The Theory of Exponential Differential Equations

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To the memory of my mother, who always encouraged my interest in mathematics but did not quite live to read these words, and to my father, who has always been wonderfully supportive.

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Abstract

This thesis is a model-theoretic study of exponential differential equations in the context of differential algebra. I define the theory of a set of differential equations and give an axiomatization for the theory of the exponential differential equations of split semiabelian varieties. In particular, this includes the theory of the equations satisfied by the usual complex exponential function and the Weierstrass $\wp$-functions.

The theory consists of a description of the algebraic structure on the solution sets together with necessary and sufficient conditions for a system of equations to have solutions. These conditions are stated in terms of a dimension theory; their necessity generalizes Ax's differential field version of Schanuel's conjecture and their sufficiency generalizes recent work of Crampin. They are shown to apply to the solving of systems of equations in holomorphic functions away from singularities, as well as in the abstract setting.

The theory can also be obtained by means of a Hrushovski-style amalgamation construction, and I give a category-theoretic account of the method.

Restricting to the usual exponential differential equation, I show that a "blurring" of Zilber's pseudo-exponentiation satisfies the same theory. I conjecture that this theory also holds for a suitable blurring of the complex exponential maps and partially resolve the question, proving the necessity but not the sufficiency of the aforementioned conditions.

As an algebraic application, I prove a weak form of Zilber's conjecture on intersections with subgroups (known as CIT) for semiabelian varieties. This in turn is used to show that the necessary and sufficient conditions are expressible in the appropriate first order language.

Contents

Chapter 1

Introduction

In this introduction I first give a detailed summary of the contents of the thesis and the dependencies between the sections. I then put the work into context by explaining how it relates to other work in model theory and in number theory. Finally, I suggest ways in which the work could be extended in the future.

1.1 Summary of the thesis

The reader is assumed to have some knowledge of model theory, but the statements of most of the main results and many of the proofs do not use any model theory explicitly, so I hope that they will be accessible to interested parties without such knowledge.

The main work of the thesis, establishing the theory of exponential differential equations, comprises chapters 4 to 7. Chapters 2 and 3 contain background material from differential algebra and algebraic geometry, and the final chapter, chapter 8, gives an application.

The aim of chapter 2 is to give the definitions which are necessary for the various ways to view differential forms which are used in the thesis. Chapter 3 gives definitions and some classification theorems for algebraic groups and their subgroups, in particular for abelian and semiabelian varieties. Invariant differential forms on a group G are defined and shown in proposition 3.20 to define group homomorphisms $G(F) \longrightarrow \Omega(F/C)$. Apart perhaps from this result and the viewpoint which leads to it and stresses it, the material in these two chapters is not new.

In chapter 4, the differential equations which are the main subject of this thesis are defined as the kernels Γ of maps arising from invariant differential forms in a differential field $\langle F; +, \cdot, D, C\rangle$. The definition is given quite generally, but the main example is as follows. The algebraic group G is given as $\mathbb{G}_a^n \times S$ where S is a semiabelian variety of dimension n, for example $\mathbb{G}_m$, the multiplicative group. In this special case, the set Γ is given directly as

$$\Gamma = \left\{ (x, y) \in \mathbb{G}_a(F) \times \mathbb{G}_m(F) \;\middle|\; \frac{Dy}{y} = Dx \right\}.$$

This is of course the differential equation satisfied if $y = e^x$ and x and y are holomorphic functions.

The model theoretic context in which these differential equations are studied is the reduct $\langle F; +, \cdot, \Gamma, C \rangle$ of a differential field, usually with the solution sets of several differential equations distinguished. For most of the thesis a set $\mathcal{S}$ of semiabelian varieties, each defined over C, is chosen, and the language used is $\mathcal{L}_{\mathcal{S}}$ which has a symbol Γ_S for each $S \in \mathcal{S}$ representing the solution set to the exponential equation for S. In chapter 4, however, a more general approach is taken. These solution sets Γ are shown to have an algebraic structure; in particular proposition 4.1 shows that they are subgroups of the relevant algebraic groups G.

The main question to answer in the study of these differential equations is which systems of equations can have solutions. This can be expressed as asking which algebraic varieties have nonempty intersection with Γ (or rather, with its non-constant points). Under the model-theoretically motivated assumption that this is controlled by a dimension theory, a heuristic argument is given describing putative necessary and sufficient conditions. These conditions are proved in chapters 5 to 7.

The necessary conditions for a system of equations to have solutions are called *Schanuel conditions*, after the conjecture of Stephen Schanuel about the usual complex exponential function, see [Lan66].

Conjecture (Schanuel's conjecture). *Let $a_1, \ldots, a_n$ be complex numbers and suppose that $\mathrm{td}_{\mathbb{Q}}(a_1, e^{a_1}, \ldots, a_n, e^{a_n}) < n$. Then there are integers $m_1, \ldots, m_n$, not all zero, such that $\sum_{i=1}^n m_i a_i = 0$ and $\prod_{i=1}^n e^{m_i a_i} = 1$.*

The main result of chapter 5 is to establish the following Schanuel condition for the exponential equation of a semiabelian variety.

Theorem (5.7). *Let F be a differential field of characteristic zero, with commuting derivations $D_1, \ldots, D_r$ and constant field C, and let S be a semiabelian variety of dimension n defined over C. Let $\Gamma \subseteq \mathbb{G}_a^n \times S$ be the solution set to the exponential differential equation of S.*

Suppose that $(x, y) \in \Gamma$ and $\mathrm{td}_C(x, y) - \mathrm{rk}\,\mathrm{Jac}(x, y) < n$. Then there is a proper algebraic subgroup H of S and constant points $\gamma \in S(C)$ and $\gamma' \in \mathbb{G}_\mathrm{a}^n(C)$ such that y lies in the coset $\gamma \oplus H$ and x lies in the coset $\gamma' + \mathrm{Log}\, H$.

The special case of this theorem when S is a power of $\mathbb{G}_\mathrm{m}$ was first proved by James Ax in [Ax71], and can be rewritten as follows, showing the close connection with Schanuel's conjecture.

Theorem (Ax). *Suppose $x_1, y_1, \ldots, x_n, y_n \in F$ satisfy $\frac{D_j y_i}{y_i} = D_j x_i$ for each $i = 1, \ldots, n$ and each $j = 1, \ldots, r$. If $\mathrm{td}_C(x_1, y_1, \ldots, x_n, y_n) - \mathrm{rk}\,\mathrm{Jac}(x_1, \ldots, x_n) < n$ then there are $m_i \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^n m_i x_i \in C$ and $\prod_{i=1}^n y_i^{m_i} \in C$.*

The proof of theorem 5.7 is based on theorem 5.4 which could be viewed as a Schanuel condition in a more general setting. Chapter 5 continues by giving examples to illustrate how this theorem can be used to derive Schanuel conditions for other equations, such as the "raising to powers" equations which are satisfied by $y = x^t$ for $t \in \mathbb{C}$.

The compactness theorem of first order model theory is then applied to the Schanuel condition to deduce a uniform Schanuel condition, theorem 5.14. This in turn is used to prove a purely algebraic result, a weak version of Boris Zilber's "conjecture on intersections with tori", as follows.

Theorem (5.15). *Let S be a semiabelian variety defined over an algebraically closed field C of characteristic zero. Let $(U_p)_{p \in P}$ be a parametric family of algebraic subvarieties of S. There is a finite family $\mathcal{J}_U^S$ of proper algebraic subgroups of S such that, for any coset $\kappa = a \oplus H$ of any algebraic subgroup H of S and any $p \in P(C)$, if X is an irreducible component of $U_p \cap \kappa$ and*

$$\dim X = (\dim U_p + \dim \kappa - \dim S) + t$$

with $t > 0$, an atypical component of the intersection, then there is $J \in \mathcal{J}_U^S$ of codimension at least t and $s \in S(C)$ such that $X \subseteq s \oplus J$.

The case where $S = \mathbb{G}_\mathrm{m}^n$ was previously proved by Zilber in [Zil02a] and Bruno Poizat in [Poi01], and is restated in more elementary terms as corollary 5.17. In the last section of chapter 5, a second proof of the Schanuel condition for semiabelian varieties is given, using a theorem of Seidenberg and another theorem of Ax.

The uniformity in the Schanuel condition implies that it is captured by the first order theory of the reducts. In fact the Schanuel condition, together with the algebraic structure of Γ, gives the universal theory T_S^0 of the reducts (in other words, the common theory of the exponential differential equations in any differential field). In chapter 6 this universal theory is given, and then an amalgamation technique is used to construct a universal domain U for $T^0_{\mathcal{S}}$. Amalgamation constructions were introduced by Fraissé in the 1950's, but it is convenient to use a more general, category-theoretic version of his amalgamation theorem. The version described and used here is essentially that due to Droste and Göbel in [DG92].

Amalgamation constructions have become very popular due to the development by Ehud Hrushovski of an amalgamation-with-predimension method in [Hru93] and [Hru92]. The Schanuel conditions can be seen to assert the positivity of a predimension function, which is the main property required to apply Hrushovski's method. Most of chapter 6 is devoted to defining the precise category to be amalgamated and proving the necessary properties. Due to useful discussions I had with Assaf Hasson, there are connections between this section and the early parts of [HH05]. For a technical reason, attention is restricted to those collections $\mathcal{S}$ containing only split semiabelian varieties, that is, products of a torus and an abelian variety. The universal domain is finally constructed and proved to be unique up to isomorphism in theorem 6.17. It is characterised as being the unique countable model of $T^0_{\mathcal{S}}$ satisfying the "strong existential closedness" condition SEC.

A consequence of Hrushovski's method is that the predimension function gives rise to a pregeometry on the amalgam U and the associated dimension function. These control the model-theoretic geometry of U, and are described in the last section of the chapter.

The first part of chapter 7 comprises a proof that the reduct of a saturated differentially closed field (a universal domain for differential fields) satisfies SEC, giving the following.

Theorem (7.1). *Let F be the countable saturated differentially closed field. Then the reduct of F to the language $\mathcal{L}_{\mathcal{S}}$ is isomorphic to the amalgam U.*

The concepts of *admissibility* and *absolute admissibility* of a variety are introduced, relating to the predimension function. A corollary of theorem 7.1 gives conditions for a system of exponential differential equations to have solutions.

Theorem (7.12). *Let S be a semiabelian variety defined over C, and let V be a subvariety of $\mathbb{G}_a^{\dim S} \times S$. If V is defined over C then a necessary and sufficient condition for there to be a nonconstant point in $\Gamma_S \cap V$ in some differential field extension is for V to be absolutely admissible.*

If V is not defined over C then a sufficient condition for a point to exist is for V to be admissible. If in addition $\mathrm{Loc}_C V$ is absolutely admissible then a nonconstant point exists.

Seidenberg's theorem shows that these conditions for solving differential equations in an abstract differential field also apply to finding solutions in a field of meromorphic functions, and theorem 7.13 explains this.

It follows from theorem 7.1 that the reduct to $\mathcal{L}_{\mathcal{S}}$ of any differentially closed field is elementarily equivalent to U, that is, they have the same first order theory. An existential closedness axiom scheme EC is given, which is a weaker version of SEC. EC uses the concept of the *normality* of a variety, a stronger property than admissibility. Normality is shown in corollary 7.17 to be a first order property, using the algebraic application 5.15 of the Schanuel condition. The scheme EC is incorporated into a complete axiomatization of the theory of the reducts.

Theorem (7.20). *Let $\mathcal{S}$ be a collection of split semiabelian varieties defined over the constant field C. The first order theory of the reduct of a differentially closed field to the language $\mathcal{L}_{\mathcal{S}}$ is axiomatized by $T_{\mathcal{S}}$ which consists of the algebraic axiom schemes A1—A7, the axiom scheme USC stating the uniform Schanuel condition, and the axiom scheme EC stating the existential closedness condition.*

The last section of chapter 7 gives some basic model-theoretic properties of the theory T, in particular showing that the reduct to $\mathcal{L}_{\mathcal{S}}$ of a differentially closed field is a proper reduct.

Chapter 8 describes an application of the main results of the thesis to the analytic geometry of the holomorphic exponential maps. Theorem 8.1 translates the uniform Schanuel condition into the analytic geometric context. A definition of an *analytically closed subfield* of $\mathbb{C}$ is then given, and it is shown that a countable such subfield exists.

The notion of "blurring" the graph of the exponential function of a semiabelian variety over a subfield of $\mathbb{C}$ is defined, and the following conjecture is made.

Conjecture (8.4). *If C is a countable subfield of $\mathbb{C}$ which is analytically closed with respect to $\mathcal{S}$, the structure $\langle \mathbb{C}; +, \cdot, C, (\mathcal{B}_S)_{S \in \mathcal{S}} \rangle$ of exponential maps blurred with respect to C is elementarily equivalent to the reduct $\langle F; +, \cdot, C, (\Gamma_S)_{S \in \mathcal{S}} \rangle$ of a differentially closed field.*

It is shown that the algebraic axioms and the uniform Schanuel condition hold, but the existentially closed condition is left open. For the usual exponential function of the multiplicative group, Boris Zilber constructed a "pseudo-exponentiation" function in [Zil05b]. This structure can also be blurred in an analogous way and the conjecture does hold here.

Theorem (8.8). *Let C be a countable subfield of the pseudo-exponential field K which is closed in the pregeometry arising from the Schanuel condition, that is, $a \in C$ iff $d(a/C) = 0$. Define the blurred graph of pseudo-exponentiation by*

$$\mathcal{B} = \{(x, y) \in (\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}})(K) \mid \mathrm{ex}(x)/y \in C\} .$$

Then the first order theory of the structure $\langle K; +, \cdot, C, \mathcal{B}\rangle$ is $T_{\mathcal{S}}$, the theory of reducts of differential fields.

Finally this suggests a natural further conjecture.

Conjecture (8.9). *The structures $\langle \mathbb{C}; +, \cdot, C, \mathcal{B}\rangle$ and $\langle K; +, \cdot, C, \mathcal{B}\rangle$, blurred complex exponentiation and blurred pseudo-exponentiation, are isomorphic.*

1.2 A broader context

The work in this thesis should be viewed in the context of Boris Zilber's programme of "analytic structures", which was its main motivation. Overviews of this programme can be found in [Zil00a] and [Zil05a], but I will briefly outline the background. Zilber made a trichotomy conjecture in [Zil84], that all uncountably categorical structures should come from one of three classical contexts: combinatorial (with trivial pregeometry), from vector spaces, or from algebraically closed fields. Hrushovski developed his amalgamation-with-predimension technique mentioned earlier in order to find counterexamples to this conjecture, which he gave in [Hru93] and [Hru92]. These examples, known as "new strongly minimal sets", were initially thought to be mathematical pathologies.

The observation that the predimension inequality used in their construction has the same form as Schanuel's conjecture led Zilber to conjecture that at least some of these structures might have prototypes arising from analytic geometry. Partly with the aim of justifying this, he has studied exponentiation in several guises in a series of papers: [Zil00b], [Zil02a], [Zil02b], [Zil03], [Zil04b], and [Zil05b]. However, many of the results rely on difficult open questions such as Schanuel's conjecture and Zilber's own conjecture on intersections with tori, CIT, discussed in [Zil02a]. The context of differential fields is one place where the questions can be resolved unconditionally, and it is also the source of proofs of the known weak forms of these questions, in particular Ax's theorem stated above which is a weak form of Schanuel's conjecture, and the corollary 5.17 of this thesis which is a weak form of CIT. These weak forms are nonetheless strong enough to prove several other related results; in particular they satisfy the needs of [Zil04c] which is sufficient motivation to look for generalizations such as those obtained in this thesis.

One application of the weak CIT is in Bruno Poizat's construction of bicoloured fields of infinite rank in [Poi01], and the recent construction by Baudisch, Hils, Martin-Pizarro, and Wagner of bicoloured fields of finite rank, known as bad fields, in [BHMPW06]. It seems likely that theorem 5.15 of this thesis can be used to produce similar bad fields with the "green" subgroup of the multiplicative group being replaced by a coloured subgroup of any other semiabelian variety.

Schanuel's conjecture itself encompasses most of what is known or conjectured positively about the transcendence theory of the complex exponential function. For example, it includes as special cases the Lindemann-Weierstrass theorem, the Gelfond-Schneider theorem, and Baker's theorem. See [Lan66, p30] or [Bak75, p120] for details. Indeed, Zilber's conjecture that complex exponentiation and his own pseudo-exponentiation are isomorphic explains why this should be so. On the other hand, as an indication of how far out of reach Schanuel's conjecture is thought to be, a trivial consequence of it would be that the numbers e and π are algebraically independent, and this has been an open question for over a century.

Ax's theorem of [Ax71] mentioned above can be viewed as a function field or power series version of Schanuel's conjecture. Chapter 8 of this thesis, in particular theorem 8.1 and proposition 8.6, shows that it also implies "generic" cases of Schanuel's conjecture. (Perhaps these could be defined to be the cases not of interest to number theorists!) Recent work of Alex Wilkie in [Wil03b] and [Wil03c] gives similar applications of Ax's theorem by a different method.

Ax generalized his own theorem to arbitrary semiabelian varieties in theorem 3 of [Ax72] although, despite the title of that paper, not in the full differential field setting but only to the setting of power series and formal groups. His statement is given as theorem 5.18 in this thesis. Although theorem 1 of that paper was apparently well known, this corollary (theorem 3) does not seem to have been widely recognised, perhaps because he omitted the proof which is not entirely straightforward. A few years later Brownawell and Kubota proved the Schanuel condition for Weierstrass $\wp$-functions (the elliptic curves case in the complex analytic setting) in [BK77] without using [Ax72]. Later still Coleman, in [Col80], improved Ax's first theorem and also gave a result for elliptic curves without mentioning either [Ax72] or [BK77].

It is perhaps also worth noting here that Bertolin recently gave a generalization of Schanuel's full conjecture to products of elliptic curves and tori and showed that it follows from the André-Grothendieck conjecture on periods of 1-motives, [Ber02]. I do not know an appropriate statement for the full number-theoretic conjecture for arbitrary semiabelian varieties.

Not myself knowing the content of Ax's paper [Ax72], I was initially trying to reprove Brownawell and Kubota's result in the abstract differential field setting, by differential algebraic arguments. In late 2004 I discovered Seidenberg's embedding theorem and realised that it could be used to transfer their result to the abstract setting directly, and this technique is published in [Kir05b]. The proof of the appropriate case of proposition 4.1 which appears there is different, based not on a geometric understanding but rather on a naive, direct calculational approach using a lemma on Weierstrassian extensions from Kolchin's paper [Kol53]. After this I succeeded in completing a direct proof which has been circulated in the preprint [Kir05a]. Still unaware of Ax's work, I turned attention to the semiabelian case, and found a proof in late 2005, which now forms much of the first half of this thesis. Just after this I did read [Ax72] and realised that my argument using Seidenberg's theorem could equally well be applied here, and this is given as an alternative proof at the end of chapter 5.

The story of the conjecture on intersections with tori, CIT, has perhaps contained almost as much duplication of work as the story of the Schanuel conditions. This conjecture has been made at least three times: by Zilber in [Zil02a], by Pink in a more general form, [Pin05], and for the case of algebraic curves by Bombieri, Masser and Zannier in [BMZ06]. Daniel Bertrand is to be thanked for bringing the three parties together. As described earlier, Zilber realised that Ax's theorem could be used together with a compactness argument to prove the weak form of CIT, and Poizat also published the proof (although not independently). Bombieri, Masser and Zannier have independently deduced the weak CIT from Ax's theorem, but using a heights argument rather than compactness, in [BMZ05].

The statements of the existentially closed condition and strong existentially closed condition come naturally out of the amalgamation-with-predimension method. In the case of the usual exponential equation for $\mathbb{G}_{\mathrm{m}}$, Cecily Crampin has proved a form of the existentially closed condition in her DPhil thesis [Cra06]. In my terminology, she has proved that if V is an absolutely normal subvariety of $\widehat{\mathbb{G}_{\mathrm{m}}^n}$ then $V \cap \Gamma_{\mathbb{G}_{\mathrm{m}}^n}$ contains a nonconstant point. My proof of theorem 7.1 was based on the ideas in her proof, although the details in my final version are somewhat different.

1.3 Further questions

In narrative terms, the natural place for this section would be at the end of the thesis, since the content may make little sense before reading the relevant chapters. In spirit however it is closer to the rest of the introduction, so I choose to place it here.

Perhaps the most obvious further work to be done is to remove the assumption in chapters 6 to 8 that the semiabelian varieties are split. The assumption is only used explicitly in chapter 6, and I believe that it is not essential for any of the results there.

Theorem 5.4 seems quite close to being a differential field version of theorem 1 of [Ax72], although I had stated and proved it before reading this paper of Ax. That theorem is on the intersection of analytic subgroups of a complex algebraic group G with algebraic subvarieties, and the main difference between it and 5.4 is that there is no assumption that G is a commutative group. It would be interesting to see whether 5.4 could be extended to noncommutative G, and also to see whether $\Gamma_{\omega,D}$ could be replaced by any differential-algebraic subgroup of G.

In another direction, there is the question of whether the methods here can be extended to other differential equations, not necessarily arising from groups and invariant differential forms. One example of this would be to consider a parametric family of semiabelian varieties over their moduli space. The analogue of Zilber's CIT in this context is Pink's extension of the André-Oort conjecture to mixed Shimura varieties, [Pin05], so perhaps a weak version of this analogous to theorem 5.15 is obtainable in this way.

I have only considered groups and differential equations defined over the constants. In §3 of [Ber06], Daniel Bertrand suggests some ways of approaching the problem for groups defined with non-constant parameters.

I have only touched very briefly on the model-theoretic properties of the theories $T_{\mathcal{S}}$. From the stability-theoretic point of view, the first question would be to find all the regular types, and in particular all the strongly minimal sets. Since $T_{\mathcal{S}}$ is a reduct of DCF_0, the question also arises of how they relate to strongly minimal sets there. One strongly minimal set is the constant field C, and Hrushovski has conjectured that any strongly minimal set in DCF_0 which is orthogonal to C is $\aleph_0$-categorical. This conjecture also applies to $T_{\mathcal{S}}$, and may be easier to address here. A more basic question is to classify the models of $T_{\mathcal{S}}$ which are isomorphic to reducts of differentially closed fields.

Chapter 8 itself contains discussions of further work to be done, and in relation to this I will only point out that conjecture 8.9 seems to be much more accessible than the conjecture that complex exponentiation satisfies Zilber's strong exponential closedness property, even assuming Schanuel's conjecture.

The observation in theorem 8.8 that pseudo-exponentiation, blurred over the substructure of dimension 0, is a model of the theory $T_{\mathcal{S}}$, may have further applications. Pseudo-exponentiation cannot naturally be studied in the first-order context because arithmetic is definable; instead the natural logical context is $L_{\omega_1\omega}$. However, the first-order theory of the blurred version is ω-stable. It may be possible to obtain analysable first-order theories from other Hrushovski-type constructions by a similar blurring method.

Chapter 2

Differential fields and forms

This chapter summarizes the properties of derivations, differentials, and differential forms used in the thesis. Proofs are omitted for the well-known results, but the details can be found in many places, for example in [Sha94a], [Eis95] and [Mar00].

The new or unusual aspects of this presentation are the definition of the map ∇, which is different from that in [Mar00], and the discussion which surrounds the viewing of a differential form as a map from a variety into $\Omega(F/C)$.

2.1 Derivations

Let F be a field, and M an F-vector space. A *derivation* of F into M is a map $F \xrightarrow{D} M$ which is additive, $D(x+y) = Dx + Dy$ for each $x, y \in F$, and satisfies the Leibniz rule, $D(xy) = xDy + yDx$ for each $x, y \in F$.

If $M = F$ then $\langle F; +, \cdot, D\rangle$ is a *differential field*. (Here "differential" is used as the adjectival form of derivation, as distinct from the usage as a noun which is defined later.) We can also consider fields with more than one derivation. In this thesis the derivations will always be taken to commute. We will only consider fields of characteristic zero.

The collection of all derivations on a field F forms an F-vector space, $\mathrm{Der}(F)$. If C is a subfield of F, then we may also consider the subspace $\mathrm{Der}(F/C)$ of $\mathrm{Der}(F)$ consisting of those derivations which vanish on C. For any derivation D on F, the *field of constants* of D is $C_D = \{x \in F \mid Dx = 0\}$. This is always a relatively algebraically closed subfield.

These definitions can be generalized naturally to the situation where C is a ring, F is a C-algebra, and M is an F-module. However, we will not use this extra generality.

Every differential field can be embedded in a universal one, a *differentially closed field*. The first order theory $\mathrm{DCF}_0^{\mathrm{n}}$ of differentially closed fields with n commuting derivations is complete and ω-stable, of Morley rank ω^n.

2.2 Kähler differentials

Given any field F and subfield C, there is a *universal derivation*, which is an F-vector space $\Omega(F/C)$ and a derivation $F \xrightarrow{d} \Omega(F/C)$ such that if $F \xrightarrow{D} M$ is any derivation, constant on C, there is a unique map D^* such that the following triangle commutes.

$$\begin{tikzcd} F \arrow[r, "d"] \arrow[dr, "D"'] & \Omega(F/C) \arrow[d, "D^*"] \\ & M \end{tikzcd}$$

The module $\Omega(F/C)$ can be constructed by the standard universal algebra method of taking the free F-vector space on the generating set $\{dx \mid x \in R\}$ and quotienting out by relations saying that the map d is additive and satisfies the Leibniz rule. The elements of $\Omega(F/C)$ are called *Kähler differentials* or just *differentials*. Thus each differential can be written (nonuniquely) in the form $\sum_{i=1}^n a_i db_i$ for some $a_i, b_i \in F$.

The dimension of the vector space $\Omega(F/C)$ is equal to the transcendence degree $\mathrm{td}(F/C)$ (provided that F has characteristic zero). Indeed, if B is a transcendence base for F over C then $\{db \mid b \in B\}$ is a linear basis of $\Omega(F/C)$. From the universal property of d, we immediately get a natural isomorphism between the dual space $\Omega(F/C)^*$ and $\mathrm{Der}(F/C)$. We can embed $\Omega(F/C)$ in its double dual space, and in this way we usually consider differentials as linear maps from $\mathrm{Der}(F/C)$ to F. When $\mathrm{td}(F/C)$ is finite we of course have $\dim \mathrm{Der}(F/C) = \dim \Omega(F/C) = \mathrm{td}(F/C)$, and $\Omega(F/C)$ is canonically isomorphic to its double dual.

Suppose that $C \subseteq E \subseteq F$ is a tower of fields. Then there are natural embeddings

$$\Omega(E/C) \hookrightarrow \Omega(E/C) \otimes_E F \hookrightarrow \Omega(F/C)$$

defined by the property $dx \mapsto dx$ in each case. The vector space $\Omega(E/C) \otimes_E F$ can be thought of as having the same basis as $\Omega(E/C)$, but with coefficients from F rather than just E. If $\omega \in \Omega(F/C)$, then we say that ω is *defined over* E iff it lies in $\Omega(E/C)$, with respect to this inclusion.

The universal property also gives a natural map

$$\Omega(F/C) \twoheadrightarrow \Omega(F/E)$$

which again is defined by $dx \mapsto dx$, although in this case it is important to realise that the two maps d are not the same, since for $x \in E$ not algebraic over C we have $dx \neq 0$ in $\Omega(F/C)$ but $dx = 0$ in $\Omega(F/E)$. The dual of this surjection is the obvious inclusion $\mathrm{Der}(F/E) \hookrightarrow \mathrm{Der}(F/C)$.

2.3 Higher differentials

The Kähler differentials defined here are Kähler 1-differentials, and for one important lemma we will need the notion of Kähler 2-differentials and the beginnings of the algebraic theory of de Rham cohomology. For this we briefly sketch the de Rham complex. For more details but a geometric viewpoint, see for example [Mat72] or [War83]. This is also mentioned in [Eis95], where it is shown to be an example of a Koszul complex. A treatment sufficient for our needs is also given in [Wil03a].

For $n \in \mathbb{N}$, define $\Omega^n(F/C)$ to be the exterior algebra $\bigwedge^n \Omega(F/C)$. This gives a graded ring $\Omega^\bullet(F/C) = \bigoplus_{n\in\mathbb{N}} \Omega^n(F/C)$. When $\mathrm{td}(F/C)$ is finite, $\Omega^n(F/C)$ is the F-vector space of all alternating n-multilinear maps from $\mathrm{Der}(F/C)^n$ to F. So $\Omega^1(F/C) = \Omega(F/C)$ and $\Omega^0(F/C) = F$. If $\mathrm{td}(F/C)$ is infinite, then $\Omega^\bullet(F/C)$ is the direct limit of the $\Omega^\bullet(F_0/C)$ such that $C \subseteq F_0 \subseteq F$ and $\mathrm{td}(F_0/C)$ is finite. We define three operations on $\Omega^\bullet(F/C)$.

The map $F \xrightarrow{d} \Omega^1(F/C)$ already defined extends to all of $\Omega^\bullet(F/C)$, and can be thought of as the coboundary map in a complex

$$0 \longrightarrow \Omega^0(F/C) \xrightarrow{d} \Omega^1(F/C) \xrightarrow{d} \Omega^2(F/C) \xrightarrow{d} \cdots \quad .$$

It is given for $\omega \in \Omega^n(F/C)$ by

$$\begin{aligned} d\omega(D_1, \ldots, D_{n+1}) = &\sum_{i=1}^{n+1} (-1)^{i+1} D_i \omega(D_1, \ldots, \hat{D}_i, \ldots, d_n) \\ &+ \sum_{0 \leqslant i < j \leqslant n} (-1)^{i+j} \omega([D_i, D_j], D_1, \ldots, \hat{D}_i \ldots, \hat{D}_j, \ldots, D_n) \end{aligned}$$

and in particular for $n = 1$ by

$$d\omega(D_1, D_2) = D_1(\omega D_2) - D_2(\omega D_1) - \omega[D_i, D_j]$$

where the Lie bracket on $\mathrm{Der}(F/C)$ is given by $[D_1, D_2]a = D_1 D_2 a - D_2 D_1 a$. It is standard to check that $d^2 = 0$ on $\Omega^\bullet(F/C)$. This complex is called the de Rham complex for F/C.

For any derivation $D \in \mathrm{Der}(F/C)$, the map $\Omega^1(F/C) \xrightarrow{D^*} F$ defined previously extends to an map $\Omega^\bullet(F/C) \xrightarrow{D^*} \Omega^\bullet(F/C)$ which is defined for $\omega \in \Omega^n(F/C)$ by

$$(D^*\omega)(D_1, \ldots, D_{n-1}) = \omega(D, D_1, \ldots, D_{n-1}).$$

This map D^* has degree -1, that is if $\omega \in \Omega^n(F/C)$ then $D^*\omega \in \Omega^{n-1}(F/C)$. By definition, d has degree $+1$. These operations can be combined into an operation of degree 0

$$L_D = D^* \circ d + d \circ D^*$$

called the *Lie derivative* of D on $\Omega^\bullet(F/C)$.

Lemma 2.1. *The Lie derivative L_D has the following properties. Let $\omega \in \Omega^1(F/C)$, $D, D' \in \mathrm{Der}(F/C)$, and $a \in F$.*

1. *L_D is C-linear.*
2. *$(L_D\omega)D' = D(\omega D') - \omega[D, D']$*
3. *$L_D(a\omega) = (Da)\omega + a(L_D\omega)$*

Proof. 1. is immediate, since d and D^* are C-linear. For 2,

$$\begin{aligned}
(L_D\omega)D' &= (D^*d\omega)D' + (d(\omega D))D' \\
&= (d\omega)(D, D') + D'(\omega D) \\
&= D(\omega D') - D'(\omega D) - \omega[D, D'] + D'(\omega D) \\
&= D(\omega D') - \omega[D, D']
\end{aligned}$$

and for 3,

$$\begin{aligned}
L_D(a\omega)D' &= D(a\omega D') - a\omega[D, D'] \\
&= (Da)\omega D' + aD(\omega D') - a\omega[D, D'] \\
&= (Da)\omega D' + a(L_D\omega)D'.
\end{aligned}$$

□

2.4 The geometric picture and differential forms

2.4.1 Varieties

I will not attempt to give complete foundations for algebraic geometry, or even a complete definition of an algebraic variety. These can be found for example in [Sha94a] and [Sha94b], and a treatment sufficient for the needs of the thesis can be found in [Mar00]. However, I will give some definitions mainly for the purpose of fixing conventions and notation.

An *affine (algebraic) variety* U is given as the zero set of a finite set of polynomials in variables $x = (x_1, \ldots, x_n)$, for some $n \in \mathbb{N}$. It is defined over a field K if all the polynomials have coefficients in K. For any field F extending K, we write $U(F)$ for the set of F-points of U, that is the set of $x \in F^n$ such that $f(x) = 0$ for all the polynomials f defining U. We endow an affine variety with its Zariski topology on each Cartesian power. An *(algebraic) variety* V is a space with a topology on each Cartesian power which has a finite atlas of charts, each chart being homeomorphic to an affine algebraic variety, such that the diagonal in V^2 is closed. More correctly, V is a functor such that $V(F)$ has these properties for each field F. The distinction between the variety V and its F-points is often blurred, but in this thesis there are often two fields, F and C, and it is important to distinguish $V(F)$ from $V(C)$.

A *regular map* from a variety V to a variety W is a function $V \xrightarrow{f} W$ which is given locally (on each chart in some atlas) as p/q where p and q are polynomial maps and q has no zeros on the chart. Such a map is necessarily continuous in the topologies on all Cartesian powers (indeed this is a defining property of the Zariski topology). A *rational map* is defined the same way, but with the denominator allowed to vanish.

Two important examples of varieties are the affine space $\mathbb{A}^1$, defined by $\mathbb{A}^1(F) = F$, and the projective space $\mathbb{P}^1$. A regular map $V \xrightarrow{f} \mathbb{A}^1$ is called a *regular function* on V, and a rational map $V \xrightarrow{f} \mathbb{A}^1$ is called a *rational function* on V.

The collection of all algebraic varieties forms a category **Var** whose morphisms are the regular maps. It is well-known that this category has finite products. The objects and morphisms of this category are all first-order definable (with parameters) in the theory ACF_0, and we sometimes consider them as definable sets.

2.4.2 Tangents

Let $U \subseteq F^n$ be an affine variety defined by polynomials in an ideal $I(U)$ of the polynomial ring $F[X_1, \ldots, X_n]$, and let $a \in U$. The *tangent space* of U at a is the affine variety given by

$$T_aU = \left\{ u \;\middle|\; \sum_{i=1}^{n} \frac{\partial p}{\partial X_i}(a)u_i = 0 \qquad \text{for each } p \in I(U) \right\}$$

which is a vector subspace of $\mathbb{A}^n$. The *tangent bundle* of an affine variety U is given by

$$TU = \{(a, u) \mid a \in U, u \in T_aU\}$$

and is easily seen to be an affine variety.

The tangent space at a point and the tangent bundle of an algebraic variety V are defined locally by means of charts. They are also algebraic varieties.

If $V \xrightarrow{f} W$ is a regular map and $a \in V$, there is a linear map $T_aV \xrightarrow{df_a} T_{f(a)}W$. If f is given locally by polynomials $(f_1, \ldots, f_m)$ then df_a is given by

$$df_a(u) = \left(\sum_{i=1}^{n} \frac{\partial f_1}{\partial X_i}(a)u_i, \ldots, \frac{\partial f_m}{\partial X_i}(a)u_i \right)$$

and we define T on morphisms by

$$\begin{array}{ccc} TV & \xrightarrow{Tf} & TW \\ (a, u) & \longmapsto & (f(a), df_a(u)). \end{array}$$

This makes T into a functor from **Var** to itself. It is easy to check that T preserves products, that is that $T(V \times W)$ is isomorphic to $TV \times TW$, naturally in V and W.

Definition 2.2. A *vector field* on $V(F)$ is a section of the tangent bundle $TV(F)$, that is, a map $V(F) \longrightarrow TV(F)$ which sends each point x into its own tangent space $T_xV(F)$.

Vector fields are usually defined on the F-points of a variety for some field F rather than on the abstract variety. They may be defined by regular or rational functions, in which case they are *regular* or *rational* vector fields.

Derivations give rise to vector fields in two different ways. Let V be an irreducible algebraic variety defined over C, let $v \in V(F)$ be generic over C and let $E = C(v)$, the subfield of F generated over C by v. Then E can be (non-canonically) identified with the function field of V, that is the field of rational functions from $V(C)$ to C, that is regular maps $V(C) \longrightarrow \mathbb{P}^1(C)$. These are also called *scalar fields* on $V(C)$.

Under this identification, $\mathrm{Der}(E/C)$ is identified with the E-vector space of rational vector fields of $V(C)$, so a derivation D corresponds to a map $V(C) \xrightarrow{X^D} TV(C)$.

For each $x \in V(C)$, this gives a tangent vector $X_x^D \in T_xV(C)$. (See any book on differential geometry, for example [War83] or [Mat72], for details.)

Again take V to be defined over C. Given any derivation $D \in \mathrm{Der}(F/C)$, we apply it to $x \in V(F)$ componentwise (on each chart) and write the result as Dx. This value is necessarily in $T_xV(F)$. (If V is not defined over C then it lies instead in the first prolongation of V, but we will only consider varieties defined over C here.) This gives an F-linear map $\mathrm{Der}(F/C) \longrightarrow T_xV$. We may also patch together these maps to get a map defined on V, which we write ∇ (suppressing the dependence on V).

$$\begin{array}{rcl} \mathrm{Der}(F/C) \times V & \xrightarrow{\nabla} & TV \\ (D, x) & \longmapsto & \nabla_D(x) = (x, Dx) \end{array}$$

If we fix D, the map ∇_D is a vector field on $V(F)$, although it does not lie in the algebraic category (it is not a regular or rational vector field).

Note that the two vector fields described here, X^D and ∇_D, are very different. ∇_D is a section of the tangent bundle of $V(F)$, whereas X^D is a section of the tangent bundle of $V(C)$. Furthermore they do not agree on $V(C)$ where they are both defined, since ∇_D is identically zero on $V(C)$.

There is one map ∇ for each variety V, but they fit together as one might hope.

Lemma 2.3. *If V and W are varieties defined over C and $V \xrightarrow{f} W$ is any regular map also defined over C then the diagram*

$$\begin{array}{ccc} V(F) & \xrightarrow{f} & W(F) \\ {\scriptstyle\nabla_D}\downarrow & & \downarrow{\scriptstyle\nabla_D} \\ TV(F) & \xrightarrow[Tf]{} & TW(F) \end{array}$$

commutes.

Proof. For any $x \in V$

$$\nabla_D(f(x)) = (f(x), D(f(x))) = (f(x), df_x(Dx)) = Tf(x, Dx) = Tf(\nabla_D(x))$$

by the definitions, using local coordinates around x, and the fact that f is defined over C. $\square$

2.4.3 Cotangents and differential forms

The dual space to the tangent space T_xV is called the *cotangent space* of V at x, and is written T_x^*V. The *cotangent bundle* of V is given as

$$T^*V = \{(x,t) \mid x \in V, t \in T_x^*V\}.$$

Informally, a *differential form* on V is a section of the cotangent bundle, but we are only interested in those sections which are related to the category of algebraic maps. Recall that a regular function on V is a regular map $V \xrightarrow{g} \mathbb{A}^1$. Applying the functor T, we get for each $x \in V$ a linear map $T_xV \xrightarrow{d_xg} T_{g(x)}\mathbb{A}^1$. The vector space $T_{g(x)}\mathbb{A}^1$ is canonically isomorphic to $\mathbb{A}^1$, and so this makes d_xg an element of T_x^*V.

Definition 2.4. A *regular differential form* ω on a variety V consists of an element $\omega_x \in T_x^*V$ for each $x \in V$ such that there is an open cover of V where, on each U in the cover, ω takes the form $\omega_x = \sum_{i=1}^n f_i(x)dg_i(x)$, where the f_i and the g_i are regular functions on U. A *rational differential form* is defined in the same way, but with the f_i allowed to be rational functions on U.

T^* also acts on regular maps. If $V \xrightarrow{f} W$ is a regular map then we define $T^*W \xrightarrow{T^*f} T^*V$ for $\omega \in T^*W$ and $(a,u) \in TV$ by $(T^*f(\omega))_a(u) = \omega_{f(a)}df_a(u)$. It is customary also to write f_* for T^*f. This makes T^* into a contravariant functor (from **Var** into the category of vector bundles over varieties).

Let V be an irreducible algebraic variety defined over C, and let E be the function field of V over C. We have seen that $\mathrm{Der}(E/C)$ can be identified with the space of rational vector fields on $V(C)$. Since $\mathrm{td}(E/C)$ is finite, $\mathrm{Der}(E/C)$ is a finite dimensional E-vector space, and so $\Omega(E/C)$ is its dual. Thus $\Omega(E/C)$ is identified with the E-vector space of rational differential 1-forms on $V(C)$.

The idea of interpreting a Kähler differential as a differential form on $V(C)$ in this way only works because E is identified with the function field of V. In general, a differential form on $V(F)$ is a sheaf of Kähler differentials from $\Omega(F/C)$, not a single Kähler differential. Indeed, let V be a variety defined over C and let ω be a regular differential form on $V(F)$. Then ω defines a function $V(F) \longrightarrow \Omega(F/C)$ as follows. Given $x \in V$, we define

$$\begin{array}{rcl} \mathrm{Der}(F/C) & \xrightarrow{\omega(x)} & F \\ D & \longmapsto & \omega_x(\nabla_D(x)) \end{array}$$

where ∇_D is the section of the tangent bundle of V defined by D. It is straightforward to check that $\omega(x)$ is a linear map, and hence an element of $\Omega(F/C)$, so, by definition, the diagram

$$\begin{array}{ccc} V(F) & \xrightarrow{\omega} & \Omega(F/C) \\ \nabla_D \big\downarrow & & \big\downarrow D^* \\ TV(F) & \xrightarrow{\omega} & F \end{array}$$

commutes. Note the difference in notation between ω_x, which is a linear map with domain T_xV, and $\omega(x)$, which is a linear map with domain $\mathrm{Der}(F/C)$. We will also call the map $V(F) \xrightarrow{\omega} \Omega(F/C)$ a differential form.

Take V to be irreducible and $v \in V(F)$, generic over C. If $E = C(v)$ and $\omega \in \Omega(E/C)$, we can define from it a differential form on $V(F)$ by means of specializations. For any $x \in V(F)$, there is a unique specialization π of E which takes v to x. The differential ω can be written as $\sum_{i=1}^n f_i dg_i$, with the $f_i, g_i \in E$ and we define the differential form by $\omega_x = \sum_{i=1}^n \pi(f_i) d\pi(g_i)$. This is well-defined. The f_i and g_i can be thought of as algebraic functions of v, in which case the differential form can be written as $\omega_x = \sum_{i=1}^n f_i(x) dg_i(x)$. Note that this construction does not involve identifying E with the function field of V, although it does involve a choice of generic point v. Also note that it defines a differential form on $V(F)$, whereas the previous construction gives a differential form on $V(C)$.

If ω is a differential form on a variety V, then we say that is is *defined over a field* K iff for every $x \in V$ we have $\omega(x) \in \Omega(K(x)/C)$. This makes use of the natural embedding of $\Omega(K(x)/C)$ into $\Omega(F/C)$. The following lemma seems to be used implicitly in the literature, but I have not found a reference for it.

Lemma 2.5. *Let V be an algebraic variety defined over C, and let ω be a regular differential form on V, defined over an extension K of C. Suppose that U is an irreducible subvariety of V (not necessarily defined over K), and $\omega(v) = 0$ for a point v of U, generic over K. Then ω vanishes on all of U.*

Proof. This is essentially a universal algebra argument. Working in an affine neighbourhood of v, $\omega(x)$ can be written as $\sum_{i=1}^n f_i(x) dg_i(x)$ for suitable regular functions f_i, g_i, defined over K and satisfying certain equations over C. When $x = v$, these equations force $\omega(v) = 0$, but every $x \in U$ and in this neighbourhood satisfies these same equations since v is generic over K, and so $\omega(x) = 0$. The set of points on which ω vanishes is Zariski-closed [Sha94a, p202], and so this extends from the neighbourhood of v to all of U. $\square$

In fact, this shows that ω vanishes on $\mathrm{Loc}_K(v)$, the locus of v, which is the smallest variety defined over K and containing v.

Chapter 3

Groups

The exponential differential equations which are the topic of this thesis will be defined in terms of invariant differential forms on commutative algebraic groups, in particular semiabelian varieties. The theory of the equations depends on a good understanding of the algebraic subgroups of commutative algebraic groups. The purpose of this chapter is to give the necessary background in these topics. Much of it is adapted from parts of [Mar00] and [Ser88].

3.1 Algebraic groups

An *algebraic group* G is an algebraic variety (also written G) together with morphisms $1 \xrightarrow{e} G$, $G \times G \xrightarrow{m} G$ and $G \xrightarrow{i} G$, where 1 is the one point variety, satisfying the usual axioms for the unit, multiplication and inverse of a group (that is, it is a group object in the category **Var**). We usually write gh rather than $m(g,h)$ and g^{-1} rather than $i(g)$. For commutative algebraic groups we shall usually write the group operation as $\oplus$, and the unary inverse and binary subtraction operations as $\ominus$.

A commutative group is of course the same thing as a $\mathbb{Z}$-module, and the scalar multiplication maps, such as $g \mapsto m(g,g)$, are morphisms. Extending this idea, for any ring R we define an *algebraic R-module* to be a commutative algebraic group G together with a morphism $G \xrightarrow{r} G$ for each $r \in R$, such that composition of the morphisms corresponds to multiplication in R, and that these operations of scalar multiplication satisfy the usual axioms for a module. Note that the ring R will not in general be a definable algebraic object, as for example with $\mathbb{Z}$.

For some examples of algebraic modules, note that F^n is an algebraic F-vector space, $\mathbb{G}_\mathrm{m}$ is a $\mathbb{Z}$-module, and an elliptic curve with complex multiplication is an algebraic $\mathbb{Z}[\tau]$-module for some imaginary quadratic number τ. For an example with R not commutative, F^n is also an algebraic $\mathrm{Mat}_{n\times n}(F)$-module.

3.1.1 Complex groups

If G is an algebraic group then its complex points $G(\mathbb{C})$ form a complex Lie group, and these can be studied using techniques from analytic geometry, topology and Lie theory. Any results concerning first order statements in the language of fields can then be transferred to algebraic groups over other algebraically closed fields of characteristic zero. For the most part I do not adopt that approach in this thesis, preferring to give algebraic proofs of algebraic statements where possible. However, the complex case gives the main motivation for the work here, which comes in particular from the covering spaces and exponential maps of commutative algebraic groups.

Let $G(\mathbb{C})$ be a commutative complex algebraic group and let V be its universal covering space, which we may identify with its Lie algebra. Since G is commutative, so is the Lie algebra, that is to say that it is essentially just a finite dimensional $\mathbb{C}$-vector space. In fact, as a complex Lie group, $G(\mathbb{C})$ can be considered as the quotient of the additive group of $\mathbb{G}_{\mathrm{a}}(\mathbb{C})^n$ by a discrete subgroup or *lattice*. This is necessarily isomorphic to $\mathbb{Z}^m$, that is, it is a free abelian group on m generators, for some $m \leqslant 2n$. If $m = 0$ then G is just $\mathbb{G}_{\mathrm{a}}(\mathbb{C})^n$. If $m = n = 1$ then G is isomorphic to the multiplicative group $\mathbb{G}_{\mathrm{m}}(\mathbb{C})$. The group G is compact iff $m = 2n$, and in this case G is called a complex torus. A complex torus may or may not be an algebraic group, but if $n = 1$ it always is and is called an *elliptic curve*.

The covering map from the universal covering space V to $G(\mathbb{C})$ is called the *exponential map* of G. It is an analytic map, and when G is commutative it is a group homomorphism. Understanding these exponential maps and the differential equations they satisfy is the main purpose of this thesis.

3.1.2 Classification

The purpose of this section is to give a description of all the commutative algebraic groups in characteristic zero. Statements are mostly taken from [Ser88] and [BL04].

Definition 3.1. A *linear algebraic group* is an algebraic group isomorphic to a group of matrices, under matrix multiplication. An *abelian variety* is an algebraic group which is a complete irreducible variety. An algebraic group of the form $\mathbb{G}_{\mathrm{a}}^n$ is called a *vector group*.

Note that $\mathbb{G}_{\mathrm{a}}$ and $\mathbb{G}_{\mathrm{m}}$ are both linear groups, the former through the representation as matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, and so every vector group is a linear group. On the other hand if a complex torus happens to be an algebraic variety then it is an abelian variety. In fact every complex abelian variety is a complex torus, and in particular is an abelian group. This was not the origin of the term abelian variety, which instead comes from the notion of abelian integrals, which were integrals studied by Abel. Indeed the terminology in this area is somewhat confusing, since in algebraic geometry the word "torus" means a power of the multiplicative group $\mathbb{G}_{\mathrm{m}}$!

Chevalley showed that every algebraic group can be constructed from a linear group and an abelian variety.

Theorem 3.2 (Chevalley's theorem). *Let G be an algebraic group. Then there is a normal algebraic subgroup L of G which is linear and such that the quotient group G/L is an abelian variety. Furthermore L is uniquely determined by these properties.*

We are interested in commutative groups. If G is commutative then necessarily L is commutative, but commutative linear groups take a particularly simple form.

Theorem 3.3 ([Ser88, pages 40, 171]). *In characteristic zero, every connected commutative linear algebraic group L is of the form $L \cong \mathbb{G}_{\mathrm{a}}^{l} \times \mathbb{G}_{\mathrm{m}}^{k}$ for some $l, k \in \mathbb{N}$.*

In arbitrary characteristic, this has to be weakened to say that $L \cong U \times \mathbb{G}_{\mathrm{m}}^{k}$, where U is a unipotent group, that is, a group isomorphic to a group of upper triangular matrices with only 1s on the diagonal.

Definition 3.4. A *semiabelian variety* is a connected algebraic group G where the linear subgroup L of the Chevalley decomposition has no unipotent part, that is, $L \cong \mathbb{G}_{\mathrm{m}}^{k}$.

We now show how to reduce a commutative algebraic group to a special form.

Definition 3.5. A (regular) homomorphism $A \xrightarrow{f} A'$ of abelian varieties is called an *isogeny* iff it is surjective and has finite kernel.

Since the kernel is a subvariety of A, it follows that A and A' have the same dimension.

Isogenies are very close to being invertible, and give a weak form of isomorphism. Indeed since they are by definition finite-to-finite correspondences, they are as good as definable bijections for transcendence theory purposes. If $A \xrightarrow{f} A'$ is an isogeny of elliptic curves then there is necessarily a dual isogeny $A' \xrightarrow{g} A$. Isogenies obviously compose, and so the existence of an isogeny between two elliptic curves defines an equivalence relation. For abelian varieties more generally there may be no dual isogeny, but the equivalence relation generated by the existence of an isogeny has a simple description.

Definition 3.6. Two abelian varieties A and A' are said to be *isogenous* iff there is a third abelian variety A'' and isogenies

$$\begin{tikzcd} & A'' & \\ A \arrow[ur, "f"] & & A' \arrow[ul, "g"'] \end{tikzcd} .$$

As remarked, this is in fact an equivalence relation.

Theorem 3.7 (Poincaré's reducibility theorem). *Let A be an abelian variety and H an algebraic subgroup of A. Then there is another algebraic subgroup J of A such that A is isogenous to $H \times J$.*

By induction on dimension, this gives the classification we want for abelian varieties. A *simple* algebraic group is an algebraic group with no proper infinite normal algebraic subgroups.

Corollary 3.8 (Poincaré's complete reducibility theorem). *Every abelian variety is isogenous to a product of powers of nonisogenous simple abelian varieties.*

We extend the notion of isogeny to other commutative algebraic groups by the same definition. Isogeny is just isomorphism for the linear part of the group, thus every commutative algebraic group (in characteristic zero) is isogenous to a group G with a decomposition

$$0 \to \mathbb{G}_{\mathrm{a}}^{l} \times \mathbb{G}_{\mathrm{m}}^{k} \to G \to \prod_{i=1}^{n} A_i^{m_i} \to 0$$

where the A_i are simple and non-isogenous. In general this sequence does not split, that is, $G \neq \mathbb{G}_{\mathrm{a}}^{l} \times \mathbb{G}_{\mathrm{m}}^{k} \times \prod_{i=1}^{n} A_i^{m_i}$.

As a generalization of the notion of groups being non-isogenous, define G_1 and G_2 to be *null-homomorphic* if the only regular homomorphisms between them are the zero homomorphisms. From Poincaré's theorems it follows that two abelian varieties A_1 and A_2 are null-homomorphic iff there are no non-trivial subgroups H_i of A_i such that H_1 is isogenous to H_2.

3.2 Algebraic subgroups

The main results in this thesis involve finding algebraic subgroups of algebraic groups, and so a description of these is essential. The tool used is a theorem due to Kolchin, from [Kol68]. He gives a classification of algebraic subgroups of products of any simple algebraic groups, but here we give the statement for commutative groups.

Theorem 3.9 (Kolchin). *Let $G_1, \ldots, G_n$ be simple commutative algebraic groups, and let H be a proper algebraic subgroup of the product $\prod_{i=1}^n G_i$. Then either*

- *For some index j the image of H under the projection onto G_j is not all of G_j; or*
- *There are l distinct indices $j_1, \ldots, j_l$ with $l \geqslant 2$ and surjective (regular) homomorphisms $G_{j_\lambda} \xrightarrow{f_\lambda} G_{j_l}$ such that $\bigoplus_{\lambda=1}^{l} f_\lambda(x_{j_\lambda}) = 0$ for each $x = (x_1, \ldots, x_n) \in H$.*

We will use the following special case.

Corollary 3.10. *Let $G_1, \ldots, G_n$ each be powers of simple commutative algebraic groups which are* null-homomorphic*, that is, there are no non-trivial regular homomorphisms $G_i \longrightarrow G_j$ for $i \neq j$.*

Suppose that H is an algebraic subgroup of $\prod_{i=1}^n G_i$. Then $H = \prod_{i=1}^n H_i$ where each H_i is an algebraic subgroup of G_i.

In order to use the theorem we need to know what homomorphisms exist between simple commutative algebraic groups, and we next collect some of these results and conclusions.

Theorem 3.11.

- *There are no nontrivial homomorphisms from $\mathbb{G}_\mathrm{a}$ to $\mathbb{G}_\mathrm{m}$ or to any abelian variety, from $\mathbb{G}_\mathrm{m}$ to $\mathbb{G}_\mathrm{a}$ or to any abelian variety, or from any abelian variety to $\mathbb{G}_\mathrm{a}$ or $\mathbb{G}_\mathrm{m}$.*
- *Any nontrivial homomorphism between simple abelian varieties is an isogeny.*
- *The endomorphism ring of $\mathbb{G}_\mathrm{a}(F)$ is the field F, and the algebraic subgroups of $\mathbb{G}_\mathrm{a}^n(F)$ are given by equations of the form $\sum_{i=1}^n a_i x_i = 0$, for $a_i \in F$.*
- *The endomorphism ring of $\mathbb{G}_\mathrm{a}^n(F)$ is the matrix ring $\mathrm{Mat}_{n\times n}(F)$ for any field F.*
- *The endomorphism ring of $\mathbb{G}_\mathrm{m}$ is $\mathbb{Z}$, and the algebraic subgroups of $\mathbb{G}_\mathrm{m}^n$ are given by equations of the form $\prod_{i=1}^n x_i^{m_i} = 1$ for $m_i \in \mathbb{Z}$.*
- *The endomorphism ring of an elliptic curve $\mathcal{E}$ is $\mathbb{Z}$ unless $\mathcal{E}$ has complex multiplication in which case it is $\mathbb{Z}[\tau]$ for some τ such that τ^2 is a negative integer. The algebraic subgroups of $\mathcal{E}^n$ are given by equations $\bigoplus_{i=1}^n a_i x_i = 0$ where the $a_i \in \mathbb{Z}$ or $\mathbb{Z}[\tau]$ as appropriate.*

- *If A is a simple abelian variety then the "field of fractions"* $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *of its endomorphism ring is a division ring.*

Proof. The first part follows from considering the torsion points of the groups. For the second part, note that the kernel of a homomorphism $A \xrightarrow{f} A'$ is an algebraic subgroup of A, which is simple, and so $\ker f$ is finite or all of A. The image is an algebraic subgroup of A' which is simple, and so it is either trivial or all of A'. The endomorphism rings of $\mathbb{G}_\mathrm{a}$ and $\mathbb{G}_\mathrm{m}$ are straightforward to see. For the endomorphism ring of an elliptic curve, see [Sil92]. The last part can be found on page 115 of [BL04]. $\square$

In abstract group theory, the first isomorphism theorem is the basic result from which much of the theory comes. An analogous statement holds for algebraic groups, although the proof is not so elementary.

Theorem 3.12. *For any algebraic group G and algebraic subgroup H, the coset space G/H has the structure of an algebraic variety and the quotient map χ_H from G to G/H is a regular map. If H is a normal subgroup, in particular if G is commutative, then G/H is an algebraic group and χ_H is a homomorphism. Furthermore, if $g \in G$ and $\chi_H(g) = c$, then g lies in the coset of H which is coded by c. That is, $\chi_H(g_1) = \chi_H(g_2)$ iff g_1 and g_2 lie in the same coset of H.*

Proof. See [Hum75, p83], [Sha94a, p190]. $\square$

The way that algebraic subgroups arise in the proofs is often by finding some subgroup and then using the following theorem to see that it is an algebraic subgroup.

Theorem 3.13 (Indecomposability theorem). *Let G be an algebraic group and let V be an irreducible subvariety of G which contains the identity e of G. Then the subgroup of G generated by V is a connected algebraic subgroup.*

A model theoretic version and proof of this can be found in [Mar02, p261], but this statement for algebraic groups is older.

3.3 Tangent bundles of groups

Proposition 3.14. *If G is an algebraic group or algebraic R-module then so is TG, with operations Tm, Tr etc., the images under T of the operations of G.*

Proof. Firstly note that each tangent space is an algebraic F-module, hence certainly an algebraic group. By the coherence theorem for product categories, we can coherently identify $(G \times G) \times G$ with G^3 etc. Since T preserves products, we can coherently identify $T(G^n)$ with $(TG)^n$ as well. Making these identifications, the commutativity of the following two diagrams asserts the associativity and inverse group axioms for G.

$$\begin{tikzcd}
G^3 \arrow[r, "(\mathrm{id}, m)"] \arrow[d, "(m, \mathrm{id})"'] & G^2 \arrow[d, "m"] \\
G^2 \arrow[r, "m"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[rr, "!"] \arrow[dd, "\Delta"'] & & 1 = G^0 \arrow[dd, "e"] \\
& G^2 \arrow[dr, "m"] & \\
G^2 \arrow[ur, "(1,i)"] \arrow[dr, "(i,1)"'] & & G \\
& G^2 \arrow[ur, "m"'] &
\end{tikzcd}$$

Here m is the multiplication on G, id is the identity map, i is the inverse, e is the unit, Δ is the diagonal map and ! is the unique map to the one point variety. Applying T to these diagrams (and implicitly using the coherent identifications) gives the axioms for TG. The unit axiom and the R-module axioms for TG follow in the same way. □

Proposition 3.15. *Suppose that G is an algebraic group or R-module which is defined over C (meaning that both the underlying variety and the group or module operations are defined over C), and let $D \in \mathrm{Der}(F/C)$. Then the map $G(F) \xrightarrow{\nabla_D} TG(F)$ is a group or R-module homomorphism.*

Proof. Apply lemma 2.3 to the multiplication, inverse, and scalar multiplication maps of G. □

It is useful to give names to the translation maps $G \xrightarrow{\lambda_g} G$ and $G \xrightarrow{\rho_g} G$, which are given for each $g, x \in G$ by $\lambda_g(x) = gx$ and $\rho_g(x) = xg$. We can express Tm explicitly in terms of these maps as follows. By definition, $Tm((g,u),(h,v)) = (gh, dm_{(g,h)}(u,v))$. Now if we express m locally by polynomials in indeterminants X in the first place and Y in the second (each a list of the appropriate length), we see that

$$\begin{aligned}
dm_{(g,h)}(u,v) &= \frac{\partial m}{\partial X}(g,h)u + \frac{\partial m}{\partial Y}(g,h)v \\
&= \frac{\partial \rho^h}{\partial X}(g)u + \frac{\partial \lambda^g}{\partial Y}(h)v \\
&= d\rho^h_g u + d\lambda^g_h v \quad ,
\end{aligned}$$

giving an explicit expression for Tm.

The group structure on an algebraic group G gives rise to a Lie algebra structure on T_eG, but we shall only consider commutative groups where this Lie algebra structure is trivial. Despite this, Lie theory plays a role in this work. If H is an algebraic subgroup of G then TH is a sub-bundle of TG, and T_eH is a linear subspace of T_eG.

Theorem 3.16 (Lie Correspondence). *If G is connected and the characteristic is zero then an algebraic subgroup H of G is determined by the subspace T_eH of T_eG.*

Proof. Suppose H_1, H_2 are two algebraic subgroups of G with $T_eH_1 = T_eH_2$. Then a field of definition of H_1, H_2 and G may be embedded into $\mathbb{C}$. Then $H_i(\mathbb{C})$ is the image of $T_eH_i(\mathbb{C})$ under the holomorphic exponential map $T_eG(\mathbb{C}) \xrightarrow{\exp} G(\mathbb{C})$, so $H_1(\mathbb{C}) = H_2(\mathbb{C})$. But H_1 and H_2 are defined over $\mathbb{C}$ which is algebraically closed, and thus $H_1 = H_2$. $\square$

If G has dimension n, then T_eG can be identified with $\mathbb{G}_\mathrm{a}^n$. For an algebraic subgroup H of G, the subspace T_eH is of course an algebraic subgroup of $\mathbb{G}_\mathrm{a}^n$, and we sometimes write this as $\operatorname{Log} H$.

Note that not every subspace or Lie subalgebra of T_eG is the logarithm of an algebraic subgroup. For example, if $G = \mathbb{G}_\mathrm{m}^2$ and H is a proper non-trivial algebraic subgroup, then $\operatorname{Log} H$ is given by an equation $ax + by = 0$ where $a, b \in \mathbb{Z}$.

3.4 Invariant differential forms

Definition 3.17. A differential form ω on an algebraic group G is said to be *left-invariant* iff for every $g \in G$ we have $T^*\lambda^g(\omega) = \omega$. Right-invariant differential forms are defined similarly with ρ^g. If G is commutative then left-invariant and right-invariant coincide, and we just say *invariant*.

Making the definition explicit, ω is left-invariant iff for all $g, h \in G$ we have

$$\omega_h u = (\lambda^g_* \omega)_h u = \omega_{gh}(d\lambda^g_h u)$$

and in particular, setting $g = h^{-1}$,

$$\omega_h u = \omega_e(d\lambda^{h^{-1}}_h u).$$

Given $\varphi \in T^*_eG$, we can define a differential form on G by $\omega_h u = \varphi(d\lambda^{h^{-1}}_h u)$. This is a regular differential form, and is left-invariant by definition, so this allows us to identify the set $\operatorname{LInv}(G)$ of left-invariant differential forms on G with $T^*_e(G)$. In particular, $\mathrm{LInv}(G)$ is a vector space of dimension $\dim G$. For a commutative group G we write the space of invariant differential forms as $\mathrm{Inv}(G)$.

The invariant differential forms on $\mathbb{G}_{\mathrm{a}}$ are multiples of dx, and those on $\mathbb{G}_{\mathrm{m}}$ are multiples of $\frac{dx}{x}$. For an elliptic curve $\mathcal{E}$ whose affine part is given by $z^2 = 4y^3 - g_2 y - g_3$, the invariant differential forms are multiples of $\frac{dy}{z}$ [Sil92, p80].

It is clear that if ω_1 and ω_2 are invariant differential forms on G_1 and G_2 respectively then $\omega_1 + \omega_2$ is an invariant differential form on $G_1 \times G_2$. For example, $\frac{dy}{y} - dx$ and $\frac{dy}{z} - dx$ are invariant differential forms on $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$ and $\mathbb{G}_{\mathrm{a}} \times \mathcal{E}$ respectively.

Invariant differential forms can be moved from one group to another via a homomorphism.

Lemma 3.18. *Suppose that G and H are algebraic groups, $G \xrightarrow{f} H$ is a regular group homomorphism, and ω is a left-invariant differential form on H. Then $f_*\omega$ is a left-invariant differential form on G.*

Proof. Let $g, h \in G$ and $u \in T_h G$. Then

$$\begin{array}{rcll}
(f_*\omega)_{gh}(d\lambda^g_h u) & = & \omega_{f(gh)}(df_{gh}(d\lambda^g_h u)) & \\
& = & \omega_{f(gh)}(d(f \circ \lambda^g)_h u) & \text{by functoriality of } T \\
& = & \omega_{f(g)f(h)}(d(\lambda^{f(g)} \circ f)_h u) & f \text{ a group homomorphism} \\
& = & \omega_{f(g)f(h)}(d\lambda^{f(g)}_{f(h)}(df_h u)) & \text{by functoriality of } T \text{ again} \\
& = & \omega_{f(h)}(df_h u) & \text{by left-invariance of } \omega \\
& = & (f_*\omega)_h u &
\end{array}$$

and so $f_*\omega$ is left-invariant. □

Corollary 3.19. *If $G \xrightarrow{f} H$ is an isogeny then the map of tangent spaces df_e gives rise to an isomorphism $\mathrm{LInv}(H) \longrightarrow \mathrm{LInv}(G)$.*

On the other hand, invariant differential forms on a commutative algebraic group can themselves be seen as group homomorphisms.

Proposition 3.20. *Suppose ω is an invariant differential form on a commutative algebraic group G, defined over C. Then the map it defines,*

$$\begin{array}{rcl}
G(F) & \longrightarrow & \Omega(F/C) \\
g & \longmapsto & \omega(g) \quad ,
\end{array}$$

is a group homomorphism.

Proof. In fact, we just suppose that ω is left- and right-invariant, and G is not necessarily commutative. Let $g, h \in G(F)$ and $D \in \mathrm{Der}(F/C)$. We write $\cdot$ for the group operation in TG. Then

$$\begin{array}{rcll}
\omega(gh)D & = & \omega_{gh}(\nabla_D(gh)) & \\
& = & \omega_{gh}(\nabla_D(g)\cdot\nabla_D(h)) & \nabla_D \text{ a group homomorphism} \\
& = & \omega_{gh}(d\rho^h_g(\nabla_D(g)) + d\lambda^g_h(\nabla_D(h))) & \text{using explicit equation for } \cdot \\
& = & \omega_{gh}(d\rho^h_g(\nabla_D(g))) + \omega_{gh}(d\lambda^g_h(\nabla_D(h))) & \text{by linearity of } \omega_{gh} \\
& = & \omega_g(\nabla_D(g)) + \omega_h(\nabla_D(h)) & \text{by invariance} \\
& = & \omega(g)D + \omega(h)D \quad , &
\end{array}$$

thus $\omega(gh) = \omega(g) + \omega(h)$ as required. $\square$

A vector field X on $G(C)$ is said to be *left-invariant* iff for all $x, y \in G(C)$, $X_{xy} = d\lambda^x_y X_y$. For convenience we recall from above that a differential 1-form ω on $G(C)$ is left-invariant iff for all $x, y \in G(C)$, $\omega_{xy} = \lambda^{x^{-1}}_* \omega_y$, that is, for any vector field X, $\omega_{xy}X_{xy} = \omega_y(d\lambda^{x^{-1}}_{xy} X_{xy})$.

Proposition 3.21. *Let G be a commutative algebraic group defined over C with function field E. Let $\omega \in \Omega(E/C)$ be an invariant differential form on $G(C)$. Then ω is a closed Kähler differential in $\Omega(F/C)$, that is, $d\omega = 0$ in $\Omega^2(F/C)$.*

Proof. $\Omega^\bullet(E/C)$ is a subcomplex of $\Omega^\bullet(F/C)$, so it is enough to show that $d\omega = 0$ in $\Omega^2(E/C)$.

The Lie algebra L of $G(C)$ is canonically isomorphic to the the space of invariant vector fields on $G(C)$, and is a C-vector space of dimension $n = \dim G$. Let $X_1, \ldots, X_n$ be a basis of L. The vector space $\mathrm{Der}(E/C)$ of all vector fields on $G(C)$ is $L \otimes_C E$, so $X_1, \ldots, X_n$ also forms an E-basis of $\mathrm{Der}(F/C)$. Let $D_1, D_2 \in \mathrm{Der}(E/C)$, say $D_1 = \sum_{i=1}^n a_i X_i$ and $D_2 = \sum_{i=1}^n b_i X_i$ with the $a_i, b_i \in E$. Then

$$\begin{array}{rcll}
d\omega(D_1, D_2) & = & d\omega(\sum_{i=1}^n a_i X_i, \sum_{i=1}^n b_i X_i) & \\
& = & \sum_{i,j} a_i b_j d\omega(X_i, X_j) & \text{by bilinearity of } d\omega \\
& = & \sum_{i,j} a_i b_j (X_i(\omega X_j) - X_j(\omega X_i) - \omega[X_i, X_j]) \quad . &
\end{array}$$

Now ω and X_i are both invariant, so for any $x, y \in G(C)$,

$$\begin{aligned}
(\omega X_j)_{xy} &= \omega_{xy}(X_j)_{xy} \\
&= \omega_y(d\lambda_x^{x^{-1}} y(X_j)_{xy}) \\
&= \omega_y(d\lambda_x^{x^{-1}} y d\lambda_y^x (X_j)_y) \\
&= \omega_y(X_j)_y \\
&= (\omega X_j)_y
\end{aligned}$$

and so ωX_j is a constant scalar field on $G(C)$. Thus $X_i(\omega X_j) = 0$, and similarly $X_j(\omega X_i) = 0$. So

$$d\omega(D_1, D_2) = -\sum_{i,j} a_i b_j \omega[X_i, X_j]$$

but [,] is the bracket on the Lie algebra of G, and G is commutative so the bracket is identically zero. So $d\omega(D_1, D_2) = 0$ for all $D_1, D_2 \in \mathrm{Der}(E/C)$, and hence $d\omega = 0$. $\square$

Chapter 4

Equations and structures

This chapter describes the differential equations under consideration, and the first order structures which represent them. The theory of these structures is outlined with proofs of the main statements deferred to later chapters.

4.1 The differential equations

We start with a motivating example. The usual complex exponential function satisfies the differential equation $\frac{d\exp(t)}{dt} = \exp(t)$ or, more generally, $\frac{d\exp(x)}{dt} = \exp(x)\frac{dx}{dt}$. If we write D for the differential operator $\frac{d}{dt}$, and write $y = \exp(x)$ this becomes simply

$$Dy = yDx \qquad \text{or} \qquad \frac{Dy}{y} = Dx \tag{4.1}$$

the latter being possible since $\exp(x) \neq 0$. Equation 4.1 is the form of the exponential equation which we consider in the context of a differential field $\langle F; +, \cdot, D, C\rangle$. In geometric terms, the variable x can be considered to come from the additive group $\mathbb{G}_\mathrm{a}$, and y comes from the multiplicative group $\mathbb{G}_\mathrm{m}$. The invariant differential forms on these groups are dx and $\frac{dy}{y}$, so we can equivalently write this equation as

$$\frac{dy}{y}D = dxD \qquad \text{or} \qquad \left(\frac{dy}{y} - dx\right)D = 0 \tag{4.2}$$

where $\frac{dy}{y} - dx$ is an invariant differential form on $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$.

We shall study differential equations of the form $\omega(g)D = 0$ where ω is an invariant differential form on a connected commutative algebraic group G, and g is a variable ranging over G. (The apparent notational inconsistency between this and the example of exponentiation is due to writing dx and $\frac{dy}{y}$ for both the invariant differential forms on $\mathbb{G}_\mathrm{a}$ and $\mathbb{G}_\mathrm{m}$ and for their images after applying them to x and y.) The aim is to study systems of equations made up of polynomial equations and differential equations of this form.

We denote the solution set of the differential equation by Γ. More precisely,

$$\Gamma_{\omega,D} = \{g \in G \mid \omega(g)D = 0\} \tag{4.3}$$

although we usually drop the subscript D, and may also drop ω where it is unambiguous. We often want to consider more than one differential form on G. If we have $\omega = (\omega_1, \ldots, \omega_s)$ then we write $\Gamma_\omega = \bigcap_{i=1}^{s} \Gamma_{\omega_i}$. Similarly we often want to consider more than one derivation, and set $D = (D_1, \ldots, D_r)$. In this case we write $\Gamma_{\omega,D} = \bigcap_{j=1}^{r} \Gamma_{\omega,D_j}$.

The main example we shall consider in detail is exponentiation on a semiabelian variety, but we also mention some other examples.

Exponentiation Let S be a semiabelian variety, defined over C, and let $n = \dim S$. Let $\xi_1, \ldots, \xi_n$ be a basis of invariant differential forms of S, which we may also take to be defined over C. Take $G = \mathbb{G}_\mathrm{a}^n \times S$, and $\omega_i = dx_i - \xi_i$ for $i = 1, \ldots, n$, where $dx_1, \ldots, dx_n$ is a basis of invariant forms on $\mathbb{G}_\mathrm{a}^n$. Take $\omega = (\omega_1, \ldots, \omega_n)$.

This is the system of differential equations satisfied by the exponential map $\mathbb{G}_\mathrm{a}^n(\mathbb{C}) \xrightarrow{\exp} S(\mathbb{C})$ of S. For $S = \mathbb{G}_\mathrm{m}$, this is the usual exponentiation given above. If S is an elliptic curve $\mathcal{E}$, with affine part given by $\{(y, z) \in F^2 \mid z^2 = 4y^3 - g_2 y - g_3\}$ for some $g_2, g_3 \in \mathbb{C}$, then the exponential map of $\mathcal{E}$ is $(\wp, \wp')$ where $\wp$ is the Weierstrass function associated with $\mathcal{E}$, and the differential equation is $Dx = \frac{Dy}{z}$.

Exponential-like equations This is a generalization of the exponential equations. Let G_1 and G_2 be connected commutative algebraic groups defined over C, of the same dimension n, and suppose they are null-homomorphic (the only regular homomorphisms between G_1 and G_2 are the zero maps). Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be a basis of invariant differential forms on G_1 and $\xi = (\xi_1, \ldots, \xi_n)$ be a basis of invariant differential forms on G_2, defined over C. Take $\omega_i = \zeta_i - \xi_i$ for $i = 1, \ldots, n$ and $\omega = (\omega_1, \ldots, \omega_n)$.

The exponential equations are of this form because $\mathbb{G}_\mathrm{a}$ is null-homomorphic to any semiabelian variety. Taking $\mathbb{G}_\mathrm{m} \times \mathcal{E}$ for an elliptic curve $\mathcal{E}$, this gives the differential equation $\frac{Dx}{x} = \frac{Dy}{z}$ satisfied by the Tate factorization of the exponential map of $\mathcal{E}$. If we take this equation for $\mathcal{E}_1 \times \mathcal{E}_2$, two nonisogenous elliptic curves, then there is no analytic function satisfying the equation, since by Chow's theorem any such function would have to be algebraic and then by [Sha94a, p192] it would have to be the composite of a homomorphism and a translation, and this is impossible since $\mathcal{E}_1$ and $\mathcal{E}_2$ are nonisogenous.

Raising to powers Let S be a *simple* semiabelian variety (either $\mathbb{G}_\mathrm{m}$ or a simple abelian variety) and take a basis $\xi = (\xi_1, \ldots, \xi_n)$ of invariant differential forms of S. Take $G = S^2$ and choose $\mu \in \mathrm{GL}_n(C)$. Define $\omega(x, y) = \mu\xi(x) - \xi(y)$.

When $S = \mathbb{G}_\mathrm{m}$, μ is just a nonzero element of C and the differential equation is $\mu\frac{Dx}{x} = \frac{Dy}{y}$, which is the differential equation satisfied by $\mu \log(x) = \log(y)$, or the multivalued function $y = x^\mu$, hence the name "raising to powers". If $\mu \in \mathbb{Q}$ then this is not interesting, and similarly for an abelian variety S it is not interesting when μ is $d\sigma_e$ for a rational endomorphism σ of S.

Logarithmic Derivatives There is an alternative way of describing all these differential equations. For any algebraic group G defined over C, and any derivation D with constants C, there is a *logarithmic derivative* DLog_G, a map $G \longrightarrow T_eG$, described in [Mar00] and [Pil04]. The raising to powers equations can be written as $\mu\,\mathrm{DLog}_S(x) = \mathrm{DLog}_S(y)$. The exponential-like equations can be written as $\mathrm{DLog}_{G_1}(x) = \mathrm{DLog}_{G_2}(y)$. To makes sense of this equation requires a choice of isomorphism between the Lie algebras T_eG_1 and T_eG_2. In the description of the equation by invariant differential forms, this isomorphism is explicitly given by the choice of a basis of invariant differential forms on each group, which is a dual basis for the Lie algebra. In effect, these bases are identified by the differential equation.

4.1.1 First order structures

Expanding the expression $\omega(g)D$ gives a differential polynomial in g, and so each Γ_ω is definable in any differential field $\langle F; +, \cdot, D, C\rangle$. In order to isolate the one equation for which Γ is the solution set, we consider the reduct structure $\langle F; +, \cdot, \Gamma, C\rangle$. We want to know which systems of equations can have solutions, and so we take F to be a (saturated) differentially closed field. In this case, we say that the theory of the reduct is the *theory of the differential equation* $\omega(g)D = 0$. More generally, one could consider the theory of any set of differential equations by adding to the language a predicate for the solution set of each equation, and then taking the reduct forgetting the derivations.

The first half of this thesis, up to chapter 5, deals with the universal part of the theory, which is given by the algebraic structure on Γ (to be described below) together with necessary conditions for a system of equations to have a solution. This applies to any differential field, so for this there is no need to assume that F is differentially closed. For the choices of ω we shall consider, the constant field C will be definable from the rest of the data. However, it is convenient to have it in the language anyway.

4.2 Algebraic structure

If we have one solution to a differential equation, that is, a point in Γ, the other solutions we automatically know about are given by the group structure described in the following propositions.

Proposition 4.1. *1. Let G be a commutative algebraic group, defined over the constant field C. Let ω be a list of invariant differential forms on G and let $\Gamma = \{g \in G \mid \omega(g)D = 0\}$. Then Γ is a subgroup of G.*

2. Suppose in addition that ω is defined over C. Then $\Gamma \supseteq G(C)$, the set of constant points of G, with equality iff ω spans the vector space $\mathrm{Inv}(G)$ *of invariant differential forms on G.*

3. For the exponential, exponential-like, and raising to powers equations, with $G = G_1 \times G_2$, if $a \in G_1$ is such that the fibre $\Gamma(a) = \{y \in G_2 \mid (a, y) \in \Gamma\}$ of the projection to G_1 is nonempty, then the fibre is a coset of the subgroup $G_2(C)$. Similarly for fibres of Γ of the projection to G_2.

Proof. For 1, Γ is the kernel of the map

$$G \xrightarrow{\omega} \Omega(F/C) \xrightarrow{D^*} F^r$$

where ω is a group homomorphism as in proposition 3.20 and D^* is the "evaluate at D" linear map. Thus Γ is a subgroup of G.

For 2, if the differential forms ω_i are defined over C then for any $g \in G$, $\omega(g) \in \Omega(C(g)/C)$. Thus for $g \in G(C)$, $\omega_i(g) \in \Omega(C/C) = \{0\}$. Hence $G(C) \subseteq \Gamma$. If ω spans $\mathrm{Inv}(G)$ then $\omega(g)$ spans $T^*_g G$, and so $g \in \Gamma$ implies that $\nabla_D(g) = 0$, that is, that $g \in G(C)$.

If $a = 0$ then, by part 2, the fibre $\Gamma(0) = G_2(C)$. If $(a, b) \in \Gamma$ then for any $y \in G_2(C)$, $(a, b \oplus y) = (a, b) \oplus (0, y)$ which is in Γ by part 1. Conversely, for any $y' \in \Gamma(a)$, $(a, b) \ominus (a, y') = (0, b \ominus y)$, and so $y' \in b \oplus G_2(C)$. □

We also need to understand how solutions to different differential equations are related. If we know about a solution to one equation then this also gives us solutions to certain other equations.

Proposition 4.2. *Let G, H be commutative algebraic groups, let ω be an invariant differential form on H, let $g \in G$, and let $G \xrightarrow{f} H$ be a regular group homomorphism. Then $g \in \Gamma_{f_*\omega}$ iff $f(g) \in \Gamma_\omega$.*

Proof. By definition, $g \in \Gamma_{f_*\omega}$ iff $(f_*\omega)(\nabla_D(g)) = 0$. But

$$(f_*\omega)(\nabla_D(g)) = \omega_{f(g)}(df_g(\nabla_D(g))) = \omega_{f(g)}(\nabla_D(f(g)))$$

by lemma 2.3, and this is zero iff $f(g) \in \Gamma_\omega$. □

Although the homomorphism f will not generally be invertible, the statement is "if and only if", allowing transfer of solutions in both directions. From this we see, for example, that understanding differential equations on one group allows us to understand the corresponding equations on an isogenous group. If we consider Γ's on several algebraic groups then this proposition shows that the group structures on each Γ can be combined into a groupoid structure.

For a commutative algebraic group G, write $\mathrm{End}(G)$ for the ring of (regular) endomorphisms of G. Then G is an algebraic $\mathrm{End}(G)$-module. For a semiabelian variety S defined over some field K, all the endomorphisms of S are also defined over K, and so $\mathrm{End}(S)$ is a well-defined ring. The endomorphisms of a vector group $\mathbb{G}_\mathrm{a}^n(F)$ are the linear maps defined over F, and so if G contains a vector group (that is, G is not a semiabelian variety) then the endomorphism ring $\mathrm{End}(G)$ depends on the field over which the endomorphisms are defined.

For any list ω of invariant differential forms on G, Γ_ω is a subgroup of G by Proposition 4.1, and hence a $\mathbb{Z}$-submodule of G. However, it may also be an R-submodule of G for a larger subring R of $\mathrm{End}(G)$. Any $\sigma \in \mathrm{End}(G)$ induces a map $T^*\sigma = \sigma_*$ on T_e^*G and hence on $\mathrm{Inv}(G)$.

Lemma 4.3. *If G is connected then the map*

$$\begin{array}{ccc} \mathrm{End}(G) & \longrightarrow & \mathrm{End}(T_e^*) \\ \sigma & \longmapsto & \sigma_* \end{array}$$

is injective. Hence regular endomorphisms of G correspond to linear endomorphisms of $\mathrm{Inv}(G)$ and, by taking the dual, to linear endomorphisms of the Lie algebra of G.

Proof. If $\sigma_* = 0$ then $\varphi(d\sigma_e(u)) = 0$ for each $\varphi \in T_e^*G$ and each $u \in T_eG$. Thus $d\sigma_e(u) = 0$ for each $u \in T_eG$ and so $d\sigma_e = 0$.

The complex realization $G(\mathbb{C})$ of G has $T_eG(\mathbb{C})$ as its universal cover, and σ lifts to the cover as the linear map $d\sigma_e$. This is the constant zero map and so, projecting back onto G, we see that σ is the zero map. □

Lemma 4.4. *Γ_ω is closed under σ iff the span Λ of ω in* $\mathrm{Inv}(G)$ *is closed under σ_*.*

Proof. Checking the definitions, it is easy to see that $(\sigma_*\omega_i)(g)D = \omega_i(\sigma(g))D$. The result follows. $\square$

Hence the largest subring of $\mathrm{End}(G)$ for which Γ_ω is a submodule is

$$R_\omega = \{\sigma \in \mathrm{End}(G) \mid \sigma_*[\Lambda] \subseteq \Lambda\}.$$

For the exponential-like equations on $G = G_1 \times G_2$, any $\sigma \in \mathrm{End}(G)$ is of the form $(\sigma_1, \sigma_2) \in \mathrm{End}(G_1) \times \mathrm{End}(G_2)$, because G_1 and G_2 are null-homomorphic. As noted, the differential equations define a bijection between $\mathrm{Inv}(G_1)$ and $\mathrm{Inv}(G_2)$ and, with lemma 4.3, this allows us to consider $\mathrm{End}(G_1)$ and $\mathrm{End}(G_2)$ both as subrings of the same ring $\mathrm{End}(\mathrm{Inv}(G_1))$.

Proposition 4.5. *For exponential-like equations, $R_\omega = \mathrm{End}(G_1) \cap \mathrm{End}(G_2)$, with respect to this identification.*

Proof. To say that $\sigma \in \mathrm{End}(G_1) \cap \mathrm{End}(G_2)$ according to this identification is the same as saying that $\sigma = (\sigma_1, \sigma_2)$ and $\sigma_{1*} = \sigma_{2*}$ on the identification of the spaces of invariant differential forms.

Now $(\sigma_{1*}, \sigma_{2*})(\omega_i) = \sigma_{1*}(\zeta_i) - \sigma_{2*}(\xi_i)$. This is in the span of ω iff the linear maps σ_{1*} and σ_{2*} agree on these vectors. Taking $i = 1, \ldots, n$, the ζ_i and ξ_i form a basis for $\mathrm{Inv}(G_1)$ and $\mathrm{Inv}(G_2)$, and so this holds for all i iff $\sigma_{1*} = \sigma_{2*}$. $\square$

In the special case of the exponential equation for a semiabelian variety S, R_ω is isomorphic to $\mathrm{End}(S)$, because the endomorphism ring of $\mathbb{G}_\mathrm{a}^n$ is the whole ring $\mathrm{End}(\mathrm{Inv}(\mathbb{G}_\mathrm{a}^n))$.

4.3 Dimension notions and conjectures

The groupoid and module structure of Γ explains how to obtain new solutions to the differential equations from a given set of solutions. The remainder of the theory must explain which systems of equations do and do not have solutions. The equations allowed in these systems are polynomial equations and the differential equations under consideration. Replacing the equations by their solution sets, this means algebraic varieties and the "differential varieties" given by Γ_ω for different ω. As will be shown later, it is enough to consider systems of equations of the form $V \cap \Gamma_\omega$ where V is an irreducible algebraic subvariety of the algebraic group G. Note that if $\Gamma_\omega \subseteq G$ then $\Gamma_\omega^n \subseteq G^n$ is also of this form, since G^n is also an algebraic group and $\Gamma_\omega^n = \Gamma_{\omega'}$ for a suitably chosen ω'.

4.3.1 A heuristic argument

We shall show that the question of when $V \cap \Gamma_\omega$ has solutions depends on a dimension theory and the related intersection theory. In model-theoretic terms, the reduct of a differentially closed field is ω-stable, which indicates that the dimension notion of Morley rank is involved. In fact the situation here is much nicer than in a general ω-stable situation, indeed even nicer than the dimension theory of algebraic geometry, because we shall get sufficient conditions for an intersection to be nonempty just from the dimension theory. We develop the theory without explicitly referring to the model-theoretic concepts lying beneath the surface. Indeed for now, we think heuristically of the dimension of a system of equations as the number of degrees of freedom, which is given by the number of variables minus the number of constraints. Eventually it will turn out that this heuristic argument is justified.

We can think of the number of constraints as being the number of equations, "counted properly". For example, the three equations

$$x = 1 \qquad y = 1 \qquad x + y = 2 \qquad\qquad (*)$$

all have different solution sets, but together form only two constraints. For linear equations, counting the number of constraints given by a system of equations amounts to calculating the rank of the matrix of coefficients. For polynomial equations it is more complicated, but we can just use the fact that the answer is the codimension of the algebraic subvariety defined by the equations. For $\Gamma_\omega \subseteq G$, the number of constraints is equal to the linear dimension of the span of ω in $\mathrm{Inv}(G)$.

For the examples under consideration, if $\dim G = 2n$ then $\dim \Gamma_\omega = n$. Counting the constraints, one would typically expect that

$$\operatorname{codim}(V \cap \Gamma_\omega) = \operatorname{codim} V + \operatorname{codim} \Gamma_\omega$$

which is the same as

$$\dim(V \cap \Gamma_\omega) = \dim V - n.$$

Taking account of a possible reduction in the number of constraints for an "atypical" situation similar to $(*)$ above, this becomes

$$\dim(V \cap \Gamma_\omega) \geqslant \dim V - n.$$

We must understand how it is possible for the number of constraints to drop when an algebraic variety intersects Γ_ω. The group and module structure of Γ_ω gives one way in which this happens. For example, if $\dim G = n$, ω consists of s independent forms and $V \subseteq G^3$ is given by $g_1 \oplus g_2 = g_3$, then, in $V \cap \Gamma_\omega^3$, the $3s$ equations given by $g_1 \in \Gamma_\omega, g_2 \in \Gamma_\omega$ and $g_3 \in \Gamma_\omega$ constitute just $2s$ further constraints, because Γ_ω is a subgroup of G. Similarly, since $G(C) \subseteq \Gamma_\omega$, if $V \models g \in G(C)$ then $g \in \Gamma_\omega$ contributes no further constraints.

Consider what this means for the exponential equation for a simple semiabelian variety S, of dimension e. Recall that $\Gamma_\omega \subseteq \mathbb{G}_\mathrm{a}^e \times S$ and $R_\omega \cong \mathrm{End}(S)$. The group $G(C)$ is an R_ω-submodule of Γ_ω, so we can consider the quotient $\Gamma_\omega/G(C)$. The R_ω-torsion points are all algebraic, and so lie in $G(C)$. Furthermore we assume that the field F is algebraically closed, and so $\Gamma_\omega/G(C)$ is a module over the "field of fractions" $K = R_\omega \otimes_\mathbb{Z} \mathbb{Q}$ of R_ω. By theorem 3.11, $\mathrm{End}(S) \otimes_\mathbb{Z} \mathbb{Q}$ is a division ring, and thus so is K, hence $\Gamma_\omega/G(C)$ is a K-vector space. Suppose that $(x_i, y_i) \in \Gamma_\omega$ for $i = 1, \ldots, n$ and that their images in $\Gamma_\omega/G(C)$ are K-linearly independent. Then there are $\sigma_i \in R_\omega$, not all zero, such that

$$\bigoplus_{i=1}^{n} \sigma_i x_i \in \mathbb{G}_\mathrm{a}^e(C) \quad \text{and} \quad \bigoplus_{i=1}^{n} \sigma_i y_i \in S(C). \qquad (\dagger)$$

From these equations and the R_ω-module structure on Γ_ω, it follows that the ne equations given by $(x_i, y_i) \in \Gamma_\omega$ for $i = 1, \ldots, n$ add only $(n-1)e$ new constraints beyond the algebraic constraints giving the K-linear dependence. In general, if the K-linear dimension of the images is k, then these equations add only ke new constraints.

Now consider the exponential equation for a semiabelian variety S which is a product of powers of non-isogoenous simple semiabelian varieties, $S = \prod_{j=1}^{m} S_j^{n_j}$. The argument above works on each factor separately, and the factors are null-homomorphic, so the groupoid structure of Γ is just the module structure on each factor. Thus if the groupoid structure acts to reduce the number of constraints, it must be from equations of the form (†) on one of the factors.

By Kolchin's description of algebraic subgroups of products of simple algebraic groups, theorem 3.9, equations (†) say that x and y lie in constant cosets of proper algebraic subgroups H_1 of $\mathbb{G}_\mathrm{a}^{ne}$ and H_2 of S^n respectively. Furthermore, H_1 and H_2 correspond to one another in the sense that their Lie algebras are equal under the identification of the Lie algebras of $\mathbb{G}_\mathrm{a}^{ne}$ and S^n given by the differential equations. This makes sense also in greater generality.

Definition 4.6. For the situation of exponential-like equations on $G = G_1 \times G_2$, define subgroups H_1 of G_1 and H_2 of G_2 to be *corresponding subgroups* iff their Lie algebras are identified under the identification of the Lie algebras of G_1 and G_2 given by the differential equation.

A subgroup of G of the form $H_1 \times H_2$ for a pair of corresponding subgroups is said to be a *special subgroup* of G.

For the exponential equation of a semiabelian variety, where $G = \mathbb{G}_a^n \times S$, define $\operatorname{Log} H$ to be the subgroup of $\mathbb{G}_a^n$ which corresponds to a subgroup H of S, and write $\widehat{H}$ for the special subgroup $\operatorname{Log} H \times H$.

Note that by theorem 3.16, each subgroup of G_2 has at most one corresponding subgroup in G_1 and vice versa. Note also that $\widehat{S}$ is isomorphic as an algebraic group to the tangent bundle TS, because S is a commutative algebraic group. This only applies to the exponential equations, not to the other examples, and so I choose not to stress the point. It also seems to give a misleading geometric picture with the groups $\mathbb{G}_a^n$ and S not playing equal roles. However, it does show that the map $S \mapsto \widehat{S}$ is a functor, and so it has a well-defined action on morphisms.

Given an irreducible subvariety V of $\mathbb{G}_a^n \times \widehat{S}$, let H be the smallest algebraic subgroup of S such that V is contained in a constant coset of $\widehat{H}$. The heuristic argument we have given says that

$$\dim(V \cap \Gamma_\omega^n) = \dim V - \dim H$$

taking into account the groupoid structure of Γ_ω. In the typical situation, H will actually be S and $\dim H = n$, which agrees with the calculation given earlier.

4.3.2 Necessary conditions for solutions

We now formalize the conclusion of the heuristic argument.

Definition 4.7. Let $V \subseteq G$ be an irreducible subvariety of G. Define the *group rank* of V to be $\operatorname{grk}(V) = \frac{1}{2}\dim H$ where H is the smallest special algebraic subgroup of G such that V is contained in a coset $\gamma \oplus H$ for some $\gamma \in G$. Define

$$\varphi(V) = \dim V - \operatorname{grk}(V).$$

For varieties V defined over C, if V lies in a coset $\gamma \oplus H$ then γ must lie in $G(C)$, as C is algebraically closed. We can also define the group rank of a point in G.

Definition 4.8. For $g \in G$, and any subfield K of F containing C, define $\operatorname{grk}_K(g) = \frac{1}{2}\dim H$ where H is the smallest special algebraic subgroup of G such that $x \in \gamma \oplus H$ for some $\gamma \in G(K)$.

Note that $\operatorname{grk}_K(g) = \operatorname{grk}(\operatorname{Loc}_K(g))$. The quantity $\varphi(V)$ is supposed to give information about when $V \cap \Gamma$ is nonempty. We define another function δ for elements of Γ, and later the definition will be extended to any elements of G.

Definition 4.9. Let $g \in \Gamma$. Define $\delta(g) = \operatorname{td}_C(g) - \operatorname{grk}_C(g)$.

Thus for $g \in \Gamma$, $\delta(g) = \varphi(\operatorname{Loc}_C(g))$ and so φ and δ look very similar, but $\varphi(V)$ is defined for varieties which may have no intersection with Γ and the definition of δ will be extended in chapter 6 to points g not lying in Γ, as the two functions will play different roles.

In a differential field, a tuple of elements can be considered to have degrees of freedom intrinsic to itself, and these must be taken into account. Let $D_1, \ldots, D_r$ be the (commuting) derivations of the differential field F, and let $x = (x_1, \ldots, x_n) \in F^n$. Define the *Jacobian matrix* of x to be

$$\operatorname{Jac}(x) = \begin{pmatrix} D_1 x_1 & \cdots & D_1 x_n \\ \vdots & & \vdots \\ D_r x_1 & \cdots & D_r x_n \end{pmatrix}$$

in the usual way. Then we consider x to have a number of degrees of freedom equal to $\operatorname{rk}\operatorname{Jac}(x)$, the rank of the Jacobian matrix. This perhaps strange-seeming convention makes sense in the context of analytic geometry, where $\operatorname{rk}\operatorname{Jac}(x)$ is the local dimension of the analytic variety for which x is a choice of local coordinate functions, and D is the appropriate list of directional derivations.

Note that $\operatorname{rk}\operatorname{Jac}(x) = 0$ iff x is constant, and if F is an ordinary differential field (that is, $r = 1$) then $\operatorname{rk}\operatorname{Jac}(x)$ is 0 or 1.

We now conjecture that in the case of semiabelian varieties, the only way in which the number of constraints in $V \cap \Gamma_\omega$ can drop (that is, that $\dim(V \cap \Gamma_\omega)$ can be strictly greater than $\dim V - n$) is due to the R_ω-module structure on Γ_ω. Since by convention a system of equations is consistent precisely when its dimension is non-negative, this gives a necessary condition for a system of equations to have a solution.

Conjecture 4.10. *If $V \subseteq G$ is an irreducible variety defined over C and $\varphi(V) < 1$ then $V \cap \Gamma$ has no nonconstant points.*

Equivalently, if $g \in \Gamma$ then $\delta(g) \geqslant \operatorname{rk}\operatorname{Jac}(g)$.

This conjecture can be made for any exponential-like equation. We call this statement the *Schanuel condition* for the differential equation.

We now translate these Schanuel conditions back into less technical language for some examples.

Exponentiation For the usual exponential function of $\mathbb{G}_\mathrm{m}$, the Schanuel condition says that if $x_1, y_1, \ldots, x_n, y_n \in F$ satisfy $\frac{D_j y_i}{y_i} = D_j x_i$ for each $i = 1, \ldots, n$ and each $j = 1, \ldots, r$, and $\mathrm{td}_C(x_1, y_1, \ldots, x_n, y_n) - \mathrm{rk}\,\mathrm{Jac}(x_1, \ldots, x_n) < n$, then there are $m_i \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^n m_i x_i \in C$ and $\prod_{i=1}^n y_i^{m_i} \in C$.

(Note that it follows from the differential equations that $\mathrm{rk}\,\mathrm{Jac}(x) = \mathrm{rk}\,\mathrm{Jac}(x, y)$.) This case is a theorem of James Ax, [Ax71], and was in fact the original motivation for the work in this thesis.

Elliptic curves For the exponential equations for an elliptic curve $\mathcal{E}$, with affine part given by $\{(y, z) \in F^2 \mid z^2 = 4y^3 - g_2 y - g_3\}$ and $\mathrm{End}(\mathcal{E}) = R$ (either $\mathbb{Z}$ or $\mathbb{Z}[\tau]$), the Schanuel condition is the following. Let $x_1, y_1, \ldots, x_n, y_n \in F$ be such that $\frac{D_j y_i}{\sqrt{4y_i^3 - g_2 y_i - g_3}} = D_j x_i$ for each i and j. Suppose that

$$\mathrm{td}_C(x_1, y_1, \ldots, x_n, y_n) - \mathrm{rk}\,\mathrm{Jac}(x_1, \ldots, x_n) < n.$$

Then there are $m_i \in R$, not all zero, such that $\sum_{i=1}^n m_i x_i \in C$ and $\bigoplus_{i=1}^n m_i y_i \in \mathcal{E}(C)$.

Semiabelian varieties For the exponential equation for any semiabelian variety S of dimension n, the Schanuel condition says:
Let $x \in \mathbb{G}_\mathrm{a}^n(F)$ and $y \in S(F)$ with $(x, y) \in \Gamma$. If $\mathrm{td}_C(x, y) - \mathrm{rk}\,\mathrm{Jac}(x) < n$ then there is a proper algebraic subgroup H of S such that y is in a constant coset of H. (It follows from the differential equations that x will be in a constant coset of $\mathrm{Log}\, H$.)

The proofs of these statements are given in chapter 5.

4.3.3 Sufficient conditions for solutions

Dimension theory arguments can also give (conjectural) sufficient conditions for the intersection $V \cap \Gamma_\omega$ to be nonempty, that is, for the corresponding system of equations to have a solution. In fact, in the differential field context we should be looking for nonconstant solutions. Typically, two subvarieties V_1 and V_2 of an n-dimensional space will have non-empty intersection when $\dim V_1 + \dim V_2 \geqslant n$. This is when the sum of the numbers of constraints is still less than the number of variables. However, there can be exceptions. For example, the two subvarieties of F^3 given by the equations $x_1 = 1$ and $x_1 = 2$ each have dimension 2, but have no intersection as the hyperplanes they define are parallel. We shall show that this sort of behaviour cannot happen in an intersection $V \cap \Gamma_\omega$.

A separate issue is that to find enough points of intersection, it may be necessary to look in the right place. For example, the two equations

$$x^2 + y^2 + z^2 = 1 \qquad \text{and} \qquad (x-2)^2 + y^2 + z^2 = 1$$

have only one point in common in $\mathbb{R}^3$, but in $\mathbb{C}^3$ they have a 1-dimensional solution set as the dimension theory suggests they should. In this example the issue is that the dimension theory for polynomial equations requires an algebraically closed field. In our situation we should be looking for solutions in a differentially closed field (or at least a differential field which has all possible solutions for the particular differential equations we are studying).

Assuming that the Schanuel conditions are true, for $V \cap \Gamma_\omega$ to have nonconstant solutions it is necessary that $\varphi(V) \geqslant 1$ and furthermore that this applies to subsystems of equations.

Indeed, for every homomorphic image H of S and homomorphism $S \xrightarrow{f} H$ we must have $\varphi_H(\widehat{f}(V)) \geqslant 1$, where φ_H is the appropriate form of φ taking H instead of S, and $\widehat{f}$ is the map $G \twoheadrightarrow \widehat{H}$ arising from f. We conjecture that if a system of equations meets this criterion then it must have (nonconstant) solutions.

Conjecture 4.11. *If $V \subseteq G$ is an irreducible algebraic variety defined over C such that $\varphi_H(\widehat{f}(V)) \geqslant 1$ for every surjective map $S \xrightarrow{f} H$, then $V \cap \Gamma_\omega$ has a nonconstant point in a differentially closed field.*

This conjecture will be proved for the exponential equations of split semiabelian varieties in chapter 7. To complete the picture of which systems of equations have solutions, it is necessary also to consider varieties defined with nonconstant parameters. We defer this to chapter 6.

Chapter 5

Schanuel conditions

In this chapter we give proofs of the Schanuel conditions, derive a uniform version and give an algebraic application. The last section of the chapter gives an alternative proof of the Schanuel condition for semiabelian varieties, using theorems of Ax and Seidenberg. It is less direct and in some ways less clear than the first proof presented here, but it is shorter and the application of Seidenberg's theorem is of independent interest.

5.1 Direct method

As before, F is a field of characteristic zero, $D = (D_1, \dots, D_r)$ is a list of derivations on F, and $C = C_D$ is the intersection of the constant fields of the D_j.

The proof is split up into a number of lemmas, both to show clearly exactly how the proof works and because the lemmas will also be used in chapter 7.

Lemma 5.1. *Let E be a subfield of F containing C, with $\mathrm{td}(E/C)$ finite. Then the F-vector space $(\Omega(E/C) \otimes_E F) \cap \mathrm{Ann}(D)$ has dimension $\mathrm{td}(E/C) - \mathrm{rk}\,\mathrm{Jac}(e)$, where e is a transcendence base for E/C.*

Proof. The E-vector space $\Omega(E/C)$ has E-linear dimension $\mathrm{td}(E/C)$, and so its extension $\Omega(E/C) \otimes_E F$ to an F-vector space has F-linear dimension $\mathrm{td}(E/C)$. Both this and $\mathrm{Ann}(D)$ are subspaces of $\Omega(F/C)$, and we show that their intersection is a subspace of $\Omega(E/C) \otimes_E F$ of codimension $\mathrm{rk}\,\mathrm{Jac}(e)$.

Consider the diagram below, where D^* is the map which comes from the universal property of d.

$$\begin{tikzcd} F \arrow[r, "d"] \arrow[dr, "D"'] & \Omega(F/C) \arrow[d, "D^*"] \\ & F^r \end{tikzcd}$$

$\mathrm{Ann}(D)$ is the kernel of the linear map D^*. The diagram restricts to

$$\begin{tikzcd} E \arrow[r, "d"] \arrow[dr, "D"'] & \Omega(E/C) \otimes_E F \arrow[d, "D^*"] \\ & F^r \end{tikzcd}$$

where again D^* is F-linear, with kernel $(\Omega(E/C) \otimes_E F) \cap \mathrm{Ann}(D)$.

The image of D^* is the image of D, which is spanned by the columns of the matrix $\mathrm{Jac}(e)$. Thus $\mathrm{rk}\,\mathrm{Jac}(e)$ is equal to the rank of the linear map D^*, which by the rank-nullity theorem is equal to the codimension of its kernel. Thus $\Omega(E/C) \otimes_E F \cap \mathrm{Ann}(D)$ has dimension $\mathrm{td}(E/C) - \mathrm{rk}\,\mathrm{Jac}(e)$, as claimed. □

This lemma allows us to find an F-linear dependence between differentials, and the next allows us to turn that into a C-linear dependence.

Lemma 5.2. *Let Δ be a subspace of $\mathrm{Der}(F/C)$ and let*

$$C_\Delta = \{x \in F \mid Dx = 0 \text{ for all } D \in \Delta\}.$$

Let $\omega_1, \ldots, \omega_n \in \Omega(F/C) \cap \mathrm{Ann}(\Delta)$ be closed *differentials, which means that $d\omega_i = 0$ in $\Omega^2(F/C)$ for each i. Suppose that the ω_i are F-linearly dependent in $\Omega(F/C)$. Then they are C_Δ-linearly dependent.*

Proof. Take $\alpha_i \in F$ such that $\sum_{i=1}^n \alpha_i \omega_i = 0$ giving a minimal F-linear dependence on the ω_i, that is, if $I = \{i \mid \alpha_i \neq 0\}$ then the F-linear dimension of $\{\omega_i \mid i \in I\}$ is $|I| - 1$. Dividing by some non-zero α_i, we may assume that for some $i = i_0$, $\alpha_{i_0} = 1$.

Applying the Lie derivative L_D for $D \in \Delta$ we get

$$\begin{aligned} 0 = L_D \sum_{i=1}^n \alpha_i \omega_i &= \sum_{i=1}^n [(D\alpha_i)\omega_i + \alpha_i L_D \omega_i] \\ &= \sum_{i=1}^n [(D\alpha_i)\omega_i + \alpha_i(dD^*\omega_i + D^* d\omega_i)] \\ &= \sum_{i=1}^n (D\alpha_i)\omega_i \end{aligned}$$

since $D^*\omega_i = d\omega_i = 0$ for each i, and using lemma 2.1. Now $\alpha_{i_0} = 1$, so $D\alpha_{i_0} = 0$ but then, by the minimality of the α_i, we have that $D\alpha_i = 0$ for every i and each $D \in \Delta$, so each $\alpha_i \in C_\Delta$. Hence the ω_i are C_Δ-linearly dependent. □

The following lemma gives the connection between linear dependence of differential forms on G and algebraic subgroups of G.

Lemma 5.3. *Let G be a commutative algebraic group defined over C, and let $\omega = (\omega_1, \ldots, \omega_n)$ be a system of linearly independent invariant differential forms on G, each defined over C.*

Suppose that $g \in G$ and that the differentials $\omega_i(g)$ are linearly dependent in $\Omega(F/C)$.

Then there is a proper algebraic subgroup H of G and a constant point $\gamma \in G(C)$ such that g lies in the coset $\gamma \oplus H$. Moreover, there is a nonzero C-linear combination η of the ω_i such that $H \subseteq \ker \eta$.

Proof. By proposition 3.21, the $\omega_i(g)$ are closed differentials, so by lemma 5.2 they are C-linearly dependent. Say $\sum_{i=1}^{n} c_i\omega_i(g) = 0$, with the $c_i \in C$, not all zero, and let $\eta = \sum_{i=1}^{n} c_i\omega_i$, an invariant differential form on G, defined over C. By proposition 3.20, η is a group homomorphism $G \longrightarrow \Omega(F/C)$, so $\ker \eta$ is a subgroup of G. The ω_i are linearly independent, so $\eta \neq 0$ and hence $\ker \eta$ is a proper subgroup of G. By construction, $g \in \ker \eta$.

Let $V = \mathrm{Loc}_C(g)$, the algebraic locus of g over C, and an algebraic subvariety of G. The field C is algebraically closed, so V has a C-point, say γ. Let $V' = \{x \ominus \gamma \mid x \in V\}$. Then V' is an irreducible algebraic variety defined over C, containing the identity e of G and having $g \ominus \gamma$ as a generic point over C.

The differential form η vanishes on $G(C)$, so

$$\eta(g \ominus \gamma) = \eta(g) - \eta(\gamma) = 0$$

and thus, by lemma 2.5, $V' \subseteq \ker \eta$. Let H be the subgroup of G generated by V'. By the indecomposability theorem, 3.13, H is an algebraic subgroup of G. It lies inside $\ker \eta$, so it is a proper algebraic subgroup. Also $g \ominus \gamma \in H$, so $g \in \gamma \oplus H$ as required. □

These three lemmas can be combined into a theorem which is essentially the Schanuel condition for a general Γ_ω where ω is any set of invariant differential forms on a commutative algebraic group. The specific results for the exponential equation on a semiabelian variety and other examples follow from this.

Theorem 5.4. *Let G be a commutative algebraic group defined over C, let $\omega = (\omega_1, \ldots, \omega_n)$ be a system of linearly independent invariant differential forms on G, each defined over C, and let $D = (D_1, \ldots, D_r) \in \mathrm{Der}(F/C)^r$ with C being the intersection of the constant fields of the D_j. Suppose that $g \in \Gamma_{\omega,D}$ (that is, $g \in G$ and for each i and j we have $\omega_i(g)D_j = 0$) and that $\mathrm{td}_C(g) - \mathrm{rk}\,\mathrm{Jac}(g) < n$.*

Then there is a proper algebraic subgroup H of G and a constant point $\gamma \in G(C)$ such that g lies in the coset $\gamma \oplus H$. Moreover, there is a nonzero C-linear combination η of the ω_i such that $H \subseteq \ker \eta$.

Proof. The differentials $\omega_i(g)$ for $i = 1, \ldots, n$ lie in $\Omega(C(g)/C)$, because the differential forms ω_i are defined over C. Thus they lie in the F-vector space $\Omega(C(g)/C) \otimes_{C(g)} F$ and they also lie in $\mathrm{Ann}(D)$. By lemma 5.1, $(\Omega(C(g)/C) \otimes_{C(g)} F) \cap \mathrm{Ann}(D)$ has linear dimension $\mathrm{td}_C(g) - \mathrm{rk}\,\mathrm{Jac}(g)$, which by assumption is $< n$, so the $\omega_i(g)$ are F-linearly dependent in $\Omega(F/C)$. Applying lemma 5.3 gives the result. □

In order to prove the Schanuel condition for the exponential differential equation of a semiabelian variety, we need more information about the subgroup H which appears in the conclusion of this theorem. In this case G decomposes as $\mathbb{G}_\mathrm{a}^n \times S$, and we need a further lemma about differential forms on such groups. In [Ax71], this lemma is proved when $S = \mathbb{G}_\mathrm{m}^n$ using an analytic method looking at the residues of the differential forms at $0 \in \mathbb{G}_\mathrm{a} \smallsetminus \mathbb{G}_\mathrm{m}$. In [BK77], this analytic method was extended to S being a product of elliptic curves and $\mathbb{G}_\mathrm{m}$, using properties of differentials of the first and second kinds. They also use a separate complex geometric argument about lattices to distinguish nonisogenous elliptic curves over $\mathbb{C}$. Here we replace all of these arguments by a simple algebraic argument using Kolchin's theorem on algebraic subgroups, 3.9.

Lemma 5.5. *Let G_1, G_2 be commutative algebraic groups defined over C which are* null-homomorphic *(the only regular homomorphisms from one to the other are the zero homomorphisms). Let ω_1, ω_2 be invariant differential forms on G_1, G_2 respectively, also defined over C, and suppose $(x, y) \in G_1 \times G_2$ is such that $\omega_1(x) = \omega_2(y)$ in $\Omega(F/C)$. Then $\omega_1(x) = 0$.*

Proof. Let $G = G_1 \times G_2$ and $\omega = \omega_1 - \omega_2$. If $\omega = 0$ then $\omega_1 = \omega_2 = 0$, so the result is trivial. Otherwise, by lemma 5.3 with $n = 1$, there is a proper algebraic subgroup H of G such that (x, y) lies in a constant coset of H and $H \subseteq \ker \omega$. By corollary 3.10 to Kolchin's theorem, $H = H_1 \times H_2$ where H_1, H_2 are subgroups of G_1, G_2 respectively. For each $h \in H_1$,

$$\omega_1(h) = \omega(h, 0) = 0$$

because $(h, 0) \in H \subseteq \ker \omega$. Thus $H_1 \subseteq \ker \omega_1$ and similarly $H_2 \subseteq \ker \omega_2$. In particular, $\omega_1(x) = \omega_2(y) = 0$. □

We also separate out one further lemma about corresponding subgroups.

Lemma 5.6. *Let $G = \mathbb{G}_a^n \times S$ and $(x, y) \in \Gamma$, the solution set of the exponential equation for S, and suppose that H is an algebraic subgroup of S. Then y lies in a constant coset of H iff x lies in a constant coset of* $\operatorname{Log} H$.

Proof. By the third part of theorem 4.1, $\Gamma/G(C)$ is the graph of a bijection

$$\frac{\mathrm{pr}_1 \Gamma}{\mathbb{G}_a^n(C)} \xrightarrow{\theta} \frac{\mathrm{pr}_2 \Gamma}{S(C)}$$

where pr_1 is the projection $G \longrightarrow \mathbb{G}_a^n$ and pr_2 is the projection $G \longrightarrow S$. By the definition (4.6) of $\operatorname{Log} H$,

$$\theta(\operatorname{Log} H + \mathbb{G}_a^n(C)) = H \oplus S(C)$$

that is, the bijection θ restricts to a bijection on pairs of corresponding subgroups. Since $(x, y) \in \Gamma$, we have $x + \mathbb{G}_a^n(C) = \theta^{-1}(y \oplus S(C))$, and so if $y \in \gamma \oplus H$ then there is $\gamma' \in \mathbb{G}_a^n(C)$ such that $x \in \gamma' + \operatorname{Log} H$, and vice versa. □

Finally we put all the results together to prove the Schanuel condition for the exponential differential equation for a semiabelian variety.

Theorem 5.7 (Schanuel condition for semiabelian varieties). *Let S be a semiabelian variety of dimension n defined over C. Let $G = \mathbb{G}_a^n \times S$ and let $\Gamma \subseteq G$ be the solution set to the exponential differential equation of S.*

Suppose that $(x, y) \in \Gamma$ and $\mathrm{td}_C(x, y) - \operatorname{rk} \operatorname{Jac}(x, y) < n$. Then there is a proper algebraic subgroup H of S and constant points $\gamma \in S(C)$ and $\gamma' \in \mathbb{G}_a^n(C)$ such that $y \in \gamma \oplus H$ and $x \in \gamma' + \operatorname{Log} H$.

Proof. $\Gamma = \Gamma_{\omega,D}$ with $\omega = (\omega_1, \ldots, \omega_n) = (\zeta_1 - \xi_1, \ldots, \zeta_n - \xi_n)$ where $(\zeta_1, \ldots, \zeta_n)$ is a basis of invariant differential 1-forms on S and $(\xi_1, \ldots, \xi_n) = (dx_1, \ldots, dx_n)$ is a basis of invariant differential 1-forms on $\mathbb{G}_a^n$, all defined over C.

By theorem 5.4, there are $c_1, \ldots, c_n \in C$, not all zero, such that if $\eta_1 = \sum_{i=1}^n c_i \xi_i$ and $\eta_2 = \sum_{i=1}^n c_i \zeta_i$ then $\eta_1(x) - \eta_2(y) = 0$. By lemma 5.5, $\eta_2(y) = 0$, but $\eta_2 \neq 0$ so by theorem 5.4 again, there is a proper algebraic subgroup H of S and $\gamma \in S(C)$ such that $y \in \gamma \oplus H$.

Applying lemma 5.6, there is $\gamma' \in \mathbb{G}_a^n(C)$ such that $x \in \gamma' + \operatorname{Log} H$. □

5.2 Raising to powers and other equations

Theorem 5.4 itself is enough to prove Schanuel conditions for the equations called "raising to powers". First we give the theorem for raising to powers in the multiplicative group $\mathbb{G}_{\mathrm{m}}$.

Theorem 5.8. *Let $\mu \in C \smallsetminus \{0\}$ and suppose for $i = 1, \ldots, n$ we have $a_i, b_i \in \mathbb{G}_{\mathrm{m}}$ such that $\mu\frac{Da_i}{a_i} = \frac{Db_i}{b_i}$, and that $\mathrm{td}_C(a, b) - \mathrm{rk}\,\mathrm{Jac}(a, b) < n$.*

Then there are $m_1, \ldots, m_{2n} \in \mathbb{Z}$, not all zero, such that $\prod_{i=1}^n a_i^{m_i} b_i^{m_{i+n}} \in C$.

Proof. Let $G = \mathbb{G}_{\mathrm{m}}^{2n}$ with coordinates $(x, y) = (x_1, y_1, \ldots, x_n, y_n)$. Consider the invariant differential forms $\omega_i(x, y) = \mu\frac{dx_i}{x_i} - \frac{dy_i}{y_i}$.

By theorem 5.4 there is a proper algebraic subgroup H of G such that (a, b) lies in a constant coset of H. By theorem 3.11, this subgroup is given by an equation of the form $\prod_{i=1}^n x_i^{m_i} y_i^{m_{i+n}} = 1$. The result follows at once. □

Note that if $\mu \in \mathbb{Q}$, say $\mu = \frac{r}{s}$ with $r, s \in \mathbb{Z}$, then it is easy to see that $a_i^r b_i^{-s} \in C$ for each i, and so this result is only of interest when μ is irrational. We think of the equations as saying "For some constant γ, $b = \gamma a^{\mu}$" and γ plays the role of the "constant of integration". Note also that we could take different powers (values of μ) for each i and the proof would remain the same.

We can consider raising to powers in other groups. For example, the same proof gives the following for a simple (semi)abelian variety A of dimension e. Now the powers are not elements of C, but invertible linear maps, that is, elements of $\mathrm{GL}_e(C)$. We phrase this statement in terms of the logarithmic derivative.

Theorem 5.9. *Let A be a simple abelian variety of dimension e, and let $\mu \in \mathrm{GL}_e(C)$. Suppose for $i = 1, \ldots, n$ we have $a_i, b_i \in A$ such that $\mu\,\mathrm{DLog}(a_i) = \mathrm{DLog}(b_i)$. Suppose also that $\mathrm{td}_C(a, b) - \mathrm{rk}\,\mathrm{Jac}(a, b) < ne$.*

Then there are $\sigma_1, \ldots, \sigma_{2n} \in \mathrm{End}(A)$, not all zero, such that

$$\bigoplus_{i=1}^{n} \sigma_i(a_i) \oplus \sigma_{i+n}(b_i) \in A(C).$$

□

The same methods can be applied to a range of similar examples where the logarithmic derivatives of two groups are equated. In general these can be proved either directly using theorem 5.4 as above, or indirectly using theorem 5.7. We give one example of the second style of proof.

Theorem 5.10. *Let $\mathcal{E}$ be an elliptic curve defined over C, say with affine part defined by $\{Z = (z, w) \in F^2 \mid w^2 = 4z^3 - g_2 z - g_3\}$. Suppose $y_1, \ldots, y_n \in \mathbb{G}_\mathrm{m}$ and $Z_1, \ldots, Z_n \in \mathcal{E}$ such that*

$$\frac{Dz_i}{w_i} = \frac{Dy_i}{y_i}$$

for each i and $\mathrm{td}_C(y, Z) - \mathrm{rk}\,\mathrm{Jac}(y) < n$.

Then there are $m_1, \ldots, m_n \in \mathbb{Z}$, not all zero, such that $\prod_{i=1}^n y_i^{m_i} \in \mathbb{G}_\mathrm{m}(C)$ and $\bigoplus_{i=1}^n m_i Z_i \in \mathcal{E}(C)$.

This is the differential equation satisfied by the Tate factorization of the exponential map for $\mathcal{E}$, and there is no need to assume that $\mathcal{E}$ does not have complex multiplication. Indeed, the proof just uses the fact that $\mathcal{E}^n$ and $\mathbb{G}_\mathrm{m}^n$ are null-homomorphic algebraic groups and every algebraic subgroup H of $\mathbb{G}_\mathrm{m}^n$ has a corresponding algebraic subgroup H' in $\mathcal{E}^n$, that is, such that $\mathrm{Log}\, H' = \mathrm{Log}\, H$.

Proof. Let $x_1, \ldots, x_n \in \mathbb{G}_\mathrm{a}$ such that $Dx_i = \frac{Dy_i}{y_i}$, and let $x_{n+i} = x_i$ for $i = 1, \ldots, n$.

Then $(x, y, Z) = (x_1, \ldots, x_{2n}, y_1, \ldots, y_n, Z_1, \ldots, Z_n) \in \mathbb{G}_\mathrm{a}^{2n} \times (\mathbb{G}_\mathrm{m}^n \times \mathcal{E}^n)$, satisfying the exponential differential equation for $\mathbb{G}_\mathrm{m}^n \times \mathcal{E}^n$, and $\mathrm{td}_C(x, y, Z) < 2n$. By theorem 5.7, there is a proper algebraic subgroup H of $\mathbb{G}_\mathrm{m}^n \times \mathcal{E}^n$ such that (y, Z) lies in a constant coset of H. By corollary 3.10, $H = H_1 \times H_2$ where H_1 is an algebraic subgroup of $\mathbb{G}_\mathrm{m}^n$ and H_2 is an algebraic subgroup of $\mathcal{E}^n$. The argument near the end of the proof of theorem 5.7 shows that H_1 is a proper subgroup, hence by theorem 3.11 there are $m_1, \ldots, m_n \in \mathbb{Z}$ such that $\prod_{i=1}^n y_i^{m_i} \in \mathbb{G}_\mathrm{m}(C)$. The argument of lemma 5.6 shows that H_2 can be taken to be the subgroup of $\mathcal{E}^n$ corresponding to H_1, and hence $\bigoplus_{i=1}^n m_i Z_i \in \mathcal{E}(C)$. $\square$

Since results of this form can be deduced from the Schanuel conditions for the exponential equations of semiabelian varieties, from now on we consider only the latter equations.

5.3 Uniformities

The compactness theorem of first order model theory can be combined with the Schanuel condition to give a *uniform* Schanuel condition. Before stating and proving that, we need a definition and some facts about definability of certain sets.

Definition 5.11. A *parametric family* $(V_p)_{p \in P}$ of subvarieties of a variety X is the collection of fibres of a subvariety $V \subseteq X \times P$ where P is a definable set, usually a variety or constructible set (a set definable without quantifiers in the field language) but it could also be a definable set in a differentially closed field. In particular, it could be the C-points of an algebraic variety.

The algebraic subgroups of $\mathbb{G}_a^n$ are uniformly definable by formulas of the form $Mx = 0$, where M ranges over the definable set of matrices $\mathrm{Mat}_{n\times n}$. In other words, the algebraic subgroups form a parametric family. However, for all other commutative algebraic groups the set of all algebraic subgroups is not uniformly definable, and for semiabelian varieties there are no infinite parametric families of algebraic subgroups at all. This lack of uniform definability in fact allows one to deduce a uniform version of the Schanuel conditions.

In the previous section we consider $g \in G$ with $\mathrm{td}_C(g) < k$ for some natural number k. This occurs when g lies in an algebraic subvariety V_c of G of dimension $< k$, defined with parameters c from C. The fibre condition of algebraic geometry gives us control on how the dimension depends on the parameters.

Lemma 5.12 (Fibre Condition). *Let $(V_p)_{p\in P}$ be a family of varieties, parametrized over a constructible set P. Then for each $k \in \mathbb{N}$, the set $\{p \in P \mid \dim V_p \geqslant k\}$ is positively definable in the field language (that is, it is a subvariety of P) and the set $\{p \in P \mid \dim V_p = k\}$ is definable.*

Proof. See for example [Sha94a, page 77]. □

A similar result holds for the rank of the Jacobian matrix. Indeed, upon close examination the main part of proof of the fibre condition is more or less this result.

Lemma 5.13. *For each algebraic variety V and for each natural number k, the set $\{x \in V \mid \mathrm{rk}\,\mathrm{Jac}(x) \leqslant k\}$ is positively definable in the language of differential fields, and the set $\{x \in V \mid \mathrm{rk}\,\mathrm{Jac}(x) = k\}$ is definable.*

Proof. V is made up of finitely many affine charts, so it is enough to consider V to be affine. For each x the Jacobian $\mathrm{Jac}(x)$ is an $r \times n$ matrix. Its rank is the largest k such that there is a $k \times k$ minor matrix with non-zero determinant. Thus $\mathrm{rk}\,\mathrm{Jac}(x) \leqslant k$ iff $\det M = 0$ for every minor matrix M of size $k+1$. The determinant is a polynomial and there are only finitely many minors, so this finite conjunction of equations is a positive first order condition on a matrix in the field language. The entries in the Jacobian are terms in the differential field language, and so we have positive definability of $\mathrm{rk}\,\mathrm{Jac}(x) \leqslant k$. The second part follows. □

Here is the uniform Schanuel condition for the exponential equation of a semiabelian variety.

Theorem 5.14 (Uniform Schanuel condition). *Let S be a semiabelian variety of dimension n, defined over C, let $G = \mathbb{G}_a^n \times S$ and let $\Gamma \subseteq G$ be the solution set to the exponential differential equation for S. For each parametric family $(V_c)_{c \in P(C)}$ of subvarieties of G, with V_c defined over $\mathbb{Q}(c)$, there is a finite set $\mathcal{H}_V^S$ of proper algebraic subgroups of S such that for each $c \in P(C)$ and each $(x, y) \in \Gamma \cap V_c$, if $\dim V_c - \operatorname{rk} \operatorname{Jac}(x, y) = n - t$ with $t > 0$, then there is $\gamma \in S(C)$ and $H \in \mathcal{H}_V^S$ of codimension at least t in S such that y lies in the coset $\gamma \oplus H$.*

Proof. The set

$$\Phi_V = \{((x, y), c) \in \Gamma \times P(C) \mid (x, y) \in V_c, \dim V_c - \operatorname{rk} \operatorname{Jac}(x, y) = n - t\}$$

is definable using the lemmas above. The set of formulas

$$\{((x, y), c) \in \Phi_V \wedge (\exists \gamma \in S(C))[y \ominus \gamma \in H]\}$$

where H ranges over all proper algebraic subgroups of S of codimension at least t is countable (as there are only countably many proper algebraic subgroups of S); in particular it is of bounded size. It is unsatisfiable by the Schanuel condition, so by the compactness theorem some finite subset of it is unsatisfiable. This gives the finite set $\mathcal{H}_V^S$. $\square$

5.4 An algebraic application

Here we give a purely algebraic result about the intersection of subvarieties and algebraic subgroups of a semiabelian variety. It is a simple corollary of the uniform Schanuel conditions, but as well as the fact that there are no parametric families of subgroups of a semiabelian variety, we use the fact that the subgroups of $\mathbb{G}_a^n$ *do* form a parametric family.

Theorem 5.15 ("Weak CIT" for semiabelian varieties). *Let S be a semiabelian variety defined over an algebraically closed field C of characteristic zero. Let $(U_p)_{p \in P}$ be a parametric family of algebraic subvarieties of S. There is a finite family $\mathcal{J}_U^S$ of proper algebraic subgroups of S such that, for any coset $\kappa = a \oplus H$ of any algebraic subgroup H of S and any $p \in P(C)$, if X is an irreducible component of $U_p \cap \kappa$ and*

$$\dim X = (\dim U_p + \dim \kappa - \dim S) + t$$

with $t > 0$, an atypical component of the intersection, then there is $J \in \mathcal{J}_U^S$ of codimension at least t and $s \in S(C)$ such that $X \subseteq s \oplus J$.

Before proving this we need to know something about the existence of solutions to the exponential differential equation for S. The complete answer will be given later, but for now we just need the following basic fact.

Lemma 5.16 (Logarithms exist). *Let F be a differential field, S a semiabelian variety defined over the constants and $y \in S(F)$. Then in some differential field extension there is $x \in \mathbb{G}_a^n$ such that $(x, y) \in \Gamma$, the solution set to the exponential differential equation for S.*

Proof. For each $i = 1, \ldots, n$ we just need to find x_i such that $D_j x_i = a_{ij}$ for the fixed elements $a_{ij} = \zeta_i(y)D_j$ of F. This can be done just by taking x to be algebraically independent over F and explicitly extending the derivations $D_1, \ldots, D_r$ to $F(x)$ as required. $\square$

Proof of theorem 5.15. Let $n = \dim S$ and define $L_{Ma} = \{x \in \mathbb{G}_a^n \mid Mx = a\}$ where M is an $n \times n$ matrix and $a \in \mathbb{G}_a^n$. So L is the parametric family of all cosets of algebraic subgroups of $\mathbb{G}_a^n$.

Suppose that X is an atypical component of $U_p \cap \kappa$ with

$$r = \dim X = (\dim U_p + \dim \kappa - \dim S) + t.$$

Let y be generic in X over C and let $D_1, \ldots, D_r$ be a basis of $\mathrm{Der}(C(y)/C)$. Then $\operatorname{rk} \mathrm{Jac}(y) = r$. Using lemma 5.16, take $x \in \mathbb{G}_a^n(F)$ with F some differential field extension such that $(x, y) \in \Gamma$, the solution set of the exponential differential equation for S. Then $\operatorname{rk} \mathrm{Jac}(x, y) = \operatorname{rk} \mathrm{Jac}(y)$. Now $y \in \kappa$, a constant coset of the algebraic subgroup H of S, so, by lemma 5.6, x lies in a constant coset of $\operatorname{Log} H$. Thus x lies in L_{Ma} for a suitable choice of $M \in \mathrm{Mat}_{n\times n}(C)$ and $a \in \mathbb{G}_a^n(C)$, with $\dim L_{Ma} = \dim \kappa$. Let $V_{Ma,p} = L_{Ma} \times U_p$. Then $(x, y) \in \Gamma \cap V_{Ma,p}$ and

$$\dim V_{Ma,p} - \operatorname{rk} \mathrm{Jac}(x, y) = \dim \kappa + \dim U_p - \dim X = \dim S - t$$

and so by theorem 5.14, there is $s \in S(C)$ and an algebraic subgroup J of S of codimension at least t from the finite set $\mathcal{H}_V^S$ such that $y \in s \oplus J$. Thus,in the notation of theorem 5.14, we may take the finite set $\mathcal{J}_U^S$ to be $\mathcal{H}_{L\times U}^S$. $\square$

This theorem is a weak version of the *Conjecture on the intersection of algebraic subgroups with subvarieties* stated by Zilber in [Zil02a], and is the natural generalization to semiabelian varieties of the version proved there and by Poizat in [Poi01] for subgroups of $\mathbb{G}_m^n$. (Subgroups of $\mathbb{G}_m^n$ are called tori, and so the conjecture is also known as the conjecture on the intersection of tori, or CIT). The proof here is in essence the same as the proof of Poizat, but simplified by using the full Schanuel condition for partial differential fields rather than just ordinary differential fields, and by stating and proving the uniform Schanuel condition separately. This version of the theorem can be restated in more elementary, less geometric terms.

Corollary 5.17. *For each $n, d, r \in \mathbb{N}$, there is $N \in \mathbb{N}$ with the following property. Suppose that $x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$ lies in an algebraic variety U defined by r polynomials of degree at most d, with coefficients in a subfield K of $\mathbb{C}$. Suppose also that x satisfies l multiplicative dependencies of the form $\prod_{i=1}^n x_i^{m_{ij}} = a_j$ with the $m_{ij} \in \mathbb{Z}$ and $a_j \in K$, and that $\operatorname{td}(K(x_1, \ldots, x_n)/K) = \dim U - l + t$, with $t > 0$.*

Then x satisfies t multiplicative dependencies with the powers m_{ij} having modulus at most N and the a_j lying in $\bar{K}$.

Proof. The subvarieties U of $\mathbb{G}_\mathrm{m}^n$ defined by r polynomials of degree at most d can be put into a single parametric family. Take $C = \bar{K}$ in 5.15. □

This statement for tori has also recently and independently been reproved by Bombieri, Masser, and Zannier in [BMZ05]. They also use Ax's theorem (the Schanuel condition for the exponential equation) but use a heights argument rather than the compactness theorem to get the natural number N. This gives them an explicit bound which cannot be obtained directly from the compactness theorem. Masser has noted (in a private communication to me) that their method should also extend to the semiabelian case.

5.5 Ax-Seidenberg method

Here I give a second proof of the Schanuel condition for the exponential equations of semiabelian varieties, using a theorem of Ax which proves a similar statement for formal groups and the theorem of Seidenberg that any finitely generated differential field can be embedded into a ring of holomorphic functions. I first put these ideas together in my paper [Kir05b], applying Seidenberg's theorem not to this theorem of Ax but to the theorem of Brownawell and Kubota [BK77], which gives the Schanuel condition for Weierstrass $\wp$-functions. It seems that neither Brownawell-Kubota nor Ax were interested in proving their results in the abstract setting of differential fields, despite Ax's original paper [Ax71] doing this for the usual exponential equation.

The theorem of Ax we use is theorem 3 of [Ax72], which is reproduced below, changing only the notation slightly to match the conventions of this thesis. Write $\widetilde{G}$ for the formalization of a group G at its identity. To explain exactly what this formalization is would take us too far afield from the topic of the thesis. See [Ax72] for an explanation.

Theorem 5.18 (Ax). *Let C be a field of characteristic zero, S a semiabelian variety defined over C, and of dimension n, and let $\mathbb{G}_{\mathrm{a}}^n \xrightarrow{\widetilde{\exp}} \widetilde{S}$ be the formal homomorphism of formal groups corresponding to the canonical isomorphism of Lie algebras. Let $x_1, \ldots, x_n \in C[\![T_1, \ldots, T_r]\!]$ be power series without constant terms. Assume that if H is an algebraic subgroup of S and if*

$$\begin{tikzcd} \mathbb{G}_{\mathrm{a}}^r \arrow[rr, "\widetilde{\exp}(x_1,\ldots,x_n)"] \arrow[drr, "\Psi"'] & & \widetilde{S} \\ & & \widetilde{H} \arrow[u, hook] \end{tikzcd}$$

commutes then $H = S$.

Then $\operatorname{td}_C(x, \exp(x)) - \operatorname{rk}\operatorname{Jac}(x) \geqslant n$.

The use of formal varieties, formal power series and formal exponentiation gives some extra generality, but also makes this statement more difficult to understand. In particular it is not immediately clear what the condition on algebraic subgroups means. Fortunately, the power of Seidenberg's theorem is such that we only need Ax's theorem for convergent complex power series, that is, for holomorphic functions, and the usual complex exponential map can be used. We will then get all the generality back from the abstract nature of differential fields. First we restate this theorem of Ax just for the complex situation. Recall that $\mathbb{C}\{\{T_1, \ldots, T_r\}\}$ is the ring of power series with non-zero radius of convergence, which is canonically isomorphic to the ring of germs of holomorphic functions at a point in $\mathbb{C}^r$, and $\mathbb{C}\langle\langle T_1, \ldots, T_r\rangle\rangle$ is its field of fractions, the field of germs of functions meromorphic at a point. We consider these as differential rings by taking the r partial differentiation operators $\frac{\partial}{\partial T_i}$.

Corollary 5.19. *Let S be a semiabelian variety defined over $\mathbb{C}$, of dimension n, and $\mathbb{G}_{\mathrm{a}}^n(\mathbb{C}) \xrightarrow{\exp} S(\mathbb{C})$ the usual complex-analytic exponential map. Let $x = (x_1, \ldots, x_n) \in \mathbb{G}_{\mathrm{a}}^n(\mathbb{C}\{\{T_1, \ldots, T_r\}\})$, with each x_i having a zero at 0, that is, the power series has no constant term. Suppose that* $\exp(x)$ *does not lie in a proper algebraic subgroup of S,*

Then $\operatorname{td}_{\mathbb{C}}(x, \exp(x)) - \operatorname{rk}\operatorname{Jac}(x) \geqslant n$.

Proof. It suffices to show that in this case, the condition on algebraic subgroups in theorem 5.18 is equivalent to $\exp(x)$ not lying in a proper algebraic subgroup of S.

Suppose $\exp(x) \in H$, a proper algebraic subgroup of S. Let $L = \mathbb{G}_a^n$ be the Lie algebra of S and let L' be the Lie-subalgebra of L corresponding to H. Then $\exp(x)$ factors as

$$\begin{tikzcd} \mathbb{G}_a^r \arrow[r, "x"] & L' \arrow[r, hook] \arrow[drr, "\exp_H"'] & L \arrow[r, "\exp_S"] & S \\ & & & H \arrow[u, hook] \end{tikzcd}$$

and so the formal map it defines has the same factorization.

Conversely, suppose that the formal map $\widetilde{\exp_S}(x)$ factors through $\widetilde{H}$. The formal map $\widetilde{\exp_H}$ is defined to be an isomorphism of formal groups $\widetilde{L'} \longrightarrow \widetilde{H}$, but $\widetilde{L'}$ is canonically isomorphic to L'. Thus for some Ψ', $\widetilde{\exp}(x)$ factors as

$$\begin{tikzcd} \mathbb{G}_a^r \arrow[r, "x"] \arrow[dr, "\Psi'"'] & L' \arrow[r, hook] \arrow[d, equal] & L \arrow[r, "\widetilde{\exp_S}"] & \widetilde{S} \\ & L' \arrow[rr, "\widetilde{\exp_H}"'] & & \widetilde{H} \arrow[u, hook] \end{tikzcd}$$

and so Ψ' and x are equal as maps $\mathbb{G}_a^r \longrightarrow L'$. In particular, for every $t \in \mathbb{G}_a^r(\mathbb{C})$, we have $x(t) \in L'$, and so $\exp_S(x(t)) \in H(\mathbb{C})$. Thus $\exp_S(x) \in H(\mathbb{C}\langle\langle T \rangle\rangle)$, as required. $\square$

This is essentially the Schanuel condition just for holomorphic functions. We use the following theorem of Seidenberg from [Sei58], [Sei69].

Theorem 5.20. *Let F be a finitely generated differential field with n commuting derivations. Then there is a connected open subset X of $\mathbb{C}^n$ and an embedding of F into the differential field of meromorphic functions on X.*

Examination of the proof of this theorem shows that the image of F is contained in the ring of *holomorphic* functions on X, as the construction goes via the usual power series, not Laurent series.

Now we put the Ax and Seidenberg theorems together to prove the Schanuel condition for differential fields. For convenience we restate it.

Theorem 5.21 (Schanuel condition for semiabelian varieties). *Let S be a semiabelian variety of dimension n defined over C. Let $G = \mathbb{G}_a^n \times S$ and let $\Gamma \subseteq G$ be the solution set to the exponential differential equation of S.*

Suppose that $(x, y) \in \Gamma$ and $\operatorname{td}_C(x, y) - \operatorname{rk} \operatorname{Jac}(x, y) < n$. Then there is a proper algebraic subgroup H of S and constant points $\gamma \in S(C)$ and $\gamma' \in \mathbb{G}_a^n(C)$ such that $y \in \gamma \oplus H$ and $x \in \gamma' + \operatorname{Log} H$.

Proof. We prove the contrapositive. Suppose that $(x, y) \in \Gamma$ and y does not lie in a constant coset of any proper algebraic subgroup of S. (By lemma 5.6, x would automatically lie in a constant coset of $\operatorname{Log} H$.) Let F_0 be a finitely generated differential subfield of F containing x and y, and over which S is defined. By Seidenberg's theorem (5.20), there is a differential ring embedding

$$F_0 \xrightarrow{\varphi} \mathbb{C}\{\{T_1, \ldots, T_r\}\}.$$

Let $f_i = \varphi(x_i) - \varphi(x_i)(0)$ for $i = 1, \ldots, n$. That is, f_i is the power series $\varphi(x_i)$ without its constant term. Then f is a holomorphic function $\mathbb{C}^r \to \mathbb{C}^n$ with a zero at 0 and $\exp_S$ is a meromorphic function on $\mathbb{C}^n$, so we may define $g = \exp_S(f) \in \mathbb{C}\langle\langle T_1, \ldots, T_r\rangle\rangle$.

The exponential function satisfies its own differential equation, and thus $(f, g) \in \Gamma$. But $(\varphi(x), \varphi(y))$ lies in Γ since (x, y) does and φ is a differential ring embedding, and $(\varphi(x)(0), 0) \in G(\mathbb{C})$, so $(f, \varphi(y)) = (\varphi(x), \varphi(y)) \ominus (\varphi(x)(0), 0)$ also lies in Γ. Thus, by part 3 of proposition 4.1, there is $\gamma \in S(\mathbb{C})$ such that $g = \varphi(y) \ominus \gamma$. Now y, and hence $\varphi(y)$, is assumed not to lie in a constant coset of a proper algebraic subgroup of S, so g does not lie in a proper algebraic subgroup. Thus, by corollary 5.19 above, $\operatorname{td}_{\mathbb{C}}(f, g) - \operatorname{rk} \operatorname{Jac}(f) \geqslant n$.

We have $(f, g) = (\varphi(x), \varphi(y)) \ominus (\varphi(x)(0), \gamma)$ and also $(\varphi(x)(0), \gamma) \in G(\mathbb{C})$, so $\operatorname{td}_{\mathbb{C}}(f, g) = \operatorname{td}_{\mathbb{C}}(\varphi(x), \varphi(y))$. Also $\varphi(F_0)$ is algebraically independent from $\mathbb{C}$ over $\varphi(F_0 \cap C)$, and so

$$\operatorname{td}(x, y/C) = \operatorname{td}(x, y/F_0 \cap C) = \operatorname{td}(\varphi(x), \varphi(y)/\varphi(F_0 \cap C)) = \operatorname{td}(f, g/\mathbb{C}).$$

Finally $\operatorname{Jac}(f) = \operatorname{Jac}(\varphi(x)) = \varphi(\operatorname{Jac}(x))$, so $\operatorname{rk} \operatorname{Jac}(x) = \operatorname{rk} \operatorname{Jac}(f)$, which gives $\operatorname{td}_C(x, y) - \operatorname{rk} \operatorname{Jac}(x) \geqslant n$. □

Chapter 6

Amalgamation constructions

6.1 The universal theory

As described in section 4.1.1, the differential equations will be considered in a reduct of a differential field. We consider only the exponential equations of semiabelian varieties. Indeed, let $\langle F; +, \cdot, D_1, \ldots, D_r, C \rangle$ be a differential field, and $\mathcal{S}$ be a collection of semiabelian varieties defined over C. We assume that $\mathcal{S}$ is closed under taking products and algebraic subgroups, and can also assume that it is closed under isogenies. For a technical reason, we also assume that $\mathcal{S}$ contains only split semiabelian varieties. It seems likely that with some more work this assumption can be dropped.

As before, for each $S \in \mathcal{S}$, let $\operatorname{Log} S = \mathbb{G}_{\mathrm{a}}^{\dim S}$, $\widehat{S} = (\operatorname{Log} S) \times S$ and Γ_S the subgroup of $\widehat{S}$ given by the solution set of the exponential differential equation for S. This requires a choice of basis of invariant differential forms for each S, and we assume that the choices are made coherently.

Add to the language constants $(c_i)_{i \in I}$ such that each $S \in \mathcal{S}$ is defined over them, and a symbol for Γ_S for each $S \in \mathcal{S}$. (We only introduce one symbol Γ_S for each S, not one for each derivation, for the reasons discussed at the end of this chapter.) Consider the reduct F to the language $\mathcal{L}_{\mathcal{S}} = \langle +, \cdot, C, (\Gamma_S)_{S \in \mathcal{S}}, (c_i)_{i \in I} \rangle$. Our first aim is to give the common theory of all such reducts.

Proposition 6.1 (Universal theory of the reducts). *The first order theory of any reduct $\langle F; +, \cdot, (\Gamma_S)_{S \in \mathcal{S}}, C \rangle$ of a differential field contains the following axioms and axiom schemes.*

A1 *F is a field of characteristic zero.*

A2 *C is a relatively algebraically closed subfield of F, each $c_i \in C$ and they satisfy the appropriate algebraic relations.*

A3 Γ_S *is a subgroup and an* $\mathrm{End}(S)$*-submodule of of the algebraic group* $\widehat{S}$.

A4 Γ_S *contains* $\widehat{S}(C)$.

A5 *The fibres of* Γ_S *in* $\operatorname{Log} S$ *and* S *are cosets of* $(\operatorname{Log} S)(C)$ *and* $S(C)$*, respectively.*

A6 $\Gamma_{S_1 \times S_2} = \Gamma_{S_1} \times \Gamma_{S_2}$.
If $S_1 \subseteq S_2$ *then* $\Gamma_{S_1} = \Gamma_{S_2} \cap G_1$.
If $S_1 \xrightarrow{f} S_2$ *is surjective then* $\widehat{f}(\Gamma_{S_1}) \subseteq \Gamma_{S_2}$.

USC *For each variety* P *and each parametric family* $(V_p)_{p \in P}$ *of algebraic subvarieties of* $\widehat{S}$*, defined over* $\mathbb{Q}$:

$$(\forall p \in P(C))(\forall g \in V_p \cap \Gamma_S)\left[\dim V_p < \dim S + 1 \to \bigvee_{H \in \mathcal{H}_V^S} \chi_{\widehat{H}}(g) \in \widehat{S}(C)\right]$$

where $\mathcal{H}_V^S$ *is the finite set of algebraic subgroups of* S *given by theorem 5.14 and* $\chi_{\widehat{H}}(x) = 0$ *is the equation (or system of equations) defining* $\widehat{H}$*. For definiteness, we may choose* $\mathcal{H}_V^S$ *to be a particular minimal finite set of subgroups for each variety* V.

Proof. A1 is by definition, A2 holds since the constant subfield of every differential field is relatively algebraically closed, and A3—A5 come from propositions 4.1 and 4.5. A6 gives the connections between Γ_S as S varies; the first two parts state that the choice of coordinates in defining the equations has been made coherently, and the third gives the groupoid structure from proposition 4.2. It is straightforward to write these axiom schemes as first order sentences in the language $\mathcal{L}_{\mathcal{S}}$.

The axiom scheme USC expresses the uniform Schanuel condition. Lemma 5.12 says that $\dim V_p < \dim S + 1$ is expressible as a first order formula in p, and with that it is clear that each axiom can be written as a first order sentence in $\mathcal{L}_{\mathcal{S}}$. □

Axioms A1—A6 here describe the algebraic structure of Γ. Given some some solutions to the differential equations (points in Γ), these say what other solutions we can get from them. The Schanuel condition puts a restriction on what systems of equations can have solutions. In terms of counting degrees of freedom, they say that an overdetermined system of equations cannot have a solution. We would also like to know which systems of equations can be solved. Up to now we have considered any differential field F, and indeed the axioms listed above are all universal axioms. (Strictly we should add constant symbols for 0 and 1, and a function symbol for multiplicative inverse, with a convention such as $0^{-1} = 0$, to make this true.) Axioms saying which systems of equations do have solutions will be existential axioms, and will thus not hold for every differential field. To answer this question we should take reducts not of any differential field but of a differentially closed field. The theory of differentially closed fields of characteristic zero is complete, and so the reducts we get from different differentially closed fields will all have the same theory.

The remainder of this chapter is devoted to the production of a non-first order theory, the first order part of which is a candidate for the theory of the reducts. In chapter 7 we will give the first order part of the theory and show that it is indeed the theory of the reducts. Of course one could just state the candidate theory and show that it holds for the reducts, but the method of finding this candidate theory is interesting in itself, and it also gives a way to show that the theory we obtain is complete.

One thing that can be said immediately is that a differentially closed field is algebraically closed. It is convenient to put this with the universal theory rather than with the other existential axioms, so we set A7 to be the axiom scheme ACF_0, and define $T^0_{\mathcal{S}}$ to be the $\mathcal{L}_{\mathcal{S}}$-theory consisting of the axiom schemes A1—A7 and USC. Unless there is reason to specify the family $\mathcal{S}$, we will just write T^0 for the theory and $\mathcal{L}$ for the language.

6.2 An amalgamation theorem

The method used to obtain a complete theory T from the incomplete theory T^0 is amalgamation. Here we make a diversion from the discussion of the differential equations which are the main topic of this thesis to give an account of the amalgamation theorem in a general category-theoretic setting, more or less following [DG92]. After this we return to apply the theorem to T^0. It turns out to be convenient to use this more general category-theoretic setting rather than the more familiar setting of amalgamating models of a universal first order theory.

Fix an infinite regular cardinal λ. In the application $\lambda = \aleph_0$ but there is no simplification to be gained from restricting to this case. Consider a category $\mathcal{K}$. A *chain* of length λ in $\mathcal{K}$ is a collection $(Z_i)_{i<\lambda}$ of objects of $\mathcal{K}$ together with arrows $Z_i \xrightarrow{\gamma_{ij}} Z_j$ for each $i \leqslant j < \lambda$, such that if $i \leqslant j \leqslant k$ then $\gamma_{jk} \circ \gamma_{ij} = \gamma_{ik}$.

An object A of $\mathcal{K}$ is λ-*small* iff for every λ-chain (Z_i, γ_{ij}) in $\mathcal{K}$ with direct limit Z, any arrow $A \xrightarrow{f} Z$ factors through the chain, that is, there is $i < \lambda$ and $A \xrightarrow{f^*} Z_i$ such that $f = \gamma_{i\lambda} \circ f^*$. For example, in the category of sets a set is $\aleph_0$-small iff it is finite. In a category of first-order structures and embeddings a structure is $\aleph_0$-small iff it is finitely generated. Write $\mathcal{K}_{<\lambda}$ for the full subcategory of $\mathcal{K}$ consisting of all the λ-small objects of $\mathcal{K}$, and $\mathcal{K}_{\leqslant\lambda}$ for the full subcategory of $\mathcal{K}$ consisting of all unions of λ-chains of λ-small objects.

Definition 6.2. We say that $\mathcal{K}$ is a λ-*amalgamation category* iff the following hold.

- Every arrow in $\mathcal{K}$ is a monomorphism.
- $\mathcal{K}$ has direct limits (unions) of chains of every ordinal length up to λ.
- $\mathcal{K}_{<\lambda}$ has at most λ objects up to isomorphism.
- For each object $A \in \mathcal{K}_{<\lambda}$ there are at most λ extensions of A in $\mathcal{K}_{<\lambda}$, up to isomorphism.
- $\mathcal{K}_{<\lambda}$ has the *amalgamation property* (AP), that is, any diagram of the form

$$\begin{tikzcd} B_1 & & B_2 \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

  can be completed to a commuting square

$$\begin{tikzcd} & C & \\ B_1 \arrow[ur] & & B_2 \arrow[ul] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

  in $\mathcal{K}_{<\lambda}$.
- $\mathcal{K}_{<\lambda}$ has the *joint embedding property* (JEP), that is, for every $B_1, B_2 \in \mathcal{K}_{<\lambda}$ there is $C \in \mathcal{K}_{<\lambda}$ and arrows

$$\begin{tikzcd} & C & \\ B_1 \arrow[ur] & & B_2 \arrow[ul] \end{tikzcd}$$

  in $\mathcal{K}_{<\lambda}$.

An *extension* of A is simply an arrow with domain A. In [DG92], Droste and Göbel consider a stronger condition than bounding the number of extensions of each A, namely that for any pair of objects A and B there are only λ arrows from A to B. This allows them to use the preexisting definition of a λ-algebroidal category, but it is not strong enough for our purposes. For example, if A and B are countable pure algebraically closed fields of transcendence degree zero then there are $2^{\aleph_0}$ embeddings of A into B, but they are all isomorphisms, and hence isomorphic extensions.

To say that an object U of $\mathcal{K}$ is $\mathcal{K}_{\leqslant\lambda}$-universal means that for every object $A \in \mathcal{K}_{\leqslant\lambda}$ there is an arrow $A \longrightarrow U$ in $\mathcal{K}$. To say that U is $\mathcal{K}_{<\lambda}$-saturated means that for any $A, B \in \mathcal{K}_{<\lambda}$ and any arrows $A \xrightarrow{f} U$ and $A \xrightarrow{g} B$ there is an arrow $B \xrightarrow{h} U$ such that $h \circ g = f$. These are just the translations into category-theoretic language of the usual model-theoretic notions.

Theorem 6.3 (Amalgamation theorem). *If $\mathcal{K}$ is a λ-amalgamation category then there is an object $U \in \mathcal{K}_{\leqslant\lambda}$ which is $\mathcal{K}_{\leqslant\lambda}$-universal and $\mathcal{K}_{<\lambda}$-saturated. Furthermore, U is unique up to isomorphism.*

Proof. The proof in [DG92] goes through, even with the slightly weaker hypothesis. □

6.3 The category $\mathcal{K}^{\triangleleft}$

Here we define the category which we will apply the amalgamation theorem to. This section consists almost entirely of definitions, although most pose as lemmas stating that the relevant definition is in fact well-defined.

Fix a countable algebraically closed field C of characteristic zero, containing the parameters c_i used in defining the groups $S \in \mathcal{S}$. To be definite, take C to have transcendence degree $\aleph_0$ over the parameters. Take $\mathcal{K}$ to be the category of models of the theory T^0 which have this given field C, with arrows being embeddings of $\mathcal{L}$-structures which fix C. We call the objects of $\mathcal{K}$ *structures*. It is convenient to apply the amalgamation theorem to this category rather than the category of all models of T^0 for two reasons. Firstly we need to deal with algebraically closed subfields, and it is more convenient to have them to start with rather than to take algebraic closures, and secondly it means we do not have to take account of C changing as we do the amalgamation.

Lemma 6.4. *The category $\mathcal{K}$ has intersections, that is, for each $B \in \mathcal{K}$, and each family $(A_i \hookrightarrow B)_{i \in I}$ of substructures of B, there is a limit $\bigcap_{i \in I} A_i \hookrightarrow B$ of the obvious* *diagram this defines. Furthermore the underlying field of this intersection is simply the intersection of the underlying fields of the substructures.*

Proof. The axiomatization of T^0 is universal, apart from the axiom scheme which says that the field is algebraically closed. The intersection of algebraically closed fields is algebraically closed, and any substructure of a model of a universal theory is also a model of that theory, so the category of models of T^0 has intersections. The intersection of extensions of C is obviously also an extension of C, and so $\mathcal{K}$ has intersections. $\square$

Using this, if $B \in \mathcal{K}$ and X is a subset of B, we can define the substructure of B generated by X by $\langle X \rangle = \bigcap \{A \hookrightarrow B \mid X \subseteq A\}$, where $A \hookrightarrow B$ means that A is a subobject of B in $\mathcal{K}$. Note that $\langle X \rangle$ depends on B (there is no quantifier elimination in the category). We say that B is *finitely generated* iff there is a finite subset X of B such that $B = \langle X \rangle$. Note that this is not the same as being finitely generated as an $\mathcal{L}$-structure. Indeed no objects of $\mathcal{K}$ are finitely generated as $\mathcal{L}$-structures since they are all algebraically closed fields.

Lemma 6.5. *An object A of $\mathcal{K}$ is finitely generated iff* $\mathrm{td}(A/C)$ *is finite. Furthermore, $\mathcal{K}_{<\aleph_0}$ consists of the finitely generated objects of $\mathcal{K}$.*

Proof. Observe that for any $A \in \mathcal{K}$ and subset X of A, $\langle X \rangle$ is simply the algebraic closure of $C \cup X$ in A. For the second part, if A is finitely generated by $x_1, \ldots, x_n$ and $A \hookrightarrow Z$ where Z is the union of an ω-chain $(Z_i)_{i<\omega}$ then each x_j lies in some $Z_{i(j)}$, so taking i greater than each $i(j)$ the embedding factors through Z_i. Conversely, if $\mathrm{td}(A/C)$ is infinite, let $X \cup \{x_j\}_{j<\omega}$ be an transcendence base for A over C, and let $Z_i = \langle X \cup \{x_j \mid j \leqslant i\}\rangle$. Then A is the union of the chain (Z_i), but is not equal to any of the Z_i. Hence it is not $\aleph_0$-small. $\square$

We write $A \subseteq_{f.g.} B$ to mean that A is a finitely generated substructure of B. From this characterization it follows that any substructure of a finitely generated structure in $\mathcal{K}$ is also finitely generated.

In section 4.3.2 we defined a function δ on points of Γ, and promised that the definition would be extended later. Here we make that extension.

Lemma 6.6. *For any extension $A \hookrightarrow B$ in $\mathcal{K}_{<\aleph_0}$ there is a maximal $S \in \mathcal{S}$ such that there is $g \in \Gamma_S(B)$, not lying in an A-coset of $\widehat{H}$ for any proper algebraic subgroup H of S. Furthermore this maximal S is uniquely defined up to isogeny.*

Proof. If $g \in \Gamma_S(B)$ and does not lie in an A-coset of $\widehat{H}$ for any proper algebraic subgroup H of S, then it does not lie in a C-coset and by the Schanuel condition axiomatized by USC, $\dim S < \mathrm{td}_C(g)$. Also $\mathrm{td}_C(g) \leqslant \mathrm{td}_C(B)$, so the dimension of S is bounded. At least one such S exists (the zero-dimensional group), and hence a maximal such S exists.

Using the assumption that $\mathcal{S}$ contains only split semiabelian varieties, S is isogenous to a product $\prod_{i=1}^{m} S_i^{n_i}$ where each S_i is a simple semiabelian variety and the S_i are non-isogenous. Let $R_i = \mathrm{End}(S_i) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, by theorem 3.11, R_i is a division ring and every algebraic subgroup of S_i is given by a system of R_i-linear equations. Thus any $(g_1, \ldots, g_k) \in \Gamma_{S_i^k}(B)$ lie in an A-coset of $\widehat{H}$ for some proper algebraic subgroup H of S iff their images are R_i-linearly dependent in the vector space $\Gamma_{S_i}(B)/\Gamma_{S_i}(A)$. This shows that n_i is equal to the R_i-linear dimension of the vector space $\Gamma_{S_i}(B)/\Gamma_{S_i}(A)$. In particular, S is determined uniquely up to isogeny. □

Definition 6.7. For an extension $A \hookrightarrow B$ in $\mathcal{K}_{<\aleph_0}$, define $S^{\max}(B/A)$ to be a maximal $S \in \mathcal{S}$ such that there is $g \in \Gamma_S(B)$, not lying in a proper A-coset of $\widehat{H}$ for any proper algebraic subgroup H of S. A point $g \in \Gamma_{S^{\max}(B/A)}$ which witnesses the maximality is said to be a *basis for* $\Gamma(B/A)$.

Note that $S^{\max}(B/A)$ is defined only up to isogeny. For $A \in \mathcal{K}_{<\aleph_0}$, define $S^{\max}(A) = S^{\max}(A/C)$.

Definition 6.8. Define the *group rank* and *predimension* of $A \in \mathcal{K}_{<\aleph_0}$ to be

$$\mathrm{grk}_C(A) = \dim S^{\max}(A) \qquad\qquad \delta(A) = \mathrm{td}(A/C) - \mathrm{grk}_C(A)$$

respectively. For any subset $X \subseteq A$, define $\mathrm{grk}_C(X) = \mathrm{grk}_C(\langle X \rangle)$ and $\delta(X) = \delta(\langle X \rangle)$.

Note that if $X = \{g\}$ for some $S \in \mathcal{S}$ and $g \in \Gamma_S$ then this definition agrees with that given in section 4.3.2. Furthermore, the Schanuel condition says precisely that $\delta(A) \geqslant 0$ for each finitely generated structure A, with equality iff $A = C$.

Definition 6.9. We say that an embedding of structures $A \hookrightarrow B$ is *strong* iff for every $X \subseteq_{f.g.} B$ we have $\delta(X \cap A) \leqslant \delta(X)$. In this case, we write the embedding as $A \lhd B$ or $A \hookrightarrow\!\!\lhd\!\!\rightarrow B$.

Lemma 6.10. *Taking all the objects of $\mathcal{K}$ with just the strong embeddings gives a subcategory $\mathcal{K}^{\lhd}$ of $\mathcal{K}$.*

Proof. It is immediate that identity embeddings are strong and the composite of strong embeddings is strong, so this defines a subcategory $\mathcal{K}^{\lhd}$ of $\mathcal{K}$, which consists of all the structures with just the strong embeddings. □

The predimension of a structure captures the notion of the "number of degrees of freedom" in the structure, the algebraic degrees of freedom minus the number of further constraints from Γ. If $A \subseteq B$ then there may constraints between elements of A which are not seen in A, but only in the larger structure B. The idea of a strong substructure $A \lhd B$ is that this does not happen. Hrushovski's original terminology was *self-sufficiency* of the substructure, which perhaps expresses this idea better.

It is customary to write strong embeddings as $A \leqslant B$, but this seems to me to be an unnecessary duplication of a common symbol and potentially confusing, so I prefer to avoid it. Note also that this is not quite the same as the usual definition of a strong embedding (see for example [Hru93]), but it is equivalent for any δ which is submodular (see below), as all predimension functions for Hrushovski-type constructions are. My definition shortens and, I believe, clarifies the presentation.

Lemma 6.11. *If $A_i \lhd B$ for each i in some index set I and $A = \bigcap_{i \in I} A_i$ is the intersection in $\mathcal{K}$, then $A \lhd B$. In particular, the category $\mathcal{K}^{\lhd}$ has intersections.*

Proof. First we show that it holds for binary intersections. Suppose $A_1, A_2 \lhd B$. Let $X \subseteq_{f.g.} A_1$. Then $\delta(X \cap (A_1 \cap A_2)) = \delta(X \cap A_2) \leqslant \delta(X)$ since $A_2 \lhd B$ and $X \subseteq_{f.g.} B$. So $A_1 \cap A_2 \lhd A_1$, but also $A_1 \lhd B$ and so $A_1 \cap A_2 \lhd B$. By induction, any finite intersection of strong substructures of B is also strong in B.

The case of an arbitrary intersection of strong subsets follows by a finite character argument. Let $X \subseteq_{f.g.} B$. Then $X \cap \bigcap_{i \in I} A_i$ is an algebraically closed subfield of X, which has finite transcendence degree. The lattice of algebraically closed subfields of X has no infinite chains, hence there is a finite subset I_0 of I such that $X \cap \bigcap_{i \in I} A_i = X \cap \bigcap_{i \in I_0} A_i$. By the above, $\bigcap_{i \in I_0} A_i \lhd B$, and so $\delta(X \cap \bigcap_{i \in I} A_i) \leqslant \delta(X)$. So $\bigcap_{i \in I} A_i \lhd B$ as required. $\square$

As with $\mathcal{K}$, the existence of intersections allows one to define the subobject *generated* by some set, and consequently the notion of a finitely generated object in $\mathcal{K}^{\lhd}$. To distinguish this notion from that in $\mathcal{K}$, we give it a different name.

Definition 6.12. If B is a structure and X is a subset of B then the *hull* of X in B is given by $\lceil X \rceil = \bigcap \{A \lhd B \mid X \subseteq A\}$.

Note that as for $\langle X \rangle$, the hull $\lceil X \rceil$ depends on B, although we do not write the dependence explicitly.

6.4 The universal structure

That completes the required definitions and we now proceed with the construction of the universal structure. The one essential fact which makes everything work is that the predimension function is *submodular*.

Lemma 6.13. *The group rank* grk_C *is modular and the predimension* δ *is submodular. That is, for any* $B \in \mathcal{K}_{<\aleph_0}$ *and any* $A_1, A_2 \subseteq B$,

$$\mathrm{grk}_C(A_1 \cup A_2) = \mathrm{grk}_C(A_1) + \mathrm{grk}_C(A_2) - \mathrm{grk}_C(A_1 \cap A_2)$$

and

$$\delta(A_1 \cup A_2) \leqslant \delta(A_1) + \delta(A_2) - \delta(A_1 \cap A_2).$$

Proof. For each simple semiabelian variety $S \in \mathcal{S}$, the vector space $\Gamma_S(\langle A_1, A_2\rangle)/\widehat{S}(C)$ has a basis given by the (necessarily disjoint) union of bases for $\Gamma_S(A_1 \cap A_2)/\widehat{S}(C)$, for $\Gamma_S(A_1)/\Gamma_S(A_1 \cap A_2)$ and for $\Gamma_S(A_2)/\Gamma_S(A_1 \cap A_2)$. Under the assumption that $\mathcal{S}$ contains no non-split semiabelian varieties, this is enough to show that the group rank grk_C is modular.

Using the modularity of group rank and the submodularity of transcendence degree,

$$\begin{aligned}
\delta(A_1 \cup A_2) &= \mathrm{td}_C(A_1 \cup A_2) - \mathrm{grk}_C(A_1 \cup A_2) \\
&= \mathrm{td}_C(A_1 \cup A_2) - [\mathrm{grk}_C(A_1) + \mathrm{grk}_C(A_2) - \mathrm{grk}_C(A_1 \cap A_2)] \\
&\leqslant [\mathrm{td}_C(A_1) + \mathrm{td}_C(A_2) - \mathrm{td}_C(A_1 \cap A_2)] \\
&\qquad\qquad -[\mathrm{grk}_C(A_1) + \mathrm{grk}_C(A_2) - \mathrm{grk}_C(A_1 \cap A_2)] \\
&\leqslant \delta(A_1) + \delta(A_2) - \delta(A_1 \cap A_2)
\end{aligned}$$

and thus δ is submodular as required. □

We now proceed with the applications of submodularity.

Lemma 6.14. *A structure* A *is* $\aleph_0$*-small in* $\mathcal{K}^{\lhd}$ *iff it is* $\aleph_0$*-small in* $\mathcal{K}$.

Recall that A is $\aleph_0$-small in $\mathcal{K}$ iff $\mathrm{td}(A/C)$ is finite.

Proof. The right to left direction is immediate.

For the left to right direction, $\mathcal{K}^{\lhd}$ has intersections, so the proof of lemma 6.5 shows that B is $\aleph_0$-small in $\mathcal{K}^{\lhd}$ iff it is finitely generated in $\mathcal{K}^{\lhd}$, that is, there is a finite subset $X \subseteq B$ such that $B = \lceil X \rceil$.

We show that if $B \in \mathcal{K}$ and $X \subseteq B$ is a finite subset then $\lceil X \rceil$ is finitely generated in $\mathcal{K}$. Consider $\{\delta(A) \mid X \subseteq A \subseteq_{f.g.} B\}$, a subset of $\mathbb{N}$. Let A be such that $\delta(A)$ is least. Then for any $Y \subseteq_{f.g.} B$,

$$0 \leqslant \delta(A \cup Y) - \delta(A) \leqslant \delta(Y) - \delta(A \cap Y)$$

with the first comparison holding by the minimality of $\delta(A)$ and the second by submodularity of δ. Thus $A \lhd B$. In particular, $\lceil X \rceil \subseteq A$, and so $\lceil X \rceil$ is finitely generated in $\mathcal{K}$. In particular it is in $\mathcal{K}_{<\aleph_0}$. $\square$

In order to apply the amalgamation theorem, we need to show that $\mathcal{K}^{\lhd}_{<\aleph_0}$ has the amalgamation property. In fact, we show more than this, which is necessary when it comes to axiomatizing the amalgam.

Proposition 6.15 (Free amalgamation). *If we have embeddings $A \lhd B_1$ and $A \hookrightarrow B_2$ in $\mathcal{K}$ then there is $E \in \mathcal{K}$ (the* free amalgam *of B_1 and B_2 over A) and embeddings $B_1 \hookrightarrow E$ and $B_2 \lhd E$ such that the square*

$$\begin{tikzcd}
& E & \\
B_1 \arrow[ur, hook] & & B_2 \arrow[ul, hook, "\lhd"'] \\
& A \arrow[ul, hook, "\lhd"] \arrow[ur, hook] &
\end{tikzcd}$$

commutes, and $E = \langle B_1, B_2 \rangle$. Furthermore, if $A \lhd B_2$ then $B_1 \lhd E$.

Proof. Let β_1, β_2 be transcendence bases of B_1, B_2 over A. As a field, take E to be the algebraic closure of the extension of A with transcendence base the disjoint union $\beta_1 \sqcup \beta_2$. This defines the field E and the embeddings $B_1 \hookrightarrow E$ and $B_2 \hookrightarrow E$ uniquely up to isomorphism, because A is algebraically closed, and so B_1 and B_2 are linearly disjoint over A in E. For each simple $S \in \mathcal{S}$, define $\Gamma_S(E)$ to be the subgroup of $\widehat{S}(E)$ generated by $\Gamma_S(B_1) \cup \Gamma_S(B_2)$. For general $S \in \mathcal{S}$, Γ_S is then determined uniquely by axiom A6 (using the assumption that $\mathcal{S}$ contains no non-split semiabelian varieties). Axioms A1—A7 hold by the construction.

Let X be a finitely generated algebraically closed substructure of E. Note that δ and grk_C were originally defined only for structures in $\mathcal{K}_{<\aleph_0}$ and we do not a priori know that $X \in \mathcal{K}_{<\aleph_0}$ because we are still to prove that USC holds. However, the definitions of δ and grk_C make sense for X because the conclusion of lemma 6.6 holds, and so $\mathrm{grk}_C(X)$ is well-defined and finite.

Let $S = S^{\max}(\langle X, B_2\rangle/B_2)$, and let g be a basis for $\Gamma(\langle X, B_2\rangle/B_2)$. Then by the construction of $\Gamma_S(E)$, there is $H \in \Gamma_S(B_1)$ such that $b = g \ominus h \in \Gamma_S(B_2)$. The group operation of S is defined over C, so certainly over B_2, and so

$$\operatorname{td}(g/B_2) = \operatorname{td}(h/B_2) = \operatorname{td}(h/A) \geqslant \operatorname{grk}_A(h)$$

with the second equation because B_1 is algebraically independent of B_2 over A and the comparison because $A \lhd B_1$.

We now show that $\operatorname{grk}_A(h) = \dim S$. If not, then there is $a \in \widehat{S}(A)$ and a proper algebraic subgroup H of S such that $h \ominus a \in \widehat{H}(B_1)$. Then $g \ominus (a \oplus b) = h \ominus a$ and $a \oplus b \in \widehat{S}(B_2)$, which contracts the fact that g is a basis for $\Gamma(\langle X, B_2\rangle/B_2)$. So $\operatorname{grk}_A(h) = \dim S$, and thus

$$\delta(\langle X, B_2\rangle/B_2) = \operatorname{td}(g/B_2) - \operatorname{grk}_{B_2}(g) \geqslant 0.$$

By submodularity, $\delta(X) - \delta(X \cap B_2) \geqslant \delta(\langle X, B_2\rangle/B_2) - \delta(B_2) \geqslant 0$. Also $X \cap B_2 \in \mathcal{K}$, and thus $\delta(X) \geqslant 0$, with equality iff $X = C$. Thus E satisfies USC, and so $E \in \mathcal{K}$, and furthermore $B_2 \lhd E$. The symmetric argument shows that if $A \lhd B_2$ then $B_1 \lhd E$. $\square$

This is the main step required to show that $\mathcal{K}^{\lhd}$ is an $\aleph_0$-amalgamation category.

Proposition 6.16. *$\mathcal{K}^{\lhd}$ is an $\aleph_0$-amalgamation category.*

Proof. Every embedding in $\mathcal{K}^{\lhd}$ is certainly a monomorphism, because $\mathcal{K}^{\lhd}$ is a concrete category and the underlying function is injective. It is also easy to see that $\mathcal{K}^{\lhd}$ has unions of chains of any ordinal length, and in particular unions of ω-chains.

Given $A \in \mathcal{K}^{\lhd}_{<\aleph_0}$, a strong extension B of A in $\mathcal{K}^{\lhd}_{<\aleph_0}$ is determined up to isomorphism by $S^{\max}(B/A)$, the algebraic locus $\operatorname{Loc}_A(g)$ of a basis g for $\Gamma(B/A)$, together with the natural number $\operatorname{td}(B/A(g))$. There are only countably many $S \in \mathcal{S}$, and only countably many algebraic varieties defined over A, so there are only countably many strong extensions of A. The structure C embeds strongly into every $B \in \mathcal{K}^{\lhd}$, so taking $A = C$ it follows in particular that $\mathcal{K}^{\lhd}_{<\aleph_0}$ has only countably many objects. The joint embedding property for $\mathcal{K}^{\lhd}_{<\aleph_0}$ follows from the amalgamation property for $\mathcal{K}^{\lhd}_{<\aleph_0}$ for the same reason, and the amalgamation property in turn is given by proposition 6.15. $\square$

Theorem 6.17. *There is a countable model U of T^0 which is universal and saturated with respect to strong embeddings. Furthermore, U is unique up to isomorphism.*

Proof. By proposition 6.16, we may apply the amalgamation theorem 6.3 to $\mathcal{K}^{\lhd}$ with $\lambda = \aleph_0$ to produce a model $U \in \mathcal{K}_{\leqslant\aleph_0}$, unique up to isomorphism. It is a union of an ω-chain of countable structures, hence is countable. Every countable model of T^0 can be strongly embedded in some $A \in \mathcal{K}_{\leqslant\aleph_0}$, by extending the constant field and taking the algebraic closure. Thus the $\mathcal{K}^{\lhd}_{\leqslant\aleph_0}$-universality of U implies that every countable model of T^0 can be strongly embedded into U. Similarly, U is saturated with respect to strong embeddings for any strong substructures of finite transcendence degree. □

The amalgam U is the unique countable model of T^0 which is $(\mathcal{K}^{\lhd})_{<\aleph_0}$-saturated (saturated for strong embeddings). We can strengthen this property slightly, which will be necessary for giving a first order axiomatization of the theory of U.

Lemma 6.18. *The amalgam U has the following Strong Existential Closedness property.*

SEC For each strong extension $A \overset{g}{\hookrightarrow\!\!\lhd} B$ in $\mathcal{K}^{\lhd}_{<\aleph_0}$ and each embedding $A \xhookrightarrow{f} U$ (not necessarily strong) there is an embedding $B \xhookrightarrow{h} U$ such that $h \circ g = f$.

Proof. Lemma 6.14 shows that $\lceil A \rceil$ is finitely generated. Let E be a finitely generated amalgam of $\lceil A \rceil$ and B over A. This exists and $\lceil A \rceil \lhd E$ by theorem 6.15. By the saturation property for strong extensions, there is an embedding of E into U, extending the embedding of $\lceil A \rceil$. This contains a strong extension of A isomorphic to B. □

6.5 The pregeometry

The geometry of the amalgam is controlled by a pregeometry, which we now describe. For any model M of T^0, in particular U, the predimension function δ gives rise to a *dimension* notion on M. The dimension function is conventially denoted d and is defined as follows.

Definition 6.19. For $X \subseteq_{\text{fin}} M$ (or even $X \subseteq M$ with $\text{td}(X/C)$ finite), define

$$d(X) = \delta(\lceil X \rceil) = \min \{\delta(XY) \mid Y \subseteq_{\text{fin}} M\}.$$

For X as above and any $A \subseteq M$, the dimension of X over A is defined to be

$$d(X/A) = \min \{d(XY) - d(Y) \mid Y \subseteq_{\text{fin}} A\}$$

Note that $d(X) = d(X/\emptyset)$, so the two definitions agree.

Lemma 6.20 (Properties of d). *Let $X, Y \subseteq_{\text{fin}} M$ and $A, B \subseteq M$.*

1. *If $X \subseteq Y$ then $d(X/A) \leqslant d(Y/A)$.*
2. *If $A \subseteq B$ then $d(X/A) \geqslant d(X/B)$.*
3. *d is submodular: $d(XY) \leqslant d(X) + d(Y) - d(X \cap Y)$.*
4. *$d(X/Y) = d(XY) - d(Y)$.*
5. *For any $x \in M$, $d(x/A) = 0$ or 1.*
6. *$d(X) = 0$ iff $X \subseteq C$.*

Proof. The first two parts are immediate from the definition. For submodularity:

$$\begin{aligned} d(XY) + d(X \cap Y) &= \delta(\lceil XY \rceil) + \delta(\lceil X \cap Y \rceil) \\ &\leqslant \delta(\lceil X \rceil \lceil Y \rceil) + \delta(\lceil X \rceil \cap \lceil Y \rceil) \\ &\leqslant \delta(\lceil X \rceil) + \delta(\lceil Y \rceil) \\ &= d(X) + d(Y) \end{aligned}$$

For 4, let $Z \subseteq Y$. Then

$$d(XY) - d(Y) \leqslant d(XZ) - d(XZ \cap Y) \leqslant d(XZ) - d(Z)$$

by submodularity and monotonicity of d. Thus the minimum value of $d(XZ) - d(Z)$ occurs when $Z = Y$.

For part 5, take $A_0 \subseteq_{\text{fin}} A$ such that $d(x/A) = d(x/A_0)$. Then

$$\begin{aligned} d(x/A_0) &= d(A_0 x) - d(A_0) \\ &= \delta(\lceil A_0 x \rceil) - \delta(\lceil A_0 \rceil) \\ &= \delta(\lceil \lceil A_0 \rceil x \rceil) - \delta(\lceil A_0 \rceil) \\ &\leqslant \delta(\lceil A_0 \rceil x) - \delta(\lceil A_0 \rceil) \\ &\leqslant \operatorname{td}(x/\lceil A_0 \rceil) \leqslant 1 \end{aligned}$$

The last part follows from the Schanuel condition. $\square$

Proposition 6.21. *The operator $\mathcal{P}M \xrightarrow{\mathrm{cl}} \mathcal{P}M$ given by $x \in \operatorname{cl} A \iff d(x/A) = 0$ is a pregeometry on M, and if $X \subseteq M$ is such that $d(X)$ is defined then $d(X)$ is equal to the dimension of X in the sense of the pregeometry.*

Proof. It is straightforward to check that cl is a closure operator with finite character. It remains to check the exchange property. Let $A \subseteq M, a, b \in M$, and $a \in \mathrm{cl}(Ab) \smallsetminus \mathrm{cl}(A)$. By finite character, there is a finite $A_0 \subseteq A$ such that $a \in \mathrm{cl}(A_0 b)$. Then $d(a/A_0) = 1$. Using part 4 of lemma 6.20, we have

$$\begin{aligned} d(b/A_0 a) &= d(A_0 ab) - d(A_0 a) \\ &= d(A_0 b) - d(A_0 a) \\ &= [d(A_0) + d(b/A_0)] - [d(A_0) + d(a/A_0)] \\ &= [d(A_0) + 1] - [d(A_0) + 1] = 0 \end{aligned}$$

and so $b \in \mathrm{cl}(Aa)$.

Finally, x is independent from A iff $d(x/A) = 1$, and so d agrees with the dimension coming from the pregeometry. □

Note that the theory T^0, and therefore the amalgam U, do not depend on how many derivations we had (provided there is at least one). That is, because Γ was taken originally to be the intersection of solutions sets of the differential equations over all derivations. An alternative would have been to take different symbols for each derivation. That would have made this chapter somewhat more difficult and did not seem worthwhile. However it would have shown the relationship between the rank of the Jacobian matrix and the dimension d arising from the pregeometry. In particular, each constant field would be closed in the pregeometry.

Chapter 7

Existential closedness and the complete theory

We show that the reduct to the language $\mathcal{L}_{\mathcal{S}}$ of a differentially closed field is elementarily equivalent to the amalgam U given in the previous chapter, and give an axiomatization of their common complete theory. For the reasons noted at the end of the previous chapter, we consider only differential fields with one derivation.

7.1 Existential closedness for differential fields

This section is devoted to proving the following theorem.

Theorem 7.1. *Let F be the countable saturated differentially closed field. Then the reduct of F to the language $\mathcal{L}_{\mathcal{S}}$ is isomorphic to the amalgam U given in the previous chapter.*

As an immediate corollary:

Corollary 7.2. *The reduct of any differentially closed field to $\mathcal{L}_{\mathcal{S}}$ is elementarily equivalent to U.*

The proof of the theorem is split into five steps, each using different ideas.

Step 1: Saturated models

Since F is countable, is a saturated differential field, and is a model of T^0, it is enough by theorem 6.17 and the subsequent lemma to show that the reduct $F\restriction_{\mathcal{L}_{\mathcal{S}}}$ satisfies the strong existential closedness property, SEC.

Suppose that $A_0 \hookrightarrow\!\!\!\lhd\!\!\rightarrow B_0$ is a strong embedding in $\mathcal{K}^{\lhd}_{<\aleph_0}$, and $A_0 \hookrightarrow F\restriction_{\mathcal{L}_{\mathcal{S}}}$ is an embedding, with A_0 identified with its image. Let $\langle A; D_0\rangle$ be the differential subfield of F generated by A_0. Then A is the algebraic closure of a finitely generated differential subfield of F, although its reduct $A\restriction_{\mathcal{L}_\mathcal{S}}$ may not be finitely generated in $\mathcal{K}$.

Let B be the free amalgam of A and B_0 over A_0, and let K be the underlying field of B. We will construct a derivation D on K such that the restriction $D\restriction_A = D_0$ and such that the reduct of $\langle K; D\rangle$ to $\mathcal{L}_\mathcal{S}$ is B. Then, since A and B are the (uniquely defined) algebraic closures of finitely generated differential fields, by saturation of F there is an embedding of K into F respecting A. Hence there is an embedding of B_0 into $F\restriction_{\mathcal{L}_\mathcal{S}}$ respecting A_0, that is, the SEC property holds.

Step 2: Admissible, free and normal varieties

As noted in the proof of proposition 6.16, the extension B of A is determined by the group $S^{\max}(B/A)$, the locus $\mathrm{Loc}_A(g) \subseteq S^{\max}(B/A)$ of a basis g for $\Gamma(B/A)$, and the natural number $t = \mathrm{td}(B/A(g))$. Suppose that b is a transcendence base for $B/A(g)$. Take $S \in \mathcal{S}$ of dimension $\geqslant t$ and extend b to an algebraically independent tuple $b' \in \mathbb{G}_\mathrm{a}^{\dim S}$. Take $s \in S$ generic over $B(b')$. Then there is a strong extension $B \hookrightarrow\!\!\!\!\!\lhd\!\!\!\!\rightarrow B'$ generated by (b', s) such that $(b', s) \in \Gamma_S$. Thus, replacing B by B' if necessary, we may assume that $t = 0$, that is, that B is generated by g over A.

Let $S = S^{\max}(B/A)$, $n = \dim S$, $G = \mathbb{G}_\mathrm{a}^n \times S$, let $g \in \Gamma_S$ be a basis for $\Gamma(B/A)$ and let $V = \mathrm{Loc}_A(g)$.

The assumptions that $B \in \mathcal{K}^{\lhd}_{<\aleph_0}$ and $A \lhd B$ give certain properties of V which we call *admissibility*. As before, we write $\widehat{H} = \mathrm{Log}\, H \times H$ for any semiabelian variety $H \in \mathcal{S}$. In particular, $G = \widehat{S}$.

Definition 7.3. An irreducible subvariety W of $\widehat{S}$ is *admissible* iff for every quotient map (surjective regular homomorphism) $S \xrightarrow{f} H$,

$$\varphi_H(\widehat{f}(W)) \geqslant 0.$$

W is *absolutely admissible* iff for each such f with $H \neq 1$,

$$\varphi_H(\widehat{f}(W)) \geqslant 1.$$

Here $\widehat{S} \xrightarrow{\widehat{f}} \widehat{H}$ is the homomorphism arising from f and φ_H is the function defined in chapter 4 to count "degrees of freedom". Recall that it is given for a subvariety U of $\widehat{H}$ by $\varphi_H(U) = \dim U - \dim J$ where J is the smallest algebraic subgroup of H such that U is contained in a coset of $\widehat{J}$.

A point $g \in \widehat{S}$ is (*absolutely*) *admissible* over a field A iff $\mathrm{Loc}_A(g)$ is (absolutely) admissible. Also define a reducible variety to be (absolutely) admissible iff at least one of its irreducible components is.

Lemma 7.4. *The variety V is admissible and $\mathrm{Loc}_C V = \mathrm{Loc}_C(g)$ is absolutely admissible.*

Proof. Since $B \in \mathcal{K}^{\lhd}$, it satisfies the Schanuel condition and so

$$\varphi_H(\mathrm{Loc}_C \widehat{f}(V)) = \varphi_H(\widehat{f}(\mathrm{Loc}_C V)) = \delta(\widehat{f}(g)/C) \geqslant 1$$

for each quotient map f, the first equality holding because S, H, and $\widehat{f}$ are defined over C and the last comparison because $\widehat{f}(g) \in \Gamma_H \smallsetminus \widehat{H}(C)$.

Note that if $W \subseteq \widehat{H}$ is any subvariety defined over A and W lies in a coset of $\widehat{J}$ then it must be an A-coset. So for each quotient map f,

$$\varphi_H(\widehat{f}(V)) = \delta(\widehat{f}(g)/A) \geqslant 0$$

because $A \lhd B$. $\square$

The assumption that $g \in \widehat{S}$ is a basis for $\Gamma(B/A)$ gives a further property of V which, following Zilber, we call *freeness*.

Definition 7.5. An irreducible subvariety W of $\widehat{S}$ is *free* iff W is not contained in a coset of $\widehat{H}$ for any proper algebraic subgroup H of S. It is *absolutely free* iff $\mathrm{pr}_S W$ is not contained in a coset of H and $\mathrm{pr}_{\mathrm{Log}\, S} W$ is not contained in a coset of $\mathrm{Log}\, H$ for any proper algebraic subgroup H of S.

A point $g \in \widehat{S}$ is (*absolutely*) *free* over a field A iff $\mathrm{Loc}_A(g)$ is (absolutely) free.

Lemma 7.6. *The variety V is free and $\mathrm{Loc}_C V$ is absolutely free.*

Proof. By the definition of a basis, g does not lie in an A-coset of $\widehat{H}$ for any proper algebraic subgroup H of S, and hence $V = \mathrm{Loc}_A(g)$ is not contained in such a coset. By lemma 5.6, the second and third conditions are equivalent for $\mathrm{Loc}_C V = \mathrm{Loc}_C(g)$, since $g \in \Gamma_S$. For both to fail would mean that V lies in a C-coset of $\widehat{H}$ which it does not, so both hold. $\square$

We also follow Zilber in defining a *normality* property.

Definition 7.7. An irreducible subvariety W of $\widehat{S}$ is *normal* iff for every quotient map $S \xrightarrow{f} H$,

$$\dim \widehat{f}(W) \geqslant \dim H$$

and *absolutely normal* iff for every such f

$$\dim \widehat{f}(W) \geqslant \dim H + 1.$$

A point $g \in \widehat{S}$ is (*absolutely*) *normal* over a field A iff $\mathrm{Loc}_A(g)$ is (absolutely) normal.

Lemma 7.8. *(Absolute) normality implies (absolute) admissiblity, and for free varieties the implication also reverses.*

Proof. The first part is immediate. The second follows because if $\widehat{f}(W)$ lies in an A-coset of $\widehat{J}$ for a proper algebraic subgroup J of H then W lies in an A-coset of $\widehat{f^{-1}(J)}$. $\square$

Step 3: Reducing to the critical case

Since $A \lhd B$, it follows that $\dim V \geqslant n$. We will now show that we may assume that $\dim V = n$, using the method of intersecting with generic hyperplanes. Suppose that $\dim V > n$.

Let the hyperplane Π_p in the affine space $\mathbb{A}^N$ be given by

$$x \in \Pi_p \quad \text{iff} \quad \sum_{i=1}^{N} p_i x_i = 1.$$

Consider the family of hyperplanes $(\Pi_p)_{p \in \mathbb{A}^N}$, which is the family of all affine hyperplanes which do not pass through the origin. From the equation defining the hyperplaces it follows that there is a duality: $a \in \Pi_p$ iff $p \in \Pi_a$.

We have $V = \mathrm{Loc}(g/A) \subseteq \widehat{S}$. Although $\widehat{S}$ will not in general be an affine variety, we may always embed it in some affine space $\mathbb{A}^N$ as a constructible set in a way which preserves the notion of algebraic dependence. (Indeed, this is essentially the definition of an algebraic variety given in chapter 2.) Choose p differentially generic in $\Pi_g(F)$ over A and replace A by A', the algebraic closure of the differential field extension generated by p over A in F. Now replace V by $V' = \mathrm{Loc}(g/A')$, the locus being meant as a subvariety of V, not of $\mathbb{A}^N$. Then $\dim V' = \dim V - 1$.

If $\widehat{S}$ is an affine group (which occurs only if $S = \mathbb{G}_{\mathrm{m}}^n$) then V' is the irreducible component of the variety $V \cap \Pi_p$ which contains g. If not, then $V \cap \Pi_p$ is considered as a constructible subset of V and since V' is Zariski-closed in V it may not be contained in Π_p. In this case the global geometric picture of intersecting V with a generic hyperplane may not be so appropriate, however the following result still holds.

Lemma 7.9. *V' is free and normal.*

To prove this we use the following lemma in the style of model-theoretic geometry which is adapted from part of a proof in [Zil04a]. The algebraic closure notion acl used here and later is $\mathrm{acl}^{\mathrm{eq}}$ in the sense of Shelah, that is, $x \in \operatorname{acl} X$ means that x is a point *in some variety* which is algebraic over X.

Lemma 7.10. *Let A be a field, let $g \in \mathbb{A}^N$ and let p be generic in Π_g over A. Suppose that h is any tuple (a point in any algebraic variety) such that $h \in \operatorname{acl}(Ag)$. Then either $g \in \operatorname{acl}(Ah)$ or $\operatorname{td}(h/Ap) = \operatorname{td}(h/A)$ (that is, h is independent of p over A).*

Proof. If g is algebraic over A then the result is trivial, so we assume not.

Let $U = \operatorname{Loc}(p/\operatorname{acl}(Ah))$. Suppose $\operatorname{td}(h/Ap) < \operatorname{td}(h/A)$. Then, by counting transcendence bases, $\dim U = \operatorname{td}(p/Ah) < \operatorname{td}(p/A) = N$, the last equation holding because $g \notin \operatorname{acl}(A)$ and so p is generic in $\mathbb{A}^N$ over A. But $\operatorname{td}(p/Ah) \geqslant \operatorname{td}(p/Ag) = N-1$ as p is generic in Π_g, an $(N-1)$-dimensional variety defined over Ag. Hence $\dim U = N-1$. Now $\operatorname{acl}(Ah) \subseteq \operatorname{acl}(Ag)$, so $U = \operatorname{Loc}(p/\operatorname{acl}(Ah) \supseteq \operatorname{Loc}(p/\operatorname{acl}(Ag)) = \Pi_g$. But $\dim U = \dim \Pi_g$ and both U and Π_g are irreducible and Zariski-closed in $\mathbb{A}^N$, so $U = \Pi_g$.

Hence Π_g is defined over $\operatorname{acl}(Ah)$, and so is the set

$$\left\{x \in \mathbb{A}^N \mid (\forall y \in \Pi_g)[x \in \Pi_y]\right\} = \{g\}.$$

Thus $g \in \operatorname{acl}(Ah)$. $\square$

Proof of lemma 7.9. For a proper algebraic subgroup H of S, let

$$\widehat{S} \xrightarrow{\chi_{\widehat{H}}} \widehat{S}/\widehat{H}$$

be the quotient map and let $h = \chi_{\widehat{H}}(g)$.

If V' lies in an A'-coset of $\widehat{H}$ then $h \in \operatorname{acl}(A') = \operatorname{acl}(Ap)$. Now g does not lie in an A-coset of $\widehat{H}$, and so $h \in \operatorname{acl}(Ap) \smallsetminus \operatorname{acl}(A)$. In particular, h is dependent on p over A. But $h \in \operatorname{acl}(Ag)$ so, by lemma 7.10, $g \in \operatorname{acl}(Ah)$, and so

$$0 = \operatorname{td}(h/Ap) = \operatorname{td}(g/Ap) = \dim V - 1.$$

Now $\dim V > n$ by assumption, so this can only happen if $n = 0$. But $n = \dim S$ by definition, and for S to have proper algebraic subgroups we must have $\dim S > 0$. So V' is free.

For normality, let $S \xrightarrow{f} H$ be a quotient map of S and let $h = \widehat{f}(g) \in \widehat{H}$. So $h \in \mathrm{acl}(Ag)$, and $\dim \hat{f}(V') = \mathrm{td}(h/A')$. If $g \in \mathrm{acl}(Ah)$ then

$$\mathrm{td}(h/A') = \mathrm{td}(g/A') = \dim V - 1 \geqslant n = \dim S \geqslant \dim H.$$

Otherwise, by lemma 7.10, $\mathrm{td}(h/A') = \mathrm{td}(h/A)$, so $\dim \widehat{f}(V') = \dim \widehat{f}(V)$ which is at least $\dim H$ by normality of V. Thus V' is normal. □

Intersecting with several hyperplanes in succession if necessary, we may thus assume that $\dim V = n$.

Step 4: Finding a derivation

We wish to consider the derivations in $\mathrm{Der}(K/C)$ which extend D_0 on A. These form a coset of the subspace $\mathrm{Der}(K/A)$ of $\mathrm{Der}(K/C)$. In order to remain working with subspaces, we follow [Pie03] in defining

$$\mathrm{Der}(K/D_0) = \{D \in \mathrm{Der}(K/C) \mid \exists \lambda \in K, D{\restriction_A} = \lambda D_0\}$$

which can be considered as the dual space of a quotient $\Omega(K/D_0)$ of $\Omega(K/C)$. This gives a sequence of inclusions

$$\mathrm{Der}(K/A) \hookrightarrow \mathrm{Der}(K/D_0) \hookrightarrow \mathrm{Der}(K/C)$$

and dually surjections

$$\Omega(K/C) \twoheadrightarrow \Omega(K/D_0) \twoheadrightarrow \Omega(K/A)$$

of K-vector spaces.

As usual, we let $g = (x, y) \in \mathrm{Log}\, S \times S$, we take bases $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ of invariant differential forms on S and $\mathrm{Log}\, S$ respectively, for each i we define $\omega_i(g) = \zeta_i(y) - \xi_i(x)$, and we take $\omega = (\omega_1, \ldots, \omega_n)$. Then the exponential equation for S is given by $\omega(g)D = 0$.

We can consider the differentials $\omega_i(g)$ in $\Omega(K/C)$, but also in $\Omega(K/D_0)$ and $\Omega(K/A)$ via the canonical surjections above.

Lemma 7.11. *The differentials $\omega_i(g)$ are K-linearly independent in $\Omega(K/A)$, and hence also in $\Omega(K/D_0)$ and $\Omega(K/C)$.*

Proof. This is essentially just the Schanuel condition. Suppose not, so the $\omega_i(g)$ are K-linearly dependent. Then by lemmas 5.3, 5.5, and 5.6, there is a proper algebraic subgroup H of S such that g lies in an A-coset of $\widehat{H}$. This contradicts the freeness of V. $\square$

The K-linear dimension of $\Omega(K/D_0)$ is equal to that of $\mathrm{Der}(K/D_0)$, which is $\dim \mathrm{Der}(K/A) + 1 = n + 1$ because $A \neq C$. Let $\Lambda = \langle \omega_1(g), \ldots, \omega_n(g) \rangle$ be the span of the $\omega_i(g)$ in $\Omega(K/C)$, with annihilator $\mathrm{Ann}(\Lambda) \subseteq \mathrm{Der}(K/C)$. The image of Λ has codimension 1 in $\Omega(K/D_0)$, so $\mathrm{Der}(K/D_0) \cap \mathrm{Ann}(\Lambda)$ has dimension 1. Let $D \in \mathrm{Der}(K/D_0) \cap \mathrm{Ann}(\Lambda)$ be nonzero. The image of Λ spans $\mathrm{Der}(K/A)$, and so $\mathrm{Der}(K/A) \cap \mathrm{Ann}(\Lambda) = \{0\}$. Hence $D{\restriction_A} = \lambda D_0$ for some non-zero λ. Replacing D by $\lambda^{-1} D$, we may assume that $\lambda = 1$, that is, D extends D_0.

Step 5: No new constants

Consider the reduct of the differential field $\langle K; +, \cdot, D, C_D \rangle$ to $\mathcal{L}_S$. We must show that this is equal to B. We show that there are no new constants for D, that is, that the constant field C_D of D is equal to C.

First note that we may assume that the extension $A \lhd B$ is *minimal* in the sense that if B' is an algebraically closed intermediate field with $A \lhd B' \lhd B$ then $B' = A$ or $B' = B$. If not, we may split the extension up into a chain of strong extensions, a finite chain since $\mathrm{td}(B/A)$ is finite, and treat each link in the chain separately. Now for any intermediate field B' we have $A \lhd B'$, since otherwise $A \ntriangleleft B$, so for every proper intermediate field B' it must be the case that $B' \ntriangleleft B$. Now $\delta(B'/A) \geqslant 0$ and $\delta(B/A) = 0$, so for $B' \ntriangleleft B$ we must have $\delta(B/B') < 0$, that is, $\mathrm{td}(B/B') < \mathrm{grk}(B/B')$.

Suppose that $C_D \neq C$. Then $C_D \nsubseteq A$. Let $K' = \mathrm{acl}(C_D, A)$. The realization of Γ in the reduct contains the realization in B by construction and thus, by the above, $\mathrm{td}(K/K') < \mathrm{grk}(K/K')$. Let $h \in \Gamma_H(K)$ be a basis for $\Gamma(K/K')$. Then by the Schanuel condition, $h \in k \oplus \widehat{J}$ for some proper algebraic subgroup J of H and some $k \in \Gamma_H(K')$. The basis h can be extended to a basis of K/A, and so there is a homomorphism $S \xrightarrow{f} H$ such that $h = \widehat{f}(g)$. By assumption, g is free over A and hence h is free over A. Now $k = \gamma \oplus a$ for some $\gamma \in H(C_D)$ and $a \in \Gamma_H(A)$, but $k \ominus a$ is also a basis for $\Gamma(K/K')$ as h is free over A, so we may assume $a = 0$. Then h lies in a C_D-coset of $\widehat{J}$, which contradicts it being a basis. Hence $C_D = C$.

The same Schanuel condition argument shows that the realization of Γ in the reduct of K is precisely the realization of Γ in B, and so the reduct is isomorphic to B as required.

That completes the proof of theorem 7.1. $\square$

The Schanuel condition can be viewed as a necessary condition for a system of differential equations to have a solution, and strong existential closedness gives a matching sufficient condition.

Theorem 7.12. *Let S be a semiabelian variety defined over C, and let V be a subvariety of $\widehat{S}$. If V is defined over C then a necessary and sufficient condition for there to be a nonconstant point in $\Gamma_S \cap V$ in some differential field extension is for V to be absolutely admissible.*

If V is not defined over C then a sufficient condition for a point to exist is for V to be admissible. If in addition $\operatorname{Loc}_C V$ is absolutely admissible then a nonconstant point exists.

Proof. The problem of finding a solution to such a system of equations can be reduced to finding a solution to a free system, by using the algebraic structure of Γ formally to find a basis for the proposed solution. This replaces $V \subseteq \widehat{S}$ by a homomorphic image $V' \subseteq \widehat{S}'$ which is free, with $\operatorname{Loc}_C V'$ absolutely free. Since V' is the image of V under a homomorphism $\widehat{S} \longrightarrow \widehat{S}'$, V' is (absolutely) admissible iff V is (absolutely) admissible. $\square$

The reduct of a differentially closed field does not have quantifier elimination in the language $\mathcal{L}_{\mathcal{S}}$, so there is no general necessary and sufficient condition when V is defined with non-constant parameters. The theory DCF_0 does have quantifier elimination, so there must be a condition which depends on what other differential equations the parameters satisfy.

This statement about differential algebra implies a result about solving differential equations in the setting of complex meromorphic functions, at least away from singularities.

Theorem 7.13. *Let Σ be a system of exponential differential equations (for split semiabelian varieties) defined over the differential field F of meromorphic functions on a domain U in $\mathbb{C}$, and suppose that solving Σ has been reduced (as it always can be) to the question of finding a point in $\Gamma_S \cap V$ for some split semiabelian variety S and an algebraic variety V defined over a differential subfield F_0 of F. Let $u \in U$ be* *a point at which no function $f \in F_0 \setminus \{0\}$ has a zero or a pole. Then the conditions of theorem 7.12 determine whether there is a solution g to Σ, meromorphic in some neighbourhood of u. In particular, if V is admissible then there is a meromorphic solution at u.*

Proof. Theorem 5.20 of Seidenberg has a refinement, given in the original papers [Sei58] and [Sei69] and explained in more detail in [Mar96], which says that the embedding of a differential field into a field of meromorphic functions can be done step by step, respecting parameters, away from singularities. This result follows at once. $\square$

7.2 Definability of normality

In the previous section the variety V was always free, so there was no need there to introduce the concept of normality as well as admissibility. In finding an axiomatization of the first order theory however we must use properties which are expressible in the first order language. Freeness and admissibility are not known to be first order, but we now show that normality is. To do this we generalize and adapt the proof in section 3 of [Zil05b].

The notion of an atypical intersection of two varieties was mentioned earlier, and we now formalize the definition.

Definition 7.14. Let U be a smooth algebraic variety, and let V, W be subvarieties of U, with $V \cap W \neq \emptyset$. The intersection $V \cap W$ is said to be *typical* (in U) iff

$$\dim(V \cap W) = \dim V + \dim W - \dim U$$

and *atypical* iff

$$\dim(V \cap W) > \dim V + \dim W - \dim U.$$

Even if V and W are irreducible, the intersection $V \cap W$ may be reducible, and its components may have different dimensions. We say that a component X of $V \cap W$ is *atypical* iff

$$\dim X > \dim V + \dim W - \dim U.$$

We also say that the *degree of atypicality* is the difference

$$\dim X - (\dim V + \dim W - \dim U).$$

Note that the intersection is typical iff $\operatorname{codim}(V \cap W) = \operatorname{codim} V + \operatorname{codim} W$, and since U is smooth the dimension of the intersection cannot be less than the typical size (assuming the intersection is nonempty). We also need the notion of an atypical image of a variety under a map, although we only define this for subvarieties of groups.

Definition 7.15. Let G be an algebraic group, H an algebraic subgroup and V an algebraic subvariety of G. Let $G \xrightarrow{q} G/H$ be the quotient map onto the coset space and write V/H for the image of V under q. This image V/H is said to be *typical* iff

$$\dim V/H = \min\{\dim G/H, \dim V\}$$

and *atypical* iff

$$\dim V/H < \min\{\dim G/H, \dim V\}.$$

The following theorem is the main result of section 3 of [Zil05b], generalized to the semiabelian case. The proof is adapted and expanded from the one in that paper, with no essential changes.

We use the fact that in the conclusion of theorem 5.15, X is a typical component of the intersection $(U_p \cap s \oplus H) \cap (\kappa \cap s \oplus H)$ in $s \oplus H$. For convenience we also choose the finite set $\mathcal{J}_W^S$ of subgroups of S given in the conclusion of that theorem to contain the trivial subgroup. The additive formula for fibres is used several times:

(AF) For an irreducible variety A and a surjective map $A \xrightarrow{f} B$,

$$\dim A = \dim B + \min_{b \in B} \dim f^{-1}(b).$$

Theorem 7.16. *Let S be a semiabelian variety and $V \subseteq \widehat{S}$ an irreducible subvariety. If V is not normal then there is $J \in \mathcal{J}_W^S$ where $W = \operatorname{pr}_S V$ such that $\dim V/\widehat{J} < \dim S/J$. That is, failure of normality is witnessed by a member of the finite set $\mathcal{J}_W^S$.*

Proof. Suppose that $V/\widehat{H} < \dim S/H$ for some algebraic subgroup H of S. If $H = 1$ is the trivial subgroup then we are done since $1 \in \mathcal{J}_W^S$, so we assume that $\dim V \geqslant \dim S$, and $H \neq 1$.

Step 1 The image W/H is atypical.

W/H is a projection of $V/\widehat{H}$, so

$$\dim W/H \leqslant \dim V/\widehat{H} < \dim S/H.$$

Thus if W/H were typical we would have $\dim W/H = \dim W$, so the fibres of the map $W \longrightarrow W/H$ would be finite. The fibres of $V \longrightarrow V/\widehat{H}$ could then have dimension at most $\dim H$, so

$$\dim V/\widehat{H} \geqslant \dim V - \dim H \geqslant \dim S - \dim H = \dim S/H$$

which contradicts the assumption. Thus W/H is atypical.

Step 2 There is $J \in \mathcal{J}_W^S$ such that

$$\dim W/J = \dim W/H - \dim J/(J \cap H) \tag{7.1}$$

and

$$\dim W/H = \dim W/(J \cap H). \tag{7.2}$$

Let $x \in W$ be generic over a field of definition of S, H and W, and let κ be the coset $x \oplus H$. Then $W \cap \kappa$ is a generic fibre of the quotient map so, by the addition formula for fibres (AF),

$$\dim W \cap \kappa = \dim W - \dim W/H$$

which is > 0 as the image is atypical. Let X be the component of $W \cap \kappa$ containing x, which must be of maximal dimension by genericity of x. Thus

$$\dim X = \dim(W \cap \kappa) = \dim W - \dim W/H \tag{7.3}$$

and by atypicality of the image

$$\dim W/H < \dim S/H = \dim S - \dim H$$

so

$$\dim X > \dim W + \dim H - \dim S.$$

Now $\dim H = \dim \kappa$ so X is an atypical component of the intersection $W \cap \kappa$ in S. By theorem 5.15 there is $J \in \mathcal{J}_W^S$ such that X is contained in the coset $\kappa' = x \oplus J$. Thus the quotient of X by $J \cap H$ is isomorphic to the quotient by H, so since X is a component of maximal dimension this implies 7.2.

By the remark above, X is a typical component of $(W \cap \kappa') \cap (\kappa \cap \kappa')$ in κ', that is

$$\dim X = \dim(W \cap \kappa') + \dim(\kappa \cap \kappa') - \dim \kappa'. \tag{7.4}$$

Let Y be the connected component of $(W \cap \kappa')$ containing X. Then 7.4 becomes

$$\dim X = \dim Y + \dim(J \cap H) - \dim J. \tag{7.5}$$

Y is a generic fibre of $W \longrightarrow W/J$, so by (AF) again,

$$\dim Y = \dim W - \dim J. \tag{7.6}$$

Substituting 7.3 and 7.6 into 7.5 gives 7.1 as required.

Let $H' = J \cap H$.

Step 3 $\dim V/\widehat{H'} < \dim S/H'$.

For $b \in W$ write $V_b \subseteq \mathbb{G}_a^n$ for the fibre of the projection $V \longrightarrow W$. The projection $\mathbb{G}_a^n / \operatorname{Log} H' \longrightarrow \mathbb{G}_a^n / \operatorname{Log} H$ has fibres of dimension $k = \dim S/H' - \dim S/H$, so for any b the fibres of the map $V_b / \operatorname{Log} H' \longrightarrow V_b / \operatorname{Log} H$ have dimension at most k. Thus

$$\dim V_b / \operatorname{Log} H' \leqslant \dim V_b / \operatorname{Log} H + k. \tag{7.7}$$

By (AF),

$$\dim V/\widehat{H'} = \dim W/H' + \min_{b \in W} \dim V_b / \operatorname{Log} H' \tag{7.8}$$

and substituting in this using 7.2 and 7.7 gives

$$\dim V/\widehat{H'} \leqslant \dim W/H + \min_{b \in W} \dim V_b / \operatorname{Log} H + k$$

which by (AF) again implies

$$\dim V/\widehat{H'} \leqslant \dim V/\widehat{H} + k < \dim S/H'$$

as required.

Step 4 $\dim V/\widehat{J} < \dim S/J$.

This is very similar to step 3. Since $H' \subseteq J$, the quotient factors as

$$V \longrightarrow V/\widehat{H'} \longrightarrow V/\widehat{J}$$

so for any $b \in W$,

$$\dim V_b / \operatorname{Log} J \leqslant \dim V_b / \operatorname{Log} H'. \tag{7.9}$$

By (AF),

$$\dim V/\widehat{J} = \dim W/J + \min_{b \in W} \dim V_b / \operatorname{Log} J \tag{7.10}$$

and using 7.1 and 7.9 this becomes

$$\dim V/\widehat{J} \leqslant \dim W/H' + \min_{b \in W} \dim V_b/\operatorname{Log} H' + (\dim S/J - \dim S/H').$$

Applying (AF) a final time with the conclusion of Step 3 gives

$$\dim V/\widehat{J} < \dim S/J$$

as required. □

It is well-known that for any parametric family of varieties $(V_p)_{p \in P}$ there is a first order formula $\operatorname{Irr}(p)$ expressing that V_p is irreducible. This together with the above shows that normality is a definable property of a variety.

Corollary 7.17. *Let S be a semiabelian variety and $(V_p)_{p \in P}$ a parametric family of subvarieties of $\widehat{S}$. The following formula expresses that V_p is normal, and is a first order formula in the language of fields.*

$$\operatorname{Irr}(p) \wedge \bigwedge_{J \in \mathcal{J}^S_{\operatorname{pr}_S V}} \dim V_p/\widehat{J} \geqslant \dim S/J$$

Proof. Normality has only been defined for irreducible varieties, and in this case Theorem 7.16 says precisely that this formula expresses normality. The finiteness of the set $\mathcal{J}^S_{\operatorname{pr}_S V}$ together with lemma 5.12 show that this is first order in the language of fields. □

7.3 The complete theory

The strong existential closedness property SEC is not expressible in a first order language, so to give an axiomatization of the first order $\mathcal{L}_{\mathcal{S}}$-theory of the reduct it is necessary to extract its first order content. We use the standard model theoretic terminology of "existential closedness" or EC for this.

EC For each $S \in \mathcal{S}$ and each normal subvariety $V \subseteq \widehat{S}$, the set $\Gamma_S \cap V$ is nonempty.

The EC and SEC properties have been written in very different language, so it is worth making the following explicit.

Lemma 7.18. *SEC implies EC.*

Proof. It is enough to consider EC for irreducible V. In such an instance of EC, if the point taken to exist in $\Gamma_S \cap V$ is generic over the parameters then it defines a strong, finitely generated extension over the parameters. SEC says that this extension must be realised in the model. $\square$

Proposition 7.19. *The property EC is expressible as an axiom scheme in the first order language $\mathcal{L}_{\mathcal{S}}$.*

Proof. Any such variety V lies in a parametric family $(V_p)_{p \in P}$, of subvarieties of $\widehat{S}$, the family defined over $\mathbb{Q}$ together with the parameters named in the language. The sentence

$$(\forall p \in P)\exists x \left[\left(\operatorname{Irr}(p) \wedge \bigwedge_{J \in \mathcal{J}^S_{\operatorname{pr}_S V}} \dim V_p/\widehat{J} \geqslant \dim S/J \right) \to \Gamma_S(x) \wedge V_p(x) \right]$$

for each $S \in \mathcal{S}$ and each family of subvarieties will do by corollary 7.17. $\square$

Theorem 7.20. *Let $\mathcal{S}$ be a collection of split semiabelian varieties defined over the constant field C. The first order theory of the reduct of a differentially closed field to the language $\mathcal{L}_{\mathcal{S}}$ is axiomatized by $T_{\mathcal{S}}$ which consists of the algebraic axiom schemes A1—A7, the axiom scheme USC stating the uniform Schanuel condition, and the axiom scheme EC stating the existential closedness condition.*

Proof. Proposition 6.1 states that the reduct of any differential field satisfies A1—A6 and USC, and a differentially closed field is algebraically closed, so satisfies A7. By theorem 7.1, the reduct of the countable saturated differentially closed field is isomorphic to the amalgam U, so by lemma 6.18 it satisfies SEC, and so by lemma 7.18 it satisfies EC. Hence, by proposition 7.19 and the completeness of the theory DCF_0, the reduct of any differentially closed field satisfies EC.

Suppose that F is a saturated, countable model of $T_{\mathcal{S}}$. We will show that F is isomorphic to the amalgam U, that is, that $T_{\mathcal{S}}$ has only one countable saturated model. It follows that $T_{\mathcal{S}}$ is complete. By the argument of step 1 of the proof of theorem 7.1, it is enough to show that F satisfies SEC.

Let $S \in \mathcal{S}$ and $V \subseteq \widehat{S}$ be irreducible, free and normal such that $\dim V = \dim S$ and $\operatorname{Loc}_C V$ is absolutely free and absolutely normal, and let W be a proper subvariety of V of codimension 1. Suppose that W is given as a subvariety of V as the union of the zero sets of the k equations $p(x) = 0$. For convenience, assume that $\mathbb{G}_{\mathrm{m}} \in \mathcal{S}$, and let $S' = S \times \mathbb{G}_{\mathrm{m}}^k$ with $\widehat{S'}$ having coordinates (x, y, z) with $x \in \widehat{S}$, $y \in \mathbb{G}_{\mathrm{m}}^k$ and $z \in \mathbb{G}_{\mathrm{a}}^k$. Extend V to the variety $V' \subseteq \widehat{S'}$ given by $V(x) \wedge p(x) = y$. Then V' is irreducible, and it is free and normal with $\operatorname{Loc}_C V'$ absolutely free and absolutely normal, since V is. By the method of step 3 of 7.1, we may assume that $\dim V' = \dim S'$.

By EC there is $(x, y, z) \in \Gamma_{S'} \cap V'$, and so in particular with no coordinate of y being zero. Thus $x \in \Gamma_S \cap (V \smallsetminus W)$. By the method of step 5 of 7.1, and the assumption that $\operatorname{Loc}_C V$ is absolutely free from C, x is (absolutely) free from C. The assumption that $\mathbb{G}_\mathrm{m} \in \mathcal{S}$ can be dropped by replacing $\mathbb{G}_\mathrm{m}^k$ by some other semiabelian variety and adapting the argument accordingly.

Since F is saturated, it follows that for any finite set of parameters, there is $x \in \Gamma_S \cap V$, generic in V over the parameters, and again free from C. By steps 2 and 3 of 7.1, this is enough to show that F satisfies SEC. Thus $F \cong U$ and $T_\mathcal{S}$ is complete. $\square$

For convenience we now give the complete axiomatization of $T_\mathcal{S}$ here.

A1 F is a field of characteristic zero.

A2 C is a relatively algebraically closed subfield of F, each $c_i \in C$ and they satisfy the appropriate algebraic relations.

A3 Γ_S is a subgroup and an $\operatorname{End}(S)$-submodule of of the algebraic group $\widehat{S}$.

A4 Γ_S contains $\widehat{S}(C)$.

A5 The fibres of Γ_S in $\operatorname{Log} S$ and S are cosets of $(\operatorname{Log} S)(C)$ and $S(C)$, respectively.

A6 $\Gamma_{S_1 \times S_2} = \Gamma_{S_1} \times \Gamma_{S_2}$.
If $S_1 \subseteq S_2$ then $\Gamma_{S_1} = \Gamma_{S_2} \cap G_1$.
If $S_1 \xrightarrow{f} S_2$ is surjective then $\widehat{f}(\Gamma_{S_1}) \subseteq \Gamma_{S_2}$.

A7 F is algebraically closed.

USC For each $S \in \mathcal{S}$, each variety P and each parametric family $(V_p)_{p \in P}$ of algebraic subvarieties of $\widehat{S}$, defined over $\mathbb{Q}$:

$$(\forall p \in P(C))(\forall g \in V_p \cap \Gamma_S)\left[\dim V_p < \dim S + 1 \to \bigvee_{H \in \mathcal{H}_V^S} \chi_{\widehat{H}}(g) \in \widehat{S}(C)\right]$$

where $\mathcal{H}_V^S$ is the finite set of algebraic subgroups of S given by theorem 5.14 and $\chi_{\widehat{H}}(x) = 0$ is the equation (or system of equations) defining $\widehat{H}$.

EC For each S, P and $(V_p)_{p\in P}$ as above:

$$(\forall p \in P)\exists x \left[\left(\mathrm{Irr}(p) \wedge \bigwedge_{J \in \mathcal{J}^S_{\mathrm{pr}_S V}} \dim V_p/\widehat{J} \geqslant \dim S/J \right) \to \Gamma_S(x) \wedge V_p(x) \right]$$

where $\mathcal{J}^S_{\mathrm{pr}_S V}$ is the finite set of algebraic subgroups of S given by theorem 5.15, in fact $\mathcal{J}^S_{\mathrm{pr}_S V} = \mathcal{H}^S_{L\times \mathrm{pr}_S V}$.

7.4 Model theoretic properties

I have not made any significant investigation into the model-theoretic properties of these reducts, but here give a few basic facts.

Proposition 7.21. *The theories $T_{\mathcal{S}}$ are ω-stable of Morley rank ω.*

Proof. They are reducts of an expansion by constants of DCF_0 which is ω-stable of rank ω. Furthermore, the theory of pairs of algebraically closed fields is a reduct of $T_{\mathcal{S}}$, and that also has Morley rank ω. Taking a reduct cannot increase the rank, so the Morley rank of $T_{\mathcal{S}}$ must be ω. □

Proposition 7.22. *The theories $T_{\mathcal{S}}$ are near model complete, that is, they have quantifier elimination to the level of existential formulas and universal formulas.*

Proof. The countable saturated model U is homogeneous for strong subsets, so if a and b are finite tuples and $\lceil a \rceil \cong \lceil b \rceil$ then $\mathrm{tp}(a) = \mathrm{tp}(b)$. The isomorphism class of $\lceil a \rceil$ is determined by which finitely generated non-strong extensions of $\langle a \rangle$ exist in U. Each of these is determined by a basis, which is a point in $\Gamma_S \cap V_a$ for some $S \in \mathcal{S}$ and some algebraic variety V_a defined over a. The formula $\exists g(\Gamma_S(g) \wedge V_a(g))$ is existential in the first order language $\mathcal{L}_{\mathcal{S}}$, and so the collection of all existential formulas and their negations (universal formulas) true of a determines $\mathrm{tp}(a)$. Hence $T_{\mathcal{S}}$ is near model complete. □

It was shown in the proof of Proposition 6.16 that there are only countably many isomorphism classes of finitely generated strong subsets, so this also gives a direct proof that $T_{\mathcal{S}}$ is ω-stable.

Proposition 7.23. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are collections of (split) semiabelian varieties which are closed under isogeny and under taking products and subgroups and $\mathcal{S}_1 \neq \mathcal{S}_2$ then the reducts to $\mathcal{L}_{\mathcal{S}_1}$ and $\mathcal{L}_{\mathcal{S}_2}$ of a differentially closed field are different.*

In particular, they are both proper reducts of DCF_0.

Proof. Suppose $S \in \mathcal{S}_1 \smallsetminus \mathcal{S}_2$, of dimension n, and let V be an absolutely free and absolutely normal subvariety of $\widehat{S}$ defined over C and of dimension $n+1$. Take a point $g \in \Gamma_S \cap V$ and take h generic in V over the parameters defining V in some saturated differentially closed field F. Then g has dimension one in the pregeometry of the reduct to $\mathcal{L}_{\mathcal{S}_1}$. However, by the Schanuel condition, g cannot be algebraically dependent on any point from $\Gamma_{S'}$ for any $S' \in \mathcal{S}_2$ so it has dimension $n+1$ in the reduct to $\mathcal{L}_{\mathcal{S}_2}$. Thus g and h have the same $\mathcal{L}_{\mathcal{S}_2}$-type, but different $\mathcal{L}_{\mathcal{S}_1}$-types, and so the reducts are different.

Given any collection $\mathcal{S}$ there is a strictly larger one, for example containing semi-abelian varieties defined over a larger constant field, and so this shows that $T_{\mathcal{S}}$ is a proper reduct of DCF_0. $\square$

Chapter 8

Complex functions

Knowing the theory of the exponential differential equations gives information about the complex exponential functions which satisfy them.

The first theorem we consider is the (uniform) Schanuel conditions in the context of complex analytic geometry. It generalizes proposition 8 of [Zil02a] to the semiabelian case, using the uniform Schanuel condition for the exponential differential equation of a semiabelian variety. The proof given in [Zil02a] is not quite correct because it uses only one derivation where more may be required, and so the "constant" produced there, although constant with respect to one particular derivation, may not actually be in $\mathbb{C}$. As with theorem 5.15, using the full Schanuel condition for partial differential fields streamlines the proof.

As usual, S is a semiabelian variety of dimension n, and G is the algebraic group $G = \mathbb{G}_{\mathrm{a}}^n \times S$. Here we consider the complex points of the groups as complex manifolds (Lie groups) and look at analytic subsets. Write $\mathcal{G}$ for the graph of the exponential function of S, that is, $\mathcal{G} = \{(x, y) \in G(\mathbb{C}) \mid y = \exp(x)\}$. As before we write $\operatorname{Log} H$ for the algebraic subgroup of $\mathbb{G}_{\mathrm{a}}^n$ corresponding to an algebraic subgroup H of S, and we write $\widehat{H} = \operatorname{Log} H \times H$.

Theorem 8.1. *Let P be an algebraic variety and $(V_p)_{p \in P(\mathbb{C})}$ be a parametric family of algebraic subvarieties of G. There is a finite collection $\mathcal{H}_V^S$ of proper algebraic subgroups of S with the following property:*

If $p \in P$ and W is a connected component of the analytic variety $\mathcal{G} \cap V_p$ with analytic dimension $\dim W$ satisfying $\dim W = (\dim V_p - n) + t$ for some $t > 0$, then there is $H \in \mathcal{H}_V^S$ of codimension at least t and $g \in G(\mathbb{C})$ such that W is contained in the coset $g \oplus \widehat{H}$.

Proof. We show that the finite collection $\mathcal{H}_V^S$ given in theorem 5.14 works here.

Let w be a regular point of W and let F be the differential field of germs at w of analytic functions on G, the differential operators being the usual $2n$ partial differentiation operators (with respect to any chosen basis). The field of constants is $\mathbb{C}$.

Let (x, y) be coordinate functions on G at w, with $x = (x_1, \ldots, x_n)$ being a basis of coordinate functions on $\mathbb{G}_a^n$ and $y = (y_1, \ldots, y_n)$ being a basis of coordinate functions on S. Then $y = \exp(x)$, and so $(x, y) \in \Gamma$, the solution set of the exponential differential equation for S. Also $(x, y) \in V_p$ and $\operatorname{rk} \operatorname{Jac}(x, y) = \dim W$, by the definition of the analytic dimension of W. By the uniform Schanuel condition (theorem 5.14), there is $H \in \mathcal{H}_V^S$ and $\gamma \in S(\mathbb{C})$ such that $y \in \gamma \oplus H$. It follows that $x \in \gamma' \oplus H'$ for some $\gamma \in \mathbb{G}_a^n(\mathbb{C})$ such that $\exp(\gamma') = \gamma$. Taking $g = (\gamma', \gamma)$, we have $(x, y) \in g \oplus \widehat{H}$. This is an algebraic relation on the coordinate functions, and so it holds on the whole component W. Thus $W \subseteq g \oplus \widehat{H}$ as required. $\square$

8.1 Analytically closed subfields

Here we make a technical definition which we use in the next section to define blurred exponentiation. It is possible that with more work, and possibly some number theoretic conjectures, this definition can be avoided. On the other hand it may be of independent interest as a means of getting some of the power of the real geometry approach to complex analysis but within the purely complex setting. Here the definition is given for the set of exponential functions of a family of semiabelian varieties, but it can easily be extended to consider any set of analytic functions.

Consider a collection $\mathcal{S}$ of semiabelian varieties defined over $\mathbb{C}$. For example, consider just the powers $\mathbb{G}_m^n$ for $n \in \mathbb{N}$. Take $\mathcal{L}_{\mathcal{S}}^f$ to be the language of fields together with function symbols for the exponential map of every semiabelian variety in $\mathcal{S}$. Take $\mathbb{C}_{\mathcal{S}}$ to be the obvious expansion of $\mathbb{C}$ in this language.

Definition 8.2. A subfield C of $\mathbb{C}$ is said to be *analytically closed* with respect to $\mathcal{S}$ iff it satisfies:

- C is the underlying field of an elementary substructure of $\mathbb{C}_{\mathcal{S}}$.
- For every analytic subset X of $\mathbb{C}^n$ which is definable in $\mathbb{C}_{\mathcal{S}}$ with parameters from C, every connected component of X has a point in C^n.

The second condition can be considered as a sort of infinitary analytic nullstellensatz, justifying the terminology "analytically closed". Note that the components of X need not themselves be definable. $\mathbb{C}$ itself is trivially analytically closed even when $\mathcal{S}$ is the collection of all semiabelian varieties defined over $\mathbb{C}$. In this case $\mathbb{C}_{\mathcal{S}}$ is an atomic structure, so has no proper elementary substructures and thus no proper analytically closed subfields. The fact we use below is that there are proper analytically closed subfields of $\mathbb{C}$ for appropriate collections $\mathcal{S}$. If $\mathcal{S}$ is countable then the downward Löwenheim-Skolem theorem gives a countable elementary substructure of $\mathbb{C}_{\mathcal{S}}$. It is possible that any elementary substructure in this language is analytically closed but, if true, this is likely to depend on arithmetic questions such as the conjecture on the intersection of tori. In any case, some of the countable subfields are analytically closed.

Lemma 8.3. *There is a countable subfield C of $\mathbb{C}$ which is analytically closed when $\mathcal{S}$ is the collection of all semiabelian varieties defined over C. Furthermore, we can find such C and $\mathcal{S}$ extending any given countable subfield C' of $\mathbb{C}$ and any countable family of semiabelian varieties.*

Proof. We proceed by induction. Let $\mathcal{S}_0$ be any countable set of semiabelian varieties defined over $\mathbb{C}$, and let C_0 be any countable elementary substructure of $\mathbb{C}_{\mathcal{S}_0}$, extending C'. In particular, each $S \in \mathcal{S}_0$ is defined over C_0.

Assume now that we have constructed a countable set $\mathcal{S}_n$ of semiabelian varieties and C_n a countable elementary substructure of $\mathbb{C}_{\mathcal{S}_n}$. Let $\mathcal{S}_{n+1}$ be the (countable) set of all semiabelian varieties defined over C_n. There are countably many analytic subsets definable in $\mathbb{C}_{\mathcal{S}_{n+1}}$ with parameters from C_n and each has only countably many components. Choose a representative of each component. Then, by the downward Löwenheim-Skolem theorem, we can find a countable elementary substructure C_{n+1} of $\mathbb{C}_{\mathcal{S}_{n+1}}$ extending C_n and containing all these representatives.

Iterating this construction, we reach a fixed point at stage ω. Take C to be $C = \bigcup_{n\in\mathbb{N}} C_n$. □

To connect this idea to other methods we make another observation about a way to construct these analytically closed fields. Let $\mathcal{S}$ be any family of semiabelian varieties defined over $\mathbb{C}$. Consider $\mathbb{C}$ not in the language $\mathcal{L}^f_{\mathcal{S}}$ but instead in the language $\mathcal{L}^r_{\mathcal{S}}$ which has the field structure, a predicate for the reals, and function symbols for all restrictions of the exponential functions of the semiabelian varieties in $\mathcal{S}$. Here a "restriction" of a function f means the product of the function with the indicator function of a box in $\mathbb{C}^n = \mathbb{R}^{2n}$ with (Gaussian) integer corners. This is the usual convention in o-minimality.

Then any elementary substructure of $\mathbb{C}$ in this language $\mathcal{L}^r_{\mathcal{S}}$ is analytically closed, because for any box B with rational coordinates, the intersection of B with an analytic variety definable in the language $\mathcal{L}^f_{\mathcal{S}}$ is definable in $\mathcal{L}^r_{\mathcal{S}}$. Thus the elementary theory says whether or not this intersection is empty, but every pair of connected components of a definable analytic set can be separated by some pair of boxes with rational corners, so every elementary substructure has a representative of every component.

The $\mathcal{L}^r_{\mathcal{S}}$ structure on $\mathbb{C}$ is interpretable in $\mathbb{R}_{\mathrm{an}}$, and so is (essentially) an o-minimal structure. It would be possible to study the analytic theory in the K-analytic setting of Peterzil and Starchenko (see for example [PS01]) but the notion of analytically closed subfield of $\mathbb{C}$ gives an alternative view which is closer to complex analysis than real analysis.

8.2 Blurred exponentiation

Studying the exponential differential equations is like studying the exponential functions themselves but with the number theoretic details "blurred" by the field of constants. Here we make this idea precise.

Fix a countable collection $\mathcal{S}$ of semiabelian varieties defined over $\mathbb{C}$, and a subfield C of $\mathbb{C}$. For each $S \in \mathcal{S}$, define the *blurred graph* with respect to C, $\mathcal{B}_S$, of the exponential function of S to be

$$\mathcal{B}_S = \{(x, y) \in (\mathbb{G}_a^n \times S)(\mathbb{C}) \mid \exp_S(x) \ominus y \in S(C)\}$$

where $n = \dim S$. We may assume that $\mathcal{S}$ is closed under isogeny and under taking products and algebraic subgroups, because the blurred graphs (or indeed graphs) of the exponential functions of these will be definable from those of the original groups.

Conjecture 8.4. *If C is a countable subfield of $\mathbb{C}$ which is analytically closed with respect to $\mathcal{S}$, the structure $\langle \mathbb{C}; +, \cdot, C, (\mathcal{B}_S)_{S\in\mathcal{S}}\rangle$ is elementarily equivalent to the reduct $\langle F; +, \cdot, C, (\Gamma_S)_{S\in\mathcal{S}}\rangle$ of a differentially closed field.*

The theory $T_{\mathcal{S}}$ of $\langle F; +, \cdot, C, (\Gamma_S)_{S\in\mathcal{S}}\rangle$ is axiomatized by A1—A7 + USC + EC. It is relatively straightforward to show that blurred exponentiation satisfies the algebraic axioms A1—A7 and the uniform Schanuel condition USC.

Proposition 8.5. *Blurred exponentiation satisfies the algebraic axioms A1—A7.*

Proof. A1, A2, A4, and A7 are immediate. The exponential function of S is an $\mathrm{End}(S)$-module homomorphism from $\mathbb{G}_{\mathrm{a}}^n$ to S, and so its graph $\mathcal{G}$ is an $\mathrm{End}(S)$-submodule of $\mathbb{G}_{\mathrm{a}}^n \times S$. The definition of $\mathcal{B}_S$, together with the fact that $G(C)$ is also an $\mathrm{End}(S)$-submodule of $G(\mathbb{C})$, gives A3, A5 and A6. $\square$

Proposition 8.6. *Blurred exponentiation satisfies USC.*

Proof. Let $(V_p)_{p\in P}$ be a parametric family of subvarieties of $G = \mathbb{G}_{\mathrm{a}}^n \times S$, defined over C. Suppose that $(a, b) \in V_p \cap \mathcal{B}_S$ for some $p \in P(\mathbb{C})$ such that $\dim V_p < n + 1$. We must show that for some $H \in \mathcal{H}_V^S$, there is $\gamma \in S(C)$ such that $b \in \gamma \oplus H$.

Let $c = \exp(a) \ominus b$ and define $V_p' = \{(x, y) \in G \mid (x, y \ominus c) \in V_p\}$. Then V_p' and V_p are definably isomorphic and so $\dim V_p' = \dim V_p < n + 1$. Let W be the connected component of $V_p' \cap \mathcal{B}_S$ containing $(a, \exp_S(a))$. If $\dim W = 0$ then W is the singleton $\{(a, \exp_S(a))\}$. Now $V_p' \cap \mathcal{B}_S$ is an analytic variety definable in $\mathbb{C}_{\mathcal{S}}$ with parameters from C, and C is analytically closed with respect to $\mathcal{S}$, so $(a, \exp_S(a)) \in G(C)$. Also $c \in S(C)$ by definition of $\mathcal{B}_S$, and so $(a, b) \in G(C)$ which is good enough.

Otherwise, $\dim W > 0 \geqslant \dim V_p' - n$ and so by theorem 8.1, there is $H \in \mathcal{H}_{V'}^S$ and $g \in G(\mathbb{C})$ such that $W \subseteq g \oplus \widehat{H}$. The subgroup $\widehat{H}$ of G is defined by some polynomial function $\chi_{\widehat{H}}(x, y) = 0$, and the coset $g \oplus \widehat{H}$ is defined by $\chi_{\widehat{H}}(x, y) = g'$ for some g' such that $g' \oplus \widehat{H} = g \oplus \widehat{H}$. By analytic closedness of C, W contains a C-point, say w, and thus $g' = \chi_{\widehat{H}}(w)$. The polynomial function $\chi_{\widehat{H}}$ is defined over C because the semiabelian variety S and all its algebraic subgroups are, and so $g' \in G(C)$. Then $g \in (g' \ominus (0, c)) \oplus \widehat{H}$. Finally, observe that V_p' is the translation of V_p by a C-point of S, and so $\mathcal{H}_{V'}^S = \mathcal{H}_V^S$. Thus g lies in a C-coset of $\widehat{H}$ for some $H \in \mathcal{H}_V^S$ as required. $\square$

The fact that C is analytically closed is not required in the definition of the structure of blurred exponentiation, nor in the proof of the algebraic axioms where it can be replaced by algebraic closedness of C and all of the $S \in \mathcal{S}$ being defined over C. It is natural to ask if analytic closedness of C is necessary for the USC axiom scheme to be true. More particularly, one would like to have a characterization of all those subfields of $\mathbb{C}$ for which USC holds. The notion of analytically closed subfield really just gives the existence of a countable such subfield. The construction is not canonical because there is no canonical choice of representatives of the connected components. However, once we have one subfield for which USC holds, it is possible to characterize all the larger subfields for which it holds.

Given such a C, define a predimension function δ on $\mathbb{C}$ exactly as in definitions 6.7 and 6.8 of chapter 6, with $\mathcal{B}_S$ in place of Γ_S. This defines a pregeometry on $\mathbb{C}$.

Proposition 8.7. *Suppose C is a subfield of $\mathbb{C}$ such that the blurred exponentiation structure $\langle \mathbb{C}; +, \cdot, C, (\mathcal{B}_S)_{S\in\mathcal{S}}\rangle$ satisfies the axioms A1—A7 and USC, and that C' is an algebraically closed intermediate field $C \subseteq C' \subseteq \mathbb{C}$. Then the exponentiation structure blurred with respect to C' satisfies USC iff C' is closed in the sense of the pregeometry defined by δ, that is, $d(a/C') = 0 \iff a \in C'$.*

Proof. Suppose there is a such that $d(a/C') = 0$ but $a \notin C'$. Then there is b extending a with $\delta(b/C') = \mathrm{td}(b/C') - \mathrm{grk}(b/C') = 0$. Take b to be minimal such, and then $b \in \gamma \oplus \mathcal{B}_S$ for some $S \in \mathcal{S}$ and some $\gamma \in \widehat{S}(C')$. By minimality, b does not lie in a C'-coset of $\widehat{H}$ for any proper algebraic subgroup H of S, and so $\mathrm{td}(b/C') = \mathrm{grk}(b/C') = \dim S$ which means that USC is not satisfied relative to C'.

Conversely, suppose that $d(a/C') = 0 \implies a \in C'$. Then, in particular, $\delta(a/C') = 0 \implies a \in C'$, which means that the Schanuel condition holds relative to C'. □

Having discussed A1—A7 and USC, conjecture 8.4 reduces to showing that the existentially closed axiom scheme EC holds. This cannot be done by the simple methods giving the other axioms because any proof would have to use the properties which define the complex field, in particular the Euclidean topology. It seems likely that the methods of Boris Zilber's paper [Zil04a] can be adapted to prove this, but at the time of writing the conjecture remains open.

8.3 Blurred pseudo-exponentiation

For the usual exponential function of $\mathbb{G}_{\mathrm{m}}$, Boris Zilber constructed in [Zil05b] a "pseudo-exponentiation" structure $\mathcal{K} = \langle K; +, \cdot, \mathrm{ex}\rangle$, where K is an algebraically closed field of characteristic zero and cardinality $2^{\aleph_0}$ and ex is a group homomorphism $\mathbb{G}_{\mathrm{a}}(K) \longrightarrow \mathbb{G}_{\mathrm{m}}(K)$. The structure $\mathcal{K}$ satisfies the obvious properties of exp together with the conclusion of Schanuel's conjecture, a corresponding existentially closed condition, and a countable closure property for the pregeometry arising from the Schanuel condition (the closure of a finite subset is countable). Furthermore, Zilber gave an $\mathcal{L}_{\omega_1\omega}(Q)$-sentence which is satisfied by this structure and which is categorical in all uncountable cardinals. See also [Mar05] for a briefer description of the structure and the theory. A natural, but very strong, conjecture is that this pseudo-exponentiation is isomorphic to the usual complex exponentiation structure $\mathbb{C}_{\mathrm{exp}}$. It seems that even the construction of pseudo-exponentiation for other semiabelian varieties is dependent on some diophantine questions, and so Zilber has not achieved this.

The conjecture that complex exponentiation and pseudo-exponentiation are isomorphic can be split into two parts: a number theoretic part and a geometric part. The number-theoretic part says that suitable subfields of $\mathcal{K}$ and $\mathbb{C}_{\mathrm{exp}}$ consisting of the "exponentially algebraic" elements are isomorphic. This is at least as strong as Schanuel's conjecture. (In fact, if Schanuel's conjecture is false then the notion of "exponentially algebraic" may not even make sense here. This is connected with the problem mentioned above of finding a minimal algebraically closed subfield of $\mathbb{C}$ for which USC holds.) The geometric part of the conjecture is that if both $\mathcal{K}$ and $\mathbb{C}_{\mathrm{exp}}$ are blurred over suitable subfields then the resulting structures are isomorphic.

Theorem 8.8. *Let C be a countable subfield of K which is closed in the pregeometry arising from the Schanuel condition, that is, $a \in C \iff d(a/C) = 0$. Define the blurred graph of pseudo-exponentiation by*

$$\mathcal{B} = \{(x, y) \in (\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}})(K) \mid \mathrm{ex}(x)/y \in C\}.$$

Then the first order theory of the structure $\langle K; +, \cdot, C, \mathcal{B}\rangle$ is $T_{\mathcal{S}}$.

Sketch proof. The algebraic axioms A1—A7 are immediate and USC is very quick, using the equivalent statement for pseudo-exponentiation and $x \in C \iff d(x/C) = 0$. The SEC property can also be seen to hold, relativizing the strong exponential closedness property given in [Mar05]. $\square$

The ultimate goal of this line of research would be to prove the geometric part of the conjecture that complex exponentiation and pseudo-exponentiation are isomorphic.

Conjecture 8.9. *The structures $\langle \mathbb{C}; +, \cdot, C, \mathcal{B}\rangle$ and $\langle K; +, \cdot, C, \mathcal{B}\rangle$, blurred complex exponentiation and blurred pseudo-exponentiation, are isomorphic.*

Conjecture 8.4 would establish elementary equivalence, and both structures have the countable closure property, so it seems likely that this conjecture would follow from the excellence property of pseudo-exponentiation.

Bibliography

[Ax71] James Ax. On Schanuel's conjectures. *Ann. of Math. (2)*, **93**:pp. 252–268, 1971.

[Ax72] James Ax. Some topics in differential algebraic geometry. I: Analytic subgroups of algebraic groups. *Amer. J. Math.*, **94**:pp. 1195–1204, 1972.

[Bak75] Alan Baker. *Transcendental number theory.* Cambridge University Press, London, 1975.

[Ber02] Cristiana Bertolin. Périodes de 1-motifs et transcendance. *J. Number Theory*, **97**(2):pp. 204–221, 2002.

[Ber06] Daniel Bertrand. Schanuel's conjecture for non-isoconstant elliptic curves over function fields, 2006. Preprint.

[BHMPW06] A. Baudisch, M. Hils, A. Martin-Pizarro, and F. O. Wagner. Die Böse Farbe, 2006. Preprint, URL `http://math.univ-lyon1.fr/~wagner/publ.html`.

[BK77] W. Dale Brownawell and K. K. Kubota. The algebraic independence of Weierstrass functions and some related numbers. *Acta Arithmetica*, **XXXIII**:pp. 111–149, 1977.

[BL04] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2004.

[BMZ05] E. Bombieri, D. Masser, and U. Zannier. Anomalous subvarieties – structure theorems and applications, 2005. Preprint.

[BMZ06] E. Bombieri, D. Masser, and U. Zannier. Intersecting curves and algebraic subgroups: conjectures and more results. *Trans. Amer. Math. Soc.*, **358**(5):pp. 2247–2257 (electronic), 2006.

[Col80] Robert F. Coleman. On a stronger version of the Schanuel-Ax theorem. *Amer. J. Math.*, **102**(4):pp. 595–624, 1980.

[Cra06] Cecily Crampin. *Reducts of differentially closed fields to fields with a relation for exponentiation*. DPhil thesis, University of Oxford, 2006.

[DG92] Manfred Droste and Rüdiger Göbel. A categorical theorem on universal objects and its application in abelian group theory and computer science. *Contemporary Mathematics*, **131 (Part 3)**:pp. 49–74, 1992.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[HH05] Assaf Hasson and Ehud Hrushovski. DMP in strongly minimal structures, 2005. Preprint.

[Hru92] Ehud Hrushovski. Strongly minimal expansions of algebraically closed fields. *Israel J. Math.*, **79**(2-3):pp. 129–151, 1992.

[Hru93] Ehud Hrushovski. A new strongly minimal set. *Ann. Pure Appl. Logic*, **62**(2):pp. 147–166, 1993.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975.

[Kir05a] Jonathan Kirby. Exponential and Weierstrass equations, April 2005. Presented at the workshop: An Introduction to Recent Applications of Model Theory, URL `http://www.newton.cam.ac.uk/preprints/NI05047.pdf`.

[Kir05b] Jonathan Kirby. A Schanuel condition for Weierstrass equations. *J. Symbolic Logic*, **70**(2):pp. 631–638 & 1023, 2005.

[Kol53] E. R. Kolchin. Galois theory of differential fields. *Amer. J. Math.*, **75**:pp. 753–824, 1953.

[Kol68] E. R. Kolchin. Algebraic groups and algebraic dependence. *Amer. J. Math.*, **90**:pp. 1151–1164, 1968.

[Lan66] Serge Lang. *Introduction to transcendental numbers*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[Mar96] David Marker. Model theory of differential fields. In David Marker, Margit Messmer, and Anand Pillay, eds., *Model Theory of Fields*, number 5 in Lecture Notes in Logic. Springer, 1996.

[Mar00] David Marker. Manin kernels. In *Connections between model theory and algebraic and analytic geometry*, volume 6 of *Quad. Mat.*, pp. 1–21. Dept. Math., Seconda Univ. Napoli, Caserta, 2000.

[Mar02] David Marker. *Model Theory: An Introduction*. Number 217 in GTM. Springer, 2002.

[Mar05] David Marker. A remark on Zilber's pseudoexponentiation, 2005. Preprint, URL `http://www.newton.cam.ac.uk/preprints/NI05008.pdf`.

[Mat72] Yozo Matsushima. *Differentiable manifolds*, volume 9 of *Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1972.

[Pie03] David Pierce. Differential forms in the model theory of differential fields. *Journal of Symbolic Logic*, **68**(3):pp. 923–945, 2003.

[Pil04] Anand Pillay. Algebraic D-groups and differential Galois theory. *Pacific J. Math.*, **216**(2):pp. 343–360, 2004.

[Pin05] Richard Pink. A combination of the conjectures of Mordell-Lang and André-Oort. In *Geometric methods in algebra and number theory*, volume 235 of *Progr. Math.*, pp. 251–282. Birkhäuser Boston, Boston, MA, 2005.

[Poi01] Bruno Poizat. L'égalité au cube. *J. Symbolic Logic*, **66**(4):pp. 1647–1676, 2001.

[PS01] Ya'acov Peterzil and Sergei Starchenko. Expansions of algebraically closed fields in o-minimal structures. *Selecta Math. (N.S.)*, **7**(3):pp. 409–445, 2001.

[Sei58] A. Seidenberg. Abstract differential algebra and the analytic case. *Proceedings of the American Mathematical Society*, **9**(1):pp. 159–164, 1958.

[Sei69] A. Seidenberg. Abstract differential algebra and the analytic case II. *Proceedings of the American Mathematical Society*, **23**(3):pp. 689–691, 1969.

[Ser88] Jean-Pierre Serre. *Algebraic groups and class fields*, volume 117 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.

[Sha94a] Igor R. Shafarevich. *Basic algebraic geometry. 1.* Springer-Verlag, Berlin, 1994.

[Sha94b] Igor R. Shafarevich. *Basic algebraic geometry. 2.* Springer-Verlag, Berlin, 1994.

[Sil92] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992.

[War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983.

[Wil03a] A. J. Wilkie. Lecture notes on Marker's lecture notes on Ax's theorem, February 2003. Logic Advanced Class, Oxford.

[Wil03b] A. J. Wilkie. A remark on Schanuel's conjecture, September 2003. Logic Advanced Class, Oxford.

[Wil03c] A. J. Wilkie. Further remarks on Schanuel's conjecture, October 2003. Logic Advanced Class, Oxford.

[Zil84] Boris Zilber. The structure of models of uncountably categorical theories. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pp. 359–368. PWN, Warsaw, 1984.

[Zil00a] Boris Zilber. Dimensions and homogeneity in mathematical structures. In *Connections between model theory and algebraic and analytic geometry*, volume 6 of *Quad. Mat.*, pp. 131–148. Dept. Math., Seconda Univ. Napoli, Caserta, 2000.

[Zil00b] Boris Zilber. Fields with pseudo-exponentiation, 2000. `arXiv:math.LO/0012023`.

[Zil02a] Boris Zilber. Exponential sums equations and the Schanuel conjecture. *J. London Math. Soc. (2)*, **65**(1):pp. 27–44, 2002.

[Zil02b] Boris Zilber. Model theory, geometry and arithmetic of the universal cover of a semi-abelian variety. In *Model theory and applications*, volume 11 of *Quad. Mat.*, pp. 427–458. Aracne, Rome, 2002.

[Zil03] Boris Zilber. Raising to powers in algebraically closed fields. *J. Math. Log.*, **3**(2):pp. 217–238, 2003.

[Zil04a] Boris Zilber. Bi-coloured fields on the complex numbers. *J. Symbolic Logic*, **69**(4):pp. 1171–1186, 2004.

[Zil04b] Boris Zilber. Covers of the multiplicative group of an algebraically closed field of characteristic zero, 2004. To appear in the Journal of the LMS, `arXiv:math.AC/0401301`.

[Zil04c] Boris Zilber. Raising to powers revisited, 2004. Preprint, URL `http://www.maths.ox.ac.uk/~zilber`.

[Zil05a] Boris Zilber. Analytic and pseudo-analytic structures. In *Logic Colloquium 2000*, volume 19 of *Lect. Notes Log.*, pp. 392–408. Assoc. Symbol. Logic, Urbana, IL, 2005. `arXiv:math.LO/0401303`.

[Zil05b] Boris Zilber. Pseudo-exponentiation on algebraically closed fields of characteristic zero. *Ann. Pure Appl. Logic*, **132**(1):pp. 67–95, 2005.