in \mathbb{D}$. In general, these objects will all be different, for example, if F is the free group functor and G the forgetful functor from groups to sets. In an idempotent adjunction however, these objects are all naturally isomorphic, since $FGF \cong F$. Similarly $GFG \cong G$. Thus an idempotent adjunction can be considered as a pair of projections (up to isomorphism) between the categories.

Proof of A.6. 1 $\implies$ 2 is immediate from the first triangle identity.

2 $\implies$ 3 follows since functors preserve isomorphisms.

3 $\implies$ 4 follows from the commutativity of the following diagram. The left triangle is the image under G of the first triangle identity and the right triangle is the second triangle identity restricted to the image of F.

$$\begin{tikzcd}
GF \arrow[r, "GF\eta"] \arrow[dr, "1_{GF}"'] & GFGF \arrow[d, "G\epsilon F"] & GF \arrow[l, "\eta GF"'] \arrow[dl, "1_{GF}"] \\
& GF &
\end{tikzcd}$$

4 $\implies$ 5 since 5 is just the restriction of 4 to the image of G.

For 5 $\implies$ 6, suppose $GF\eta G = \eta GFG$. Then in the following diagram the outer square commutes by naturality of η and the upper triangle is an image of the second triangle identity. So the lower triangle also commutes, i.e. $\eta G \circ G\epsilon = 1_{GFG}$.

$$\begin{tikzcd} GFG \arrow[r, "GF\eta G"] \arrow[d, "G\epsilon"'] \arrow[dr, "1_{GFG}"] & GFGFG \arrow[d, "GFG\epsilon"] \\ G \arrow[r, "\eta G"'] & GFG \end{tikzcd}$$

We know that $G\epsilon \circ \eta G = 1_{GFG}$ by the second triangle identity, so $G\epsilon$ is an isomorphism. The implications

$$6 \implies 7 \implies 8 \implies 9 \implies 10 \implies 1$$

follow from those we have proved by duality. □

A.8 Theorem. *Let* $\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D}$ ($\perp$) *be an idempotent adjunction. Then it restricts to an equivalence of categories between* $\mathcal{I}(F)$ *and* $\mathcal{I}(G)$. *Furthermore,* $\mathcal{I}(G)$ *is reflective in* $\mathbb{C}$ *with reflector the composite* GF*, and* $\mathcal{I}(F)$ *is coreflective in* $\mathbb{D}$ *with coreflector* FG.

Proof. An adjunction is an equivalence iff its unit and counit are natural isomorphisms, i.e. iff each of their components is an isomorphism. Let $A \in \mathcal{I}(G)$. Then we have an isomorphism $A \xrightarrow{i} GB$ for some $B \in \mathbb{D}$ and, by assumption, $\eta_G B$ is an isomorphism. By naturality of η,

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "i"'] & GFA \\ GB \arrow[r, "\eta_{GB}"'] & GFGB \arrow[u, "GFi^{-1}"'] \end{tikzcd}$$

commutes, and so η_A is an isomorphism. Essentially the same argument shows that the counit is a natural isomorphism.

To show that GF is a reflector for $\mathcal{I}(G)$ in $\mathbb{C}$, we show that it has the appropriate universal property. Let $A \in \mathbb{C}$, $B \in \mathcal{I}(G)$ and $A \xrightarrow{f} B$ in $\mathbb{C}$. Firstly, suppose that $B = GD$ for some $D \in \mathbb{D}$. Then by the universal property of the adjunction $F \dashv G$, there is $FA \xrightarrow{\bar{f}} D$ such that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & GFA \arrow[d, "G\bar{f}"] \\ & GD \end{tikzcd}$$

commutes. So we have existence. For uniqueness, suppose we have

$$\begin{tikzcd} A \arrow[r, "\eta_A"] & GFA \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & GD \end{tikzcd}$$

with $g\eta_A = h\eta_A$. Then $(GFg) \circ (GF\eta_A) = (GFh) \circ (GF\eta_A)$, but $GF\eta_A$ is an isomorphism, so $GFg = GFh$. Then, by naturality of ϵ,

$$\begin{tikzcd} GFGFA \arrow[r, "GFg = GFh"] \arrow[d, "G\epsilon_{FA}"'] & GFGD \arrow[d, "G\epsilon_D"] \\ GFA \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & GD \end{tikzcd}$$

commutes serially. But $G\epsilon_{FA}$ is an isomorphism, so $g = h$. So we have a unique $f^\sharp$ such that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & GFA \arrow[d, "f^\sharp"] \\ & GD \end{tikzcd}$$

commutes. Now for a general $B \in \mathcal{I}(G)$, we have an isomorphism $B \xrightarrow{i} GD$ for some $D \in \mathbb{D}$. Then

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & GFA \arrow[d, "(if)^\sharp"] \arrow[dl, "f^\sharp"] \\ B \arrow[r, "i"'] & GD \end{tikzcd}$$

must commute, so we must have $f^\sharp = i^{-1} \circ (if)^\sharp$, which gives both existence and uniqueness.

The fact that FG is a coreflector for $\mathcal{I}(F)$ in $\mathbb{D}$ follows by duality. $\square$

The general adjoint functor theorem gives conditions for the existence of a left adjoint to a given functor. It is useful in conditions where the existence of a left adjoint is suspected but where is difficult or impossible to construct it explicitly. We state a special case of it below.

A.9 Theorem. *Let $\mathbb{D} \hookrightarrow \mathbb{C}$ be a full subcategory, where $\mathbb{D}$ is locally small and has all small products and equalizers. Suppose that for each $A \in \mathbb{C}$ there is a small subset $\mathbb{S} \subseteq \mathrm{ob}\,\mathbb{D}$ such that each arrow $A \xrightarrow{f} B$ in $\mathbb{C}$ where $B \in \mathbb{D}$ factors as*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & S \arrow[ur, "h"'] & \end{tikzcd}$$

for some $S \in \mathbb{S}$ and some arrows g, h in $\mathbb{C}$. Then $\mathbb{D}$ is a reflective subcategory of $\mathbb{C}$.

Proof. See [8, pp 113-121]. $\square$

Appendix B

Reference tables

The calculations done in this thesis are summarised in two reference tables here. The first contains the non-Hausdorff spaces from Counterexamples in Topology and gives information about the separation properties, sobriety, the type of specialization topology if any, and the reflections calculated. The second contains the seven spaces which are T_1 but not T_2, giving their packedness and, for comparison, their compactness.

The following abbreviations are used in the first table. P = presober, S = sober, ST = specialization topology. For the four properties, 1 indicates that the space has the property, 0 that it does not. The blanks for the partition topology indicate that the properties depend on the particular partition space. In the specialization topology column, 0 indicates that the topology is not a specialization topology, $\mathcal{W}$ that it is a $\mathcal{W}$-topology, Υ that it is an Alexandrov topology, and 1 that it is some other specialization topology. For the reflections, X indicates that the reflection is the space itself, **1** that it is the one point space, D that it is a discrete space, $\mathbb{R}$ is the real numbers with the metric topology, R_0 indicates that the reflection is the same as the T_0 reflection, and X^+ indicates the construction described in 2.13. In the case of a blank, refer to chapter 8 for a description of the space.

Table B.1: Sobriety, specialization and reflections

		Properties					Reflections		
No.	Name	T_0	T_1	P	S	ST	R_0	R_1	S
1-3	Discrete	1	1	1	1	Υ	X	X	X
4	Indiscrete	0	0	1	0	Υ	D	D	D
5	Partition			1		Υ	D	D	D
6	Odd-even partition	0	0	1	0	Υ	D	D	D
7	Deleted integer	0	0	1	0	Υ	D	D	D
8-11	Particular point	1	0	1	1	Υ	X	**1**	X
13-15	Excluded point	1	0	1	1	Υ	X	**1**	X
17	Either-Or	1	0	1	1	Υ	X	D	X
18-19	Cofinite	1	1	0	0	$\mathcal{W}$	X	X	X^+
20	Cocountable	1	1	0	0	1	X	X	X^+
21	Double pointed cocountable	0	0	0	0	1		R_0	${R_0}^+$
22	Cocompact on $\mathbb{R}$	1	1	0	0	0	X	X	X^+
27	Modified Fort	1	1	1	1	0	X	X	X
35	1-pt compactification of $\mathbb{Q}$	1	1	1	1	0	X	X	X
50	"Right order" on $\mathbb{R}$	1	0	0	0	$\mathcal{W}$	X	**1**	X^+
52	Nested interval	0	0	1	0	$\mathcal{W}$,Υ		**1**	R_0
53	Overlapping interval	1	0	1	1	$\mathcal{W}$	X	**1**	X
54	Interlocking interval	0	0	0	0	$\mathcal{W}$		**1**	${R_0}^+$
55	Hjalmar-Ekdal	1	0	1	1	Υ	X	D	X
56	Prime Ideal	1	0	1	1	$\mathcal{W}$	X	**1**	X
57	Divisor	1	0	1	1	Υ	X	**1**	X
62	Double pointed $\mathbb{R}$	0	0	1	0	0	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
73	Telophase	1	1	1	1	0	X	X	X
99	Maximal compact	1	1	1	1	0	X	X	X
121	Integer broom	1	1	1	1	0	X	X	

Table B.2: Compactness and Packedness

No.	Name	Compact	Packed
18-19	Cofinite	1	0
20	Cocountable	0	1
22	Co-compact on $\mathbb{R}$	1	0
27	Modified Fort	1	0
35	1-pt compactification of $\mathbb{Q}$	1	1
73	Telophase	1	0
99	Maximal compact	1	1

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